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Abstract

The present paper develops a new Instrumental Variables (IV) estimator for spatial, dynamic panel data models with interactive effects under large N and T asymptotics. For this class of models, the only approaches available in the literature are based on quasi-maximum likelihood estimation. The approach put forward in this paper is appealing from both a theoretical and a practical point of view for a number of reasons. Firstly, the proposed IV estimator is linear in the parameters of interest and it is computationally inexpensive. Secondly, the IV estimator is free from asymptotic bias. In contrast, existing QML estimators suffer from incidental parameter bias, depending on the magnitude of unknown parameters. Thirdly, the IV estimator retains the attractive feature of Method of Moments estimation in that it can accommodate endogenous regressors, so long as external exogenous instruments are available. The IV estimator is consistent and asymptotically normal as $N, T \to \infty$, such that $N/T \to c$, where $0 < c < \infty$. The proposed methodology is employed to study the determinants of risk attitude of banking institutions. The results of our analysis provide evidence that the more risk-sensitive capital regulation that was introduced by the Dodd-Frank Act in 2011 has succeeded in influencing banks' behaviour in a substantial manner.

JEL classification: C33; C36; C38; C55; G21.

Key Words: Panel data, instrumental variables, state dependence, social interactions, common factors, large N and T asymptotics, bank risk behavior; capital regulation.

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1 Introduction

Economic behavior is intrinsically dynamic; that is, it is influenced by past own behaviour. This phenomenon, commonly described as "state dependence", is due to habit formation, costs of adjustment and economic slack, among other factors. The importance of state dependence has been recognised in the panel data literature since its infancy.[1]

More recently, it has been forcefully pointed out that, in addition to state dependence, economic behaviour is also subject to network effects, and social interactions among individual agents (see e.g. the pioneering work of Case (1991) and Manski (1993)). At the same time, economic agents inhabit common economic environments, and therefore their behaviour is subject to aggregate shocks, which may be due to shifts in technology and productivity, changes in preferences and tastes, to mention only a few. In the former case, economic agents' own behaviour is influenced by the behaviour of other agents, possibly their peers. In the latter case, agents' own behaviour is influenced by economy-wide shocks.

In panel data analysis, state dependence is commonly characterised using dynamic models; social interactions are modelled using spatial econometric techniques, as described e.g. in Kelejian and Piras (2017); and aggregate shocks are typically captured by common factors, also known as "interactive effects" (Sarafidis and Wansbeek (2012, 2020)).

The present paper develops a new Instrumental Variables (IV) estimator for spatial, dynamic panel data models with interactive effects under large N and T asymptotics, where N denotes the number of cross-sectional units and T denotes the number of time series observations. For this class of models, the only approaches available in the literature are based on quasi-maximum likelihood estimation (QMLE); see Shi and Lee (2017) and Bai and Li (2018). The approach put forward in this paper is appealing both from a theoretical and from a practical point of view for a number of reasons.

Firstly, the proposed IV estimator is linear in the parameters of interest and it is computationally inexpensive. In contrast, QML estimators are nonlinear and require estimation of the Jacobian matrix of the likelihood function, which may be subject to a high level of numerical complexity in spatial models with N large; see e.g. Section 12.3.2 in Lee and Yu

[1]See e.g. the seminar papers by Balestra and Nerlove (1966), Anderson and Hsiao (1982) and Arellano and Bond (1991). A recent overview of this literature is provided by Bun and Sarafidis (2015).

(2015). To provide some indication of the likely computational gains of our method, in the Monte Carlo section of this paper we found that the total length of time taken to estimate 2,000 replications of the model when $N = T = 200$, was roughly 4.5 minutes for IV and 4.5 hours for QMLE. Hence in this specific design, QMLE was 60 times slower than IV.[2]

Secondly, the proposed IV approach is free from asymptotic bias. In contrast, existing QML estimators suffer from incidental parameter bias, depending on the sample size and the magnitude of unknown parameters of the data generating process (DGP). Unfortunately, approximate procedures aiming to re-center the limiting distribution of these estimators using first-order bias correction can fail to fully remove the bias in finite samples, which can lead to severe size distortions, as confirmed in our Monte Carlo study.

Last, the proposed estimator retains the attractive feature of Method of Moments estimation in that it can potentially accommodate endogenous regressors, so long as external exogenous instruments are available. Even in cases where such instruments are not easy to find, our approach provides a framework for testing for endogeneity, based on the overidentifying restrictions test statistic. In contrast, the exogeneity restriction is difficult to verify within MLE and so it is typically taken for granted.

There is substantial literature on dynamic panels under large N and T asymptotics (e.g. Hahn and Kuersteiner (2002) and Alvarez and Arellano (2003), among others). More recently, several new methods have been developed to control for unobserved shocks, common factors and strong cross-sectional dependence; see e.g. Chudik and Pesaran (2015), Everaert and De Groote (2016), Moon and Weidner (2017), Juodis et al. (2020) and Norkute et al. (2020). However, none of these papers considers spatial interactions and endogenous network effects.

There is also substantial literature on spatial panel data analysis and social interactions, which, however, mostly ignores the potential presence of common unobserved shocks. Some notable contributions include Yu et al. (2008), Korniotis (2010), Debarsy et al. (2012) and Lee and Yu (2014), among others.

The present paper sits on the intersection of the above two strands of literature. Despite the fact that such intersection is highly relevant for the analysis of economic behaviour, the

[2]This ratio appears to decrease (increase) roughly exponentially with smaller (larger) values of N.

field is fairly new in the econometrics literature and, as such, it is sparse.

We put forward a two-step IV estimation approach that extends the methodology of Norkute et al. (2020) to the case of panel data models with spatial interactions, in addition to state dependence and interactive effects. The main results in our paper cannot be deduced from those in Norkute et al. (2020). Our two-step procedure can be outlined as follows: in the first step, the common factors in the exogenous covariates are projected out using principal components analysis, as in Bai (2003). Next, the slope parameters are estimated using standard IV regression, which makes use of instruments constructed from defactored regressors. In the second step, the entire model is defactored based on factors extracted from the first step residuals. Subsequently, an IV regression is implemented again using the same instruments.

The strategy above requires that the covariates used to construct instruments are strictly exogenous with respect to the purely idiosyncratic error term. That is, endogeneity arises primarily due to non-zero correlations between the regressors and the common factor component. Otherwise, the proposed approach requires the use of external instruments, which are exogenous with respect to the idiosyncratic disturbance, although they can be potentially correlated with the common factor component.

The proposed IV estimator is consistent and asymptotically normally distributed as $N, T \to \infty$ such that $N/T \to c$, where $0 < c < \infty$. Moreover, the proposed estimator does not have asymptotic bias in either cross-sectional or time series dimension. The main intuition of this result lies in that we extract factor estimates from two sets of information that are mutually independent, namely the exogenous covariates and the regression residuals. Therefore, there is no correlation between the regressors and the estimation error of the interactive fixed effects obtained in the second step. In addition, the proposed estimator is not subject to "Nickell bias" that arises with QML-type estimators in dynamic panel data models.

The underlying assumption behind our approach is that the covariates of the model are subject to a linear common factor structure. While this poses certain restrictions on the DGP from a statistical point of view, there exist several economic theories and plenty of evidence that provide support for such assumption (see e.g. Favero et al. (2005) and Heckman et al.

(2006)). Furthermore, this assumption has been frequently employed in both econometrics and statistics literature (see e.g. Pesaran et al. (2013), Bai and Li (2013), Westerlund and Urbain (2015) and Hansen and Liao (2018), among many others.) Notably, the factors that hit the regressors can be entirely different to those that enter into the regression disturbance.

We study the determinants of risk attitude of banking institutions, with emphasis on the impact of increased capital regulation. To the best of our knowledge, this is the first paper in the literature that estimates state dependence and endogenous network effects, while controlling for unobserved aggregate shocks. The results bear important policy implications and provide evidence that the more risk-sensitive capital regulation introduced by the Dodd-Frank Act in 2011 has succeeded in influencing banks' behaviour in a substantial manner. A Stata program (ado file) that computes our approach is under preparation and will be made available to the community in due course.

The remainder of this paper is organised as follows. Section 2 describes the model and the main idea behind the proposed method. Section 3 lists the set of assumptions employed and derives the large sample properties of the proposed IV estimator. Section 4 examines the finite sample performance of the estimator and confirms that it performs well. Section 5 presents the empirical illustration. A final section concludes. Proofs of the main results are documented in the Online Appendix of the paper.

Throughout, we denote the largest and the smallest eigenvalues of the $N \times N$ matrix $\mathbf{A} = (a_{ij})$ by $\mu_{\max}(\mathbf{A})$ and $\mu_{\min}(\mathbf{A})$, respectively, its trace by $\text{tr}(\mathbf{A}) = \sum_{i=1}^{N} a_{ii}$, its column sum norm by $\|\mathbf{A}\|_1 = \max_{1 \le j \le N} \sum_{i=1}^{N} |a_{ij}|$, its Frobenius norm by $\|\mathbf{A}\| = \sqrt{\text{tr}(\mathbf{A}'\mathbf{A})}$, and its row sum norm by $\|\mathbf{A}\|_\infty = \max_{1 \le i \le N} \sum_{j=1}^{N} |a_{ij}|$. The projection matrix on $\mathbf{A}$ is $\mathbf{P_A} = \mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'$ and $\mathbf{M_A} = \mathbf{I} - \mathbf{P_A}$. C is a generic positive constant large enough, $\delta_{NT}^2 = \min\{N, T\}$. We use $N, T \to \infty$ to denote that N and T pass to infinity jointly.

2 Model and Two-Step Estimation Approach

We consider the following spatial dynamic panel data model with exogenous covariates:

$$y_{it} = \psi \sum_{j=1}^{N} w_{ij} y_{jt} + \rho y_{it-1} + \mathbf{x}_{it}' \boldsymbol{\beta} + \boldsymbol{\varphi}_i^{0\prime} \mathbf{h}_t^0 + \varepsilon_{it}, \tag{2.1}$$

$i = 1, 2, \ldots, N$, $t = 1, 2, \ldots, T$, where y_{it} denotes the observation on the dependent variable for individual unit i at time period t, and $\mathbf{x}_{it}$ is a $k \times 1$ vector of regressors with slope coefficients $\boldsymbol{\beta}$. The spatial variable $\sum_{j=1}^{N} w_{ij} y_{jt}$ picks up endogenous network effects, with corresponding parameter ψ. w_{ij} denotes the (i, j)th element of the $N \times N$ spatial weights matrix $\mathbf{W}_N$, which is assumed to be known. The lagged dependent variable captures dynamic or temporal effects.[3] The error term of the model is composite: $\mathbf{h}_t^0$ and $\boldsymbol{\varphi}_i^0$ denote $r_y \times 1$ vectors of latent factors and factor loadings respectively, and ε_{it} is an idiosyncratic error.

To ensure that the covariates are endogenous to the factor component, we assume that

$$\mathbf{x}_{it} = \boldsymbol{\Gamma}_i^{0\prime} \mathbf{f}_t^0 + \mathbf{v}_{it}, \tag{2.2}$$

where $\mathbf{f}_t^0$ denotes a $r_x \times 1$ vector of latent factors, $\boldsymbol{\Gamma}_i^0$ denotes an $r_x \times k$ factor loading matrix, while $\mathbf{v}_{it}$ is an idiosyncratic disturbance of dimension $k \times 1$. Note that $\mathbf{h}_t^0$ and $\mathbf{f}_t^0$ can be identical, share some common factors, or they can be completely different but may be mutually correlated. Similarly, $\boldsymbol{\varphi}_i^0$ and $\boldsymbol{\Gamma}_i^0$ can be mutually correlated.[4]

In the context of spatial panels, the above structure of the covariates has also been studied by Bai and Li (2013). The main difference between these two specifications is that the model in Eq. (2.1) allows for dynamics through the lagged dependent variable, and the covariates in Eq. (2.2) are not necessarily driven by the same factors as those entering into the error term of y. This has an appealing generality in that, in practice, the common shocks that hit y and X may not be identical.

Stacking the T observations for each i yields

$$\begin{aligned} \mathbf{y}_i &= \psi \mathbf{Y} \mathbf{w}_i + \rho \mathbf{y}_{i,-1} + \mathbf{X}_i \boldsymbol{\beta} + \mathbf{H}^0 \boldsymbol{\varphi}_i^0 + \boldsymbol{\varepsilon}_i; \\ \mathbf{X}_i &= \mathbf{F}^0 \boldsymbol{\Gamma}_i^0 + \mathbf{V}_i, \end{aligned} \tag{2.3}$$

where $\mathbf{y}_i = (y_{i1}, \ldots, y_{iT})'$, $\mathbf{y}_{i,-1} = (y_{i0}, \ldots, y_{i,T-1})'$ and $\boldsymbol{\varepsilon}_i = (\varepsilon_{i1}, \cdots, \varepsilon_{iT})'$ denote $T \times 1$ vectors, $\mathbf{X}_i = (\mathbf{x}_{i1}, \cdots, \mathbf{x}_{iT})'$ and $\mathbf{V}_i = (\mathbf{v}_{i1}, \ldots, \mathbf{v}_{iT})'$ are matrices of order $T \times k$, while

[3] It is straightforward enough to extend this model by adding a spatial-time lag, as e.g. in Shi and Lee (2017). We do not explicitly consider this specification here in order to simplify the exposition. The theory developed in the present paper remains valid for this case, with only minor modifications. Simulation results for this specification are reported in Section 4. Furthermore, exogenous network effects, e.g. through an additional term $\sum_{j=1}^{N} w_{ij} \mathbf{x}_{jt}' \boldsymbol{\delta}$, and further lagged values of y_{it} can also be allowed in a straightforward manner without affecting the main derivations of the paper.

[4] Without loss of generality, r_y and r_x are treated as known. In practice, the number of factors can be estimated consistently using e.g. the information criteria of Bai and Ng (2002), or the eigenvalue ratio test of Ahn and Horenstein (2013). The results of the Monte Carlo section indicate that these methods provide quite accurate estimates in our design.

$\mathbf{H}^0 = (\mathbf{h}_1^0, \cdots, \mathbf{h}_T^0)'$ and $\mathbf{F}^0 = (\mathbf{f}_1^0, \cdots, \mathbf{f}_T^0)'$ are of dimensions $T \times r_y$ and $T \times r_x$, respectively. Finally, $\mathbf{Y} = (\mathbf{y}_1, \cdots, \mathbf{y}_T)'$ denotes a $T \times N$, matrix and the $N \times 1$ vector $\mathbf{w}_i$ represents the ith row of $\mathbf{W}_N$.

The model in Eq. (2.3) can be written more succinctly as follows:

$$\mathbf{y}_i = \mathbf{C}_i \boldsymbol{\theta} + \mathbf{u}_i, \tag{2.4}$$

where $\mathbf{C}_i = (\mathbf{Y}\mathbf{w}_i, \mathbf{y}_{i,-1}, \mathbf{X}_i)$, $\boldsymbol{\theta} = (\psi, \rho, \boldsymbol{\beta}')'$ and $\mathbf{u}_i = \mathbf{H}^0 \boldsymbol{\varphi}_i^0 + \boldsymbol{\varepsilon}_i$.

Let $\mathbf{X}_{i,-\tau} \equiv L^\tau \mathbf{X}_i$, where L^τ denotes the time series lag operator of order τ. We shall make use of the convention $\mathbf{X}_{i,-0} = \mathbf{X}_i$. Our estimation approach involves two steps. In the first step, the common factors in $\mathbf{X}_{i,-\tau}$ are asymptotically eliminated using principal component analysis, as advanced by Bai (2003). Next, instruments are constructed using defactored covariates. The resulting first-step IV estimator of $\boldsymbol{\theta}$ is consistent. In the second step, the entire model is defactored based on estimated factors extracted from the first step IV residuals. Subsequently, a second IV regression is implemented, using the same instruments as in step one.

In particular, define $\widehat{\mathbf{F}}_{-\tau}$ as $\sqrt{T}$ times the eigenvectors corresponding to the r_x largest eigenvalues of the $T \times T$ matrices $(NT)^{-1} \sum_{i=1}^N \mathbf{X}_{i,-\tau} \mathbf{X}'_{i,-\tau}$, for $\tau = 0, 1$. Let $\mathbf{M}_{\widehat{F}_{-\tau}} = \mathbf{I}_T - \widehat{\mathbf{F}}_{-\tau} \left(\widehat{\mathbf{F}}'_{-\tau} \widehat{\mathbf{F}}_{-\tau} \right)^{-1} \widehat{\mathbf{F}}'_{-\tau}$ denote $T \times T$ matrices that project onto the orthogonal complement of $\widehat{\mathbf{F}}_{-\tau}$, $\tau = 0, 1$.

The matrix of instruments is formulated as follows:

$$\widehat{\mathbf{Z}}_i = \left(\mathbf{M}_{\widehat{F}} \mathbf{X}_i, \quad \mathbf{M}_{\widehat{F}_{-1}} \mathbf{X}_{i,-1}, \quad \sum_{j=1}^N w_{ij} \mathbf{M}_{\widehat{F}} \mathbf{X}_j \right), \tag{2.5}$$

which is of dimension $T \times 3K$.[5]

The first-step IV estimator of $\boldsymbol{\theta}$ is defined as:

$$\widehat{\boldsymbol{\theta}} = \left(\widehat{\mathbf{A}}' \widehat{\mathbf{B}}^{-1} \widehat{\mathbf{A}} \right)^{-1} \widehat{\mathbf{A}}' \widehat{\mathbf{B}}^{-1} \widehat{\mathbf{c}}_y, \tag{2.6}$$

[5] More instruments can be used with respect to further lags of $\mathbf{X}_i$ or spatial lags $\sum_{j=1}^N w_{ij} \mathbf{X}_{j,-\tau}$, for $\tau \geq 1$. Instruments constructed from powers of the spatial weights matrix can also be used, such as $\sum_{j=1}^N w_{ij}^{(\ell)} \mathbf{X}_j$, for $\ell = 2, 3, \ldots$, where $w_{ij}^{(\ell)}$ denotes the (i,j)th element of the $N \times N$ spatial weights matrix $\mathbf{W}_N^\ell$, which is defined as the product matrix taking $\mathbf{W}_N$ and multiplying it by itself ℓ-times. It is well documented in the literature that including a larger number of instruments may render the IV estimator more efficient, although such practice can also potentially magnify small sample bias. In principle, one could devise a lag selection procedure for optimising the bias-variance trade-off for the IV estimator, as per Okui (2009); however, we leave this avenue for future research. The present paper assumes that both $\tau \geq 1$ and $\ell \geq 1$ are small and do not depend on T.

where

$$\widehat{\mathbf{A}} = \frac{1}{NT}\sum_{i=1}^{N}\widehat{\mathbf{Z}}_i'\mathbf{C}_i; \quad \widehat{\mathbf{B}} = \frac{1}{NT}\sum_{i=1}^{N}\widehat{\mathbf{Z}}_i'\widehat{\mathbf{Z}}_i; \quad \widehat{\mathbf{c}}_y = \frac{1}{NT}\sum_{i=1}^{N}\widehat{\mathbf{Z}}_i'\mathbf{y}_i\,.$$

Under certain regularity conditions, $\widehat{\boldsymbol{\theta}}$ is consistent (see Theorem 3.1 in Section 3), although asymptotically biased. Rather than bias-correcting this estimator, we put forward a second-step estimator, which is free from asymptotic bias and is potentially more efficient.

Remark 2.1 Since our approach makes use of the defactored covariates as instruments, identification of the autoregressive and spatial parameters requires that *at least one* element of $\boldsymbol{\beta}$ is not equal to zero. Otherwise, it is easily seen that identification of ρ and ψ is not possible since the lagged and spatial defactored covariates become irrelevant instruments. We believe that this requirement is mild, especially compared to the restriction that all of the elements in $\boldsymbol{\beta}$ are non-zero. Moreover, this restriction is common in estimation of spatial models using Method of Moments, see e.g. Kelejian and Prucha (2007). Note that it is not necessary to know a priori which covariates have non-zero coefficients, since by construction IV regression does not require all instruments to be relevant to all endogenous regressors.

To implement the second step, we estimate the space spanned by $\mathbf{H}^0$ from the first step IV residuals, i.e. $\hat{\mathbf{u}}_i = \mathbf{y}_i - \mathbf{C}_i\hat{\boldsymbol{\theta}}$. In specific, let $\widehat{\mathbf{H}}$ be defined as $\sqrt{T}$ times the eigenvectors corresponding to the r_y largest eigenvalues of the $T \times T$ matrix $(NT)^{-1}\sum_{i=1}^{N}\hat{\mathbf{u}}_i\hat{\mathbf{u}}_i'$.

The proposed second-step IV estimator for $\boldsymbol{\theta}$ is defined as follows:

$$\widetilde{\boldsymbol{\theta}} = (\widetilde{\mathbf{A}}'\widetilde{\mathbf{B}}^{-1}\widetilde{\mathbf{A}})^{-1}\widetilde{\mathbf{A}}'\widetilde{\mathbf{B}}^{-1}\widetilde{\mathbf{c}}_y \tag{2.7}$$

where

$$\widetilde{\mathbf{A}} = \frac{1}{NT}\sum_{i=1}^{N}\widehat{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{C}_i\,, \widetilde{\mathbf{B}} = \frac{1}{NT}\sum_{i=1}^{N}\widehat{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\widehat{\mathbf{Z}}_i\,, \tilde{\mathbf{c}}_y = \frac{1}{NT}\sum_{i=1}^{N}\widehat{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{y}_i\,.$$

Section 3 shows that the second-step IV estimator is normally distributed and correctly centered around the true value.

Remark 2.2 The validity of the procedure above crucially hangs on the assumption that $\mathbf{X}_i$ is strictly exogenous with respect to $\boldsymbol{\varepsilon}_i$. Violations of such restriction are detectable using the overidentifying restrictions test statistic, which is readily available within our framework. When strict exogeneity of $\mathbf{X}_i$ fails, identification of the model parameters requires the use of external instruments. These instruments can still be correlated with the common factor component, although they need to be exogenous with respect to $\boldsymbol{\varepsilon}_i$. The theoretical analysis of our approach based on external instruments remains exactly identical, with $\mathbf{X}_i$ in Eq. (2.5) replaced by the external instruments. As it is common practice in the literature (e.g. Robertson and Sarafidis (2015) and Kuersteiner and Prucha (2018)), in what follows we do not explicitly account for this possibility in order to avoid the cost of additional notation to separate covariates that can be used as instruments from those that cannot. Finite sample results for a model with endogenous regressors are provided in the Monte Carlo section.

A particularly useful diagnostic within IV estimation is the so-called overidentifying restrictions (J) test statistic. In our context, this test is expected to pick up potential violations of exogeneity of the defactored covariates with respect to the idiosyncratic error in the DGP for y, ε_{it}. The J test statistic is given by

$$J = \frac{1}{NT}\left(\sum_{i=1}^{N}\widetilde{\mathbf{u}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\widehat{\mathbf{Z}}_i\right)\widehat{\boldsymbol{\Omega}}^{-1}\left(\sum_{i=1}^{N}\widehat{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\widetilde{\mathbf{u}}_i\right) \tag{2.8}$$

where $\widetilde{\mathbf{u}}_i = \mathbf{y}_i - \mathbf{C}_i\widetilde{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\Omega}} = \widetilde{\sigma}_\varepsilon^2\widetilde{\mathbf{B}}$ with $\widetilde{\sigma}_\varepsilon^2 = \sum_{i=1}^{N}\widetilde{\mathbf{u}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\widetilde{\mathbf{u}}_i/NT$.

3 Asymptotic properties

The following assumptions are employed throughout the paper:

Assumption A (idiosyncratic error in y) *The disturbances ε_{it} are independently distributed across i and over t, with mean zero, $\mathbb{E}(\varepsilon_{it}^2) = \sigma_\varepsilon^2 > 0$ and $\mathbb{E}|\varepsilon_{it}|^{8+\delta} \leq C < \infty$ for some $\delta > 0$.*

Assumption B (idiosyncratic error in x) *The idiosyncratic error in the DGP for $\mathbf{x}_{it}$ satisfies the following conditions:*

1. *$\mathbf{v}_{it}$ is group-wise independent from ε_{it};*
2. *$\mathbb{E}(\mathbf{v}_{it}) = 0$ and $\mathbb{E}\|\mathbf{v}_{it}\|^{8+\delta} \leq C < \infty$;*
3. *Let $\boldsymbol{\Sigma}_{ij,st} \equiv \mathbb{E}\left(\mathbf{v}_{is}\mathbf{v}_{jt}'\right)$. We assume that there exist $\bar{\sigma}_{ij}$ and $\tilde{\sigma}_{st}$, $\|\boldsymbol{\Sigma}_{ij,st}\| \leq \bar{\sigma}_{ij}$ for all (s,t), and $\|\boldsymbol{\Sigma}_{ij,st}\| \leq \tilde{\sigma}_{st}$ for all (i,j), such that*

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\bar{\sigma}_{ij} \leq C < \infty\,, \frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\sigma}_{st} \leq C < \infty\,, \frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{s=1}^{T}\sum_{t=1}^{T}\|\boldsymbol{\Sigma}_{ij,st}\| \leq C < \infty\,.$$

4. *For every (s,t), $\mathbb{E}\|N^{-1/2}\sum_{i=1}^{N}(\mathbf{v}_{is}\mathbf{v}_{it}' - \boldsymbol{\Sigma}_{ii,st})\|^4 \leq C < \infty$.*
5. *The largest eigenvalue of $\mathbb{E}\left(\mathbf{V}_i\mathbf{V}_i'\right)$ is bounded uniformly in i and T.*

6. *For any h, we have*

$$\frac{1}{N}\sum_{i_1=1}^{N}\sum_{j_1=1}^{N}\sum_{i_2=1}^{N}\sum_{j_2=1}^{N}|w_{i_1j_1}||w_{i_2j_2}|\Big\|\frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\mathrm{cov}(\mathbf{v}_{hs}\otimes\mathbf{v}_{j_2s},\mathbf{v}_{ht}\otimes\mathbf{v}_{j_1t})\Big\|\leq C$$

7. *For any s, we have*

$$\mathbb{E}\left\|\frac{1}{\sqrt{NT}}\sum_{h=1}^{N}\sum_{t=1}^{T}\left[\mathbf{v}_{hs}'\mathbf{v}_{ht}-\mathbb{E}\left(\mathbf{v}_{hs}'\mathbf{v}_{ht}\right)\right]\mathbf{f}_t^0\right\|^2\leq C$$

8. $$\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{s_1=1}^{T}\sum_{s_2=1}^{T}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\left\|\mathrm{cov}\left(\mathbf{v}_{is_1}'\mathbf{v}_{is_2},\mathbf{v}_{jt_1}'\mathbf{v}_{jt_2}\right)\right\|\leq C$$

Assumption C (factors) $\mathbb{E}\|\mathbf{f}_t^0\|^4\leq C<\infty$, *$T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\xrightarrow{p}\mathbf{\Sigma}_F>0$ as $T\to\infty$ for some non-random positive definite matrix $\mathbf{\Sigma}_F$. $\mathbb{E}\|\mathbf{h}_t^0\|^4\leq C<\infty$, $T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\xrightarrow{p}\mathbf{\Sigma}_H>0$ as $T\to\infty$ for some non-random positive definite matrix $\mathbf{\Sigma}_H$. $\mathbf{f}_t^0$ and $\mathbf{h}_t^0$ are group-wise independent from $\mathbf{v}_{it}$ and ε_{it}.*

Assumption D (loadings) $\mathbf{\Gamma}_i^0\sim\mathrm{i.i.d}(\mathbf{0},\mathbf{\Sigma}_\Gamma)$, $\boldsymbol{\varphi}_i^0\sim\mathrm{i.i.d}(\mathbf{0},\mathbf{\Sigma}_\varphi)$, *where $\mathbf{\Sigma}_\Gamma$ and $\mathbf{\Sigma}_\varphi$ are positive definite. $\mathbb{E}\left\|\mathbf{\Gamma}_i^0\right\|^4\leq C<\infty$, $\mathbb{E}\left\|\boldsymbol{\varphi}_i^0\right\|^4\leq C<\infty$. In addition, $\mathbf{\Gamma}_i^0$ and $\boldsymbol{\varphi}_i^0$ are independent groups from ε_{it}, $\mathbf{v}_{it}$, $\mathbf{f}_t^0$ and $\mathbf{h}_t^0$.*

Assumption E (weighting matrix) *The weights matrix $\mathbf{W}_N$ satisfies that*

1. *All diagonal elements of $\mathbf{W}_N$ are zeros;*

2. *The matries $\mathbf{W}_N$ and $\mathbf{I}_N-\psi\mathbf{W}_N$ are invertible;*

3. *The row and column sums of the matrices $\mathbf{W}_N$ and $\left(\mathbf{I}_N-\psi\mathbf{W}_N\right)^{-1}$ are bounded uniformly in absolute value.*

4. $$\sum_{\ell=0}^{\infty}\left\|[\rho(\mathbf{I}_N-\psi\mathbf{W}_N)^{-1}]^\ell\right\|_\infty\leq C\,;\quad\sum_{\ell=0}^{\infty}\left\|[\rho(\mathbf{I}_N-\psi\mathbf{W}_N)^{-1}]^\ell\right\|_1\leq C$$

Assumption F (identification) *We assume that*

1. *$\overline{\mathbf{A}}=plim_{N,T\to\infty}\frac{1}{NT}\sum_{i=1}^{N}\mathbf{Z}_i'\mathbf{C}_i$ is fixed with full column rank, and $\overline{\mathbf{B}}=plim_{N,T\to\infty}\frac{1}{NT}\sum_{i=1}^{N}\mathbf{Z}_i'\mathbf{Z}_i$ is fixed and positive definite.*

2. *$\mathbb{E}\left\|T^{-1}\mathbf{Z}_i'\mathbf{Z}_i\right\|^{2+2\delta}\leq C<\infty$ and $\mathbb{E}\left\|T^{-1}\mathbf{Z}_i'\mathbf{C}_i\right\|^{2+2\delta}\leq C<\infty$ for all i and T.*

The assumptions above merit some discussion. Assumption A is in line with existing spatial literature (see e.g. Lee and Yu (2014)) and is imposed mainly for simplicity. In particular, in practice ε_{it} can be heterogeneously distributed across both i and t. However, as it commonly the case in a large body of the panel data literature based on Method of Moments estimation, we do not consider such generalizations in order to avoid unnecessary notational complexity.

Assumption B implies that the covariates of the model, $\mathbf{x}_{it}$, are strictly exogenous with respect to ε_{it}, i.e. $E\left(\varepsilon_{it}|\mathbf{x}'_{is}\right) = 0$ for all t and s. This assumption is often employed in the panel data literature with common factor residuals when both N and T are large (see e.g. Pesaran (2006) and Bai (2009)). Assumption B implies that defactored covariates are valid instruments for the endogenous variables of the model. In addition, Assumption B allows for cross-sectional and time series heteroskedasticity, as well as autocorrelation in $\mathbf{v}_{it}$. Note that, unlike with ε_{it}, here it is important to allow explicitly for this more general setup because, conditional on $\mathbf{F}^0$, the dynamics in $\mathbf{X}_i$ are solely driven by $\mathbf{V}_i$.

Assumptions C and D are standard in the principal components literature; see e.g. Bai (2003), among others. Assumption C permits correlations between $\mathbf{f}_t^0$ and $\mathbf{h}_t^0$, and within each one of them. Assumption D allows for possible non-zero correlations between $\boldsymbol{\varphi}_i^0$ and $\boldsymbol{\Gamma}_i^0$, and within each one of them. Since for each i y_{it} and $\mathbf{x}_{it}$ can be affected by common shocks in a related manner, it is potentially important to allow for this possibility in practice.

Assumption E is standard in the spatial literature, see e.g. Kelejian and Prucha (2010). In particular, Assumption E.1 is just a normalisation of the model and implies that no individual is viewed as its own neighbour. Assumption E.2 implies that there is no dominant unit in the sample, i.e. an individual unit that is asymptotically, for N large, correlated with all remaining individuals. Assumptions E.3-E.4 concern the parameter space of ψ and are discussed in detail Kelejian and Prucha (2010, Sec. 2.2). Notice that the assumptions above do not depend on a particular ordering of the data, which can be arbitrary so long as Assumption E holds true. Moreover, $\mathbf{W}_N$ is not required to be row normalized. Although it is convenient to work with a row-normalised weighting matrix, in some applications, especially those analysing social interactions and network structures, row normalisation might not always be appropriate.

Last, Assumption F ensures IV-based identification, see e.g. Wooldridge (2002, Ch. 5).

The asymptotic properties of the one-step estimator are determined primarily by those of $\widehat{\mathbf{Z}}_i'\mathbf{u}_i/\sqrt{NT}$. Thus, the following proposition is useful:

Proposition 3.1 *Under Assumptions A-F, we have*

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\widehat{\mathbf{Z}}_i'\mathbf{u}_i=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbb{Z}_i'\mathbf{u}_i+\sqrt{\frac{T}{N}}\mathbf{b}_1+\sqrt{\frac{N}{T}}\mathbf{b}_2+o_p(1),$$

where $\mathbb{Z}_i=\left(\sum_{j=1}^{N}w_{ij}\mathbf{M}_{\mathbf{F}}\boldsymbol{\mathcal{X}}_j,\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{F}_{-1}^0}\boldsymbol{\mathcal{X}}_{i,-1},\mathbf{M}_{\mathbf{F}^0}\boldsymbol{\mathcal{X}}_i\right)$ *with* $\boldsymbol{\mathcal{X}}_i=\mathbf{X}_i-\frac{1}{N}\sum_{\ell=1}^{N}\mathbf{X}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_i^0$, $\boldsymbol{\mathcal{X}}_{i,-1}=\mathbf{X}_{i,-1}-\frac{1}{N}\sum_{\ell=1}^{N}\mathbf{X}_{\ell,-1}\boldsymbol{\Gamma}_\ell^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_i^0,\boldsymbol{\Upsilon}^0=N^{-1}\sum_{i=1}^{N}\boldsymbol{\Gamma}_i^0\boldsymbol{\Gamma}_i^{0\prime}$, *while* $\mathbf{b}_1=(\mathbf{b}_{11}',\mathbf{b}_{12}',\mathbf{b}_{13}')'$ *and* $\mathbf{b}_2=(\mathbf{b}_{21}',\mathbf{b}_{22}',\mathbf{b}_{23}')'$.[6]

Based on the above proposition, Theorem 3.1 establishes convergence in probability of the one-step IV estimator, $\widehat{\boldsymbol{\theta}}$.

Theorem 3.1 *Under Assumptions A-F, as $N,T\to\infty$ such that $N/T\to c$, where $0<c<\infty$, we have*

$$\sqrt{NT}\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)=O_p(1)\,. \tag{3.1}$$

Asymptotic normality follows through by using similar arguments as for $\widetilde{\boldsymbol{\theta}}$ below. To save space, we do not derive this property explicitly here because $\widehat{\boldsymbol{\theta}}$ is mainly used to estimate $\mathbf{H}$.

Since the asymptotic properties of the two-step estimator are determined primarily by those of $\widehat{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i/\sqrt{NT}$, in what follows we focus on this particular term. The formal analysis is provided as a proposition below.

Proposition 3.2 *Under Assumptions A-F, we have*

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\widehat{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{Z}_i'\boldsymbol{\varepsilon}_i+O_p\left(\frac{1}{\delta_{NT}}\right)+O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right) \tag{3.2}$$

As we see from Proposition 3.2, the estimation effect in $\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\widehat{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i$ can be ignored asymptotically. Since $\boldsymbol{\varepsilon}_i$ is independent of $\mathbf{Z}_i$ and $\mathbf{H}^0$ with zero mean, the limiting distribution of $\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\widehat{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i$ is centered at zero. Hence, the asymptotic normality result can be

[6] See Eq. (A.87) and (A.88) in the Online Appendix of the paper for explicit expressions of these bias terms. To save space, we do not report these expressions here, given also that we do not bias-correct the first-step estimator.

readily obtained by applying the central limit theorem for martingale differences in Kelejian and Prucha (2001).

The following theorem establishes consistency and asymptotic normality for $\widetilde{\boldsymbol{\theta}}$.

Theorem 3.2 *Under Assumptions A-F, as $N, T \to \infty$ such that $N/T \to c$, where $0 < c < \infty$, we have*

$$\sqrt{NT}\left(\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\right) \xrightarrow{d} N\left(\mathbf{0}, \boldsymbol{\Psi}\right)$$

where $\boldsymbol{\Psi} = \sigma_\varepsilon^2 \left(\mathbf{A}_0' \mathbf{B}_0^{-1} \mathbf{A}_0\right)^{-1}$, $\mathbf{A}_0 = \text{plim}_{N,T\to\infty} \mathbf{A}$, $\mathbf{B}_0 = \text{plim}_{N,T\to\infty} \mathbf{B}$, *with*

$$\mathbf{A} = \frac{1}{NT}\sum_{i=1}^{N} \mathbf{Z}_i' \mathbf{C}_i \,, \mathbf{B} = \frac{1}{NT}\sum_{i=1}^{N} \mathbf{Z}_i' \mathbf{Z}_i \,.$$

Moreover, $\widetilde{\boldsymbol{\Psi}} - \boldsymbol{\Psi} \xrightarrow{p} \mathbf{0}$ *as* $N, T \to \infty$, *where*

$$\widetilde{\boldsymbol{\Psi}} = \tilde{\sigma}_\varepsilon^2 \left(\widetilde{\mathbf{A}}' \widetilde{\mathbf{B}}^{-1} \widetilde{\mathbf{A}}\right)^{-1} \,.$$

Note that $\tilde{\boldsymbol{\theta}}$ is asymptotically unbiased. This is in stark contrast with existing QMLE estimators available for spatial panels, which require bias correction. The main intuition of this result lies in that within our approach, factor estimates are extracted from two sets of information that are mutually independent, the exogenous covariates and the regression residuals. Therefore, there is no correlation between the regressors and the estimation error of the interactive fixed effects obtained in the second step of our procedure.[7]

The limiting distribution of the overidentifying restrictions test statistic is established in the following theorem:

Theorem 3.3 *Under Assumptions A-F, as $N, T \to \infty$ such that $N/T \to c$, where $0 < c < \infty$, we have*

$$J \xrightarrow{d} \chi_\nu^2$$

where $\nu = 3k - (k + 2)$.

4 Monte Carlo Experiments

We investigate the finite sample behaviour of the proposed approach by means of Monte Carlo experiments. We shall focus on the mean, bias, RMSE, empirical size and power of the t-test.

[7] For the case of a static panel with no spatial lags, Cui et al. (2020) provide a detailed technical comparison between the present methodology and the one developed by Bai (2009).

4.1 Design

We consider the following spatial dynamic panel data model:

$$y_{it} = \alpha_i + \rho y_{it-1} + \psi \sum_{j=1}^{N} w_{ij} y_{jt} + \sum_{\ell=1}^{k} \beta_\ell x_{\ell it} + u_{it}; \quad u_{it} = \alpha_i + \sum_{s=1}^{r_y} \varphi_{si}^0 f_{s,t}^0 + \varepsilon_{it}, \tag{4.1}$$

$i = 1, ..., N$, $t = -49, ..., T$, where

$$f_{s,t}^0 = \rho_{fs} f_{s,t-1}^0 + (1 - \rho_{fs}^2)^{1/2} \zeta_{s,t}, \tag{4.2}$$

with $\zeta_{s,t} \sim i.i.d.N(0,1)$ for $s = 1, ..., r_y$. We set $\rho_{fs} = 0.5$ $\forall s$, $k = 2$ and $r_y = 3$.

The spatial weighting matrix, $\mathbf{W}_N = [w_{ij}]$ is an invertible rook matrix of circular form (see Kappor et al. (2007)), such that its ith row, $1 < i < N$, has non-zero entries in positions $i-1$ and $i+1$, whereas the non-zero entries in rows 1 and N are in positions $(1,2)$, $(1,N)$, and $(N,1)$, $(N,N-1)$, respectively. This matrix is row normalized so that all of its nonzero elements are equal to $1/2$.

The idiosyncratic error, ε_{it}, is non-normal and heteroskedastic across both i and t, such that $\varepsilon_{it} = \varsigma_\varepsilon \sigma_{it}(\epsilon_{it} - 1)/\sqrt{2}$, $\epsilon_{it} \sim i.i.d.\chi_1^2$, with $\sigma_{it}^2 = \eta_i \phi_t$, $\eta_i \sim i.i.d.\chi_2^2/2$, and $\phi_t = t/T$ for $t = 0, 1, ..., T$ and unity otherwise.

The process for the covariates is given by

$$x_{\ell it} = \mu_{\ell i} + \sum_{s=1}^{r_x} \gamma_{\ell si}^0 f_{s,t}^0 + v_{\ell it}; \; i = 1, 2, ..., N; \quad t = -49, -48, ..., T, \tag{4.3}$$

for $\ell = 1, 2$. We set $r_x = 2$. Thus, the first two factors in u_{it}, f_{1t}^0, f_{2t}^0, also drive the DGP for $x_{\ell it}$, $\ell = 1, 2$. However, f_{3t}^0 does not enter into the DGP of the covariates directly. Observe that, using notation of earlier sections, $\mathbf{f}_t^0 = (f_{1t}^0, f_{2t}^0)'$, and $\mathbf{h}_t^0 = (f_{1t}^0, f_{2t}^0, f_{3t}^0)'$.

The idiosyncratic errors in the covariates are serially correlated, such that

$$v_{\ell it} = \rho_{\upsilon,\ell} v_{\ell it-1} + (1 - \rho_{\upsilon,\ell}^2)^{1/2} \varpi_{\ell it}; \; \varpi_{\ell it} \sim i.i.d.N(0, \varsigma_\upsilon^2), \tag{4.4}$$

for $\ell = 1, 2$. We set $\rho_{\upsilon,\ell} = \rho_\upsilon = 0.5$ for all ℓ.

All individual-specific effects and factor loadings are generated as correlated and mean-zero random variables. In particular, the individual-specific effects are drawn as

$$\alpha_i \sim i.i.d.N(0, (1-\rho)^2); \quad \mu_{\ell i} = \rho_{\mu,\ell} \alpha_i + (1 - \rho_{\mu,\ell}^2)^{1/2} \omega_{\ell i}, \tag{4.5}$$

where $\omega_{\ell i} \sim i.i.d.N(0, (1-\rho)^2)$, for $\ell = 1, 2$. We set $\rho_{\mu,\ell} = 0.5$ for $\ell = 1, 2$.

The factor loadings in u_{it} are generated as $\varphi_{si}^0 \sim i.i.d.N(0,1)$ for $s = 1, ..., r_y (= 3)$, and the factor loadings in x_{1it} and x_{2it} are drawn as

$$\gamma_{1si}^0 = \rho_{\gamma,1s} \varphi_{3i}^0 + (1 - \rho_{\gamma,1s}^2)^{1/2} \xi_{1si}; \; \xi_{1si} \sim i.i.d.N(0,1); \tag{4.6}$$

$$\gamma_{2si}^{0} = \rho_{\gamma,2s}\gamma_{si}^{0} + (1-\rho_{\gamma,2s}^{2})^{1/2}\xi_{2si};\ \xi_{2si} \sim i.i.d.N(0,1); \tag{4.7}$$

respectively, for $s = 1, ..., r_x = 2$. The process in Eq. (4.6) allows the factor loadings to $f_{1,t}^0$ and $f_{2,t}^0$ in x_{1it} to be correlated with the factor loadings corresponding to the factor that does not enter into the DGP of the covariates, i.e. $f_{3,t}^0$. On the other hand, Eq. (4.7) ensures that the factor loadings to $f_{1,t}^0$ and $f_{2,t}^0$ in x_{2it} are correlated with the factor loadings corresponding to the same factors in u_{it}, $f_{1,t}^0$ and $f_{2,t}^0$. We consider $\rho_{\gamma,11} = \rho_{\gamma,12} \in \{0, 0.5\}$, whilst $\rho_{\gamma,21} = \rho_{\gamma,22} = 0.5$.

It is straightforward to see that the average variance of ε_{it} depends only on ς_ε^2. Let π_u denote the proportion of the average variance of u_{it} that is due to ε_{it}. That is, we define $\pi_u := \varsigma_\varepsilon^2/\left(r_y + \varsigma_\varepsilon^2\right)$. Thus, for example, $\pi_u = 3/4$ means that the variance of the idiosyncratic error accounts for 75% of the total variance in u. In this case most of the variation in the total error is due to the idiosyncratic component and the factor structure has relatively minor significance.

Solving in terms of ς_ε^2 yields

$$\varsigma_\varepsilon^2 = \frac{\pi_u}{(1-\pi_u)} r_y. \tag{4.8}$$

We set ς_ε^2 such that $\pi_u \in \{1/4, 3/4\}$.[8]

We define the signal-to-noise ratio (SNR) conditional on the factor structure, the individual-specific effects and the spatial lag, as follows:

$$SNR := \frac{\text{var}\left[\left(y_{it} - \varepsilon_{it}\right)|\mathcal{L}\right]}{\overline{\text{var}}\left(\varepsilon_{it}\right)} = \frac{\left(\frac{\beta_1^2+\beta_2^2}{1-\rho^2}\right)\varsigma_\upsilon^2 + \frac{\varsigma_\varepsilon^2}{1-\rho^2} - \varsigma_\varepsilon^2}{\varsigma_\varepsilon^2}, \tag{4.9}$$

where $\mathcal{L}$ is the information set that contains the factor structure, the individual-specific effects and the spatial lag[9], whereas $\overline{\text{var}}\left(\varepsilon_{it}\right)$ is the overall average of $E\left(\varepsilon_{it}^2\right)$ over i and t. Solving for ς_υ^2 yields

$$\varsigma_\upsilon^2 = \varsigma_\varepsilon^2\left[SNR - \frac{\rho^2}{1-\rho^2}\right]\left(\frac{\beta_1^2+\beta_2^2}{1-\rho^2}\right)^{-1}. \tag{4.10}$$

We set $SNR = 4$, which lies with the range $\{3, 9\}$ considered by the simulation study of Bun and Kiviet (2006) and Juodis and Sarafidis (2018).

We set $\rho = 0.4$, $\psi = 0.25$, and $\beta_1 = 3$ and $\beta_2 = 1$, following Bai (2009).

[8]These values of π_u are motivated by the results in Sargent and Sims (1977), in which they find that two common factors explain 86% of the variation in unemployment rate and 26% of the variation in residential construction.

[9]The reason for conditioning on these variables is that they influence both the composite error of y_{it}, as well as the covariates.

In addition to the model provided in Eq. (4.1), we also consider an augmented model that includes a spatial-time lag as an additional covariate, that is,

$$y_{it} = \alpha_i + \rho y_{it-1} + \psi \sum_{j=1}^{N} w_{ij} y_{jt} + \sum_{\ell=1}^{k} \beta_\ell x_{\ell it} + \psi_1 \sum_{j=1}^{N} w_{ij} y_{jt-1} + u_{it}. \tag{4.11}$$

We study the optimal two-step IV estimator, defined in Eq. (2.7), based on the following set of instruments stack in the $(T \times 4k)$ matrix $\hat{\mathbf{Z}}_i$:

$$\hat{\mathbf{Z}}_i = \left(\mathbf{M}_{\widehat{\mathbf{F}}} \underline{\mathbf{X}}_i, \quad \mathbf{M}_{\widehat{\mathbf{F}}_{-1}} \underline{\mathbf{X}}_{i,-1}, \quad \mathbf{M}_{\widehat{\mathbf{F}}} \sum_{j=1}^{N} w_{ij} \underline{\mathbf{X}}_j, \quad \mathbf{M}_{\widehat{\mathbf{F}}_{-1}} \sum_{j=1}^{N} w_{ij} \underline{\mathbf{X}}_{j,-1} \right), \tag{4.12}$$

where $\underline{\mathbf{X}}_i \equiv \mathbf{X}_i - \overline{\mathbf{X}}$, i.e. the covariates are cross-sectionally demeaned in order to control for individual-specific fixed effects. Thus, the above set of instruments employs contemporaneous as well as lagged defactored (and spatial) covariates.[10]

In order to allow for cross-section and time series heteroskedasticity, the variance estimator for the two-step IV procedure is given by

$$\widetilde{\boldsymbol{\Psi}} = \left(\widetilde{\mathbf{A}}' \widetilde{\mathbf{B}}^{-1} \widetilde{\mathbf{A}} \right)^{-1} \widetilde{\mathbf{A}}' \widetilde{\mathbf{B}}^{-1} \widehat{\boldsymbol{\Omega}} \widetilde{\mathbf{B}}^{-1} \widetilde{\mathbf{A}} \left(\widetilde{\mathbf{A}}' \widetilde{\mathbf{B}}^{-1} \widetilde{\mathbf{A}} \right)^{-1}, \tag{4.13}$$

with

$$\widehat{\boldsymbol{\Omega}} = \frac{1}{NT} \sum_{i=1}^{N} \hat{\mathbf{Z}}_i' \mathbf{M}_{\widehat{\mathbf{F}}} \hat{\mathbf{u}}_i \hat{\mathbf{u}}_i' \mathbf{M}_{\widehat{\mathbf{F}}} \hat{\mathbf{Z}}_i, \tag{4.14}$$

and $\hat{\mathbf{u}}_i = \mathbf{y}_i - \mathbf{C}_i \hat{\boldsymbol{\theta}}$.

The J test statistic we use is as in Eq. (2.8) with $\widehat{\boldsymbol{\Omega}}$ replaced by the expression above.

In order to check the performance of the IV estimator with endogenous covariates, as well as the power of the J test, we consider the situation where $\mathbf{x}_{it}$ and ε_{it} are contemporaneously correlated. In particular, for $\ell = 1$ the DGP in Eq. (4.3) is replaced by the following one:

$$x_{1it} = \mu_{1i} + \sum_{s=1}^{r_x} \gamma_{1si}^0 f_{s,t}^0 + v_{1it} + 0.5\varepsilon_{it}; \; i = 1, 2, ..., N; \quad t = -49, -48, ..., T. \tag{4.15}$$

This implies that x_{1it} is endogenous with respect to ε_{it}, while x_{2it} remains strictly exogenous.[11] We construct a single external instrument, denoted as x_{3it}, which is given by

$$x_{3it} = \mu_{3i} + \sum_{s=1}^{r_x} \gamma_{3si}^0 f_{s,t}^0 + v_{1it} + \vartheta v_{3it}; \; i = 1, 2, ..., N; \quad t = -49, -48, ..., T, \tag{4.16}$$

[10] We have also explored the performance of two additional IV estimators. The first one omits the lagged spatial defactored covariates, i.e. it excludes the last term in Eq. (4.12) from the set of instruments. The second one replaces the lagged spatial defactored covariates with contemporaneous defactored covariates that arise from the square of the spatial weighting matrix, as e.g. in Kelejian and Prucha (2007), p. 143. In the baseline model ($\psi_1 = 0$), all three estimators perform similarly. However, when $\psi_1 \neq 0$, the performance of two IV estimators that do not make use of lagged spatial defactored covariates as instruments, deteriorates substantially. This is mainly because ψ_1 is weakly identified in this case. In order to facilitate the comparison of the results obtained for the two DGP's in Eq. (4.1) and Eq. (4.11), we shall focus on the IV estimator that makes use of the instruments in Eq. (4.12). The results for the remaining IV estimators are available from the authors upon request.

[11] The power of the J statistic is expected to be higher when both covariates are endogenous.

where μ_{3i}, v_{3it} and γ_{3si}^0 are generated as in Eqs. (4.5), (4.4) and (4.7) respectively. The value of ϑ is set such that the correlation between $(v_{1it} + 0.5\varepsilon_{it})$ in Eq. (4.15) and $(v_{1it} + \vartheta v_{3it})$ in Eq. (4.16) equals 0.5.

