Technical Appendix to "Optimal Fiscal and Monetary Policy under Sectorial Heterogeneity"

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1 Appendix A - The Firms' Problem

Noting that $\theta > 1$, FOC from firms' optimization problem is given by:

$$E_t \sum_{j=t}^{\infty} \alpha_k^{j-t} \Theta_{t,j} \frac{\partial \Psi_j \left(p_{k,t}\left(z\right),\ .\right)}{\partial p_{k,t}\left(z\right)} = 0; \tag{1}$$

taking derivatives and dividing resulting expression by $1-\theta$

$$E_t \sum_{j=t}^{\infty} \alpha_k^{j-t} \Theta_{t,j} \frac{p_{k,t}(z)}{P_{k,t+j}}^{-\theta} Y_{k,j} \{(1-\tau_{k,j}) + \frac{\theta}{1-\theta} \frac{w_{k,j}(z)}{a_{k,j}} \frac{P_j}{P_{k,j}} \frac{P_{k,j}}{p_{k,t}(z)}\} = 0;$$

using expression in the main text for labor supply, production function and discount factor:

$$E_t \sum_{j=t}^{\infty} (\alpha_k \beta)^{j-t} \frac{C_j^{-\sigma}}{P_j} \frac{p_{k,t}(z)}{P_{k,j}}^{-\theta} Y_{k,j} \{(1-\tau_{k,j}) + \\ + \mu_{k,j}^{w} \frac{\theta\lambda}{1-\theta} \frac{y_{k,j}(z)^{\nu}}{C_j^{-\sigma}} \frac{1}{a_{k,j}^{\nu+1}} \frac{P_j}{P_{k,j}} \frac{P_{k,j}}{p_{k,t}(z)}\} = 0;$$

using expression for demand for good z in terms of sectorial aggregates and isolating terms $p_{k,t}(z)/P_{k,t}$.

$$\frac{p_{k,t}(z)}{P_{k,t}}^{1+\theta\nu} E_t \sum_{j=t}^{\infty} (\alpha_k \beta)^{j-t} \frac{C_j^{-\sigma}}{P_j} \frac{P_{k,t}}{P_{k,j}}^{-\theta} Y_{k,j} \left(1-\tau_{k,j}\right) = \\ \mu_{k,j}^{w} \frac{\theta\lambda}{\theta-1} m_k^{-\nu} E_t \sum_{j=t}^{\infty} (\alpha_k \beta)^{j-t} \frac{P_{k,j}}{P_{k,t}}^{1+\theta(\nu+1)} \frac{Y_{k,j}}{a_{k,j}}^{\nu+1} \frac{1}{P_{k,j}}$$

$$\frac{p_{k,t}(z)}{P_{k,t}}^{1+\theta\nu} = \frac{\frac{\theta\lambda}{\theta-1} m_k^{-\nu} E_t \sum_{j=t}^{\infty} \left(\alpha_k \beta\right)^{j-t} \mu_{k,j}^{w} \frac{P_{k,j}}{P_{k,t}}^{\theta(\nu+1)} \frac{Y_{k,j}}{a_{k,j}}^{\nu+1}}{E_t \sum_{j=t}^{\infty} \left(\alpha_k \beta\right)^{j-t} \left(1-\tau_{k,j}\right) C_j^{-\sigma} \frac{P_{k,j}}{P_{k,t}}^{\theta-1} p_{k,j} Y_{k,j}} \tag{2}$$

2 Appendix B - Steady State

This section shows that there is a steady state characterized by zero inflation and constant values for all variables, where exogenous disturbances also assume constant values, that is: $\bar{\xi} = \{\bar{G},\ \bar{a}_{k,t}\}$, where $\bar{G} > 0$ and $\bar{a}_{k,t} = 1$, all k. We focus particular attention to a steady state with positive real debt at maturity, that is $b_{-1}^{*} = \bar{b}^{*} > \dot{0}$, price dispersion equals one, $\Delta_{k,-1} = 1$, and relative price also equals one, $p_{k,-1} = 1$, all k. While $\bar{b}^{*}$ is arbitrary, it is nonetheless subject to a upper bound. To see this, take the government budget constraint, which in steady state is given by:

$$(1-\beta)\bar{b}^* = \sum_{k=1}^{K} \bar{\tau}_k \bar{Y}_k - \bar{G}. \tag{3}$$

Assuming debt and government expenses are non-zero in steady state imply $\bar{\tau}_k > 0$, for some k. Also, given $p_{k,-1} = 1$ and zero inflation, all k, then $\bar{p}_k = 1$. From demand for sectorial output in terms of aggregate output, $\bar{Y}_k = m_k \bar{Y}$, which imply (3) becomes

$$(1-\beta)\bar{b}^* + \bar{G} = \bar{\tau}\bar{Y}, \tag{4}$$

where $\bar{\tau} = \sum_{k=1}^{K} m_k \bar{\tau}_k$, once steady state values are properly replaced. From firms' maximizing conditions in the main text and taking into account that $\bar{\Pi}_k = 1$,

$$\bar{K}_k = \bar{F}_k;$$

using definitions

$$\frac{\theta\lambda}{\theta-1}\bar{\mu}_k^w m_k^{-\nu}\bar{Y}_k^{\nu} = (1-\bar{\tau}_k)\left(\bar{C}\right)^{-\sigma}, \tag{5}$$

where we have used the fact that $\bar{p}_k = 1$, $\bar{a}_k = 1$, and $\bar{Y}_k = m_k\bar{Y}$. Sectorial tax rate is given by

$$\bar{\tau}_k = 1 - \frac{\theta\lambda}{\theta-1}\bar{\mu}_k^w \left(\bar{C}\right)^{\sigma} \bar{Y}^{\nu}, \tag{6}$$

which only depends of aggregate variables and sector specific parameter $\bar{\mu}_k^w$. We assume that steady-state wage markup is the same across sectors, that is $\bar{\mu}_k^w = \bar{\mu}^w$, all k. In this case, steady-state distortive tax rates are the same across sectors, that is

$$\bar{\tau}_k = \bar{\tau}, \tag{7}$$

all k, which is positive whenever

$$\frac{\theta\lambda}{\theta-1}\bar{\mu}^w \left(\bar{C}\right)^{\sigma} < 1,$$

or

$$\bar{C} < \left(\frac{\theta-1}{\theta\lambda}\frac{1}{\bar{\mu}^w}\right)^{\sigma},$$

once one considers an always-possible normalization $\bar{Y} = 1$; that is, the level of consumption over GDP should not be too high. Considering in a more concrete fashion, for the parameter values used in our calibration, that is $\theta = 10$, $\lambda = .98$,

$\bar{\mu}^w = 1.05$, and $\sigma = 2$, the steady state value for $\bar{C}$ should not be larger than 76% of the GDP. We believe it does not represent a significant restriction. Equations

$$\frac{\theta\lambda}{\theta - 1}\bar{\mu}^w \bar{Y}^\nu = (1 - \bar{\tau})\left(\bar{Y} - \bar{G}\right)^{-\sigma} \tag{8}$$

and (4) define the aggregate output level in steady state as well as the aggregate tax rate. In (8) steady state output $\bar{Y}$ is a negative function of steady state aggregate tax rate, industry and wage markups and a positive function of steady state government purchase. If $\bar{\tau}$ equals unity, than $\bar{Y}$ is zero and so the government revenue. Then, it should be the case that $\bar{\tau} < 1$. Once $\bar{\tau}$ is bounded above, so should be $\bar{G}$ and $\bar{b}^*$ for (4) to hold. On the other hand, as $\bar{G}$ and $\bar{b}^*$ are both greater than zero by hypothesis, then it is clear that $\bar{\tau} > 0$. Let $\bar{Y}_1(\bar{\tau})$ and $\bar{Y}_2(\bar{\tau})$ be the aggregate output defined respectively by equations (8) and (4), both functions of aggregate tax rate. From (8), one should notice that $\bar{Y}_1(\bar{\tau})$ approaches $\bar{G}$ as $\bar{\tau}$ approaches unity. In this case, there should be a range of aggregate tax rates $0 < \bar{\tau}' < \bar{\tau}^* < \bar{\tau}'' < 1$ such as $\bar{Y}_1(\bar{\tau}^*) > \bar{G}/\bar{\tau}^*$, provided $\bar{G}$ are small enough. Fixing $\bar{G}$ and $\bar{b}^*$, (4) uniquely determines $\bar{Y}_2$, for any $\bar{\tau}^{**} \in (0, 1)$. For $\bar{b}^*$ small enough, than it should be that $Y_2(\bar{\tau}^{**}) < Y_1(\bar{\tau}^*)$ for some values of $\bar{\tau}^{**} \in (0, 1)$. If, however, $\bar{\tau}^{**}$ is too small, then $Y_2(\bar{\tau}^{**}) > Y_1(\bar{\tau}^*)$.By continuity, it should be the case that $Y_2(\bar{\tau}) = Y_1(\bar{\tau})$ for a $\bar{\tau} \in (0, 1)$, which determine the aggregate level of tax rate and output. Using (7), above conditions hold for every k. Finally, from definitions for K_k, F_k and government budget constraint in recursive terms one can define steady state values for $\bar{K}_k$, $\bar{F}_k$ and $\bar{W}$, which complete the characterization of the steady state values.

Define the set of commitments $X_t = \{K_{k,t},\ F_{k,t},\ W_t\}$, all k, and let X_0 be the set of initial commitments that make policy optimal form a timeless perspective. We wish to characterize a steady state by a constant policy and set of initial commitments, constant debt level and tax rates, constant aggregate and sectorial outputs, relative prices as sectorial price dispersions equal to their initial values, that is: one. The centralized policy maker chooses a sequence of $\mathcal{X}_t = \{\Pi_t,\ \Pi_{k,t},\ Y_t,\ Y_{k,t},\ F_{k,t},\ K_{k,t},\ W_t,\ \Delta_{k,t},\ \tau_{k,t},\ b_t^*,\ p_{k,t}\}$, all k, for $t \geq t_0$ in order to maximize

$$U_t \equiv E_{t_0} \sum_{t=t_0}^{\infty} \beta^{j-t_0} \left[u\left(Y_t, \xi_t\right) - \sum_{k=1}^{K} m_k v\left(Y_{k,t}, \xi_t\right) \Delta_{k.t} \right], \tag{9}$$

where

$$u(Y_t, \xi_t) \equiv \frac{(Y_t - G_t)^{1-\sigma}}{1 - \sigma}$$

$$v\left(Y_{k,t}, \xi_t\right) \equiv \frac{\lambda}{1 + \nu} \left[\frac{Y_{k,t}}{m_k a_{k,t}} \right]^{1+\nu}$$

subject to:

$$\Delta_{k.t} = \alpha_k \Pi_{k,t}^{\theta(1+\nu)} \Delta_{k.t-1} + (1-\alpha_k)\left(\frac{1-\alpha_k \Pi_{k,t}^{\theta-1}}{1-\alpha_k}\right)^{\frac{\theta(1+\nu)}{\theta-1}} \tag{10}$$

$$K_{k,t}\left(\frac{1-\alpha_k \Pi_{k,t}^{\theta-1}}{1-\alpha_k}\right)^{\frac{1+\theta\nu}{1-\theta}} = F_{k,t} \tag{11}$$

$$\Pi_t^{1-\eta} = \sum_{k=1}^{K} m_k \left(\Pi_{k,t} p_{k,t-1}\right)^{1-\eta} \tag{12}$$

$$p_{k,t} = \frac{\Pi_{k,t}}{\Pi_t} p_{k,t-1} \tag{13}$$

$$p_{k,t}^{\eta} = m_k \frac{Y_t}{Y_{k,t}} \tag{14}$$

$$W_t = (Y_t - G_t)^{-\sigma} [\sum_{k=1}^{K} \tau_{k,t} p_{k,t} Y_{k,t} - G_t] + \beta E_t \left[W_{t+1}\right] \tag{15}$$

and the definitions:

$$K_{k,t} = \frac{\theta\lambda}{\theta-1} \bar{\mu}^w m_k^{-\nu} \frac{Y_{k,t}}{a_{k,t}}^{\nu+1} + \alpha_k \beta E_t \left[\Pi_{k,t+1}^{\theta(\nu+1)} K_{k,t+1}\right] \tag{16}$$

$$F_{k,t} = (1-\tau_{k,t})(Y_t - G_t)^{-\sigma} p_{k,t} Y_{k,t} + \alpha_k \beta E_t \left[\Pi_{k,t+1}^{\theta-1} F_{k,t+1}\right] \tag{17}$$

$$W_t = \frac{(Y_t - G_t)^{-\sigma}}{\Pi_t} b_{t-1}^{*} \tag{18}$$

and taking as given the initial commitments X_0 and the initial conditions $\mathcal{I}_{-1} = \{b_{-1}^{*},\ \Delta_{k,-1},\ p_{k,-1}\}$ for every k and $t \geq t_0$. In order to impose constant commitments $X_0 = \bar{X}$ we consider additional restrictions such as the first order conditions for the problem in $t = t_0$ are equivalent to the first order conditions for a generic $t > 0$. Let $\phi_{k,t}^1$,$\phi_{k,t}^2$,ϕ_t^3, $\phi_{k,t}^4$, $\phi_{k,t}^5$, ϕ_t^6, $\phi_{k,t}^7$, $\phi_{k,t}^8$, ϕ_t^9 be the Lagrange multipliers corresponding to equations (10) to (18). In order to complete the proof, we need to show that first order conditions for the indicated steady state are satisfied for time-invariant Lagrange multipliers. The first order conditions to the maximization problem are the following.

With respect $\Delta_{k,t}$

$$-m_k v\left(Y_{k,t}, \xi_t\right) + \phi_{k,t}^1 - \phi_{k,t+1}^1 \beta \alpha_k \Pi_{k,t+1}^{\theta(1+\nu)} = 0.$$

With respect to $K_{k,t}$

$$\phi^2_{k,t}\left[\frac{1-\alpha_k\Pi^{\theta-1}_{k,t}}{1-\alpha_k}\right]^{\frac{1+\theta\nu}{1-\theta}}+\phi^7_{k,t}-\phi^7_{k,t-1}\alpha_k\Pi^{\theta(\nu+1)}_{k,t}=0.$$

With respect to $F_{k,t}$

$$-\phi^2_{k,t}+\phi^8_{k,t}-\phi^8_{k,t-1}\alpha_k\Pi^{\theta-1}_{k,t}=0.$$

With respect to W_t

$$\phi^6_t-\phi^6_{t-1}+\phi^9_t=0$$

With respect to $\tau_{k,t}$

$$-\phi^6_t+\phi^8_{k,t}=0.$$

With respect to b^*_t

$$\phi^9_t=0.$$

With respect to $\Pi_{k,t}$

$$\begin{aligned}
&-\phi^1_{k,t}\alpha_k\theta(1+\nu)\Pi^{\theta(1+\nu)-1}_{k,t}\Delta_{k.t-1}+\\
&\quad-\phi^1_{k,t}(1-\alpha_k)\frac{\theta(1+\nu)}{\theta-1}\left[\frac{1-\alpha_k\Pi^{\theta-1}_{k,t}}{1-\alpha_k}\right]^{\frac{\theta(1+\nu)}{\theta-1}-1}\left[\frac{-\alpha_k(\theta-1)\Pi^{\theta-2}_{k,t}}{1-\alpha_k}\right]+\\
&\quad+\phi^2_{k,t}K_{k,t}\frac{\theta(1+\nu)}{\theta-1}\left[\frac{1-\alpha_k\Pi^{\theta-1}_{k,t}}{1-\alpha_k}\right]^{\frac{\theta(1+\nu)}{\theta-1}-1}\left[\frac{-\alpha_k(\theta-1)\Pi^{\theta-2}_{k,t}}{1-\alpha_k}\right]+\\
&-\phi^3_t m_k p^{1-\eta}_{k,t-1}(1-\eta)\Pi^{-\eta}_{k,t}-\phi^4_{k,t}p_{k,t-1}/\Pi_t-\phi^7_{k,t-1}\alpha_k\theta(\nu+1)\Pi^{\theta(\nu+1)-1}_{k,t}K_{k,t}\\
&\qquad-\phi^8_{k,t-1}\alpha_k(\theta-1)\Pi^{\theta-2}_{k,t}F_{k,t}=0.
\end{aligned}$$

With respect to Π_t

$$\phi^3_t(1-\eta)\Pi^{-\eta}_t+\sum_{k=1}^{K}\phi^4_{k,t}\frac{\Pi_{k,t}p_{k,t-1}}{\Pi^2_t}+\phi^9_t\frac{(Y_t-G_t)^{-\sigma}}{\Pi^2_t}b^*_{t-1}=0.$$

With respect to Y_t

$$u_Y(Y_t,\xi_t) - \sum_{k=1}^{K}\phi_{k,t}^5\frac{m_k}{Y_{k,t}}+$$
$$+\phi_t^6\sigma\left(Y_t-G_t\right)^{-\sigma-1}[\sum_{k=1}^{K}\tau_{k,t}p_{k,t}Y_{k,t}-G_t]+$$
$$+\phi_{k,t}^8\left(1-\tau_{k,t}\right)\sigma\left(Y_t-G_t\right)^{-\sigma-1}p_{k,t}Y_{k,t}+$$
$$+\phi_t^9\frac{\sigma\left(Y_t-G_t\right)^{-\sigma-1}}{\Pi_t}b_{t-1}^*=0.$$

