The real solution of the Weitzman-Gollier Puzzle

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Abstract: The Weitzman-Gollier Puzzle centered on the question of whether certainty equivalent discount rates should be growing or declining functions of time in capital markets with perfectly autocorrelated stochastic interest rates. Absent a convincing solution of the puzzle in the context of risk neutrality, most of the literature trying to reconcile the two approaches appealed to the notion of risk-aversion, and many claim having solved the puzzle while endorsing the notion of declining discount rates (DDRs). This note proves that the DDR recommendation results from the fallacy of ignoring that the expectation of the inverses is not equal to the inverse of the expectation and shows how incorrect CERs can be computed from correct ones and vice versa. Consequently, the Weitzman-Gollier Puzzle is not a puzzle, but an insidious, long undetected mistake.

Keywords: Weitzman-Gollier Puzzle; Declining discount rates; Discounting

JEL classification: D61; H43

The literature on the Weitzman-Gollier puzzle, based on Weitzman (1998 and 2001) and Gollier (2004), centered on the question of whether certainty equivalent discount rates should be growing or declining functions of time in capital markets with perfectly autocorrelated stochastic interest rates. Declining certainty equivalents (CERs) can be derived from the expected value of discount factors while growing ones can be derived from the expected value of compound factors.

The respective CERs are the following:

$$r^{w}(t) = -\left({}^{1}/_{t}\right)\ ln\left(\sum_i p_i e^{-r_i t}\right) \tag{1}$$

$$r^{*}(t) = \left({}^{1}/_{t}\right)\ ln\left(\sum_i p_i e^{r_i t}\right) \tag{2}$$

where the r_i are all possible positive interest rates with probabilities p_i of occurring. The r_i are constant through time t, making interest rates perfectly autocorrelated.

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Space does not permit reviewing the voluminous literature that this conundrum engendered, but the sense of dismay felt by many is apparent in Ben Groom *et al* (2005):

> "So, confusingly, whereas in the absence of uncertainty the two decision criteria are equivalent, once uncertainty regarding the discount rate is introduced the appropriate discount rate for use in CBA depends upon whether we choose ENPV or ENFV as our decision criterion. In the former case, discount rates are declining and in the latter they are rising through time. It is not immediately clear which of these criteria is correct."

This echoes Pazner and Razin (1975), which concludes "as the two criteria discussed here are equally likely, on *a priori* grounds, to be used as guides to investment decision making, and as their use may provide different rankings of investment prospects, the question arises as to what is the correct way to approach the problem in general."

Because $r^w(t)$ is derived from a certain future value (FV) and a stochastic present value (PV), and $r^*(t)$ is derived from a certain present value (PV) and a stochastic FV, Gollier (2004) stated that "Taking the expected net future value is equivalent to assuming that all risks will be borne by the future generation. […] Using the expected net present value implicitly means that it is the current generation who bears the risk." This is a strange remark, given the assumption of risk-neutrality implicit in the fundamental papers of the puzzle. But Gollier (2016) went further: "the risk-neutrality assumption underlying the two discounting rules is technically incompatible with an uncertain interest (or discount) rate […] Thus, in order to reconcile the basic ingredient of the gamma discounting approach (*i.e.*, uncertain interest rates with economic theory), a model with a risk-averse representative agent must be considered."

For reasons like these, and absent a convincing solution of the puzzle in the context of risk neutrality, most of the literature trying to reconcile the two calculation methods appeals to the notion of risk-aversion, and this is the basis on which Gollier and Weitzman (2010) claimed to have solved the puzzle[1] and endorsed the notion of DDRs.

This note does not follow that approach, however, but rather addresses the question in its original context, assuming risk neutrality.

Gollier *et al* (2008) presents a numerical example of a present value calculated according to the definition of expected discount factor $A(t)$ proposed by Weitzman (1998):

[1] A claim disputed in Szekeres (2017).

$$A(t) \stackrel{\text{def}}{=} \sum p_i e^{-r_i t} \tag{3}$$

"The rate could be either 3 percent or 5 percent with equal probability. Note that the average expected rate is 4 percent (=0.5*0.03+0.5*0.05). In this case, the expected PV of €1,000 received after *t* years is $0.5*1000*e^{-0.03t}+0.5*1000*e^{-0.05t}$." In Table B1 of Gollier *et al* (2008) the result of this calculation is given as €28.2625 for *t* =100.