To obtain a consistent IV estimator, the matrix of instruments is revised as follows:

$$\hat{\mathbf{Z}}_i = \left(\mathbf{M}_{\widehat{\mathbf{F}}} \underline{\widetilde{\mathbf{X}}}_i, \quad \mathbf{M}_{\widehat{\mathbf{F}}_{-1}} \underline{\widetilde{\mathbf{X}}}_{i,-1}, \quad \mathbf{M}_{\widehat{\mathbf{F}}} \sum_{j=1}^{N} w_{ij} \underline{\widetilde{\mathbf{X}}}_j, \quad \mathbf{M}_{\widehat{\mathbf{F}}_{-1}} \sum_{j=1}^{N} w_{ij} \underline{\widetilde{\mathbf{X}}}_{j,-1} \right) \tag{4.17}$$

where $\underline{\widetilde{\mathbf{X}}}_i = (\underline{\mathbf{x}}_{3i}, \underline{\mathbf{x}}_{2i})$ is of dimension $T \times 2$.

As a benchmark, we also consider the bias-corrected QMLE estimator proposed by Shi and Lee (2017).[12] While the QMLE estimator is not consistent when some covariates are correlated with the idiosyncratic error, it is of interest to see the extent to which its performance is affected in this case.

In terms of the sample size, we consider three cases. Case I specifies $N = 100\tau$ and $T = 25\tau$ for $\tau = 1, 2, 4$. This implies that while N and T increase by multiples of 2, the ratio N/T remains equal to 4 in all circumstances. Case II specifies $T = 100\tau$ with $N = 25\tau$ for $\tau = 1, 2, 4$. Therefore, $N/T = 1/4$, as both N and T grow. Finally, Case III sets $N = T = 50\tau$, $\tau = 1, 2, 4$. These choices allow us to consider different combinations of (N, T) in relatively small and large sample sizes.

All results are obtained based on 2,000 replications, and all tests are conducted at the 5% significance level. For the power of the "t-test", we specify $H_0 : \rho = \rho^0 + 0.1$ (or $H_0 : \psi = \psi^0 + 0.1$, and $H_0 : \beta_\ell = \beta_\ell^0 + 0.1$ for $\ell = 1, 2$) against two sided alternatives, where $\rho^0, \psi^0, \beta_1^0, \beta_2^0$ denote the true parameter values.

4.2 Results

Tables 4.1–4.4 report simulation results for the baseline model in Eq. (4.1) for $\pi_u = 3/4$. Results for $\pi_u = 1/4$ can be found in Online Appendix C. "Mean" denotes the average value of the estimated parameters across 2,000 replications. Similarly, "RMSE" represents the average squared deviation of the estimated parameter from its true value across 2,000 samples. "ARB" denotes absolute relative bias, which is defined as $ARB \equiv \left(|\widehat{\theta}_\ell - \theta_\ell| / \theta_\ell \right) 100$, where θ_ℓ denotes the ℓth entry of $\boldsymbol{\theta} = (\psi, \rho, \boldsymbol{\beta}')'$. Size-corrected power is reported, based on the 2.5% and 97.5%

[12] We are grateful to Wei Shi and Lung-fei Lee for providing us the computational algorithm for the QMLE estimator.

quantiles of the empirical distribution of the t-ratio under the null hypothesis.[13]

As we can see from all four tables, for both IV and QMLE the values obtained for the Mean are close to the true parameters in most cases. Moreover, as predicted by theory, RMSE declines steadily with larger values of N and T, roughly at the rate of $\sqrt{NT}$. Therefore, in what follows we shall mainly focus in our discussion on relative RMSE performance, ARB and size properties of the two estimators.

Table 4.1 presents results for the autoregressive parameter, ρ. QMLE outperforms IV in terms of RMSE, which reflects the higher efficiency of maximum likelihood/least-squares compared to IV. However, QMLE exhibits substantial ARB and thereby it is severely size-distorted. Note that both ARB and size distortions tend to become smaller as the sample size increases, albeit at a slow rate when $N/T = 4$. In contrast, IV has little ARB and good size properties in most cases, with some mild distortions observed only when N is small.[14]

Table 4.1: Baseline Model. Results for $\rho = 0.4$, $\pi_u = 3/4$

	IV					QMLE				
					Case I: $N = 100\tau$, $T = 25\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.400	.017	.082	.065	1.00	.390	.014	2.52	.361	1.00
2	.400	.007	.104	.059	1.00	.396	.006	1.08	.293	1.00
4	.400	.004	.054	.052	1.00	.398	.003	.551	.286	1.00
					Case II: $N = 25\tau$, $T = 100\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.400	.014	.116	.084	1.00	.400	.008	.169	.138	1.00
2	.400	.007	.033	.066	1.00	.400	.004	.087	.085	1.00
4	.400	.003	.008	.048	1.00	.400	.002	.080	.079	1.00
					Case III: $N = 50\tau$, $T = 50\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.400	.015	.074	.052	1.00	.397	.009	.870	.146	1.00
2	.400	.007	.026	.056	1.00	.398	.005	.493	.121	1.00
4	.400	.003	.017	.053	1.00	.399	.002	.225	.109	1.00

Table 4.2 focuses on the spatial parameter, ψ. One noticeable difference compared to Table 4.1 is that in this case IV appears to outperform QMLE in terms of RMSE, albeit the difference decreases substantially as the sample size gets larger. As before, IV is subject to

[13] The size-adjusted power is employed in the present experimental study because of size distortions, which otherwise make the power comparison between the two estimators difficult.

[14] This is in contrast to the PC estimator developed by Bai (2009), who shows that the estimator suffers from asymptotic bias. Cui et al. (2020) analyse this result further.

some small size distortion when N is small, which however tends to be eliminated quickly as N grows. QMLE is severely size-distorted.

Table 4.2: Baseline Model. Results for $\psi = 0.25$, $\pi_u = 3/4$

	IV					QMLE				
	Case I: $N = 100\tau$, $T = 25\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.250	.019	.119	.062	1.00	.250	.033	.216	.392	1.00
2	.250	.008	.112	.051	1.00	.250	.015	.159	.258	1.00
4	.250	.004	.080	.052	1.00	.250	.006	.113	.134	1.00
	Case II: $N = 25\tau$, $T = 100\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.250	.017	.054	.094	1.00	.250	.023	.156	.156	1.00
2	.250	.008	.009	.062	1.00	.250	.010	.226	.078	1.00
4	.250	.004	.016	.054	1.00	.250	.005	.007	.070	1.00
	Case III: $N = 50\tau$, $T = 50\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.251	.017	.218	.076	1.00	.250	.025	.026	.213	1.00
2	.250	.008	.039	.060	1.00	.250	.011	.092	.139	1.00
4	.250	.004	.012	.054	1.00	.250	.005	.001	.068	1.00

Tables 4.3–4.4 report results for β_1 and β_2, respectively. The results are qualitatively no different from those in Table 4.2, with one exception: when either N or T is small, size-adjusted power appears to be relatively lower. Moreover, IV often appears to have higher power than QMLE in moderate sample sizes.

Table 4.3: Baseline Model. Results for $\beta_1 = 3$, $\pi_u = 3/4$

	IV					QMLE				
	Case I: $N = 100\tau$, $T = 25\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	3.00	.057	.019	.058	.405	3.01	.065	.312	.209	.395
2	3.00	.026	.030	.060	.952	3.01	.026	.204	.087	.982
4	3.00	.013	.022	.051	1.00	3.00	.013	.112	.087	1.00
	Case II: $N = 25\tau$, $T = 100\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	3.00	.052	.021	.082	.465	3.00	.087	.037	.404	.198
2	3.00	.026	.001	.061	.957	3.00	.034	.033	.211	.877
4	3.00	.004	.008	.054	1.00	3.00	.015	.003	.134	1.00
	Case III: $N = 50\tau$, $T = 50\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	3.00	.056	.046	.056	.459	3.00	.065	.144	.236	.370
2	3.00	.027	.004	.067	.931	3.00	.027	.121	.117	.965
4	3.00	.012	.004	.050	1.00	3.00	.012	.045	.085	1.00

Table 4.4: Baseline Model. Results for $\beta_2 = 1$, $\pi_u = 3/4$

	IV					QMLE				
	Case I: $N = 100\tau$, $T = 25\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	1.00	.066	.947	.109	.350	1.06	.093	6.41	.415	.345
2	1.00	.025	.087	.057	.981	1.01	.029	1.35	.142	.982
4	1.00	.012	.039	.055	1.00	1.00	.012	.379	.079	1.00
	Case II: $N = 25\tau$, $T = 100\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	1.00	.049	.043	.086	.465	1.07	.118	7.10	.531	.179
2	1.00	.025	.043	.074	.972	1.02	.040	1.81	.306	.876
4	1.00	.012	.003	.059	1.00	1.00	.014	.361	.134	1.00
	Case III: $N = 50\tau$, $T = 50\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	1.00	.050	.222	.063	.539	1.05	.086	5.18	.369	.311
2	1.00	.024	.023	.052	.985	1.01	.028	1.15	.156	.987
4	1.00	.012	.023	.050	1.00	1.00	.012	.228	.090	1.00

Table 4.5 reports simulation results corresponding to the augmented spatial panel data model in Eq. (4.11), which includes a spatial-time lag. As argued in Section 2, the proposed IV estimator remains consistent and asymptotically normal in this model. Table 4.5 focuses on ψ_1. The results for the remaining coefficients are similar to those obtained for the model without a spatial-time lag and can be found in Online Appendix C. In most cases IV has negligible bias with fairly accurate size properties. On the other hand, QMLE tends to be biased and size-distorted unless both N and T are relatively large. QMLE outperforms IV in terms of RMSE.

Table 4.5: **Model with spatial-time lag. Results for** $\psi_1 = 0.20$, $\pi_u = 3/4$

	IV					QMLE				
	Case I: $N = 100\tau$, $T = 25\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.200	.030	.128	.059	.920	.204	.017	1.98	.167	1.00
2	.200	.014	.128	.056	1.00	.200	.008	.839	.091	1.00
4	.200	.007	.013	.049	1.00	.200	.004	.612	.095	1.00
	Case II: $N = 25\tau$, $T = 100\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.199	.028	.456	.087	.925	.198	.020	.895	.248	1.00
2	.200	.014	.090	.058	1.00	.200	.009	.028	.173	1.00
4	.200	.007	.150	.060	1.00	.200	.004	.052	.089	1.00
	Case III: $N = 50\tau$, $T = 50\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.199	.028	.434	.070	.922	.201	.016	.475	.137	1.00
2	.200	.014	.069	.067	1.00	.201	.008	.478	.111	1.00
4	.200	.007	.059	.063	1.00	.200	.004	.168	.068	1.00

Table 4.6 reports simulation results for the baseline model in Eq. (4.11) with x_1 endogenous, as in Eq. (4.15). The matrix of instruments is given by Eq. (4.17). To save space, we only report results for the coefficient of the endogenous variable, β_1. The IV estimator performs well. In comparison to Table 4.3, two main differences are noteworthy: firstly, for small values of τ the RMSE of the IV estimator increases by a multiple scalar that ranges between 1.3-1.7; secondly, the power of the t-test decreases substantially for small values of either N or T. These results are not surprising given that the correlation between the defactored regressor x_1 and the instrument drops by a half. For QMLE, ARB ranges between $35\% - 40\%$, and the size of the t-test equals 1 under all circumstances. These results show that QMLE is quite sensitive when some of the covariates are endogenous. As far as the remaining parameters are concerned (not reported here), the estimate of the autoregressive coefficient appears to be more sensitive to endogeneity than the estimate of the spatial parameter. In particular, the bias of the estimate of ρ fluctuates around 6.7%. Furthermore, the size of the estimator is close to 1 in all cases.[15] This implies that detecting possible violations of exogeneity of the regressors is very important in practice.

[15] These simulation results are available upon request.

Table 4.6: Model with endogenous covariate. Results for $\beta_1 = 3$, $\pi_u = 3/4$

	IV					QMLE				
	Case I: $N = 100\tau$, $T = 25\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	3.01	.077	.269	.075	.312	4.19	1.19	39.5	1.00	.340
2	3.00	.034	.049	.051	.831	4.14	1.14	38.0	1.00	.662
4	3.00	.017	.014	.050	1.00	4.12	1.12	37.4	1.00	.964
	Case II: $N = 25\tau$, $T = 100\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	3.00	.067	.123	.083	.359	4.06	1.06	35.2	1.00	.198
2	3.00	.033	.040	.064	.837	4.07	1.07	35.5	1.00	.356
4	3.00	.017	.028	.055	1.00	4.09	1.09	36.3	1.00	.544
	Case III: $N = 50\tau$, $T = 50\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	3.00	.069	.184	.065	.359	4.11	1.11	37.0	1.00	.236
2	3.00	.034	.035	.063	.812	4.11	1.11	36.9	1.00	.468
4	3.00	.017	.021	.054	1.00	4.10	1.10	36.7	1.00	.792

Table 4.7 summarises the finite sample behaviour for the overidentifying restrictions test. Panel A and Panel B provide the rejection frequencies corresponding to the IV estimator that makes use of the matrix of instruments given by Eq. (4.12). Columns '**I**', '**II**' and '**III**' correspond to $N = 100\tau$, $T = 25\tau$, $N = 25\tau$, $T = 100\tau$ and $N = 50\tau$, $T = 50\tau$, respectively, for $\tau = 1, 2, 4$.

As we can see, the size of the J-test is close to its nominal level (5%) in most cases, with some minor size distortions when N is small. On the other hand, the J-test appears to have substantial power when the exogeneity of (a subset of) the instruments is violated. For example, the power of the test for $N = T = 50$ and $N = T = 100$ is 30.9% and 88.9% respectively.

Panel C provides results on the empirical size of the J-test when the instruments employed are given by Eq. (4.17). In general, some mild distortions are observed only for N small.

Table 4.7: Size and power performance for the J test statistic

	Panel A			Panel B			Panel C		
	size			power			size		
τ	**I**	**II**	**III**	**I**	**II**	**III**	**I**	**II**	**III**
1	.054	.083	.068	.388	.515	.513	.070	.092	.073
2	.055	.063	.067	.978	.989	.988	.063	.074	.062
4	.049	.053	.051	1.00	1.00	1.00	.049	.058	.048

Finally, a desirable aspect of our approach is that it is linear in the parameters and therefore it is computationally inexpensive. Obviously, the exact computational gains depend on the sample size. As an indication, when $N = T = 50$, the total length of time taken to estimate 2,000 replications of the model was roughly 20 seconds for IV and 740 seconds for QMLE. On the other hand, when $N = T = 200$ the corresponding figures were roughly 4.5 minutes and 4.5 hours for IV and QMLE, respectively.[16] Therefore, QMLE was 37 (60) times slower than IV in these specific designs.[17]

5 An Analysis of Bank Attitude Towards Risk

We study the determinants of risk attitude of banking institutions, with emphasis on the impact of increased capital regulation over the past decade or so, via the introduction of the Dodd-Frank Act[18], which also coincided with the establishment of Basel III internationally. We employ panel data from a random sample of 350 U.S. banking institutions, each one observed over 56 time periods, namely 2006:Q1-2019:Q4. All data are publicly available and they have been downloaded from the Federal Deposit Insurance Corporation (FDIC) website.[19]

We pay focus on the spatial interactions of banking behavior, in order to account for banking linkages and endogenous network effects. Furthermore, we control for unobserved common shocks, such as the recent economic recession that took place during the period 2007-2009, which was accompanied by rapidly falling housing prices. To the best of our knowledge, this

[16]The simulation experiments have been implemented using Matlab on a single CPU with Core i7-6700 @ 3.40 GHz and 16 GB RAM. The algorithm is currently being written as an ado file in Stata 15 and it will be made available to all Stata users on the web.

[17]Simulation results for the case $\pi_u = 1/4$ are reported in the Online Appendix C of the paper. The results are qualitatively similar in terms of a comparison between IV and QMLE. However, in general, both estimators perform a bit better than in the case where $\pi_u = 3/4$. In addition, the Online Appendix C of the paper reports results for the case where the purely idiosyncratic error term of the model is normally distributed and homoskedastic, and $\pi_u = 3/4$; see Tables C.11-C.14. Again, the results are qualitatively similar in terms of the relative performance of IV and QMLE, and therefore the conclusions remain unchanged.

[18]The Dodd-Frank Act (DFA) is a US federal law enacted during 2010, aiming "to promote the financial stability of the United States by improving accountability and transparency in the financial system, to end "too big to fail", to protect the American taxpayer by ending bailouts, to protect consumers from abusive financial services practices, and for other purposes"; see https://www.cftc.gov/sites/default/files/idc/groups/public/@swaps/documents/file/hr4173_enrolledbill.pdf. In a nutshell, the DFA has instituted a new failure-resolution regime, which seeks to ensure that losses resulting from bad decisions by managers are absorbed by equity and debt holders, thus potentially reducing moral hazard.

[19]See https://www.fdic.gov/.

is the first paper in the literature that estimates state dependence and endogenous network effects and controls for the impact of unobserved aggregate shocks.

5.1 Model Specification

We estimate the same regression model as in Eq. (2.1) or (4.1), for $i = 1, \ldots, 350$, and $t = 1, \ldots, 56$, using the following variables:

$y_{it} \equiv NPL_{it}$, which denotes the ratio of non-performing loans to total loans for bank i at time period t. This is a popular measure of bank risk.[20] Higher values of the NPL ratio indicate that banks ex-ante took higher lending risk and therefore they have accumulated ex-post more bad loans;

$x_{1it} \equiv INEFF_{it}$ denotes the time-varying operational inefficiency of bank i at period t, which has been constructed based on a cost frontier model using a translog functional form, two outputs and three inputs[21];

$x_{2it} \equiv CAR_{it}$ stands for "capital adequacy ratio", which is proxied by the ratio of Tier 1 (core) capital over risk-weighted assets;

$x_{3it} \equiv SIZE_{it}$ is proxied by the natural logarithm of banks' total assets;

$x_{4it} \equiv BUFFER_{it}$ denotes the amount of capital buffer, and it is computed by subtracting from the core capital (leverage) ratio the value of the minimum regulatory capital ratio (8%);

$x_{5it} \equiv PROFITABILITY_{it}$ is proxied by the return on equity (ROE), defined as annualized net income expressed as a percentage of average total equity on a consolidated basis;

[20] An alternative such measure is the ratio of risk-weighted assets to total assets. This involves multiplying the amount of different types of assets by the standardised risk weight associated with each type of assets. However, this measure has been criticised because it can be easily manipulated e.g. by engaging in regulatory capital arbitrage; see Vallascas and Hangendorff (2013).

[21] In particular, following Altunbas et al. (2007), we specify

$$\begin{aligned} lnTC_{it} = &\sum_{h=1}^{3} \gamma_h lnP_{hit} + \sum_{h=1}^{2} \delta_h lnY_{hit} + 0.5 \sum_{m=1}^{2} \sum_{n=1}^{2} \mu_{mn} lnY_{mit} lnY_{nit} \\ &+ \sum_{m=1}^{3} \sum_{n=1}^{3} \pi_{mn} lnP_{mit} lnP_{nit} + \sum_{m=1}^{2} \sum_{n=1}^{3} \xi_{mn} lnY_{mit} lnP_{nit} + \epsilon_i + \tau_t + \upsilon_{it}, \end{aligned} \tag{5.1}$$

where TC represents total cost, Y_1 and Y_2 denote two outputs, net loans and securities, respectively. The former is defined as gross loans minus reserves for loan loss provision. The latter is the sum of securities held to maturity and securities held for sale. P_1, P_2 and P_3 denote three input prices, namely the price of capital, price of labor and price of loanable funds. The model above is estimated using two-way fixed effects regression. The bank-specific, time-varying operational inefficiency component is captured by the sum of the two fixed effects, i.e. $\epsilon_i + \tau_t$.

$x_{6it} \equiv QUALITY_{it}$ represents the quality of banks' assets and is computed as the total amount of loan loss provisions (LLP) expressed as a percentage of assets. Thus, a higher level of loan loss provisions indicates lower quality;

$x_{7it} \equiv LIQUIDITY_{it}$ is proxied by the loan-to-deposit (LTD) ratio. The main idea is that if this ratio is too high, banks may not have enough liquidity to meet unforeseen funding requirements, and vice versa;

$x_{8it} \equiv PRESSURE_{it}$ represents "institutional pressure" and is binary. In specific, it takes the value of unity if a bank has a capital buffer that is less than or equal to the $10th$ percentile of the distribution of capital buffer in any given period, and zero otherwise.

Finally, the error term is composite; η_i captures bank-specific effects, $\mathbf{h}_t^0$ is a $r_y \times 1$ vector of unobserved common shocks with corresponding loadings given by $\boldsymbol{\varphi}_i^0$ and ε_{it} is a purely idiosyncratic error. Note that r_y is unknown.

Some discussion on the interpretation of the parameters is noteworthy. The autoregressive coefficient, ρ, reflects costs of adjustment that prevent banks from achieving optimal risk levels instantaneously (Shrieves and Dahl (1992)). The coefficient of the spatial lag, ψ, captures endogenous links within a network model of interconnected bank balance sheets.

β_ℓ, for $\ell = 1, \ldots, k$, denote the slope coefficients of the model. β_1 captures the effect of operational inefficiency on problem loans. There are two competing hypotheses that predict opposite scenarios in regards to this effect: the so-called "bad management hypothesis" advocates that lower cost efficiency leads to an increase in the number of problematic loans. In particular, managers' failure to control costs sufficiently, can result in poor monitoring of loans and thereby higher default rates (see e.g. Fiordelisi et al. (2011)). In contrast, the so-called "skimping hypothesis" posits that banks may achieve low costs by under-spending on loan underwriting and monitoring, which brings about a larger volume of problem loans (see e.g. Tan and Floros (2013)). Thus β_1 could be either positive or negative depending on which hypothesis appears to be supported by the data.

β_2 measures the effect of capital adequacy on bank risk. Several theories predict that changes in capital levels and bank risk are positively related to one another. For example, a standard view is that since the value of expected bankruptcy costs is an increasing function of the probability of bankruptcy, banks would tend to increase (decrease) capital levels when they increase (decrease) asset portfolio risk, and conversely.[22]

β_3 measures the effect of size on risk-taking behavior. Under the "too-big-to-fail hypothesis", large banks, knowing they are systematically important, may count on public bailout in periods of financial distress.

Capital buffer theory postulates that for a value-maximizing bank, incentives to increase asset risk decline as its capital increases. That is, more stringent capital requirements reduce the gains to a bank from increasing the risk of its asset portfolio (see e.g. Furlong and Keely (1989)). Thus, β_4 is expected to be negative. The same argument applies for the coefficient of institutional pressure, β_8.

Finally, the direction of the effects of profitability (ROE), asset quality and liquidity on bank risk behavior, β_5, β_6 and β_7, is ultimately an empirical question. For example, standard theory suggests that higher bank profitability dissuades bank risk-taking because profitable banks stand to lose more shareholder value if downside risks realize (Keeley (1990)). On the other hand, in the presence of leverage constraints, more profitable banks can borrow more and engage in risky side activities on a larger scale (Martynova et al. (2019)).

The spatial weights matrix has been constructed following the methodology of Fernandez (2011). In particular, let

$$d_{ij} = \sqrt{2\,(1 - \rho_{ij})}, \tag{5.2}$$

where ρ_{ij} denotes Spearman's correlation coefficient between banks i and j, corresponding to a specific financial indicator observed over t time periods. Then, the (i, j)-element of the $N \times N$ spatial weights matrix, $\mathbf{W}_N$, is defined as $w_{ij} = exp(-d_{ij})$. Thus, more distant observations take a smaller weight. Each of the rows of $\mathbf{W}_N$ has been divided by the sum of its corresponding elements so that $\sum_j w_{ij} = 1$ for all j. Finally, the diagonal elements of $\mathbf{W}_N$ are set equal to zero in order to ensure that no individual is treated as its own neighbor.

We make use of two financial indicators to construct weights, namely the debt ratio, defined as total liabilities over total assets, and the dividend yield, defined as the dividend

[22] This theory is mainly relevant for banks whose optimum capital ratio is in excess of the regulatory minimum levels. Alternative theories supporting a positive value of β_2 are discussed by Shrieves and Dahl (1992).

over market price per share.

5.2 Estimation

The model is estimated using the second-step IV estimator put forward in the present paper. $INEFF$ is treated as endogenous with respect to ε_{it} due to reverse causality. Reverse causality arises because higher levels of risk imply additional costs and managerial efforts incurred by banks in order to improve existing loan underwriting and monitoring procedures. To tackle reverse causality we instrument $INEFF$ using the ratio of interest expenses paid on deposits over the value of total deposits. Higher values of this variable indicate lower levels of cost efficiency, all other things being equal.[23]

The remaining covariates are treated as exogenous with respect to ε_{it}. However, these covariates can be potentially correlated with the common factor component, $\boldsymbol{\varphi}_i'\mathbf{h}_t^0$, in which case they are endogenous with respect to the total error term, u_{it}. Therefore, we instrument these covariates using the corresponding defactored regressors. The matrix of instruments is

$$\hat{\mathbf{Z}}_i = \left(\mathbf{M}_{\widehat{\mathbf{F}}}\widetilde{\underline{\mathbf{X}}}_i, \quad \mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\widetilde{\underline{\mathbf{X}}}_{i,-1}, \quad \mathbf{M}_{\widehat{\mathbf{F}}}\sum_{j=1}^{N} w_{ij}\widetilde{\underline{\mathbf{X}}}_j, \quad \mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\sum_{j=1}^{N} w_{ij}\widetilde{\underline{\mathbf{X}}}_{j,-1} \right), \tag{5.3}$$

where $\widetilde{\underline{\mathbf{X}}}_i = (\tilde{\underline{\mathbf{x}}}_{1i}, \underline{\mathbf{x}}_{2i}, \ldots, \underline{\mathbf{x}}_{8i})$ is of order $T \times 8$, with $\underline{\mathbf{x}}_{\ell i} = \mathbf{x}_{\ell i} - \overline{\mathbf{x}}_{\ell}$ and $\mathbf{x}_{\ell i}$ is a vector of order $T \times 1$ that denotes the ℓth covariate corresponding to β_ℓ, for $\ell = 2, \ldots, k$. $\tilde{\underline{\mathbf{x}}}_{1i}$ denotes the external instrument used to identify the effect of cost inefficiency. Thus, we make use of 32 moment conditions in total, and with 10 parameters the number of degrees of freedom equals 22. Such degree of overidentification is important in order to enhance identification even if some covariates end up not being statistically significant.

The projection matrix $\mathbf{M}_{\widehat{\mathbf{F}}}$ is computed based on $\hat{r}_x$ factors estimated from $(NT)^{-1}\sum_{i=1}^{N}(\tilde{\underline{\mathbf{x}}}_{1i}, \underline{\mathbf{x}}_{4i}, \underline{\mathbf{x}}_{8i})$ $\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}$ and $\mathbf{M}_{\widehat{\mathbf{H}}}$ are computed in a similar manner. The number of factors is estimated using the eigenvalue ratio test of Ahn and Horenstein (2013). The variance estimator for the two-step IV procedure is given by Eqs. (4.13)-(4.14).

Following Debarsy et al. (2012), we distinguish between direct, indirect and total effects.

[23] The correlation between these two variables in the sample equals 0.22.

[24] This choice is due to the fact that using all covariates to estimate factors resulted in rejecting the model based on the J-test.

In particular, stacking the N observations for each t in Eq. (2.1) yields

$$\begin{aligned} \mathbf{y}_{(t)} &= \rho \mathbf{y}_{(t-1)} + \psi \mathbf{W}_N \mathbf{y}_{(t)} + \sum_{\ell=1}^{k} \beta_\ell \mathbf{x}_{\ell(t)} + \mathbf{u}_{(t)}; \\ \mathbf{u}_{(t)} &= \left(\boldsymbol{\eta} + \boldsymbol{\Phi}^0 \mathbf{h}_t^0 + \boldsymbol{\varepsilon}_{(t)} \right), \end{aligned} \tag{5.4}$$

where $\mathbf{y}_{(t)}$ is of dimension $N \times 1$, and similarly for the remaining variables. $\boldsymbol{\Phi}^0 = (\boldsymbol{\varphi}_1^0, \dots, \boldsymbol{\varphi}_N^0)'$, denotes an $N \times r_y$ matrix of factor loadings.

Solving the model above yields

$$\mathbf{y}_{(t)} = [\mathbf{I}_N(1-\rho L) - \psi \mathbf{W}_N]^{-1} \left(\sum_{\ell=1}^{k} \beta_\ell \mathbf{x}_{\ell(t)} \right) + [\mathbf{I}_N(1-\rho L) - \psi \mathbf{W}_N]^{-1} \mathbf{u}_{(t)}. \tag{5.5}$$

The matrix of partial derivatives of the expected value of $\mathbf{y}$ with respect to the ℓth covariate in the long-run is given by:

$$\left[\frac{\partial E(\mathbf{y})}{\partial x_{\ell 1}} \dots \frac{\partial E(\mathbf{y})}{\partial x_{\ell N}} \right] = [\mathbf{I}_N(1-\rho) - \psi \mathbf{W}_N]^{-1} \beta_\ell \mathbf{I}_N. \tag{5.6}$$

Following LeSage and Pace (2009), we define the direct effect as the average of the diagonal elements in the matrix above. The indirect effect is defined as the average of the sum of the column entries other than those on the main diagonal. The total effect is the sum of the two effects. The matrix of partial derivatives of the expected value of $\mathbf{y}$ with respect to the ℓth covariate in the short-run is obtained from 5.6 by setting $\rho = 0$.

5.3 Results

Column "Full" in Table 5.1 below reports results for the entire period of the sample, i.e. 2006:Q1-2019:Q4.[25] Columns "Basel I-II" and "DFA" present results for two different subperiods, namely 2006:Q1-2010:Q4 and 2011:Q1-2019:Q4 respectively. The first subsample corresponds to the Basel I-II regulatory framework and includes the financial crisis period (2007-2009). The second subsample corresponds to the Dodd-Frank Act.

In regards to Column "Full", we can see that the autoregressive and spatial parameters are statistically significant and similar in magnitude, which provides evidence for both state dependence and endogenous network linkages.

The coefficient of operational inefficiency is positive and statistically significant, providing support for the "bad management hypothesis" instead of the "skimping hypothesis". This outcome is consistent with Williams (2004). The effect of capital adequacy ratio on bank

[25] Tables D.1-D.2 in the Online Supplement report additional robustness results in terms of different specifications and/or different estimation approaches.

risk is positive and statistically significant at the 5% level. On the other hand, bank size appears not to be associated with risky attitude from a statistical point of view. This finding per se is in contrast with the "too-big-to-fail hypothesis". However, as we shall shortly see from the discussion of the results reported in Columns "Basel I-II"-"DFA", the bank size effect appears to be large and statistically significant when the model is re-estimated during 2006:Q1-2010:Q4 only.

Capital buffer has a negative and significant effect on risk attitude, which is consistent with capital buffer theory. Asset quality (or lack of thereof) appears to have a strong positive effect on risk attitude, which is in line with the findings of Aggarwal and Jacques (2001), who show that banks with higher levels of loan loss provision also have a larger proportion of risky assets in their portfolios. Similarly, liquidity (or lack of thereof) appears to exert a strong positive effect on risk. Not surprisingly, banks with less liquid assets face more risk.

Profitability does not appear to exert a statistically significant effect on risk attitude.[26] Finally, conditional on capital buffer levels, the effect of institutional pressure is not statistically significant, although the sign of the coefficient is plausible.

Columns "Basel I-II"-"DFA" present results for two different subperiods. Some major differences are worth noting. First of all, the size effect is much larger in magnitude during the period under Basel I-II, and remains statistically significant at the 1% level. This implies that the "too-big-to-fail hypothesis", or moral hazard-type behavior in general, was indeed prevalent before the financial crisis hit, and up to 2010. However, the introduction of the DFA appears to largely alleviate this problem. In particular, the effect of size becomes small and is no longer statistically significant. This is consistent with the findings of Zhu et al. (2020), who show that bank behavior during the DFA provides support to Gibrat's "Law", which postulates that the size of a bank and its growth rate are independent.

Secondly, the effect of operational inefficiency appears to be much larger during the period under the Basel I-II than that under the DFA. Similarly, quality and liquidity of portfolios exert a much stronger effect on risk-taking behavior during the period under Basel I-II than DFA. That is, banks with more liquid and higher quality assets are willing to take on more

[26] However, this result changes when profitability is proxied using an alternative measure, namely the return on assets (ROA) as opposed to ROE. This outcome is documented in Table D.1.

risk during Basel I-II but not so during the DFA. Finally, it appears that more profitable banks are less willing to take on more risk during the DFA, whereas there seems to be no effect during Basel I-II.

These results bear important policy implications and provide evidence that the more risk-sensitive capital regulation introduced by the DFA framework has succeeded in influencing banks' behaviour in a substantial manner. This conclusion is contrary to the findings of Ding and Sickles (2019), who infer that the effectiveness of the DFA may be limited.

Table 5.2 below reports direct, indirect and total effects, which have been computed as described in Eq. (5.6) with $\rho = 0$.[27] Total effects are simply the sum of direct and indirect effects. Panel A corresponds to the full sample, i.e. the period spanning 2006:Q1-2019:Q4. In this panel, the direct effects are identical to the estimated coefficients reported in Column (1) of Table 5.1. Direct and indirect effects appear to be of similar magnitude. In particular, roughly speaking, around 55% of the total effects can be attributed to the direct ones, and 45% is due to the indirect effects.

The results change substantially when the sample is split into two subperiods. In particular, for the first subsample (Panel B), the direct effects appear to be larger, contributing roughly three quarters of the total effect. In contrast, for the second subsample (Panel C), direct effects contribute about 48% of the total effect, which is of similar magnitude with the finding obtained from the full sample.

[27] The long-run results are qualitatively identical and so we do not provide them here to save space. They are available upon request.

Table 5.1: **Results for different subperiods**

	Full	**Basel I-II**	**DFA**
$\widehat{\rho}$ (AR parameter)	0.405*** (0.060)	0.388*** (0.070)	0.413*** (0.128)
$\widehat{\psi}$ (spatial parameter)	0.449*** (0.104)	0.255** (0.109)	0.535** (0.270)
$\widehat{\beta}_1$ (inefficiency)	0.331*** (0.086)	0.584** (0.296)	0.196* (0.104)
$\widehat{\beta}_2$ (CAR)	0.011** (0.005)	0.030*** (0.011)	0.008* (0.004)
$\widehat{\beta}_3$ (size)	0.031 (0.072)	0.871*** (0.328)	0.020 (0.178)
$\widehat{\beta}_4$ (buffer)	-0.033** (0.015)	-0.028 (0.025)	-0.015 (0.015)
$\widehat{\beta}_5$ (profitability)	-0.002 (0.002)	-0.002 (0.004)	-0.010** (0.004)
$\widehat{\beta}_6$ (quality)	0.224*** (0.035)	0.239*** (0.042)	0.001 (0.077)
$\widehat{\beta}_7$ (liquidity)	1.438*** (0.213)	2.714*** (0.531)	0.937*** (0.358)
$\widehat{\beta}_8$ (inst. pressure)	-0.022 (0.041)	0.014 (0.066)	-0.021 (0.057)
$\widehat{r}_y$	1	1	1
$\widehat{r}_x$	2	1	1
J-test	28.649 [0.156]	30.205 [0.114]	28.937 [0.147]

Notes: Column "Full" reports results obtained from the full sample. Column "Basel I-II" reports results for the first subsample that spans 2006:Q1-2010:Q4. This is the period under Basel I-II. Column "DFA" reports results for the second subsample that spans 2011:Q1-2019:Q4. Standard errors in parentheses. *** $p < 0.01$, ** $p < 0.05$, * $p < 0.10$. *p-values in square brackets.*

Table 5.2: Decomposition of effects

Panel A: Full sample

Full sample	**Direct**	**Indirect**	**Total**
inefficiency	0.331	0.265	0.596
CAR	0.011	.008	0.019
size	0.031	0.025	0.056
buffer	-0.033	-0.026	-0.059
profitability	-0.002	-0.002	-0.004
quality	0.224	0.179	0.404
liquidity	1.438	1.157	2.591
inst. pressure	-0.022	-0.018	-0.040

Panel B: Basel I-II

	Direct	**Indirect**	**Total**
inefficiency	0.584	0.197	0.782
CAR	0.030	0.010	0.040
size	0.871	0.197	0.782
buffer	-0.028	-0.009	-0.037
profitability	-0.001	-0.001	-0.002
quality	0.239	0.081	0.320
liquidity	2.714	0.917	3.631
inst. pressure	0.014	0.005	0.019

Panel C: DFA

	Direct	**Indirect**	**Total**
inefficiency	0.196	0.220	0.417
CAR	0.008	0.008	0.015
size	0.020	0.022	0.042
buffer	-0.015	-0.017	-0.032
profitability	-0.010	-0.011	-0.021
quality	0.001	0.001	0.002
liquidity	0.937	1.052	1.991
inst. pressure	-0.021	-0.023	-0.044

Notes: See the discussion in the main text on the computation of direct and indirect effects. Basel I-II spans the period 2006:Q1-2010:Q4 ($T = 21$). DFA spans the period 2011:Q1-2019:Q4 ($T = 35$).

6 Concluding Remarks

This paper develops a new IV estimator for spatial, dynamic panel data models with interactive effects under large N and T asymptotics. The proposed estimator is computationally inexpensive and straightforward to implement. Moreover, it is free from asymptotic bias in either cross-sectional or time series dimension. Last, the proposed estimator retains the attractive feature of Method of Moments estimation in that it can potentially accommodate endogenous regressors, so long as external exogenous instruments are available.

Simulation results show that the proposed IV estimator performs well in finite samples, that is, it has negligible bias and produces credible inferences in all cases considered. We have applied our methodology to study the determinants of risk attitude of banking institutions, with emphasis on the impact of increased capital regulation over the past decade or so. The results of our analysis bear important policy implications and provide evidence that the more risk-sensitive capital regulation that was introduced by the Dodd-Frank framework in 2011 has succeeded in influencing banks' behaviour in a substantial manner.

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Online Supplement to

“IV Estimation of Spatial Dynamic Panels with Interactive Effects: Large Sample Theory and an Application on Bank Attitude Toward Risk”

by Guowei Cui, Vasilis Sarafidis and Takashi Yamagata

Online Appendix A: Proofs of main theoretical results

Throughout the appendix, we use C to denote a generic finite constant large enough, which need not to be the same at each appearance. Denote the projection matrix $\mathbf{P_A} = \mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'$ and the residual maker $\mathbf{M_A} = \mathbf{I} - \mathbf{P_A}$ for a matrix $\mathbf{A}$. Let $\mathbf{\Xi}$ be $r_x \times r_x$ diagonal matrix that consist of the first r_x largest eigenvalues of the $T \times T$ matrix $(NT)^{-1}\sum_{i=1}^{N}\mathbf{X}_i\mathbf{X}_i'$. Then by the definition of eigenvalues and $\widehat{\mathbf{F}}$, $\widehat{\mathbf{F}}\mathbf{\Xi} = (NT)^{-1}\sum_{i=1}^{N}\mathbf{X}_i\mathbf{X}_i'\widehat{\mathbf{F}}$. It's easy to show that $\mathbf{\Xi}$ is invertible following the proof of Lemma A.3 in Bai (2003). Then

$$\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{F}^0\mathbf{\Gamma}_i^0\mathbf{V}_i'\widehat{\mathbf{F}}\mathbf{\Xi}^{-1} + \frac{1}{NT}\sum_{i=1}^{N}\mathbf{V}_i\mathbf{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\widehat{\mathbf{F}}\mathbf{\Xi}^{-1} + \frac{1}{NT}\sum_{i=1}^{N}\mathbf{V}_i\mathbf{V}_i'\widehat{\mathbf{F}}\mathbf{\Xi}^{-1} \tag{A.1}$$

and

$$\hat{\mathbf{f}}_t - \mathbf{R}'\mathbf{f}_t^0 = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Xi}^{-1}\widehat{\mathbf{F}}'\mathbf{V}_i\mathbf{\Gamma}_i^{0\prime}\mathbf{f}_t^0 + \frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Xi}^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0\mathbf{\Gamma}_i^0\mathbf{v}_{it} + \frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Xi}^{-1}\widehat{\mathbf{F}}'\mathbf{V}_i\mathbf{v}_{it} \tag{A.2}$$

where $\mathbf{R} = (NT)^{-1}\sum_{i=1}^{N}\mathbf{\Gamma}_i^0\mathbf{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\widehat{\mathbf{F}}\mathbf{\Xi}^{-1}$. Following the proof of Lemma A.3 in Bai (2003) again, we can show that $\mathbf{R}$ is invertible.

Let $\mathbf{\Xi}_L$ be a $r_x \times r_x$ diagonal matrix that consists of the first r_x largest eigenvalues of the $T \times T$ matrix $(NT)^{-1}\sum_{i=1}^{N}\mathbf{X}_{i,-1}\mathbf{X}_{i,-1}'$. Then by the definition of eigenvalues and $\widehat{\mathbf{F}}_{-1}$, $\widehat{\mathbf{F}}_{-1}\mathbf{\Xi}_L = (NT)^{-1}\sum_{i=1}^{N}\mathbf{X}_{i,-1}\mathbf{X}_{i,-1}'\widehat{\mathbf{F}}_{-1}$. It's easy to show that $\mathbf{\Xi}_L$ is invertible following the proof of Lemma A.3 in Bai (2003). Then

$$\begin{aligned}&\widehat{\mathbf{F}}_{-1} - \mathbf{F}_{-1}^0\mathfrak{R}\\ =&\frac{1}{NT}\sum_{i=1}^{N}\mathbf{F}_{-1}^0\mathbf{\Gamma}_i^0\mathbf{V}_{i,-1}'\widehat{\mathbf{F}}_{-1}\mathbf{\Xi}_L^{-1} + \frac{1}{NT}\sum_{i=1}^{N}\mathbf{V}_{i,-1}\mathbf{\Gamma}_i^{0\prime}\mathbf{F}_{-1}^{0\prime}\widehat{\mathbf{F}}_{-1}\mathbf{\Xi}_L^{-1} + \frac{1}{NT}\sum_{i=1}^{N}\mathbf{V}_{i,-1}\mathbf{V}_{i,-1}'\widehat{\mathbf{F}}_{-1}\mathbf{\Xi}_L^{-1}\end{aligned} \tag{A.3}$$

where $\mathfrak{R} = (NT)^{-1}\sum_{i=1}^{N}\mathbf{\Gamma}_i^0\mathbf{\Gamma}_i^{0\prime}\mathbf{F}_{-1}^{0\prime}\widehat{\mathbf{F}}_{-1}\mathbf{\Xi}_L^{-1}$. Following the proof of Lemma A.3 in Bai (2003) again, we can show that $\mathfrak{R}$ is invertible.

Lemma A.1 *Under Assumptions B to D, we have*

$$
\begin{aligned}
&(a)\quad T^{-1}\|\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\|^2 = O_p(\delta_{NT}^{-2})\,, T^{-1}\|\widehat{\mathbf{F}}_{-1}-\mathbf{F}^0_{-1}\mathfrak{R}\|^2 = O_p(\delta_{NT}^{-2})\,,\\
&(b)\quad T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{F}^0 = O_p(\delta_{NT}^{-2})\,, T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{F}^0_{-1} = O_p(\delta_{NT}^{-2})\,,\\
&\qquad T^{-1}(\widehat{\mathbf{F}}_{-1}-\mathbf{F}^0_{-1}\mathfrak{R})'\mathbf{F}^0 = O_p(\delta_{NT}^{-2})\,, T^{-1}(\widehat{\mathbf{F}}_{-1}-\mathbf{F}^0_{-1}\mathfrak{R})'\mathbf{F}^0_{-1} = O_p(\delta_{NT}^{-2})\,,\\
&(c)\quad T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\widehat{\mathbf{F}} = O_p(\delta_{NT}^{-2})\,, T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\widehat{\mathbf{F}}_{-1} = O_p\left(\delta_{NT}^{-2}\right)\,,\\
&\qquad T^{-1}(\widehat{\mathbf{F}}_{-1}-\mathbf{F}^0_{-1}\mathfrak{R})'\widehat{\mathbf{F}} = O_p(\delta_{NT}^{-2})\,, T^{-1}(\widehat{\mathbf{F}}_{-1}-\mathbf{F}^0_{-1}\mathfrak{R})'\widehat{\mathbf{F}}_{-1} = O_p(\delta_{NT}^{-2})\,,\\
&(d)\quad T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{H}^0 = O_p(\delta_{NT}^{-2})\,, T^{-1}(\widehat{\mathbf{F}}_{-1}-\mathbf{F}^0_{-1}\mathfrak{R})'\mathbf{H}^0 = O_p(\delta_{NT}^{-2})\,,\\
&(e)\quad \mathbf{\Xi} = O_p(1)\,, \mathbf{R} = O_p(1)\,, \mathbf{\Xi}^{-1} = O_p(1)\,, \mathbf{R}^{-1} = O_p(1)\,,\\
&\qquad \mathbf{\Xi}_L = O_p(1)\,, \mathfrak{R} = O_p(1)\,, \mathbf{\Xi}_L^{-1} = O_p(1)\,, \mathfrak{R}^{-1} = O_p(1)\\
&(f)\quad \mathbf{R}\mathbf{R}' - (T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1} = O_p(\delta_{NT}^{-2})\,, \mathfrak{R}\mathfrak{R}' - (T^{-1}\mathbf{F}^{0\prime}_{-1}\mathbf{F}^0_{-1})^{-1} = O_p\left(\delta_{NT}^{-2}\right)\,,\\
&(g)\quad \mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0} = O_p(\delta_{NT}^{-1})\,, \mathbf{M}_{\widehat{\mathbf{F}}_{-1}} - \mathbf{M}_{\mathbf{F}^0_{-1}} = O_p(\delta_{NT}^{-1})\,,\\
&(h)\quad N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^{00}\mathbf{V}_\ell'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}) = O_p(N^{-1/2}T^{1/2}) + O_p(\delta_{NT}^{-2}T^{1/2})\,,\\
&\qquad N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^{00}\mathbf{V}_{\ell,-1}'(\widehat{\mathbf{F}}_{-1}-\mathbf{F}^0_{-1}\mathfrak{R}) = O_p(N^{-1/2}T^{1/2}) + O_p(\delta_{NT}^{-2}T^{1/2})\,,\\
&(i)\quad \|N^{-1/2}T^{-1}\sum_{h=1}^{N}\left(\mathbf{V}_h\mathbf{V}_h' - \mathbb{E}\left(\mathbf{V}_h\mathbf{V}_h'\right)\right)\mathbf{F}^0\| = O_p(1)\,,\\
&(j)\quad \|N^{-1/2}T^{-1}\sum_{h=1}^{N}\mathbf{F}^{0\prime}\left(\mathbf{V}_h\mathbf{V}_h' - \mathbb{E}\left(\mathbf{V}_h\mathbf{V}_h'\right)\right)\mathbf{F}^0\| = O_p(1)\,,\\
&(k)\quad \|N^{-1/2}T^{-1}\sum_{h=1}^{N}\left(\mathbf{V}_h\mathbf{V}_h' - \mathbb{E}\left(\mathbf{V}_h\mathbf{V}_h'\right)\right)\mathbf{H}^0\| = O_p(1)\,,\\
&(l)\quad \|N^{-1/2}T^{-1}\sum_{h=1}^{N}\mathbf{F}^{0\prime}\left(\mathbf{V}_h\mathbf{V}_h' - \mathbb{E}\left(\mathbf{V}_h\mathbf{V}_h'\right)\right)\mathbf{H}^0\| = O_p(1)\,,\\
&(m)\quad \|N^{-1}T^{-1/2}\sum_{\ell=1}^{N}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\| = O(1)\,,\\
&(n)\quad N^{-1/2}T^{-1}\sum_{\ell=1}^{N}\left[\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right] = O_p(1)\,.
\end{aligned}
$$

Proof of Lemma A.1. For the proofs of (a) to (f), and (h), see Proof of Lemma 4 in Supplemental Material, Norkute et al. (2020). For (g), we decompose the left hand side term as

$$
\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0} = -\frac{\widehat{\mathbf{F}}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'}{T} - \frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\mathbf{R}'\mathbf{F}^{0\prime}}{T} - \frac{1}{T}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}' - \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right)\mathbf{F}^{0\prime}
$$

then it will be bounded in norm by

$$
\left\|\frac{\widehat{\mathbf{F}}}{\sqrt{T}}\right\|\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\| + \|\mathbf{R}\|\left\|\frac{\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\| + \left\|\frac{\mathbf{F}^0}{\sqrt{T}}\right\|^2\left\|\mathbf{R}\mathbf{R}' - \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\| = O_p\left(\delta_{NT}^{-1}\right)
$$

with Lemmas A.1 (a), (e), (f) and the facts that $\left\|T^{-1/2}\widehat{\mathbf{F}}\right\|^2 = r_x$ and $\mathbb{E}\left\|T^{-1/2}\mathbf{F}^0\right\|^2 \le C$ by Assumption C. For (i), we have

$$\left\|\frac{1}{\sqrt{NT}}\sum_{h=1}^{N}\left(\mathbf{V}_h\mathbf{V}_h'-\mathbb{E}\left(\mathbf{V}_h\mathbf{V}_h'\right)\right)\mathbf{F}^0\right\| = \left(\frac{1}{T}\sum_{s=1}^{T}\left\|\frac{1}{\sqrt{NT}}\sum_{h=1}^{N}\sum_{t=1}^{T}\left[\mathbf{v}_{hs}'\mathbf{v}_{ht}-\mathbb{E}\left(\mathbf{v}_{hs}'\mathbf{v}_{ht}\right)\right]\mathbf{f}_t^0\right\|^2\right)^{1/2}$$
$$=O_p(1)$$

since $\mathbb{E}\left\|\frac{1}{\sqrt{NT}}\sum_{h=1}^{N}\sum_{t=1}^{T}\left[\mathbf{v}_{hs}'\mathbf{v}_{ht}-\mathbb{E}\left(\mathbf{v}_{hs}'\mathbf{v}_{ht}\right)\right]\mathbf{f}_t^0\right\|^2 \leq C$ by Assumption B7. For (j), it holds because

$$\mathbb{E}\left\|\frac{1}{\sqrt{NT}}\sum_{h=1}^{N}\mathbf{F}^{0\prime}\left(\mathbf{V}_h\mathbf{V}_h'-\mathbb{E}\left(\mathbf{V}_h\mathbf{V}_h'\right)\right)\mathbf{F}^0\right\|^2 \leq C$$

which can be proved following the proof of Lemma A.2(i) in Bai (2009) with Assumption B8. The proofs of (k) and (l) are similar to those of (i) and (j), respectively.