With respect to $Y_{k,t}$

$$-m_k v_{Y_k}\left(Y_{k,t},\xi_t\right)\Delta_{k.t}+\phi_{k,t}^5\frac{m_kY_t}{Y_{k,t}^2}+$$
$$-\phi_t^6\left(Y_t-G_t\right)^{-\sigma}\tau_{k,t}p_{k,t}-\phi_{k,t}^7\frac{\theta\lambda}{\theta-1}\bar{\mu}^w(\nu+1)m_k^{-\nu}\frac{Y_{k,t}}{a_{k,t}}^{\nu}\frac{1}{a_{k,t}}$$
$$-\phi_{k,t}^8\left(1-\tau_{k,t}\right)\left(Y_t-G_t\right)^{-\sigma}p_{k,t}=0.$$

With respect to $p_{k,t}$

$$-\phi_{k,t+1}^3 m_k(1-\eta)\beta\Pi_{k,t+1}^{1-\eta}\left(p_{k,t}\right)^{-\eta}+$$
$$+\phi_{k,t}^4-\phi_{k,t+1}^4\beta\frac{\Pi_{k,t+1}}{\Pi_{t+1}}+\phi_{k,t}^6p_{k,t}^{\eta}+$$
$$-\phi_{k,t}^7\left(Y_t-G_t\right)^{-\sigma}\tau_{k,t}Y_{k,t}+$$
$$-\phi_{k,t}^{10}\left(1-\tau_{k,t}\right)\left(Y_t-G_t\right)^{-\sigma}Y_{k,t}=0.$$

From FOC with respect to $\Delta_{k,t}$

$$\phi_k^1=\frac{\lambda}{1+\nu}\frac{m_k^{-\nu}\bar{Y}_k^{1+\nu}}{1-\beta\alpha_k}, \tag{19}$$

which solves for ϕ_k^1, all k. From FOC with respect to $K_{k,t}$ and $F_{k,t}$:

$$\phi_k^2=-\phi_k^7(1-\alpha_k), \tag{20}$$

$$\phi_k^2=\phi_k^8(1-\alpha_k), \tag{21}$$

which imply

$$-\phi_k^7=\phi_k^8. \tag{22}$$

Optimality with respect to W_t yields

$$\phi^9=0. \tag{23}$$

From FOC with respect to $\tau_{k,t}$ and b_t^* yield, respectively

$$\phi^6 = \phi_k^8, \tag{24}$$

and

$$\phi^9 = 0. \tag{25}$$

From FOC with respect to Π_t

$$(1-\eta)\phi^3 = -\sum_{k=1}^{K} \phi_k^4. \tag{26}$$

From FOC with respect to $\Pi_{k,t}$

$$- \phi_k^2 \bar{K}_k \frac{\alpha_k \theta(1+\nu)}{1-\alpha_k} - \phi^3 m_k(1-\eta) - \phi_k^4 + \\ - \phi_k^7 \alpha_k \theta(\nu+1)\bar{K}_k - \phi_k^8 \alpha_k(\theta-1)\bar{F}_k = 0;$$

using the relation (22)

$$-\phi_k^2 \frac{\alpha_k(\theta-1)}{1-\alpha_k} \bar{K}_k - \phi^3 m_k(1-\eta) - \phi_k^4 = 0,$$

$$-\sum_{k=1}^{K} \phi_k^2 \frac{\alpha_k(\theta-1)}{1-\alpha_k} \bar{K}_k = 0, \tag{27}$$

where $\bar{K}_k$ is given by:

$$\bar{K}_k = \frac{1}{1-\alpha_k\beta} \frac{\theta\lambda}{\theta-1} \bar{\mu}^w m_k^{-\nu} \bar{Y}_{k,t}{}^{\nu+1}. \tag{28}$$

FOC with respect to Y_t, and using (25)

$$u_Y(\bar{Y},\bar{\xi}) - \sum_{k=1}^{K} \phi_k^5 \frac{m_k}{\bar{Y}_k} + \phi^6 \sigma \left(\bar{Y} - \bar{G}\right)^{-\sigma-1} [\sum_{k=1}^{K} \bar{\tau}_k \bar{Y}_k - \bar{G}] \\ + \sum_{k=1}^{K} \phi_k^8 \left(1-\bar{\tau}_k\right) \sigma \left(\bar{Y} - \bar{G}\right)^{-\sigma-1} \bar{Y}_k = 0.$$

Using (24)

$$u_Y(\bar{Y},\bar{\xi}) - \frac{1}{\bar{Y}} \sum_{k=1}^{K} \phi_k^5 + \phi^6 \sigma \left(\bar{Y} - \bar{G}\right)^{-\sigma-1} [\sum_{k=1}^{K} \bar{Y}_k - \bar{G}] = 0,$$

$$\left(\bar{Y}-\bar{G}\right)^{-\sigma}+\phi^6\sigma\left(\bar{Y}-\bar{G}\right)^{-\sigma}=\frac{1}{\bar{Y}}\sum_{k=1}^{K}\phi_k^5. \tag{29}$$

FOC with respect to $Y_{k,t}$

$$-m_k^{-\nu}\lambda\bar{Y}_k^{\nu}+\phi_k^5\frac{1}{\bar{Y}_k}-\phi^6[\frac{\theta\lambda(\nu+1)}{\theta-1}\bar{\mu}^w m_k^{-\nu}\bar{Y}_k{}^{\nu}+\left(\bar{Y}-\bar{G}\right)^{-\sigma}]=0, \tag{30}$$

multiplying by m_k and using the definition for $\bar{Y}_k$

$$-m_k\lambda\bar{Y}^{\nu}+\phi_k^5\frac{1}{\bar{Y}}-\phi^6[\frac{\theta\lambda(\nu+1)}{\theta-1}\bar{\mu}^w m_k\bar{Y}^{\nu}+m_k\left(\bar{Y}-\bar{G}\right)^{-\sigma}]=0,$$

summing across sectors and using the relation (29) yields

$$-\lambda\bar{Y}^{\nu}+\left(\bar{Y}-\bar{G}\right)^{-\sigma}=\phi^6[\frac{\theta\lambda(\nu+1)}{\theta-1}\bar{\mu}^w\bar{Y}^{\nu}+(1-\sigma)\left(\bar{Y}-\bar{G}\right)^{-\sigma}].$$

It is then possible to establish the steady state value of ϕ^6 only as function of aggregate variables:

$$\phi^6=\frac{\left(\bar{Y}-\bar{G}\right)^{-\sigma}-\lambda\bar{Y}^{\nu}}{\frac{\theta\lambda(\nu+1)}{\theta-1}\bar{\mu}^w\bar{Y}^{\nu}+(1-\sigma)\left(\bar{Y}-\bar{G}\right)^{-\sigma}}, \tag{31}$$

Having determined the value for ϕ^6 allows us to determine $\sum_{k=1}^{K}\phi_k^5$ using (29), ϕ_k^5 using (30), ϕ_k^8 using (24), and ϕ_k^7 and ϕ_k^2 using respectively (22) and (21).

FOCs with respect to $p_{k,t}$ yields

$$\begin{aligned}&-\phi^3 m_k(1-\eta)\beta+\phi_k^4(1-\beta)+\phi_k^5-\\&\qquad-\phi^6\left(\bar{Y}-\bar{G}\right)^{-\sigma}\bar{\tau}_k\bar{Y}_k-\phi_k^8\left(1-\bar{\tau}_k\right)\left(\bar{Y}-\bar{G}\right)^{-\sigma}\bar{Y}_k=0,\end{aligned}$$

using (24)

$$-\phi^3 m_k(1-\eta)\beta+\phi_k^4(1-\beta)+\phi_k^5-\phi^6\left(\bar{Y}-\bar{G}\right)^{-\sigma}\bar{Y}m_k=0, \tag{32}$$

summing across sectors

$$-\phi^3(1-\eta)\beta+(1-\beta)\sum_{k=1}^{K}\phi_k^4+\sum_{k=1}^{K}\phi_k^5-\phi^6\left(\bar{Y}-\bar{G}\right)^{-\sigma}\bar{Y}=0, \tag{33}$$

using (26) and (29)

$$-\frac{(1-\eta)\phi^3}{\bar{Y}} + \left(\bar{Y}-\bar{G}\right)^{-\sigma} = \phi^6(1-\sigma)\left(\bar{Y}-\bar{G}\right)^{-\sigma}, \tag{34}$$

which solves for ϕ^3 as we use as a function only of aggregate variables, as we use (31). In this case, $\sum_{k=1}^{K}\phi_k^4$ can be determined by (26). Finally, ϕ_k^4 can be determined using (32). It follows the system is just determined which completes the proof.

3 Appendix C - Second Order Approximation to Utility Function

3.1 Second Order Approximation of Utility Function

We start with a second order Taylor expansion of the representative consumer's welfare function, along the lines of Woodford (2003).

$$U_0 \equiv E_0 \sum_{t=0}^{\infty} \beta^t \left[u\left(Y_t, \xi_t\right) - \sum_{k=1}^{K} m_k v\left(Y_{k,t}, \xi_t\right) \Delta_{k.t} \right], \tag{35}$$

where

$$u(Y_t, \xi_t) = \frac{(Y_t - G_t)^{1-\sigma}}{1-\sigma}, \tag{36}$$

$$v\left(Y_{k,t}, \xi_t\right) = \frac{\lambda}{1+\nu} \left[\frac{Y_{k,t}}{m_k a_{k,t}} \right]^{1+\nu}, \tag{37}$$

and where ξ_t refers to the full vector of random disturbances, as in Benigno and Woodford (2003). We start by working with $u\left(Y_t, \xi_t\right)$. A second order Taylor expansion over original expression yields

$$u\left(Y_t, \xi_t\right) = u_Y(\bar{Y}, \bar{\xi})(Y_t - \bar{Y}) + \frac{1}{2} u_{YY}(\bar{Y}, \bar{\xi})(Y_t - \bar{Y})^2 + \\ + u_{Y\xi}(\bar{Y}, \bar{\xi})(Y_t - \bar{Y})(\xi_t - \bar{\xi}) + tips + O_p^3,$$

where the term *tips* refers to terms independent of policy hereafter.

$$u\left(Y_t, \xi_t\right) = u_Y(\bar{Y}, \bar{\xi})\bar{Y}[\frac{(Y_t - \bar{Y})}{\bar{Y}} + \frac{1}{2} \frac{u_{YY}(\bar{Y}, \bar{\xi})\bar{Y}}{u_Y(\bar{Y}, \bar{\xi})} \frac{(Y_t - \bar{Y})^2}{\bar{Y}^2} + \\ + \frac{u_{Y\xi}(\bar{Y}, \bar{\xi})\bar{Y}}{u_Y(\bar{Y}, \bar{\xi})} \frac{(Y_t - \bar{Y})}{\bar{Y}} \frac{(\xi_t - \bar{\xi})}{\bar{Y}}] + tips + O_p^3.$$

Define hereafter, for any variable X_t

$$\tilde{X}_t \equiv \frac{X_t - \bar{X}}{\bar{X}}, \tag{38}$$

and

$$\hat{X}_t \equiv \log \frac{X_t}{\bar{X}}. \tag{39}$$

It is know that the following relation holds up to second order:

$$\tilde{X}_t \simeq \hat{X}_t + \frac{1}{2}\hat{X}_t^2. \tag{40}$$

Given the functional form assumed for the utility function, we have:

$$u\left(Y_t, \xi_t\right) = \bar{C}^{-\sigma}\bar{Y}[\tilde{Y}_t - \frac{\sigma}{2}\frac{\bar{Y}}{\bar{C}}\tilde{Y}_t^2 + \sigma\frac{\bar{Y}}{\bar{C}}\tilde{Y}_t\tilde{\xi}_t] + tips + O_p^3,$$

where $\hat{\xi}_t$ represents the absolute deviation *over GDP*. As G_t is the only random disturbance considered in this case, than it is clear that

$$\tilde{\xi}_t = \tilde{G}_t = \frac{G_t - \bar{G}}{\bar{Y}},$$

as an exception to (38). We define the ratio of consumption over output

$$s_C = \frac{\bar{C}}{\bar{Y}}, \tag{41}$$

and use (40), yielding

$$u\left(Y_t, \xi_t\right) = \bar{C}^{-\sigma}\bar{Y}[\hat{Y}_t + \frac{1}{2}\hat{Y}_t^2(1 - \sigma s_C^{-1}) + \sigma s_C^{-1}\hat{Y}_t\hat{G}_t] + tips + O_p^3. \tag{42}$$

A second order Taylor expansion of $v\left(Y_{k,t}, \xi_t\right)\Delta_{k.t}$ around steady state values yield

$$\begin{aligned} v\left(Y_{k,t}, \xi_t\right)\Delta_{k.t} = {} & v\left(\bar{Y}_k, \bar{\xi}\right)(\Delta_{k.t} - 1) + v_{Y_k}\left(\bar{Y}_k, \bar{\xi}\right)(Y_{k,t} - \bar{Y}_k) + \\ & + \frac{1}{2}v_{Y_kY_k}\left(\bar{Y}_k, \bar{\xi}\right)(Y_{k,t} - \bar{Y}_k)^2 + v_{Y_k}\left(\bar{Y}_k, \bar{\xi}\right)(Y_{k,t} - \bar{Y}_k)(\Delta_{k.t} - 1) \\ & + v_{Y_k\xi}\left(\bar{Y}_k, \bar{\xi}\right)(Y_{k,t} - \bar{Y}_k)(\xi_t - \bar{\xi}) + v_{\xi}\left(\bar{Y}_k, \bar{\xi}\right)(\Delta_{k.t} - 1)(\xi_t - \bar{\xi}) + \\ & + tips + O_p^3. \end{aligned} \tag{43}$$

Considering that in this component of utility function, the vector ξ_t contains only non-zero terms for disturbances $a_{k,t}$ and that $\bar{a}_k = 1$, all k, then

$$\tilde{a}_{k,t} = a_{k,t} - 1,$$

and also

$$\tilde{\Delta}_{k,t} = \Delta_{k,t} - 1.$$

Expression above (43) simplifies to

$$\begin{aligned} v\left(Y_{k,t},\xi_t\right)\Delta_{k.t} &= v\left(\bar{Y}_k,\bar{\xi}\right)\tilde{\Delta}_{k.t} + v_{Y_k}\left(\bar{Y}_k,\bar{\xi}\right)\bar{Y}_k(\hat{Y}_{k,t} + \frac{1}{2}\hat{Y}_{k,t}^2) + \\ &+\frac{1}{2}v_{Y_kY_k}\left(\bar{Y}_k,\bar{\xi}\right)\bar{Y}_k^2(\hat{Y}_{k,t}^2) + v_{Y_k}\left(\bar{Y}_k,\bar{\xi}\right)\bar{Y}_k(\hat{Y}_{k,t})\tilde{\Delta}_{k.t} + \\ &+v_{Y_k\xi}\left(\bar{Y}_k,\bar{\xi}\right)\bar{Y}_k(\hat{Y}_{k,t}\hat{a}_{k,t}) + v_{\xi}\left(\bar{Y}_k,\bar{\xi}\right)\tilde{\Delta}_{k.t}(\hat{a}_{k,t}) + \\ &+tips + O_p^3, \end{aligned} \tag{44}$$

where we have used the relation (40) for both $\hat{a}_{k,t}$ and $\hat{Y}_{k,t}$. Using the definition for $\Delta_{k,t}$ one can show that $\tilde{\Delta}_{k,t}$ is a term of second order. In this sense, interactions between $\tilde{\Delta}_{k,t}$ and $\hat{a}_{k,t}$ or $\tilde{\Delta}_{k,t}$ and $\hat{Y}_{k,t}$ can be ignored for they are of no importance up to second order. To see this, recall that

$$\Delta_{k.t} = m_k^{-1}\int_{m_k} \frac{p_{k,t}(z)}{P_{k,t}}^{-\theta(1+\nu)} dz. \tag{45}$$

Define:

$$m_k^{-1}\int_{m_k} \frac{p_{k,t}(z)}{P_{k,t}}^{-\theta(1+\nu)} = m_k^{-1}\int_{m_k} q_{k,t}^{-\frac{\theta(1+\nu)}{1-\theta}} dz. \tag{46}$$

First order Taylor expansion of $q_{k,t}^{-\frac{\theta(1+\nu)}{1-\theta}}$, yields:

$$q_{k,t}^{-\frac{\theta(1+\nu)}{1-\theta}} = \bar{q}_k^{-\frac{\theta(1+\nu)}{1-\theta}} + \frac{\theta(1+\nu)}{\theta-1}\bar{q}_k^{-\frac{\theta(1+\nu)}{1-\theta}-1}(q_{k,t}-\bar{q}_k) + O_p^2, \tag{47}$$

Steady state values for prices imply, for every k:

$$\bar{q}_k = \frac{\bar{p}_k(z)}{\bar{P}_k}^{1-\theta} = 1. \tag{48}$$

Using integrals

$$m_k^{-1}\int_{m_k} q_{k,t}^{-\frac{\theta(1+\nu)}{1-\theta}} dz = 1 + \frac{\theta(1+\nu)}{\theta-1}(m_k^{-1}\int_{m_k} q_{k,t}dz - 1) + O_p^2. \tag{49}$$