If €28.2625 is the expected present value of €1,000 received after 100 years, then, according to the definition of present value, €28.2625 should compound back to €1,000, in expected value terms, using the same interest rate probabilities. Verifying, we get $0.5*28.2625*e^{0.03t} + 0.5*28.2625*e^{0.05t} = 2{,}381.0972$ when $t = 100$.

Therefore €28.2625 is *not* the expected present value (EPV) of €1,000 under the conditions stated. As shown in Szekeres (2013) the correct EPV can be readily derived from the FV, using the definition of present value:

$$EPV(t)\ \sum p_i e^{r_i t} \stackrel{\text{def}}{=} 1{,}000 \tag{4}$$

Applying this to the above numerical example we get:

$$EPV\ (0.5e^{0.03\ 100} + 0.5e^{0.05\ 100}) \equiv 1{,}000 \tag{5}$$

$$EPV = \frac{1{,}000}{0.5e^{0.03\ 100}+0.5e^{0.05\ 100}} = 11.8695 \tag{6}$$

Verifying, we get $0.5*11.8695*e^{0.03t} + 0.5*11.8695*e^{0.05t} = 1{,}000$ when $t = 100$.

Thus, the correct EPV is not the one calculated by Gollier *et al* (2008), but rather 11.8695. Definition (3) is a fallacy: *it assumes that the expectation of the inverses equals the inverse of the expectation*, which is what defines present value, whereas in reality

$$\sum p_i e^{-rt} \neq \frac{1}{\sum p_i e^{rt}} \tag{7}$$

Correctly calculated *EPV(t)* and *EFV(t)* pairs will always be congruent and will always be related to each other by the following expression:

$$EPV(t) = \frac{EFV(t)}{\sum p_i e^{r_i t}} \tag{8}$$

No value other than the above *EPV(t)* is the present value of *EFV(t).* Weitzman's *A(t),* which uses a different computational procedure, is therefore not the present value of *EFV(t).* For a conceptual interpretation of what Weitzman's *A(t)* actually computes, see Szekeres (2019).

Expression (3) overstates correct present values. Pazner and Razin (1975) shows that

$$\sum p_i e^{-rt} > \frac{1}{\sum p_i e^{rt}} \tag{9}$$

We can use a well-known statistical relationship to measure the difference between the correct and incorrect ways of computing EPVs. Let random variable X be e^{rt} and random variable Y be $1/e^{rt}$. The expected values of X and Y relate as follows:

$$E[XY] = E[X]E[Y] + cov(X, Y) \tag{10}$$

As $E[XY] = 1$ because Y is the reciprocal of X, we can rewrite (10) as follows:

$$E[Y] = \frac{1 - cov(X,Y)}{E[X]} \tag{11}$$

Which becomes the following if we replace X and Y by what they stand for:

$$\sum p_i e^{-rt} = \frac{1 - cov(e^{rt},\ e^{-rt})}{\sum p_i e^{rt}} \tag{12}$$

This is illustrated in the following table, with data taken from Scenario B of Gollier *et al* (2008):

Table 1

Relationship between the correct and incorrect present values of €1

t	E[exp(-rt]]	E[exp(*rt*]]	Correct EPV	Cov (e^{rt}, e^{-rt})	Incorrect EPV
(1)	(2)	(3)	(4)	(5)	(6)
1	0.960837	1.040862815	0.960741	-0.0001	0.960837
10	0.673674	1.499290039	0.666982	-0.01003	0.673674
50	0.152608	8.332091516	0.120018	-0.27154	0.152608
100	0.028263	84.24934801	0.01187	-1.3811	0.028263
150	0.005831	949.0297729	0.001054	-4.53383	0.005831

200	0.001262	11214.94729	8.92E-05	-13.1541	0.001262
300	6.19E-05	1638560.228	6.1E-07	-100.358	6.19E-05
400	3.07E-06	242663975.1	4.12E-09	-744.74	3.07E-06

Column (1) shows the years displayed in Table B1 of Gollier *et al* (2008), column (2) contains the latter's corresponding Scenario B EPVs divided by 1,000, to make the future value equal to €1; column (3) contains the corresponding compound factors; column (4) contains the reciprocals of the values in column (3), which are therefore the correct EPVs. Column (5) contains the covariances as defined in the text above, while column (6) contains the incorrect results predicted by equation (12). Notice that column (6) values equal column (2) values, calculated according to equation (3).