Consider (m). Since

$$\frac{1}{N}\sum_{i=1}^{N}\left|\mathbb{E}\left(\mathbf{v}_{is}'\mathbf{v}_{it}\right)\right| \leq \frac{1}{N}\sum_{i=1}^{N}\left(\mathbb{E}\left(\mathbf{v}_{is}'\mathbf{v}_{is}\right)\mathbb{E}\left(\mathbf{v}_{it}'\mathbf{v}_{it}\right)\right)^{1/2}$$
$$\leq\left(\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(\mathbf{v}_{is}'\mathbf{v}_{is}\right)\frac{1}{N}\sum_{j=1}^{N}\mathbb{E}\left(\mathbf{v}_{jt}'\mathbf{v}_{jt}\right)\right)^{1/2} \leq C$$

we have

$$\left\|\frac{1}{N\sqrt{T}}\sum_{\ell=1}^{N}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right\|^2 = \frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\left|\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(\mathbf{v}_{is}'\mathbf{v}_{it}\right)\right|^2$$
$$\leq\frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\frac{1}{N}\sum_{i=1}^{N}\left|\mathbb{E}\left(\mathbf{v}_{is}'\mathbf{v}_{it}\right)\right|\frac{1}{N}\sum_{j=1}^{N}\left|\mathbb{E}\left(\mathbf{v}_{js}'\mathbf{v}_{jt}\right)\right| \leq \frac{C}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\frac{1}{N}\sum_{i=1}^{N}\left|\mathbb{E}\left(\mathbf{v}_{is}'\mathbf{v}_{it}\right)\right| \leq \frac{C}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\sigma}_{st} \leq C^2\,,$$

by Assumption B3.

Consider (n), we can derive that

$$\mathbb{E}\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\left[\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\right\|^2 = \mathbb{E}\left(\frac{1}{T^2}\sum_{s=1}^{T}\sum_{t=1}^{T}\left[\frac{1}{\sqrt{N}}\sum_{\ell=1}^{N}\left(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}-\mathbb{E}\left(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}\right)\right)\right]^2\right)$$
$$=\frac{1}{T^2}\sum_{s=1}^{T}\sum_{t=1}^{T}\mathbb{E}\left[\frac{1}{\sqrt{N}}\sum_{\ell=1}^{N}\left(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}-\mathbb{E}\left(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}\right)\right)\right]^2 \leq C\,,$$

by Assumption B4. Then we can prove part (n) by Markov inequality. This completes the proof. □

Lemma A.2 *Under Assumptions A to D, for all i, we have*

$$
\begin{aligned}
&(a)\quad \mathbb{E}\left\|\frac{1}{\sqrt{T}}\boldsymbol{\varepsilon}_i\right\|^2=\sigma_\varepsilon^2\,,\\
&(b)\quad \mathbb{E}\left\|\frac{1}{\sqrt{T}}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\right\|^2\le C\,,\\
&(c)\quad \mathbb{E}\left\|\frac{1}{NT}\sum_{\ell=1}^N\boldsymbol{\varepsilon}_i'\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{F}^0\right\|^2\le C\,,\\
&(d)\quad \mathbb{E}\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^N\boldsymbol{\varepsilon}_i'\left[\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\mathbf{F}^0\right\|^2\le C\,,\\
&(e)\quad \mathbb{E}\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^N\boldsymbol{\varepsilon}_i'\left[\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\right\|^2\le C\,,\\
&(f)\quad \mathbb{E}\left\|\frac{1}{\sqrt{T}}\mathbf{V}_i'\mathbf{F}^0\right\|^2\le C\,,
\end{aligned}
$$

Proof of Lemma A.2. (a) can be derived easily with Assumption A.

Consider (b). with Assumptions A and C, we have

$$
\mathbb{E}\left\|\frac{1}{\sqrt{T}}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\right\|^2=\mathbb{E}\left(\frac{1}{T}\sum_{s=1}^T\sum_{t=1}^T\varepsilon_{is}\varepsilon_{it}\mathbf{f}_s^{0\prime}\mathbf{f}_t^0\right)=\frac{1}{T}\sum_{s=1}^T\sum_{t=1}^T\mathbb{E}\left(\varepsilon_{is}\varepsilon_{it}\right)\mathbb{E}\left(\mathbf{f}_s^{0\prime}\mathbf{f}_t^0\right)=\frac{\sigma_\varepsilon^2}{T}\sum_{t=1}^T\mathbb{E}\left\|\mathbf{f}_t^0\right\|^2\le C.
$$

Consider (c). By Cauchy-Schwardz inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left\|\frac{1}{NT}\sum_{\ell=1}^N\boldsymbol{\varepsilon}_i'\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{F}^0\right\|^2=\mathbb{E}\left\|\frac{1}{T}\sum_{s=1}^T\sum_{t=1}^T\left(\frac{1}{N}\sum_{l=1}^N\mathbb{E}\left(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}\right)\right)\varepsilon_{is}\mathbf{f}_t^0\right\|^2\\
\le&\frac{1}{T^2}\sum_{s_1=1}^T\sum_{t_1=1}^T\sum_{s_2=1}^T\sum_{t_2=1}^T\left|\frac{1}{N}\sum_{\ell=1}^N\mathbb{E}\left(\mathbf{v}_{\ell s_1}'\mathbf{v}_{\ell t_1}\right)\right|\left|\frac{1}{N}\sum_{\ell=1}^N\mathbb{E}\left(\mathbf{v}_{\ell s_2}'\mathbf{v}_{\ell t_2}\right)\right|\mathbb{E}\left(\left\|\varepsilon_{is_1}\mathbf{f}_{t_1}^0\right\|\left\|\varepsilon_{is_2}\mathbf{f}_{t_2}^0\right\|\right)\\
\le&\frac{1}{T^2}\sum_{s_1=1}^T\sum_{t_1=1}^T\sum_{s_2=1}^T\sum_{t_2=1}^T\left|\frac{1}{N}\sum_{\ell=1}^N\mathbb{E}\left(\mathbf{v}_{\ell s_1}'\mathbf{v}_{\ell t_1}\right)\right|\left|\frac{1}{N}\sum_{\ell=1}^N\mathbb{E}\left(\mathbf{v}_{\ell s_2}'\mathbf{v}_{\ell t_2}\right)\right|\left(\mathbb{E}\varepsilon_{is_1}^4\mathbb{E}\varepsilon_{is_2}^4\mathbb{E}\left\|\mathbf{f}_{t_1}^0\right\|^4\mathbb{E}\left\|\mathbf{f}_{t_2}^0\right\|^4\right)^{1/4}\\
\le&C\times\left(\frac{1}{T}\sum_{s=1}^T\sum_{t=1}^T\left|\frac{1}{N}\sum_{\ell=1}^N\mathbb{E}\left(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}\right)\right|\right)^2\le C\times\left(\frac{1}{T}\sum_{s=1}^T\sum_{t=1}^T\tilde{\sigma}_{st}\right)^2\le C^3\,,
\end{aligned}
$$

by Assumptions A, B3, and C.

Consider (d). We have

$$
\begin{aligned}
&\mathbb{E}\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^N\boldsymbol{\varepsilon}_i'\left[\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\mathbf{F}^0\right\|^2=\mathbb{E}\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^N\sum_{s=1}^T\sum_{t=1}^T\left[\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}-\mathbb{E}\left(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}\right)\right]\mathbf{f}_t^0\varepsilon_{is}\right\|^2\\
=&\frac{1}{NT^2}\sum_{\ell_1=1}^N\sum_{\ell_2=1}^N\sum_{s_1=1}^T\sum_{s_2=1}^T\sum_{t_1=1}^T\sum_{t_2=1}^T\mathbb{E}\left(\left[\mathbf{v}_{\ell_1s_1}'\mathbf{v}_{\ell_1t_1}-\mathbb{E}\left(\mathbf{v}_{\ell_1s_1}'\mathbf{v}_{\ell_1t_1}\right)\right]\left[\mathbf{v}_{\ell_2s_2}'\mathbf{v}_{\ell_2t_2}-\mathbb{E}\left(\mathbf{v}_{\ell_2s_2}'\mathbf{v}_{\ell_2t_2}\right)\right]\mathbf{f}_{t_1}^{0\prime}\mathbf{f}_{t_2}^0\varepsilon_{is_1}\varepsilon_{is_2}\right)\\
=&\frac{1}{NT^2}\sum_{\ell_1=1}^N\sum_{\ell_2=1}^N\sum_{s_1=1}^T\sum_{s_2=1}^T\sum_{t_1=1}^T\sum_{t_2=1}^T\mathbb{E}\left(\left[\mathbf{v}_{\ell_1s_1}'\mathbf{v}_{\ell_1t_1}-\mathbb{E}\left(\mathbf{v}_{\ell_1s_1}'\mathbf{v}_{\ell_1t_1}\right)\right]\left[\mathbf{v}_{\ell_2s_2}'\mathbf{v}_{\ell_2t_2}-\mathbb{E}\left(\mathbf{v}_{\ell_2s_2}'\mathbf{v}_{\ell_2t_2}\right)\right]\right)\mathbb{E}\left(\mathbf{f}_{t_1}^{0\prime}\mathbf{f}_{t_2}^0\right)\mathbb{E}\left(\varepsilon_{is_1}\varepsilon_{is_2}\right)\\
\le&\frac{1}{NT^2}\sum_{\ell_1=1}^N\sum_{\ell_2=1}^N\sum_{s_1=1}^T\sum_{s_2=1}^T\sum_{t_1=1}^T\sum_{t_2=1}^T\left|\operatorname{cov}\left(\mathbf{v}_{\ell_1s_1}'\mathbf{v}_{\ell_1t_1},\mathbf{v}_{\ell_2s_2}'\mathbf{v}_{\ell_2t_2}\right)\right|\left(\mathbb{E}\left\|\mathbf{f}_{t_1}^0\right\|^2\mathbb{E}\left\|\mathbf{f}_{t_2}^0\right\|^2\right)^{1/2}\sigma_\varepsilon^2\\
\le&\frac{\sigma_\varepsilon^2C}{NT^2}\sum_{\ell_1=1}^N\sum_{\ell_2=1}^N\sum_{s_1=1}^T\sum_{s_2=1}^T\sum_{t_1=1}^T\sum_{t_2=1}^T\left|\operatorname{cov}\left(\mathbf{v}_{\ell_1s_1}'\mathbf{v}_{\ell_1t_1},\mathbf{v}_{\ell_2s_2}'\mathbf{v}_{\ell_2t_2}\right)\right|\le\sigma_\varepsilon^2C^2
\end{aligned}
$$

by Assumptions A, B8, and C.

Consider (e). We have

$$\mathbb{E}\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\left[\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\right\|^2=\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\sum_{s=1}^{T}\varepsilon_{is}\left[\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}-\mathbb{E}\left(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}\right)\right]\right\|^2\leq C$$

Consider (f). We have

$$\mathbb{E}\left\|\frac{1}{\sqrt{T}}\mathbf{V}_i'\mathbf{F}\right\|^2=\frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\operatorname{tr}\left[\mathbb{E}\left(\mathbf{v}_{is}\mathbf{v}_{it}'\right)\right]\mathbb{E}\left(\mathbf{f}_s^{0\prime}\mathbf{f}_t^0\right)\leq\frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\left\|\mathbb{E}\mathbf{v}_{is}\mathbf{v}_{it}'\right\|\left(\mathbb{E}\left\|\mathbf{f}_s^0\right\|^2\mathbb{E}\left\|\mathbf{f}_t^0\right\|^2\right)^{1/2}$$
$$\leq\frac{C}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\sigma}_{st}\leq C^2$$

by Assumptions B3 and C. This completes the proof. □

Lemma A.3 *Under Assumptions A to D, we have*

$$(a)\quad N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i-N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i=O_p(\delta_{NT}^{-1}),$$
$$(b)\quad N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N}w_{ij}w_{i\ell}\mathbf{X}_j'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_\ell-N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N}w_{ij}w_{i\ell}\mathbf{X}_j'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_\ell=O_p(\delta_{NT}^{-1}),$$
$$(c)\quad N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{X}_j'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{X}_{i,-1}-N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{X}_j'\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{F}^0_{-1}}\mathbf{X}_{i,-1}=O_p(\delta_{NT}^{-1})$$
$$(d)\quad N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{X}_j'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i-N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{X}_j'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i=O_p(\delta_{NT}^{-1}),$$
$$(e)\quad N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{X}_{i,-1}'\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{X}_{i,-1}-N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{X}_{i,-1}'\mathbf{M}_{\mathbf{F}^0_{-1}}\mathbf{X}_{i,-1}=O_p(\delta_{NT}^{-1}),$$
$$(f)\quad N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{X}_{i,-1}'\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i-N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{X}_{i,-1}'\mathbf{M}_{\mathbf{F}^0_{-1}}\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i=O_p(\delta_{NT}^{-1}),$$
$$(g)\quad N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{w}_i'\mathbf{Y}'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_j-N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{w}_i'\mathbf{Y}'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_j=O_p(\delta_{NT}^{-1}),$$
$$(h)\quad N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{Y}'\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{X}_{i,-1}-N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{Y}'\mathbf{M}_{\mathbf{F}^0_{-1}}\mathbf{X}_{i,-1}=O_p(\delta_{NT}^{-1}),$$
$$(i)\quad N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{Y}'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i-N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{w}_i'\mathbf{Y}'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i=O_p(\delta_{NT}^{-1}),$$
$$(j)\quad N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{y}_{i,-1}'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_j-N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{y}_{i,-1}'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_j=O_p(\delta_{NT}^{-1}),$$
$$(k)\quad N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{y}_{i,-1}'\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{X}_{i,-1}-N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{y}_{i,-1}'\mathbf{M}_{\mathbf{F}^0_{-1}}\mathbf{X}_{i,-1}=O_p(\delta_{NT}^{-1}),$$
$$(l)\quad N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{y}_{i,-1}'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i-N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{y}_{i,-1}'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i=O_p(\delta_{NT}^{-1}),$$
$$(m)\quad N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{X}_{i,-1}'\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{X}_i-N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{X}_{i,-1}'\mathbf{M}_{\mathbf{F}^0_{-1}}\mathbf{X}_i=O_p(\delta_{NT}^{-1}).$$

Proof of Lemma A.3. For (a), please see Lemma 6 in Norkute et al. (2020). For (b), the left hand side is bounded in norm by

$$
\begin{aligned}
&\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N}w_{ij}w_{i\ell}\mathbf{X}_j'\left(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\right)\mathbf{X}_\ell\right\| \\
\leq&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N}|w_{ij}||w_{i\ell}|\left\|\frac{\mathbf{X}_j}{\sqrt{T}}\right\|\left\|\frac{\mathbf{X}_\ell}{\sqrt{T}}\right\|\left\|\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\right\| \\
=&O_p\left(\frac{1}{\delta_{NT}}\right)\times\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N}|w_{ij}||w_{i\ell}|\left\|\frac{\mathbf{X}_j}{\sqrt{T}}\right\|\left\|\frac{\mathbf{X}_\ell}{\sqrt{T}}\right\|
\end{aligned}
$$

by Lemma A.1 (g). With Assumptions B, C and D, it's easy to show that

$$
\mathbb{E}\left\|\mathbf{x}_{it}\right\|^4\leq C \tag{A.4}
$$

for all i and t. In addition, we can derive that

$$
\begin{aligned}
&\mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N}|w_{ij}||w_{i\ell}|\left\|\frac{\mathbf{X}_j}{\sqrt{T}}\right\|\left\|\frac{\mathbf{X}_\ell}{\sqrt{T}}\right\|\right)=\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N}|w_{ij}||w_{i\ell}|\mathbb{E}\left(\left\|\frac{\mathbf{X}_j}{\sqrt{T}}\right\|\left\|\frac{\mathbf{X}_\ell}{\sqrt{T}}\right\|\right) \\
\leq&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N}|w_{ij}||w_{i\ell}|\left(\mathbb{E}\left\|\frac{\mathbf{X}_j}{\sqrt{T}}\right\|^2\mathbb{E}\left\|\frac{\mathbf{X}_\ell}{\sqrt{T}}\right\|^2\right)^{1/2} \\
=&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N}|w_{ij}||w_{i\ell}|\left(\left(\frac{1}{T}\sum_{s=1}^{T}\mathbb{E}\left\|\mathbf{x}_{js}\right\|^2\right)\left(\frac{1}{T}\sum_{s=1}^{T}\mathbb{E}\left\|\mathbf{x}_{\ell s}\right\|^2\right)\right)^{1/2} \\
\leq&\frac{C}{N}\sum_{i=1}^{N}\left(\sum_{j=1}^{N}|w_{ij}|\right)\left(\sum_{\ell=1}^{N}|w_{i\ell}|\right)\leq C^3
\end{aligned} \tag{A.5}
$$

by Assumption E3 and the equation (A.4). Then by Markov inequality, we derive that

$$
\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N}|w_{ij}||w_{i\ell}|\left\|\frac{\mathbf{X}_j}{\sqrt{T}}\right\|\left\|\frac{\mathbf{X}_\ell}{\sqrt{T}}\right\|=O_p(1)
$$

Combining the above equations, we can derive that (b). Consider (c), we can bound the left hand side term as

$$
\begin{aligned}
&\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{X}_j'\left(\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}-\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{F}^0_{-1}}\right)\mathbf{X}_{i,-1}\right\| \\
=&\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{X}_j'\mathbf{M}_{\widehat{\mathbf{F}}}\left(\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}-\mathbf{M}_{\mathbf{F}^0_{-1}}\right)\mathbf{X}_{i,-1}\right\|+\left\|\frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{X}_j'\left(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\right)\mathbf{M}_{\mathbf{F}^0_{-1}}\mathbf{X}_{i,-1}\right\| \\
\leq&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{X}_j'\mathbf{M}_{\widehat{\mathbf{F}}}}{\sqrt{T}}\right\|\left\|\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}-\mathbf{M}_{\mathbf{F}^0_{-1}}\right\|\left\|\frac{\mathbf{X}_{i,-1}}{\sqrt{T}}\right\|+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{X}_j}{\sqrt{T}}\right\|\left\|\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\right\|\left\|\frac{\mathbf{M}_{\mathbf{F}^0_{-1}}\mathbf{X}_{i,-1}}{\sqrt{T}}\right\| \\
\leq&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{X}_j'}{\sqrt{T}}\right\|\left\|\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}-\mathbf{M}_{\mathbf{F}^0_{-1}}\right\|\left\|\frac{\mathbf{X}_{i,-1}}{\sqrt{T}}\right\|+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{X}_j}{\sqrt{T}}\right\|\left\|\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\right\|\left\|\frac{\mathbf{X}_{i,-1}}{\sqrt{T}}\right\| \\
=&O_p\left(\frac{1}{\delta_{NT}}\right)\times\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{X}_j}{\sqrt{T}}\right\|\left\|\frac{\mathbf{X}_{i,-1}}{\sqrt{T}}\right\|+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{X}_j}{\sqrt{T}}\right\|\left\|\frac{\mathbf{X}_{i,-1}}{\sqrt{T}}\right\|\right)
\end{aligned}
$$

Similar to the equation (A.5), we can show that

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{X}_j}{\sqrt{T}}\right\|\left\|\frac{\mathbf{X}_{i,-1}}{\sqrt{T}}\right\|=O_p(1)$$

with Markov inequality. Then we have (c). The cases (d) to (m) can be proved similarly. This completes the proof. □

Lemma A.4 *Under Assumptions A to E, we have*

$$(a)\quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^0\right\|\left\|\boldsymbol{\varphi}_i^0\right\|=O_p(1)$$

$$(b)\quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^0\right\|\left\|\boldsymbol{\varphi}_i^0\right\|=O_p(1)$$

$$(c)\quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^0\right\|\left\|\frac{\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|=O_p(1)$$

$$(d)\quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^0\right\|\left\|\frac{\mathbf{u}_i}{\sqrt{T}}\right\|=O_p(1)$$

$$(e)\quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^0\right\|\left\|\frac{1}{\sqrt{T}}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\right\|=O_p(1)\,,\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^0\right\|\left\|\frac{1}{\sqrt{T}}\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i\right\|=O_p(1)$$

$$(f)\quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^0\right\|\left\|\frac{1}{NT}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{F}^0\right\|=O_p(1)$$

$$(g)\quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^0\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{T}\boldsymbol{\varepsilon}_i'\left[\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\mathbf{F}^0\right\|=O_p(1)$$

$$(h)\quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^0\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{T}\boldsymbol{\varepsilon}_i'\left[\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\right\|=O_p(1)$$

$$(i)\quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^0\right\|\left\|\frac{1}{T}\left(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1}\right)'\boldsymbol{\varepsilon}_i\right\|=O_p\left(\delta_{NT}^{-2}\right)$$

Proof of Lemma A.4. Consider (a). By Cauchy-Schwardz inequality, we have

$$\begin{aligned}\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^0\right\|\left\|\boldsymbol{\varphi}_i^0\right\|\right|&=\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\mathbb{E}\left(\left\|\mathbf{\Gamma}_j^0\right\|\left\|\boldsymbol{\varphi}_i^0\right\|\right)\\ \leq&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left(\mathbb{E}\left\|\mathbf{\Gamma}_j^0\right\|^2\mathbb{E}\left\|\boldsymbol{\varphi}_i^0\right\|^2\right)^{1/2}\leq\frac{C}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\leq C^2\end{aligned}$$

Then we derive that

$$\mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^0\right\|\left\|\boldsymbol{\varphi}_i^0\right\|\geq M\right)\leq\frac{\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^0\right\|\left\|\boldsymbol{\varphi}_i^0\right\|\right|}{M}\leq\frac{C^2}{M}$$

by the Markov inequality. Thus, for any $\epsilon>0$, there exists $M=C^2/\epsilon$, such that

$$\mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^0\right\|\left\|\boldsymbol{\varphi}_i^0\right\|\geq M\right)\leq\epsilon\,.$$

Then we complete the proof of (a). Following the proof of part (a), we can prove part (b) with $\mathbb{E}\left\|\boldsymbol{\varphi}_i\right\|^2\leq C$, and prove part (c) with Lemma A.2 (a).

Consider (d). Note that $T^{-1/2}\mathbf{F}^0 = O_p(1)$ and $T^{-1/2}\mathbf{H}^0 = O_p(1)$. With Lemmas A.4 (a) to (c), we can prove (d) easily with the definition of $\mathbf{u}_i$.

Following the proof part of (a), we can prove part (e) with Lemma A.2 (b), and we can prove part (f) with Lemma A.2 (c).

Consider (g). We can show that

$$
\begin{aligned}
&\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{T}\boldsymbol{\varepsilon}_i'\left[\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\mathbf{F}^0\right\|\right| \\
=&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\mathbb{E}\left(\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{T}\boldsymbol{\varepsilon}_i'\left[\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\mathbf{F}^0\right\|\right) \\
\leq&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left(\mathbb{E}\left\|\boldsymbol{\Gamma}_j^0\right\|^2\mathbb{E}\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{T}\boldsymbol{\varepsilon}_i'\left[\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\mathbf{F}^0\right\|^2\right)^{1/2} \\
\leq&\frac{C}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}| \leq C^2
\end{aligned}
$$

by Assumption D and Lemma A.2 (d).

Following the proof of (g), we can prove part (h) with Assumption D and Lemma A.2 (e).

Consider (i). With the equation (A.1), we have

$$
\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\frac{1}{T}\left(\mathbf{F}^0 - \widehat{\mathbf{F}}\mathbf{R}^{-1}\right)'\boldsymbol{\varepsilon}_i\right\| \\
\leq&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\boldsymbol{\varepsilon}_i'\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{F}^0\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\widehat{\mathbf{F}}\boldsymbol{\Xi}^{-1}\mathbf{R}^{-1}\right\| + \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\boldsymbol{\varepsilon}_i'\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\mathbf{F}^{0\prime}\widehat{\mathbf{F}}\boldsymbol{\Xi}^{-1}\mathbf{R}^{-1}\right\| \\
&+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\boldsymbol{\varepsilon}_i'\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{V}_\ell\mathbf{V}_\ell'\widehat{\mathbf{F}}\boldsymbol{\Xi}^{-1}\mathbf{R}^{-1}\right\| \\
\leq&\frac{1}{\sqrt{NT}}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\frac{1}{\sqrt{T}}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\right\|\right)\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0\right\|\left\|\mathbf{R}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\left\|\mathbf{R}^{-1}\right\| \\
&+\frac{1}{\sqrt{NT}}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\frac{1}{\sqrt{T}}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\right\|\right)\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\right\|\left\|\frac{1}{\sqrt{T}}\left(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}\right)\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\left\|\mathbf{R}^{-1}\right\| \\
&+\frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\right\|\left\|\frac{\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\frac{\widehat{\mathbf{F}}}{\sqrt{T}}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\left\|\mathbf{R}^{-1}\right\| \\
&+\frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\frac{\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\left\|\frac{1}{N\sqrt{T}}\sum_{\ell=1}^{N}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right\|\left\|\frac{\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\left\|\mathbf{R}^{-1}\right\| \\
&+\frac{1}{\sqrt{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\frac{\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\left[\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\right\|\left\|\frac{\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\left\|\mathbf{R}^{-1}\right\| \\
&+\frac{1}{T}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\boldsymbol{\varepsilon}_i'\frac{1}{NT}\sum_{\ell=1}^{N}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{F}^0\right\|\left\|\mathbf{R}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\left\|\mathbf{R}^{-1}\right\| \\
&+\frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\boldsymbol{\varepsilon}_i'\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\left[\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\mathbf{F}^0\right\|\left\|\mathbf{R}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\left\|\mathbf{R}^{-1}\right\| \\
=&O_p\left(\delta_{NT}^{-2}\right)
\end{aligned}
$$

by Lemma A.1 (a) and A.1 (e), Lemma A.4 (c), (e), (f) and (g). This completes the proof. □

Lemma A.5 *Under Assumptions A to D, we have*

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
=&-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^{0}\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
&-\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i + O_p\left(\frac{1}{\delta_{NT}^2}\right) + O_p\left(\sqrt{\frac{T}{N^3}}\right)
\end{aligned}
$$

Proof of Lemma A.5. First, we have

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1})'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
=&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\frac{1}{NT}\sum_{\ell=1}^{N}\mathbf{R}^{-1\prime}\boldsymbol{\Xi}^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\frac{1}{NT}\sum_{\ell=1}^{N}\mathbf{R}^{-1\prime}\boldsymbol{\Xi}^{-1}\widehat{\mathbf{F}}'\mathbf{F}^{0}\boldsymbol{\Gamma}_\ell^{0}\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\frac{1}{NT}\sum_{\ell=1}^{N}\mathbf{R}^{-1\prime}\boldsymbol{\Xi}^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
=&\mathbb{B}_1+\mathbb{B}_2+\mathbb{B}_3
\end{aligned}
$$

We consider the term $\mathbb{B}_1$. By Lemma A.1 (h), we have

$$
\begin{aligned}
\frac{1}{NT}\sum_{\ell=1}^{N}\widehat{\mathbf{F}}'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime} =&\mathbf{R}'\frac{1}{NT}\sum_{\ell=1}^{N}\mathbf{F}^{0\prime}\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime} + \frac{1}{NT}\sum_{\ell=1}^{N}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime} \\
=&O_p\left(\frac{1}{\sqrt{NT}}\right)+O_p\left(\frac{1}{N}\right)+O_p\left(\frac{1}{\sqrt{N}\delta_{NT}^2}\right).
\end{aligned}
\tag{A.6}
$$

Note that that $\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F}^0=\mathbf{M}_{\widehat{\mathbf{F}}}\left(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1}\right)$ and $\mathbf{M}_{\widehat{\mathbf{F}}}=\mathbf{I}_T-T^{-1}\widehat{\mathbf{F}}\widehat{\mathbf{F}}'$. Given the equation (A.6), we can derive that

$$
\mathbb{B}_1 = -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\frac{1}{NT}\sum_{\ell=1}^{N}\mathbf{R}^{-1\prime}\boldsymbol{\Xi}^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\left(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1}\right)'\mathbf{u}_i \tag{A.7}
$$

$$
-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{R}^{-1\prime}\boldsymbol{\Xi}^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\left(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1}\right)'\widehat{\mathbf{F}}\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{u}_i \tag{A.8}
$$

$$
-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{R}^{-1\prime}\boldsymbol{\Xi}^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\left(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1}\right)'\widehat{\mathbf{F}}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\mathbf{u}_i \tag{A.9}
$$

$$
=\mathbb{B}_{1.1}+\mathbb{B}_{1.2}+\mathbb{B}_{1.3}.
$$

$$
\|\mathbb{B}_{1.1}\| \leq \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\frac{1}{NT}\sum_{\ell=1}^{N}\mathbf{R}^{-1\prime}\boldsymbol{\Xi}^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\left(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1}\right)'\mathbf{H}^0\boldsymbol{\varphi}_i^0\right\| \tag{A.10}
$$

$$
+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\frac{1}{NT}\sum_{\ell=1}^{N}\mathbf{R}^{-1\prime}\boldsymbol{\Xi}^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\left(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1}\right)'\boldsymbol{\varepsilon}_i\right\| \tag{A.11}
$$

$$\leq \sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|\mathbf{R}^{-1\prime}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\left\|\frac{1}{NT}\sum_{\ell=1}^{N}\widehat{\mathbf{F}}'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\right\|\left\|\frac{\left(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1}\right)'\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\varphi}_i^0\right\| \quad (A.12)$$

$$+\sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|\mathbf{R}^{-1\prime}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\left\|\frac{1}{NT}\sum_{\ell=1}^{N}\widehat{\mathbf{F}}'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\right\|\left\|\frac{\left(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1}\right)'\boldsymbol{\varepsilon}_i}{T}\right\| \quad (A.13)$$

$$= o_p\left(\frac{\sqrt{NT}}{\delta_{NT}^2}\right)$$

by Lemma A.4 (a) (i) and Lamma B.1 (b).

$$\|\mathbb{B}_{1.2}\| \leq \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{R}^{-1\prime}\boldsymbol{\Xi}^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\left(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1}\right)'\widehat{\mathbf{F}}\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i\right\| \quad (A.14)$$

$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{R}^{-1\prime}\boldsymbol{\Xi}^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\left(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1}\right)'\widehat{\mathbf{F}}\mathbf{R}'\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\right\| \quad (A.15)$$

$$\leq \sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|\mathbf{R}^{-1\prime}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\left\|\frac{1}{NT}\sum_{\ell=1}^{N}\widehat{\mathbf{F}}'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\right\|\left\|\frac{\left(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1}\right)'\widehat{\mathbf{F}}}{T}\right\|\left\|\mathbf{R}'\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\varphi}_i\right\| \quad (A.16)$$

$$+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|\mathbf{R}^{-1\prime}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\left\|\frac{1}{NT}\sum_{\ell=1}^{N}\widehat{\mathbf{F}}'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\right\|\left\|\frac{\left(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1}\right)'\widehat{\mathbf{F}}}{T}\right\|\left\|\mathbf{R}'\right\|\left\|\frac{\mathbf{F}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\| \quad (A.17)$$

$$= o_p\left(\frac{\sqrt{NT}}{\delta_{NT}^2}\right)$$

by Lemma A.4 (a) (e) and Lamma A.1 (c).

$$\|\mathbb{B}_{1.3}\| \leq \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{R}^{-1\prime}\boldsymbol{\Xi}^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\left(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1}\right)'\widehat{\mathbf{F}}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\mathbf{H}^0\boldsymbol{\varphi}_i\right\| \quad (A.18)$$

$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{R}^{-1\prime}\boldsymbol{\Xi}^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\left(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1}\right)'\widehat{\mathbf{F}}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\boldsymbol{\varepsilon}_i\right\| \quad (A.19)$$

$$\leq \sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|\mathbf{R}^{-1\prime}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\left\|\frac{1}{NT}\sum_{\ell=1}^{N}\widehat{\mathbf{F}}'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\right\|\left\|\frac{\left(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1}\right)'\widehat{\mathbf{F}}}{T}\right\|\left\|\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\varphi}_i\right\| \quad (A.20)$$

$$+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|\mathbf{R}^{-1\prime}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\left\|\frac{1}{NT}\sum_{\ell=1}^{N}\widehat{\mathbf{F}}'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\right\|\left\|\frac{\left(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1}\right)'\widehat{\mathbf{F}}}{T}\right\|\left\|\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\varepsilon_i}{T}\right\| \quad (A.21)$$

$$= o_p\left(\frac{\sqrt{NT}}{\delta_{NT}^4}\right)$$

by Lemma A.4 (a) (i), Lemma A.1 (c) and Lemma B.1 (b). Thus, $\|\mathbb{B}_1\| = o_p\left(\frac{\sqrt{NT}}{\delta_{NT}^2}\right)$.

Consider the term $\mathbb{B}_2$. With the definition of $\mathbf{R}$, $\mathbb{B}_2$ can be reformulated as

$$\mathbb{B}_2 = -\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i$$

Consider the term $\mathbb{B}_3$. With the definition of $\mathbf{R}$, $\mathbb{B}_3$ can be written as

$$\mathbb{B}_3 = -\frac{1}{N^{3/2}T^{3/2}} \sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i$$

Combining the above three terms, we can complete the proof. □

Lemma A.6 *Under Assumptions A to D, we have*

$$\begin{aligned}
&-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\\
=&-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i\\
&+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{V}_h}{T}\boldsymbol{\Gamma}_h^{0\prime}\right)(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\frac{\mathbf{F}^{0\prime}\mathbf{u}_i}{T}+O_p\left(\frac{1}{\delta_{NT}}\right)+O_p\left(\frac{\sqrt{T}}{N}\right)
\end{aligned}$$

Proof of Lemma A.6. Note that $\mathbf{M}_{\mathbf{F}^0}-\mathbf{M}_{\widehat{\mathbf{F}}}=\mathbf{P}_{\widehat{\mathbf{F}}}-\mathbf{P}_{\mathbf{F}^0}$ and $\mathbf{P}_{\widehat{\mathbf{F}}}=T^{-1}\widehat{\mathbf{F}}\widehat{\mathbf{F}}'$. We can derive that

$$\begin{aligned}
&-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i-\Big(-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i\Big)\\
=&\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{u}_i\\
&+\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\\
&+\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\\
&+\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0\big(\mathbf{R}\mathbf{R}'-(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\big)\mathbf{F}^{0\prime}\mathbf{u}_i\\
=&\mathbb{C}_1+\mathbb{C}_2+\mathbb{C}_3+\mathbb{C}_4
\end{aligned}$$

We first consider the last three terms. Consider the term $\mathbb{C}_2$. By Lemmas A.1 (b) and (d), Lemmas A.4 (a), (b) and (i), we can derive that

$$\begin{aligned}
\|\mathbb{C}_2\|\le&\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\,\|\boldsymbol{\Gamma}_j^0\|\,\|\boldsymbol{\varphi}_i^0\|\right)\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0\right\|\left\|\frac{1}{T}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\mathbf{H}^{\mathbf{0}}\right\|\|(\boldsymbol{\Upsilon}^0)^{-1}\|\,\|\mathbf{R}\|\\
&+\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\,\|\boldsymbol{\Gamma}_j^0\|\left\|\frac{1}{T}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\boldsymbol{\varepsilon}_i\right\|\right)\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0\right\|\|(\boldsymbol{\Upsilon}^0)^{-1}\|\,\|\mathbf{R}\|\\
=&O_p\left(\delta_{NT}^{-2}\right)
\end{aligned}$$

given the fact that $(NT)^{-1/2}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0=O_p(1)$ and $\mathbf{u}_i=\mathbf{H}^0\boldsymbol{\varphi}_i^0+\boldsymbol{\varepsilon}_i$.

Consider the term $\mathbb{C}_3$. Since $\mathbf{u}_i = \mathbf{H}^0\boldsymbol{\varphi}_i^0 + \boldsymbol{\varepsilon}_i$, we can derive that

$$
\begin{aligned}
\|\mathbb{C}_3\| =& \left\| \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N w_{ij} \boldsymbol{\Gamma}_j^{0\prime} (\boldsymbol{\Upsilon}^0)^{-1} \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \frac{1}{T} \left(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}\right) \left(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}\right)' \mathbf{u}_i \right\| \\
\leq & \left(\frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N |w_{ij}| \, \|\boldsymbol{\Gamma}_j^0\| \, \|\boldsymbol{\varphi}_i^0\| \right) \left\| \frac{1}{\sqrt{NT}} \sum_{\ell=1}^N \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \left(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}\right) \right\| \left\| \frac{1}{T} \left(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}\right)' \mathbf{H}^0 \right\| \|(\boldsymbol{\Upsilon}^0)^{-1}\| \\
& + \left(\frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N |w_{ij}| \, \|\boldsymbol{\Gamma}_j^0\| \left\| \frac{1}{T} \left(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}\right)' \boldsymbol{\varepsilon}_i \right\| \right) \left\| \frac{1}{\sqrt{NT}} \sum_{\ell=1}^N \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \left(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}\right) \right\| \|(\boldsymbol{\Upsilon}^0)^{-1}\| \\
=& O_p\left(\delta_{NT}^{-2}\right) \times \left[O_p\left(\sqrt{\frac{T}{N}}\right) + O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right) \right] = O_p\left(\sqrt{\frac{T}{N^3}}\right) + O_p\left(\frac{1}{\sqrt{T^3}}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right)
\end{aligned}
$$

by Lemmas A.1 (b), (d) and (h) and Lemmas A.4 (a), (b) and (i).

Consider the term $\mathbb{C}_4$. We have

$$
\begin{aligned}
& \left\| \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N w_{ij} \boldsymbol{\Gamma}_j^{0\prime} (\boldsymbol{\Upsilon}^0)^{-1} \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \frac{1}{T} \mathbf{F}^0 \left(\mathbf{R}\mathbf{R}' - \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1} \right) \mathbf{F}^{0\prime} \mathbf{u}_i \right\| \\
\leq & \left(\frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N |w_{ij}| \, \|\boldsymbol{\Gamma}_j^0\| \left\| \frac{1}{\sqrt{T}} \mathbf{u}_i \right\| \right) \left\| \frac{1}{\sqrt{NT}} \sum_{\ell=1}^N \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \mathbf{F}^0 \right\| \left\| \mathbf{R}\mathbf{R}' - \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1} \right\| \left\| \frac{\mathbf{F}^0}{\sqrt{T}} \right\| \|(\boldsymbol{\Upsilon}^0)^{-1}\| \\
=& O_p\left(\delta_{NT}^{-2}\right)
\end{aligned}
$$

by Lemma A.1 (f) and Lemma A.4 (d).

We consider the term $\mathbb{C}_1$. Because $(\hat{\mathbf{f}}_s - \mathbf{R}'\mathbf{f}_s^0)'\mathbf{R}'\mathbf{f}_t^0 = \mathbf{f}_t^{0\prime}\mathbf{R}(\hat{\mathbf{f}}_s - \mathbf{R}'\mathbf{f}_s^0)$ is scalar and commutable, we have

$$
\begin{aligned}
\mathbb{C}_1 =& \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N w_{ij} \boldsymbol{\Gamma}_j^{0\prime} (\boldsymbol{\Upsilon}^0)^{-1} \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \frac{1}{T} \left(\widehat{\mathbf{F}}\mathbf{R}^{-1} - \mathbf{F}^0\right) \mathbf{R}\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{u}_i \\
=& \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N w_{ij} \boldsymbol{\Gamma}_j^{0\prime} (\boldsymbol{\Upsilon}^0)^{-1} \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \frac{1}{T} \left(\widehat{\mathbf{F}}\mathbf{R}^{-1} - \mathbf{F}^0\right) \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1} \mathbf{F}^{0\prime}\mathbf{u}_i \\
& + \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N w_{ij} \boldsymbol{\Gamma}_j^{0\prime} (\boldsymbol{\Upsilon}^0)^{-1} \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \frac{1}{T} \left(\widehat{\mathbf{F}}\mathbf{R}^{-1} - \mathbf{F}^0\right) \left(\mathbf{R}\mathbf{R}' - \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1} \right) \mathbf{F}^{0\prime}\mathbf{u}_i \\
=& \mathbb{C}_{1.1} + \mathbb{C}_{1.2}
\end{aligned}
$$

We first consider the term $\mathbb{C}_{1.2}$. We have

$$
\begin{aligned}
& \left\| \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N w_{ij} \boldsymbol{\Gamma}_j^{0\prime} (\boldsymbol{\Upsilon}^0)^{-1} \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \frac{1}{T} \left(\widehat{\mathbf{F}}\mathbf{R}^{-1} - \mathbf{F}^0\right) \left(\mathbf{R}\mathbf{R}' - \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1} \right) \mathbf{F}^{0\prime}\mathbf{u}_i \right\| \\
\leq & \left(\frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N |w_{ij}| \, \|\boldsymbol{\Gamma}_j^0\| \left\| \frac{1}{\sqrt{T}} \mathbf{u}_i \right\| \right) \left\| \frac{1}{\sqrt{NT}} \sum_{\ell=1}^N \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \left(\widehat{\mathbf{F}}\mathbf{R}^{-1} - \mathbf{F}^0\right) \right\| \left\| \mathbf{R}\mathbf{R}' - \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1} \right\| \left\| \frac{\mathbf{F}^0}{\sqrt{T}} \right\| \|(\boldsymbol{\Upsilon}^0)^{-1}\| \\
=& O_p\left(\delta_{NT}^{-2}\right) \times \left[O_p\left(\sqrt{\frac{T}{N}}\right) + O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right) \right] = O_p\left(\sqrt{\frac{T}{N^3}}\right) + O_p\left(\frac{1}{\sqrt{T^3}}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right)
\end{aligned}
$$

by Lemmas A.1 (f), (h) and Lemma A.4 (d).

We consider the term $\mathbb{C}_{1.1}$. By the equation (A.1), we have

$$
\begin{aligned}
&\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\frac{1}{T}\left(\widehat{\mathbf{F}}\mathbf{R}^{-1}-\mathbf{F}^0\right)\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\\
=&\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\frac{1}{NT^2}\sum_{h=1}^{N}\mathbf{F}^0\mathbf{\Gamma}_h^0\mathbf{V}_h'\widehat{\mathbf{F}}\left(\frac{\mathbf{F}^{0\prime}\widehat{\mathbf{F}}}{T}\right)^{-1}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\\
&+\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\frac{1}{NT}\sum_{h=1}^{N}\mathbf{V}_h\mathbf{\Gamma}_h^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\\
&+\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\frac{1}{NT^2}\sum_{h=1}^{N}\mathbf{V}_h\mathbf{V}_h'\widehat{\mathbf{F}}\left(\frac{\mathbf{F}^{0\prime}\widehat{\mathbf{F}}}{T}\right)^{-1}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\\
=&\mathbb{C}_{1.1.1}+\mathbb{C}_{1.1.2}+\mathbb{C}_{1.1.3}
\end{aligned}
$$

The term $\mathbb{C}_{1.1.1}$ can be decomposed as

$$
\begin{aligned}
&\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\frac{1}{NT^2}\sum_{h=1}^{N}\mathbf{F}^0\mathbf{\Gamma}_h^0\mathbf{V}_h'\mathbf{F}^0\mathbf{R}\left(\frac{\mathbf{F}^{0\prime}\widehat{\mathbf{F}}}{T}\right)^{-1}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\\
&+\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\frac{1}{NT^2}\sum_{h=1}^{N}\mathbf{F}^0\mathbf{\Gamma}_h^0\mathbf{V}_h'\left(\widehat{\mathbf{F}}-\mathbf{F}^{00}\mathbf{R}\right)\left(\frac{\mathbf{F}^{0\prime}\widehat{\mathbf{F}}}{T}\right)^{-1}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\\
=&\mathbb{C}_{1.1.1.1}+\mathbb{C}_{1.1.1.2}
\end{aligned}
$$

The term $\mathbb{C}_{1.1.1.1}$ is bounded in norm by

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\,\|\mathbf{\Gamma}_j^0\|\left\|\frac{1}{\sqrt{T}}\mathbf{u}_i\right\|\right)\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0\right\|^2\left\|\left(\frac{\mathbf{F}^{0\prime}\widehat{\mathbf{F}}}{T}\right)^{-1}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^0}{\sqrt{T}}\right\|\|\mathbf{R}\|\,\|(\mathbf{\Upsilon}^0)^{-1}\|^2\\
=&O_p\left(\frac{1}{\sqrt{NT}}\right)
\end{aligned}
$$

by Lemma A.1 (e) and Lemma A.4 (d). Similarly, we can show the term $\mathbb{C}_{1.1.1.2}$ is bounded in norm by

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\,\|\mathbf{\Gamma}_j^0\|\left\|\frac{1}{\sqrt{T}}\mathbf{u}_i\right\|\right)\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{h=1}^{N}\mathbf{\Gamma}_h^0\mathbf{V}_h'\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\widehat{\mathbf{F}}}{T}\right)^{-1}\right\|\\
&\times\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^0}{\sqrt{T}}\right\|\|(\mathbf{\Upsilon}^0)^{-1}\|^2\\
=&O_p\left(\frac{1}{\sqrt{NT}}\right)\times\left[O_p\left(\sqrt{\frac{T}{N}}\right)+O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right)\right]=O_p\left(\frac{1}{N}\right)+O_p\left(\frac{1}{\sqrt{N}\delta_{NT}^2}\right)
\end{aligned}
$$

by Lemma A.1 (h) and Lemma A.4 (d). For the term $\mathbb{C}_{1.1.2}$, we can reformulate it as

$$
\begin{aligned}
&\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\frac{1}{NT}\sum_{h=1}^{N}\mathbf{V}_h\mathbf{\Gamma}_h^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\\
=&\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\mathbf{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{V}_h}{T}\mathbf{\Gamma}_h^{0\prime}\right)(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\frac{\mathbf{F}^{0\prime}\mathbf{u}_i}{T}
\end{aligned}
$$

For the term $\mathbb{C}_{1.1.3}$, we can decompose it as

$$\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\frac{1}{NT}\sum_{h=1}^{N}\mathbb{E}\left(\mathbf{V}_h\mathbf{V}_h'\right)\mathbf{F}^{00}\mathbf{R}\left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\frac{\mathbf{F}^{0\prime}\mathbf{u}_i}{T}$$

$$+\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\frac{1}{NT}\sum_{h=1}^{N}\left(\mathbf{V}_h\mathbf{V}_h'-\mathbb{E}\left(\mathbf{V}_h\mathbf{V}_h'\right)\right)\mathbf{F}^{0}\mathbf{R}\left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\frac{\mathbf{F}^{0\prime}\mathbf{u}_i}{T}$$

$$+\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\frac{1}{NT}\sum_{h=1}^{N}\mathbb{E}\left(\mathbf{V}_h\mathbf{V}_h'\right)\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\frac{\mathbf{F}^{0\prime}\mathbf{u}_i}{T}$$

$$+\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\frac{1}{NT}\sum_{h=1}^{N}\left(\mathbf{V}_h\mathbf{V}_h'-\mathbb{E}\left(\mathbf{V}_h\mathbf{V}_h'\right)\right)\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\frac{\mathbf{F}^{0\prime}\mathbf{u}_i}{T}$$

which are bounded by

$$\begin{aligned}
&\Big(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\frac{1}{\sqrt{T}}\mathbf{u}_i\right\|\Big)\left\|\left(\frac{\mathbf{F}^{0\prime}\widehat{\mathbf{F}}}{T}\right)^{-1}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^0}{\sqrt{T}}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|^2\|\mathbf{R}\| \\
&\quad\times\Bigg(\frac{1}{\sqrt{T}}\Big\|\frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\Big(\frac{1}{\sqrt{N}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{v}_{\ell s}\Big)\mathbf{f}_t^{0\prime}\mathbb{E}\Big(\frac{1}{N}\sum_{h=1}^{N}\mathbf{v}_{hs}'\mathbf{v}_{ht}\Big)\Big\| \\
&\qquad+\frac{1}{\sqrt{N}}\Big\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\Big\|\Big\|\frac{1}{\sqrt{NT}}\sum_{h=1}^{N}\left(\mathbf{V}_h\mathbf{V}_h'-\mathbb{E}\left(\mathbf{V}_h\mathbf{V}_h'\right)\right)\mathbf{F}^0\Big\| \\
&\qquad+\Big\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\Big\|\Big\|\frac{1}{N\sqrt{T}}\sum_{h=1}^{N}\mathbb{E}\left(\mathbf{V}_h\mathbf{V}_h'\right)\Big\|\Big\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\Big\| \\
&\qquad+\sqrt{\frac{T}{N}}\Big\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\Big\|\left\|\frac{1}{\sqrt{NT}}\Big(\sum_{h=1}^{N}\mathbf{V}_h\mathbf{V}_h'-\mathbb{E}\left(\mathbf{V}_h\mathbf{V}_h'\right)\Big)\right\|\Big\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\Big\|\Bigg) \\
=&O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right)+O_p\left(\frac{1}{\sqrt{N}}\right)
\end{aligned}$$

by Lemmas A.1 (a), (i), (m) and (n) and Lemma A.4 (d), and the fact that

$$\left\|\frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{N}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{v}_{\ell s}\right)\mathbf{f}_t^{0\prime}\mathbb{E}\left(\frac{1}{N}\sum_{h=1}^{N}\mathbf{v}_{hs}'\mathbf{v}_{ht}\right)\right\|=O_p\left(1\right)$$

because

$$\begin{aligned}
&\mathbb{E}\left\|\frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\left(\frac{1}{\sqrt{N}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{v}_{\ell s}\right)\mathbf{f}_t^{0\prime}\mathbb{E}\left(\frac{1}{N}\sum_{h=1}^{N}\mathbf{v}_{hs}'\mathbf{v}_{ht}\right)\right\| \\
\leq&\frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\left(\mathbb{E}\left\|\frac{1}{\sqrt{N}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{v}_{\ell s}\right\|^2\right)^{1/2}\left(\mathbb{E}\left\|\mathbf{f}_t^0\right\|^2\right)^{1/2}\left|\mathbb{E}\left(\frac{1}{N}\sum_{h=1}^{N}\mathbf{v}_{hs}'\mathbf{v}_{ht}\right)\right| \\
\leq&\frac{C^{1/2}}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}\operatorname{tr}\left(\Sigma_{\ell j,ss}\mathbb{E}(\boldsymbol{\Gamma}_j^{0\prime}\boldsymbol{\Gamma}_\ell^0)\right)\right)^{1/2}\left|\mathbb{E}\left(\frac{1}{N}\sum_{h=1}^{N}\mathbf{v}_{hs}'\mathbf{v}_{ht}\right)\right| \\
\leq&\frac{C^{1/2}k}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}\|\Sigma_{\ell j,ss}\|\left(\mathbb{E}\left\|\boldsymbol{\Gamma}_j^0\right\|^2\mathbb{E}\left\|\boldsymbol{\Gamma}_\ell^0\right\|^2\right)^{1/2}\right)^{1/2}\left|\mathbb{E}\left(\frac{1}{N}\sum_{h=1}^{N}\mathbf{v}_{hs}'\mathbf{v}_{ht}\right)\right| \\
\leq&\frac{Ck}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}\bar{\sigma}_{\ell s}\right)^{1/2}\left|\mathbb{E}\left(\frac{1}{N}\sum_{h=1}^{N}\mathbf{v}_{hs}'\mathbf{v}_{ht}\right)\right|\leq\frac{C^{3/2}k}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\left|\mathbb{E}\left(\frac{1}{N}\sum_{h=1}^{N}\mathbf{v}_{hs}'\mathbf{v}_{ht}\right)\right|\leq C^{5/2}k
\end{aligned}$$

by Assumptions B3, C and D.

Combining the above terms, we can derive that

$$\begin{aligned}
&-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
=&-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i \\
&+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{V}_h}{T}\boldsymbol{\Gamma}_h^{0\prime}\right)(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\frac{\mathbf{F}^{0\prime}\mathbf{u}_i}{T}+O_p\left(\frac{1}{\delta_{NT}}\right)+O_p\left(\frac{\sqrt{T}}{N}\right)
\end{aligned}$$

This completes the proof. □

Lemma A.7 *Under Assumptions A to D, we have*

$$\begin{aligned}
&\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
=&\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\widehat{\mathbf{F}}'\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i+O_p\left(\sqrt{T}\delta_{NT}^{-2}\right)+O_p\left(T^{-1/2}\right)
\end{aligned}$$

if $T/N^2\to 0$ as $N,T\to\infty$.