Noting further that

$$m_k^{-1}\int_{m_k} q_{k,t} = m_k^{-1}\int_{m_k} \frac{p_{k,t}(z)}{P_{k,t}}^{1-\theta} = 1, \tag{50}$$

due to the definition of sectorial price index, then we have:

$$\tilde{\Delta}_{k,t} = \Delta_{k,t} - 1 = O_p^2.$$

Hence, expression (44) simplifies to

$$v\left(Y_{k,t}, \xi_t\right) \Delta_{k.t} = v\left(\bar{Y}_k, \bar{\xi}\right) \tilde{\Delta}_{k.t} + v_{Y_k}\left(\bar{Y}_k, \bar{\xi}\right) \bar{Y}_k (\hat{Y}_{k,t} + \frac{1}{2}\hat{Y}_{k,t}^2) +$$
$$\frac{1}{2} v_{Y_k Y_k}\left(\bar{Y}_k, \bar{\xi}\right) \bar{Y}_k^2 (\hat{Y}_{k,t}^2) + v_{Y_k \xi}\left(\bar{Y}_k, \bar{\xi}\right) \bar{Y}_k (\hat{Y}_{k,t} \hat{a}_{k,t}) +$$
$$+ \, tips + O_p^3,$$

$$v\left(Y_{k,t}, \xi_t\right) \Delta_{k.t} = \lambda \left[\frac{Y_{k,t}}{m_k}\right]^{1+\nu} \{\frac{\hat{\Delta}_{k.t}}{1+\nu} + \hat{Y}_{k,t} + \frac{1+\nu}{2}\hat{Y}_{k,t}^2 - (1+\nu)\hat{Y}_{k,t}\hat{a}_{k,t}\} + tips + O_p^3. \tag{51}$$

where we have used the relation

$$\tilde{\Delta}_{k,t} = \hat{\Delta}_{k,t} + \frac{1}{2}\hat{\Delta}_{k,t}^2 + O_p^3,$$

which simplifies to

$$\tilde{\Delta}_{k,t} = \hat{\Delta}_{k,t} + O_p^3,$$

once one notice that $\hat{\Delta}_{k,t}^2$ is of higher order than O_p^2. Using a second order Taylor expansion over the law of motion for sectorial price dispersion given by

$$\Delta_{k.t} = \alpha_k \Pi_{k,t}^{\theta(1+\nu)} \Delta_{k.t-1} + (1-\alpha_k) \left(\frac{1-\alpha_k \Pi_{k,t}^{\theta-1}}{(1-\alpha_k)}\right)^{\frac{\theta(1+\nu)}{\theta-1}}, \tag{52}$$

yields

$$\hat{\Delta}_{k.t} = \alpha_k \hat{\Delta}_{k.t-1} + \frac{1}{2}\frac{\alpha_k}{(1-\alpha_k)}\theta(1+\nu)(1+\theta\nu)\hat{\Pi}_{k,t}^2 + O_p^3,$$

where interactions between $\hat{\Delta}_{k.t-1}$ and $\hat{\Pi}_{k.t}$ have been explicitly considered as of third order. Using the relation:

$$\hat{\Pi}_{k,t} = \pi_{k,t} + \frac{1}{2}\pi_{k,t}^2. \tag{53}$$

We have, up to second order,

$$\hat{\Delta}_{k.t} = \alpha_k \hat{\Delta}_{k.t-1} + \frac{1}{2}\frac{\alpha_k}{(1-\alpha_k)}\theta(1+\nu)(1+\theta\nu)\pi_{k,t}^2 + O_p^3,$$

where $\pi_{k,t}$ is the percent variation of sectorial price level, or best known as sectorial inflation, $\pi_{k,t} = \log P_{k,t}/P_{k,t-1}$. Interacting backwards yields

$$\hat{\Delta}_{k.t} = \alpha_k^{t-1}\hat{\Delta}_{k.-1} + \frac{1}{2}\frac{\alpha_k}{(1-\alpha_k)}\theta(1+\nu)(1+\theta\nu)\sum_{j=0}^{t}\alpha_k^{t-j}\pi_{k,j}^2 + O_p^3, \tag{54}$$

while we consider the sectorial price dispersion in the remote past as a "term independent of policy". Further considering that it is possible to change positions of sums over t and k on (51), that is

$$\sum_{t=0}^{\infty}\beta^t\sum_{k=1}^{K}m_k\lambda\left[\frac{Y_{k,t}}{m_k}\right]^{1+\nu}\{\frac{1}{1+\nu}\hat{\Delta}_{k.t}\} = \sum_{k=1}^{K}m_k\lambda\left[\frac{Y_{k,t}}{m_k}\right]^{1+\nu}\{\frac{1}{1+\nu}\sum_{t=0}^{\infty}\beta^t\hat{\Delta}_{k.t}\}.$$

Reordering terms, one can find that

$$\sum_{t=0}^{\infty}\beta^t\hat{\Delta}_{k.t} = \frac{1}{2}\frac{\alpha_k}{(1-\alpha_k)(1-\alpha_k\beta)}\theta(1+\nu)(1+\theta\nu)\sum_{t=0}^{\infty}\beta^t\pi_{k,t}^2 + tips + O_p^3. \tag{55}$$

Substituting (55) over (51) yields

$$\begin{aligned} v\left(Y_{k,t},\xi_t\right)\Delta_{k.t} = \lambda\left[\frac{Y_{k,t}}{m_k}\right]^{1+\nu} & \{\frac{1}{2}\frac{\alpha_k\theta(1+\theta\nu)}{(1-\alpha_k)(1-\alpha_k\beta)}\pi_{k,t}^2 + \hat{Y}_{k,t} + \\ & + \frac{1+\nu}{2}\hat{Y}_{k,t}^2 - (1+\nu)\hat{Y}_{k,t}\hat{a}_{k,t}\} + tips + O_p^3, \end{aligned}$$

where we have benefited from the possibility of swapping sums of t and k. Using (16) and (17) in the text, one can show that the following relation holds in steady state:

$$\frac{\theta\lambda}{\theta-1}\bar{\mu}^w m_k^{-\nu}\bar{Y}_k^{\nu} = (1-\bar{\tau}_k)\left(\bar{C}\right)^{-\sigma}.$$

It follows that

$$\lambda\left[\frac{Y_{k,t}}{m_k}\right]^{1+\nu} = [\bar{C}^{-\sigma}\bar{Y}](1-\Phi),$$

where

$$(1-\Phi) \equiv \frac{\theta-1}{\theta}\frac{(1-\bar{\tau})}{\bar{\mu}^w},$$

where the last equality is due to relation (7). These last definitions lead to (35) being approximated up to second order by the following expression:

$$
\begin{aligned}
U_{t_0} \quad = \quad & \Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{ \hat{Y}_t + \frac{(1-\tilde{\sigma})}{2} \hat{Y}_t^2 + \tilde{\sigma} \hat{Y}_t \hat{G}_t + \\
& - \sum_{k=1}^{K} m_k (1-\Phi) [\frac{\theta}{\kappa_k} \frac{\pi_{k,t}^2}{2} + \hat{Y}_{k,t} + \frac{1+\nu}{2} \hat{Y}_{k,t}^2 + \\
& -(1+\nu) \hat{Y}_{k,t} \hat{a}_{k,t}] \} + tips + O_p^3,
\end{aligned} \tag{56}
$$

where

$$\Omega \equiv \bar{C}^{-\sigma} \bar{Y}, \tag{57}$$

$$\kappa_k \equiv \frac{(1-\alpha_k)(1-\alpha_k \beta)}{(1+\theta\nu)\alpha_k}, \tag{58}$$

$$\tilde{\sigma} \equiv \sigma s_C^{-1} \tag{59}$$

and

$$(1-\Phi) \equiv \frac{\theta-1}{\theta} \frac{(1-\bar{\tau})}{\bar{\mu}^w}, \tag{60}$$

as above.

3.2 Second Order Approximation to AS Equation

The starting point is the expression for the sectorial non-linear Phillips Curve, given by:

$$\left(\frac{1-\alpha_k \Pi_{k,t}^{\theta-1}}{(1-\alpha_k)} \right)^{\frac{1+\theta\nu}{\theta-1}} = \frac{F_{k,t}}{K_{k,t}}. \tag{61}$$

We define $V_{k,t}$ as

$$V_{k,t} = \frac{1-\alpha_k \Pi_{k,t}^{\theta-1}}{(1-\alpha_k)}. \tag{62}$$

Applying logs yield the exact approximation:

$$\left[\frac{1+\theta\nu}{\theta-1} \right] \hat{V}_{k,t} = \hat{F}_{k,t} - \hat{K}_{k,t}, \tag{63}$$

where we used the definition (39). Using a second order Taylor expansion over

$\hat{V}_{k,t}$:

$$\hat{V}_{k.t} = -\frac{\alpha_k(\theta-1)}{(1-\alpha_k)}(\Pi_{k,t}) - \frac{1}{2}\frac{\alpha_k(\theta-1)}{(1-\alpha_k)^2}\{(\theta-1)-(1-\alpha)\}(\Pi_{k,t}^2) + O_p^3. \quad (64)$$

Using (53), one obtain:

$$\hat{V}_{k.t} = -\frac{\alpha_k(\theta-1)}{(1-\alpha_k)}\left[\pi_{k,t} + \frac{1}{2}\frac{(\theta-1)}{(1-\alpha_k)}\pi_{k,t}^2\right] + O_p^3. \quad (65)$$

Considering the expression for $K_{k,t}$ given by (16), define for convenience

$$\Pi_{k,t,s} = \frac{P_{k,s}}{P_{k,t}}, \quad (66)$$

where $s \geq t$ is some date in the future and $P_{k,t}$ the aggregate price level in sector k in period t. We use a second order Taylor expansion over

$$K_{k,t} = \frac{\theta\lambda}{\theta-1}m_k^{-\nu}E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}\,\mu_{k,t}^w\Pi_{k,t,j}^{\theta(\nu+1)}\frac{Y_{k,j}}{a_{k,j}}^{\nu+1},$$

yields

$$\tilde{K}_{k,t} = (1-\beta\alpha_k)E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}\{\hat{k}_{k,j} + \frac{1}{2}\hat{k}_{k,j}^2\} + O_p^3 \quad (67)$$

where the term $\hat{k}_{k,t}$ can be defined as

$$\hat{k}_{k,j} = \theta(1+\nu)\pi_{k,t,j} + (1+\nu)\hat{Y}_{k,j} - (1+\nu)\hat{a}_{k,j} + \hat{\mu}_{k,t}^w, \quad (68)$$

as we have used the relation in (40) for variables $\tilde{\Pi}_{k,t,j}$, $\tilde{Y}_{k,t}$ and $\tilde{a}_{k,t}$. Using the same relation applied for $\tilde{K}_{k,t}$ yields

$$\hat{K}_{k,t} + \frac{1}{2}\hat{K}_{k,t}^2 = (1-\beta\alpha_k)E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}\{\hat{k}_{k,j} + \frac{1}{2}\hat{k}_{k,j}^2\} + O_p^3. \quad (69)$$

Taking the expression in the text for $F_{k,t}$ given by (17), we define the net revenue factor as

$$\Gamma_{k,t} \equiv 1 - \tau_{k,t}. \quad (70)$$

Applying (70) and (66) over (17) yields

$$F_{k,t} = E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}\,\Gamma_{k,j}C_j^{-\sigma}\Pi_{k,t,j}^{\theta-1}p_{k,j}Y_{k,j}. \quad (71)$$

We apply a second order Taylor expansion over (17) which yields

$$\tilde{F}_{k,t} = (1 - \beta\alpha_k) E_t \sum_{j=t}^{\infty} (\alpha_k \beta)^{j-t} \{\hat{f}_{k,j} + \frac{1}{2}\hat{f}_{k,j}^2\} + O_p^3, \tag{72}$$

where $\tilde{F}_{k,t}$ follows (38) and we define $\hat{f}_{k,t}$ as

$$\hat{f}_{k,j} = \hat{\Gamma}_{k,j} - \sigma\hat{C}_j + \hat{Y}_{k,j} + \hat{p}_{k,j} + (\theta - 1)\,\pi_{k,t,j}, \tag{73}$$

where hat variables correspond to their definitions in (39). More explicitly,

$$\hat{\Gamma}_{k,t} = \log \frac{1 - \tau_{k,t}}{1 - \bar{\tau}} \tag{74}$$

and, as above,

$$\pi_{k,t,j} = \log \frac{P_{k,j}}{P_{k,t}}. \tag{75}$$

Also, from (40), we have

$$\hat{F}_{k,t} + \frac{1}{2}\hat{F}_{k,t}^2 = (1 - \beta\alpha_k) E_t \sum_{j=t}^{\infty} (\alpha_k \beta)^{j-t} \{\hat{f}_{k,j} + \frac{1}{2}\hat{f}_{k,j}^2\} + O_p^3. \tag{76}$$

We can subtract (65) from (76) yielding

$$\begin{aligned} \hat{F}_{k,t} - \hat{K}_{k,t} = (1 - \beta\alpha_k) E_t \sum_{j=t}^{\infty} (\alpha_k \beta)^{j-t} \{\left[\hat{f}_{k,j} - \hat{k}_{k,j}\right] + \frac{1}{2}\left[\hat{f}_{k,j}^2 - \hat{k}_{k,j}^2\right]\} + \\ - \frac{1}{2}\{\hat{F}_{k,t}^2 - \hat{K}_{k,t}^2\} + O_p^3. \end{aligned} \tag{77}$$

Also

$$\begin{aligned} \hat{F}_{k,t} + \hat{K}_{k,t} = (1 - \beta\alpha_k) E_t \sum_{j=t}^{\infty} (\alpha_k \beta)^{j-t} \{\left[\hat{f}_{k,j} + \hat{k}_{k,j}\right] + \frac{1}{2}\left[\hat{f}_{k,j}^2 + \hat{k}_{k,j}^2\right]\} + \\ - \frac{1}{2}\{\hat{F}_{k,t}^2 + \hat{K}_{k,t}^2\} + O_p^3. \end{aligned} \tag{78}$$

We can multiply this last expression by (63), which yields:

$$\hat{F}_{k,t}^2 - \hat{K}_{k,t}^2 = \left[\frac{1+\theta\nu}{\theta-1}\right] \hat{V}_{k,t}(1-\beta\alpha_k) E_t \sum_{j=t}^{\infty} (\alpha_k\beta)^{j-t} \left\{ \left[\hat{f}_{k,j} + \hat{k}_{k,j}\right] \right\} + O_p^3. \quad (79)$$

Replacing this last expression over (77) and (63)

$$\left[\frac{1+\theta\nu}{\theta-1}\right] \hat{V}_{k,t} = -\frac{1}{2}\left[\frac{1+\theta\nu}{\theta-1}\right] \hat{V}_{k,t}(1-\beta\alpha_k) E_t \sum_{j=t}^{\infty} (\alpha_k\beta)^{j-t} \left\{ \left[\hat{f}_{k,j} + \hat{k}_{k,j}\right] \right\} + \quad (80)$$

$$+(1-\beta\alpha_k) E_t \sum_{j=t}^{\infty} (\alpha_k\beta)^{j-t} \left\{ \left[\hat{f}_{k,j} - \hat{k}_{k,j}\right] + \frac{1}{2}\left[\hat{f}_{k,j} - \hat{k}_{k,j}\right]\left[\hat{f}_{k,j} + \hat{k}_{k,j}\right] \right\} + O_p^3.$$

Using the definitions for $\hat{f}_{k,t}$ and $\hat{k}_{k,t}$, we have

$$\hat{f}_{k,j} - \hat{k}_{k,j} = \hat{\Gamma}_{k,j} - \sigma\hat{C}_j - \nu\hat{Y}_{k,j} + \hat{p}_{k,j} - (1+\theta\nu)\pi_{k,t,j} + (1+\nu)\hat{a}_{k,j} - \hat{\mu}_{k,t}^w,$$

$$\hat{f}_{k,j} + \hat{k}_{k,j} = \hat{\Gamma}_{k,j} - \sigma\hat{C}_j + (2+\nu)\hat{Y}_{k,j} + \hat{p}_{k,j} + [(\theta-1)+\theta(1+\nu)]\pi_{k,t,j} - (1+\nu)\hat{a}_{k,j} + \hat{\mu}_{k,t}^w,$$

For convenience, we can also define

$$\hat{f}_{k,j} + \hat{k}_{k,j} = \hat{X}_{k,j} + [(\theta-1)+\theta(1+\nu)]\pi_{k,t,j}, \quad (81)$$

where

$$\hat{X}_{k,j} \equiv \hat{\Gamma}_{k,j} - \sigma\hat{C}_j + (2+\nu)\hat{Y}_{k,j} + \hat{p}_{k,j} - (1+\nu)\hat{a}_{k,j} + \hat{\mu}_{k,t}^w \quad (82)$$

and also

$$\hat{f}_{k,j} - \hat{k}_{k,j} = z_{k,j} - (1+\theta\nu)\pi_{k,t,j}, \quad (83)$$

where

$$z_{k,j} = \hat{\Gamma}_{k,j} - \sigma\hat{C}_j - \nu\hat{Y}_{k,j} + \hat{p}_{k,j} + (1+\nu)\hat{a}_{k,j} - \hat{\mu}_{k,t}^w. \quad (84)$$

Replacing above expressions over (80)

$$\left[\frac{1+\theta\nu}{\theta-1}\right]\hat{V}_{k,t} = (1-\beta\alpha_k)E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}\left\{[z_{k,j}-(1+\theta\nu)\pi_{k,t,j}]+\right.$$

$$\left.+\frac{1}{2}[z_{k,j}-(1+\theta\nu)\pi_{k,t,j}][\hat{X}_{k,j}+[(\theta-1)+\theta(1+\nu)]\pi_{k,t,j}]\right\}$$

$$-\frac{1}{2}\left[\frac{1+\theta\nu}{\theta-1}\right]\hat{V}_{k,t}(1-\beta\alpha_k)E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}\{\hat{X}_{k,j}+[(\theta-1)+\theta(1+\nu)]\pi_{k,t,j}\}+O_p^3.$$

Define

$$Z_{k,t} \equiv E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}\left\{\left[\hat{X}_{k,j}+[(\theta-1)+\theta(1+\nu)]\pi_{k,t,j}\right]\right\}. \tag{85}$$