For either the correctly or incorrectly calculated present values CERs can be computed by the following expression:

$$CER(t) = -\frac{ln(EPV(t))}{t} \qquad (13)$$

The CERs corresponding to the correct and incorrect EPVs are plotted in the following Figure 1. These correspond to the values in columns (4) and (2) of Table 1, respectively.

Figure 1
Correct and Incorrect Certainty Equivalent Rates

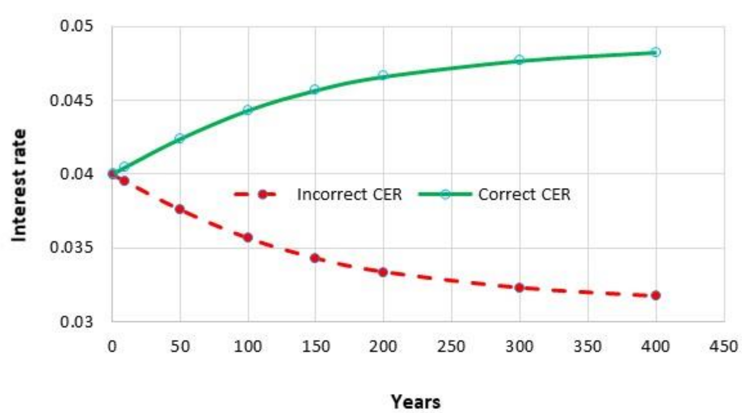

Figure B2 in Gollier *et al* (2008) shows CERs corresponding to its Scenario B that are like the incorrect CER plotted above. The discrepancy between correct and incorrect CERs is due to the to the incorrect definition of expected present value on which the latter are based. If interest rates are perfectly autocorrelated, CERs are a growing, not a declining function of time.

As the foregoing relationship between declining and growing CERs will always hold for any probability distribution of perfectly correlated interest rates, correct CERs can always be calculated from incorrect CERs (and vice versa) using expressions (12) and (13). We can unequivocally state, therefore, that their discrepancy is not a puzzle, but the predictable consequence of ignoring the fact that the expectation of the inverses is not equal to the inverse of the expectation. Consequently, the Weitzman-Gollier Puzzle is not a puzzle, but an insidious, long undetected mistake.

REFERENCES

Gollier, C. (2004) Maximizing the expected net future value as an alternative strategy to gamma discounting. Finance Research Letters Volume 1, Issue 2, June 2004, Pages 85-89. https://www.sciencedirect.com/science/article/pii/S1544612304000315

Gollier, Christian, Phoebe Koundouri, Theologos Pantelidis (2008) "Declining Discount Rates: Economic Justifications and Implications for Long-Run Policy" *Economic Policy*, Volume 23, Issue 56, 1 October 2008, Pages 758–795, https://doi.org/10.1111/j.1468-0327.2008.00211.

Gollier, Christian and Martin L. Weitzman (2010). "How Should the Distant Future Be Discounted When Discount Rates are Uncertain?" *Economics Letters,* 107 350–353. http://scholar.harvard.edu/weitzman/publications/how-should-distant-future-be-discounted-when-discount-rates-are-uncertain

Gollier, C. (2016) Gamma discounters are short-termist, *Journal of Public Economics*, Volume 142, Pages 83-90, ISSN 0047-2727, https://doi.org/10.1016/j.jpubeco.2016.08.006.

Pazner, E. A., Razin, A. (1975) On expected value vs. expected future value. *The Journal of Finance*, 30(3): 875-878.

Szekeres, Szabolcs (2013). The "Weitzman-Gollier puzzle" is not a paradox but a mistake, and it is most likely moot. *Open Science Repository Economics*, Online (open-access), e23050448. http://www.open-science-repository.com/economics-23050448.html

Szekeres, Szabolcs (2017) Checking Gollier and Weitzman's solution of the "Weitzman–Gollier puzzle". Economics Discussion Papers, No 2017-11, *Kiel Institute for the World Economy.* http://www.economics-ejournal.org/economics/discussionpapers/2017-11

Szekeres, Szabolcs (2019) Why expected discount factors yield incorrect expected present values. *Journal of Applied Economic Sciences*, Volume XIV, Spring, 1(63): 282-292. http://cesmaa.org/Extras/JAESArchive

Weitzman, M.L. (1998), "Why the far distant future should be discounted at its lowest possible rate." *Journal of Environmental Economics and Management*, 36, 201-208.

Weitzman, M. L. (2001) Gamma discounting. *American Economic Review* 91:260–271