Proof of Lemma A.7. Denote the left-hand-side of the above equation

$$\mathbb{C}_5=\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\widehat{\mathbf{F}}'\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i.$$

Since $\mathbf{M}_{\widehat{\mathbf{F}}}=\mathbf{I}_T-T^{-1}\widehat{\mathbf{F}}\widehat{\mathbf{F}}'$ and $\mathbf{u}_i=\mathbf{H}^0\boldsymbol{\varphi}_i^0+\boldsymbol{\varepsilon}_i$,

$$\mathbb{C}_5=\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\widehat{\mathbf{F}}'\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{H}^0\boldsymbol{\varphi}_i^0 \tag{A.22}$$

$$+\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\widehat{\mathbf{F}}'\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\boldsymbol{\varepsilon}_i \tag{A.23}$$

$$-\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\widehat{\mathbf{F}}'\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\widehat{\mathbf{F}}\frac{\widehat{\mathbf{F}}'\mathbf{u}_i}{T} \tag{A.24}$$

$$=\mathbb{C}_{5.1}+\mathbb{C}_{5.2}-\mathbb{C}_{5.3}.$$

$$\|\mathbb{C}_{5.1}\|\leq\left\|\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\mathbf{R}'\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{H}^0\boldsymbol{\varphi}_i^0\right\| \tag{A.25}$$

$$+\left\|\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{H}^0\boldsymbol{\varphi}_i^0\right\| \tag{A.26}$$

$$\leq\frac{1}{NT^{1/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^{0\prime}\right\|\left\|(\mathbf{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\|\mathbf{R}\|\left\|\frac{\sum_{\ell=1}^{N}\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{H}^0}{\sqrt{N}T}\right\|\left\|\boldsymbol{\varphi}_i^0\right\| \tag{A.27}$$

$$+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^{0\prime}\right\|\left\|(\mathbf{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|\left\|\frac{\sum_{\ell=1}^{N}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{H}^0}{\sqrt{N}T}\right\|\left\|\boldsymbol{\varphi}_i^0\right\| \tag{A.28}$$

$$=O_p\left(T^{-1/2}\right)+O_p\left(\delta_{NT}^{-1}\right)$$

by Lemma A.1 (a) (k) (l) and Lemma A.4 (a).

$$\|\mathbb{C}_{5.2}\|\leq\left\|\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\mathbf{R}'\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\boldsymbol{\varepsilon}_i\right\| \tag{A.29}$$

$$+\left\|\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\boldsymbol{\varepsilon}_i\right\| \tag{A.30}$$

$$\leq\frac{1}{NT^{1/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^{0\prime}\right\|\left\|(\mathbf{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\|\mathbf{R}\|\left\|\frac{\sum_{\ell=1}^{N}\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\boldsymbol{\varepsilon}_i}{\sqrt{N}T}\right\|\left\|\boldsymbol{\varphi}_i^0\right\| \tag{A.31}$$

$$+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^{0\prime}\right\|\left\|(\mathbf{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|\left\|\frac{\sum_{\ell=1}^{N}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\boldsymbol{\varepsilon}_i}{\sqrt{N}T}\right\|\left\|\boldsymbol{\varphi}_i^0\right\| \tag{A.32}$$

$$=O_p\left(T^{-1/2}\right)+O_p\left(\delta_{NT}^{-1}\right)$$

by Lemma A.1 (a), Lemma A.2 (d) (e) and Lemma A.4 (a).

$$\mathbb{C}_{5.3}=\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\mathbf{R}'\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0\mathbf{R}\frac{\widehat{\mathbf{F}}'\mathbf{u}_i}{T} \tag{A.33}$$

$$+\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\mathbf{R}'\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\frac{\widehat{\mathbf{F}}'\mathbf{u}_i}{T} \tag{A.34}$$

$$+\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0\mathbf{R}\frac{\widehat{\mathbf{F}}'\mathbf{u}_i}{T} \tag{A.35}$$

$$+\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\frac{\widehat{\mathbf{F}}'\mathbf{u}_i}{T} \tag{A.36}$$

$$=\mathbb{C}_{5.3.1}+\mathbb{C}_{5.3.2}+\mathbb{C}_{5.3.3}+\mathbb{C}_{5.3.4}.$$

Noting that

$$\frac{\widehat{\mathbf{F}}'\mathbf{u}_i}{T}=\mathbf{R}'\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\boldsymbol{\varphi}_i^0+\mathbf{R}'\frac{\mathbf{F}^{0\prime}\varepsilon_i}{T}+\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\mathbf{H}^0}{T}\boldsymbol{\varphi}_i^0+\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\varepsilon_i}{T},$$

$$
\begin{aligned}
\|\mathbb{C}_{5.3.1}\| \leq & \left\| \frac{1}{N^{3/2}T^{3/2}} \sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\mathbf{R}'\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0\mathbf{R}\mathbf{R}'\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\boldsymbol{\varphi}_i^0 \right\| && \text{(A.37)}\\
& + \left\| \frac{1}{N^{3/2}T^{3/2}} \sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\mathbf{R}'\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0\mathbf{R}\mathbf{R}'\frac{\mathbf{F}^{0\prime}\varepsilon_i}{T} \right\| && \text{(A.38)}\\
& + \left\| \frac{1}{N^{3/2}T^{3/2}} \sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\mathbf{R}'\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0\mathbf{R}\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\mathbf{H}^0}{T}\boldsymbol{\varphi}_i^0 \right\| && \text{(A.39)}\\
& + \left\| \frac{1}{N^{3/2}T^{3/2}} \sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\mathbf{R}'\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0\mathbf{R}\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\varepsilon_i}{T} \right\|
\end{aligned}
$$

$$
\begin{aligned}
& \leq \frac{1}{NT^{1/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\sum_{\ell=1}^{N}\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0}{\sqrt{N}T}\right\| && \text{(A.40)}\\
& \times\|\mathbf{R}\|^3\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\varphi}_i^0\right\| && \text{(A.41)}\\
& + \frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\sum_{\ell=1}^{N}\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0}{\sqrt{N}T}\right\| && \text{(A.42)}\\
& \times\|\mathbf{R}\|^3\left\|\frac{\mathbf{F}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\| && \text{(A.43)}\\
& + \frac{1}{NT^{1/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\sum_{\ell=1}^{N}\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0}{\sqrt{N}T}\right\| && \text{(A.44)}\\
& \times\|\mathbf{R}\|^2\left\|\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\varphi}_i^0\right\| && \text{(A.45)}\\
& + \frac{1}{NT^{1/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\sum_{\ell=1}^{N}\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0}{\sqrt{N}T}\right\| && \text{(A.46)}\\
& \times\|\mathbf{R}\|^2\left\|\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\varepsilon_i}{T}\right\| && \text{(A.47)}\\
& = O_p\left(T^{-1/2}\right)
\end{aligned}
$$

by Lemma A.1 (a), Lemma A.2 (j) and Lemma A.4 (a) (e) (i).

$$
\begin{aligned}
& \|\mathbb{C}_{5.3.2}\| && \text{(A.48)}\\
\leq & \left\| \frac{1}{N^{3/2}T^{3/2}} \sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\mathbf{R}'\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\mathbf{R}'\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\boldsymbol{\varphi}_i^0 \right\| && \text{(A.49)}\\
& + \left\| \frac{1}{N^{3/2}T^{3/2}} \sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\mathbf{R}'\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\mathbf{R}'\frac{\mathbf{F}^{0\prime}\varepsilon_i}{T} \right\| && \text{(A.50)}\\
& + \left\| \frac{1}{N^{3/2}T^{3/2}} \sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\mathbf{R}'\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\mathbf{H}^0}{T}\boldsymbol{\varphi}_i^0 \right\| && \text{(A.51)}
\end{aligned}
$$

$$
+\left\|\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\mathbf{R}'\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\varepsilon_i}{T}\right\|
$$

$$
\leq\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^{0\prime}\right\|\left\|(\mathbf{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\sum_{\ell=1}^{N}\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)}{\sqrt{N}T}\right\| \tag{A.52}
$$

$$
\times\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|\|\mathbf{R}\|^2\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\varphi}_i^0\right\| \tag{A.53}
$$

$$
+\frac{1}{N\sqrt{T}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^{0\prime}\right\|\left\|(\mathbf{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\sum_{\ell=1}^{N}\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)}{\sqrt{N}T}\right\| \tag{A.54}
$$

$$
\times\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|\|\mathbf{R}\|^2\left\|\frac{\mathbf{F}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\| \tag{A.55}
$$

$$
+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^{0\prime}\right\|\left\|(\mathbf{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\sum_{\ell=1}^{N}\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)}{\sqrt{N}T}\right\| \tag{A.56}
$$

$$
\times\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|\|\mathbf{R}\|\left\|\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\varphi}_i^0\right\| \tag{A.57}
$$

$$
+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^{0\prime}\right\|\left\|(\mathbf{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\sum_{\ell=1}^{N}\mathbf{F}^{0\prime}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)}{\sqrt{N}T}\right\| \tag{A.58}
$$

$$
\times\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|\|\mathbf{R}\|\left\|\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\varepsilon_i}{T}\right\| \tag{A.59}
$$

$$
=O_p\left(\delta_{NT}^{-1}\right)
$$

by Lemma A.1 (a), Lemma A.2 (i) and Lemma A.4 (a) (e) (i).

$$
\left\|\mathbb{C}_{5.3.3}\right\| \tag{A.60}
$$

$$
\leq\left\|\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0\mathbf{R}\mathbf{R}'\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\boldsymbol{\varphi}_i^0\right\| \tag{A.61}
$$

$$
+\left\|\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0\mathbf{R}\mathbf{R}'\frac{\mathbf{F}^{0\prime}\varepsilon_i}{T}\right\| \tag{A.62}
$$

$$
+\left\|\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0\mathbf{R}\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\mathbf{H}^0}{T}\boldsymbol{\varphi}_i^0\right\| \tag{A.63}
$$

$$
+\left\|\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0\mathbf{R}\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\varepsilon_i}{T}\right\|
$$

$$
\leq\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\mathbf{\Gamma}_j^{0\prime}\right\|\left\|(\mathbf{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|\left\|\frac{\sum_{\ell=1}^{N}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0}{\sqrt{N}T}\right\| \tag{A.64}
$$

$$
\times\|\mathbf{R}\|^2\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\varphi}_i^0\right\| \tag{A.65}
$$

$$+\frac{1}{N\sqrt{T}}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|\left\|\frac{\sum_{\ell=1}^{N}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0}{\sqrt{N}T}\right\| \tag{A.66}$$

$$\times\left\|\mathbf{R}\right\|^2\left\|\frac{\mathbf{F}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\| \tag{A.67}$$

$$+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|\left\|\frac{\sum_{\ell=1}^{N}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0}{\sqrt{N}T}\right\| \tag{A.68}$$

$$\times\left\|\mathbf{R}\right\|\left\|\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\varphi}_i^0\right\| \tag{A.69}$$

$$+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|\left\|\frac{\sum_{\ell=1}^{N}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0}{\sqrt{N}T}\right\| \tag{A.70}$$

$$\times\left\|\mathbf{R}\right\|\left\|\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\varepsilon_i}{T}\right\| \tag{A.71}$$

$$=O_p\left(\delta_{NT}^{-1}\right)$$

by Lemma A.1 (a), Lemma A.2 (i) and Lemma A.4 (a) (e) (i).

$$\left\|\mathbb{C}_{5.3.4}\right\| \tag{A.72}$$

$$\leq\left\|\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\mathbf{R}'\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\boldsymbol{\varphi}_i^0\right\| \tag{A.73}$$

$$+\left\|\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\mathbf{R}'\frac{\mathbf{F}^{0\prime}\varepsilon_i}{T}\right\| \tag{A.74}$$

$$+\left\|\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\mathbf{H}^0}{T}\boldsymbol{\varphi}_i^0\right\| \tag{A.75}$$

$$+\left\|\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\varepsilon_i}{T}\right\|$$

$$\leq\sqrt{T}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\sum_{\ell=1}^{N}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)}{\sqrt{N}T}\right\| \tag{A.76}$$

$$\times\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|^2\left\|\mathbf{R}\right\|^2\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\varphi}_i^0\right\| \tag{A.77}$$

$$+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\sum_{\ell=1}^{N}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)}{\sqrt{N}T}\right\| \tag{A.78}$$

$$\times\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|^2\left\|\mathbf{R}\right\|^2\left\|\frac{\mathbf{F}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\| \tag{A.79}$$

$$+\sqrt{T}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\sum_{\ell=1}^{N}\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)}{\sqrt{N}T}\right\| \tag{A.80}$$

$$\times\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|^2\|\mathbf{R}\|\left\|\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\varphi}_i^0\right\| \tag{A.81}$$

$$+\sqrt{T}\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\sum_{\ell=1}^N\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)}{\sqrt{N}T}\right\| \tag{A.82}$$

$$\times\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|^2\|\mathbf{R}\|\left\|\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\varepsilon_i}{T}\right\| \tag{A.83}$$

$$=O_p\left(\sqrt{T}/\delta_{NT}^{-2}\right)$$

by Lemma A.1 (a), Lemma A.2 (n) and Lemma A.4 (a) (e) (i). Thus we conclude $\mathbb{C}_5=O_p\left(\sqrt{T}/\delta_{NT}^{-2}\right)+O_p\left(T^{-1/2}\right)$ as required.□

Lemma A.8 *Under Assumptions A to D, we have*

$$\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^N\sum_{\ell=1}^N\sum_{j=1}^N w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\widehat{\mathbf{F}}'\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i$$
$$=\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^N\sum_{\ell=1}^N\sum_{j=1}^N w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i+O_p\left(\frac{1}{\sqrt{T}}\right)+O_p\left(\frac{\sqrt{N}}{T}\right)$$

if $T/N^2\to 0$ as $N,T\to\infty$.

Proof of Lemma A.8. Note that by Assumptions B5, we have

$$\left\|\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\right\|\le\mu_{\max}\left(\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\left\|\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\right\|\le C\left\|\mathbf{u}_i\right\|$$

and

$$\left\|\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\left(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\right)\mathbf{u}_i\right\|\le\mu_{\max}\left(\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\left\|\left(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\right)\mathbf{u}_i\right\|$$
$$\le C\left\|\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\right\|\left\|\mathbf{u}_i\right\|=O_p(\delta_{NT}^{-1})\left\|\mathbf{u}_i\right\|$$

In addition,

$$\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\frac{\widehat{\mathbf{F}}}{\sqrt{T}}'-\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\frac{\mathbf{F}^0}{\sqrt{T}}'\right\|=\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\frac{\widehat{\mathbf{F}}}{\sqrt{T}}'-\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\frac{\widehat{\mathbf{F}}}{\sqrt{T}}'\mathbf{P}_{\mathbf{F}^0}\right\|$$
$$=\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\frac{\widehat{\mathbf{F}}}{\sqrt{T}}'\mathbf{M}_{\mathbf{F}^0}\right\|=\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\frac{\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'}{\sqrt{T}}\mathbf{M}_{\mathbf{F}^0}\right\|\le\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|=O_p\left(\delta_{NT}^{-1}\right)$$

By substracting and adding terms, we have

$$
\begin{aligned}
&\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\widehat{\mathbf{F}}'\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\\
&-\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i\\
=&\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left[\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\widehat{\mathbf{F}}'-\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\right]\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\\
&+\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\left(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\right)\mathbf{u}_i
\end{aligned}
$$

which are bounded in norm by

$$
\begin{aligned}
&\sqrt{\frac{N}{T}}\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\frac{\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i}{\sqrt{T}}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\frac{\widehat{\mathbf{F}}'}{\sqrt{T}}-\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\frac{\mathbf{F}^{0\prime}}{\sqrt{T}}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\\
&+\sqrt{\frac{N}{T}}\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\frac{\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\left(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\right)\mathbf{u}_i}{\sqrt{T}}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^0}{\sqrt{T}}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\\
\leq&\sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\frac{\mathbf{u}_i}{\sqrt{T}}\right\|\left\|\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\frac{\widehat{\mathbf{F}}'}{\sqrt{T}}-\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\frac{\mathbf{F}^{0\prime}}{\sqrt{T}}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\\
&+\frac{1}{\delta_{NT}}\sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\frac{\mathbf{u}_i}{\sqrt{T}}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^0}{\sqrt{T}}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\\
=&O_p\left(\frac{\sqrt{N}}{\delta_{NT}\sqrt{T}}\right)
\end{aligned}
$$

by the above three facts and Lemma A.4 (d). This completes the proof. □

Lemma A.9 *Under Assumptions A to D, we have*

$$(a)\quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j\right\|\|\boldsymbol{\varphi}_i^0\| = O_p(1)$$

$$(b)\quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{F}^0\right\|\|\boldsymbol{\varphi}_i^0\| = O_p(1)$$

$$(c)\quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{F}^0\right\|\|\boldsymbol{\varphi}_i^0\| = O_p(1)$$

$$(d)\quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{F}^0\right\|\left\|\frac{1}{\sqrt{T}}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\right\| = O_p(1)$$

$$(e)\quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j\right\|\left\|\frac{1}{T}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\boldsymbol{\varepsilon}_i\right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right)$$

$$(f)\quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{F}^0\right\|\left\|\frac{1}{T}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\boldsymbol{\varepsilon}_i\right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right)$$

$$(g)\quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{T}\mathbf{V}_j'\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\right\|\|\boldsymbol{\varphi}_i^0\| = O_p\left(\frac{1}{\delta_{NT}}\right)$$

$$(h)\quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{T}\mathbf{V}_j'\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\right\|\|\boldsymbol{\varphi}_i^0\| = O_p\left(\frac{1}{\delta_{NT}}\right)$$

$$(i)\quad \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{T}\mathbf{V}_j'\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\right\|\left\|\frac{1}{T}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\boldsymbol{\varepsilon}_i\right\| = O_p\left(\frac{1}{\delta_{NT}^3}\right)$$

Proof of Lemma A.9. With Assumption B1 and Lemma A.2 (f), we can easily prove parts (a)-(d) by following the proof of Lemma A.4 (a).

Parts (e) and (f) can be proved similar to the proof of A.4 (i).

Consider (g). With the equation (A.1), we have

$$
\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{1}{T}\mathbf{V}_j'\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\right\| \\
\leq&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\mathbf{V}_j'\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{F}^0\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\widehat{\mathbf{F}}\boldsymbol{\Xi}^{-1}\right\|+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\mathbf{V}_j'\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\mathbf{F}^{0\prime}\widehat{\mathbf{F}}\boldsymbol{\Xi}^{-1}\right\| \\
&+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\mathbf{V}_j'\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{V}_\ell\mathbf{V}_\ell'\widehat{\mathbf{F}}\boldsymbol{\Xi}^{-1}\right\| \\
\leq&\frac{1}{\sqrt{NT}}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{F}^0\right\|\right)\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0\right\|\left\|\mathbf{R}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\| \\
&+\frac{1}{\sqrt{NT}}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{F}^0\right\|\right)\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\right\|\left\|\frac{1}{\sqrt{T}}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\right\|\left\|\boldsymbol{\Xi}^{-1}\right\| \\
&+\frac{1}{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{1}{T}\sum_{\ell=1}^{N}\mathbb{E}(\mathbf{V}_j'\mathbf{V}_\ell)\boldsymbol{\Gamma}_\ell^{0\prime}\right\|\left\|\frac{\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\frac{\widehat{\mathbf{F}}}{\sqrt{T}}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\| \\
&+\frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\left(\mathbf{V}_j'\mathbf{V}_\ell-\mathbb{E}(\mathbf{V}_j'\mathbf{V}_\ell)\right)\boldsymbol{\Gamma}_\ell^{0\prime}\right\|\left\|\frac{\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\frac{\widehat{\mathbf{F}}}{\sqrt{T}}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\| \\
&+\frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{1}{N\sqrt{T}}\sum_{\ell=1}^{N}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right\|\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\| \\
&+\frac{1}{\sqrt{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\left[\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\right\|\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\| \\
&+\frac{1}{T}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{1}{NT}\sum_{\ell=1}^{N}\mathbf{V}_j\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{F}^0\right\|\left\|\mathbf{R}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\| \\
&+\frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\left[\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\mathbf{F}^0\right\|\left\|\mathbf{R}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\| \\
=&O_p\left(\frac{1}{\delta_{NT}}\right)
\end{aligned}
$$

by Lemmas A.1 (i), (m) and (n) and Lemma A.9 (a). In the proof of (i), we replace $\boldsymbol{\varphi}_i$ by $T^{-1}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\boldsymbol{\varepsilon}_i$, and use Lemma A.9 (e) and Lemma A.9 (f). This completes the proof. □

Lemma A.10 *Under Assumptions A to D, we have*

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
=&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i \\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{NT}\sum_{h=1}^{N}\mathbf{V}_h\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i+O_p\left(\frac{1}{\delta_{NT}}\right)+O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)
\end{aligned}
$$

Proof of Lemma A.10. Since $\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}=-T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}-T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'-T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'-$

$T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}$, we have

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\left(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\right)\mathbf{u}_i\\
=&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{T}(\widehat{\mathbf{F}}\mathbf{R}^{-1}-\mathbf{F}^0)\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{T}(\widehat{\mathbf{F}}\mathbf{R}^{-1}-\mathbf{F}^0)\left(\mathbf{R}\mathbf{R}'-\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right)\mathbf{F}^{0\prime}\mathbf{u}_i\\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{T}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{T}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{T}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right)\mathbf{F}^{0\prime}\mathbf{u}_i\\
=&\mathbb{D}_1+\mathbb{D}_2+\mathbb{D}_3+\mathbb{D}_4+\mathbb{D}_5
\end{aligned}
$$

We first consider the last four terms. By Lemma A.1 (f) and Lemmas A.9 (g), (h), we have

$$
\begin{aligned}
\|\mathbb{D}_2\|=&\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{T}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}^{-1}\left(\mathbf{R}\mathbf{R}'-\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right)\mathbf{F}^{0\prime}\mathbf{u}_i\right\|\\
\leq&\sqrt{NT}\times\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{T}\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\right\|\|\boldsymbol{\varphi}_i^0\|\left\|\frac{1}{T}\mathbf{F}^{0\prime}\mathbf{H}^0\right\|\left\|\mathbf{R}\mathbf{R}'-\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\|\mathbf{R}^{-1}\|\\
&+\sqrt{N}\times\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{T}\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\right\|\left\|\frac{1}{\sqrt{T}}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\right\|\left\|\mathbf{R}\mathbf{R}'-\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\|\mathbf{R}^{-1}\|\\
=&O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)
\end{aligned}
$$

For $\mathbb{D}_3$, we have

$$
\begin{aligned}
\|\mathbb{D}_3\|=&\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{T}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\left(\mathbf{H}^0\boldsymbol{\varphi}_i^0+\boldsymbol{\varepsilon}_i\right)\right\|\\
\leq&\sqrt{N}\times\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{F}^0\right\|\|\boldsymbol{\varphi}_i^0\|\left\|\frac{1}{T}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{H}^0\right\|\|\mathbf{R}\|\\
&+\sqrt{N}\times\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{F}^0\right\|\left\|\frac{1}{T}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\boldsymbol{\varepsilon}_i\right\|\|\mathbf{R}\|\\
=&O_p\left(\frac{\sqrt{N}}{\delta_{NT}^2}\right)
\end{aligned}
$$

by Lemmas A.1 (b), (d) and (e) and Lemmas A.9 (b), (c) and (f).

For $\mathbb{D}_4$, we have

$$\begin{aligned}
\|\mathbb{D}_4\| \leq & \sqrt{NT} \times \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}| \left\|\frac{1}{T}\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\right\| \|\boldsymbol{\varphi}_i^0\| \left\|\frac{1}{T}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{H}^0\right\| \\
& + \sqrt{NT} \times \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}| \left\|\frac{1}{T}\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\right\| \left\|\frac{1}{T}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\boldsymbol{\varepsilon}_i\right\| \\
= & O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)
\end{aligned}$$

by Lemmas A.1 (b), (d) and Lemmas A.9 (g), (h) and (i).

For $\mathbb{D}_5$, we have

$$\begin{aligned}
\|\mathbb{D}_5\| = & \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j'\frac{1}{T}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right)\mathbf{F}^{0\prime}\left(\mathbf{H}^0\boldsymbol{\varphi}_i^0+\boldsymbol{\varepsilon}_i\right)\right\| \\
\leq & \sqrt{N} \times \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}| \left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{F}^0\right\| \|\boldsymbol{\varphi}_i^0\| \left\|\frac{1}{T}\mathbf{F}^{0\prime}\mathbf{H}^0\right\| \left\|\mathbf{R}\mathbf{R}'-\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\| \\
& + \sqrt{\frac{N}{T}} \times \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}| \left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{F}^0\right\| \left\|\frac{1}{\sqrt{T}}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\right\| \left\|\mathbf{R}\mathbf{R}'-\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\| \\
= & O_p\left(\frac{\sqrt{N}}{\delta_{NT}^2}\right)
\end{aligned}$$

by Lemma A.1 (f) and Lemmas A.9 (b), (c) and (d).

Now we consider the term $\mathbb{D}_1$. With (A.1), we can decompose the term $\mathbb{D}_1$ as follows

$$\begin{aligned}
& -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j'\frac{1}{T}(\widehat{\mathbf{F}}\mathbf{R}^{-1}-\mathbf{F}^0)\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i \\
= & -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j'\frac{1}{NT}\sum_{h=1}^{N}\mathbf{V}_h\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i \\
& -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j'\frac{1}{NT^2}\sum_{h=1}^{N}\mathbf{F}^0\boldsymbol{\Gamma}_h^0\mathbf{V}_h'\widehat{\mathbf{F}}\left(\frac{\mathbf{F}^{0\prime}\widehat{\mathbf{F}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i \\
& -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\frac{1}{NT^2}\sum_{h=1}^{N}\mathbb{E}\left(\mathbf{V}_j'\mathbf{V}_h\right)\mathbf{V}_h'\widehat{\mathbf{F}}\left(\frac{\mathbf{F}^{0\prime}\widehat{\mathbf{F}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i \\
& -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\frac{1}{NT^2}\sum_{h=1}^{N}\left(\mathbf{V}_j'\mathbf{V}_h-\mathbb{E}\left(\mathbf{V}_j'\mathbf{V}_h\right)\right)\mathbf{V}_h'\widehat{\mathbf{F}}\left(\frac{\mathbf{F}^{0\prime}\widehat{\mathbf{F}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i \\
= & \mathbb{D}_{1.1}+\mathbb{D}_{1.2}+\mathbb{D}_{1.3}+\mathbb{D}_{1.4}
\end{aligned}$$

We consider the last three terms $\mathbb{D}_{1.2}$, $\mathbb{D}_{1.3}$, and $\mathbb{D}_{1.4}$. For $\mathbb{D}_{1.2}$, we can derive that

$$
\begin{aligned}
\|\mathbb{D}_{1.2}\| = & \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \mathbf{V}_j' \frac{1}{NT^2} \sum_{h=1}^N \mathbf{F}^0 \boldsymbol{\Gamma}_h^0 \mathbf{V}_h' \widehat{\mathbf{F}} \left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}^0)^{-1} \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \mathbf{F}^{0\prime}\mathbf{u}_i \right\| \\
\leq & \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \mathbf{V}_j' \frac{1}{NT^2} \sum_{h=1}^N \mathbf{F}^0 \boldsymbol{\Gamma}_h^0 \mathbf{V}_h' \mathbf{F}^0\mathbf{R} \left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}^0)^{-1} \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \mathbf{F}^{0\prime}\mathbf{u}_i \right\| \\
& + \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \mathbf{V}_j' \frac{1}{NT^2} \sum_{h=1}^N \mathbf{F}^0 \boldsymbol{\Gamma}_h^0 \mathbf{V}_h' \left(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}\right) \left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}^0)^{-1} \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \mathbf{F}^{0\prime}\mathbf{u}_i \right\| \\
\leq & \frac{1}{\sqrt{T}} \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N |w_{ij}| \left\| \frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}} \right\| \|\boldsymbol{\varphi}_i^0\| \left\| \frac{1}{\sqrt{NT}} \sum_{h=1}^N \boldsymbol{\Gamma}_h^0 \mathbf{V}_h' \mathbf{F}^0 \right\| \left\| \left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left\| \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T} \right\| \|(\boldsymbol{\Upsilon}^0)^{-1}\| \|\mathbf{R}\| \\
& + \frac{1}{T} \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N |w_{ij}| \left\| \frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}} \right\| \left\| \frac{\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}} \right\| \left\| \frac{1}{\sqrt{NT}} \sum_{h=1}^N \boldsymbol{\Gamma}_h^0 \mathbf{V}_h' \mathbf{F}^0 \right\| \left\| \left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left\| \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \right\| \|(\boldsymbol{\Upsilon}^0)^{-1}\| \|\mathbf{R}\| \\
& + \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N |w_{ij}| \left\| \frac{\mathbf{V}_j'}{\sqrt{T}} \right\| \|\boldsymbol{\varphi}_i^0\| \left\| \frac{1}{\sqrt{NT}} \sum_{h=1}^N \boldsymbol{\Gamma}_h^0 \mathbf{V}_h' \right\| \left\| \left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left\| \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T} \right\| \left\| \frac{\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}}{\sqrt{T}} \right\| \|(\boldsymbol{\Upsilon}^0)^{-1}\| \\
& + \frac{1}{\sqrt{T}} \frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N |w_{ij}| \left\| \frac{\mathbf{V}_j'}{\sqrt{T}} \right\| \left\| \frac{\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}} \right\| \left\| \frac{1}{\sqrt{NT}} \sum_{h=1}^N \boldsymbol{\Gamma}_h^0 \mathbf{V}_h' \right\| \left\| \left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T} \right)^{-1} \right\| \left\| \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \right\| \left\| \frac{\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}}{\sqrt{T}} \right\| \|(\boldsymbol{\Upsilon}^0)^{-1}\| \\
= & O_p\left(\frac{1}{\delta_{NT}}\right)
\end{aligned}
$$

by Lemmas A.1 (a), (b), (c) and (d).

For $\mathbb{D}_{1.3}$, we can decompose it as follows

$$
\begin{aligned}
& - \frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \frac{1}{NT^2} \sum_{h=1}^N \mathbb{E}\left(\mathbf{V}_j'\mathbf{V}_h\right) \mathbf{V}_h' \widehat{\mathbf{F}} \left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}^0)^{-1} \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \mathbf{F}^{0\prime}\mathbf{u}_i \\
= & - \frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \frac{1}{NT^2} \sum_{h=1}^N \mathbb{E}\left(\mathbf{V}_j'\mathbf{V}_h\right) \mathbf{V}_h' \mathbf{F}^0\mathbf{R} \left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}^0)^{-1} \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \mathbf{F}^{0\prime}\mathbf{u}_i \\
& - \frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \frac{1}{NT^2} \sum_{h=1}^N \mathbb{E}\left(\mathbf{V}_j'\mathbf{V}_h\right) \mathbf{V}_h' \left(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}\right) \left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}^0)^{-1} \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \mathbf{F}^{0\prime}\mathbf{u}_i \\
= & - \frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \frac{1}{NT^2} \sum_{h=1}^N \mathbb{E}\left(\mathbf{V}_j'\mathbf{V}_h\right) \mathbf{V}_h' \mathbf{F}^0\mathbf{R} \left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}^0)^{-1} \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0 \\
& - \frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \frac{1}{NT^2} \sum_{h=1}^N \mathbb{E}\left(\mathbf{V}_j'\mathbf{V}_h\right) \mathbf{V}_h' \mathbf{F}^0\mathbf{R} \left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}^0)^{-1} \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i \\
& - \frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \frac{1}{NT^2} \sum_{h=1}^N \mathbb{E}\left(\mathbf{V}_j'\mathbf{V}_h\right) \mathbf{V}_h' \left(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}\right) \left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}^0)^{-1} \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0 \\
& - \frac{1}{\sqrt{NT}} \sum_{i=1}^N \sum_{j=1}^N w_{ij} \frac{1}{NT^2} \sum_{h=1}^N \mathbb{E}\left(\mathbf{V}_j'\mathbf{V}_h\right) \mathbf{V}_h' \left(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}\right) \left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}^0)^{-1} \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i
\end{aligned}
$$

which are bounded in norm by

$$
\begin{aligned}
&\frac{1}{\sqrt{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\mathbb{E}\left(\frac{1}{T}\mathbf{V}_j'\mathbf{V}_h\right)\right\|\left\|\frac{\mathbf{V}_h'\mathbf{F}^0}{\sqrt{T}}\right\|\|\mathbf{R}\|\left\|\left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T}\right)^{-1}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\|(\boldsymbol{\Upsilon}^0)^{-1}\| \\
&+\frac{1}{\sqrt{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}|w_{ij}|\left\|\frac{\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\left\|\mathbb{E}\left(\frac{1}{T}\mathbf{V}_j'\mathbf{V}_h\right)\right\|\left\|\frac{\mathbf{V}_h'\mathbf{F}^0}{\sqrt{T}}\right\|\|\mathbf{R}\|\left\|\left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T}\right)^{-1}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\|(\boldsymbol{\Upsilon}^0)^{-1}\|\left\|\frac{\mathbf{F}^0}{\sqrt{T}}\right\| \\
&+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\mathbb{E}\left(\frac{1}{T}\mathbf{V}_j'\mathbf{V}_h\right)\right\|\left\|\frac{\mathbf{V}_h}{\sqrt{T}}\right\|\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T}\right)^{-1}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\|(\boldsymbol{\Upsilon}^0)^{-1}\| \\
&+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}|w_{ij}|\left\|\frac{\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\left\|\mathbb{E}\left(\frac{1}{T}\mathbf{V}_j'\mathbf{V}_h\right)\right\|\left\|\frac{\mathbf{V}_h}{\sqrt{T}}\right\|\left\|\frac{\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}}{\sqrt{T}}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T}\right)^{-1}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\left\|\frac{\mathbf{F}^0}{\sqrt{T}}\right\|\right\|\|(\boldsymbol{\Upsilon}^0)^{-1}\|
\end{aligned}
$$

With the fact that $\left\|\mathbb{E}\left(T^{-1}\mathbf{V}_j'\mathbf{V}_h\right)\right\| \leq \bar{\sigma}_{jh}$ by Assumption B3, the above terms can be further bounded in norm by

$$
\begin{aligned}
&\frac{1}{\sqrt{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\bar{\sigma}_{jh}\left\|\frac{\mathbf{V}_h'\mathbf{F}^0}{\sqrt{T}}\right\|\times O_p(1) \\
&+\frac{1}{\sqrt{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}|w_{ij}|\left\|\frac{\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\bar{\sigma}_{jh}\left\|\frac{\mathbf{V}_h'\mathbf{F}^0}{\sqrt{T}}\right\|\times O_p(1) \\
&+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\bar{\sigma}_{jh}\left\|\frac{\mathbf{V}_h}{\sqrt{T}}\right\|\times O_p\left(\frac{1}{\delta_{NT}}\right) \\
&+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}|w_{ij}|\left\|\frac{\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\bar{\sigma}_{jh}\left\|\frac{\mathbf{V}_h}{\sqrt{T}}\right\|\times O_p\left(\frac{1}{\delta_{NT}}\right)
\end{aligned}
\tag{A.84}
$$

Easily, we can show that

$$
\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\bar{\sigma}_{jh}\left\|\frac{\mathbf{V}_h'\mathbf{F}^0}{\sqrt{T}}\right\| = O_p(1)
$$

since

$$
\begin{aligned}
&\mathbb{E}\left|\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\bar{\sigma}_{jh}\left\|\frac{\mathbf{V}_h'\mathbf{F}^0}{\sqrt{T}}\right\|\right| = \mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\bar{\sigma}_{jh}\left\|\frac{\mathbf{V}_h'\mathbf{F}^0}{\sqrt{T}}\right\|\right) \\
=&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}|w_{ij}|\bar{\sigma}_{jh}\mathbb{E}\left\|\boldsymbol{\varphi}_i^0\right\|\mathbb{E}\left\|\frac{\mathbf{V}_h'\mathbf{F}^0}{\sqrt{T}}\right\| \leq \frac{C^2}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}|w_{ij}|\bar{\sigma}_{jh} = \frac{C^2}{N}\sum_{j=1}^{N}\sum_{h=1}^{N}\bar{\sigma}_{jh}\left(\sum_{i=1}^{N}|w_{ij}|\right) \\
\leq&\frac{C^3}{N}\sum_{j=1}^{N}\sum_{h=1}^{N}\bar{\sigma}_{jh} \leq C^4
\end{aligned}
$$

Similarly, we can show that

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\bar{\sigma}_{jh}\left\|\frac{\mathbf{V}_h'\mathbf{F}^0}{\sqrt{T}}\right\| = O_p(1)$$

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}|w_{ij}|\left\|\frac{\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\bar{\sigma}_{jh}\left\|\frac{\mathbf{V}_h'\mathbf{F}^0}{\sqrt{T}}\right\| = O_p(1)$$

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\bar{\sigma}_{jh}\left\|\frac{\mathbf{V}_h}{\sqrt{T}}\right\| = O_p(1)$$

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}|w_{ij}|\left\|\frac{\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\bar{\sigma}_{jh}\left\|\frac{\mathbf{V}_h}{\sqrt{T}}\right\| = O_p(1)$$

Combining the above equations and the equation (A.84), we can derive that

$$\mathbb{D}_{1.3} = O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\sqrt{\frac{T}{N\delta_{NT}^2}}\right) = O_p\left(\frac{1}{\sqrt{N}}\right) + O_p\left(\frac{\sqrt{T}}{N}\right)$$

For $\mathbb{D}_{1.4}$, it can be decomposed as

$$\begin{aligned}
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\frac{1}{NT^2}\sum_{h=1}^{N}\left(\mathbf{V}_j'\mathbf{V}_h-\mathbb{E}\left(\mathbf{V}_j'\mathbf{V}_h\right)\right)\mathbf{V}_h'\widehat{\mathbf{F}}\left(\frac{\mathbf{F}^{0\prime}\widehat{\mathbf{F}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\\
=&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\frac{1}{NT^2}\sum_{h=1}^{N}\left(\mathbf{V}_j'\mathbf{V}_h-\mathbb{E}\left(\mathbf{V}_j'\mathbf{V}_h\right)\right)\mathbf{V}_h'\mathbf{F}^0\mathbf{R}\left(\frac{\mathbf{F}^{0\prime}\widehat{\mathbf{F}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\frac{1}{NT^2}\sum_{h=1}^{N}\left(\mathbf{V}_j'\mathbf{V}_h-\mathbb{E}\left(\mathbf{V}_j'\mathbf{V}_h\right)\right)\mathbf{V}_h'\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\left(\frac{\mathbf{F}^{0\prime}\widehat{\mathbf{F}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\\
=&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\frac{1}{NT}\sum_{h=1}^{N}\left(\mathbf{V}_j'\mathbf{V}_h-\mathbb{E}\left(\mathbf{V}_j'\mathbf{V}_h\right)\right)\mathbf{V}_h'\mathbf{F}^0\mathbf{R}\left(\frac{\mathbf{F}^{0\prime}\widehat{\mathbf{F}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\boldsymbol{\varphi}_i^0\\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\frac{1}{NT^2}\sum_{h=1}^{N}\left(\mathbf{V}_j'\mathbf{V}_h-\mathbb{E}\left(\mathbf{V}_j'\mathbf{V}_h\right)\right)\mathbf{V}_h'\mathbf{F}^0\mathbf{R}\left(\frac{\mathbf{F}^{0\prime}\widehat{\mathbf{F}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\frac{1}{NT}\sum_{h=1}^{N}\left(\mathbf{V}_j'\mathbf{V}_h-\mathbb{E}\left(\mathbf{V}_j'\mathbf{V}_h\right)\right)\mathbf{V}_h'\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\left(\frac{\mathbf{F}^{0\prime}\widehat{\mathbf{F}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\boldsymbol{\varphi}_i^0\\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\frac{1}{NT^2}\sum_{h=1}^{N}\left(\mathbf{V}_j'\mathbf{V}_h-\mathbb{E}\left(\mathbf{V}_j'\mathbf{V}_h\right)\right)\mathbf{V}_h'\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\left(\frac{\mathbf{F}^{0\prime}\widehat{\mathbf{F}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\\
=&\mathbb{D}_{1.4.1}+\mathbb{D}_{1.4.2}+\mathbb{D}_{1.4.3}+\mathbb{D}_{1.4.4}.
\end{aligned}$$

Consider the term $\mathbb{D}_{1.4.1}$. With Lemma A.2 (f), we have $N^{-1}\sum_{h=1}^{N}\mathbb{E}\left\|T^{-1/2}\sum_{s=1}^{T}\mathbf{v}_{hs}\mathbf{f}_s^{0\prime}\right\|^2\leq C$. In addition, we have

$$\begin{aligned}
&\frac{1}{N}\sum_{h=1}^{N}\mathbb{E}\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\boldsymbol{\varphi}_i^{0\prime}\otimes w_{ij}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'-\mathbb{E}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'\right)\right)\right\|^2\\
=&\frac{1}{N}\sum_{h=1}^{N}\mathbb{E}\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\mathrm{vec}\left(w_{ij}\boldsymbol{\varphi}_i^{0\prime}\otimes\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'-\mathbb{E}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'\right)\right)\right)\right\|^2\\
=&\frac{1}{N}\sum_{h=1}^{N}\mathbb{E}\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}w_{ij}\boldsymbol{\varphi}_i^{0}\otimes\mathrm{vec}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'-\mathbb{E}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'\right)\right)\right\|^2\\
=&\frac{1}{N}\sum_{h=1}^{N}\mathbb{E}\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}w_{ij}\boldsymbol{\varphi}_i^{0}\otimes\left(\mathbf{v}_{ht}\otimes\mathbf{v}_{jt}-\mathbb{E}\left(\mathbf{v}_{ht}\otimes\mathbf{v}_{jt}\right)\right)\right\|^2\\
=&\frac{1}{N^2T}\sum_{h=1}^{N}\sum_{i_1=1}^{N}\sum_{j_1=1}^{N}\sum_{i_2=1}^{N}\sum_{j_2=1}^{N}\sum_{s=1}^{T}\sum_{t=1}^{T}w_{i_1j_1}w_{i_2j_2}\mathbb{E}\left(\boldsymbol{\varphi}_{i_2}^{0\prime}\boldsymbol{\varphi}_{i_1}^{0}\right)\times\mathrm{tr}\left(\mathrm{cov}\left(\mathbf{v}_{hs}\otimes\mathbf{v}_{j_2s},\mathbf{v}_{ht}\otimes\mathbf{v}_{j_1t}\right)\right)\\
\leq&\frac{1}{N^2}\sum_{h=1}^{N}\sum_{i_1=1}^{N}\sum_{j_1=1}^{N}\sum_{i_2=1}^{N}\sum_{j_2=1}^{N}|w_{i_1j_1}||w_{i_2j_2}|\left(\mathbb{E}\left\|\boldsymbol{\varphi}_{i_2}^{0}\right\|^2\mathbb{E}\left\|\boldsymbol{\varphi}_{i_1}^{0}\right\|^2\right)^{1/2}\times\left|\mathrm{tr}\left(\frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\mathrm{cov}\left(\mathbf{v}_{hs}\otimes\mathbf{v}_{j_2s},\mathbf{v}_{ht}\otimes\mathbf{v}_{j_1t}\right)\right)\right|\\
\leq&\frac{C}{N^2}\sum_{h=1}^{N}\sum_{i_1=1}^{N}\sum_{j_1=1}^{N}\sum_{i_2=1}^{N}\sum_{j_2=1}^{N}|w_{i_1j_1}||w_{i_2j_2}|\left\|\frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}\mathrm{cov}\left(\mathbf{v}_{hs}\otimes\mathbf{v}_{j_2s},\mathbf{v}_{ht}\otimes\mathbf{v}_{j_1t}\right)\right\|\leq C^2
\end{aligned}$$

Then we can derive that

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{N}\sum_{h=1}^{N}\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\boldsymbol{\varphi}_i^{0\prime}\otimes w_{ij}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'-\mathbb{E}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'\right)\right)\right\|\left\|\frac{1}{\sqrt{T}}\sum_{s=1}^{T}\mathbf{v}_{hs}\mathbf{f}_s^{0\prime}\right\|\right)\\
=&\frac{1}{N}\sum_{h=1}^{N}\mathbb{E}\left(\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\boldsymbol{\varphi}_i^{0\prime}\otimes w_{ij}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'-\mathbb{E}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'\right)\right)\right\|\left\|\frac{1}{\sqrt{T}}\sum_{s=1}^{T}\mathbf{v}_{hs}\mathbf{f}_s^{0\prime}\right\|\right)\\
\leq&\frac{1}{N}\sum_{h=1}^{N}\left(\mathbb{E}\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\boldsymbol{\varphi}_i^{0\prime}\otimes w_{ij}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'-\mathbb{E}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'\right)\right)\right\|^2\mathbb{E}\left\|\frac{1}{\sqrt{T}}\sum_{s=1}^{T}\mathbf{v}_{hs}\mathbf{f}_s^{0\prime}\right\|^2\right)^{1/2}\qquad\text{(A.85)}\\
\leq&\left(\frac{1}{N}\sum_{h=1}^{N}\mathbb{E}\left\|\frac{1}{\sqrt{T}}\sum_{s=1}^{T}\mathbf{v}_{hs}\mathbf{f}_s^{0\prime}\right\|^2\right)^{1/2}\left(\frac{1}{N}\sum_{h=1}^{N}\mathbb{E}\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\boldsymbol{\varphi}_i^{0\prime}\otimes w_{ij}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'-\mathbb{E}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'\right)\right)\right\|^2\right)^{1/2}\\
\leq&C
\end{aligned}$$

Then from the equation (A.85), we have

$$\frac{1}{N}\sum_{h=1}^{N}\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\boldsymbol{\varphi}_i^{0\prime}\otimes w_{ij}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'-\mathbb{E}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'\right)\right)\right\|\left\|\frac{1}{\sqrt{T}}\sum_{s=1}^{T}\mathbf{v}_{hs}\mathbf{f}_s^{0\prime}\right\|=O_p(1)$$

by Markov inequality. Thus, for the term $\mathbb{D}_{1.4.1}$, we can derive that

$$\begin{aligned}
&\left\|\mathbb{D}_{1.4.1}\right\| \\
=&\left\|\operatorname{vec}\left(\frac{1}{\sqrt{T}}\frac{1}{N}\sum_{h=1}^{N}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\left(\mathbf{V}_j'\mathbf{V}_h-\mathbb{E}\left(\mathbf{V}_j'\mathbf{V}_h\right)\right)\times\left(\frac{1}{\sqrt{T}}\mathbf{V}_h'\mathbf{F}^0\right)\right.\right.\\
&\left.\left.\times\mathbf{R}\left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\times\boldsymbol{\varphi}_i^0\right)\right\|\\
=&\left\|\frac{1}{\sqrt{T}}\frac{1}{N}\sum_{h=1}^{N}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\boldsymbol{\varphi}_i^{0\prime}\otimes w_{ij}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'-\mathbb{E}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'\right)\right)\right)\right.\\
&\left.\times\left[\left(\mathbf{R}\left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right)'\otimes\mathbf{I}_k\right]\times\operatorname{vec}\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}\mathbf{v}_{hs}\mathbf{f}_s^{0\prime}\right)\right\|\\
\leq&\frac{1}{\sqrt{T}}\frac{1}{N}\sum_{h=1}^{N}\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\boldsymbol{\varphi}_i^{0\prime}\otimes w_{ij}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'-\mathbb{E}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'\right)\right)\right\|\left\|\frac{1}{\sqrt{T}}\sum_{s=1}^{T}\mathbf{v}_{hs}\mathbf{f}_s^{0\prime}\right\|\\
&\times\left\|\mathbf{R}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\hat{\mathbf{F}}}{T}\right)^{-1}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\left\|\mathbf{I}_k\right\|\\
=&O_p\left(\frac{1}{\sqrt{T}}\right)\frac{1}{N}\sum_{h=1}^{N}\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\boldsymbol{\varphi}_i^{0\prime}\otimes w_{ij}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'-\mathbb{E}\left(\mathbf{v}_{jt}\mathbf{v}_{ht}'\right)\right)\right\|\left\|\frac{1}{\sqrt{T}}\sum_{s=1}^{T}\mathbf{v}_{hs}\mathbf{f}_s^{0\prime}\right\|\\
=&O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}$$

with the above equation, where the third and the fourth equations use the fact that $\operatorname{vec}(\mathbf{ABC})=(\mathbf{C}'\otimes\mathbf{A})\operatorname{vec}(\mathbf{B})$ for any comfortable matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. Similarly, we can show that the term $\mathbb{D}_{1.4.2}$ is $O_p\left(T^{-1/2}\right)$.

For the term $\mathbb{D}_{1.4.3}$, we have

$$
\begin{aligned}
\|\mathbb{D}_{1.4.3}\| = & \left\| \mathrm{vec} \left(\frac{1}{N} \sum_{h=1}^{N} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \left(\mathbf{V}_j' \mathbf{V}_h - \mathbb{E}\left(\mathbf{V}_j' \mathbf{V}_h \right) \right) \frac{\mathbf{V}_h'}{\sqrt{T}} \frac{\widehat{\mathbf{F}} - \mathbf{F}^0 \mathbf{R}}{\sqrt{T}} \right. \right. \\
& \left. \left. \times \left(\frac{\mathbf{F}^{0\prime} \widehat{\mathbf{F}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}^0)^{-1} \left(\frac{\mathbf{F}^{0\prime} \mathbf{F}^0}{T} \right)^{-1} \frac{\mathbf{F}^{0\prime} \mathbf{H}^0}{T} \boldsymbol{\varphi}_i^0 \right) \right\| \\
= & \left\| \frac{1}{N} \sum_{h=1}^{N} \left(\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} \boldsymbol{\varphi}_i^{0\prime} \otimes w_{ij} \left(\mathbf{v}_{jt} \mathbf{v}_{ht}' - \mathbb{E}\left(\mathbf{v}_{jt} \mathbf{v}_{ht}' \right) \right) \right) \right. \\
& left. \quad \left. \times \left[\left(\frac{\widehat{\mathbf{F}} - \mathbf{F}^0 \mathbf{R}}{\sqrt{T}} \left(\frac{\mathbf{F}^{0\prime} \widehat{\mathbf{F}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}^0)^{-1} \left(\frac{\mathbf{F}^{0\prime} \mathbf{F}^0}{T} \right)^{-1} \frac{\mathbf{F}^{0\prime} \mathbf{H}^0}{T} \right)' \otimes \mathbf{I}_k \right] \mathrm{vec} \left(\frac{1}{\sqrt{T}} \mathbf{V}_h \right) \right\| \\
\leq & \frac{1}{N} \sum_{h=1}^{N} \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} \boldsymbol{\varphi}_i^{0\prime} \otimes w_{ij} \left(\mathbf{v}_{jt} \mathbf{v}_{ht}' - \mathbb{E}\left(\mathbf{v}_{jt} \mathbf{v}_{ht}' \right) \right) \right\| \left\| \frac{1}{\sqrt{T}} \mathbf{V}_h \right\| \left\| \frac{\widehat{\mathbf{F}} - \mathbf{F}^0 \mathbf{R}}{\sqrt{T}} \right\| \\
& \times \left\| \left(\frac{\mathbf{F}^{0\prime} \widehat{\mathbf{F}}}{T} \right)^{-1} \right\| \|(\boldsymbol{\Upsilon}^0)^{-1}\| \left\| \left(\frac{\mathbf{F}^{0\prime} \mathbf{F}^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}^{0\prime} \mathbf{H}^0}{T} \right\| \|\mathbf{I}_k\| \\
= & O_p\left(\frac{1}{\delta_{NT}} \right) \frac{1}{N} \sum_{h=1}^{N} \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} \boldsymbol{\varphi}_i^{0\prime} \otimes w_{ij} \left(\mathbf{v}_{jt} \mathbf{v}_{ht}' - \mathbb{E}\left(\mathbf{v}_{jt} \mathbf{v}_{ht}' \right) \right) \right\| \left(\frac{1}{T} \sum_{s=1}^{T} \|\mathbf{v}_{hs}\|^2 \right)^{1/2} \\
= & O_p\left(\frac{1}{\delta_{NT}} \right)
\end{aligned}
$$

since

$$
\frac{1}{N} \sum_{h=1}^{N} \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{t=1}^{T} \boldsymbol{\varphi}_i^{0\prime} \otimes w_{ij} \left(\mathbf{v}_{jt} \mathbf{v}_{ht}' - \mathbb{E}\left(\mathbf{v}_{jt} \mathbf{v}_{ht}' \right) \right) \right\| \left(\frac{1}{T} \sum_{s=1}^{T} \|\mathbf{v}_{hs}\|^2 \right)^{1/2} = O_p(1)
$$

The equality can be proved similar to the equation (A.85).