We can replace in the expression above and get:

$$\left[\frac{1+\theta\nu}{\theta-1}\right]\hat{V}_{k,t} = (1-\beta\alpha_k)E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}\{[z_{k,j}-(1+\theta\nu)\pi_{k,t,j}]\}+$$

$$+\frac{1}{2}(1-\beta\alpha_k)E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}[z_{k,j}-(1+\theta\nu)\pi_{k,t,j}]\left[\hat{X}_{k,j}+[(\theta-1)+\theta(1+\nu)]\pi_{k,t,j}\right]\}$$

$$-\frac{1}{2}(1-\beta\alpha_k)\left[\frac{1+\theta\nu}{\theta-1}\right]\hat{V}_{k,t}Z_{k,t}+O_p^3.$$

Taking the lead, multiplying by $\alpha_k\beta$ and then subtracting from expression above yields:

$$\frac{(1+\theta\nu)}{(\theta-1)(1-\beta\alpha_k)}\left[\hat{V}_{k,t}-\alpha_k\beta E_t\hat{V}_{k,t+1}\right] = E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}\{[z_{k,j}-(1+\theta\nu)\pi_{k,t,j}]\}+$$

$$-(\alpha_k\beta)E_t\sum_{j=t+1}^{\infty}(\alpha_k\beta)^{j-t-1}\{[z_{k,j}-(1+\theta\nu)\pi_{k,t+1,j}]\}+$$

$$+\frac{1}{2}E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}[z_{k,j}-(1+\theta\nu)\pi_{k,t,j}]\left[\hat{X}_{k,j}+[(\theta-1)+\theta(1+\nu)]\pi_{k,t,j}\right]\}+$$

$$-(\alpha_k\beta)\frac{1}{2}E_t\sum_{j=t+1}^{\infty}(\alpha_k\beta)^{j-t-1}[z_{k,j}-(1+\theta\nu)\pi_{k,t+1,j}]\left[\hat{X}_{k,j}+[(\theta-1)+\theta(1+\nu)]\pi_{k,t+1,j}\right]\}+$$

$$-\frac{1}{2}\left[\frac{1+\theta\nu}{\theta-1}\right]\left[\hat{V}_{k,t}Z_{k,t}-\alpha_k\beta E_t\hat{V}_{k,t+1}Z_{k,t+1}\right]+O_p^3.$$

Using the facts that $\pi_{k,t,t} = 0$, $\pi_{k,t,j} - \pi_{k,t+1,j} = \pi_{k,t,t+1} = \pi_{k,t+1}$ and also that

$$\pi^2_{k,t+1,j} - \pi^2_{k,t,j} = \pi^2_{k,t+1} - 2\pi_{k,t+1}\pi_{k,t,j}, \tag{86}$$

one gets

$$\frac{(1+\theta\nu)}{(\theta-1)(1-\beta\alpha_k)}\left[\hat{V}_{k,t} - \alpha_k\beta E_t\hat{V}_{k,t+1}\right] = z_{k,t} - [1+\theta\nu]\frac{\alpha_k\beta}{(1-\alpha_k\beta)}E_t\pi_{k,t+1} + \frac{1}{2}\{z_{k,t}\hat{X}_{k,t}\} +$$
$$+\frac{1}{2}(\alpha_k\beta)E_t\sum_{j=t+1}^{\infty}(\alpha_k\beta)^{j-t-1}\{z_{k,j}[(\theta-1)+\theta(1+\nu)](\pi_{k,t+1}) - [1+\theta\nu](\pi_{k,t+1})\hat{X}_{k,j} +$$
$$+[1+\theta\nu][(\theta-1)+\theta(1+\theta\nu)](\pi^2_{k,t+1} - 2\pi_{k,t+1}\pi_{k,t,j})\} +$$
$$-\frac{1}{2}\frac{1+\theta\nu}{\theta-1}\left[\hat{V}_{k,t}Z_{k,t} - \alpha_k\beta E_t\hat{V}_{k,t+1}Z_{k,t+1}\right] + O_p^3.$$

Noticing that

$$\pi_{k,t,j} = \pi_{k,t+1} + \pi_{k,t+1,j},$$

expression above simplifies to

$$\frac{(1+\theta\nu)}{(\theta-1)(1-\beta\alpha_k)}\left[\hat{V}_{k,t} - \alpha_k\beta E_t\hat{V}_{k,t+1}\right] = z_{k,t} - (1+\theta\nu)\frac{\alpha_k\beta}{(1-\alpha_k\beta)}E_t\pi_{k,t+1} + \frac{1}{2}z_{k,t}\hat{X}_{k,t} +$$
$$-\frac{1}{2}(1+\theta\nu)[(\theta-1)+\theta(1+\nu)]\frac{\alpha_k\beta}{(1-\alpha_k\beta)}E_t\pi^2_{k,t+1}$$
$$+\frac{1}{2}[(\theta-1)+\theta(1+\nu)](\alpha_k\beta)(\pi_{k,t+1})E_t\sum_{j=t+1}^{\infty}(\alpha_k\beta)^{j-t-1}\{z_{k,j} - (1+\theta\nu)(\pi_{k,t+1,j})\}$$
$$-\frac{1}{2}(1+\theta\nu)(\alpha_k\beta)(\pi_{k,t+1})E_t\sum_{j=t+1}^{\infty}(\alpha_k\beta)^{j-t-1}\{\hat{X}_{k,j} + [(\theta-1)+\theta(1+\nu)](\pi_{k,t+1,j})\}$$
$$-\frac{1}{2}\left[\frac{1+\theta\nu}{\theta-1}\right]\left[\hat{V}_{k,t}Z_{k,t} - \alpha_k\beta E_t\hat{V}_{k,t+1}Z_{k,t+1}\right] + O_p^3.$$

Using the definition for $Z_{k,t}$, expression simplifies to

$$
\begin{aligned}
\frac{(1+\theta\nu)}{(\theta-1)(1-\beta\alpha_k)}\left[\hat{V}_{k,t}-\alpha_k\beta E_t\hat{V}_{k,t+1}\right] &= z_{k,t}-(1+\theta\nu)\frac{\alpha_k\beta}{(1-\alpha_k\beta)}E_t\pi_{k,t+1}+\frac{1}{2}z_{k,t}\hat{X}_{k,t} \\
&-\frac{1}{2}(1+\theta\nu)[(\theta-1)+\theta(1+\nu)]\frac{\alpha_k\beta}{(1-\alpha_k\beta)}E_t\pi^2_{k,t+1} \\
&+\frac{1}{2}(1+\theta\nu)[(\theta-1)+\theta(1+\nu)]\frac{\alpha_k\beta}{(\theta-1)(1-\beta\alpha_k)}E_t\left[\hat{V}_{k,t+1}\pi_{k,t+1}\right] \\
&-\frac{1}{2}(1+\theta\nu)\left(\alpha_k\beta\right)E_t[\pi_{k,t+1}Z_{k,t+1}]-\frac{1}{2}\frac{(1+\theta\nu)}{(\theta-1)}\left[\hat{V}_{k,t}Z_{k,t}-\alpha_k\beta E_t\hat{V}_{k,t+1}Z_{k,t+1}\right]+O^3_p,
\end{aligned}
$$

where we have used the fact that, from (80) and from the definition of $\hat{f}_{k,t}-\hat{k}_{k,t}$:

$$
\frac{(1+\theta\nu)}{(\theta-1)(1-\beta\alpha_k)}\hat{V}_{k,t}(\pi_{k,t+1})=(\pi_{k,t+1})E_t\sum_{j=t+1}^{\infty}\left(\alpha_k\beta\right)^{j-t-1}\{z_{k,j}-(1+\theta\nu)(\pi_{k,t,j})\}+O^3_p. \tag{87}
$$

We can use the definition for $\hat{V}_{k,t}$ in (65) and replace above, also discharging the terms O^3_p or of higher order.

$$
\begin{aligned}
-\kappa_k^{-1}\left[\pi_{k,t}+\frac{1}{2}\frac{(\theta-1)}{(1-\alpha_k)}\pi^2_{k,t}-\alpha_k\beta E_t\pi_{k,t+1}-\frac{1}{2}\frac{(\theta-1)}{(1-\alpha_k)}\alpha_k\beta E_t\pi^2_{k,t+1}\right] &= \\
z_{k,t}+\frac{1}{2}z_{k,t}\hat{X}_{k,t}-(1+\theta\nu)\frac{\alpha_k\beta}{(1-\alpha_k\beta)}E_t\pi_{k,t+1}+ \\
-\frac{1}{2}[(\theta-1)+\theta(1+\nu)]\frac{\beta}{\kappa_k}E_t\pi^2_{k,t+1}+ \\
-\frac{1}{2}(1+\theta\nu)\left(\alpha_k\beta\right)E_t[\pi_{k,t+1}Z_{k,t+1}]+ \\
+\frac{1}{2}\frac{(1+\theta\nu)\alpha_k}{(1-\alpha_k)}\left[\pi_{k,t}Z_{k,t}-\alpha_k\beta E_t[\pi_{k,t+1}Z_{k,t+1}]\right]+O^3_p,
\end{aligned}
$$

where we have defined κ_k as

$$
\kappa_k=\frac{(1-\alpha_k)(1-\alpha_k\beta)}{(1+\theta\nu)\alpha_k}. \tag{88}
$$

Further simplification yields

$$
\begin{aligned}
-\kappa_k^{-1}\pi_{k,t} - \frac{1}{2}\kappa_k^{-1}\frac{(\theta-1)}{(1-\alpha_k)}\pi_{k,t}^2 - \frac{1}{2}\frac{(1+\theta\nu)\alpha_k}{(1-\alpha_k)}\pi_{k,t}Z_{k,t}& \\
= z_{k,t} + \frac{1}{2}z_{k,t}\hat{X}_{k,t} - \kappa_k^{-1}\beta E_t\pi_{k,t+1}& \\
-\frac{1}{2}\kappa_k^{-1}\{\frac{(\theta-1)}{(1-\alpha_k)} + \theta(1+\nu)\}\beta E_t\pi_{k,t+1}^2& \\
-\frac{1}{2}\frac{(1+\theta\nu)\alpha_k}{(1-\alpha_k)}\beta E_t[\pi_{k,t+1}Z_{k,t+1}] + O_p^3.&
\end{aligned}
$$

Multiplying both sides for $-\kappa_k$ allow us to write above expression as

$$
\mathcal{V}_{k,t} = -\kappa_k\{z_{k,t} + \frac{1}{2}z_{k,t}\hat{X}_{k,t}\} + \frac{\theta(1+\nu)}{2}\pi_{k,t}^2 + \beta E_t\mathcal{V}_{k,t+1} + O_p^3. \tag{89}
$$

where:

$$
\mathcal{V}_{k,t} = \pi_{k,t} + \frac{1}{2}\{\frac{(\theta-1)}{(1-\alpha_k)} + \theta(1+\nu)\}\pi_{k,t}^2 + \frac{1}{2}\frac{\kappa_k\alpha_k}{(1-\alpha_k)}[\pi_{k,t}Z_{k,t}] \tag{90}
$$

and $z_{k,t}$, $\hat{X}_{k,t}$ and $Z_{k,t}$ are give, respectively, by (84), (82), (85). A second order Taylor expansion of $\log(1-\tau_{k,t})$ allows us to relate (74) with the original tax rate variables:

$$
\log(1-\tau_{k,t}) = \log(1-\bar{\tau}) - \frac{\bar{\tau}}{1-\bar{\tau}}\tilde{\tau}_{k,t} - \frac{1}{2}\frac{\bar{\tau}^2}{(1-\bar{\tau})^2}\tilde{\tau}_{k,t}^2 + O_p^3,
$$

$$
\hat{\Gamma}_{k,t} = -\delta\hat{\tau}_{k,t} - \frac{\delta}{(1-\bar{\tau})}\frac{1}{2}\hat{\tau}_{k,t}^2 + O_p^3,
$$

where

$$
\delta = \frac{\bar{\tau}}{1-\bar{\tau}}. \tag{91}
$$

Also, a log-linearization of

$$
C_t = Y_t - G_t
$$

yields

$$
\hat{C}_t = s_C^{-1}\hat{Y}_t - s_C^{-1}\hat{G}_t + \frac{1}{2}s_C^{-1}(1-s_C^{-1})\hat{Y}_t^2 - \frac{1}{2}s_C^{-1}(1+s_C^{-1})\hat{G}_t^2 + s_C^{-2}\hat{Y}_t\hat{G}_t + O_p^3, \tag{92}
$$

where

$$
s_C = \bar{C}/\bar{Y}.
$$

Using both results, one can redefine $z_{k,t}$ and $\hat{X}_{k,t}$ as:

$$\begin{aligned} \hat{X}_{k,t} &= -\delta\hat{\tau}_{k,t} - \frac{1}{2}\frac{\delta}{(1-\bar{\tau})}\hat{\tau}_{k,t}^2 + (2+\nu)\hat{Y}_{k,t} + \hat{p}_{k,t} - (1+\nu)\hat{a}_{k,t} + \hat{\mu}_{k,t}^w \\ &\quad -\tilde{\sigma}\{\hat{Y}_t - \hat{G}_t + \frac{1}{2}(1-s_C^{-1})Y_t^2 + s_C^{-1}\hat{Y}_t\hat{G}_t\} + tips + O_p^3, \end{aligned} \tag{93}$$

and

$$\begin{aligned} z_{k,t} &= -\delta\hat{\tau}_{k,t} - \frac{1}{2}\frac{\delta}{(1-\bar{\tau})}\hat{\tau}_{k,t}^2 - \nu\hat{Y}_{k,t} + \hat{p}_{k,t} + (1+\nu)\hat{a}_{k,t} - \hat{\mu}_{k,t}^w \\ &\quad -\tilde{\sigma}\{\hat{Y}_t - \hat{G}_t + \frac{1}{2}(1-s_C^{-1})Y_t^2 + s_C^{-1}\hat{Y}_t\hat{G}_t\} + tips + O_p^3, \end{aligned} \tag{94}$$

where $\tilde{\sigma}$ is defined as in (59) and also noting that $\hat{p}_{k,t}$ relates to sectorial and aggregate outputs following

$$\hat{p}_{k,t} = \eta^{-1}(\hat{Y}_t - \hat{Y}_{k,t}).$$

Finally, (89) can be generally expressed as

$$\mathcal{V}_{k,t} = E_{t_0}\sum_{j=t}^{\infty}\beta^{j-t}\{-\kappa_k[z_{k,t} + \frac{1}{2}z_{k,t}\hat{X}_{k,t}] + \frac{\theta(1+\nu)}{2}\pi_{k,t}^2\} + tips + O_p^3 \tag{95}$$

where $\mathcal{V}_{k,t}$ is defined in (90), $\hat{X}_{k,t}$ in (93) and $z_{k,t}$ in (94). One could finally note that a first order approximation to (63) yields the known Phillips Curve of the form:

$$\begin{aligned} \pi_{k,t} &= \kappa_k\{(\tilde{\sigma} - \eta^{-1})\hat{Y}_t + (\nu + \eta^{-1})\hat{Y}_{k,t} + \delta\hat{\tau}_{k,t} \\ &\quad -\tilde{\sigma}\hat{G}_t - (1+\nu)\hat{a}_{k,t} + \hat{\mu}_{k,t}^w\} + \beta E_t\pi_{k,t+1} + O_p^2. \end{aligned} \tag{96}$$

3.3 Second Order Approximation to the Budget Constraint

We approximate the intertemporal government budget restriction by a second order Taylor expansion. We take the definition of government's intertemporal budget constraint in the text

$$W_t = E_t\sum_{j=t}^{\infty}\beta^{j-t}C_j^{-\sigma}s_j, \tag{97}$$

where W_t is defined as

$$W_t = \frac{C_t^{-\sigma}}{\Pi_t} b_{t-1}^*, \tag{98}$$

and b_t^* as the real value at maturity of government debt in terms of one-period riskless bond, or $b_t^* = R_t b_t$, and s_t is given by

$$s_t = \sum_{k=1}^{K} \tau_{k,t} p_{k,t} Y_{k,t} - G_t. \tag{99}$$

Expanding (97) yields:

$$\tilde{W}_t = (1-\beta) E_t \sum_{j=t}^{\infty} \beta^{j-t} \{-\sigma \tilde{C}_t + \tilde{s}_t + \frac{1}{2}\sigma(\sigma+1)\tilde{C}_t^2 - \sigma \tilde{C}_t \tilde{s}_t\} + O_p^3, \tag{100}$$

where tilde variables are defined in (38) and where we have used the relation

$$\bar{W} = \frac{\bar{C}^{-\sigma} \bar{s}}{1-\beta}. \tag{101}$$

We can use relation (40) in order to simplify equation above to:

$$\tilde{W}_t = (1-\beta) E_t \sum_{j=t}^{\infty} \beta^{j-t} \{-\sigma \hat{C} + \tilde{s}_t + \frac{1}{2}\sigma^{-2} \hat{C}_t^2 - \sigma \hat{C} \tilde{s}_t\} + O_p^3, \tag{102}$$

where hat variables are defined as in (39). In this sense, $\tilde{W}_t$ can be defined in terms of log variables using the relation given (40). Using logs over (98), $\hat{W}_t$ can be defined as:

$$\hat{W}_t = \hat{b}_{t-1}^* - \sigma \hat{C}_t - \pi_t, \tag{103}$$

where hat variables are defined as log deviations from steady state levels. Once

$$\tilde{W} = \hat{W} + \frac{1}{2}\hat{W} + O_p^3, \tag{104}$$

holds, we have:

$$\tilde{W}_t = \hat{b}_{t-1}^* - \sigma \hat{C}_t - \pi_t + \frac{1}{2}(\hat{b}_{t-1}^* - \sigma \hat{C}_t - \pi_t)^2 + O_p^3. \tag{105}$$