Similarly, we can derive that the term $\mathbb{D}_{1.4.2}$ and $\mathbb{D}_{1.4.4}$ are $O_p\left(\delta_{NT}^{-1}\right)$. Combining the above terms $\mathbb{D}_{1.4.1}$ to $\mathbb{D}_{1.4.4}$, we derive that $\mathbb{D}_{1.4} = O_p\left(\delta_{NT}^{-1}\right)$. This completes the proof. □

Proof of Proposition 3.1. With the definition of $\widehat{\mathbf{Z}}_i$, we have

$$
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \widehat{\mathbf{Z}}_i' \mathbf{u}_i = \begin{pmatrix} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{X}_j' \mathbf{M}_{\widehat{\mathbf{F}}} \mathbf{u}_i \\ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{X}_{i,-1}' \mathbf{M}_{\widehat{\mathbf{F}}_{-1}} \mathbf{u}_i \\ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{X}_i' \mathbf{M}_{\widehat{\mathbf{F}}} \mathbf{u}_i \end{pmatrix} \tag{A.86}
$$

By Norkute et al. (2020), we have

$$
\begin{pmatrix} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \boldsymbol{\mathcal{X}}_{i,-1}' \mathbf{M}_{\mathbf{F}_{-1}^0} \mathbf{u}_i \\ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \boldsymbol{\mathcal{X}}_i' \mathbf{M}_{\mathbf{F}^0} \mathbf{u}_i \end{pmatrix} + \sqrt{\frac{T}{N}} \begin{pmatrix} \mathbf{b}_{12} \\ \mathbf{b}_{13} \end{pmatrix} + \sqrt{\frac{N}{T}} \begin{pmatrix} \mathbf{b}_{22} \\ \mathbf{b}_{23} \end{pmatrix} + o_p(1)
$$

where

$$
\begin{aligned}
\mathbf{b}_{12} &= -\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\boldsymbol{\mathcal{V}}_{i,-1}'\mathbf{V}_{j,-1}}{T}\boldsymbol{\Gamma}_j'(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}_{-1}'\mathbf{F}_{-1}}{T}\right)^{-1}\frac{\mathbf{F}_{-1}'\mathbf{u}_i}{T} \\
\mathbf{b}_{13} &= -\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\boldsymbol{\mathcal{V}}_{i}'\mathbf{V}_{j}}{T}\boldsymbol{\Gamma}_j'(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{0}}{T}\right)^{-1}\frac{\mathbf{F}^{0\prime}\mathbf{u}_i}{T} \\
\mathbf{b}_{22} &= -\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}_{-1}'\mathbf{F}_{-1}}{T}\right)^{-1}\mathbf{F}_{-1}'\boldsymbol{\Sigma}_{-1}\mathbf{M}_{\mathbf{F}^0_{-1}}\mathbf{u}_i \\
\mathbf{b}_{23} &= -\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{0}}{T}\right)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i
\end{aligned}
\tag{A.87}
$$

with $\boldsymbol{\mathcal{X}}_i = \mathbf{X}_i - \frac{1}{N}\sum_{\ell=1}^{N}\mathbf{X}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_i$, $\boldsymbol{\mathcal{X}}_{i,-1} = \mathbf{X}_{i,-1} - \frac{1}{N}\sum_{\ell=1}^{N}\mathbf{X}_{\ell,-1}\boldsymbol{\Gamma}_\ell^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_i$, $\boldsymbol{\mathcal{V}}_i = \mathbf{V}_i - \frac{1}{N}\sum_{\ell=1}^{N}\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_i$, $\boldsymbol{\mathcal{V}}_{i,-1} = \mathbf{V}_{i,-1} - \frac{1}{N}\sum_{\ell=1}^{N}\mathbf{V}_{\ell,-1}\boldsymbol{\Gamma}_\ell^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_i$, $\boldsymbol{\Upsilon} = N^{-1}\sum_{i=1}^{N}\boldsymbol{\Gamma}_i\boldsymbol{\Gamma}_i'$. In addition, $\boldsymbol{\Sigma} = N^{-1}\sum_{\ell=1}^{N}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)$ and $\boldsymbol{\Sigma}_{-1} = N^{-1}\sum_{l=1}^{N}\mathbb{E}\left(\mathbf{V}_{\ell,-1}\mathbf{V}_{\ell,-1}'\right)$.

Combining Lemmas A.5, A.6, A.7, A.8 and A.10, we can derive that

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{X}_j'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
=&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i - \frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^{0}\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i \\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{NT}\sum_{h=1}^{N}\mathbf{V}_h\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{0}}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i \\
&+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^{0}\frac{\mathbf{V}_\ell'\mathbf{V}_h}{T}\boldsymbol{\Gamma}_h^{0\prime}\right)(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{0}}{T}\right)^{-1}\frac{\mathbf{F}^{0\prime}\mathbf{u}_i}{T} \\
&-\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{0}}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i + o_p(1) \\
=&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\mathcal{X}}_j'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i - \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\mathcal{V}}_j'\frac{1}{NT}\sum_{h=1}^{N}\mathbf{V}_h\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{0}}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i \\
&-\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{0}}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i + o_p(1) \\
=&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\mathcal{X}}_j'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i + \sqrt{\frac{T}{N}}\mathbf{b}_{11} + \sqrt{\frac{N}{T}}\mathbf{b}_{21} + o_p(1)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{b}_{11} &= -\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{h=1}^{N}w_{ij}\frac{\boldsymbol{\mathcal{V}}_j'\mathbf{V}_h}{T}\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{0}}{T}\right)^{-1}\frac{\mathbf{F}^{0\prime}\mathbf{u}_i}{T} \\
\mathbf{b}_{21} &= -\frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{0}}{T}\right)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i
\end{aligned}
\tag{A.88}
$$

Combining the terms $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, we have

$$
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\widehat{\mathbf{Z}}_i'\mathbf{u}_i = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbb{Z}_i'\mathbf{u}_i + \sqrt{\frac{T}{N}}\mathbf{b}_1 + \sqrt{\frac{N}{T}}\mathbf{b}_2 + o_p(1)
$$

where $\mathbb{Z}_i = \left(\sum_{j=1}^N w_{ij}\mathbf{M}_{\mathbf{F}^0}\boldsymbol{\mathcal{X}}_j, \mathbf{M}_{\mathbf{F}^0_{-1}}\boldsymbol{\mathcal{X}}_{i,-1}, \mathbf{M}_{\mathbf{F}^0}\boldsymbol{\mathcal{X}}_i\right)$, $\mathbf{b}_1 = (\mathbf{b}_{11}', \mathbf{b}_{12}', \mathbf{b}_{13}')'$ and $\mathbf{b}_2 = (\mathbf{b}_{21}', \mathbf{b}_{22}', \mathbf{b}_{23}')'$. This completes the proof. □

Proof of Theorem 3.1

We have

$$\sqrt{NT}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right) = \left(\widehat{\mathbf{A}}'\widehat{\mathbf{B}}^{-1}\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{A}}'\widehat{\mathbf{B}}^{-1}\frac{1}{\sqrt{NT}}\sum_{i=1}^N \widehat{\mathbf{Z}}_i'\mathbf{u}_i.$$

By Lemma A.3, $\widehat{\mathbf{A}} - \overline{\mathbf{A}} \xrightarrow{p} \mathbf{0}$ and $\widehat{\mathbf{B}} - \overline{\mathbf{B}} \xrightarrow{p} \mathbf{0}$, and $\widehat{\mathbf{B}}^{-1} - \overline{\mathbf{B}}^{-1} \xrightarrow{p} \mathbf{0}$ by continuous mapping theorem, thus, $\widehat{\mathbf{A}}'\widehat{\mathbf{B}}^{-1}\widehat{\mathbf{A}} - \overline{\mathbf{A}}'\overline{\mathbf{B}}^{-1}\overline{\mathbf{A}} \xrightarrow{p} \mathbf{0}$. Under Assumption F $\overline{\mathbf{A}}'\overline{\mathbf{B}}^{-1}\overline{\mathbf{A}}$ is positive definite which implies $\left(\widehat{\mathbf{A}}'\widehat{\mathbf{B}}^{-1}\widehat{\mathbf{A}}\right)^{-1}\widehat{\mathbf{A}}'\widehat{\mathbf{B}}^{-1} = O_p(1)$. By Proposition 3.1 $\frac{1}{\sqrt{NT}}\sum_{i=1}^N \widehat{\mathbf{Z}}_i'\mathbf{u}_i = \frac{1}{\sqrt{NT}}\sum_{i=1}^N \mathbb{Z}_i'\left(\mathbf{H}^0\boldsymbol{\varphi}_i^0 + \boldsymbol{\varepsilon}_i\right) + \sqrt{\frac{T}{N}}\mathbf{b}_1 + \sqrt{\frac{N}{T}}\mathbf{b}_2 + o_p(1)$. First, due to the independence between $\boldsymbol{\varepsilon}_i$ and $\mathbb{Z}_i$, a suitable central limit theorem ensures that $\frac{1}{\sqrt{NT}}\sum_{i=1}^N \mathbb{Z}_i'\boldsymbol{\varepsilon}_i = O_p(1)$. Consider the first component of $\frac{1}{\sqrt{NT}}\sum_{i=1}^N \mathbb{Z}_i'\mathbf{H}^0\boldsymbol{\varphi}_i^0$. Recalling $\mathbb{Z}_i = \left(\sum_{j=1}^N w_{ij}\mathbf{M}_{\mathbf{F}^0}\mathcal{V}_j, \mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{F}^0_{-1}}\mathcal{V}_{i,-1}, \mathbf{M}_{\mathbf{F}^0}\mathcal{V}_i\right)$ where $\mathcal{V}_i = \mathbf{V}_i - \frac{1}{N}\sum_{\ell=1}^N \mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_i^0$, $\mathcal{V}_{i,-1} = \mathbf{V}_{i,-1} - \frac{1}{N}\sum_{\ell=1}^N \mathbf{V}_{\ell,-1}\boldsymbol{\Gamma}_\ell^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_i^0$, and noting $\mathbf{M}_{\mathbf{F}^0} = \mathbf{M}_{\mathbf{F}} = \frac{\mathbf{F}\mathbf{F}'}{T}$, $\frac{1}{\sqrt{NT}}\sum_{i=1}^N \frac{\sum_{j=1}^N w_{ij}\mathcal{V}_j'\mathbf{F}}{\sqrt{T}}\frac{\mathbf{F}'\mathbf{H}^0}{T}\boldsymbol{\varphi}_i^0 = O_p(1)$ due to the independence between $\mathcal{V}_j$ and $\mathbf{F}$ and cross-sectional independence between $\sum_{j=1}^N w_{ij}\mathcal{V}_j$ and $\boldsymbol{\varphi}_i^0$. In a similar manner, we can show that other components in $\frac{1}{\sqrt{NT}}\sum_{i=1}^N \mathbb{Z}_i'\mathbf{H}^0\boldsymbol{\varphi}_i^0$ and the bias terms $\mathbf{b}_1$ and $\mathbf{b}_2$are $O_p(1)$. Together with the condition $N/T \to c$ where $0 < c \leq \infty$ as $N, T \to \infty$, we have $\sqrt{NT}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right) = O_p(1)$ as required. □

Online Appendix B: Proofs of the main theoretical results for the two-step IV estimator

Let $\boldsymbol{\Xi}$ be $r_y \times r_y$ diagonal matrix that consist of the first r_y largest eigenvalues of the $T \times T$ matrix $(NT)^{-1} \sum_{i=1}^{N} \widehat{\mathbf{u}}_i \widehat{\mathbf{u}}_i'$. Then by the definition of eigenvalues and $\widehat{\mathbf{H}}$, $\widehat{\mathbf{H}}\boldsymbol{\Xi} = (NT)^{-1} \sum_{i=1}^{N} \widehat{\mathbf{u}}_i \widehat{\mathbf{u}}_i' \widehat{\mathbf{H}}$. It's easy to show that $\boldsymbol{\Xi}$ is invertible following the proof of Proposition A.1 (i) in Bai (2009) given the covergence rate of $\hat{\boldsymbol{\theta}}$ to $\boldsymbol{\theta}$ is $\sqrt{NT}$. Then

$$\begin{aligned}
&\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}} \\
=& \frac{1}{NT} \sum_{i=1}^{N} \mathbf{C}_i \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\right) \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\right)' \mathbf{C}_i' \widehat{\mathbf{H}} \boldsymbol{\Xi}^{-1} + \frac{1}{NT} \sum_{i=1}^{N} \mathbf{C}_i \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\right) \mathbf{u}_i' \widehat{\mathbf{H}} \boldsymbol{\Xi}^{-1} + \frac{1}{NT} \sum_{i=1}^{N} \mathbf{u}_i \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\right)' \mathbf{C}_i' \widehat{\mathbf{H}} \boldsymbol{\Xi}^{-1} \\
&+ \frac{1}{NT} \sum_{i=1}^{N} \mathbf{H}^0 \boldsymbol{\varphi}_i^0 \boldsymbol{\varepsilon}_i' \widehat{\mathbf{H}} + \frac{1}{NT} \sum_{i=1}^{N} \boldsymbol{\varepsilon}_i \boldsymbol{\varphi}_i^{0\prime} \mathbf{H}^{0\prime} \widehat{\mathbf{H}} \boldsymbol{\Xi}^{-1} + \frac{1}{NT} \sum_{i=1}^{N} \boldsymbol{\varepsilon}_i \boldsymbol{\varepsilon}_i' \widehat{\mathbf{H}} \boldsymbol{\Xi}^{-1}
\end{aligned} \tag{B.1}$$

where $\boldsymbol{\mathcal{R}} = \frac{1}{NT} \sum_{i=1}^{N} \boldsymbol{\varphi}_i^0 \boldsymbol{\varphi}_i^{0\prime} \mathbf{H}^{0\prime} \widehat{\mathbf{H}} \boldsymbol{\Xi}^{-1}$. Following the proof of Proposition A.1 (ii) in Bai (2009), we can show that $\boldsymbol{\mathcal{R}}$ is invertible.

Lemma B.1 *Under Assumptions B to D, we have*

$$\begin{aligned}
&(a) \quad T^{-1} \left\| \widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}} \right\|^2 = O_p\left(\delta_{NT}^{-2}\right), \\
&(b) \quad T^{-1} \left(\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}}\right)' \mathbf{H}^0 = O_p\left(\delta_{NT}^{-2}\right), T^{-1} \left(\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}}\right)' \mathbf{F}^0 = O_p\left(\delta_{NT}^{-2}\right), \\
&(c) \quad T^{-1} \left(\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}}\right)' \widehat{\mathbf{H}} = O_p\left(\delta_{NT}^{-2}\right), \\
&(e) \quad \boldsymbol{\Xi} = O_p(1), \boldsymbol{\mathcal{R}} = O_p(1), \boldsymbol{\Xi}^{-1} = O_p(1), \boldsymbol{\mathcal{R}}^{-1} = O_p(1). \\
&(f) \quad \boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1} = O_p\left(\delta_{NT}^{-2}\right), \\
&(g) \quad \mathbf{M}_{\widehat{\mathbf{H}}} - \mathbf{M}_{\mathbf{H}^0} = O_p\left(\delta_{NT}^{-1}\right), \\
&(h) \quad \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{N} \boldsymbol{\varphi}_\ell \boldsymbol{\varepsilon}_\ell' \left(\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}}\right) = O_p\left(\sqrt{\frac{T}{N}}\right) + O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right), \\
&(i) \quad \left\| \frac{1}{\sqrt{NT}} \sum_{h=1}^{N} \left(\boldsymbol{\varepsilon}_h \boldsymbol{\varepsilon}_h' - \mathbb{E}\left(\boldsymbol{\varepsilon}_h \boldsymbol{\varepsilon}_h'\right)\right) \mathbf{H}^0 \right\| = O_p(1), \\
&(j) \quad \left\| \frac{1}{\sqrt{NT}} \sum_{h=1}^{N} \mathbf{H}^{0\prime} \left(\boldsymbol{\varepsilon}_h \boldsymbol{\varepsilon}_h' - \mathbb{E}\left(\boldsymbol{\varepsilon}_h \boldsymbol{\varepsilon}_h'\right)\right) \mathbf{H}^0 \right\| = O_p(1), \\
&(m) \quad \left\| \frac{1}{N\sqrt{T}} \sum_{\ell=1}^{N} \mathbb{E}\left(\boldsymbol{\varepsilon}_\ell \boldsymbol{\varepsilon}_\ell'\right) \right\| = O(1), \\
&(n) \quad \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{N} \left[\boldsymbol{\varepsilon}_\ell \boldsymbol{\varepsilon}_\ell' - \mathbb{E}\left(\boldsymbol{\varepsilon}_\ell \boldsymbol{\varepsilon}_\ell'\right)\right] = O_p(1).
\end{aligned}$$

Proof of Lemma B.1. We can follow the way of the proof of Lemma A.1 to prove this lemma. Thus, we omitted the details. □

Lemma B.2 *Under Assumptions B to D, we have*

$$\frac{1}{NT} \sum_{i=1}^{N} \widehat{\mathbf{Z}}_i' \mathbf{M}_{\widehat{\mathbf{H}}} \mathbf{C}_i - \frac{1}{NT} \sum_{i=1}^{N} \mathbf{Z}_i' \mathbf{M}_{\mathbf{H}^0} \mathbf{C}_i = O_p\left(\delta_{NT}^{-1}\right)$$

$$\frac{1}{NT} \sum_{i=1}^{N} \widehat{\mathbf{Z}}_i' \mathbf{M}_{\widehat{\mathbf{H}}} \widehat{\mathbf{Z}}_i - \frac{1}{NT} \sum_{i=1}^{N} \mathbf{Z}_i' \mathbf{M}_{\mathbf{H}^0} \mathbf{Z}_i = O_p\left(\delta_{NT}^{-1}\right)$$

Proof of Lemma B.2. With Lemma B.1 (g), we can follow the way of the proof of Lemma A.3 to prove this lemma. Thus, we omitted the details. □

Lemma B.3 *Under Assumptions A to D, we have*

(a) $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j\right\|\|\boldsymbol{\varphi}_i^0\| = O_p(1)$

(b) $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{H}^0\right\|\|\boldsymbol{\varphi}_i^0\| = O_p(1)$, $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{F}^0\right\|\|\boldsymbol{\varphi}_i^0\| = O_p(1)$

(c) $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{H}^0\right\|\left\|\frac{1}{\sqrt{T}}\boldsymbol{\varepsilon}_i'\mathbf{H}^0\right\| = O_p(1)$, $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{F}^0\right\|\left\|\frac{1}{\sqrt{T}}\boldsymbol{\varepsilon}_i'\mathbf{H}^0\right\| = O_p(1)$,

$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{F}^0\right\|\left\|\frac{1}{\sqrt{T}}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\right\| = O_p(1)$

(d) $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j\right\|\left\|\frac{1}{T}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)'\boldsymbol{\varepsilon}_i\right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right)$

(e) $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{H}^0\right\|\left\|\frac{1}{T}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)'\boldsymbol{\varepsilon}_i\right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right)$

$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{F}^0\right\|\left\|\frac{1}{T}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)'\boldsymbol{\varepsilon}_i\right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right)$

$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\boldsymbol{\varepsilon}_j'\mathbf{H}^0\right\|\left\|\frac{1}{T}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)'\boldsymbol{\varepsilon}_i\right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right)$

$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\boldsymbol{\varepsilon}_j'\mathbf{F}^0\right\|\left\|\frac{1}{T}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)'\boldsymbol{\varepsilon}_i\right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right)$

$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\boldsymbol{\varepsilon}_j'\mathbf{H}^0\right\|\left\|\frac{1}{T}\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)'\boldsymbol{\varepsilon}_i\right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right)$

(f) $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{T}\mathbf{V}_j'\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)\right\|\|\boldsymbol{\varphi}_i^0\| = O_p\left(\frac{1}{\delta_{NT}}\right)$, $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{T}\mathbf{V}_j'\left(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\right)\right\|\|\boldsymbol{\varphi}_i^0\| = O_p\left(\frac{1}{\delta_{NT}}\right)$

(g) $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{T}\mathbf{V}_j'\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)\right\|\left\|\frac{1}{T}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)'\boldsymbol{\varepsilon}_i\right\| = O_p\left(\frac{1}{\delta_{NT}^3}\right)$

(h) $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\|\boldsymbol{\Gamma}_j^0\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i}{T}\right\| = O_p\left(\frac{1}{\delta_{NT}^2}\right)$

(i) $\left\|\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{V}_h}{T}\boldsymbol{\Gamma}_h^{0\prime}\right\| = O_p(1)$

Proof of Lemma B.3. Following the proof of Lemma A.9 (a), (b) and (d), we can easily prove parts (a)-(c), respectively.

Consider (d). Since

$$
\begin{aligned}
&\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1}-\mathbf{H}^0\\
=&\frac{1}{NT}\sum_{i=1}^{N}\mathbf{C}_i\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)'\mathbf{C}_i'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}+\frac{1}{NT}\sum_{i=1}^{N}\mathbf{C}_i\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)\mathbf{u}_i'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\\
&+\frac{1}{NT}\sum_{i=1}^{N}\mathbf{u}_i\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)'\mathbf{C}_i'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}+\frac{1}{NT}\sum_{i=1}^{N}\mathbf{H}^0\boldsymbol{\varphi}_i^0\boldsymbol{\varepsilon}_i'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\\
&+\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i\boldsymbol{\varphi}_i^{0\prime}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}+\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}
\end{aligned}
$$

We have

$$
\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{1}{T}\left(\mathbf{H}^0-\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1}\right)'\varepsilon_i\right\| = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{1}{T}\varepsilon_i'\left(\mathbf{H}^0-\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1}\right)\right\| \\
\le&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\varepsilon_i'\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{C}_\ell\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)'\mathbf{C}_\ell'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\right\| \\
&+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\varepsilon_i'\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{C}_\ell\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)\mathbf{u}_\ell'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\right\| \\
&+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\varepsilon_i'\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{u}_\ell\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)'\mathbf{C}_\ell'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\right\| \\
&+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\varepsilon_i'\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{H}^0\boldsymbol{\varphi}_l^0\varepsilon_\ell'\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1}\boldsymbol{\mathcal{R}}^{-1}\right\|+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\varepsilon_i'\frac{1}{NT^2}\sum_{\ell=1}^{N}\varepsilon_\ell\boldsymbol{\varphi}_l'\mathbf{H}^{0\prime}\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1}\boldsymbol{\mathcal{R}}^{-1}\right\| \\
&+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\varepsilon_i'\frac{1}{NT^2}\sum_{\ell=1}^{N}\varepsilon_\ell\varepsilon_\ell'\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1}\boldsymbol{\mathcal{R}}^{-1}\right\| \\
\le&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{\varepsilon_i}{\sqrt{T}}\right\|\left(\frac{1}{N}\sum_{\ell=1}^{N}\left\|\frac{\mathbf{C}_\ell}{\sqrt{T}}\right\|^2\right)\left\|\frac{\widehat{\mathbf{H}}}{\sqrt{T}}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1}\right\|\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\|\left\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right\|^2 \\
&+\frac{2}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{\varepsilon_i}{\sqrt{T}}\right\|\left(\frac{1}{N}\sum_{\ell=1}^{N}\left\|\frac{\mathbf{C}_\ell}{\sqrt{T}}\right\|\left\|\frac{\mathbf{u}_\ell}{\sqrt{T}}\right\|\right)\left\|\frac{\widehat{\mathbf{H}}}{\sqrt{T}}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1}\right\|\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\|\left\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right\| \\
&+\frac{1}{\sqrt{NT}}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{1}{\sqrt{T}}\varepsilon_i'\mathbf{H}^0\right\|\right)\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\varphi}_l^0\varepsilon_\ell'\mathbf{H}^0\right\|\|\boldsymbol{\mathcal{R}}\|\|\boldsymbol{\Xi}^{-1}\|\|\boldsymbol{\mathcal{R}}^{-1}\| \\
&+\frac{1}{\sqrt{NT}}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{1}{\sqrt{T}}\varepsilon_i'\mathbf{H}^0\right\|\right)\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\varphi}_l^0\varepsilon_\ell'\right\|\left\|\frac{1}{\sqrt{T}}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)\right\|\|\boldsymbol{\Xi}^{-1}\|\|\boldsymbol{\mathcal{R}}^{-1}\| \\
&+\frac{1}{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{\ell=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{1}{T}\mathbb{E}(\varepsilon_i'\varepsilon_\ell)\right\|\|\boldsymbol{\varphi}_l^0\|\left\|\frac{\mathbf{H}^0}{\sqrt{T}}\right\|\left\|\frac{\widehat{\mathbf{H}}}{\sqrt{T}}\right\|\|\boldsymbol{\Xi}^{-1}\|\|\boldsymbol{\mathcal{R}}^{-1}\| \\
&+\frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\left(\varepsilon_i'\varepsilon_\ell-\mathbb{E}(\varepsilon_i'\varepsilon_\ell)\right)\boldsymbol{\varphi}_l'\right\|\left\|\frac{\mathbf{H}^0}{\sqrt{T}}\right\|\left\|\frac{\widehat{\mathbf{H}}}{\sqrt{T}}\right\|\|\boldsymbol{\Xi}^{-1}\|\|\boldsymbol{\mathcal{R}}^{-1}\| \\
&+\frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{\varepsilon_i}{\sqrt{T}}\right\|\left\|\frac{1}{N\sqrt{T}}\sum_{\ell=1}^{N}\mathbb{E}\left(\varepsilon_\ell\varepsilon_\ell'\right)\right\|\left\|\frac{\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}}{\sqrt{T}}\right\|\|\boldsymbol{\Xi}^{-1}\|\|\boldsymbol{\mathcal{R}}^{-1}\| \\
&+\frac{1}{\sqrt{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{\varepsilon_i}{\sqrt{T}}\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\left[\varepsilon_\ell\varepsilon_\ell'-\mathbb{E}\left(\varepsilon_\ell\varepsilon_\ell'\right)\right]\right\|\left\|\frac{\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}}{\sqrt{T}}\right\|\|\boldsymbol{\Xi}^{-1}\|\|\boldsymbol{\mathcal{R}}^{-1}\| \\
&+\frac{1}{T}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\varepsilon_i'\frac{1}{NT}\sum_{\ell=1}^{N}\mathbb{E}\left(\varepsilon_\ell\varepsilon_\ell'\right)\mathbf{H}^0\right\|\|\boldsymbol{\mathcal{R}}\|\|\boldsymbol{\Xi}^{-1}\|\|\boldsymbol{\mathcal{R}}^{-1}\| \\
&+\frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{\varepsilon_i}{\sqrt{T}}\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\left[\varepsilon_\ell\varepsilon_\ell'-\mathbb{E}\left(\varepsilon_\ell\varepsilon_\ell'\right)\right]\mathbf{H}^0\right\|\|\boldsymbol{\mathcal{R}}\|\|\boldsymbol{\Xi}^{-1}\|\|\boldsymbol{\mathcal{R}}^{-1}\| \\
=&O_p\left(\delta_{NT}^{-2}\right)
\end{aligned}
$$

by Lemma A.1 (a) and A.1 (e), Lemma A.4 (c), (e), (g) and (h). Similarly, we can prove (e).

Consider the first result in (f). We have

$$
\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{1}{T}\mathbf{V}_j'\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)\right\|\\
\leq&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\mathbf{V}_j'\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{C}_\ell\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)'\mathbf{C}_\ell'\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1}\right\|\\
&+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\mathbf{V}_j'\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{C}_\ell\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)\mathbf{u}_\ell'\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1}\right\|\\
&+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\mathbf{V}_j'\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{u}_\ell\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)'\mathbf{C}_\ell'\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1}\right\|\\
&+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\mathbf{V}_j'\frac{1}{NT^2}\sum_{\ell=1}^{N}\mathbf{H}^0\boldsymbol{\varphi}_l^0\boldsymbol{\varepsilon}_\ell'\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1}\right\|\\
&+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\mathbf{V}_j'\frac{1}{NT^2}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_\ell\boldsymbol{\varphi}_l'\mathbf{H}^{0\prime}\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1}\right\|+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\mathbf{V}_j'\frac{1}{NT^2}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_\ell\boldsymbol{\varepsilon}_\ell'\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1}\right\|\\
\leq&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left(\frac{1}{N}\sum_{\ell=1}^{N}\left\|\frac{\mathbf{C}_\ell}{\sqrt{T}}\right\|^2\right)\left\|\frac{\widehat{\mathbf{H}}}{\sqrt{T}}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\left\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right\|^2\\
&+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left(\frac{1}{N}\sum_{\ell=1}^{N}\left\|\frac{\mathbf{C}_\ell}{\sqrt{T}}\right\|\left\|\frac{\mathbf{u}_\ell}{\sqrt{T}}\right\|\right)\left\|\frac{\widehat{\mathbf{H}}}{\sqrt{T}}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\left\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right\|\\
&+\frac{1}{\sqrt{NT}}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{H}^0\right\|\right)\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\varphi}_l^0\boldsymbol{\varepsilon}_\ell'\mathbf{H}^0\right\|\left\|\boldsymbol{\mathcal{R}}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\\
&+\frac{1}{\sqrt{NT}}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{H}^0\right\|\right)\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\varphi}_l^0\boldsymbol{\varepsilon}_\ell'\right\|\left\|\frac{1}{\sqrt{T}}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\\
&+\frac{1}{\sqrt{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_\ell\boldsymbol{\varphi}_l'\right\|\left\|\frac{\mathbf{H}^0}{\sqrt{T}}\right\|\left\|\frac{\widehat{\mathbf{H}}}{\sqrt{T}}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\\
&+\frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{1}{N\sqrt{T}}\sum_{\ell=1}^{N}\mathbb{E}\left(\boldsymbol{\varepsilon}_\ell\boldsymbol{\varepsilon}_\ell'\right)\right\|\left\|\frac{\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}}{\sqrt{T}}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\\
&+\frac{1}{\sqrt{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\left[\boldsymbol{\varepsilon}_\ell\boldsymbol{\varepsilon}_\ell'-\mathbb{E}\left(\boldsymbol{\varepsilon}_\ell\boldsymbol{\varepsilon}_\ell'\right)\right]\right\|\left\|\frac{\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}}{\sqrt{T}}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\\
&+\frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{1}{N\sqrt{T}}\sum_{\ell=1}^{N}\mathbb{E}\left(\boldsymbol{\varepsilon}_\ell\boldsymbol{\varepsilon}_\ell'\right)\right\|\left\|\frac{\mathbf{H}^0}{\sqrt{T}}\right\|\left\|\boldsymbol{\mathcal{R}}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\\
&+\frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\left[\boldsymbol{\varepsilon}_\ell\boldsymbol{\varepsilon}_\ell'-\mathbb{E}\left(\boldsymbol{\varepsilon}_\ell\boldsymbol{\varepsilon}_\ell'\right)\right]\mathbf{H}^0\right\|\left\|\boldsymbol{\mathcal{R}}\right\|\left\|\boldsymbol{\Xi}^{-1}\right\|\\
=&O_p\left(\frac{1}{\delta_{NT}}\right)
\end{aligned}
$$

by Lemmas A.1 (i), (m) and (n) and Lemma B.3 (a). The other variant is derived in a similar manner.

Replacing $\boldsymbol{\varphi}_i^0$ by $\frac{1}{T}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)'\boldsymbol{\varepsilon}_i$, with Lemma B.3 (d) and (e), we can prove (g). Following the way of proof of

Lemma A.4 (i), we can prove (h). Now consider (i).

$$
\begin{aligned}
&\left\|\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\mathbf{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{V}_h}{T}\mathbf{\Gamma}_h^{0\prime}\right\| \\
\leq&\left\|\frac{1}{NT}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\sum_{t=1}^{T}\mathbf{\Gamma}_\ell^0 E(\mathbf{V}_{\ell t}\mathbf{V}_{ht}')\mathbf{\Gamma}_h^{0\prime}\right\|+\left\|\frac{1}{NT}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\sum_{t=1}^{T}\mathbf{\Gamma}_\ell^0\Big(\mathbf{V}_{\ell t}\mathbf{V}_{ht}'-E(\mathbf{V}_{\ell t}\mathbf{V}_{ht}')\Big)\mathbf{\Gamma}_h^{0\prime}\right\| \\
\leq&\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\|\mathbf{\Gamma}_\ell^0\|\|\mathbf{\Gamma}_h^0\|\bar{\sigma}_{\ell h}+\frac{1}{\sqrt{T}}\left\|\frac{1}{N\sqrt{T}}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\sum_{t=1}^{T}\mathbf{\Gamma}_\ell^0\Big(\mathbf{V}_{\ell t}\mathbf{V}_{ht}'-E(\mathbf{V}_{\ell t}\mathbf{V}_{ht}')\Big)\mathbf{\Gamma}_h^{0\prime}\right\|=O_p(1)
\end{aligned}
$$

since

$$
\mathbb{E}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\|\mathbf{\Gamma}_\ell^0\|\|\mathbf{\Gamma}_h^0\|\bar{\sigma}_{\ell h}\right)\leq\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\left(\mathbb{E}\|\mathbf{\Gamma}_\ell^0\|^2\mathbb{E}\|\mathbf{\Gamma}_h^0\|^2\right)^{1/2}\bar{\sigma}_{\ell h}\leq\frac{C}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\bar{\sigma}_{\ell h}\leq C^2
$$

by Assumption B3.

Lemma B.4 *Under Assumptions A to D, we have*

$$
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i+O_p\left(\frac{1}{\delta_{NT}}\right)+O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)
$$

Proof of Lemma B.4. Since $\mathbf{M}_{\mathbf{H}^0}\mathbf{u}_i=\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i$ and

$$
\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}-\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{H}^0}=\mathbf{M}_{\mathbf{F}^0}\left(\mathbf{M}_{\widehat{\mathbf{H}}}-\mathbf{M}_{\mathbf{H}^0}\right)+\left(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\right)\mathbf{M}_{\mathbf{H}^0}+\left(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\right)\left(\mathbf{M}_{\widehat{\mathbf{H}}}-\mathbf{M}_{\mathbf{H}^0}\right)
$$

we have

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\
=&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\mathbf{M}_{\mathbf{F}^0}\left(\mathbf{M}_{\widehat{\mathbf{H}}}-\mathbf{M}_{\mathbf{H}^0}\right)\mathbf{u}_i+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\left(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\right)\mathbf{M}_{\mathbf{H}^0}\mathbf{u}_i \\
&+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\left(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\right)\left(\mathbf{M}_{\widehat{\mathbf{H}}}-\mathbf{M}_{\mathbf{H}^0}\right)\mathbf{u}_i \\
=&\mathbb{H}_1+\mathbb{H}_2+\mathbb{H}_3
\end{aligned}
$$

Now we consider the term $\mathbb{H}_1$. Since $\mathbf{M}_{\widehat{\mathbf{H}}} - \mathbf{M}_{\mathbf{H}^0} = -T^{-1}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime} - T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})' - T^{-1}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})' - T^{-1}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right)\mathbf{H}'^0$, we have

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^N\sum_{j=1}^N w_{ij}\mathbf{V}_j'\mathbf{M}_{\mathbf{F}^0}\left(\mathbf{M}_{\widehat{\mathbf{H}}} - \mathbf{M}_{\mathbf{H}^0}\right)\mathbf{u}_i \\
=&-\frac{1}{\sqrt{NT}}\sum_{i=1}^N\sum_{j=1}^N w_{ij}\mathbf{V}_j'\frac{1}{T}(\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1} - \mathbf{H}^0)\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i \\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^N\sum_{j=1}^N w_{ij}\mathbf{V}_j'\frac{1}{T}(\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1} - \mathbf{H}^0)\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{u}_i \\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^N\sum_{j=1}^N w_{ij}\mathbf{V}_j'\frac{1}{T}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i - \frac{1}{\sqrt{NT}}\sum_{i=1}^N\sum_{j=1}^N w_{ij}\mathbf{V}_j'\frac{1}{T}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i \\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^N\sum_{j=1}^N w_{ij}\mathbf{V}_j'\frac{1}{T}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{u}_i \\
&+\frac{1}{\sqrt{NT}}\sum_{i=1}^N\sum_{j=1}^N w_{ij}\mathbf{V}_j'\mathbf{F}^0\left(\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\mathbf{F}^{0\prime}\frac{1}{T}(\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1} - \mathbf{H}^0)\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i \\
&+\frac{1}{\sqrt{NT}}\sum_{i=1}^N\sum_{j=1}^N w_{ij}\mathbf{V}_j'\mathbf{F}^0\left(\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\mathbf{F}^{0\prime}\frac{1}{T}(\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1} - \mathbf{H}^0)\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{u}_i \\
&+\frac{1}{\sqrt{NT}}\sum_{i=1}^N\sum_{j=1}^N w_{ij}\mathbf{V}_j'\mathbf{F}^0\left(\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\mathbf{F}^{0\prime}\frac{1}{T}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i \\
&+\frac{1}{\sqrt{NT}}\sum_{i=1}^N\sum_{j=1}^N w_{ij}\mathbf{V}_j'\mathbf{F}^0\left(\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\mathbf{F}^{0\prime}\frac{1}{T}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i \\
&+\frac{1}{\sqrt{NT}}\sum_{i=1}^N\sum_{j=1}^N w_{ij}\mathbf{V}_j'\mathbf{F}^0\left(\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\mathbf{F}^{0\prime}\frac{1}{T}\mathbf{H}\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{u}_i \\
=&\mathbb{H}_{1.1} + \mathbb{H}_{1.2} + \mathbb{H}_{1.3} + \mathbb{H}_{1.4} + \mathbb{H}_{1.5} + \mathbb{H}_{1.6} + \mathbb{H}_{1.7} + \mathbb{H}_{1.8} + \mathbb{H}_{1.9}
\end{aligned}
$$

We first consider the terms $\mathbb{H}_{1.2}$ to $\mathbb{H}_{1.9}$. Note that $\mathbf{u}_i = \mathbf{H}^0\boldsymbol{\varphi}_i^0 + \boldsymbol{\varepsilon}_i$. For $\mathbb{H}_{1.2}$, we have

$$
\begin{aligned}
\|\mathbb{H}_{1.2}\| =& \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^N\sum_{j=1}^N w_{ij}\mathbf{V}_j'\frac{1}{T}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}^{-1}\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{u}_i\right\| \\
\le&\sqrt{NT}\times\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N|w_{ij}|\left\|\frac{1}{T}\mathbf{V}_j'(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})\right\|\|\boldsymbol{\varphi}_i^0\|\left\|\frac{1}{T}\mathbf{H}^{0\prime}\mathbf{H}^0\right\|\left\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\|\|\boldsymbol{\mathcal{R}}^{-1}\| \\
&+\sqrt{N}\times\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N|w_{ij}|\left\|\frac{1}{T}\mathbf{V}_j'(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})\right\|\left\|\frac{1}{\sqrt{T}}\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i\right\|\left\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\|\|\boldsymbol{\mathcal{R}}^{-1}\| \\
=&O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)
\end{aligned}
$$

by Lemma A.1 (f) and Lemmas A.9 (g), (h).

For $\mathbb{H}_{1.3}$, we have

$$\begin{aligned}\|\mathbb{H}_{1.3}\| =& \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j' \frac{1}{T}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\left(\mathbf{H}^0\boldsymbol{\varphi}_i^0+\boldsymbol{\varepsilon}_i\right)\right\| \\ \leq& \sqrt{N}\times\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{H}^0\right\|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{1}{T}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}\right\|\|\boldsymbol{\mathcal{R}}\| \\ &+\sqrt{N}\times\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{H}^0\right\|\left\|\frac{1}{T}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i\right\|\|\boldsymbol{\mathcal{R}}\| \\ =& O_p\left(\frac{\sqrt{N}}{\delta_{NT}^2}\right)\end{aligned}$$

by Lemmas A.1 (b), (d) and (e) and Lemmas A.9 (b), (c) and (f).

For $\mathbb{H}_{1.4}$, we have

$$\begin{aligned}\|\mathbb{H}_{1.4}\| \leq& \sqrt{NT}\times\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{T}\mathbf{V}_j'(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\right\|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{1}{T}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}\right\| \\ &+\sqrt{NT}\times\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{T}\mathbf{V}_j'(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\right\|\left\|\frac{1}{T}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i\right\| \\ =& O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)\end{aligned}$$

by Lemmas A.1 (b), (d) and Lemmas A.9 (g), (h) and (i).

For $\mathbb{H}_{1.5}$, we have

$$\begin{aligned}\|\mathbb{H}_{1.5}\| =& \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{T}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right)\mathbf{H}^{0\prime}\left(\mathbf{H}^0\boldsymbol{\varphi}_i^0+\boldsymbol{\varepsilon}_i\right)\right\| \\ \leq& \sqrt{N}\times\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{H}^0\right\|\left\|\boldsymbol{\varphi}_i^0\right\|\left\|\frac{1}{T}\mathbf{H}^{0\prime}\mathbf{H}^0\right\|\left\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\| \\ &+\sqrt{\frac{N}{T}}\times\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{T}}\mathbf{V}_j'\mathbf{H}^0\right\|\left\|\frac{1}{\sqrt{T}}\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i\right\|\left\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\| \\ =& O_p\left(\frac{\sqrt{N}}{\delta_{NT}^2}\right)\end{aligned}$$

by Lemma A.1 (f) and Lemmas A.9 (b), (c) and (d).

For $\mathbb{H}_{1.6}$, we have

$$\begin{aligned}\|\mathbb{H}_{1.6}\| =& \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\mathbf{F}^0\left(\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\mathbf{F}^{0\prime}\frac{1}{T}(\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1}-\mathbf{H}^0)\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\left(\mathbf{H}^0\boldsymbol{\varphi}_i^0+\boldsymbol{\varepsilon}_i\right)\right\| \\ \leq& \sqrt{N}\times\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\boldsymbol{\varphi}_i^0\right\|\times\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}(\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1}-\mathbf{H}^0)}{T}\right\| \\ &+\sqrt{\frac{N}{T}}\times\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\times\left\|\frac{\mathbf{F}^{0\prime}(\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1}-\mathbf{H}^0)}{T}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\| \\ =& O_p\left(\frac{\sqrt{N}}{\delta_{NT}^2}\right)\end{aligned}$$

by Lemma B.1 (b), Lemmas A.9 (b)-(c) and Lemma B.3 (c). Similarly, we can show that $\mathbb{H}_{1.7} = O_p\left(\sqrt{N}\delta_{NT}^{-4}\right)$ and $\mathbb{H}_{1.10} = O_p\left(\sqrt{N}\delta_{NT}^{-2}\right)$.

For the term $\mathbb{H}_{1.8}$, we have

$$\begin{aligned}
\|\mathbb{H}_{1.8}\| =& \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j'\mathbf{F}^0\left(\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\mathbf{F}^{0\prime}\frac{1}{T}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\left(\mathbf{H}^0\boldsymbol{\varphi}_i^0+\boldsymbol{\varepsilon}_i\right)\right\| \\
\leq&\sqrt{N}\times\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\boldsymbol{\varphi}_i^0\right\|\times\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\mathcal{R}}\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0}{T}\right\| \\
&+\sqrt{N}\times\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\frac{1}{T}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i\right\|\times\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\mathcal{R}}\| \\
=&O_p\left(\frac{\sqrt{N}}{\delta_{NT}^2}\right)
\end{aligned}$$

by Lemmas A.9 (b) and (c), Lemma B.3 (e). Similarly, we can show that $\mathbb{H}_{1.9} = O_p\left(\sqrt{N}\delta_{NT}^{-4}\right)$.

Now we consider the term $\mathbb{H}_{1.1}$. Since

$$\begin{aligned}
&\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1}-\mathbf{H}^0 \\
=&\frac{1}{NT}\sum_{i=1}^{N}\mathbf{C}_i\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)'\mathbf{C}_i'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}+\frac{1}{NT}\sum_{i=1}^{N}\mathbf{C}_i\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)\mathbf{u}_i'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1} \\
&+\frac{1}{NT}\sum_{i=1}^{N}\mathbf{u}_i\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)'\mathbf{C}_i'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}+\frac{1}{NT}\sum_{i=1}^{N}\mathbf{H}^0\boldsymbol{\varphi}_i^0\boldsymbol{\varepsilon}_i'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1} \\
&+\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i\boldsymbol{\varphi}_i^{0\prime}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}+\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}
\end{aligned}$$

we can decompose the term $\mathbb{H}_{1.1}$ as follows

$$\begin{aligned}
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{T}(\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1}-\mathbf{H}^0)\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i \\
=&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{NT}\sum_{h=1}^{N}\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i \\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{NT^2}\sum_{h=1}^{N}\mathbf{H}^0\boldsymbol{\varphi}_h^0\boldsymbol{\varepsilon}_h'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i \\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{NT^2}\sum_{h=1}^{N}\boldsymbol{\varepsilon}_h\boldsymbol{\varepsilon}_h'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i \\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{NT^2}\sum_{h=1}^{N}\mathbf{C}_h\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)'\mathbf{C}_h'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i \\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{NT^2}\sum_{h=1}^{N}\mathbf{C}_h\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)\mathbf{u}_h'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i \\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{NT^2}\sum_{h=1}^{N}\mathbf{u}_h\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)'\mathbf{C}_h'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i \\
=&\mathbb{H}_{1.1.1}+\mathbb{H}_{1.1.2}+\mathbb{H}_{1.1.3}+\mathbb{H}_{1.1.4}+\mathbb{H}_{1.1.5}+\mathbb{H}_{1.1.6}
\end{aligned}$$

We consider the last five terms $\mathbb{H}_{1.1.2}$ to $\mathbb{H}_{1.1.6}$. For $\mathbb{H}_{1.1.2}$, we can derive that

$$\begin{aligned}
\|\mathbb{H}_{1.1.2}\| =& \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij} \mathbf{V}_j' \frac{1}{NT^2} \sum_{h=1}^{N} \mathbf{H}^0 \boldsymbol{\varphi}_h^0 \boldsymbol{\varepsilon}_h' \widehat{\mathbf{H}} \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}_{\varphi}^0)^{-1} \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \mathbf{H}^{0\prime}\mathbf{u}_i \right\| \\
\leq & \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij} \mathbf{V}_j' \frac{1}{NT^2} \sum_{h=1}^{N} \mathbf{H}^0 \boldsymbol{\varphi}_h^0 \boldsymbol{\varepsilon}_h' \mathbf{H}^0 \boldsymbol{\mathcal{R}} \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}_{\varphi}^0)^{-1} \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \mathbf{H}^{0\prime}\mathbf{u}_i \right\| \\
&+ \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij} \mathbf{V}_j' \frac{1}{NT^2} \sum_{h=1}^{N} \mathbf{H}^0 \boldsymbol{\varphi}_h^0 \boldsymbol{\varepsilon}_h' \left(\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}} \right) \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}_{\varphi}^0)^{-1} \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \mathbf{H}^{0\prime}\mathbf{u}_i \right\| \\
\leq & \frac{1}{\sqrt{T}} \frac{1}{N} \sum_{i=1}^{N}\sum_{j=1}^{N} |w_{ij}| \left\| \frac{\mathbf{V}_j'\mathbf{H}^0}{\sqrt{T}} \right\| \|\boldsymbol{\varphi}_i^0\| \left\| \frac{1}{\sqrt{NT}} \sum_{h=1}^{N} \boldsymbol{\varphi}_h^0 \boldsymbol{\varepsilon}_h' \mathbf{H}^0 \right\| \left\| \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T} \right)^{-1} \right\| \left\| \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right\| \|(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\| \|\boldsymbol{\mathcal{R}}\| \\
&+ \frac{1}{T} \frac{1}{N} \sum_{i=1}^{N}\sum_{j=1}^{N} |w_{ij}| \left\| \frac{\mathbf{V}_j'\mathbf{H}^0}{\sqrt{T}} \right\| \left\| \frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}} \right\| \left\| \frac{1}{\sqrt{NT}} \sum_{h=1}^{N} \boldsymbol{\varphi}_h^0 \boldsymbol{\varepsilon}_h' \mathbf{H}^0 \right\| \left\| \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T} \right)^{-1} \right\| \left\| \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \right\| \|(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\| \|\boldsymbol{\mathcal{R}}\| \\
&+ \frac{1}{N} \sum_{i=1}^{N}\sum_{j=1}^{N} |w_{ij}| \left\| \frac{\mathbf{V}_j'\mathbf{H}^0}{\sqrt{T}} \right\| \|\boldsymbol{\varphi}_i^0\| \left\| \frac{1}{\sqrt{NT}} \sum_{h=1}^{N} \boldsymbol{\varphi}_h^0 \boldsymbol{\varepsilon}_h' \right\| \left\| \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T} \right)^{-1} \right\| \left\| \frac{\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}}{\sqrt{T}} \right\| \|(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\| \\
&+ \frac{1}{\sqrt{T}} \frac{1}{N} \sum_{i=1}^{N}\sum_{j=1}^{N} |w_{ij}| \left\| \frac{\mathbf{V}_j'\mathbf{H}^0}{\sqrt{T}} \right\| \left\| \frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}} \right\| \left\| \frac{1}{\sqrt{NT}} \sum_{h=1}^{N} \boldsymbol{\varphi}_h^0 \boldsymbol{\varepsilon}_h' \right\| \left\| \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T} \right)^{-1} \right\| \left\| \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \right\| \left\| \frac{\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}}{\sqrt{T}} \right\| \|(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\| \\
=& O_p\left(\frac{1}{\delta_{NT}} \right)
\end{aligned}$$

by Lemmas A.1 (a), (b), (c) and (d).

Consider the term $\mathbb{H}_{1.1.3}$. With the fact that $\mathbb{E}\left(\mathbf{V}_j'\boldsymbol{\varepsilon}_h\right) = 0$, following the way of the proof of the term $\mathbb{D}_{1.4}$, we can show that $\mathbb{H}_{1.1.3} = O_p\left(\delta_{NT}^{-1}\right)$.