We should also define $\tilde{s}_t$ in terms of log deviations from steady state levels. Taking a second order Taylor expansion over (99) yields:

$$s_d \tilde{s}_t = \sum_{k=1}^{K} m_k \bar{\tau} [(\hat{\tau}_k + \hat{p}_{k,t} + \hat{Y}_{k,t}) + \frac{1}{2}(\hat{\tau}_k + \hat{p}_{k,t} + \hat{Y}_{k,t})^2] - \hat{G}_t - \frac{1}{2}\hat{G}_t^2 + O_p^3, \tag{106}$$

where hat variables are log deviations from steady state values and we have used the relation in (40) for $\tilde{\tau}_{k,t}$, $\tilde{Y}_{k,t}$ and $\tilde{G}_t$ as well as $\bar{Y}_k = m_k\bar{Y}$. The term s_d is defined as

$$s_d \equiv \frac{\bar{s}}{\bar{Y}}, \tag{107}$$

where

$$\bar{s} = \sum_{k=1}^{K} \bar{\tau}\bar{Y}_k - \bar{G} = \bar{\tau}\bar{Y} - \bar{G}. \tag{108}$$

Finally, for mathematical convenience, we choose to redefine (102) by multiplying both sides by s_d:

$$\tilde{\mathcal{W}}_{t_0} \equiv s_d\tilde{W}_t = (1-\beta)E_t\sum_{j=t}^{\infty}\beta^{j-t}\{-\sigma s_d\hat{C} + s_d\tilde{s}_t + \frac{1}{2}\sigma^{-2}\hat{C}_t^2 - \sigma\hat{C}s_d\tilde{s}_t\} + O_p^3. \tag{109}$$

Hence, the second order approximation for the intertemporal budget constraint can be obtained by replacing (92), (105), (106) into (109). One can notice that a first order approximation yields:

$$\begin{aligned}
\hat{b}_{t-1}^* - \tilde{\sigma}(\hat{Y}_t - \hat{G}_t) - \pi_t = \\
(1-\beta)E_t\sum_{j=t}^{\infty}\beta^{j-t}\{s_d^{-1}\sum_{k=1}^{K} m_k\bar{\tau}[\hat{\tau}_k + \hat{p}_{k,t} + \hat{Y}_{k,t}] + \\
+ (\tilde{\sigma} - s_d^{-1})\hat{G}_t - \tilde{\sigma}\hat{Y}_t\} + tips + O_p^2,
\end{aligned}$$

where, as underlined elsewhere, $\hat{p}_{k,t}$ is a function of sectorial and overall outputs and $\tilde{\sigma}$ and s_d are, respectively, defined in (59) and (107).

3.4 Aggregate and Sectorial Output Relation

Sectorial demand expressed in,

$$p_{k,t}^{\eta} = m_k\frac{Y_t}{Y_{k,t}}, \tag{110}$$

when log-linearized, yields

$$\hat{p}_{k,t} = \eta^{-1}(\hat{Y}_t - \hat{Y}_{k,t}). \tag{111}$$

which establishes an exact (inverse) relation between sector relative price and sector relative product. Also, by substituting (110) and

$$p_{k,t} = \frac{\Pi_{k,t}}{\Pi_t} p_{k,t-1} \tag{112}$$

over

$$\Pi_t^{1-\eta} = \sum_{k=1}^{K} m_k \left(\Pi_{k,t} p_{k,t-1} \right)^{1-\eta}, \tag{113}$$

one gets

$$Y_t^{(\eta-1)/\eta} = \sum_{k=1}^{K} m_k^{1/\eta} Y_{k,t}^{(\eta-1)/\eta}, \tag{114}$$

which relates aggregate and sectorial outputs. Log linearization of (114) yields

$$\hat{Y}_t + \frac{1}{2}(1-\eta^{-1})\hat{Y}_t^2 = \sum_{k=1}^{K} m_k \hat{Y}_{k,t} + \frac{1}{2}(1-\eta^{-1}) \sum_{k=1}^{K} m_k \hat{Y}_{k,t}^2 + O_p^3. \tag{115}$$

4 Appendix D - Elimination of Linear Terms

4.1 Matrix Notation

We start by defining

$$x_t' = \left[\begin{array}{cccccccccc} \hat{Y}_t & \hat{Y}_{1,t} & ... & \hat{Y}_{K,t} & \pi_{1,t} & ... & \pi_{K,t} & \hat{\tau}_{1,t} & ... & \hat{\tau}_{K,t} \end{array} \right] \tag{116}$$

and

$$\xi_t' = \left[\begin{array}{ccccccc} \hat{G}_t & \hat{a}_{1,t} & ... & \hat{a}_{K,t} & \hat{\mu}_{1,t}^w & ... & \hat{\mu}_{K,t}^w \end{array} \right]. \tag{117}$$

For notational convenience, we also define the following terms:

$$\upsilon \equiv 1 + \nu, \tag{118}$$

$$\omega_\eta \equiv 1 - \eta^{-1}, \tag{119}$$

$$\chi \equiv \nu + \eta^{-1}, \tag{120}$$

$$\tilde{\sigma} \equiv \sigma s_C^{-1}, \tag{121}$$

$$\varsigma \equiv \tilde{\sigma} - \eta^{-1}, \tag{122}$$

$$\delta \equiv \frac{\bar{\tau}}{1-\bar{\tau}} \tag{123}$$

and

$$1 - s_C^{-1} = -\frac{\bar{Y} - \bar{C}}{\bar{C}} \equiv -\omega_C, \tag{124}$$

in addition to the terms defined elsewhere:

$$s_C \equiv \bar{C}/\bar{Y}, \tag{125}$$

$$s_d \equiv \bar{s}/\bar{Y}. \tag{126}$$

Using the definitions above, expression in (56) can be written in matrix notation as

$$U_{t_0} \equiv \Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{A_x' x_t - \frac{1}{2} x_t' A_{xx} x_t - x_t' A_\xi \xi_t\} + tips + O_p^3, \tag{127}$$

where A_x, A_{xx}, and A_ξ are, respectively, $(3K+1) \times 1$, $(3K+1) \times (3K+1)$ and $(3K+1) \times (2K+1)$ matrices, such as:

$$A_x' = \begin{bmatrix} 1 & -m_1(1-\Phi) & ... & -m_K(1-\Phi) & 0 & ... & 0 & 0 & ... & 0 \end{bmatrix}, \tag{128}$$

$$A_{xx} = \begin{bmatrix} A_{xx}^{11} & 0 & 0 & 0 \\ 0 & A_{xx}^{22} & 0 & 0 \\ 0 & 0 & A_{xx}^{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{129}$$

where A_{xx}^{11} is a 1×1 matrix such as

$$A_{xx}^{11} = -(1-\tilde{\sigma}),$$

A_{xx}^{22} is a $K \times K$ diagonal matrix such as its typical k^{th} element is

$$\left(A_{xx}^{22}\right)_{kk} = m_k(1-\Phi)\upsilon,$$

A_{xx}^{33} is a $K \times K$ diagonal matrix such as its typical k^{th} element is

$$\left(A_{xx}^{33}\right)_{kk} = \frac{m_k(1-\Phi)}{\kappa_k}\theta,$$

and

$$A_\xi = \begin{bmatrix} A_\xi^{11} & 0 & 0 \\ 0 & A_\xi^{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{130}$$

where

$$A_\xi^{11} = -\tilde{\sigma}$$

and A_ξ^{22} is a $K \times K$ diagonal matrix such as its typical k^{th} element is

$$\left(A_\xi^{22}\right)_{kk} = -m_k(1-\Phi)\upsilon,$$

and where we have observed the following definitions:

$$\Omega \equiv \bar{C}^{-\sigma}\bar{Y},$$

$$\kappa_k \equiv \frac{(1-\alpha_k)(1-\alpha_k\beta)}{(1+\theta\nu)\alpha_k},$$

$$(1-\Phi) \equiv \frac{\theta-1}{\theta}\frac{(1-\bar{\tau})}{\bar{\mu}^w}.$$

The Sectorial Phillips Curve expressed in (95) can also be written in matrix notation. We start by substituting expressions for $\hat{p}_{k,t}$ into definitions for $z_{k,t}$ and $\hat{X}_{k,t}$, underlined in (94) and (93). Our aim is to separate quadratic and linear terms. Quadratic and linear terms of random disturbances are placed into *tips*. After some manipulation one obtains:

$$\mathcal{V}_{k,t_0} = E_{t_0}\sum_{j=t_0}^{\infty}\beta^{j-t_0}\{C_{x,k}'x_t + \frac{1}{2}x_t'C_{xx,k}x_t + x_t'C_{\xi,k}\xi_t\} + tips + O_p^3, \tag{131}$$

for a generic sector k. As in (127), matrices $C_{x,k}$, $C_{xx,k}$, and $C_{\xi,k}$ have, respectively, dimension $(3K+1)\times 1$, $(3K+1)\times(3K+1)$ and $(3K+1)\times(2K+1)$, such as:

$$C_{x,k}' = \begin{bmatrix} C_{x,k}^{11\prime} & C_{x,k}^{12\prime} & 0 & C_{x,k}^{14\prime} \end{bmatrix}, \tag{132}$$

where $C_{x,k}^{11\prime}$ is 1×1 matrix such as

$$C_{x,k}^{11\prime} = \kappa_k\varsigma$$

every k, $C_{x,k}^{12\prime}$ is $1\times K$ matrix such as

$$\left(C_{x,k}^{12\prime}\right)_{1k} = \kappa_k\chi$$

and zeros elsewhere, $C^{14\prime}_{x,k}$ is $1 \times K$ matrix such as

$$\left(C^{14\prime}_{x,k}\right)_{1k} = \kappa_k \delta$$

and zeros elsewhere; and

$$C_{xx.k} = \begin{bmatrix} C^{11}_{xx.k} & C^{12}_{xx.k} & 0 & C^{14}_{xx.k} \\ C^{21}_{xx.k} & C^{22}_{xx.k} & 0 & C^{24}_{xx.k} \\ 0 & 0 & C^{33}_{xx.k} & 0 \\ C^{41}_{xx.k} & C^{42}_{xx.k} & 0 & C^{44}_{xx.k} \end{bmatrix} \tag{133}$$

such that $C^{11}_{xx,k}$ is 1×1 matrix

$$C^{11}_{xx,k} = -\kappa_k[\tilde{\sigma}\omega_C + \varsigma^2]$$

for every k, $C^{12}_{xx,k}$ is $1 \times K$ matrix such that

$$\left(C^{12}_{xx,k}\right)_{1k} = \kappa_k \varsigma \omega_\eta$$

and zeros elsewhere, all k, and $C^{12\prime}_{xx,k} = C^{21}_{xx,k}$; $C^{14}_{xx,k}$ is $1 \times K$ matrix, such as

$$\left(C^{14}_{xx,k}\right)_{1k} = -\kappa_k \varsigma \delta$$

and zero otherwise, for all k, and $C^{41}_{xx,k} = C^{14\prime}_{xx,k}$; $C^{22}_{xx,k}$ is $K \times K$ diagonal matrix such that, all k,

$$\left(C^{22}_{xx,k}\right)_{kk} = \chi \kappa_k (\upsilon + \omega_\eta)$$

$C^{33}_{xx,k}$ is $K \times K$ diagonal matrix such that, for all k,

$$\left(C^{33}_{xx,k}\right)_{kk} = \theta \upsilon$$

$C^{24}_{xx,k}$ is $K \times K$ diagonal matrix such as

$$\left(C^{24}_{xx,k}\right)_{kk} = \kappa_k \delta \omega_\eta$$

all k, $C^{44}_{xx,k}$ is $K \times K$ diagonal matrix such as

$$\left(C^{44}_{xx,k}\right)_{kk} = \kappa_k \delta$$

for every k, and $C^{42}_{xx,k} = C^{24\prime}_{xx,k}$. Also, matrix $C_{\xi,k}$ can be defined as

$$C_{\xi,k} = \begin{bmatrix} C^{11}_{\xi,k} & 0 & 0 \\ C^{21}_{\xi,k} & C^{22}_{\xi,k} & C^{23}_{\xi,k} \\ 0 & 0 & 0 \\ C^{41}_{\xi,k} & 0 & 0 \end{bmatrix} \tag{134}$$

where $C^{11}_{\xi,k}$ is 1×1 matrix, such that

$$C^{11}_{\xi,k} = \kappa_k[\omega_C + \tilde{\sigma} + \omega_\eta]\tilde{\sigma}$$

for every k; $C^{21}_{\xi,k}$ is a $K\times 1$ matrix, such as

$$\left(C^{21}_{\xi,k}\right)_{1k} = -\kappa_k\omega_\eta\tilde{\sigma}$$

and zero elsewhere, $C^{22}_{\xi,k}$ is $K\times K$ diagonal matrix such that

$$\left(C^{22}_{\xi,k}\right)_{kk} = -\kappa_k\upsilon^2$$

and zero elsewhere, $C^{23}_{\xi,k}$ is $K\times K$ diagonal matrix such that

$$\left(C^{23}_{\xi,k}\right)_{kk} = \kappa_k\upsilon$$

and zero otherwise, $C^{41}_{\xi,k}$ is $K\times 1$ matrix such that

$$\left(C^{41}_{\xi,k}\right)_{k1} = \kappa_k\delta\tilde{\sigma}^{-1}$$

and zero elsewhere. We recall the definition for δ as

$$\delta \equiv \frac{\bar{\tau}}{1-\bar{\tau}}$$

in addition to the definitions from (118) to (126).

The government budget constraint can also be simplified in matrix notation. Taking expression given in (102), we eliminate references for $\hat{p}_{k,t}$, and replace $\hat{C}_t$ and $\tilde{s}_t$ for their expressions in terms of endogenous variables x_t and exogenous processes ξ_t. Grouping linear and quadratic terms, yields:

$$\tilde{\mathcal{W}}_{t_0} = (1-\beta)E_{t_0}\sum_{j=t_0}^{\infty}\beta^{j-t_0}\{B'_x x_t + \frac{1}{2}x'_t B_{xx}x_t + x'_t B_\xi \xi_t\} + tips + O_p^3 \quad (135)$$

where, as in (127) and (131), matrices B_x, B_{xx}, and B_ξ are, respectively, of dimensions $(3K+1)\times 1$, $(3K+1)\times(3K+1)$ and $(3K+1)\times(K+1)$, such as:

$$B'_x = \left[\begin{array}{ccccccccc} -\tilde{\sigma}s_d + \bar{\tau}\eta^{-1} & \omega_\eta m_1\bar{\tau} & \dots & \omega_\eta m_K\bar{\tau} & 0 & \dots & 0 & m_1\bar{\tau} & \dots & m_K\bar{\tau}\end{array}\right], \quad (136)$$

$$B_{xx} = \left[\begin{array}{cccc} B^{11}_{xx} & B^{12}_{xx} & 0 & B^{14}_{xx} \\ B^{21}_{xx} & B^{22}_{xx} & 0 & B^{24}_{xx} \\ 0 & 0 & 0 & 0 \\ B^{41}_{xx} & B^{42}_{xx} & 0 & B^{44}_{xx}\end{array}\right], \quad (137)$$

such as B^{11}_{xx} is 1×1 matrix such as

$$B^{11}_{xx} = \tilde{\sigma}s_d(\omega_C + \tilde{\sigma}) - \varsigma\eta^{-1}\bar{\tau},$$

for every k, B^{12}_{xx} is $1\times K$ matrix such as

$$\left(B_{xx}^{12}\right)_{1k} = -\varsigma\omega_\eta m_k\bar{\tau},$$

every k, $B_{xx}^{12\prime} = B_{xx}^{21}$, B_{xx}^{14} is $1 \times K$ matrix, such as

$$\left(B_{xx}^{14}\right)_{1k} = -\varsigma m_k\bar{\tau},$$

every k, $B_{xx}^{41} = C_{xx}^{14\prime}$; B_{xx}^{22} is $K \times K$ diagonal matrix that

$$\left(B_{xx}^{22}\right)_{kk} = \omega_\eta^2 m_k\bar{\tau},$$

every k, B_{xx}^{24} is $K \times K$ diagonal matrix such as

$$\left(B_{xx}^{24}\right)_{kk} = \omega_\eta m_k\bar{\tau},$$

every k, and $B_{xx}^{42} = B_{xx}^{24\prime}$; and B_{xx}^{44} is a is $K \times K$ diagonal matrix, such that

$$\left(B_{xx}^{44}\right)_{kk} = m_k\bar{\tau},$$

every k. Also:

$$B_\xi = \begin{bmatrix} B_\xi^{11} & 0 & 0 \\ B_\xi^{21} & 0 & 0 \\ 0 & 0 & 0 \\ B_\xi^{41} & 0 & 0 \end{bmatrix}, \tag{138}$$

where B_ξ^{11} is a 1×1 matrix such that

$$B_\xi^{11} = \tilde{\sigma}\eta^{-1}\bar{\tau} - \tilde{\sigma}s_d(s_C^{-1} - \tilde{\sigma}),$$

for every k, B_ξ^{21} is a $K \times 1$ matrix such that

$$\left(B_\xi^{21}\right)_{k1} = \tilde{\sigma}\omega_\eta m_k\bar{\tau},$$

every k, B_ξ^{41} is a $K \times 1$ matrix such as

$$\left(B_\xi^{41}\right)_{k1} = \tilde{\sigma} m_k\bar{\tau},$$

every k.