For the term $\mathbb{H}_{1.1.4}$, we have

$$\begin{aligned}
\|\mathbb{H}_{1.1.4}\| \leq & \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij} \mathbf{V}_j' \frac{1}{NT} \sum_{h=1}^{N} \mathbf{C}_h \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\right) \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\right)' \mathbf{C}_h' \widehat{\mathbf{H}} \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}_{\varphi}^0)^{-1} \boldsymbol{\varphi}_i^0 \right\| \\
&+ \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij} \mathbf{V}_j' \frac{1}{NT^2} \sum_{h=1}^{N} \mathbf{C}_h \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\right) \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\right)' \mathbf{C}_h' \widehat{\mathbf{H}} \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}_{\varphi}^0)^{-1} \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i \right\| \\
\leq & \sqrt{NT} \times \frac{1}{N} \sum_{i=1}^{N}\sum_{j=1}^{N} |w_{ij}| \left\| \frac{\mathbf{V}_j}{\sqrt{T}} \right\| \|\boldsymbol{\varphi}_i^0\| \times \frac{1}{N} \sum_{h=1}^{N} \left\| \frac{\mathbf{C}_h}{\sqrt{T}} \right\|^2 \left\| \boldsymbol{\theta} - \hat{\boldsymbol{\theta}} \right\|^2 \|(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\| \left\| \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T} \right)^{-1} \right\| \left\| \frac{\widehat{\mathbf{H}}}{\sqrt{T}} \right\| \\
&+ \sqrt{N} \times \frac{1}{N} \sum_{i=1}^{N}\sum_{j=1}^{N} |w_{ij}| \left\| \frac{\mathbf{V}_j}{\sqrt{T}} \right\| \left\| \frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}} \right\| \times \frac{1}{N} \sum_{h=1}^{N} \left\| \frac{\mathbf{C}_h}{\sqrt{T}} \right\|^2 \left\| \boldsymbol{\theta} - \hat{\boldsymbol{\theta}} \right\|^2 \|(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\| \left\| \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T} \right)^{-1} \right\|^2 \left\| \frac{\widehat{\mathbf{H}}}{\sqrt{T}} \right\| \\
=& O_p\left(\frac{1}{\sqrt{NT}} \right)
\end{aligned}$$

For the term $\mathbb{H}_{1.1.5}$, we have

$$\begin{aligned}
&\left\|-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{NT^2}\sum_{h=1}^{N}\mathbf{C}_h\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)\mathbf{u}_h'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^{0})^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^{0}}{T}\right)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i\right\| \\
\leq&\left\|\mathrm{vec}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{N}\sum_{h=1}^{N}\mathbf{C}_h\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_{\varphi}^{0})^{-1}\boldsymbol{\varphi}_i^{0}\right)\right\| \\
&+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{NT}\sum_{h=1}^{N}\mathbf{C}_h\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_{\varphi}^{0})^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^{0}}{T}\right)^{-1}\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i\right\| \\
&+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{NT}\sum_{h=1}^{N}\mathbf{C}_h\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)\boldsymbol{\varepsilon}_h'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^{0})^{-1}\boldsymbol{\varphi}_i^{0}\right\| \\
&+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\frac{1}{NT^2}\sum_{h=1}^{N}\mathbf{C}_h\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)\boldsymbol{\varepsilon}_h'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^{0})^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^{0}}{T}\right)^{-1}\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i\right\| \\
\leq&\left\|\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\varphi}_i^{0\prime}\otimes\mathbf{V}_j'\right)\mathrm{vec}\left(\frac{1}{N}\sum_{h=1}^{N}\mathbf{C}_h\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_{\varphi}^{0})^{-1}\right)\right\| \\
&+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\times\frac{1}{N}\sum_{h=1}^{N}\left\|\frac{\mathbf{C}_h}{\sqrt{T}}\right\|\|\boldsymbol{\varphi}_h^{0}\|\times\|(\boldsymbol{\Upsilon}_{\varphi}^{0})^{-1}\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^{0}}{T}\right)^{-1}\right\|\left\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right\| \\
&+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\|\boldsymbol{\varphi}_i^{0}\|\times\|(\boldsymbol{\Upsilon}_{\varphi}^{0})^{-1}\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^{0}}{T}\right)^{-1}\right\|\left\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right\| \\
&\quad\times\left(\frac{1}{N}\sum_{h=1}^{N}\left\|\frac{\mathbf{C}_h}{\sqrt{T}}\right\|\left\|\frac{\boldsymbol{\varepsilon}_h'\mathbf{H}^{0}}{\sqrt{T}}\right\|\|\mathcal{R}\|+\sqrt{T}\times\frac{1}{N}\sum_{h=1}^{N}\left\|\frac{\mathbf{C}_h}{\sqrt{T}}\right\|\left\|\frac{\boldsymbol{\varepsilon}_h}{\sqrt{T}}\right\|\left\|\frac{\widehat{\mathbf{H}}-\mathbf{H}^{0}\mathcal{R}}{\sqrt{T}}\right\|\right) \\
&+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j}{\sqrt{T}}\right\|\left\|\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\times\frac{1}{N}\sum_{h=1}^{N}\left\|\frac{\mathbf{C}_h}{\sqrt{T}}\right\|\left\|\frac{\boldsymbol{\varepsilon}_h}{\sqrt{T}}\right\| \\
&\quad\times\left\|\frac{\widehat{\mathbf{H}}}{\sqrt{T}}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1}\right\|\|(\boldsymbol{\Upsilon}_{\varphi}^{0})^{-1}\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^{0}}{T}\right)^{-1}\right\|\left\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right\| \\
\leq&\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\varphi}_i^{0\prime}\otimes\mathbf{V}_j'\right\|\frac{1}{N}\sum_{h=1}^{N}\left\|\frac{\mathbf{C}_h}{\sqrt{T}}\right\|\|\boldsymbol{\varphi}_h^{0}\|\,\|(\boldsymbol{\Upsilon}_{\varphi}^{0})^{-1}\|\times\sqrt{T}\left\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right\|+O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right)+O_p\left(\frac{\sqrt{N}}{\delta_{NT}^2}\right) \\
=&O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right)+O_p\left(\frac{\sqrt{N}}{\delta_{NT}^2}\right)
\end{aligned}$$

For the term $\mathbb{H}_{1.1.6}$, we have

$$
\begin{aligned}
&\left\| -\frac{1}{\sqrt{NT}} \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j' \frac{1}{NT^2}\sum_{h=1}^{N} \mathbf{u}_h \left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)' \mathbf{C}_h'\widehat{\mathbf{H}} \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1} (\boldsymbol{\Upsilon}_{\varphi}^0)^{-1} \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1} \mathbf{H}^{0\prime}\mathbf{u}_i \right\| \\
\leq & \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j' \frac{1}{NT}\sum_{h=1}^{N} \mathbf{H}^0\boldsymbol{\varphi}_h^0 \left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)' \mathbf{C}_h'\widehat{\mathbf{H}} \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1} (\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\boldsymbol{\varphi}_i^0 \right\| \\
&+ \left\| \frac{1}{NT}\sum_{h=1}^{N} \frac{1}{\sqrt{NT}} \sum_{i=1}^{N}\sum_{j=1}^{N} \operatorname{vec}\left((w_{ij}\mathbf{V}_j'\boldsymbol{\varepsilon}_h) \left(\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)' \mathbf{C}_h'\widehat{\mathbf{H}} \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1} (\boldsymbol{\Upsilon}_{\varphi}^0)^{-1} \right) \boldsymbol{\varphi}_i^0 \right) \right\| \\
&+ \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j' \frac{1}{NT^2}\sum_{h=1}^{N} \mathbf{H}^0\boldsymbol{\varphi}_h^0 \left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)' \mathbf{C}_h'\widehat{\mathbf{H}} \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1} (\boldsymbol{\Upsilon}_{\varphi}^0)^{-1} \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1} \mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i \right\| \\
&+ \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j' \frac{1}{NT^2}\sum_{h=1}^{N} \boldsymbol{\varepsilon}_h \left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)' \mathbf{C}_h'\widehat{\mathbf{H}} \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1} (\boldsymbol{\Upsilon}_{\varphi}^0)^{-1} \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1} \mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i \right\| \\
\leq & \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} |w_{ij}| \left\|\frac{\mathbf{V}_j\mathbf{H}}{\sqrt{T}}\right\| \|\boldsymbol{\varphi}_i^0\| \frac{1}{N}\sum_{h=1}^{N} \|\boldsymbol{\varphi}_h^0\| \left\|\frac{\mathbf{C}_h}{\sqrt{T}}\right\| \left\|\frac{\widehat{\mathbf{H}}}{\sqrt{T}}\right\| \left\| \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1} \right\| \|(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\| \times \sqrt{N} \left\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right\| \\
&+ \left\| \sqrt{N}\frac{1}{N}\sum_{h=1}^{N} \left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\varphi}_i^{0\prime} \otimes \frac{\mathbf{V}_j'\boldsymbol{\varepsilon}_h}{\sqrt{T}} \right) \operatorname{vec}\left(\left(\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right)' \frac{\mathbf{C}_h'\widehat{\mathbf{H}}}{T} \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1} (\boldsymbol{\Upsilon}_{\varphi}^0)^{-1} \right) \right\| \\
&+ \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} |w_{ij}| \left\|\frac{\mathbf{V}_j\mathbf{H}}{\sqrt{T}}\right\| \left\|\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\| \frac{1}{N}\sum_{h=1}^{N} \|\boldsymbol{\varphi}_h^0\| \left\|\frac{\mathbf{C}_h}{\sqrt{T}}\right\| \left\|\frac{\widehat{\mathbf{H}}}{\sqrt{T}}\right\| \left\| \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1} \right\| \|(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\| \left\| \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1} \right\| \\
&\times \sqrt{\frac{N}{T}} \left\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right\| \\
&+ \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} |w_{ij}| \left\|\frac{\mathbf{V}_j\mathbf{H}}{\sqrt{T}}\right\| \left\|\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\| \frac{1}{N}\sum_{h=1}^{N} \left\|\frac{\boldsymbol{\varepsilon}_h}{\sqrt{T}}\right\| \left\|\frac{\mathbf{C}_h}{\sqrt{T}}\right\| \left\|\frac{\widehat{\mathbf{H}}}{\sqrt{T}}\right\| \left\| \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1} \right\| \|(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\| \left\| \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1} \right\| \\
&\times \sqrt{N} \left\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right\| \\
\leq & O_p\left(\frac{\sqrt{N}}{\delta_{NT}^2}\right) + \frac{1}{N}\sum_{h=1}^{N} \left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} |w_{ij}| \|\boldsymbol{\varphi}_i^0\| \left\|\frac{\mathbf{V}_j'\boldsymbol{\varepsilon}_h}{\sqrt{T}}\right\| \right) \left\|\frac{\mathbf{C}_h}{\sqrt{T}}\right\| \times \left\|\frac{\widehat{\mathbf{H}}}{\sqrt{T}}\right\| \left\| \left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1} \right\| \|(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\| \times \sqrt{N} \left\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right\| \\
= & O_p\left(\frac{\sqrt{N}}{\delta_{NT}^2}\right)
\end{aligned}
$$

Combining the above terms, we can show that

$$
\mathbb{H}_1 = -\frac{1}{\sqrt{NT}} \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j' \frac{1}{NT}\sum_{h=1}^{N} \boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1} \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1} \mathbf{H}^{0\prime}\mathbf{u}_i + O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)
$$

Now we consider the term $\mathbb{H}_2$. Since $\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0} = -T^{-1}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime} - T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' - T^{-1}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' - T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}' - \left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}$, we have

$$
\mathbb{H}_2 = \frac{1}{\sqrt{NT}} \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j' \left(\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0}\right)\mathbf{M}_{\mathbf{H}^0}\mathbf{u}_i \tag{B.2}
$$

$$
= -\frac{1}{\sqrt{NT}} \sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j' T^{-1}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{M}_{\mathbf{H}^0}\mathbf{u}_i \tag{B.3}
$$

$$-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j' T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{M}_{\mathbf{H}^0}\mathbf{u}_i \tag{B.4}$$

$$-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j' T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{M}_{\mathbf{H}^0}\mathbf{u}_i \tag{B.5}$$

$$-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j' T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}\mathbf{M}_{\mathbf{H}^0}\mathbf{u}_i \tag{B.6}$$

$$= \mathbb{H}_{2.1}+\mathbb{H}_{2.2}+\mathbb{H}_{2.3}+\mathbb{H}_{2.4}$$

$$\|\mathbb{H}_{2.1}\| = \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j' T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{M}_{\mathbf{H}^0}\varepsilon_i\right\| \tag{B.7}$$

$$\leq \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j' T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}\varepsilon_i\right\| \tag{B.8}$$

$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j' T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{H}^0\left(\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\mathbf{H}^{0\prime}\varepsilon_i\right\| \tag{B.9}$$

$$\leq \sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{F}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\|\left\|T^{-1}\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\right\|\|\mathbf{R}'\| \tag{B.10}$$

$$+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{H}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\|\left\|T^{-1}\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\right\|\|\mathbf{R}'\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\| \tag{B.11}$$

$$= O_p\left(\frac{\sqrt{N}}{\delta_{NT}^2}\right)$$

by Lemma B.3 (e). In similar manner,

$$\|\mathbb{H}_{2.2}\| = \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j' T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{M}_{\mathbf{H}^0}\mathbf{u}_i\right\| \tag{B.12}$$

$$\leq \sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\varepsilon_i}{T}\right\|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\|\mathbf{R}\| \tag{B.13}$$

$$+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{H}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\|\left\|T^{-1}\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\right\|\|\mathbf{R}'\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\| \tag{B.14}$$

$$= O_p\left(\frac{\sqrt{N}}{\delta_{NT}^2}\right).$$

$$\|\mathbb{H}_{2.3}\| = \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j' T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{M}_{\mathbf{H}^0}\mathbf{u}_i\right\| \tag{B.15}$$

$$\leq \sqrt{NT}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|T^{-1}\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\right\|\left\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\varepsilon_i\right\| \tag{B.16}$$

$$+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{T}\right\|\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{H}}{T}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{H}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\| \tag{B.17}$$

$$= O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)$$

by Lemma B.1 (b) and Lemma B.3 (e) (g).

$$
\begin{aligned}
\|\mathbb{H}_{2.4}\| &= \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} \mathbf{F}^0 \left(\mathbf{R}\mathbf{R}' - \left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1} \right) \mathbf{F}^{0\prime} \mathbf{M}_{\mathbf{H}^0} \mathbf{u}_i \right\| && \text{(B.18)} \\
&\le \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} \mathbf{F}^0 \left(\mathbf{R}\mathbf{R}' - \left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1} \right) \mathbf{F}^{0\prime} \varepsilon_i \right\| && \text{(B.19)} \\
&\quad + \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} \mathbf{F}^0 \left(\mathbf{R}\mathbf{R}' - \left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1} \right) \mathbf{F}^{0\prime} \mathbf{H}^0 \left(\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1} \mathbf{H}^{0\prime} \varepsilon_i \right\| && \text{(B.20)} \\
&\le \sqrt{\frac{N}{T}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |w_{ij}| \left\| \frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}} \right\| \left\| \mathbf{R}\mathbf{R}' - \left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1} \right\| \left\| \frac{\mathbf{F}^{0\prime}\varepsilon_i}{\sqrt{T}} \right\| && \text{(B.21)} \\
&\quad + \sqrt{\frac{N}{T}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |w_{ij}| \left\| \frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}} \right\| \left\| \mathbf{R}\mathbf{R}' - \left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1} \right\| \left\| \frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T} \right\| \left\| \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1} \right\| \left\| \frac{\mathbf{H}^{0\prime}\varepsilon_i}{\sqrt{T}} \right\| && \text{(B.22)} \\
&= O_p\left(\sqrt{\frac{N}{T}} \frac{1}{\delta_{NT}^2} \right)
\end{aligned}
$$

by Lemma B.1 (f) and B.3 (c). To sum up, $\|\mathbb{H}_2\| = O_p\left(\frac{\sqrt{N}}{\delta_{NT}^2}\right) + O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)$.

Now we consider the term $\mathbb{H}_3$. Using $\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0} = -T^{-1}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime} - T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}} - \mathbf{F}\mathbf{R})^{0\prime} - T^{-1}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' - T^{-1}\mathbf{F}^0 \left(\mathbf{R}\mathbf{R}' - \left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1} \right) \mathbf{F}^{0\prime}$ and $\mathbf{M}_{\widehat{\mathbf{H}}} - \mathbf{M}_{\mathbf{H}^0} = -T^{-1}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime} - T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})' - T^{-1}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})' - T^{-1}\mathbf{H}^0 \left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1} \right)$ we have

$$
\begin{aligned}
\mathbb{H}_3 &= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' \left(\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0} \right) \left(\mathbf{M}_{\widehat{\mathbf{H}}} - \mathbf{M}_{\mathbf{H}^0} \right) \mathbf{u}_i && \text{(B.23)} \\
&= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} (\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}) \mathbf{R}'\mathbf{F}^{0\prime} T^{-1} (\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}) \boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime} \mathbf{u}_i && \text{(B.24)} \\
&\quad + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} (\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}) \mathbf{R}'\mathbf{F}^{0\prime} T^{-1} \mathbf{H}^0\boldsymbol{\mathcal{R}} (\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})' \mathbf{u}_i && \text{(B.25)} \\
&\quad + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} (\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}) \mathbf{R}'\mathbf{F}^{0\prime} T^{-1} (\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}) (\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})' \mathbf{u}_i && \text{(B.26)} \\
&\quad + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} (\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}) \mathbf{R}'\mathbf{F}^{0\prime} T^{-1} \mathbf{H}^0 \left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}\right)^{-1} \right) \mathbf{H}^{0\prime} \mathbf{u}_i \\
&\quad + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} \mathbf{F}^0\mathbf{R} (\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' T^{-1} (\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}) \boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime} \mathbf{u}_i && \text{(B.27)} \\
&\quad + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} \mathbf{F}^0\mathbf{R} (\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' T^{-1} \mathbf{H}^0\boldsymbol{\mathcal{R}} (\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})' \mathbf{u}_i && \text{(B.28)} \\
&\quad + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} \mathbf{F}^0\mathbf{R} (\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' T^{-1} (\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}) (\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})' \mathbf{u}_i && \text{(B.29)} \\
&\quad + \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} \mathbf{F}^0\mathbf{R} (\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' T^{-1} \mathbf{H}^0 \left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}\right)^{-1} \right) \mathbf{H}^{0\prime} \mathbf{u}_i
\end{aligned}
$$

$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}'\mathbf{u}_i \tag{B.30}$$

$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i \tag{B.31}$$

$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i \tag{B.32}$$

$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{u}_i$$

$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\mathbf{u}_i \tag{B.33}$$

$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i \tag{B.34}$$

$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i \tag{B.35}$$

$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{u}_i$$

$$= \mathbb{H}_{3.1.1}+\mathbb{H}_{3.1.2}+\mathbb{H}_{3.1.3}+\mathbb{H}_{3.1.4}+ \tag{B.36}$$

$$\mathbb{H}_{3.2.1}+\mathbb{H}_{3.2.2}+\mathbb{H}_{3.2.3}+\mathbb{H}_{3.2.4}+ \tag{B.37}$$

$$\mathbb{H}_{3.3.1}+\mathbb{H}_{3.3.2}+\mathbb{H}_{3.3.3}+\mathbb{H}_{3.3.4}+ \tag{B.38}$$

$$\mathbb{H}_{3.4.1}+\mathbb{H}_{3.4.2}+\mathbb{H}_{3.4.3}+\mathbb{H}_{3.4.4}.$$

$$\|\mathbb{H}_{3.1.1}\| = \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\mathbf{u}_i\right\| \tag{B.39}$$

$$\le \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0\right\| \tag{B.40}$$

$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i\right\|$$

$$\le \sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|T^{-1}\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\right\|\|\mathbf{R}'\|\left\|T^{-1}\mathbf{F}^{0\prime}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\right\|\|\boldsymbol{\mathcal{R}}'\|\left\|\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i^0\| \tag{B.41}$$

$$+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|T^{-1}\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\right\|\|\mathbf{R}'\|\left\|T^{-1}\mathbf{F}^{0\prime}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\right\|\left\|\frac{\mathbf{H}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\| \tag{B.42}$$

$$= O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)$$

by Lemma B.1 (b) and Lemma B.3 (e) (f).

$$\|\mathbb{H}_{3.1.2}\| = \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i\right\| \tag{B.43}$$

$$\leq \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} (\widehat{\mathbf{F}} - \mathbf{F}^0 \mathbf{R}) \mathbf{R}' \mathbf{F}^{0\prime} T^{-1} \mathbf{H}^0 \boldsymbol{\mathcal{R}} (\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}})' \mathbf{H} \varphi_i \right\| \tag{B.44}$$

$$+ \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} (\widehat{\mathbf{F}} - \mathbf{F}^0 \mathbf{R}) \mathbf{R}' \mathbf{F}^{0\prime} T^{-1} \mathbf{H}^0 \boldsymbol{\mathcal{R}} (\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}})' \varepsilon_i \right\|$$

$$\leq \sqrt{NT} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |w_{ij}| \left\| \frac{\mathbf{V}_j' (\widehat{\mathbf{F}} - \mathbf{F}^0 \mathbf{R})}{T} \right\| \left\| \frac{\mathbf{F}^{0\prime} \mathbf{H}^0}{T} \right\| \|\boldsymbol{\mathcal{R}}\|^2 \left\| \frac{(\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}})' \mathbf{H}^0}{T} \right\| \|\boldsymbol{\varphi}_i\| \tag{B.45}$$

$$+ \sqrt{NT} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |w_{ij}| \left\| \frac{\mathbf{V}_j' (\widehat{\mathbf{F}} - \mathbf{F}^0 \mathbf{R})}{T} \right\| \left\| \frac{\mathbf{F}^{0\prime} \mathbf{H}^0}{T} \right\| \|\boldsymbol{\mathcal{R}}\|^2 \left\| \frac{(\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}})' \varepsilon_i}{T} \right\| \tag{B.46}$$

$$= O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3} \right)$$

by Lemma B.1 (b) and Lemma B.3 (f) (g).

$$\|\mathbb{H}_{3.1.3}\| = \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} (\widehat{\mathbf{F}} - \mathbf{F}^0 \mathbf{R}) \mathbf{R}' \mathbf{F}^{0\prime} T^{-1} (\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}}) (\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}})' \mathbf{u}_i \right\| \tag{B.47}$$

$$\leq \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} (\widehat{\mathbf{F}} - \mathbf{F}^0 \mathbf{R}) \mathbf{R}' \mathbf{F}^{0\prime} T^{-1} (\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}}) (\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}})' \mathbf{H} \varphi_i \right\| \tag{B.48}$$

$$+ \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} (\widehat{\mathbf{F}} - \mathbf{F}^0 \mathbf{R}) \mathbf{R}' \mathbf{F}^{0\prime} T^{-1} (\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}}) (\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}})' \varepsilon_i \right\|$$

$$\leq \sqrt{NT} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |w_{ij}| \left\| \frac{\mathbf{V}_j' (\widehat{\mathbf{F}} - \mathbf{F}^0 \mathbf{R})}{T} \right\| \|\mathbf{R}'\| \left\| \frac{\mathbf{F}^{0\prime} (\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}})}{T} \right\| \left\| \frac{(\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}})' \mathbf{H}^0}{T} \right\| \|\varphi_i\| \tag{B.49}$$

$$+ \sqrt{NT} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |w_{ij}| \left\| \frac{\mathbf{V}_j' (\widehat{\mathbf{F}} - \mathbf{F}^0 \mathbf{R})}{T} \right\| \|\mathbf{R}'\| \left\| \frac{\mathbf{F}^{0\prime} (\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}})}{T} \right\| \left\| \frac{(\widehat{\mathbf{H}} - \mathbf{H}^0 \boldsymbol{\mathcal{R}})' \varepsilon_i}{T} \right\| \tag{B.50}$$

$$= O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^5} \right)$$

by Lemma B.1 (b) and Lemma B.3 (f) (g).

$$\|\mathbb{H}_{3.1.4}\| = \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} (\widehat{\mathbf{F}} - \mathbf{F}^0 \mathbf{R}) \mathbf{R}' \mathbf{F}^{0\prime} T^{-1} \mathbf{H}^0 \left(\boldsymbol{\mathcal{R}} \boldsymbol{\mathcal{R}}' - \left(T^{-1} \mathbf{H}^{0\prime} \mathbf{H}^0 \right)^{-1} \right) \mathbf{H}^{0\prime} \mathbf{u}_i \right\| \tag{B.51}$$

$$\leq \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} (\widehat{\mathbf{F}} - \mathbf{F}^0 \mathbf{R}) \mathbf{R}' \mathbf{F}^{0\prime} T^{-1} \mathbf{H}^0 \left(\boldsymbol{\mathcal{R}} \boldsymbol{\mathcal{R}}' - \left(T^{-1} \mathbf{H}^{0\prime} \mathbf{H}^0 \right)^{-1} \right) \mathbf{H}^{0\prime} \mathbf{H} \varphi_i \right\| \tag{B.52}$$

$$+ \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{V}_j' T^{-1} (\widehat{\mathbf{F}} - \mathbf{F}^0 \mathbf{R}) \mathbf{R}' \mathbf{F}^{0\prime} T^{-1} \mathbf{H}^0 \left(\boldsymbol{\mathcal{R}} \boldsymbol{\mathcal{R}}' - \left(T^{-1} \mathbf{H}^{0\prime} \mathbf{H}^0 \right)^{-1} \right) \mathbf{H}^{0\prime} \varepsilon_i \right\|$$

$$\leq \sqrt{NT} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |w_{ij}| \left\| \frac{\mathbf{V}_j' (\widehat{\mathbf{F}} - \mathbf{F}^0 \mathbf{R})}{T} \right\| \|\mathbf{R}'\| \left\| \frac{\mathbf{F}^{0\prime} \mathbf{H}^0}{T} \right\| \left\| \boldsymbol{\mathcal{R}} \boldsymbol{\mathcal{R}}' - \left(T^{-1} \mathbf{H}^{0\prime} \mathbf{H}^0 \right)^{-1} \right\| \left\| \frac{\mathbf{H}^{0\prime} \mathbf{H}^0}{T} \right\| \|\boldsymbol{\varphi}_i\| \tag{B.53}$$

$$+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{T}\right\|\|\mathbf{R}'\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right\|\left\|\frac{\mathbf{H}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\| \tag{B.54}$$

$$=O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)$$

by Lemma B.1 (f) and Lemma B.3 (e).

$$\|\mathbb{H}_{3.2.1}\|=\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\mathbf{u}_i\right\| \tag{B.55}$$

$$\leq\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0\right\| \tag{B.56}$$

$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\varepsilon_i\right\|$$

$$\leq\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\|\mathbf{R}\|\left\|(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})/\sqrt{T}\right\|\left\|(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})/\sqrt{T}\right\|\|\boldsymbol{\mathcal{R}}'\|\left\|\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i\| \tag{B.57}$$

$$+\sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\|\mathbf{R}\|\left\|(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})/\sqrt{T}\right\|\left\|(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})/\sqrt{T}\right\|\|\boldsymbol{\mathcal{R}}'\|\left\|\frac{\mathbf{H}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\| \tag{B.58}$$

$$=O_p\left(\frac{\sqrt{N}}{\delta_{NT}^2}\right)$$

by Lemma A.1 (a), Lemma B.1 (a), Lemma B.3 (b) (c).

$$\|\mathbb{H}_{3.2.2}\|=\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}\mathbf{H}\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i\right\| \tag{B.59}$$

$$\leq\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0\boldsymbol{\varphi}_i^0\right\| \tag{B.60}$$

$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\varepsilon_i\right\|$$

$$\leq\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\|\mathbf{R}\|\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{H}^0}{T}\right\|\|\boldsymbol{\mathcal{R}}\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i^0\| \tag{B.61}$$

$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\|\mathbf{R}\|\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{H}^0}{T}\right\|\|\boldsymbol{\mathcal{R}}\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\varepsilon_i}{T}\right\| \tag{B.62}$$

$$=O_p\left(\frac{\sqrt{N}}{\delta_{NT}^4}\right)$$

by Lemma B.1 (b) and Lemma B.3 (b) (e).

$$\|\mathbb{H}_{3.2.3}\|=\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0\boldsymbol{\varphi}_i\right\| \tag{B.63}$$

$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\varepsilon_i\right\| \tag{B.64}$$

$$\leq \sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\|\mathbf{R}\|\left\|(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})/\sqrt{T}\right\|\left\|(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})/\sqrt{T}\right\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i^0\| \tag{B.65}$$

$$+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\|\mathbf{R}\|\left\|(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})/\sqrt{T}\right\|\left\|(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})/\sqrt{T}\right\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\varepsilon_i}{T}\right\| \tag{B.66}$$

$$= O_p\left(\frac{\sqrt{N}}{\delta_{NT}^4}\right)$$

by Lemma A.1 (a), Lemma B.1 (a) (b) and Lemma B.3 (b) (e).

$$\|\mathbb{H}_{3.2.4}\| \leq \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0\right\| \tag{B.67}$$

$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}\right)^{-1}\right)\mathbf{H}^{0\prime}\varepsilon_i\right\| \tag{B.68}$$

$$\leq \sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\|\mathbf{R}\|\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right\|\left\|\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i\| \tag{B.69}$$

$$+\sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\|\mathbf{R}\|\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right\|\left\|\frac{\mathbf{H}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\| \tag{B.70}$$

$$= O_p\left(\frac{\sqrt{N}}{\delta_{NT}^4}\right)$$

by Lemma B.1 (b) (f) and Lemma B.3 (b) (c).

$$\|\mathbb{H}_{3.3.1}\| \leq \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i\right\| \tag{B.71}$$

$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}'\varepsilon_i\right\| \tag{B.72}$$

$$\leq \sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{T}\right\|\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{\sqrt{T}}\right\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})}{\sqrt{T}}\right\|\|\boldsymbol{\mathcal{R}}'\|\left\|\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i\| \tag{B.73}$$

$$+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{T}\right\|\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{\sqrt{T}}\right\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})}{\sqrt{T}}\right\|\|\boldsymbol{\mathcal{R}}'\|\left\|\frac{\mathbf{H}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\| \tag{B.74}$$

$$= O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)$$

by Lemma A.1 (a), Lemma B.1 (a), Lemma B.3 (e) (f).

$$\|\mathbb{H}_{3.3.2}\| \leq \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0\boldsymbol{\varphi}_i\right\| \tag{B.75}$$

$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\varepsilon_i\right\| \tag{B.76}$$

$$\leq \sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{T}\right\|\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{H}^0}{T}\right\|\|\boldsymbol{\mathcal{R}}\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i^0\| \tag{B.77}$$

$$+\sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{T}\right\|\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{H}^0}{T}\right\|\|\boldsymbol{\mathcal{R}}\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\varepsilon_i}{T}\right\| \tag{B.78}$$

$$= O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^5}\right)$$

by Lemma B.1 (b) and Lemma B.3 (f) (g).

$$\|\mathbb{H}_{3.3.3}\| \leq \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0\boldsymbol{\varphi}_i^0\right\| \tag{B.79}$$

$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\varepsilon_i\right\| \tag{B.80}$$

$$\leq \sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{T}\right\|\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{\sqrt{T}}\right\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})}{\sqrt{T}}\right\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i\| \tag{B.81}$$

$$+\sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{T}\right\|\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{\sqrt{T}}\right\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})}{\sqrt{T}}\right\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\varepsilon_i}{T}\right\| \tag{B.82}$$

$$= O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^5}\right)$$

by Lemma A.1 (a), Lemma B.1 (a), Lemma B.3 (f) (g).

$$\|\mathbb{H}_{3.3.4}\| \leq \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i\right\| \tag{B.83}$$

$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right)\mathbf{H}^{0\prime}\varepsilon_i\right\| \tag{B.84}$$

$$\leq \sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{T}\right\|\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right\|\left\|\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i\| \tag{B.85}$$

$$+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{T}\right\|\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right\|\left\|\frac{\mathbf{H}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\| \tag{B.86}$$

$$= O_p\left(\frac{\sqrt{NT}}{\delta^5}\right)$$

by Lemma B.1 (b) (f), Lemma B.3 (e) (f).

$$\|\mathbb{H}_{3.4.1}\| \leq \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0\right\| \tag{B.87}$$

$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\varepsilon_i\right\| \tag{B.88}$$

$$\leq \sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})}{T}\right\|\|\boldsymbol{\mathcal{R}}'\|\left\|\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i^0\| \tag{B.89}$$

$$+\sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})}{T}\right\|\|\boldsymbol{\mathcal{R}}'\|\left\|\frac{\mathbf{H}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\| \tag{B.90}$$

$$= O_p\left(\frac{\sqrt{N}}{\delta_{NT}^4}\right)$$

by Lemma A.1 (f), Lemma B.1 (b) and Lemma B.3 (b) (c).

$$\|\mathbb{H}_{3.4.2}\| \leq \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0\boldsymbol{\varphi}_i^0\right\| \tag{B.91}$$

$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\varepsilon_i\right\| \tag{B.92}$$

$$\leq \sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\mathcal{R}}\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i^0\| \tag{B.93}$$

$$+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\mathcal{R}}\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\varepsilon_i}{T}\right\| \tag{B.94}$$

$$= O_p\left(\frac{\sqrt{N}}{\delta_{NT}^4}\right)$$

by Lemma A.1 (f), Lemma B.1 (b) and Lemma B.3 (b) (e).

$$\|\mathbb{H}_{3.4.3}\| \leq \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0\boldsymbol{\varphi}_i\right\| \tag{B.95}$$

$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\varepsilon_i\right\| \tag{B.96}$$

$$\leq \sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})}{T}\right\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i\| \tag{B.97}$$

$$+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})}{T}\right\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\varepsilon_i}{T}\right\| \tag{B.98}$$

$$= O_p\left(\frac{\sqrt{N}}{\delta_{NT}^6}\right)$$

by Lemma A.1 (f), Lemma B.1 (b), Lemma B.3 (b) (e).

$$\|\mathbb{H}_{3.4.4}\| \leq \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0\right\| \tag{B.99}$$

$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right)\mathbf{H}^{0\prime}\varepsilon_i\right\| \tag{B.100}$$

$$\leq \sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\left\|\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right\|\left\|\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i^0\| \tag{B.101}$$

$$+\sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\left\|\frac{\mathbf{V}_j'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\left\|\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right\|\left\|\frac{\mathbf{H}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\| \tag{B.102}$$

$$
= O_p\left(\frac{\sqrt{N}}{\delta_{NT}^4}\right)
$$

by Lemma A.1 (f), Lemma B.1 (f) and Lemma B.3 (b) (c).

Combining the above terms $\mathbb{H}_{3.1.1}$ to $\mathbb{H}_{3.4.4}$, we derive that $\mathbb{H}_3 = O_p\left(\sqrt{NT}\delta_{NT}^{-3}\right)$.

Then, we have

$$
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i + \mathbb{H}_{1.1.1} + O_p\left(\sqrt{NT}\delta_{NT}^{-3}\right)
$$

Now we consider the term $\mathbb{H}_{1.1.1}$, then we have

$$
\begin{aligned}
\mathbb{H}_{1.1.1} =& -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j'\frac{1}{NT}\sum_{h=1}^{N}\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i \\
=& -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j'\frac{1}{N}\sum_{h=1}^{N}\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0 - \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j'\frac{1}{NT}\sum_{h=1}^{N}\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i \\
=&\mathbb{H}_{1.1.1.1} + \mathbb{H}_{1.1.1.2}
\end{aligned}
$$

Since

$$
\begin{aligned}
&\mathbb{E}\left\|-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\varphi}_i^{0\prime}\otimes\left(\frac{1}{N}\sum_{h=1}^{N}\sum_{t=1}^{T}\mathbf{v}_{jt}\boldsymbol{\varepsilon}_{ht}\boldsymbol{\varphi}_h^{0\prime}\right)\right\|^2 \\
=&\frac{1}{N^3T}\sum_{i_1=1}^{N}\sum_{j_1=1}^{N}\sum_{h_1=1}^{N}\sum_{t_1=1}^{T}\sum_{i_2=1}^{N}\sum_{j_2=1}^{N}\sum_{h_2=1}^{N}\sum_{t_2=1}^{T} w_{i_1j_1}w_{i_2j_2}\mathbb{E}(\mathbf{v}_{j_2t_2}'\mathbf{v}_{j_1t_1})\mathbb{E}(\boldsymbol{\varepsilon}_{h_1t_1}\boldsymbol{\varepsilon}_{h_2t_2})\mathbb{E}\left(\boldsymbol{\varphi}_{i_1}^{0\prime}\boldsymbol{\varphi}_{i_2}^{0}\boldsymbol{\varphi}_{h_1}^{0\prime}\boldsymbol{\varphi}_{h_2}^{0}\right) \\
=&\frac{\sigma_\varepsilon^2}{N^3T}\sum_{i_1=1}^{N}\sum_{j_1=1}^{N}\sum_{h=1}^{N}\sum_{t=1}^{T}\sum_{i_2=1}^{N}\sum_{j_2=1}^{N} w_{i_1j_1}w_{i_2j_2}\mathbb{E}(\mathbf{v}_{j_2t_2}'\mathbf{v}_{j_1t_1})\mathbb{E}\left(\boldsymbol{\varphi}_{i_1}^{0\prime}\boldsymbol{\varphi}_{i_2}^{0}\boldsymbol{\varphi}_{h}^{0\prime}\boldsymbol{\varphi}_{h}^{0}\right) \\
\leq&\frac{\sigma_\varepsilon^2}{N^2}\sum_{i_1=1}^{N}\sum_{j_1=1}^{N}\sum_{i_2=1}^{N}\sum_{j_2=1}^{N}|w_{i_1j_1}||w_{i_2j_2}|\bar{\sigma}_{j_1j_2}\left(\mathbb{E}\|\boldsymbol{\varphi}_{i_1}^0\|^4\mathbb{E}\|\boldsymbol{\varphi}_{i_2}^0\|^4\right)^{1/4}\times\frac{1}{N}\sum_{h=1}^{N}\left(\mathbb{E}\|\boldsymbol{\varphi}_h^0\|^4\right)^{1/2} \\
\leq&\frac{\sigma_\varepsilon^2C}{N^2}\sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\left(\sum_{i_1=1}^{N}|w_{i_1j_1}|\right)\left(\sum_{i_2=1}^{N}|w_{i_2j_2}|\right)\bar{\sigma}_{j_1j_2} \leq \frac{\sigma_\varepsilon^2C^3}{N^2}\sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\bar{\sigma}_{j_1j_2} \leq \frac{\sigma_\varepsilon^2C^4}{N}
\end{aligned}
$$

then, for the term $\mathbb{H}_{1.1.1.1}$, we have

$$
\begin{aligned}
\mathbb{H}_{1.1.1.1} =& \operatorname{vec}\left(-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j'\frac{1}{N}\sum_{h=1}^{N}\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0\right) \\
=& -\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\varphi}_i^{0\prime}\otimes\left(\frac{1}{N}\sum_{h=1}^{N}\sum_{t=1}^{T}\mathbf{v}_{jt}\boldsymbol{\varepsilon}_{ht}\boldsymbol{\varphi}_h^{0\prime}\right)\operatorname{vec}((\boldsymbol{\Upsilon}_\varphi^0)^{-1}) = O_p\left(\frac{1}{\sqrt{N}}\right)
\end{aligned}
$$

The term $\mathbb{H}_{1.1.1.2}$ is bounded in norm by

$$
\begin{aligned}
&\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j'\frac{1}{NT}\sum_{h=1}^{N}\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i\right\| \\
=&\frac{1}{\sqrt{T}}\left\|\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\left(\frac{1}{\sqrt{NT}}\sum_{h=1}^{N}\sum_{t=1}^{T}\mathbf{V}_{jt}\boldsymbol{\varepsilon}_{ht}\boldsymbol{\varphi}_h^{0\prime}\right)(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\| \\
\leq&\frac{1}{\sqrt{T}}\times\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\frac{1}{\sqrt{NT}}\sum_{h=1}^{N}\sum_{t=1}^{T}\mathbf{V}_{jt}\boldsymbol{\varepsilon}_{ht}\boldsymbol{\varphi}_h^{0\prime}\right\|\left\|\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\times\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\| = O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}
$$

This completes the proof. □

Lemma B.5 *Under Assumptions A to D, we have*

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i \\
=&-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^{0}\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i \\
&-\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i + O_p\left(\frac{1}{\delta_{NT}^2}\right) + O_p\left(\sqrt{\frac{T}{N^3}}\right)
\end{aligned}
$$

Proof of Lemma B.5. We can follow the way of the proof of Lemma A.5 to prove this lemma. Thus, we omitted the details. □

Lemma B.6 *Under Assumptions A to D, we have*

$$
\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i = O_p\left(\frac{1}{\delta_{NT}}\right) + O_p\left(\frac{\sqrt{N}}{T}\right) + O_p\left(\frac{\sqrt{T}}{N}\right)
$$

Proof of Lemma B.6. We can follow the way of the proof of Lemma A.7 to show that

$$
\begin{aligned}
&\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i \\
=&\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\widehat{\mathbf{F}}'\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i + O_p\left(\frac{1}{\delta_{NT}}\right) + O_p\left(\frac{\sqrt{T}}{N}\right)
\end{aligned}
$$

To proceed, we can follow the way of the proof of Lemma A.8 to show that

$$
\begin{aligned}
&\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}'\mathbf{F}^0}{T}\right)^{-1}\widehat{\mathbf{F}}'\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i && \text{(B.103)} \\
=&\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i + O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{\sqrt{N}}{T}\right) && \text{(B.104)}
\end{aligned}
$$

We write

$$
\begin{aligned}
\mathbb{H}_4 &= \frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i && \text{(B.105)} \\
&= \frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\left(\mathbf{I}_T - \mathbf{P}_{\mathbf{H}^0} - \mathbf{P}_{\mathbf{F}^0} + \mathbf{P}_{\mathbf{H}^0}\mathbf{P}_{\mathbf{F}^0}\right)\boldsymbol{\varepsilon}_i && \text{(B.106)} \\
&= \mathbb{H}_{4.1} + \mathbb{H}_{4.2} + \mathbb{H}_{4.3} + \mathbb{H}_{4.4}.
\end{aligned}
$$

First consider $\mathbb{H}_{4.1}$.

$$
\begin{aligned}
\mathrm{vec}\left(\mathbb{H}_{4.1}\right) &= \mathrm{vec}\left(\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\boldsymbol{\Gamma}_\ell^{0}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\boldsymbol{\varepsilon}_i\right) && \text{(B.107)} \\
&= \mathrm{vec}\left(\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\mathbf{f}_s\mathbb{E}\left(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}\right)\varepsilon_{it}\right) && \text{(B.108)} \\
&= \frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}\sum_{s=1}^{T}\sum_{t=1}^{T} w_{ij}\mathbb{E}\left(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}\right)\varepsilon_{it}\mathbf{f}_s^{0\prime}\otimes\boldsymbol{\Gamma}_j^{0\prime}\mathrm{vec}\left((\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}}{T}\right)^{-1}\right) && \text{(B.109)}
\end{aligned}
$$

$$= \frac{1}{N^{3/2}T^{3/2}} \sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}\sum_{s=1}^{T}\sum_{t=1}^{T} w_{ij}\mathbb{E}\left(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}\right)\varepsilon_{it}\mathbf{f}_s^{0\prime}\otimes\mathbf{\Gamma}_j^{0\prime}\times O_p(1)\,.$$

Since

$$\begin{aligned}
&\mathbb{E}\left\|\frac{1}{N^{3/2}T^{3/2}} \sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}\sum_{s=1}^{T}\sum_{t=1}^{T} w_{ij}\mathbb{E}\left(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}\right)\varepsilon_{it}\mathbf{f}_s^{0\prime}\otimes\mathbf{\Gamma}_j^{0\prime}\right\|^2\\
=&\mathrm{tr}\Bigg(\frac{1}{N^3T^3}\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\sum_{s_1=1}^{T}\sum_{s_2=1}^{T}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T} w_{i_1j_1}w_{i_2j_2}\mathbb{E}\left(\mathbf{v}_{\ell_1s_1}'\mathbf{v}_{\ell_1t_1}\right)\mathbb{E}\left(\mathbf{v}_{\ell_2s_2}'\mathbf{v}_{\ell_2t_2}\right)\\
&\quad\times\mathbb{E}(\varepsilon_{i_1t_1}\varepsilon_{i_2t_2})\mathbb{E}\left(\mathbf{f}_{s_1}^{0\prime}\mathbf{f}_{s_2}^0\right)\mathbb{E}\left(\mathbf{\Gamma}_{j_1}^{0\prime}\mathbf{\Gamma}_{j_2}^0\right)\Bigg)\\
=&\mathrm{tr}\left(\frac{\sigma_\varepsilon^2}{N^3T^3}\sum_{i=1}^{N}\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\sum_{s_1=1}^{T}\sum_{s_2=1}^{T}\sum_{t=1}^{T} w_{ij_1}w_{ij_2}\mathbb{E}\left(\mathbf{v}_{\ell_1s_1}'\mathbf{v}_{\ell_1t}\right)\mathbb{E}\left(\mathbf{v}_{\ell_2s_2}'\mathbf{v}_{\ell_2t}\right)\mathbb{E}\left(\mathbf{f}_{s_1}^{0\prime}\mathbf{f}_{s_2}^0\right)\mathbb{E}\left(\mathbf{\Gamma}_{j_1}^{0\prime}\mathbf{\Gamma}_{j_2}^0\right)\right)\\
\le&\frac{\sigma_\varepsilon^2r_xC^2}{N^3T^3}\sum_{i=1}^{N}\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\sum_{s_1=1}^{T}\sum_{s_2=1}^{T}\sum_{t=1}^{T}|w_{ij_1}||w_{ij_2}||\mathbb{E}\left(\mathbf{v}_{\ell_1s_1}'\mathbf{v}_{\ell_1t}\right)||\mathbb{E}\left(\mathbf{v}_{\ell_2s_2}'\mathbf{v}_{\ell_2t}\right)|\\
\le&\frac{\sigma_\varepsilon^2r_xkC^2}{T^2}\times\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{j_1=1}^{N}|w_{ij_1}|\right)\left(\sum_{j_2=1}^{N}|w_{ij_2}|\right)\times\frac{1}{T}\sum_{t=1}^{T}\left(\sum_{s_1=1}^{T}\tilde{\sigma}_{s_1t}\right)\left(\sum_{s_2=1}^{T}\tilde{\sigma}_{s_2t}\right)\le\frac{\sigma_\varepsilon^2r_xkC^4}{T^2}
\end{aligned}$$

by Lemma A.1 (m) and Assumption B3. Then we have

$$\frac{1}{N^{3/2}T^{3/2}} \sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}\sum_{s=1}^{T}\sum_{t=1}^{T} w_{ij}\mathbb{E}\left(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}\right)\varepsilon_{it}\mathbf{f}_s^{0\prime}\otimes\mathbf{\Gamma}_j^{0\prime} = O_p\left(\frac{1}{T}\right)$$

and we conclude $\mathbb{H}_{4.1} = O_p\left(\frac{1}{T}\right)$. Next, in a similar manner we obtain

$$\begin{aligned}
\mathrm{vec}\left(\mathbb{H}_{4.2}\right) &= \mathrm{vec}\left(\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{\Gamma}_j^{0\prime}\mathbf{\Gamma}_\ell^0\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{P}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i\right) &\text{(B.110)}\\
&= \mathrm{vec}\left(\frac{1}{N^{1/2}T^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T} w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\left(\frac{1}{NT}\sum_{\ell=1}^{N}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{H}^0\right)\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{h}_t^0\varepsilon_{it}\right) &\text{(B.111)}\\
&= \frac{1}{N^{1/2}T^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T} w_{ij}\varepsilon_{it}\mathbf{h}_t^{0\prime}\otimes\mathbf{\Gamma}_j^{0\prime}\mathrm{vec}\left((\mathbf{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\left(\frac{1}{NT}\sum_{\ell=1}^{N}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{H}^0\right)\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right) &\text{(B.112)}\\
&= \frac{1}{N^{1/2}T^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T} w_{ij}\varepsilon_{it}\mathbf{h}_t^{0\prime}\otimes\mathbf{\Gamma}_j^{0\prime}\times O_p(1).
\end{aligned}$$

Similar to $\mathbb{H}_{4.1}$ we can show that

$$\frac{1}{N^{1/2}T^{3/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T} w_{ij}\varepsilon_{it}\mathbf{h}_t^{0\prime}\otimes\mathbf{\Gamma}_j^{0\prime} = O_p\left(\frac{1}{T}\right) \tag{B.113}$$

so that $\mathbb{H}_{4.2} = O_p\left(\frac{1}{T}\right)$. In a similar way to $\mathbb{H}_{4.2}$, we can show $\mathbb{H}_{4.3}$ and $\mathbb{H}_{4.4}$ are $O_p\left(\frac{1}{T}\right)$. Combining the above facts, we derive that $\mathbb{H}_4 = O_p\left(\frac{1}{T}\right)$, which completes the proof. □

Lemma B.7 *Under Assumptions A to D, we have*

$$
\begin{aligned}
&-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i\\
=&-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i+O_p\left(\frac{1}{\delta_{NT}}\right)+O_p\left(\frac{\sqrt{T}}{N}\right)
\end{aligned}
$$

Proof of Lemma B.7. We can follow the way of the proof of Lemma A.6 to show that

$$
\begin{aligned}
&\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i\\
=&\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\mathbf{\Gamma}_\ell^0\frac{\mathbf{V}_\ell^\prime\mathbf{V}_h}{T}\mathbf{\Gamma}_h^{0\prime}\right)(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\frac{\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i}{T}+O_p\left(\frac{1}{\delta_{NT}}\right)+O_p\left(\frac{\sqrt{T}}{N}\right)
\end{aligned}
\tag{B.114}
$$

Now we consider the first term on the right hand side in (B.114). Note that $\mathbf{M}_{\mathbf{H}^0}\mathbf{u}_i=\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i$ and $\mathbf{M}_{\mathbf{H}^0}-\mathbf{M}_{\widehat{\mathbf{H}}}=\mathbf{P}_{\widehat{\mathbf{H}}}-\mathbf{P}_{\mathbf{H}^0}$. We can derive that

$$
\begin{aligned}
&-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i-\left(-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i\right)\\
=&\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{M}_{\mathbf{F}^0}\mathbf{P}_{\widehat{\mathbf{H}}}\mathbf{u}_i-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{M}_{\mathbf{F}^0}\mathbf{P}_{\mathbf{H}^0}\mathbf{u}_i\\
=&\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\frac{1}{T}\mathbf{M}_{\mathbf{F}^0}\widehat{\mathbf{H}}\widehat{\mathbf{H}}^\prime\mathbf{u}_i-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{M}_{\mathbf{F}^0}\mathbf{P}_{\mathbf{H}^0}\mathbf{u}_i\\
=&\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\frac{1}{T}\mathbf{M}_{\mathbf{F}^0}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)\boldsymbol{\mathcal{R}}^\prime\mathbf{H}^{0\prime}\mathbf{u}_i\\
&+\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\frac{1}{T}\mathbf{M}_{\mathbf{F}^0}\mathbf{H}^0\boldsymbol{\mathcal{R}}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)^\prime\mathbf{u}_i\\
&+\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\frac{1}{T}\mathbf{M}_{\mathbf{F}^0}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)^\prime\mathbf{u}_i\\
&+\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{\Gamma}_j^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\frac{1}{T}\mathbf{M}_{\mathbf{F}^0}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}^\prime-\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{u}_i\\
=&\mathbb{I}_1+\mathbb{I}_2+\mathbb{I}_3+\mathbb{I}_4
\end{aligned}
$$

We first consider the last three terms. Consider the term $\mathbb{I}_2$. By Lemmas A.1 (b) and (d), Lemmas A.4 (a), (b) and (i), we can derive that

$$
\begin{aligned}
\|\mathbb{I}_2\| \leq & \left\| \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij} \mathbf{\Gamma}_j^{0\prime} (\mathbf{\Upsilon}^0)^{-1} \mathbf{\Gamma}_\ell^0 \mathbf{V}_\ell^\prime \frac{1}{T} \mathbf{M}_{\mathbf{F}^0} \mathbf{H}^0 \boldsymbol{\mathcal{R}} \left(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}\right)^\prime \mathbf{H}^0 \boldsymbol{\varphi}_i^0 \right\| \\
& + \left\| \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij} \mathbf{\Gamma}_j^{0\prime} (\mathbf{\Upsilon}^0)^{-1} \mathbf{\Gamma}_\ell^0 \mathbf{V}_\ell^\prime \frac{1}{T} \mathbf{M}_{\mathbf{F}^0} \mathbf{H}^0 \boldsymbol{\mathcal{R}} \left(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}\right)^\prime \boldsymbol{\varepsilon}_i \right\| \\
\leq & \left(\frac{1}{N} \sum_{i=1}^{N}\sum_{j=1}^{N} |w_{ij}| \left\|\mathbf{\Gamma}_j^0\right\| \left\|\boldsymbol{\varphi}_i^0\right\| \right) \left\| \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{N} \mathbf{\Gamma}_\ell^0 \mathbf{V}_\ell^\prime \mathbf{M}_{\mathbf{F}^0} \mathbf{H}^0 \right\| \left\| \frac{1}{T} \left(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}\right)^\prime \mathbf{H}^0 \right\| \left\|(\mathbf{\Upsilon}^0)^{-1}\right\| \left\|\boldsymbol{\mathcal{R}}\right\| \\
& + \left(\frac{1}{N} \sum_{i=1}^{N}\sum_{j=1}^{N} |w_{ij}| \left\|\mathbf{\Gamma}_j^0\right\| \left\| \frac{1}{T} \left(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}\right)^\prime \boldsymbol{\varepsilon}_i \right\| \right) \left\| \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{N} \mathbf{\Gamma}_\ell^0 \mathbf{V}_\ell^\prime \mathbf{M}_{\mathbf{F}^0} \mathbf{H}^0 \right\| \left\|(\mathbf{\Upsilon}^0)^{-1}\right\| \left\|\boldsymbol{\mathcal{R}}\right\| \\
= & O_p\left(\delta_{NT}^{-2}\right)
\end{aligned}
$$

given the fact that $(NT)^{-1/2} \sum_{\ell=1}^N \mathbf{\Gamma}_\ell^0 \mathbf{V}_\ell^\prime \mathbf{M}_{\mathbf{F}^0} \mathbf{H}^0 = (NT)^{-1/2} \sum_{\ell=1}^N \mathbf{\Gamma}_\ell^0 \mathbf{V}_\ell^\prime \mathbf{H}^0 - (NT)^{-1/2} \sum_{\ell=1}^N \mathbf{\Gamma}_\ell^0 \mathbf{V}_\ell^\prime \mathbf{F}^0 \times \left(\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1} \mathbf{F}^{0\prime}\mathbf{H}^0 = O_p(1)$.

Consider the term $\mathbb{I}_3$. We can derive that

$$
\begin{aligned}
\|\mathbb{I}_3\| \leq & \left\| \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij} \mathbf{\Gamma}_j^{0\prime} (\mathbf{\Upsilon}^0)^{-1} \mathbf{\Gamma}_\ell^0 \mathbf{V}_\ell^\prime \frac{1}{T} \mathbf{M}_{\mathbf{F}^0} \left(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}\right) \left(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}\right)^\prime \mathbf{H}^0 \boldsymbol{\varphi}_i^0 \right\| \\
+ & \left\| \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij} \mathbf{\Gamma}_j^{0\prime} (\mathbf{\Upsilon}^0)^{-1} \mathbf{\Gamma}_\ell^0 \mathbf{V}_\ell^\prime \frac{1}{T} \mathbf{M}_{\mathbf{F}^0} \left(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}\right) \left(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}\right)^\prime \boldsymbol{\varepsilon}_i \right\| \\
\leq & \left(\frac{1}{N} \sum_{i=1}^{N}\sum_{j=1}^{N} |w_{ij}| \left\|\mathbf{\Gamma}_j^0\right\| \left\|\boldsymbol{\varphi}_i^0\right\| \right) \left\| \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{N} \mathbf{\Gamma}_\ell^0 \mathbf{V}_\ell^\prime \mathbf{M}_{\mathbf{F}^0} \left(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}\right) \right\| \left\| \frac{1}{T} \left(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}\right)^\prime \mathbf{H}^0 \right\| \left\|(\mathbf{\Upsilon}^0)^{-1}\right\| \\
+ & \left(\frac{1}{N} \sum_{i=1}^{N}\sum_{j=1}^{N} |w_{ij}| \left\|\mathbf{\Gamma}_j^0\right\| \left\| \frac{1}{T} \left(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}\right)^\prime \boldsymbol{\varepsilon}_i \right\| \right) \left\| \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{N} \mathbf{\Gamma}_\ell^0 \mathbf{V}_\ell^\prime \mathbf{M}_{\mathbf{F}^0} \left(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}\right) \right\| \left\|(\mathbf{\Upsilon}^0)^{-1}\right\| \\
= & O_p\left(\delta_{NT}^{-2}\right) \times \left[O_p\left(\sqrt{\frac{T}{N}}\right) + O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right) \right] = O_p\left(\sqrt{\frac{T}{N^3}}\right) + O_p\left(\frac{1}{\sqrt{T^3}}\right) + O_p\left(\frac{1}{\sqrt{NT}}\right)
\end{aligned}
$$

by Lemmas A.1 (b), (d) and (h) and Lemmas A.4 (a), (b) and (i), and the fact that

$$
\begin{aligned}
& \left\| \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{N} \mathbf{\Gamma}_\ell^0 \mathbf{V}_\ell^\prime \mathbf{M}_{\mathbf{F}^0} \left(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}\right) \right\| \\
\leq & \left\| \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{N} \mathbf{\Gamma}_\ell^0 \mathbf{V}_\ell^\prime \left(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}\right) \right\| + \left\| \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{N} \mathbf{\Gamma}_\ell^0 \mathbf{V}_\ell^\prime \mathbf{F}^0 \right\| \left\| \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1} \right\| \left\| \frac{\mathbf{F}^{0\prime}\left(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}\right)}{T} \right\| \\
= & O_p\left(\sqrt{\frac{T}{N}}\right) + O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right)
\end{aligned}
$$

Consider the term $\mathbb{I}_4$. We have

$$
\begin{aligned}
& \left\| \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N} w_{ij} \mathbf{\Gamma}_j^{0\prime} (\mathbf{\Upsilon}^0)^{-1} \mathbf{\Gamma}_\ell^0 \mathbf{V}_\ell^\prime \frac{1}{T} \mathbf{M}_{\mathbf{F}^0} \mathbf{H}^0 \left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}^\prime - \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1} \right) \mathbf{H}^{0\prime} \mathbf{u}_i \right\| \\
\leq & \left(\frac{1}{N} \sum_{i=1}^{N}\sum_{j=1}^{N} |w_{ij}| \left\|\mathbf{\Gamma}_j^0\right\| \left\| \frac{1}{\sqrt{T}} \mathbf{u}_i \right\| \right) \left\| \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{N} \mathbf{\Gamma}_\ell^0 \mathbf{V}_\ell^\prime \mathbf{M}_{\mathbf{F}^0} \mathbf{H}^0 \right\| \left\| \boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}^\prime - \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1} \right\| \left\| \frac{\mathbf{H}^0}{\sqrt{T}} \right\| \left\|(\mathbf{\Upsilon}^0)^{-1}\right\| \\
= & O_p\left(\delta_{NT}^{-2}\right)
\end{aligned}
$$

by Lemma A.1 (f) and Lemma A.4 (d).

We consider the term $\mathbb{I}_1$. We have

$$\begin{aligned}
\mathbb{I}_1 =& \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N w_{ij} \boldsymbol{\Gamma}_j^{0\prime} (\boldsymbol{\Upsilon}^0)^{-1} \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \frac{1}{T} \mathbf{M}_{\mathbf{F}^0} \left(\widehat{\mathbf{H}} \boldsymbol{\mathcal{R}}^{-1} - \mathbf{H}^0 \right) \boldsymbol{\mathcal{R}} \boldsymbol{\mathcal{R}}' \mathbf{H}^{0\prime} \mathbf{u}_i \\
=& \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N w_{ij} \boldsymbol{\Gamma}_j^{0\prime} (\boldsymbol{\Upsilon}^0)^{-1} \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \frac{1}{T} \mathbf{M}_{\mathbf{F}^0} \left(\widehat{\mathbf{H}} \boldsymbol{\mathcal{R}}^{-1} - \mathbf{H}^0 \right) \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \mathbf{H}^{0\prime} \mathbf{u}_i \\
&+ \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N w_{ij} \boldsymbol{\Gamma}_j^{0\prime} (\boldsymbol{\Upsilon}^0)^{-1} \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \frac{1}{T} \mathbf{M}_{\mathbf{F}^0} \left(\widehat{\mathbf{H}} \boldsymbol{\mathcal{R}}^{-1} - \mathbf{H}^0 \right) \left(\boldsymbol{\mathcal{R}} \boldsymbol{\mathcal{R}}' - \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \right) \mathbf{H}^{0\prime} \mathbf{u}_i \\
=& \mathbb{I}_{1.1} + \mathbb{I}_{1.2}
\end{aligned}$$

We first consider the term $\mathbb{I}_{1.2}$. We have

$$\begin{aligned}
&\left\| \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N w_{ij} \boldsymbol{\Gamma}_j^{0\prime} (\boldsymbol{\Upsilon}^0)^{-1} \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \frac{1}{T} \mathbf{M}_{\mathbf{F}^0} \left(\widehat{\mathbf{H}} \boldsymbol{\mathcal{R}}^{-1} - \mathbf{H}^0 \right) \left(\boldsymbol{\mathcal{R}} \boldsymbol{\mathcal{R}}' - \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \right) \mathbf{H}^{0\prime} \mathbf{u}_i \right\| \\
\leq& \left(\frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N |w_{ij}| \, \|\boldsymbol{\Gamma}_j^0\| \left\| \frac{1}{\sqrt{T}} \mathbf{u}_i \right\| \right) \left\| \frac{1}{\sqrt{NT}} \sum_{\ell=1}^N \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \mathbf{M}_{\mathbf{F}^0} \left(\widehat{\mathbf{H}} \boldsymbol{\mathcal{R}}^{-1} - \mathbf{H}^0 \right) \right\| \left\| \boldsymbol{\mathcal{R}} \boldsymbol{\mathcal{R}}' - \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{H}^0}{\sqrt{T}} \right\| \|(\boldsymbol{\Upsilon}^0)^{-1}\| \\
=& O_p\left(\delta_{NT}^{-2}\right) \times \left[O_p\left(\sqrt{\frac{T}{N}} \right) + O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2} \right) \right] = O_p\left(\sqrt{\frac{T}{N^3}} \right) + O_p\left(\frac{1}{\sqrt{T^3}} \right) + O_p\left(\frac{1}{\sqrt{NT}} \right)
\end{aligned}$$

by Lemmas A.1 (f), (h) and Lemma A.4 (d).

We consider the term $\mathbb{I}_{1.1}$. By the equation (B.1), we have

$$\begin{aligned}
&\frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N w_{ij} \boldsymbol{\Gamma}_j^{0\prime} (\boldsymbol{\Upsilon}^0)^{-1} \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \mathbf{M}_{\mathbf{F}^0} \frac{1}{T} \left(\widehat{\mathbf{H}} \boldsymbol{\mathcal{R}}^{-1} - \mathbf{H}^0 \right) \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \mathbf{H}^{0\prime} \mathbf{u}_i \\
=& \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N w_{ij} \boldsymbol{\Gamma}_j^{0\prime} (\boldsymbol{\Upsilon}^0)^{-1} \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \mathbf{M}_{\mathbf{F}^0} \frac{1}{NT^2} \sum_{h=1}^N \mathbf{H}^0 \boldsymbol{\varphi}_h^0 \boldsymbol{\varepsilon}_h' \widehat{\mathbf{H}} \left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}_\varphi^0)^{-1} \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \mathbf{H}^{0\prime} \mathbf{u}_i \\
&+ \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N w_{ij} \boldsymbol{\Gamma}_j^{0\prime} (\boldsymbol{\Upsilon}^0)^{-1} \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \mathbf{M}_{\mathbf{F}^0} \frac{1}{NT} \sum_{h=1}^N \boldsymbol{\varepsilon}_h \boldsymbol{\varphi}_h^{0\prime} (\boldsymbol{\Upsilon}_\varphi^0)^{-1} \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \mathbf{H}^{0\prime} \mathbf{u}_i \\
&+ \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N w_{ij} \boldsymbol{\Gamma}_j^{0\prime} (\boldsymbol{\Upsilon}^0)^{-1} \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \mathbf{M}_{\mathbf{F}^0} \frac{1}{NT^2} \sum_{h=1}^N \boldsymbol{\varepsilon}_h \boldsymbol{\varepsilon}_h' \widehat{\mathbf{H}} \left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}_\varphi^0)^{-1} \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \mathbf{H}^{0\prime} \mathbf{u}_i \\
&+ \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N w_{ij} \boldsymbol{\Gamma}_j^{0\prime} (\boldsymbol{\Upsilon}^0)^{-1} \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \mathbf{M}_{\mathbf{F}^0} \frac{1}{NT^2} \sum_{h=1}^N \mathbf{C}_h \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}} \right) \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}} \right)' \mathbf{C}_h' \widehat{\mathbf{H}} \left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}_\varphi^0)^{-1} \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \mathbf{H}^{0\prime} \mathbf{u}_i \\
&+ \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N w_{ij} \boldsymbol{\Gamma}_j^{0\prime} (\boldsymbol{\Upsilon}^0)^{-1} \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \mathbf{M}_{\mathbf{F}^0} \frac{1}{NT^2} \sum_{h=1}^N \mathbf{C}_h \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}} \right) \mathbf{u}_h' \widehat{\mathbf{H}} \left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}_\varphi^0)^{-1} \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \mathbf{H}^{0\prime} \mathbf{u}_i \\
&+ \frac{1}{N^{3/2}T^{1/2}} \sum_{i=1}^N \sum_{\ell=1}^N \sum_{j=1}^N w_{ij} \boldsymbol{\Gamma}_j^{0\prime} (\boldsymbol{\Upsilon}^0)^{-1} \boldsymbol{\Gamma}_\ell^0 \mathbf{V}_\ell' \mathbf{M}_{\mathbf{F}^0} \frac{1}{NT^2} \sum_{h=1}^N \mathbf{u}_h \left(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}} \right)' \mathbf{C}_h' \widehat{\mathbf{H}} \left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T} \right)^{-1} (\boldsymbol{\Upsilon}_\varphi^0)^{-1} \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \mathbf{H}^{0\prime} \mathbf{u}_i \\
=& \mathbb{I}_{1.1.1} + \mathbb{I}_{1.1.2} + \mathbb{I}_{1.1.3} + \mathbb{I}_{1.1.4} + \mathbb{I}_{1.1.5} + \mathbb{I}_{1.1.6}
\end{aligned}$$

For the term $\mathbb{I}_{1.1.1}$, we have

$$
\begin{aligned}
&\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\frac{1}{NT^2}\sum_{h=1}^{N}\mathbf{M}_{\mathbf{F}^0}\mathbf{H}^0\boldsymbol{\varphi}_h^0\boldsymbol{\varepsilon}_h^\prime\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i\\
=&\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\frac{1}{NT^2}\sum_{h=1}^{N}\mathbf{M}_{\mathbf{F}^0}\mathbf{H}^0\boldsymbol{\varphi}_h^0\boldsymbol{\varepsilon}_h^\prime\mathbf{H}^0\boldsymbol{\mathcal{R}}\left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i\\
&+\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\frac{1}{NT^2}\sum_{h=1}^{N}\mathbf{M}_{\mathbf{F}^0}\mathbf{H}^0\boldsymbol{\varphi}_h^0\boldsymbol{\varepsilon}_h^\prime\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)\left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i\\
=&\mathbb{I}_{1.1.1.1}+\mathbb{I}_{1.1.1.2}
\end{aligned}
$$

The term $\mathbb{I}_{1.1.1.1}$ is bounded in norm by

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\,\|\boldsymbol{\Gamma}_j^0\|\left\|\frac{1}{\sqrt{T}}\mathbf{u}_i\right\|\right)\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{M}_{\mathbf{F}^0}\mathbf{H}^0\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{h=1}^{N}\boldsymbol{\varphi}_h^0\boldsymbol{\varepsilon}_h^\prime\mathbf{H}^0\right\|\\
&\times\left\|\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{H}^0}{\sqrt{T}}\right\|\|\boldsymbol{\mathcal{R}}\|\,\|(\boldsymbol{\Upsilon}^0)^{-1}\|\,\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\|\\
=&O_p\left(\frac{1}{\sqrt{NT}}\right)
\end{aligned}
$$

by Lemma A.1 (e) and Lemma A.4 (d). Similarly, we can show the term $\mathbb{I}_{1.1.1.2}$ is bounded in norm by

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\,\|\boldsymbol{\Gamma}_j^0\|\left\|\frac{1}{\sqrt{T}}\mathbf{u}_i\right\|\right)\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{M}_{\mathbf{F}^0}\mathbf{H}^0\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{h=1}^{N}\boldsymbol{\varphi}_h^0\boldsymbol{\varepsilon}_h^\prime\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}\right\|\\
&\times\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{H}^0}{\sqrt{T}}\right\|\|(\boldsymbol{\Upsilon}^0)^{-1}\|\,\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\|\\
=&O_p\left(\frac{1}{\sqrt{NT}}\right)\times\left[O_p\left(\sqrt{\frac{T}{N}}\right)+O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right)\right]=O_p\left(\frac{1}{N}\right)+O_p\left(\frac{1}{\sqrt{N}\delta_{NT}^2}\right)
\end{aligned}
$$

by Lemma A.1 (h) and Lemma A.4 (d). For the term $\mathbb{I}_{1.1.2}$, we can reformulate it as

$$
\begin{aligned}
&\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{M}_{\mathbf{F}^0}\frac{1}{NT}\sum_{h=1}^{N}\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i\\
=&\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell^\prime\mathbf{M}_{\mathbf{F}^0}\boldsymbol{\varepsilon}_h}{T}\boldsymbol{\varphi}_h^{0\prime}\right)(\boldsymbol{\Upsilon}^0)_\varphi^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\frac{\mathbf{H}^{0\prime}\mathbf{u}_i}{T}
\end{aligned}
$$

For the term $\mathbb{I}_{1.1.3}$, with the fact that $\mathbb{E}\left(\boldsymbol{\varepsilon}_h\boldsymbol{\varepsilon}_h'\right)=\sigma_\varepsilon^2\mathbf{I}_T$, we can decompose it as

$$
\begin{aligned}
&\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^N\sum_{\ell=1}^N\sum_{j=1}^N w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\frac{1}{NT^2}\sum_{h=1}^N\boldsymbol{\varepsilon}_h\boldsymbol{\varepsilon}_h'\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i\\
=&\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^N\sum_{\ell=1}^N\sum_{j=1}^N w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\frac{1}{T}\mathbf{H}^0\boldsymbol{\mathcal{R}}\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\frac{\mathbf{H}^{0\prime}\mathbf{u}_i}{T}\\
&+\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^N\sum_{\ell=1}^N\sum_{j=1}^N w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\frac{1}{NT}\sum_{h=1}^N\left(\boldsymbol{\varepsilon}_h\boldsymbol{\varepsilon}_h'-\mathbb{E}\left(\boldsymbol{\varepsilon}_h\boldsymbol{\varepsilon}_h'\right)\right)\mathbf{H}^0\boldsymbol{\mathcal{R}}\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}\\
&\quad\times(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\frac{\mathbf{H}^{0\prime}\mathbf{u}_i}{T}\\
&+\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^N\sum_{\ell=1}^N\sum_{j=1}^N w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\frac{1}{T}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\frac{\mathbf{H}^{0\prime}\mathbf{u}_i}{T}\\
&+\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^N\sum_{\ell=1}^N\sum_{j=1}^N w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\frac{1}{NT}\sum_{h=1}^N\left(\boldsymbol{\varepsilon}_h\boldsymbol{\varepsilon}_h'-\mathbb{E}\left(\boldsymbol{\varepsilon}_h\boldsymbol{\varepsilon}_h'\right)\right)\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}\\
&\quad\times(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\frac{\mathbf{H}^{0\prime}\mathbf{u}_i}{T}
\end{aligned}
$$

which are bounded in norm by

$$
\begin{aligned}
&\frac{1}{T}\left(\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\frac{1}{\sqrt{T}}\mathbf{u}_i\right\|\right)\left(\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^N\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{H}\right\|+\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^N\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\right)\\
&\quad\times\left\|\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{H}^0}{\sqrt{T}}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\right\|\left\|\boldsymbol{\mathcal{R}}\right\|\\
&+\frac{1}{\sqrt{N}}\left(\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\frac{1}{\sqrt{T}}\mathbf{u}_i\right\|\right)\left(\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^N\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\right\|+\frac{1}{\sqrt{T}}\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^N\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^0}{\sqrt{T}}\right\|\right)\\
&\quad\times\left\|\frac{1}{\sqrt{NT}}\sum_{h=1}^N\left(\boldsymbol{\varepsilon}_h\boldsymbol{\varepsilon}_h'-\mathbb{E}\left(\boldsymbol{\varepsilon}_h\boldsymbol{\varepsilon}_h'\right)\right)\mathbf{H}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{H}^0}{\sqrt{T}}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\right\|\left\|\boldsymbol{\mathcal{R}}\right\|\\
&+\frac{1}{\sqrt{T}}\left(\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\frac{1}{\sqrt{T}}\mathbf{u}_i\right\|\right)\left(\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^N\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\right\|+\frac{1}{\sqrt{T}}\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^N\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^0}{\sqrt{T}}\right\|\right)\\
&\quad\times\left\|\frac{\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)}{\sqrt{T}}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{H}^0}{\sqrt{T}}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\right\|\left\|\boldsymbol{\mathcal{R}}\right\|\\
&+\sqrt{\frac{T}{N}}\left(\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N|w_{ij}|\left\|\boldsymbol{\Gamma}_j^0\right\|\left\|\frac{1}{\sqrt{T}}\mathbf{u}_i\right\|\right)\left(\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^N\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\right\|+\frac{1}{\sqrt{T}}\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^N\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^0}{\sqrt{T}}\right\|\right)\\
&\quad\times\left\|\frac{1}{\sqrt{NT}}\left(\sum_{h=1}^N\boldsymbol{\varepsilon}_h\boldsymbol{\varepsilon}_h'-\mathbb{E}\left(\boldsymbol{\varepsilon}_h\boldsymbol{\varepsilon}_h'\right)\right)\right\|\left\|\frac{\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)}{\sqrt{T}}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{H}^0}{\sqrt{T}}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\right\|\left\|\boldsymbol{\mathcal{R}}\right\|\\
=&O_p\left(\frac{\sqrt{T}}{N}\right)+O_p\left(\frac{1}{\sqrt{N}}\right)+O_p\left(\frac{1}{T}\right)
\end{aligned}
$$

by Lemmas A.1 (a), (i), (m) and (n) and Lemma A.4 (d).

For the term $\mathbb{I}_{1.1.5}$, with the fact that $\|\mathbf{M}_{\mathbf{F}^0}\mathbf{C}_h\| \leq \|\mathbf{C}_h\|$, we have

$$
\begin{aligned}
&\|\mathbb{I}_{1.1.5}\| \\
\leq&\sqrt{T}\left\|\boldsymbol{\theta}-\hat{\boldsymbol{\theta}}\right\| \times \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\,\|\boldsymbol{\Gamma}_j^0\|\left\|\frac{\mathbf{u}_i}{\sqrt{T}}\right\| \\
&\times\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\right\|\frac{1}{N}\sum_{h=1}^{N}\left\|\frac{\mathbf{C}_h}{\sqrt{T}}\right\|\left\|\frac{\mathbf{u}_h}{\sqrt{T}}\right\|\left\|\frac{\widehat{\mathbf{H}}}{\sqrt{T}}\right\|\left\|\frac{\mathbf{H}^0}{\sqrt{T}}\right\|\|(\boldsymbol{\Upsilon}^0)^{-1}\|\left\|\left(\frac{\mathbf{H}^{0\prime}\hat{\mathbf{H}}}{T}\right)^{-1}\right\|\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\| \\
=&O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right)
\end{aligned}
$$

Similarly, we can derive that $\mathbb{I}_{1.1.6} = O_p\left(\sqrt{T}\delta_{NT}^{-4}\right)$ and $\mathbb{I}_{1.1.4} = O_p\left(\sqrt{T}\delta_{NT}^{-2}\right)$.

For the term $\mathbb{I}_{1.1.2}$, we have

$$
\begin{aligned}
\mathbb{I}_{1.1.2} =& \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\boldsymbol{\varepsilon}_h}{T}\boldsymbol{\varphi}_h^{0\prime}\right)(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\frac{\mathbf{H}^{0\prime}\mathbf{u}_i}{T} \\
=&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\boldsymbol{\varepsilon}_h}{T}\boldsymbol{\varphi}_h^{0\prime}\right)(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0 \\
&+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\boldsymbol{\varepsilon}_h}{T}\boldsymbol{\varphi}_h^{0\prime}\right)(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{T}
\end{aligned}
$$

which is bounded in norm by

$$
\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\,\|\boldsymbol{\Gamma}_j^0\|\,\|\boldsymbol{\varphi}_i^0\|\,\|(\boldsymbol{\Upsilon}^0)^{-1}\|\,\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\|\left\|\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\boldsymbol{\varepsilon}_h}{T}\boldsymbol{\varphi}_h^{0\prime}\right\| \\
&+\frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\,\|\boldsymbol{\Gamma}_j^0\|\left\|\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\|(\boldsymbol{\Upsilon}^0)^{-1}\|\,\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\|\left\|\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\boldsymbol{\varepsilon}_h}{T}\boldsymbol{\varphi}_h^{0\prime}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\| \\
=&O_p(1)\times\left\|\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\boldsymbol{\varepsilon}_h}{T}\boldsymbol{\varphi}_h^{0\prime}\right\| = O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left\|\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\boldsymbol{\varepsilon}_h}{T}\boldsymbol{\varphi}_h^{0\prime}\right\| = \left\|\frac{1}{NT}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\sum_{t=1}^{T}\boldsymbol{\Gamma}_\ell^0\mathbf{v}_{\ell t}\varepsilon_{ht}\boldsymbol{\varphi}_h^{0\prime} - \frac{1}{NT^2}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\sum_{h=1}^{N}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}\right\| \\
\leq&\left\|\frac{1}{NT}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\sum_{t=1}^{T}\boldsymbol{\Gamma}_\ell^0\mathbf{v}_{\ell t}\varepsilon_{ht}\boldsymbol{\varphi}_h^{0\prime}\right\| + \frac{1}{T}\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{h=1}^{N}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}\right\| = O_p\left(\frac{1}{\sqrt{T}}\right)
\end{aligned}
$$

given the facts that

$$
\begin{aligned}
&\mathbb{E}\left(\mathbb{E}\left(\left\|\frac{1}{NT}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\sum_{t=1}^{T}\boldsymbol{\Gamma}_{\ell}^{0}\mathbf{v}_{\ell t}\varepsilon_{ht}\boldsymbol{\varphi}_{h}^{0\prime}\right\|^{2}\middle|\{\boldsymbol{\Gamma}_{i}^{0},\boldsymbol{\varphi}_{i}^{0}\}_{i=1}^{N}\right)\right)\\
=&\mathbb{E}\left(\mathbb{E}\left(\left\|\frac{1}{NT}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\sum_{t=1}^{T}\operatorname{vec}\left(\boldsymbol{\Gamma}_{\ell}^{0}\mathbf{v}_{\ell t}\varepsilon_{ht}\boldsymbol{\varphi}_{h}^{0\prime}\right)\right\|^{2}\middle|\{\boldsymbol{\Gamma}_{i}^{0},\boldsymbol{\varphi}_{i}^{0}\}_{i=1}^{N}\right)\right)\\
=&\mathbb{E}\left(\mathbb{E}\left(\left\|\frac{1}{NT}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\sum_{t=1}^{T}\left(\boldsymbol{\varphi}_{h}^{0}\otimes\boldsymbol{\Gamma}_{\ell}^{0}\right)\mathbf{v}_{\ell t}\varepsilon_{ht}\right\|^{2}\middle|\{\boldsymbol{\Gamma}_{i}^{0},\boldsymbol{\varphi}_{i}^{0}\}_{i=1}^{N}\right)\right)\\
=&\mathbb{E}\left(\operatorname{tr}\left(\frac{1}{N^{2}T^{2}}\sum_{\ell_1=1}^{N}\sum_{h_1=1}^{N}\sum_{t_1=1}^{T}\sum_{\ell_2=1}^{N}\sum_{h_2=1}^{N}\sum_{t_2=1}^{T}\left(\boldsymbol{\varphi}_{h_1}^{0}\otimes\boldsymbol{\Gamma}_{\ell_1}^{0}\right)\mathbb{E}\left(\mathbf{v}_{\ell_1 t}\mathbf{v}_{\ell_2 t}'\right)\mathbb{E}\left(\boldsymbol{\varepsilon}_{h_1t_1}\boldsymbol{\varepsilon}_{h_2t_2}\right)\left(\boldsymbol{\varphi}_{h_2}^{0\prime}\otimes\boldsymbol{\Gamma}_{\ell_2}^{0\prime}\right)\right)\right)\\
=&\mathbb{E}\left(\operatorname{tr}\left(\frac{\sigma_{\varepsilon}^{2}}{N^{2}T^{2}}\sum_{\ell_1=1}^{N}\sum_{h=1}^{N}\sum_{t=1}^{T}\sum_{\ell_2=1}^{N}\left(\boldsymbol{\varphi}_{h}^{0}\otimes\boldsymbol{\Gamma}_{\ell_1}^{0}\right)\mathbb{E}\left(\mathbf{v}_{\ell_1 t}\mathbf{v}_{\ell_2 t}'\right)\left(\boldsymbol{\varphi}_{h}^{0\prime}\otimes\boldsymbol{\Gamma}_{\ell_2}^{0\prime}\right)\right)\right)\\
\leq&\frac{\sigma_{\varepsilon}^{2}r}{T}\left(\frac{1}{N^{2}}\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\sum_{h=1}^{N}\mathbb{E}\left(\left\|\boldsymbol{\varphi}_{h}^{0}\right\|^{2}\left\|\boldsymbol{\Gamma}_{\ell_1}^{0}\right\|\left\|\boldsymbol{\Gamma}_{\ell_2}^{0}\right\|\right)\times\frac{1}{T}\sum_{t=1}^{T}\left\|\mathbb{E}\left(\mathbf{v}_{\ell_1 t}\mathbf{v}_{\ell_2 t}'\right)\right\|\right)\\
\leq&\frac{\sigma_{\varepsilon}^{2}r}{T}\frac{1}{N^{2}}\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\sum_{h=1}^{N}\left(\mathbb{E}\left\|\boldsymbol{\varphi}_{h}^{0}\right\|^{4}\right)^{1/2}\left(\mathbb{E}\left\|\boldsymbol{\Gamma}_{\ell_1}^{0}\right\|^{4}\right)^{1/4}\left(\mathbb{E}\left\|\boldsymbol{\Gamma}_{\ell_2}^{0}\right\|^{4}\right)^{1/4}\times\frac{1}{T}\sum_{t=1}^{T}\left\|\mathbb{E}\left(\mathbf{v}_{\ell_1 t}\mathbf{v}_{\ell_2 t}'\right)\right\|\\
\leq&\frac{\sigma_{\varepsilon}^{2}rC}{T}\frac{1}{N}\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\bar{\sigma}_{\ell_1\ell_2}\leq\frac{\sigma_{\varepsilon}^{2}rC^{2}}{T}
\end{aligned}
$$

by Assumption B3.

Combining the above terms, for the first term on the right hand side in (B.114), we can derive that

$$
\begin{aligned}
&-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_{j}^{0\prime}(\boldsymbol{\Upsilon}^{0})^{-1}\boldsymbol{\Gamma}_{\ell}^{0}\mathbf{V}_{\ell}'\mathbf{M}_{\mathbf{F}^{0}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_{i}\\
=&-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_{j}^{0\prime}(\boldsymbol{\Upsilon}^{0})^{-1}\boldsymbol{\Gamma}_{\ell}^{0}\mathbf{V}_{\ell}'\mathbf{M}_{\mathbf{F}^{0}}\mathbf{M}_{\mathbf{H}^{0}}\boldsymbol{\varepsilon}_{i}+O_{p}\left(\frac{1}{\delta_{NT}}\right)+O_{p}\left(\frac{\sqrt{T}}{N}\right)
\end{aligned}
\tag{B.115}
$$

Next, we consider the second term on the right hand side in (B.114). It can be written as

$$
\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_{j}^{0\prime}\boldsymbol{\Gamma}_{\ell}^{0}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_{\ell}^{0}\mathbf{V}_{\ell}'\mathbf{V}_{h}T\boldsymbol{\Gamma}_{h}^{0\prime}\right)(\boldsymbol{\Upsilon}^{0})^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{0}}{T}\right)^{-1}T^{-1}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_{i}\tag{B.116}
$$

$$
=\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_{j}^{0\prime}\boldsymbol{\Gamma}_{\ell}^{0}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_{\ell}^{0}\mathbf{V}_{\ell}'\mathbf{V}_{h}T\boldsymbol{\Gamma}_{h}^{0\prime}\right)(\boldsymbol{\Upsilon}^{0})^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{0}}{T}\right)^{-1}T^{-1}\mathbf{F}^{0\prime}\mathbf{M}_{\mathbf{H}^{0}}\mathbf{u}_{i}\tag{B.117}
$$

$$
+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_{j}^{0\prime}\boldsymbol{\Gamma}_{\ell}^{0}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_{\ell}^{0}\frac{\mathbf{V}_{\ell}'\mathbf{V}_{h}}{T}\boldsymbol{\Gamma}_{h}^{0\prime}\right)(\boldsymbol{\Upsilon}^{0})^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{0}}{T}\right)^{-1}T^{-1}\mathbf{F}^{0\prime}\left(\mathbf{M}_{\widehat{\mathbf{H}}}-\mathbf{M}_{\mathbf{H}^{0}}\right)\mathbf{u}_{i}\tag{B.118}
$$

$$
=\mathbb{I}_{5}+\mathbb{I}_{6}.
$$

Using
$\mathbf{M}_{\widehat{\mathbf{H}}}-\mathbf{M}_{\mathbf{H}^{0}}=-T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^{0}\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}-T^{-1}\mathbf{H}^{0}\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^{0}\boldsymbol{\mathcal{R}})'-T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^{0}\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^{0}\boldsymbol{\mathcal{R}})'-T^{-1}\mathbf{H}^{0}\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^{0}\right)^{-1}\right)\mathbf{H}^{0\prime}$,
we have

$$
\mathbb{I}_{6}=\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_{j}^{0\prime}(\boldsymbol{\Upsilon}^{0})^{-1}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_{\ell}^{0}\frac{\mathbf{V}_{\ell}'\mathbf{V}_{h}}{T}\boldsymbol{\Gamma}_{h}^{0\prime}\right)(\boldsymbol{\Upsilon}^{0})^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{0}}{T}\right)^{-1}T^{-2}\mathbf{F}^{0\prime}\left(\widehat{\mathbf{H}}-\mathbf{H}^{0}\mathcal{R}\right)\mathcal{R}'\mathbf{H}^{0\prime}\mathbf{u}_{i}\tag{B.119}
$$

$$+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\boldsymbol{\Gamma}_\ell^0\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{V}_h}{T}\boldsymbol{\Gamma}_h^{0\prime}\right)(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}T^{-2}\mathbf{F}^{0\prime}\mathbf{H}^0\mathcal{R}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\mathcal{R}\right)'\mathbf{u}_i \tag{B.120}$$

$$+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\boldsymbol{\Gamma}_\ell^0\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{V}_h}{T}\boldsymbol{\Gamma}_h^{0\prime}\right)(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{\mathbf{0}}}{T}\right)^{-1}T^{-2}\mathbf{F}^{0\prime}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\mathcal{R}\right)\left(\widehat{\mathbf{H}}-\mathbf{H}^0\mathcal{R}\right)'\mathbf{u}_i \tag{B.121}$$

$$+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\boldsymbol{\Gamma}_\ell^0\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{V}_h}{T}\boldsymbol{\Gamma}_h^{0\prime}\right)(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{\mathbf{0}}}{T}\right)^{-1}T^{-2}\mathbf{F}^{0\prime}\mathbf{H}^0\left(\mathcal{R}\mathcal{R}'-\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{u}_i \tag{B.122}$$

$$=\mathbb{I}_{6.1}+\mathbb{I}_{6.2}+\mathbb{I}_{6.3}+\mathbb{I}_{6.4}.$$

$$\|\mathbb{I}_{6.1}\|\le\left\|\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{V}_h}{T}\boldsymbol{\Gamma}_h^{0\prime}\right)(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}T^{-2}\mathbf{F}^{0\prime}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\mathcal{R}\right)\mathcal{R}'\mathbf{H}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0\right\|$$

$$+\left\|\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\boldsymbol{\Gamma}_\ell^0\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{V}_h}{T}\boldsymbol{\Gamma}_h^{0\prime}\right)(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{\mathbf{0}}}{T}\right)^{-1}T^{-2}\mathbf{F}^{0\prime}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\mathcal{R}\right)\mathcal{R}'\mathbf{H}^{0\prime}\varepsilon_i\right\|$$

$$\le\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{V}_h}{T}\boldsymbol{\Gamma}_h^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\mathcal{R}\right)}{T}\right\|$$

$$\times\|\mathcal{R}'\|\left\|\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\varphi}_i^0\right\|$$

$$+\sqrt{\frac{1}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{V}_h}{T}\boldsymbol{\Gamma}_h^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}\left(\widehat{\mathbf{H}^0}-\mathbf{H}^0\mathcal{R}\right)}{T}\right\|\|\mathcal{R}'\|\left\|\frac{\mathbf{H}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\|$$

$$=O_p\left(\sqrt{\frac{T}{N}}\frac{1}{\delta_{NT}^2}\right)$$

by Lemma A.4 (a) (b) (e), Lemma B.1 (b) and Lemma B.3 (i).

$$\|\mathbb{I}_{6.2}\|\le\left\|\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{V}_h}{T}\boldsymbol{\Gamma}_h^{0\prime}\right)(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}T^{-2}\mathbf{F}^{0\prime}\mathbf{H}^0\mathcal{R}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\mathcal{R}\right)'\mathbf{H}^0\boldsymbol{\varphi}_i\right\|$$

$$+\left\|\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}\boldsymbol{\Gamma}_\ell^0\left(\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{V}_h}{T}\boldsymbol{\Gamma}_h^{0\prime}\right)(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{\mathbf{0}}}{T}\right)^{-1}T^{-2}\mathbf{F}^{0\prime}\mathbf{H}^0\mathcal{R}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\mathcal{R}\right)'\varepsilon_i\right\|$$

$$\le\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{V}_h}{T}\boldsymbol{\Gamma}_h^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\|\mathcal{R}\|$$

$$\times\left\|\frac{\left(\widehat{\mathbf{H}}-\mathbf{H}\mathcal{R}\right)'\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\varphi}_i^0\right\|$$

$$+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\left\|\boldsymbol{\Gamma}_j^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\frac{1}{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\frac{\mathbf{V}_\ell'\mathbf{V}_h}{T}\boldsymbol{\Gamma}_h^{0\prime}\right\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\|\mathcal{R}\|\left\|\frac{\left(\widehat{\mathbf{H}}-\mathbf{H}^0\mathcal{R}\right)'\varepsilon_i}{T}\right\|$$

$$=O_p\left(\sqrt{\frac{T}{N}}\frac{1}{\delta_{NT}^2}\right)$$

by Lemma A.4 (a) (b), Lemma B.1 (b) and Lemma B.3 (h) (i).

$$
\begin{aligned}
\|\mathbb{I}_{6.3}\| &\leq \left\| \sqrt{\frac{T}{N}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{\Gamma}_j^{0\prime} (\mathbf{\Upsilon}^0)^{-1} \left(\frac{1}{N} \sum_{\ell=1}^{N} \sum_{h=1}^{N} \mathbf{\Gamma}_\ell^0 \frac{\mathbf{V}_\ell' \mathbf{V}_h}{T} \mathbf{\Gamma}_h^{0\prime} \right) (\mathbf{\Upsilon}^0)^{-1} \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} T^{-2} \mathbf{F}^{0\prime} \left(\widehat{\mathbf{H}} - \mathbf{H}^0 \mathcal{R} \right) \left(\widehat{\mathbf{H}} - \mathbf{H}^0 \mathcal{R} \right)' \mathbf{H}^0 \boldsymbol{\varphi}_i^0 \right\| \\
&+ \left\| \sqrt{\frac{T}{N}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{\Gamma}_j^{0\prime} \mathbf{\Gamma}_\ell^0 \left(\frac{1}{N} \sum_{\ell=1}^{N} \sum_{h=1}^{N} \mathbf{\Gamma}_\ell^0 \frac{\mathbf{V}_\ell' \mathbf{V}_h}{T} \mathbf{\Gamma}_h^{0\prime} \right) (\mathbf{\Upsilon}^0)^{-1} \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{\mathbf{0}}}{T} \right)^{-1} T^{-2} \mathbf{F}^{0\prime} \left(\widehat{\mathbf{H}} - \mathbf{H}^0 \mathcal{R} \right) \left(\widehat{\mathbf{H}} - \mathbf{H}^0 \mathcal{R} \right)' \varepsilon_i \right\| \\
&\leq \sqrt{\frac{T}{N}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |w_{ij}| \, \|\mathbf{\Gamma}_j^{0\prime}\| \, \|(\mathbf{\Upsilon}^0)^{-1}\| \left\| \frac{1}{N} \sum_{\ell=1}^{N} \sum_{h=1}^{N} \mathbf{\Gamma}_\ell^0 \frac{\mathbf{V}_\ell' \mathbf{V}_h}{T} \mathbf{\Gamma}_h^{0\prime} \right\| \|(\mathbf{\Upsilon}^0)^{-1}\| \left\| \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}^{0\prime} \left(\widehat{\mathbf{H}} - \mathbf{H}^0 \mathcal{R} \right)}{T} \right\| \\
&\times \left\| \frac{\left(\widehat{\mathbf{H}} - \mathbf{H}^0 \mathcal{R} \right)' \mathbf{H}^0}{T} \right\| \|\boldsymbol{\varphi}_i^0\| \\
&+ \sqrt{\frac{T}{N}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |w_{ij}| \, \|\mathbf{\Gamma}_j^{0\prime}\| \, \|(\mathbf{\Upsilon}^0)^{-1}\| \left\| \frac{1}{N} \sum_{\ell=1}^{N} \sum_{h=1}^{N} \mathbf{\Gamma}_\ell^0 \frac{\mathbf{V}_\ell' \mathbf{V}_h}{T} \mathbf{\Gamma}_h^{0\prime} \right\| \|(\mathbf{\Upsilon}^0)^{-1}\| \left\| \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}^{0\prime} \left(\widehat{\mathbf{H}} - \mathbf{H}^0 \mathcal{R} \right)}{T} \right\| \\
&\times \left\| \frac{\left(\widehat{\mathbf{H}} - \mathbf{H}^0 \mathcal{R} \right)' \varepsilon_i}{T} \right\| \\
&= O_p \left(\sqrt{\frac{T}{N}} \frac{1}{\delta_{NT}^4} \right)
\end{aligned}
$$

by Lemma A.4 (a) (b), Lemma B.1 (b) and Lemma B.3 (h) (i).

$$
\begin{aligned}
&\|\mathbb{I}_{6.4}\| \\
&\leq \left\| \sqrt{\frac{T}{N}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{\Gamma}_j' (\mathbf{\Upsilon}^0)^{-1} \left(\frac{1}{N} \sum_{\ell=1}^{N} \sum_{h=1}^{N} \mathbf{\Gamma}_\ell^0 \frac{\mathbf{V}_\ell' \mathbf{V}_h}{T} \mathbf{\Gamma}_h^{0\prime} \right) (\mathbf{\Upsilon}^0)^{-1} \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} T^{-2} \mathbf{F}^{0\prime}\mathbf{H}^0 \left(\mathcal{R}\mathcal{R}' - \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \right) \mathbf{H}^{0\prime}\mathbf{H}^0 \boldsymbol{\varphi}_i^0 \right\| \\
&+ \left\| \sqrt{\frac{T}{N}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{\Gamma}_j^{0\prime} \mathbf{\Gamma}_\ell^0 \left(\frac{1}{N} \sum_{\ell=1}^{N} \sum_{h=1}^{N} \mathbf{\Gamma}_\ell^0 \frac{\mathbf{V}_\ell' \mathbf{V}_h}{T} \mathbf{\Gamma}_h^{0\prime} \right) (\mathbf{\Upsilon}^0)^{-1} \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^{\mathbf{0}}}{T} \right)^{-1} T^{-2} \mathbf{F}^{0\prime}\mathbf{H}^0 \left(\mathcal{R}\mathcal{R}' - \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \right) \mathbf{H}^{0\prime} \varepsilon_i \right\| \\
&\leq \sqrt{\frac{T}{N}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |w_{ij}| \, \|\mathbf{\Gamma}_j^{0\prime}\| \, \|(\mathbf{\Upsilon}^0)^{-1}\| \left\| \frac{1}{N} \sum_{\ell=1}^{N} \sum_{h=1}^{N} \mathbf{\Gamma}_\ell^0 \frac{\mathbf{V}_\ell' \mathbf{V}_h}{T} \mathbf{\Gamma}_h^{0\prime} \right\| \|(\mathbf{\Upsilon}^0)^{-1}\| \left\| \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T} \right\| \left\| \mathcal{R}\mathcal{R}' - \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right\| \\
&+ \sqrt{\frac{T}{N}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |w_{ij}| \, \|\mathbf{\Gamma}_j^{0\prime}\| \, \|(\mathbf{\Upsilon}^0)^{-1}\| \left\| \frac{1}{N} \sum_{\ell=1}^{N} \sum_{h=1}^{N} \mathbf{\Gamma}_\ell^0 \frac{\mathbf{V}_\ell' \mathbf{V}_h}{T} \mathbf{\Gamma}_h^{0\prime} \right\| \|(\mathbf{\Upsilon}^0)^{-1}\| \left\| \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T} \right\| \left\| \mathcal{R}\mathcal{R}' - \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{H}^{0\prime}\varepsilon_i}{T} \right\| \\
&= O_p\left(\sqrt{\frac{T}{N}} \frac{1}{\delta_{NT}^2} \right)
\end{aligned}
$$

by Lemma A.4 (a) (b) (e), Lemma B.1 (f) and Lemma B.3 (i). Therefore, noting that T/N tends to a finite positive constant we conclude $\mathbb{I}_6 = O_p\left(\delta_{NT}^{-2}\right)$. Next

$$
\begin{aligned}
\|\mathbb{I}_5\| &= \left\| \sqrt{\frac{T}{N}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \mathbf{\Gamma}_j^{0\prime} (\mathbf{\Upsilon}^0)^{-1} \left(\frac{1}{N} \sum_{\ell=1}^{N} \sum_{h=1}^{N} \mathbf{\Gamma}_\ell^0 \frac{\mathbf{V}_\ell' \mathbf{V}_h}{T} \mathbf{\Gamma}_h^{0\prime} \right) \mathbf{\Upsilon}^{-1} \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} T^{-1} \mathbf{F}^{0\prime} \mathbf{M}_{\mathbf{H}^0} \mathbf{u}_i \right\| \\
&\leq \sqrt{\frac{1}{N}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |w_{ij}| \, \|\mathbf{\Gamma}_j^{0\prime}\| \, \|(\mathbf{\Upsilon}^0)^{-1}\| \left\| \frac{1}{N} \sum_{\ell=1}^{N} \sum_{h=1}^{N} \mathbf{\Gamma}_\ell^0 \frac{\mathbf{V}_\ell' \mathbf{V}_h}{T} \mathbf{\Gamma}_h^{0\prime} \right\| \|(\mathbf{\Upsilon}^0)^{-1}\| \left\| \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}^{0\prime}\varepsilon_i}{\sqrt{T}} \right\| \\
&+ \sqrt{\frac{1}{N}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} |w_{ij}| \, \|\mathbf{\Gamma}_j^{0\prime}\| \, \|(\mathbf{\Upsilon}^0)^{-1}\| \left\| \frac{1}{N} \sum_{\ell=1}^{N} \sum_{h=1}^{N} \mathbf{\Gamma}_\ell^0 \frac{\mathbf{V}_\ell' \mathbf{V}_h}{T} \mathbf{\Gamma}_h^{0\prime} \right\| \|(\mathbf{\Upsilon}^0)^{-1}\| \left\| \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T} \right\| \left\| \left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{H}^{0\prime}\varepsilon_i}{\sqrt{T}} \right\| \\
&= O_p\left(N^{-1/2} \right)
\end{aligned}
$$

by Lemma A.4 (e) and Lemma B.3 (i). Thus, the second term on the right hand side in (B.114) is $O_p\left(T^{1/2}N^{-1/2}\delta_{NT}^{-2}\right)$. Consequently, with (B.115), we complete the proof. □

Lemma B.8 *Under Assumptions A to D, we have*

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}\mathbf{F}_{-1}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i$$
$$=-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}^{\prime}\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i$$
$$-\frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left(\frac{\widehat{\mathbf{F}}_{-1}^{\prime}\mathbf{F}_{-1}^0}{T}\right)^{-1}\widehat{\mathbf{F}}_{-1}^{\prime}\mathbf{V}_{\ell,-1}\mathbf{V}_{\ell,-1}^{\prime}\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i+O_p\left(\delta_{NT}^{-2}\right)+O_p\left(\sqrt{\frac{T}{N^3}}\right)$$

Proof of Lemma B.8. First, we have

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{F}_{-1}^0-\widehat{\mathbf{F}}_{-1}\mathfrak{R}^{-1})^{\prime}\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i$$
$$=-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}\frac{1}{NT}\sum_{\ell=1}^{N}\mathfrak{R}^{-1\prime}\mathbf{\Xi}_L^{-1}\widehat{\mathbf{F}}_{-1}^{\prime}\mathbf{V}_{\ell,-1}\mathbf{\Gamma}_\ell^{0\prime}\mathbf{F}_{-1}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i$$
$$-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}\frac{1}{NT}\sum_{\ell=1}^{N}\mathfrak{R}^{-1\prime}\mathbf{\Xi}_L^{-1}\widehat{\mathbf{F}}_{-1}^{\prime}\mathbf{F}_{-1}^0\mathbf{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}^{\prime}\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i$$
$$-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}\frac{1}{NT}\sum_{\ell=1}^{N}\mathfrak{R}^{-1\prime}\mathbf{\Xi}_L^{-1}\widehat{\mathbf{F}}_{-1}^{\prime}\mathbf{V}_{\ell,-1}\mathbf{V}_{\ell,-1}^{\prime}\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i$$
$$=\mathbb{J}_1+\mathbb{J}_2+\mathbb{J}_3$$

We consider the term $\mathbb{J}_1$. By Lemma A.1 (h), we have

$$\begin{aligned}\frac{1}{NT}\sum_{\ell=1}^{N}\widehat{\mathbf{F}}_{-1}^{\prime}\mathbf{V}_{\ell,-1}\mathbf{\Gamma}_\ell^{0\prime}=&\mathfrak{R}^{\prime}\frac{1}{NT}\sum_{\ell=1}^{N}\mathbf{F}_{-1}^{0\prime}\mathbf{V}_{\ell,-1}\mathbf{\Gamma}_\ell^{0\prime}+\frac{1}{NT}\sum_{\ell=1}^{N}\left(\widehat{\mathbf{F}}_{-1}-\mathbf{F}_{-1}^0\mathfrak{R}\right)^{\prime}\mathbf{V}_{\ell,-1}\mathbf{\Gamma}_\ell^{0\prime}\\=&O_p\left(\frac{1}{\sqrt{NT}}\right)+O_p\left(\frac{1}{N}\right)+O_p\left(\frac{1}{\sqrt{N}\delta_{NT}^2}\right).\end{aligned}\tag{B.123}$$

Note that that $\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{H}^0 = \mathbf{M}_{\widehat{\mathbf{H}}}\left(\mathbf{H}^0 - \widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1}\right)$. Given the equation (B.123), we can derive that

$$
\begin{aligned}
\|\mathbb{J}_1\| =& \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^N \boldsymbol{\Gamma}_i^{0\prime} \frac{1}{NT} \sum_{\ell=1}^N \mathfrak{R}^{-1\prime}\boldsymbol{\Xi}_L^{-1}\widehat{\mathbf{F}}'_{-1}\mathbf{V}_{\ell,-1}\boldsymbol{\Gamma}_\ell^{0\prime}\left(\mathbf{F}_{-1}^0 - \widehat{\mathbf{F}}_{-1}\mathfrak{R}^{-1}\right)' \mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i \right\| \\
\leq& \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^N \boldsymbol{\Gamma}_i^{0\prime} \frac{1}{NT} \sum_{\ell=1}^N \mathfrak{R}^{-1\prime}\boldsymbol{\Xi}_L^{-1}\widehat{\mathbf{F}}'_{-1}\mathbf{V}_{\ell,-1}\boldsymbol{\Gamma}_\ell^{0\prime}\left(\mathbf{F}_{-1}^0 - \widehat{\mathbf{F}}_{-1}\mathfrak{R}^{-1}\right)' \mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\left(\mathbf{H}^0 - \widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1}\right)\boldsymbol{\varphi}_i^0 \right\| \\
&+ \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^N \boldsymbol{\Gamma}_i^{0\prime} \frac{1}{NT} \sum_{\ell=1}^N \mathfrak{R}^{-1\prime}\boldsymbol{\Xi}_L^{-1}\widehat{\mathbf{F}}'_{-1}\mathbf{V}_{\ell,-1}\boldsymbol{\Gamma}_\ell^{0\prime}\left(\mathbf{F}_{-1}^0 - \widehat{\mathbf{F}}_{-1}\mathfrak{R}^{-1}\right)' \mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\boldsymbol{\varepsilon}_i \right\| \\
\leq& \sqrt{NT}\left(\frac{1}{N}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\,\|\boldsymbol{\varphi}_i^0\|\right)\|\mathfrak{R}^{-1\prime}\|\,\|\boldsymbol{\Xi}_L^{-1}\|\left\|\frac{1}{NT}\sum_{\ell=1}^N \widehat{\mathbf{F}}'_{-1}\mathbf{V}_{\ell,-1}\boldsymbol{\Gamma}_\ell^{0\prime}\right\|\left\|\frac{\mathbf{F}_{-1}^0 - \widehat{\mathbf{F}}_{-1}\mathfrak{R}^{-1}}{\sqrt{T}}\right\|\left\|\frac{\mathbf{H}^0 - \widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1}}{\sqrt{T}}\right\| \\
&+ \sqrt{NT}\left(\frac{1}{N}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\left\|\frac{1}{T}\left(\mathbf{F}_{-1}^0 - \widehat{\mathbf{F}}_{-1}\mathfrak{R}^{-1}\right)'\boldsymbol{\varepsilon}_i\right\|\right)\|\mathfrak{R}^{-1\prime}\|\,\|\boldsymbol{\Xi}_L^{-1}\|\left\|\frac{1}{NT}\sum_{\ell=1}^N \widehat{\mathbf{F}}'_{-1}\mathbf{V}_{\ell,-1}\boldsymbol{\Gamma}_\ell^{0\prime}\right\| \\
&+ \sqrt{NT}\left(\frac{1}{N}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\left\|\frac{\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\right)\|\mathfrak{R}^{-1\prime}\|\,\|\boldsymbol{\Xi}_L^{-1}\|\left\|\frac{1}{NT}\sum_{\ell=1}^N \widehat{\mathbf{F}}'_{-1}\mathbf{V}_{\ell,-1}\boldsymbol{\Gamma}_\ell^{0\prime}\right\|\left\|\frac{1}{T}\left(\mathbf{F}_{-1}^0 - \widehat{\mathbf{F}}_{-1}\mathfrak{R}^{-1}\right)'\widehat{\mathbf{F}}_{-1}\right\| \\
&\times\left(\left\|\frac{\widehat{\mathbf{F}}_{-1}}{\sqrt{T}}\right\| + \left\|\frac{\widehat{\mathbf{F}}_{-1}\mathbf{P}_{\widehat{\mathbf{H}}}}{\sqrt{T}}\right\|\right) \\
&+ \sqrt{NT}\left(\frac{1}{N}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\left\|\frac{\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\right)\|\mathfrak{R}^{-1\prime}\|\,\|\boldsymbol{\Xi}_L^{-1}\|\left\|\frac{1}{NT}\sum_{\ell=1}^N \widehat{\mathbf{F}}'_{-1}\mathbf{V}_{\ell,-1}\boldsymbol{\Gamma}_\ell^{0\prime}\right\|\left\|\frac{1}{T}\left(\mathbf{F}_{-1}^0 - \widehat{\mathbf{F}}_{-1}\mathfrak{R}^{-1}\right)'\widehat{\mathbf{H}}\right\|\left\|\frac{\widehat{\mathbf{H}}}{\sqrt{T}}\right\| \\
=& \sqrt{NT}O_p\left(\delta_{NT}^{-2}\right)\left[O_p\left(\frac{1}{\sqrt{NT}}\right) + O_p\left(\frac{1}{N}\right) + O_p\left(\frac{1}{\sqrt{N}\delta_{NT}^2}\right)\right] = O_p\left(\delta_{NT}^{-2}\right) + O_p\left(\sqrt{\frac{T}{N^3}}\right)
\end{aligned}
$$

by Lemmas A.1 (b), (c) and (d) and Lemmas A.4 (a), (b), (c) and (i). If $T/N^3 \to 0$ as $N, T \to \infty$, then $\mathbb{J}_1 = O_p(1)$.