Finally, (115) can be expressed in matrix notation as

$$0 = \sum_{j=t}^{\infty} \beta^{j-t}\{H_x' x_t + \frac{1}{2}x_t' H_{xx} x_t\} + O_p^3 \tag{139}$$

where we have used the fact that the definition for aggregate output in terms of its sectorial counterparts expressed in (115) is valid at all dates. Matrices H_x and H_{xx} have, respectively, dimension $(3K+1) \times 1$ and $(3K+1) \times (3K+1)$, such as:

$$H'_x = \left[\begin{array}{cccccccccc} 1 & -m_1 & ... & -m_K & 0 & ... & 0 & 0 & ... & 0 \end{array}\right], \tag{140}$$

$$H_{xx} = \omega_\eta \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & H_{xx}^{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{141}$$

where H_{xx}^{22} is a $K \times K$ diagonal matrix such as

$$\left(H_{xx}^{22}\right)_{kk} = -m_k,$$

for every k.

4.2 Elimination of Linear Terms

In order to eliminate linear terms in (127), we need to find a set a multipliers $\vartheta_C^1, ..., \vartheta_C^K, \vartheta_B, \vartheta_H$, such as

$$\vartheta_C^1 C_x^{1\prime} + ... + \vartheta_C^K C_x^{K\prime} + \vartheta_B B'_x + \vartheta_H H'_x = A'_x \tag{142}$$

By solving the linear system of equations, one gets the following set of solution:

$$\vartheta_B = -\frac{\Phi}{\Upsilon} \tag{143}$$

$$\vartheta_H = 1 - \Xi\frac{\Phi}{\Upsilon} \tag{144}$$

and, for every k,

$$\vartheta_C^k = \frac{m_k(1-\bar{\tau})}{\kappa_k}\frac{\Phi}{\Upsilon} \tag{145}$$

where we have used the fact that $\bar{\tau} = \bar{\tau}_k$, all k, and defined:

$$\Phi \equiv 1 - \frac{\theta - 1}{\theta}\frac{(1-\bar{\tau})}{\bar{\mu}^w},$$

$$\Upsilon \equiv (\varsigma + \chi)(1-\bar{\tau}) + \tilde{\sigma} s_d - \bar{\tau}, \tag{146}$$

$$\Xi \equiv \varsigma(1-\bar{\tau}) + \tilde{\sigma} s_d - \bar{\tau}\eta^{-1} \tag{147}$$

and where s_d stand for

$$s_d \equiv \frac{\bar{s}}{\bar{Y}}.$$

Hence, using relations (127), (131), (135), (139) and (142) one can write:

$$E_{t_0}\sum_{j=t_0}^{\infty}\beta^{j-t_0}A_x'x_t = E_{t_0}\sum_{j=t_0}^{\infty}\beta^{j-t_0}[\sum_{k=1}^{K}\vartheta_C^k C_x^{k\prime}+\vartheta_B B_x'+\vartheta_H H_x']x_t \tag{148}$$

$$= -E_{t_0}\sum_{j=t_0}^{\infty}\beta^{j-t_0}\{\frac{1}{2}x_t'D_{xx}x_t + x_t'D_\xi\xi_t\} + \sum_{k=1}^{K}\vartheta_C^k\mathcal{V}_{k,t_0} + \frac{\vartheta_B\tilde{W}_{t_0}}{(1-\beta)}$$

where

$$D_{xx} = \sum_{k=1}^{K}\vartheta_C^k C_{xx,k} + \vartheta_B B_{xx} + \vartheta_H H_{xx}$$

and

$$D_\xi = \sum_{k=1}^{K}\vartheta_C^k C_\xi^k + \vartheta_B B_\xi$$

We use this last relations in order to rewrite (127)

$$U_{t_0} = \Omega E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\{A_x'x_t - \frac{1}{2}x_t'A_{xx}x_t - x_t'A_\xi\xi_t\} + tips + O_p^3 \tag{149}$$

as

$$U_{t_0} = -\Omega E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\{\frac{1}{2}x_t'[A_{xx}+D_{xx}]x_t + x_t'[A_\xi + D_\xi]\xi_t\} +$$

$$+\sum_{k=1}^{K}\vartheta_C^k\mathcal{V}_{k,t_0} + \frac{\vartheta_B\tilde{W}_{t_0}}{(1-\beta)} + tips + O_p^3$$

$$U_{t_0} \equiv -\Omega E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\{\frac{1}{2}x_t'Q_{xx}x_t + x_t'Q_\xi\xi_t\} + T_{t_0} + tips + O_p^3 \tag{150}$$

where

$$T_{t_0} = \Omega\{\sum_{k=1}^{K}\vartheta_C^k\mathcal{V}_{k,t_0} + \frac{\vartheta_B\tilde{W}_{t_0}}{(1-\beta)}\} \tag{151}$$

is a vector of predetermined variables and where Q_{xx} and Q_ξ can be defined, respectively, as

$$Q_{xx}=\left[\begin{array}{cccc} Q_{xx}^{11} & Q_{xx}^{12} & 0 & Q_{xx}^{14} \\ Q_{xx}^{21} & Q_{xx}^{22} & 0 & Q_{xx}^{24} \\ 0 & 0 & Q_{xx}^{33} & 0 \\ Q_{xx}^{41} & Q_{xx}^{42} & 0 & Q_{xx}^{44} \end{array}\right], \tag{152}$$

where Q_{xx}^{11} is a 1×1 matrix such as

$$Q_{xx}^{11}=-(1-\tilde{\sigma})-[\tilde{\sigma}\omega_C+\varsigma^2](1-\bar{\tau})\frac{\Phi}{\Upsilon}+ \\ -\frac{\Phi}{\Upsilon}[\tilde{\sigma}s_d(\omega_C+\tilde{\sigma})-\varsigma\eta^{-1}\bar{\tau}]+(1-\Xi\frac{\Phi}{\Upsilon})\omega_\eta,$$

Q_{xx}^{22} is a $K\times K$ diagonal matrix such as, for a generic k diagonal element,

$$\left(Q_{xx}^{22}\right)_{kk}=m_k\{(1-\Phi)\upsilon+\chi(\upsilon+\omega_\eta)(1-\bar{\tau})\frac{\Phi}{\Upsilon}-\omega_\eta^2\bar{\tau}\frac{\Phi}{\Upsilon}-(1-\Xi\frac{\Phi}{\Upsilon})\omega_\eta\},$$

Q_{xx}^{33} is a $K\times K$ diagonal matrix such as, for a generic k diagonal element,

$$\left(Q_{xx}^{33}\right)_{kk}=\theta\kappa_k^{-1}m_k\{(1-\Phi)+\frac{\Phi}{\Upsilon}(1-\bar{\tau})\upsilon\},$$

Q_{xx}^{44} is a $K\times K$ null matrix, once

$$\left(Q_{xx}^{44}\right)_{kk}=-\frac{\Phi}{\Upsilon}m_k(1-\bar{\tau})\delta+\frac{\Phi}{\Upsilon}m_k\bar{\tau}=0,$$

Q_{xx}^{12} a $1\times K$ such as its typical k^{th}-column element is

$$\left(Q_{xx}^{12}\right)_{1k}=\frac{\Phi}{\Upsilon}\varsigma\omega_\eta m_k,$$

and $Q_{xx}^{21}=Q_{xx}^{12\prime}$; Q_{xx}^{14} a $1\times K$ null matrix once, for any k^{th}-column element,

$$\left(Q_{xx}^{14}\right)_{1k}=\frac{\Phi}{\Upsilon}\varsigma m_k\{(1-\bar{\tau})\delta-\bar{\tau}\}=0,$$

and $Q_{xx}^{41}=Q_{xx}^{14\prime}$; and, finally, Q_{xx}^{24} is a $K\times K$ null matrix such as, for every k diagonal element,

$$\left(Q_{xx}^{24}\right)_{kk}=\frac{\Phi}{\Upsilon}\omega_\eta[m_k(1-\bar{\tau})\delta-m_k\bar{\tau}]=0,$$

and $Q_{xx}^{42}=Q_{xx}^{24\prime}$. In the same fashion, we define the matrix Q_ξ as

$$Q_\xi = \begin{bmatrix} Q_\xi^{11} & 0 & 0 \\ Q_\xi^{21} & Q_\xi^{22} & Q_\xi^{23} \\ 0 & 0 & 0 \\ Q_\xi^{41} & 0 & 0 \end{bmatrix}, \tag{153}$$

where Q_ξ^{11} is a 1×1 matrix such as

$$Q_\xi^{11} = -\tilde{\sigma} + [\omega_C + \tilde{\sigma} + \omega_\eta]\tilde{\sigma}(1-\bar{\tau})\frac{\Phi}{\Upsilon} - \frac{\Phi}{\Upsilon}[\tilde{\sigma}\eta^{-1}\bar{\tau} - \tilde{\sigma}s_d(s_C^{-1} - \tilde{\sigma})],$$

Q_ξ^{22} is a $K \times K$ diagonal matrix such as, for a generic k diagonal element,

$$\left(Q_\xi^{22}\right)_{kk} = -m_k\{(1-\Phi)\upsilon + \frac{\Phi}{\Upsilon}(1-\bar{\tau})\upsilon^2\},$$

Q_ξ^{21} a $K \times 1$ dimension matrix such as its typical k^{th}-line element is

$$\left(Q_\xi^{21}\right)_{k1} = -\omega_\eta\tilde{\sigma}\frac{\Phi}{\Upsilon}m_k(1-\bar{\tau}) - \frac{\Phi}{\Upsilon}\tilde{\sigma}\omega_\eta m_k\bar{\tau} = -m_k\omega_\eta\tilde{\sigma}\frac{\Phi}{\Upsilon},$$

Q_ξ^{23} a $K \times K$ diagonal matrix such as its typical k^{th}-line element is

$$\left(Q_\xi^{23}\right)_{k1} = m_k\frac{\Phi}{\Upsilon}(1-\bar{\tau})\upsilon,$$

and Q_ξ^{41} a $K \times 1$ dimension matrix of null elements once its typical k^{th}-line element is given by

$$\left(Q_\xi^{41}\right)_{k1} = \frac{\Phi}{\Upsilon}\tilde{\sigma}\{m_k(1-\bar{\tau})\delta - m_k\bar{\tau}\} = 0.$$

As in Benigno and Woodford (2003) and Ferrero (2005), references to sector tax rates have been eliminated. These are important for welfare considerations only to the extent they influence the wedge between desired and actual levels of sectorial and aggregate outputs. Only references to sectorial inflation measures, sectorial and aggregate outputs remain, which imply (150) can be simplified further by getting rid-off tax rates references and by separating terms referring to sectorial and overall outputs from references to sectorial inflation. Proceeding in such fashion yields

$$U_{t_0} = -\frac{\Omega}{2}E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\{x_{y,t}'\tilde{Q}_y x_{y,t} + 2x_{y,t}'\tilde{Q}_\xi\xi_t + x_{\pi,t}'\tilde{Q}_\pi x_{\pi,t}\} + T_{t_0} + tips + O_p^3, \tag{154}$$

where $x_{y,t}$ is a $K+1 \times 1$ vector containing only references to aggregate and sectorial outputs measures, or

$$x'_{y,t} = \left[\begin{array}{cccc} \hat{Y}_t & \hat{Y}_{1,t} & ... & \hat{Y}_{K,t} \end{array}\right],$$

$x_{\pi,t}$ is a $K \times 1$ vector containing only sectorial inflation measures, or

$$x'_{\pi,t} = \left[\begin{array}{ccc} \pi_{1,t} & ... & \pi_{K,t} \end{array}\right],$$

and $\tilde{Q}_y$, $\tilde{Q}_\xi$ and $\tilde{Q}_\pi$ are given, respectively, by:

$$\tilde{Q}_y = \left[\begin{array}{cc} Q_{xx}^{11} & Q_{xx}^{12} \\ Q_{xx}^{21} & Q_{xx}^{22} \end{array}\right],$$

$$\tilde{Q}_\pi = \left[\begin{array}{c} Q_{xx}^{33} \end{array}\right],$$

$$\tilde{Q}_\xi = \left[\begin{array}{ccc} Q_\xi^{11} & 0 & 0 \\ Q_\xi^{21} & Q_\xi^{22} & Q_\xi^{23} \end{array}\right],$$

where accurate specifications for submatrices Q_{xx}^{ij} and Q_ξ^{ij} are given in (152) and (153). From (154), we now focus on the term

$$x'_{y,t}\tilde{Q}_y x_{y,t} = q_y Y_t^2 + \sum_{k=1}^{K} m_k q_{y_k} Y_{k,t}^2 + 2\sum_{k=1}^{K} m_k q_{y,y_k} Y_t Y_{k,t}, \tag{155}$$

where q terms are defined according to

$$q_y = -(1-\tilde{\sigma})-[\tilde{\sigma}\omega_C+\varsigma^2](1-\bar{\tau})\frac{\Phi}{\Upsilon}-\frac{\Phi}{\Upsilon}[\tilde{\sigma}s_d(\omega_C+\tilde{\sigma})-\varsigma\eta^{-1}\bar{\tau}]+(1-\Xi\frac{\Phi}{\Upsilon})\omega_\eta, \tag{156}$$

$$q_{y_k} = (1-\Phi)\upsilon + \chi(\upsilon+\omega_\eta)(1-\bar{\tau})\frac{\Phi}{\Upsilon} - \omega_\eta^2\bar{\tau}\frac{\Phi}{\Upsilon} - (1-\Xi\frac{\Phi}{\Upsilon})\omega_\eta, \tag{157}$$

$$q_{y,y_k} = \frac{\Phi}{\Upsilon}\varsigma\omega_\eta. \tag{158}$$

Under the assumption that wage markups is steady state as well as markups over marginal costs are the same across sectors ($\bar{\mu}_k^w = \bar{\mu}^w$ and $\theta_k = \theta$) , q coefficients are all independent of k. We use the following proposition in order to simplify (155) further:

Proposition 1 *The following expression relating sum of sectorial output variances and covariances of sectorial outputs and aggregate output is of third order:*

$$\hat{Y}_t \sum_{k=1}^{K} m_k \hat{Y}_{k,t} - \sum_{k=1}^{K} m_k \hat{Y}_{k,t}^2 = O_p^3.$$

Proof. On one hand, from (114)

$$\hat{Y}_t - \sum_{k=1}^{K} m_k \hat{Y}_{k,t} = \frac{(1-\eta^{-1})}{2}(\sum_{k=1}^{K} m_k \hat{Y}_{k,t}^2 - \hat{Y}_t^2) + O_p^3. \tag{159}$$

On the other hand, from the definition of sectorial demand it is possible to establish the following exact relation:

$$\hat{p}_{k,t} = \eta^{-1}(\hat{Y}_t - \hat{Y}_{k,t}). \tag{160}$$

Summing across sectors yields:

$$\sum_{k=1}^{K} m_k \hat{p}_{k,t} = \eta^{-1}(\hat{Y}_t - \sum_{k=1}^{K} m_k \hat{Y}_{k,t}). \tag{161}$$

From the definition of aggregate price level in terms of sectorial prices:

$$1 = \sum_{k=1}^{K} m_k p_{k,t}^{1-\eta}. \tag{162}$$

Log-approximation on (162) yields:

$$\sum_{k=1}^{K} m_k \hat{p}_{k,t} = \frac{1}{2}(1-\eta) \sum_{k=1}^{K} m_k \hat{p}_{k,t}^2 + O_p^3.$$

One can use (160) and (161) in order to replace for $\hat{p}_{k,t}$, which yields:

$$\hat{Y}_t - \sum_{k=1}^{K} m_k \hat{Y}_{k,t} = -\frac{(1-\eta^{-1})}{2}(\hat{Y}_t^2 - 2\hat{Y}_t \sum_{k=1}^{K} m_k \hat{Y}_{k,t} + \sum_{k=1}^{K} m_k \hat{Y}_{k,t}^2) + O_p^3. \tag{163}$$

Comparing (159) and (163) yields the result. ■

Given proposition above, (155) is equivalent to:

$$x'_{y,t} \tilde{Q}_y x_{y,t} = q_y Y_t^2 + q'_{y_k} \sum_{k=1}^{K} m_k Y_{k,t}^2 + O_p^3, \tag{164}$$

where:

$$q'_{y_k} = q_{y_k} + 2q_{y,y_k}.$$

We now focus on the second term of (154), containing the interactions between endogenous variables and exogenous processes:

$$x'_{y,t} \tilde{Q}_\xi \xi_t = q_{yG} \hat{Y}_t \hat{G}_t + q_{y_k G} \sum_{k=1}^{K} m_k Y_{k,t} \hat{G}_t + \sum_{k=1}^{K} m_k \hat{Y}_{k,t} [q_{y_k a_k} \hat{a}_{k,t} + q_{y_k \mu_k} \hat{\mu}_{k,t}]. \tag{165}$$

where coefficients defined as

$$q_{yG} = -\tilde{\sigma} + [\omega_C + \tilde{\sigma} + \omega_\eta]\tilde{\sigma}(1-\bar{\tau})\frac{\Phi}{\Upsilon} - \frac{\Phi}{\Upsilon}\tilde{\sigma}[\eta^{-1}\bar{\tau} - s_d(s_C^{-1} - \tilde{\sigma})], \tag{166}$$

$$q_{y_k a_k} = -(1-\Phi)\upsilon - \frac{\Phi}{\Upsilon}(1-\bar{\tau})\upsilon^2, \tag{167}$$

$$q_{y_k G} = -\omega_\eta \tilde{\sigma}\frac{\Phi}{\Upsilon}, \tag{168}$$

$$q_{y_k \mu_k} = \frac{\Phi}{\Upsilon}(1-\bar{\tau})\upsilon \tag{169}$$

are all independent of sector-specific characteristics.

Proposition 2 *The following expression is, at least, of second order:*

$$\hat{Y}_t - \sum_{k=1}^{K} m_k \hat{Y}_{k,t} = O_p^2.$$

Proof. Follows directly from (115). ■

From above, the following holds:

Proposition 3 *The following expression holds:*

$$[\hat{Y}_t - \sum_{k=1}^{K} m_k Y_{k,t}]\hat{G}_t = O_p^3.$$

Proof. From proposition above plus the fact that all exogenous processes are O_p^1. ■

From (165), one can use above to get:

$$x'_{y,t}\tilde{Q}_\xi \xi_t = \sum_{k=1}^{K} m_k Y_{k,t}[q'_{y_k G}\hat{G}_t + q_{y_k a_k}\hat{a}_{k,t} + q_{y_k \mu_k}\hat{\mu}_{k,t}] + O_p^3, \tag{170}$$

where

$$q'_{y_k G} = q_{yG} + q_{y_k G}.$$

We now focus our attention on (164). The following lemma can help us simplify the expression even further.

Proposition 4 *The following expression is of third order:*

$$\hat{Y}_t^2 - \sum_{k=1}^{K} m_k \hat{Y}_{k,t}^2 = O_p^3.$$

Proof. From the first proposition:

$$\hat{Y}_t \sum_{k=1}^{K} m_k \hat{Y}_{k,t} - \sum_{k=1}^{K} m_k \hat{Y}_{k,t}^2 = O_p^3. \tag{171}$$

From the second proposition:

$$\hat{Y}_t - \sum_{k=1}^{K} m_k \hat{Y}_{k,t} = O_p^2. \tag{172}$$

Replacing (172) over (171) yields:

$$\hat{Y}_t^2 - \sum_{k=1}^{K} m_k \hat{Y}_{k,t}^2 = O_p^3,$$

once we notice that $\hat{Y}_t O_p^2$ is O_p^3. ■

From (164):

$$x'_{y,t}\tilde{Q}_y x_{y,t} = q_y[Y_t^2 - \sum_{k=1}^{K} m_k Y_{k,t}^2] + [q'_{y_k} + q_y] \sum_{k=1}^{K} m_k Y_{k,t}^2 \tag{173}$$

Applying the last Proposition above:

$$x'_{y,t}\tilde{Q}_y x_{y,t} = q''_{y_k} \sum_{k=1}^{K} m_k Y_{k,t}^2 + O_p^3, \tag{174}$$

where

$$q''_{y_k} = q'_{y_k} + q_y.$$

Replacing (170) and (174) over (154) yields the expression for the second order approximation for the utility function:

$$U_{t_0} = -\frac{\Omega}{2} E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{\lambda_{y_k} \sum_{k=1}^{K} m_k y_{k,t}^2 + \sum_{k=1}^{K} m_k \lambda_{k,\pi} \pi_{k,t}^2\} + T_{t_0} + tips + O_p^3,$$

where

$$y_{k,t} = \hat{Y}_{k,t} - \hat{Y}^*_{k,t}$$

and

$$-\hat{Y}^*_{k,t} = \lambda^{-1}_{y_k}[(q_{yG} + q_{y_k G})\hat{G}_t + q_{y_k a_k}\hat{a}_{k,t} + q_{y_k \mu_k}\hat{\mu}_{k,t}] \tag{175}$$

all k, and, most importantly,

$$\lambda_{y_k} \equiv q_{y_k} + 2q_{y,y_k} + q_y, \tag{176}$$

$$\lambda_{k,\pi} \equiv \theta\kappa_k^{-1}\{(1-\Phi) + \frac{\Phi}{\Upsilon}(1-\bar{\tau})\upsilon\} \tag{177}$$

while terms such as q_{y_k}, q_y , and q_{y,y_k} are defined from (156) to (158) and terms such as q_{yG}, $q_{y_k G}$, $q_{y_k a_k}$ and $q_{y_k \mu_k}$ are defined from (166) to (169).

5 Appendix E - Concavity

The concavity properties of the second order quadratic approximation for the utility function depend largely on the parameter values chosen. We are particular interested in determining the set of conditions that allow the second order approximation to yield a unique solution to the approximated Ramsey problem. Sufficient condition for concavity can be obtained if λ-coefficients defined in the last section are positive. We start out by considering the coefficients of sectorial inflation:

$$\lambda_{k,\pi} = \theta\kappa_k^{-1}m_k\{(1-\Phi) + \frac{\Phi}{\Upsilon}(1-\bar{\tau})\upsilon\} > 0,$$

all k, which holds if

$$(1-\Phi) + \frac{\Phi}{\Upsilon}(1-\bar{\tau})\upsilon > 0.$$

The terms $1-\Phi$ and $1-\bar{\tau}$ will always be positive provided a upper bound for tax rates in steady state. Considering also the implausibility of negative values for the inverse of Frisch elasticity, then $\upsilon > 0$. Φ is bounded bellow by $\bar{\tau}$, which is always great than zero. A sufficient condition for $\lambda_{k,\pi} > 0$ is having a set of parameter values such as $\Upsilon > 0$, or

$$(\varsigma + \chi)(1-\bar{\tau}) + \tilde{\sigma}s_d > \bar{\tau},$$

which will always hold provided tax rates are not excessively high and once we consider that ν, $\tilde{\sigma}$ and s_d are all positive.

Having considered the conditions upon which the coefficients over inflation variance are positive, we turn now to the conditions that ensure that the coefficients over sectorial output variances are also positive. We carry out a numerical analysis of the sensibility of values of λ_{y_k} under the baseline calibration, largely based of Rotemberge and Woodford (1998), Benigno and Woodford (2003) and Ferrero (2005). That is characterized by: a wage markup in steady state ($\bar{\mu}_k$) of 5%, a λ set to .98, a within sector elasticity of substitution (θ) of 10, a government expenses over GDP ($\bar{G}/\bar{Y}$) of 25%, β of 99%, which corresponds to steady state interest rate of 4.1% year, a government primary surplus over GDP ($\bar{s}$) of 2.5%, a Frisch elasticity of labor supply (ν) of .47, a coefficient of risk aversion (σ) of 2 and a cross-sector elasticity of substitution (η) of 4.5. The two graphs below present sufficient conditions for concavity (i.e.: $\lambda_{y_k} > 0$) of the linear-quadratic approximation to the utility function as some key parameter values change. In the first graph we contrast different values for the elasticity of substitution across sectors with steady state tax rate levels, while keeping the other parameters confined to the basic calibration. Steady state taxation level is confined between 5% to 50% of GDP, for a constant primary surplus of 2.5%. One should note that either changes in steady state taxation levels nor changes in the elasticity of substitution across sectors affect sufficient conditions for concavity in a significant extent. Concavity fails only when η is close to zero.

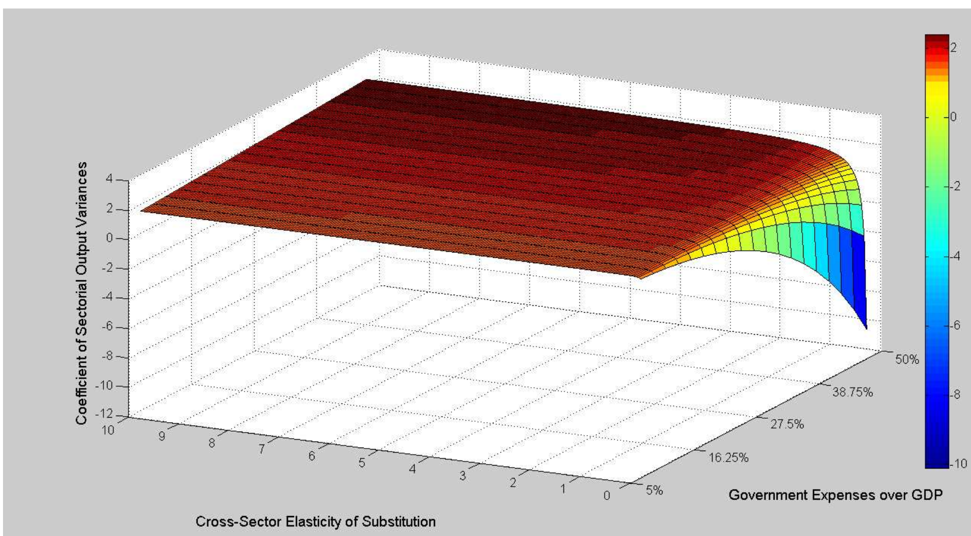


Figure1: Sufficient Conditions for Concavity as a function of Cross-Sector Elasticity of Substitution and Steady State Tax Rate.

The following graph explores sufficient conditions for concavity for a variety of different values on the degree of risk aversion and on the elasticity of substitution across-sectors, while we fix the steady state tax rate level at 25% of GDP and a primary surplus of 2.5%. Other parameter values equal those of the baseline calibration. Concavity of utility function is attained for reasonable parameters of risk aversion and substitution elasticity amongst goods from different sectors.

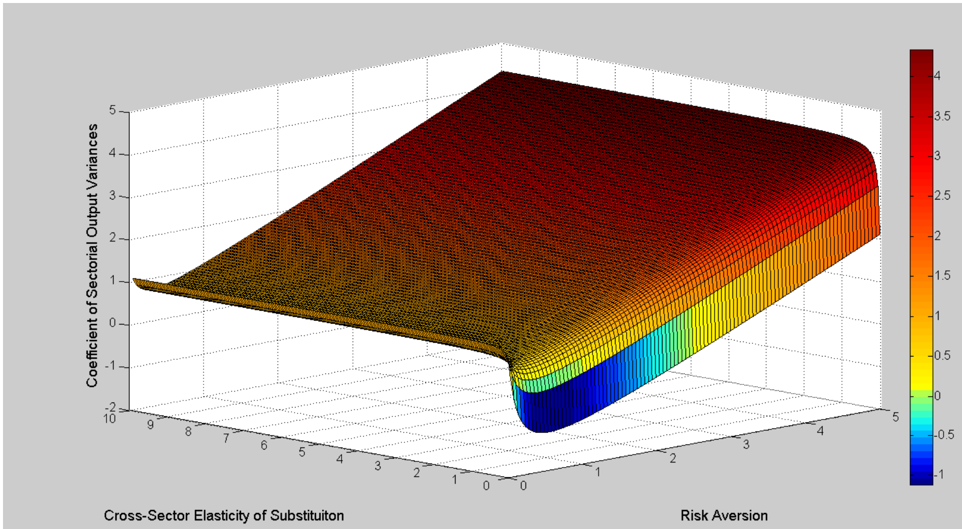


Figure 2: Sufficient Conditions for Concavity as a function of Cross-Sector Elasticity of Substitution and Risck Aversion

6 Appendix F - Log-linear Approximation of Restrictions

6.1 Definition of Target Variables

Explicitly using the assumption that sector specific tax rates as well as wage markups in steady state are the same across sectors, we can define the target level of aggregate output using (175):

$$-\hat{Y}^*_{k,t} = \lambda^{-1}_{y_k}[(q_{yG} + q_{y_kG})\hat{G}_t + q_{y_ka_k}\hat{a}_{k,t} + q_{y_k\mu_k}\hat{\mu}_{k,t}], \tag{178}$$

and

$$-\hat{Y}^*_t = \lambda^{-1}_{y_k}[(q_{yG} + q_{y_kG})\hat{G}_t + q_{y_ka_k}\hat{a}_t + q_{y_k\mu_k}\hat{\mu}_t] \tag{179}$$

where coefficients q are defined elsewhere and $\bar{a}_t$ and $\bar{\mu}_t$ are respectively defined as:

$$\hat{a}_t = \sum_{k=1}^{K} m_k \hat{a}_{k,t}$$

and

$$\hat{\mu}_t^w = \sum_{k=1}^{K} m_k \hat{\mu}_{k,t}^w.$$

6.2 Aggregate supply and cost-push disturbance term

We take the first order terms of AS equation in (95), valid for all k.

$$\begin{aligned}\pi_{k,t} &= \kappa_k\{(\tilde{\sigma} - \eta^{-1})\hat{Y}_t + (\nu + \eta^{-1})\hat{Y}_{k,t} + \delta\hat{\tau}_{k,t} \\ &\quad -\tilde{\sigma}\hat{G}_t - (1+\nu)\hat{a}_{k,t} + \hat{\mu}_{k,t}^w\} + \beta E_t \pi_{k,t+1} + O_p^2.\end{aligned}$$

Adding and subtracting, respectively, the terms referring to overall and sectorial output targets with the appropriate coefficients yield

$$\pi_{k,t} = \kappa_k\{(\tilde{\sigma} - \eta^{-1})y_t + (\nu + \eta^{-1})y_{k,t} + \delta(\hat{\tau}_{k,t} - \hat{\tau}_{k,t}^*)\} + \beta E_t \pi_{k,t+1} + u_{k,t}, \quad (180)$$

for every k, where the definition for the cost-push term $u_{k,t}$ is given by

$$u_{k,t} = \kappa_k[1 - (\nu + \eta^{-1})\lambda_{y_k}^{-1} q_{y_k \mu_k}]\hat{\mu}_{k,t}^w \quad (181)$$

and

$$\begin{aligned}-\delta\hat{\tau}_{k,t}^* &= -[(\tilde{\sigma} + \nu)\lambda_{y_k}^{-1}(q_{yG} + q_{y_k G}) + \tilde{\sigma}]\hat{G}_t - (\tilde{\sigma} - \eta^{-1})\lambda_{y_k}^{-1} q_{y_k \mu_k}\hat{\mu}_t^w \quad (182) \\ &\quad -(\tilde{\sigma} - \eta^{-1})\lambda_{y_k}^{-1} q_{y_k a_k}\hat{a}_t - [(\nu + \eta^{-1})\lambda_{y_k}^{-1} q_{y_k a_k} + (1+\nu)]\hat{a}_{k,t}.\end{aligned}$$

can be understood as the target level for distortive taxation in sector k. Averaging across sectors allows us to determine the generalized aggregate first order approximation for the AS equation (Phillips Curve), similar to Carvalho (2006).

$$\pi_t = \sum_{k=1}^{K} m_k \kappa_k\{(\tilde{\sigma} - \eta^{-1})y_t + (\nu + \eta^{-1})y_{k,t} + \delta(\hat{\tau}_{k,t} - \hat{\tau}_{k,t}^*) + u_{k,t}\} + \beta E_t \pi_{t+1} \quad (183)$$

6.3 Budget Constraint and fiscal disturbance term

We start by taking a first order approximation to expression (109), yielding

$$\hat{b}_{t-1}^* - \tilde{\sigma}(\hat{Y}_t - \hat{G}_t) - \pi_t = (1-\beta)\sum_{t=t_0}^{\infty}\beta^{t-t_0}\{b_y\hat{Y}_t + \bar{\tau}s_d^{-1}\sum_{k=1}^{K} m_k[\hat{\tau}_k + \omega_\eta \hat{Y}_{k,t}] + b_G\hat{G}_t\}, \quad (184)$$

where we have defined for convenience the terms b_y and b_G, respectively, as

$$b_y \equiv s_d^{-1}\bar{\tau}\eta^{-1} - \tilde{\sigma},$$

and

$$b_G \equiv \tilde{\sigma} - s_d^{-1}.$$

Expression (184) can be written in recursive terms. Using the definition for aggregate output in terms of sectorial outputs and the definitions for target variables given in (175) and (182), we get:

$$\hat{b}_{t-1}^{*} - \tilde{b}_y y_t - \pi_t + \zeta_t = (1-\beta)\bar{\tau}s_d^{-1}\sum_{k=1}^{K} m_k(\hat{\tau}_{k,t} - \hat{\tau}_{k,t}^{*}) + \beta E_t[\hat{b}_t^{*} - \tilde{\sigma}y_{t+1} - \pi_{t+1}], \tag{185}$$

where

$$\tilde{b}_y \equiv \tilde{\sigma} + (1-\beta)(b_y + \bar{\tau}\omega_\eta s_d^{-1}),$$

and

$$\zeta_t \equiv [\tilde{\sigma} - (1-\beta)b_G]\hat{G}_t - \tilde{b}_y Y_t^{*} - (1-\beta)\bar{\tau}s_d^{-1}\sum_{k=1}^{K} m_k[\hat{\tau}_{k,t}^{*}] + \tilde{\sigma}\beta E_t(\hat{Y}_{t+1}^{*} - \hat{G}_{t+1}), \tag{186}$$

is a combination of exogenous processes. ζ_t can be redefining in terms of structural shocks as

$$\zeta_t = \omega_1^G \hat{G}_t + \omega_1^a \hat{a}_t + \omega_1^\mu \hat{\mu}_t^w - \omega_2^G E_t\hat{G}_{t+1} - \omega_2^a E_t\hat{a}_{t+1} - \omega_2^\mu E_t\hat{\mu}_{t+1}^w, \tag{187}$$

where

$$\omega_1^G \equiv \tilde{\sigma} - (1-\beta)b_G + \tilde{b}_y\lambda_{y_k}^{-1}(q_{yG} + q_{y_kG}) - (1-\beta)(1-\bar{\tau})s_d^{-1}[(\tilde{\sigma}+\nu)\lambda_{y_k}^{-1}(q_{yG} + q_{y_kG}) + \tilde{\sigma}],$$

$$\omega_2^G \equiv \beta\tilde{\sigma}[1 + \lambda_{y_k}^{-1}(q_{yG} + q_{y_kG})],$$

$$\omega_1^a \equiv \tilde{b}_y\lambda_{y_k}^{-1}q_{y_ka_k} - (1-\beta)(1-\bar{\tau})s_d^{-1}[(\tilde{\sigma}+\nu)\lambda_{y_k}^{-1}q_{y_ka_k} + (1+\nu)],$$

$$\omega_2^a \equiv \tilde{\sigma}\beta\lambda_{y_k}^{-1}q_{y_ka_k},$$

$$\omega_1^\mu \equiv \lambda_{y_k}^{-1}q_{y_k\mu_k}[\tilde{b}_y - (1-\beta)(1-\bar{\tau})s_d^{-1}(\tilde{\sigma} - \eta^{-1})]$$

and

$$\omega_2^\mu \equiv \tilde{\sigma}\beta\lambda_{y_k}^{-1}q_{y_k\mu_k}.$$

6.4 Aggregate and Sectorial Output Relation

First order approximation to (115) yields:

$$\hat{Y}_t = \sum_{k=1}^{K} m_k \hat{Y}_{k,t},$$

which can be redefined in terms of deviation from aggregate and sectorial output targets, yielding

$$y_t = \sum_{k=1}^{K} m_k y_{k,t}.$$

6.5 Euler Equation and Equilibrium Interest Rate

Taking the first order approximation of the Euler equation in the main text yields

$$\hat{R}_t = \tilde{\sigma} E_t \Delta \hat{Y}_{t+1} - \tilde{\sigma} E_t \Delta \hat{G}_{t+1} + E_t \pi_{t+1} + O_p^2,$$

where we have used the relation in (92) to substitute for $\hat{C}_t$ in terms of $\hat{Y}_t$ and $\hat{G}_t$. Expressing equilibrium interest rates in terms of aggregate output gap by using definition in (175), which yields

$$\begin{aligned} \hat{R}_t = \tilde{\sigma} E_t \Delta y_{t+1} + E_t \pi_{t+1} - \tilde{\sigma}[\lambda_{y_k}^{-1}(q_{yG} + q_{y_k G}) + 1] E_t \Delta \hat{G}_{t+1} + \\ - \tilde{\sigma} \lambda_{y_k}^{-1} q_{y_k a_k} E_t \Delta \hat{a}_{t+1} - \tilde{\sigma} \lambda_{y_k}^{-1} q_{y_k \mu_k} E_t \Delta \hat{\mu}_{t+1} + O_p^2, \end{aligned}$$

7 Appendix G - Optimal Solution with Commitment

For simplicity, define:

$$\check{\tau}_{k,t} \equiv \hat{\tau}_{k,t} - \hat{\tau}^*_{k,t}.$$

Setting up the Lagrangian:

$$\max_{\substack{\{\pi_t,\pi_{1,t},\ldots,\pi_{1,t},\\ y_t,y_{1,t},\ldots,y_{K,t}\\ ,\check{\tau}_{1,t},\ldots,\check{\tau}_{K,t},\hat{b}^*_t\}}} \frac{1}{2}E_{t_0}\{\sum_{t=t_0}^{\infty}\beta^{t-t_0}\{\lambda_{y_k}\sum_{k=1}^{K}m_k y_{k,t}^2+\sum_{k=1}^{K}m_k\lambda_{k,\pi}\pi_{k,t}^2+$$

$$+2m_1M^{\pi}_{1,t}\{\pi_{1,t}-\kappa_1[(\tilde{\sigma}-\eta^{-1})y_t+(\nu+\eta^{-1})y_{1,t}+\delta\check{\tau}_{1,t}]-\beta\pi_{1,t+1}-u_{1,t}\}+...$$

$$+2m_KM^{\pi}_{K,t}\{\pi_{K,t}-\kappa_K[(\tilde{\sigma}-\eta^{-1})y_t+(\nu+\eta^{-1})y_{K,t}+\delta\check{\tau}_{K,t}]-\beta\pi_{K,t+1}-u_{K,t}\}+$$

$$+2M^b_t\{\hat{b}^*_{t-1}-\tilde{b}_y y_t-\pi_t-(1-\beta)\bar{\tau}s_d^{-1}\sum_{k=1}^{K}m_k\check{\tau}_{k,t}-\beta E_t[\hat{b}^*_t-\tilde{\sigma}y_{t+1}-\pi_{t+1}]+\zeta_t\}+$$

$$+2M^y_t[y_t-\sum_{k=1}^{K}m_k y_{k,t}]+2M^{\pi}_t[\pi_t-\sum_{k=1}^{K}m_k\pi_{k,t}]\}\}$$

$$+2\sum_{k=1}^{K}m_kM^{\pi}_{k,-1}[-\pi_{k,0}]+2M^b_{-1}[\pi_0]+2M^b_{-1}[\tilde{\sigma}y_0]$$

where M^x_t denotes the multiplier of equation referred to variable x and where the last line correspond to the preconditions that allow the problem to be valid for all $t\geq 0$. As usual for a cashless economy case, the Euler equation defining equilibrium interest rate as a function of exogenous shocks and evolution of aggregate product is not relevant, serving only to determine the equilibrium interest rates once optimal paths for sectorial outputs and inflations as well as tax rates and debt level are already chosen. FOCs are given by:

With respect to $\pi_{t,k}$

$$\lambda_{\pi,k}\pi_{t,k}+M^{\pi}_{k,t}-M^{\pi}_{k,t-1}=M^{\pi}_t. \tag{188}$$

With respect to π_t

$$M^{\pi}_t=M^b_t-M^b_{t-1}. \tag{189}$$

With respect to $\check{\tau}_{k,t}$

$$M^{\pi}_{k,t}=-M^b_t\frac{(1-\bar{\tau})(1-\beta)}{\kappa_k}s_d^{-1}. \tag{190}$$

With respect to y_t

$$-\sum_{k=1}^{K}m_kM^{\pi}_{k,t}\kappa_k(\tilde{\sigma}-\eta^{-1})-M^b_t\tilde{b}_y+M^b_{t-1}\tilde{\sigma}+M^y_t=0. \tag{191}$$

With respect to $y_{k,t}$

$$\lambda_{y_k}y_{k,t}-M^{\pi}_{k,t}[\kappa_k(\nu+\eta^{-1})]-M^y_t=0. \tag{192}$$

With respect to b^*_t

$$M_t^b = E_t M_{t+1}^b, \tag{193}$$

plus the problem's constraints. Substituting (189) and (190) into (188) yields

the law of motion to sectorial inflation in terms of debt Lagrange Multiplier M_t^b, that is:

$$\pi_{k,t} = \psi_k^\pi (M_t^b - M_{t-1}^b), \tag{194}$$

where

$$\psi_k^\pi \equiv \lambda_{\pi,k}^{-1} \left[1 + \frac{(1-\beta)(1-\bar{\tau})s_d^{-1}}{\kappa_k} \right].$$

From (191),

$$M_t^y = \tilde{\Phi}_1 M_t^b - \tilde{\Phi}_2 M_{t-1}^b, \tag{195}$$

where

$$\tilde{\Phi}_1 = \tilde{b}_y - (1-\bar{\tau})(1-\beta)s_d^{-1}(\tilde{\sigma} - \eta^{-1}),$$

$$\tilde{\Phi}_2 = \tilde{\sigma}.$$

Taking (192), replacing for $M_{k,t}^\pi$ from (190) and isolating for $y_{k,t}$ yields

$$y_{k,t} = \varphi_1 M_t^b - \varphi_2 M_{t-1}^b, \tag{196}$$

where

$$\varphi_1 \equiv \lambda_{y_k}^{-1}[\tilde{\Phi}_1 - (1-\bar{\tau})(1-\beta)s_d^{-1}(\nu + \eta^{-1})],$$

$$\varphi_2 \equiv \lambda_{y_k}^{-1} \tilde{\Phi}_2,$$

which establishes a relation between sectorial output and aggregate variables.

Summing up across sectors yields the aggregate output in terms of debt Lagrange Multiplier:

$$y_t = \Sigma_1 M_t^b - \Sigma_2 M_{t-1}^b, \tag{197}$$

where we defined coefficients Σ_1 and Σ_2, respectively as:

$$\Sigma_1 \equiv \varphi_1,$$

$$\Sigma_2 \equiv \varphi_2.$$

where definitions for φ coefficients are give elsewhere.

We now use (197), (196) and (194) over the sectorial Phillips Curve in order to establish the law of motion for tax rates in each sector:

$$
\begin{aligned}
&\psi_k^\pi(M_t^b - M_{t-1}^b) - \kappa_k(\tilde{\sigma} - \eta^{-1})(\Sigma_1 M_t^b - \Sigma_2 M_{t-1}^b) + \\
&- \kappa_k(\nu + \eta^{-1})(\varphi_1 M_t^b - \varphi_2 M_{t-1}^b) - \kappa_k \delta \check{\tau}_{k,t} - \beta\psi_k^\pi(E_t M_{t+1}^b - M_t^b) = u_{k,t}.
\end{aligned}
$$

Using (193):

$$
\begin{aligned}
\check{\tau}_{k,t} = (\kappa_k\delta)^{-1}\{&\psi_k^\pi(M_t^b - M_{t-1}^b) - \kappa_k(\tilde{\sigma} - \eta^{-1})(\Sigma_1 M_t^b - \Sigma_2 M_{t-1}^b) + \\
&- \kappa_k(\nu + \eta^{-1})(\varphi_1 M_t^b - \varphi_2 M_{t-1}^b) - u_{k,t}\}
\end{aligned}
$$

or

$$
\check{\tau}_{k,t} = \phi_{k,1} M_t^b - \phi_{k,2} M_{t-1}^b - (\kappa_k\delta)^{-1} u_{k,t}, \tag{198}
$$

where we have defined:

$$
\phi_{k,1} = (\kappa_k\delta)^{-1}\{\psi_k^\pi - \kappa_k(\tilde{\sigma} - \eta^{-1})\Sigma_1 - \kappa_k(\nu + \eta^{-1})\varphi_1\},
$$

$$
\phi_{k,2} = (\kappa_k\delta)^{-1}\{\psi_k^\pi - \kappa_k(\tilde{\sigma} - \eta^{-1})\Sigma_2 - \kappa_k(\nu + \eta^{-1})\varphi_2\}.
$$

Finally, we considering government constraint. We use (197), (196), (194), and (198) to get:

$$
\begin{aligned}
&\hat{b}_{t-1}^* - \tilde{b}_y[\Sigma_1 M_t^b - \Sigma_2 M_{t-1}^b] - (M_t^b - M_{t-1}^b)\sum_{k=1}^K m_k \psi_k^\pi + \\
&\quad - (1-\beta)s_d^{-1}\bar{\tau}\sum_{k=1}^K m_k[\phi_{k,1}M_t^b - \phi_{k,2}M_{t-1}^b - (\kappa_k\delta)^{-1}u_{k,t}] + \\
&\quad - \beta[\hat{b}_t^* - \tilde{\sigma}(\Sigma_1 M_t^b - \Sigma_2 M_{t-1}^b) - (E_t M_{t+1}^b - M_t^b)\sum_{k=1}^K m_k\psi_k^\pi] + \zeta_t = 0
\end{aligned}
$$

Using (193) and by isolating terms M_t^b and M_{t-1}^b, it is possible to establish the law of motion for debt value at maturity in terms of debt Lagrange Multiplier and exogenous shocks:

$$
\hat{b}_t^* = \beta^{-1}\hat{b}_{t-1}^* - \tilde{\Omega}_1 M_t^b + \tilde{\Omega}_2 M_{t-1}^b + \sum_{k=1}^K m_k \digamma_k u_{k,t} + \beta^{-1}\zeta_t, \tag{199}
$$

where we have defined

$$
\tilde{\Omega}_1 \equiv \beta^{-1}\{(\tilde{b}_y - \beta\tilde{\sigma})\Sigma_1 + \psi^\pi + (1-\beta)s_d^{-1}\bar{\tau}\sum_{k=1}^K m_k\phi_{k,1}\},
$$

$$\tilde{\Omega}_2 \equiv \beta^{-1}\{(\tilde{b}_y - \beta\tilde{\sigma})\Sigma_2 + \psi^\pi + (1-\beta)s_d^{-1}\bar{\tau}\sum_{k=1}^{K} m_k \phi_{k,2}\},$$

$$\digamma_k \equiv \beta^{-1}(1-\beta)s_d^{-1}\bar{\tau}(\kappa_k\delta)^{-1}$$

and

$$\psi^\pi \equiv \sum_{k=1}^{K} m_k \psi_k^\pi.$$

7.1 Determinacy under optimal policy

In order to prove determinacy we take the set of expressions resulting from solving the set of first order conditions and restrictions applied for the problem above. In this sense, we take (193), (194), (197), (196), (198), and (199) , and write in terms of the following system of equations:

$$\Gamma_0 E_t z_{t+1} = \Gamma_1 z_t + \varepsilon_{t+1}, \tag{200}$$

where the vector for the system's variables can be described as:

$$z_t = \begin{bmatrix} M_t^b \\ M_{t-1}^b \\ \pi_{t-1} \\ \vec{\pi}_{k,t-1} \\ y_{t-1} \\ \vec{y}_{k,t-1} \\ \vec{\tau}_{k,t-1} \\ b_{t-1}^* \end{bmatrix},$$

where notation $\vec{x}_{k,t}$ refers to the full set of sectorial variables x. This disposition

of variables allow to write Γ_1 as an inferior triangular matrices whose eigenvalues lie in the main diagonal. Matrix Γ_1 can be defined as

$$\Gamma_1 = \begin{bmatrix} 1 & * & * & * & * & * & * & * \\ 1 & 0 & * & * & * & * & * & * \\ \psi^\pi & -\psi^\pi & 0 & * & * & * & * & * \\ \vec{\psi}_k^\pi & -\vec{\psi}_k^\pi & 0 & 0 & * & * & * & * \\ \Sigma_1 & -\Sigma_2 & 0 & 0 & 0 & * & * & * \\ \vec{\varphi}_1 & -\vec{\varphi}_2 & 0 & 0 & 0 & 0 & * & * \\ \vec{\phi}_{k,1} & -\vec{\phi}_{k,2} & 0 & 0 & 0 & 0 & 0 & * \\ -\tilde{\Omega}_1 & \tilde{\Omega}_2 & 0 & 0 & 0 & 0 & 0 & \beta^{-1} \end{bmatrix}$$

and there are two non-zero eigenvalues, 1, stable, and β^{-1}, unstable. Matrix Γ_1 is an identity matrix, whose eigenvalues equal one. We can redefine (200) as

$$E_t z_{t+1} = \Gamma_0^{-1}\Gamma_1 z_t + \Gamma_0^{-1}\varepsilon_{t+1}. \tag{201}$$

Because Γ_0 is identity, Γ_0^{-1} is identity, and therefore $\Gamma_0^{-1}\Gamma_1$ has the same eigenvalues of Γ_1, one unstable, and other stable. As they match the number of forward looking and backward looking variables, this fact alone allows us to establish determinacy for (200).

Finally, it is relevant to notice that under commitment, optimal solution imply that policy is conducted such a way that:

$$E_t \pi_{k,t+1} = 0, \tag{202}$$

every k. It is somewhat a more strict condition than for an economy with homogeneous stickiness. In order to see this, we take leads in (194), apply expectation and use relation (193). In its turn, (202) for every k imply the same behavior for aggregate inflation, or:

$$E_t \pi_{t+1} = 0. \tag{203}$$

Also, for very k, (194) and (196) imply

$$\Delta y_{k,t} = \frac{\varphi_1}{\psi_k^\pi}\pi_{k,t} - \frac{\varphi_2}{\psi_k^\pi}\pi_{k,t-1} \tag{204}$$

and the aggregate relation

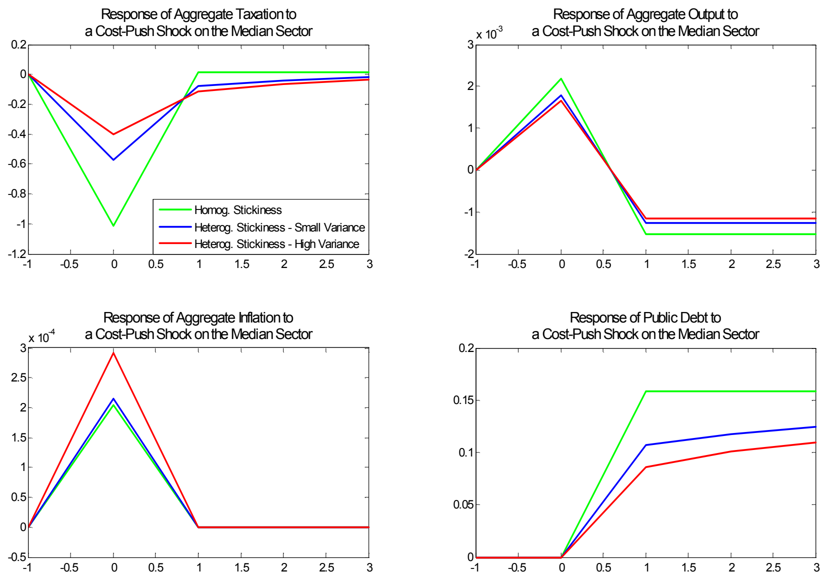
$$\Delta y_t = \frac{\Sigma_1}{\psi^\pi}\pi_t - \frac{\Sigma_2}{\psi^\pi}\pi_{t-1} \tag{205}$$

Using (198), we define optimal sectorial taxation as a function of date t sectorial inflation and output, as well as aggregate output.

$$\hat{\tau}_{k,t} = (\kappa_k\delta)^{-1}\{\pi_{k,t} - \kappa_k\varsigma y_t - \kappa_k\chi y_{k,t} - u_{k,t}\} \tag{206}$$

8 Appendix H: Cost-Push - Homogeneous Taxation

The following figure presents the response of aggregate variables to a cost-push shock in the median stickiness sector:



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