With the definition of $\mathfrak{R}$, $\mathbb{J}_2$ can be reformulated as

$$
\mathbb{J}_2 = -N^{-3/2}T^{-1/2}\sum_{i=1}^N\sum_{\ell=1}^N \boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}'_{\ell,-1}\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i
$$

and $\mathbb{J}_3$ can be written as

$$
\mathbb{J}_3 = -N^{-3/2}T^{-3/2}\sum_{i=1}^N\sum_{\ell=1}^N \boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\widehat{\mathbf{F}}'_{-1}\mathbf{F}_{-1}^0)^{-1}\widehat{\mathbf{F}}'_{-1}\mathbf{V}_{\ell,-1}\mathbf{V}'_{\ell,-1}\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i
$$

Combining the above three terms, we can complete the proof. □

Lemma B.9 *Under Assumptions A to D, we have*

$$
\begin{aligned}
&-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^N\sum_{\ell=1}^N \boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}'_{\ell,-1}\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i \\
=&-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^N\sum_{\ell=1}^N \boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}'_\ell\mathbf{M}_{\mathbf{F}_{-1}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i + O_p\left(\frac{1}{\delta_{NT}}\right) + O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right)
\end{aligned}
$$

Proof of Lemma B.9. Note that $\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i = \mathbf{M}_{\mathbf{H}^0}\mathbf{u}_i$, we can derive that

$$
\begin{aligned}
& -\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i - \left(-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\mathbf{M}_{\mathbf{F}^0_{-1}}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i\right) \\
=& -\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\left(\mathbf{M}_{\widehat{\mathbf{F}}_{-1}} - \mathbf{M}_{\mathbf{F}^0_{-1}}\right)\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\
& -\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\mathbf{M}_{\mathbf{F}^0_{-1}}\left(\mathbf{M}_{\widehat{\mathbf{H}}} - \mathbf{M}_{\mathbf{H}^0}\right)\mathbf{u}_i \\
& -\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\left(\mathbf{M}_{\widehat{\mathbf{F}}_{-1}} - \mathbf{M}_{\mathbf{F}^0_{-1}}\right)\left(\mathbf{M}_{\widehat{\mathbf{H}}} - \mathbf{M}_{\mathbf{H}^0}\right)\mathbf{u}_i \\
=& \mathbb{K}_1 + \mathbb{K}_2 + \mathbb{K}_3
\end{aligned}
$$

Now we consider the term $\mathbb{K}_1$. Then

$$
\begin{aligned}
& -\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'(\mathbf{M}_{\widehat{\mathbf{F}}_{-1}} - \mathbf{M}_{\mathbf{F}^0_{-1}})\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\
=& \frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'(\widehat{\mathbf{F}}_{-1} - \mathbf{F}^0_{-1}\mathfrak{R})\mathfrak{R}'\mathbf{F}^{0\prime}_{-1}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\
& + \frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\mathbf{F}^0_{-1}\mathfrak{R}(\widehat{\mathbf{F}}_{-1} - \mathbf{F}^0_{-1}\mathfrak{R})'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\
& + \frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'(\widehat{\mathbf{F}}_{-1} - \mathbf{F}^0_{-1}\mathfrak{R})(\widehat{\mathbf{F}}_{-1} - \mathbf{F}^0_{-1}\mathfrak{R})'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\
& + \frac{1}{N^{3/2}T^{3/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\mathbf{F}^0_{-1}\big(\mathfrak{R}\mathfrak{R}' - (T^{-1}\mathbf{F}^{0\prime}_{-1}\mathbf{F}^0_{-1})^{-1}\big)\mathbf{F}^{0\prime}_{-1}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\
=& \mathbb{K}_{1.1} + \mathbb{K}_{1.2} + \mathbb{K}_{1.3} + \mathbb{K}_{1.4}
\end{aligned}
$$

The term $\mathbb{K}_{1.1}$ is bounded in norm by

$$
\begin{aligned}
\|\mathbb{K}_{1.1}\| \leq & \frac{1}{N}\sum_{i=1}^{N}\left\|\boldsymbol{\Gamma}_i^0\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\right\|\left\|\frac{\widehat{\mathbf{F}}_{-1} - \mathbf{F}^0_{-1}\mathfrak{R}}{\sqrt{T}}\right\|\|\mathfrak{R}\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}_{-1}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\| \\
\leq & \frac{1}{N}\sum_{i=1}^{N}\left\|\boldsymbol{\Gamma}_i^0\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\right\|\left\|\frac{\widehat{\mathbf{F}}_{-1} - \mathbf{F}^0_{-1}\mathfrak{R}}{\sqrt{T}}\right\|\|\mathfrak{R}\|\left\|(\boldsymbol{\Upsilon}^0)^{-1}\right\| \\
& \times\left(\left\|\frac{\mathbf{F}^{0\prime}_{-1}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\| + \left\|\frac{\mathbf{F}^{0\prime}_{-1}\mathbf{F}^0}{T}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\| + \left\|\frac{\mathbf{F}^{0\prime}_{-1}\mathbf{H}^0}{T}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^{0\prime}}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\right) \\
= & O_p\left(\frac{1}{\delta_{NT}}\right)
\end{aligned}
$$

Similarly, with the fact that $\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\mathbf{F}^0_{-1} = O_p(1)$, we can show that $\mathbb{K}_{1.4} = O_p\left(T^{-1/2}\delta_{NT}^{-2}\right)$.

For the term $\mathbb{K}_{1.2}$, we have

$$\|\mathbb{K}_{1.2}\| \leq \frac{1}{N}\sum_{i=1}^{N}\left\|\mathbf{\Gamma}_i^0\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\mathbf{F}_{-1}^0\right\|\|\mathfrak{R}\|\left\|(\mathbf{\Upsilon}^0)^{-1}\right\|\left\|\frac{\left(\widehat{\mathbf{F}}_{-1}-\mathbf{F}_{-1}^0\mathfrak{R}\right)'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i}{T}\right\|$$

$$\begin{aligned}
\leq&\frac{1}{N}\sum_{i=1}^{N}\left\|\mathbf{\Gamma}_i^0\right\|\left\|\frac{\left(\widehat{\mathbf{F}}_{-1}-\mathbf{F}_{-1}^0\mathfrak{R}\right)'\boldsymbol{\varepsilon}_i}{T}\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\mathbf{F}_{-1}^0\right\|\|\mathfrak{R}\|\left\|(\mathbf{\Upsilon}^0)^{-1}\right\|\\
&+\frac{1}{N}\sum_{i=1}^{N}\left\|\mathbf{\Gamma}_i^0\right\|\left\|\frac{\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i}{T}\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\mathbf{F}_{-1}^0\right\|\|\mathfrak{R}\|\left\|(\mathbf{\Upsilon}^0)^{-1}\right\|\left\|\frac{\left(\widehat{\mathbf{F}}_{-1}-\mathbf{F}_{-1}^{00}\mathfrak{R}\right)\mathbf{F}^{\mathbf{0}}}{T}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\\
&+\frac{1}{N}\sum_{i=1}^{N}\left\|\mathbf{\Gamma}_i^0\right\|\left\|\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{T}\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\mathbf{F}_{-1}^0\right\|\|\mathfrak{R}\|\left\|(\mathbf{\Upsilon}^0)^{-1}\right\|\\
&\times\left(\left\|\frac{\left(\widehat{\mathbf{F}}_{-1}-\mathbf{F}_{-1}^0\mathfrak{R}\right)\mathbf{H}}{T}\right\|+\left\|\frac{\left(\widehat{\mathbf{F}}_{-1}-\mathbf{F}_{-1}^0\mathfrak{R}\right)\mathbf{F}^0}{T}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\|\right)\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^{0\prime}}{T}\right)^{-1}\right\|\\
=&O_p\left(\frac{1}{\delta_{NT}^2}\right)
\end{aligned}$$

Replacing $\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\mathbf{F}_{-1}^0\right\|$ by $\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\right\|\left\|\widehat{\mathbf{F}}_{-1}-\mathbf{F}_{-1}^0\mathfrak{R}\right\|$, we can follow the proof of $\mathbb{K}_{1.2}$ to show that $\mathbb{K}_{1.3}=O_p\left(T^{1/2}\delta_{NT}^{-3}\right)$. Thus, the term $\mathbb{K}_1=O_p\left(T^{1/2}\delta_{NT}^{-3}\right)+O_p\left(\delta_{NT}^{-1}\right)$.

Now we consider the term $\mathbb{K}_2$. Then

$$\begin{aligned}
&-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\mathbf{M}_{\mathbf{F}_{-1}^0}\left(\mathbf{M}_{\widehat{\mathbf{H}}}-\mathbf{M}_{\mathbf{H}^0}\right)\mathbf{u}_i\\
=&\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\mathbf{M}_{\mathbf{F}_{-1}^0}\frac{1}{T}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\mathbf{u}_i\\
&+\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\mathbf{M}_{\mathbf{F}_{-1}^0}\frac{1}{T}\mathbf{H}\boldsymbol{\mathcal{R}}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)'\mathbf{u}_i\\
&+\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\mathbf{M}_{\mathbf{F}_{-1}^0}\frac{1}{T}\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)\left(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\right)'\mathbf{u}_i\\
&+\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\mathbf{M}_{\mathbf{F}_{-1}^0}\frac{1}{T}\mathbf{H}\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{u}_i\\
=&\mathbb{K}_{3.1}+\mathbb{K}_{3.2}+\mathbb{K}_{3.3}+\mathbb{K}_{3.4}
\end{aligned}$$

Following the proofs of the terms $\mathbb{I}_1$ to $\mathbb{I}_4$, we can show that $\mathbb{K}_3=O_p(\delta_{NT}^{-1})+O_p(T^{1/2}\delta_{NT}^{-2})$.

For the term $\mathbb{K}_3$, we have

$$\begin{aligned}
&\left\|\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\left(\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}-\mathbf{M}_{\mathbf{F}_{-1}^0}\right)\left(\mathbf{M}_{\widehat{\mathbf{H}}}-\mathbf{M}_{\mathbf{H}^0}\right)\boldsymbol{\varepsilon}_i\right\|\\
\leq&\sqrt{T}\times\frac{1}{N}\sum_{i=1}^{N}\left\|\mathbf{\Gamma}_i^0\right\|\left\|\frac{\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_{\ell,-1}'\right\|\left\|\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}-\mathbf{M}_{\mathbf{F}_{-1}^0}\right\|\left\|\mathbf{M}_{\widehat{\mathbf{H}}}-\mathbf{M}_{\mathbf{H}^0}\right\|\left\|(\mathbf{\Upsilon}^0)^{-1}\right\|\\
=&O_p\left(\frac{\sqrt{T}}{\delta_{NT}^2}\right)
\end{aligned}$$

This completes the proof. □

Lemma B.10 *Under Assumptions A to D, we have*

$$
\begin{aligned}
&N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\widehat{\mathbf{F}}_{-1}'\mathbf{F}_{-1}^0)^{-1}\widehat{\mathbf{F}}_{-1}'\mathbf{V}_{\ell,-1}\mathbf{V}_{\ell,-1}'\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i\\
=&O_p(\delta_{NT}^{-1})+O_p(N^{1/2}\delta_{NT}^{-2})+O_p(T^{1/2}\delta_{NT}^{-2})
\end{aligned}
$$

Proof of Lemma B.10. Following the way of the proof of Lemma B.6, we can prove this lemma. Thus, we omit the details. □

Lemma B.11 *Under Assumptions A to D, we have*

$$
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{M}_{\mathbf{F}_{-1}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i+O_p\left(\frac{1}{\delta_{NT}}\right)+O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)
$$

Proof of Lemma B.11. Following the way of the proof of Lemma B.4, we can show that

$$
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i+O_p\left(\frac{1}{\delta_{NT}}\right)+O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)
$$

Then, similar to the proof of the term $\mathbb{H}_2$, we can prove that

$$
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{M}_{\mathbf{F}_{-1}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i+O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)
$$

This completes the proof. □

Proof of Proposition 3.2. The term $N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\widehat{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i$, is equal to

$$
\begin{pmatrix}
N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{X}_j'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i\\
N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{X}_{i,-1}'\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i\\
N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i
\end{pmatrix}
\tag{B.124}
$$

Consider the first term in (B.124). By Lemmas B.4, B.5, B.6 and B.7 and the fact that $\mathbf{M}_{\mathbf{H}^0}\mathbf{X}_j=\mathbf{M}_{\mathbf{H}^0}\mathbf{V}_j$, we can derive that

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{X}_j'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i\\
=&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i+O_p\left(\frac{1}{\delta_{NT}}\right)+O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)\\
=&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\mathcal{X}}_j'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i+O_p\left(\frac{1}{\delta_{NT}}\right)+O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)
\end{aligned}
$$

where $\boldsymbol{\mathcal{X}}_j=\mathbf{X}_j-\frac{1}{N}\sum_{\ell=1}^{N}\mathbf{X}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_j^0$.

Consider the second term in (B.124). By Lemmas B.8, B.9, B.10 and B.11, we have

$$
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_{i,-1}'\mathbf{M}_{\widehat{\mathbf{F}}_{-1}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\boldsymbol{\mathcal{X}}_{i,-1}'\mathbf{M}_{\mathbf{F}_{-1}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i+O_p\left(\frac{1}{\delta_{NT}}\right)+O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)
$$

Similarly, for the third term, we can show that

$$
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\boldsymbol{\mathcal{X}}_i'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i+O_p\left(\frac{1}{\delta_{NT}}\right)+O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)
$$

then

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\widehat{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i=\begin{pmatrix}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\mathcal{X}}_j'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i\\ \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\boldsymbol{\mathcal{X}}_{i,-1}'\mathbf{M}_{\mathbf{F}^0_{-1}}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i\\ \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\boldsymbol{\mathcal{X}}_i'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i\end{pmatrix}+O_p\left(\frac{1}{\delta_{NT}}\right)+O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)$$

To proceed, we consider another asymptotic representation. Since

$$\begin{aligned}
&\mathbb{E}\left\|\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}w_{ij}\boldsymbol{\varepsilon}_{it}\mathbf{v}_{\ell t}'\boldsymbol{\Gamma}_\ell^{0\prime}\otimes\boldsymbol{\Gamma}_j^{0\prime}\right\|^2\\
=&\frac{1}{N^3T}\sum_{i_1=1}^{N}\sum_{\ell_1=1}^{N}\sum_{j_1=1}^{N}\sum_{t_1=1}^{T}\sum_{i_2=1}^{N}\sum_{\ell_2=1}^{N}\sum_{j_2=1}^{N}\sum_{t_2=1}^{T}\mathbb{E}\left(w_{i_1j_1}w_{i_2j_2}\boldsymbol{\varepsilon}_{i_1t_1}\boldsymbol{\varepsilon}_{i_2t_2}\mathbf{v}_{\ell_1t_1}'\boldsymbol{\Gamma}_{\ell_1}^{0\prime}\boldsymbol{\Gamma}_{\ell_2}^{0\prime}\mathbf{v}_{\ell_2t_2}\mathrm{tr}\left(\boldsymbol{\Gamma}_{j_1}^{0\prime}\boldsymbol{\Gamma}_{j_2}^{0}\right)\right)\\
=&\frac{1}{N^3T}\sum_{i_1=1}^{N}\sum_{\ell_1=1}^{N}\sum_{j_1=1}^{N}\sum_{t_1=1}^{T}\sum_{i_2=1}^{N}\sum_{\ell_2=1}^{N}\sum_{j_2=1}^{N}\sum_{t_2=1}^{T}\mathbb{E}\left(w_{i_1j_1}w_{i_2j_2}\boldsymbol{\varepsilon}_{i_1t_1}\boldsymbol{\varepsilon}_{i_2t_2}\left(\mathbf{v}_{\ell_2t_2}'\otimes\mathbf{v}_{\ell_1t_1}'\right)\otimes\mathrm{vec}\left(\boldsymbol{\Gamma}_{\ell_1}^{0\prime}\boldsymbol{\Gamma}_{\ell_2}^{0\prime}\right)\mathrm{tr}\left(\boldsymbol{\Gamma}_{j_1}^{0\prime}\boldsymbol{\Gamma}_{j_2}^{0}\right)\right)\\
=&\frac{\sigma_\varepsilon^2}{N^3T}\sum_{i=1}^{N}\sum_{\ell_1=1}^{N}\sum_{j_1=1}^{N}\sum_{t=1}^{T}\sum_{\ell_2=1}^{N}\sum_{j_2=1}^{N}w_{ij_1}w_{ij_2}\mathbb{E}\left(\mathbf{v}_{\ell_2t}'\otimes\mathbf{v}_{\ell_1t}'\right)\mathbb{E}\left(\mathrm{vec}\left(\boldsymbol{\Gamma}_{\ell_1}^{0\prime}\boldsymbol{\Gamma}_{\ell_2}^{0\prime}\right)\mathrm{tr}\left(\boldsymbol{\Gamma}_{j_1}^{0\prime}\boldsymbol{\Gamma}_{j_2}^{0}\right)\right)\\
\leq&\frac{\sqrt{r_x}\sigma_\varepsilon^2}{N^3T}\sum_{i=1}^{N}\sum_{\ell_1=1}^{N}\sum_{j_1=1}^{N}\sum_{t=1}^{T}\sum_{\ell_2=1}^{N}\sum_{j_2=1}^{N}|w_{ij_1}||w_{ij_2}|\left\|\mathbb{E}\left(\mathbf{v}_{\ell_2t}\mathbf{v}_{\ell_1t}'\right)\right\|\left(\mathbb{E}\left\|\boldsymbol{\Gamma}_{\ell_1}^0\right\|^4\mathbb{E}\left\|\boldsymbol{\Gamma}_{\ell_2}^0\right\|^4\mathbb{E}\left\|\boldsymbol{\Gamma}_{j_1}^0\right\|^4\mathbb{E}\left\|\boldsymbol{\Gamma}_{j_2}^0\right\|^4\right)^{1/4}\\
\leq&\frac{\sqrt{r_x}\sigma_\varepsilon^2C}{N}\times\frac{1}{N}\sum_{i=1}^{N}\left(\sum_{j_1=1}^{N}|w_{ij_1}|\right)\left(\sum_{j_2=1}^{N}|w_{ij_2}|\right)\times\frac{1}{N}\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\bar{\sigma}_{\ell_1\ell_2}\leq\frac{\sqrt{r_x}\sigma_\varepsilon^2C^3}{N}
\end{aligned}$$

by Assumptions B3 and E3. Then, we can derive that

$$\begin{aligned}
&\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i=\mathrm{vec}\left(\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i\right)\\
=&\mathrm{vec}\left(\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\boldsymbol{\varepsilon}_i\right)\\
&-\mathrm{vec}\left(\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{H}^0(\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i\right)\\
=&\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}w_{ij}\boldsymbol{\varepsilon}_{it}\mathbf{v}_{\ell t}'\boldsymbol{\Gamma}_\ell^{0\prime}\otimes\boldsymbol{\Gamma}_j^{0\prime}\mathrm{vec}((\boldsymbol{\Upsilon}^0)^{-1})\\
&-\frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}w_{ij}(\mathbf{h}_t^{0\prime}\boldsymbol{\varepsilon}_{it})\otimes\boldsymbol{\Gamma}_j^{0\prime}\mathrm{vec}\left((\boldsymbol{\Upsilon}^0)^{-1}\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{H}^0\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right)\\
=&\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}w_{ij}\boldsymbol{\varepsilon}_{it}\mathbf{v}_{\ell t}'\boldsymbol{\Gamma}_\ell^{0\prime}\otimes\boldsymbol{\Gamma}_j^{0\prime}\times O_p(1)+O_p\left(\frac{1}{\sqrt{NT}}\right)=O_p\left(\frac{1}{\sqrt{N}}\right)
\end{aligned}$$

.

Similarly, we can show that $\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\mathbf{P}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i = O_p(T^{-1/2})$. Then

$$\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\mathcal{X}}_j'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\
=&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\boldsymbol{\varepsilon}_i - \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\mathbf{P}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i - \frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}w_{ij}\boldsymbol{\Gamma}_j^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\
=&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\boldsymbol{\varepsilon}_i + O_p\left(\frac{1}{\delta_{NT}}\right)
\end{aligned}$$

Following the way of the proof of the above term, we can prove that

$$\begin{aligned}
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\boldsymbol{\mathcal{X}}_{i,-1}'\mathbf{M}_{\mathbf{F}_{-1}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_{i,-1}'\boldsymbol{\varepsilon}_i + O_p\left(\frac{1}{\delta_{NT}}\right) \\
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\boldsymbol{\mathcal{X}}_i'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'\boldsymbol{\varepsilon}_i + O_p\left(\frac{1}{\delta_{NT}}\right)
\end{aligned}$$

Thus, we have

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\widehat{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i = \begin{pmatrix}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\boldsymbol{\varepsilon}_i \\ \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_{i,-1}'\boldsymbol{\varepsilon}_i \\ \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'\boldsymbol{\varepsilon}_i\end{pmatrix} + O_p\left(\frac{1}{\delta_{NT}}\right) + O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)$$

With the above proof, we can also show that

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\widehat{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i = \begin{pmatrix}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{X}_j'\mathbf{M}_{\mathbf{F}^0}\boldsymbol{\varepsilon}_i \\ \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_{i,-1}'\mathbf{M}_{\mathbf{F}_{-1}^0}\boldsymbol{\varepsilon}_i \\ \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\boldsymbol{\varepsilon}_i\end{pmatrix} + O_p\left(\frac{1}{\delta_{NT}}\right) + O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)$$

This completes the proof. □

Proof of Theorem 3.2. Substituting $\mathbf{y}_i = \mathbf{C}_i\boldsymbol{\theta} + \mathbf{u}_i$ into $\widetilde{\mathbf{c}}_y$ and multiplying by $\sqrt{NT}$ we have

$$\sqrt{NT}(\widetilde{\boldsymbol{\theta}} - \boldsymbol{\theta}) = (\widetilde{\mathbf{A}}'\widetilde{\mathbf{B}}^{-1}\widetilde{\mathbf{A}})^{-1}\widetilde{\mathbf{A}}'\widetilde{\mathbf{B}}^{-1}\cdot N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\widetilde{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i\,.$$

With Lemmas A.3 and B.2, we have

$$\begin{aligned}
\widetilde{\mathbf{A}} - \mathbf{A} &= N^{-1}T^{-1}\sum_{i=1}^{N}\widehat{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{C}_i - N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\mathbf{M}_{\mathbf{H}^0}\mathbf{C}_i = O_p(\delta_{NT}^{-1})\,, \\
\widehat{\mathbf{B}} - \mathbf{B} &= N^{-1}T^{-1}\sum_{i=1}^{N}\widehat{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\widehat{\mathbf{Z}}_i - N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\mathbf{M}_{\mathbf{H}^0}\mathbf{Z}_i = O_p(\delta_{NT}^{-1})\,.
\end{aligned} \tag{B.125}$$

Since $\mathbf{Z}_i = (\sum_{j=1}^{N}w_{ij}\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_j, \mathbf{M}_{\mathbf{F}_{-1}^0}\mathbf{X}_{i,-1}, \mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)$, we have

$$N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\mathbf{P}_{\mathbf{H}^0}\mathbf{C}_i = \begin{pmatrix}N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}w_{ij}\mathbf{V}_j'\left(\mathbf{P}_{\mathbf{H}^0} - \mathbf{P}_{\mathbf{F}^0}\right)\mathbf{C}_i \\ N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{V}_{i,-1}'\left(\mathbf{P}_{\mathbf{H}^0} - \mathbf{P}_{\mathbf{F}_{-1}^0}\mathbf{P}_{\mathbf{H}^0}\right)\mathbf{C}_i \\ N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{V}_i'\left(\mathbf{P}_{\mathbf{H}^0} - \mathbf{P}_{\mathbf{F}^0}\right)\mathbf{C}_i\end{pmatrix}$$

The first block in the above matrix is $O_p(T^{-1/2})$, since it is bounded in norm by

$$
\begin{aligned}
&\|N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j'\mathbf{P}_{\mathbf{H}^0}\mathbf{C}_i\| + \|N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N} w_{ij}\mathbf{V}_j'\mathbf{P}_{\mathbf{F}^0}\mathbf{C}_i\| \\
\leq & T^{-1/2}\cdot N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\|T^{-1/2}\mathbf{V}_j'\mathbf{H}^0\|\|T^{-1/2}\mathbf{C}_i\|\cdot\|T^{-1/2}\mathbf{H}^0\|\|(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\| \\
&+ T^{-1/2}\cdot N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}|w_{ij}|\|T^{-1/2}\mathbf{V}_j'\mathbf{F}^0\|\|T^{-1/2}\mathbf{C}_i\|\cdot\|T^{-1/2}\mathbf{F}^0\|\|(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\| \\
= & O_p(T^{-1/2}),
\end{aligned}
$$

with the similar argument, the other blocks can be proved to be $O_p(T^{-1/2})$. Then

$$
N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\mathbf{P}_{\mathbf{H}^0}\mathbf{C}_i = O_p(T^{-1/2}). \tag{B.126}
$$

A similar derivation gives

$$
N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{Z}_i'\mathbf{P}_{\mathbf{H}^0}\mathbf{Z}_i = O_p(T^{-1/2}). \tag{B.127}
$$

By Proposition 3.2, (B.125)(B.126), and (B.127), we obtain

$$
\sqrt{NT}.(\widetilde{\boldsymbol{\theta}}-\boldsymbol{\theta}) = (\mathbf{A}'\mathbf{B}^{-1}\mathbf{A})^{-1}\mathbf{A}'\mathbf{B}^{-1}\cdot N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{Z}_i'\boldsymbol{\varepsilon}_i + O_p(\delta_{NT}^{-1}) + O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3}).
$$

As $N, T \to \infty$ with $N/T^2 \to 0$ and $T/N^2 \to 0$, the central limit theorem of the martingale difference in Kelejian and Prucha (2001) can be applicable. Then

$$
\sqrt{NT}(\widetilde{\boldsymbol{\theta}}-\boldsymbol{\theta}) \xrightarrow{d} N(\mathbf{0}, \boldsymbol{\Psi}).
$$

With the equation (B.125), it's sufficient to prove $\tilde{\sigma}_\varepsilon^2$ is the consistent estimator of σ_ε^2. Note that

$$
\mathbf{M}_{\widehat{\mathbf{H}}}(\mathbf{y}_i - \mathbf{C}_i\tilde{\boldsymbol{\theta}}) = \mathbf{C}_i(\boldsymbol{\theta}-\tilde{\boldsymbol{\theta}}) + \mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{H}^0\boldsymbol{\varphi}_i^0 + \mathbf{M}_{\widehat{\mathbf{H}}}\boldsymbol{\varepsilon}_i,
$$

then

$$
\begin{aligned}
&\tilde{\sigma}_\varepsilon^2 - \sigma_\varepsilon^2 \\
= & N^{-1}T^{-1}\sum_{i=1}^{N}(\boldsymbol{\theta}-\tilde{\boldsymbol{\theta}})'\mathbf{C}_i'\mathbf{C}_i(\boldsymbol{\theta}-\tilde{\boldsymbol{\theta}}) \\
&+ 2N^{-1}T^{-1}\sum_{i=1}^{N}(\boldsymbol{\theta}-\tilde{\boldsymbol{\theta}})'\mathbf{C}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{H}^0\boldsymbol{\varphi}_i^0 + 2N^{-1}T^{-1}\sum_{i=1}^{N}(\boldsymbol{\theta}-\tilde{\boldsymbol{\theta}})'\mathbf{C}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\boldsymbol{\varepsilon}_i \\
&+ N^{-1}T^{-1}\sum_{i=1}^{N}\boldsymbol{\varphi}_i^{0\prime}\mathbf{H}^{0\prime}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{H}^0\boldsymbol{\varphi}_i^0 \\
&+ 2N^{-1}T^{-1}\sum_{i=1}^{N}\boldsymbol{\varphi}_i^{0\prime}\mathbf{H}^{0\prime}\mathbf{M}_{\widehat{\mathbf{H}}}\boldsymbol{\varepsilon}_i - N^{-1}T^{-1}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{P}_{\widehat{\mathbf{H}}}\boldsymbol{\varepsilon}_i + N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{t=1}^{T}(\varepsilon_{it}^2 - \sigma_\varepsilon^2).
\end{aligned}
$$

The first term can be easily shown both to be $O_p(\delta_{NT}^{-4})$. The last term is $O_p(N^{-1/2}T^{-1/2})$. Since $\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{H}^0 = \mathbf{M}_{\widehat{\mathbf{H}}}(\mathbf{H}^0 - \widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1})$, the second term is bounded in norm by

$$
N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{C}_i\|\|\boldsymbol{\varphi}_i^0\|\cdot\|T^{-1/2}(\mathbf{H}^0 - \widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1})\|\|\boldsymbol{\theta}-\tilde{\boldsymbol{\theta}}\| = O_p(N^{-1/2}T^{-1/2}\delta_{NT}^{-1}) + O_p(\delta_{NT}^{-4}),
$$

the fourth term is bounded in norm by

$$N^{-1}\sum_{i=1}^{N}\|\boldsymbol{\varphi}_i^0\|^2\cdot\|T^{-1/2}(\mathbf{H}^0-\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1})\|^2=O_p(\delta_{NT}^{-2})\,,$$

the fifth term is bounded in norm by

$$2N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\boldsymbol{\varepsilon}_i\|\|\boldsymbol{\varphi}_i^0\|\cdot\|T^{-1/2}(\mathbf{H}^0-\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1})\|=O_p(\delta_{NT}^{-1})\,.$$

The third term is bounded in norm by

$$2N^{-1}T^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{C}_i\|\|T^{-1/2}\boldsymbol{\varepsilon}_i\|\cdot\|\boldsymbol{\theta}-\tilde{\boldsymbol{\theta}}\|=O_p(N^{-1/2}T^{-1/2})+O_p(\delta_{NT}^{-3})\,,$$

the sixth term is bounded in norm by

$$N^{-1}T^{-2}\sum_{i=1}^{N}\|\boldsymbol{\varepsilon}_i'\widehat{\mathbf{H}}\|^2\leq N^{-1}T^{-1}\sum_{i=1}^{N}\|T^{-1/2}\boldsymbol{\varepsilon}_i'\mathbf{H}^0\|^2\cdot\|\boldsymbol{\mathcal{R}}\|^2+N^{-1}T^{-1}\sum_{i=1}^{N}\|\boldsymbol{\varepsilon}_i\|^2\cdot\|T^{-1/2}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\|^2\,,$$

which is $O_p\left(\delta_{NT}^{-2}\right)$. Collecting the above terms, we have $\tilde{\sigma}_\varepsilon^2-\sigma_\varepsilon^2=O_p(\delta_{NT}^{-1})$. Thus, we complete the proof. □

Proof of Theorem 3.3. Noting $\widetilde{\mathbf{u}}_i=\mathbf{u}_i-\mathbf{C}_i\left(\widetilde{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)$ we have $\frac{1}{\sqrt{NT}}\sum_{i=1}^N\widehat{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\widetilde{\mathbf{u}}_i=\frac{1}{\sqrt{NT}}\sum_{i=1}^N\widehat{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i-\widetilde{\mathbf{A}}\sqrt{NT}\left(\widetilde{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)$. Since $\sqrt{NT}\left(\widetilde{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)=\left(\mathbf{A}'\mathbf{B}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}'\mathbf{B}^{-1}\frac{1}{\sqrt{NT}}\sum_{i=1}^N\mathbf{Z}_i'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i+o_p(1)$ by Proposition 3.2 and defining $\mathbf{L}=\boldsymbol{\Omega}^{-1/2}\mathbf{A}$ we have $\hat{\boldsymbol{\Omega}}^{-1/2}\frac{1}{\sqrt{NT}}\sum_{i=1}^N\widehat{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\widetilde{\mathbf{u}}_i=\mathbf{M}_L\boldsymbol{\Omega}^{-1/2}\frac{1}{\sqrt{NT}}\sum_{i=1}^N\mathbf{Z}_i'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i+o_p(1)$ with $\mathbf{M}_L=\mathbf{I}_{3k}-\mathbf{L}\left(\mathbf{L}'\mathbf{L}\right)^{-1}\mathbf{L}'$ whose rank is $\nu=3k-(\ k+2)$, which yields $\frac{1}{NT}\sum_{i=1}^N\widetilde{\mathbf{u}}'\mathbf{M}_{\widehat{\mathbf{H}}}\widehat{\mathbf{Z}}_i\hat{\boldsymbol{\Omega}}_{NT}^{-1}\sum_{i=1}^N\widehat{\mathbf{Z}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\widetilde{\mathbf{u}}\xrightarrow{d}\chi_\nu^2$ as required. Thus, we complete the proof. □

Online Appendix C: Simulation results

Table C.1: Baseline DGP. Simulation results for $\rho = 0.4$, $\pi_u = 1/4$

	IV					QMLE				
					Case I: $N = 100\tau$, $T = 25\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.400	.010	.057	.068	1.00	.398	.004	.483	.139	1.00
2	.400	.004	.008	.058	1.00	.399	.002	.181	.101	1.00
4	.400	.002	.007	.052	1.00	.400	.001	.099	.112	1.00
					Case II: $N = 25\tau$, $T = 100\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.400	.008	.010	.099	1.00	.400	.003	.085	.082	1.00
2	.400	.004	.002	.068	1.00	.400	.002	.043	.053	1.00
4	.400	.002	.004	.065	1.00	.400	.001	.015	.066	1.00
					Case III: $N = 50\tau$, $T = 50\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.400	.008	.021	.075	1.00	.399	.003	.195	.098	1.00
2	.400	.004	.011	.058	1.00	.400	.002	.089	.070	1.00
4	.400	.018	.002	.053	1.00	.400	.001	.035	.070	1.00

Notes: The data generating process is $y_{it} = \psi \sum_{j=1}^{N} w_{ij} y_{jt} + \alpha_i + \rho y_{it-1} + \sum_{\ell=1}^{2} \beta_\ell x_{\ell it} + u_{it}$, $u_{it} = \sum_{s=1}^{3} \gamma_{si}^0 f_{st}^0 + \varepsilon_{it}$, $x_{\ell it} = \mu_{\ell i} + \sum_{s=1}^{2} \gamma_{\ell si}^0 f_{st}^0 + v_{\ell it}$ $\ell = 1, 2$; $i = 1, .., N; t = -50, .., T$ and the first 50 observations are discarded; $f_{st}^0 = \rho_{fs} f_{st-1}^0 + (1 - \rho_{fs}^2)^{1/2} \zeta_{st}$, $\zeta_{st} \sim i.i.d.N(0, 1)$, $\gamma_{si}^0 \sim i.i.d.N(0, 1)$ for $s = 1, 2, 3$, $\varepsilon_{it} = \varsigma_\varepsilon \sigma_{it} (\epsilon_{it} - 1)/\sqrt{2}$, $\epsilon_{it} \sim i.i.d.\chi_1^2$ with $\sigma_{it}^2 = \eta_i \varphi_t$, $\eta_i \sim i.i.d.\chi_2^2/2$, and $\varphi_t = t/T$ for $t = 0, 1, ..., T$ and unity otherwise; $\gamma_{\ell si}^0 = \rho_{\gamma,1s} \gamma_{3i}^0 + (1 - \rho_{\gamma,1s}^2)^{1/2} \xi_{1si}$, $\gamma_{2si}^0 = \rho_{\gamma,2s} \gamma_{si}^0 + (1 - \rho_{\gamma,2s}^2)^{1/2} \xi_{2si}$, $\xi_{\ell si} \sim i.i.d.N(0, 1)$, $v_{\ell it} = \rho_{\upsilon,\ell} v_{\ell it-1} + (1 - \rho_{\upsilon,\ell}^2)^{1/2} \varpi_{\ell it}$, $\varpi_{\ell it} \sim i.i.d.N(0, \varsigma_\upsilon^2 \sigma_{\varpi_{\ell,i}}^2)$, $\sigma_{\varpi_{\ell,i}}^2 \sim i.i.d.U\,[0.5, 1.5]$ for $\ell = 1, 2$, $s = 1, 2$. We set $\{\rho, \psi, \beta_1, \beta_2\} = \{0.4, 0.25, 3, 1\}$. In addition, we specify $\rho_{fs} = \rho_{\gamma,2s} = \rho_{\upsilon,\ell} = 0.5$ and $\rho_{\gamma,1s} = 0.0$ for all ℓ, s. Case I specifies $N = 100\tau$ and $T = 25\tau$ for $\tau = 1, 2, 4$. Case II specifies $N = 25\tau$ and $T = 100\tau$ for $\tau = 1, 2, 4$. Case III specifies $N = T = 50\tau$ for $\tau = 1, 2, 4$.

Table C.2: Baseline DGP. Simulation results for $\psi = 0.25$, $\pi_u = 1/4$

	IV					QMLE				
					Case I: $N = 100\tau$, $T = 25\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.250	.011	.009	.059	1.00	.250	.013	.054	.109	1.00
2	.250	.005	.020	.040	1.00	.250	.006	.090	.069	1.00
4	.250	.002	.024	.055	1.00	.250	.003	.052	.065	1.00
					Case II: $N = 25\tau$, $T = 100\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.250	.010	.020	.068	1.00	.250	.015	.069	.220	1.00
2	.250	.004	.020	.071	1.00	.250	.006	.045	.128	1.00
4	.250	.002	.010	.051	1.00	.250	.003	.006	.081	1.00
					Case III: $N = 50\tau$, $T = 50\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.250	.010	.087	.078	1.00	.250	.013	.020	.119	1.00
2	.250	.013	.011	.057	1.00	.250	.006	.043	.098	1.00
4	.250	.002	.011	.063	1.00	.250	.003	.003	.055	1.00

Notes: See notes of Table C.1.

Table C.3: **Baseline DGP. Simulation results for** $\beta_1 = 3$, $\pi_u = 1/4$

	IV					QMLE				
	Case I: $N = 100\tau$, $T = 25\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	3.00	.032	.022	.055	.872	3.00	.032	.064	.108	.871
2	3.00	.014	.010	.058	1.00	3.00	.014	.041	.069	1.00
4	3.00	.007	.008	.056	1.00	3.00	.007	.018	.063	1.00
	Case II: $N = 25\tau$, $T = 100\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	3.00	.030	.013	.105	.866	3.00	.034	.236	.189	.847
2	3.00	.015	.003	.179	.957	3.00	.015	.016	.112	1.00
4	3.00	.007	.011	.060	1.00	3.00	.007	.006	.075	1.00
	Case III: $N = 50\tau$, $T = 50\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	3.00	.030	.006	.063	.880	3.00	.030	.043	.102	.917
2	3.00	.013	.002	.058	1.00	3.00	.015	.029	.088	1.00
4	3.00	.007	.002	.052	1.00	3.00	.007	.004	.067	1.00

Notes: The data generating process is $y_{it} = \psi \sum_{j=1}^{N} w_{ij} y_{jt} + \alpha_i + \rho y_{it-1} + \sum_{\ell=1}^{2} \beta_\ell x_{\ell it} + u_{it}$, $u_{it} = \sum_{s=1}^{3} \gamma_{si}^0 f_{st}^0 + \varepsilon_{it}$, $x_{\ell it} = \mu_{\ell i} + \sum_{s=1}^{2} \gamma_{\ell si}^0 f_{st}^0 + v_{\ell it}$ $\ell = 1, 2$; $i = 1, .., N; t = -50, .., T$ and the first 50 observations are discarded; $f_{st}^0 = \rho_{fs} f_{st-1}^0 + (1 - \rho_{fs}^2)^{1/2} \zeta_{st}$, $\zeta_{st} \sim i.i.d.N(0, 1)$, $\gamma_{si}^0 \sim i.i.d.N(0, 1)$ for $s = 1, 2, 3$, $\varepsilon_{it} = \varsigma_\varepsilon \sigma_{it}(\epsilon_{it} - 1)/\sqrt{2}$, $\epsilon_{it} \sim i.i.d.\chi_1^2$ with $\sigma_{it}^2 = \eta_i \varphi_t$, $\eta_i \sim i.i.d.\chi_2^2/2$, and $\varphi_t = t/T$ for $t = 0, 1, ..., T$ and unity otherwise; $\gamma_{\ell si}^0 = \rho_{\gamma,1s} \gamma_{3i}^0 + (1 - \rho_{\gamma,1s}^2)^{1/2} \xi_{1si}$, $\gamma_{2si}^0 = \rho_{\gamma,2s} \gamma_{si}^0 + (1 - \rho_{\gamma,2s}^2)^{1/2} \xi_{2si}$, $\xi_{\ell si} \sim i.i.d.N(0, 1)$, $v_{\ell it} = \rho_{\upsilon,\ell} v_{\ell it-1} + (1 - \rho_{\upsilon,\ell}^2)^{1/2} \varpi_{\ell it}$, $\varpi_{\ell it} \sim i.i.d.N(0, \varsigma_\upsilon^2 \sigma_{\varpi_{\ell,i}}^2)$, $\sigma_{\varpi_{\ell,i}}^2 \sim i.i.d.U[0.5, 1.5]$ for $\ell = 1, 2$, $s = 1, 2$. We set $\{\rho, \psi, \beta_1, \beta_2\} = \{0.4, 0.25, 3, 1\}$. In addition, we specify $\rho_{fs} = \rho_{\gamma,2s} = \rho_{\upsilon,\ell} = 0.5$ and $\rho_{\gamma,1s} = 0.0$ for all ℓ, s. Case I specifies $N = 100\tau$ and $T = 25\tau$ for $\tau = 1, 2, 4$. Case II specifies $N = 25\tau$ and $T = 100\tau$ for $\tau = 1, 2, 4$. Case III specifies $N = T = 50\tau$ for $\tau = 1, 2, 4$.

Table C.4: **Baseline DGP. Simulation results for** $\beta_2 = 1$, $\pi_u = 1/4$

	IV					QMLE				
	Case I: $N = 100\tau$, $T = 25\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	1.00	.033	.034	.065	.861	1.00	.032	.206	.113	.912
2	1.00	.014	.016	.058	1.00	1.00	.014	.070	.083	1.00
4	1.00	.007	.008	.055	1.00	1.00	.007	.040	.061	1.00
	Case II: $N = 25\tau$, $T = 100\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	1.00	.029	.013	.094	.912	1.00	.034	.781	.182	.852
2	1.00	.013	.030	.058	1.00	1.00	.015	.065	.116	1.00
4	1.00	.007	.001	.057	1.00	1.00	.007	.007	.074	1.00
	Case III: $N = 50\tau$, $T = 50\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	1.00	.029	.121	.074	.928	1.00	.030	.232	.117	.933
2	1.00	.013	.018	.058	1.00	1.00	.013	.081	.069	1.00
4	1.00	.007	.015	.056	1.00	1.00	.007	.002	.070	1.00

Notes: See notes of Table C.3.

Table C.5: DGP with spatial-time lag. Simulation results for $\rho = 0.4$.

Panel A	IV					QMLE				
$\pi_u = 1/4$					**Case I:** $N = 100\tau$, $T = 25\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.401	.014	.087	.058	1.00	.398	.004	.461	.134	1.00
2	.400	.007	.117	.054	1.00	.399	.002	.174	.098	1.00
4	.400	.003	.058	.052	1.00	.400	.001	.095	.104	1.00
					Case II: $N = 25\tau$, $T = 100\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.400	.013	.050	.077	1.00	.400	.004	.074	.111	1.00
2	.400	.006	.021	.064	1.00	.400	.002	.032	.055	1.00
4	.400	.003	.018	.049	1.00	.400	.001	.012	.056	1.00
					Case III: $N = 50\tau$, $T = 50\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.400	.014	.067	.062	1.00	.399	.004	.181	.106	1.00
2	.400	.007	.018	.055	1.00	.400	.002	.090	.071	1.00
4	.400	.003	.001	.058	1.00	.400	.001	.031	.057	1.00

Panel B	IV					QMLE				
$\pi_u = 3/4$					**Case I:** $N = 100\tau$, $T = 25\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.401	.018	.014	.066	1.00	.390	.014	2.40	.338	1.00
2	.400	.008	.103	.060	1.00	.396	.006	1.03	.259	1.00
4	.400	.004	.056	.054	1.00	.398	.003	.531	.262	1.00
					Case II: $N = 25\tau$, $T = 100\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.400	.016	.032	.088	1.00	.401	.009	.184	.169	1.00
2	.400	.007	.010	.062	1.00	.400	.004	.057	.098	1.00
4	.400	.004	.012	.050	1.00	.400	.002	.073	.078	1.00
					Case III: $N = 50\tau$, $T = 50\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.400	.017	.006	.078	1.00	.397	.010	.836	.158	1.00
2	.400	.008	.002	.059	1.00	.398	.005	.486	.119	1.00
4	.400	.004	.008	.051	1.00	.399	.002	.211	.094	1.00

Notes: The data generating process is $y_{it} = \psi \sum_{j=1}^{N} w_{ij} y_{jt} + \psi_1 \sum_{j=1}^{N} w_{ij} y_{jt-1} + \alpha_i + \rho y_{it-1} + \sum_{\ell=1}^{2} \beta_\ell x_{\ell it} + u_{it}$, $u_{it} = \sum_{s=1}^{3} \gamma_{si}^0 f_{st}^0 + \varepsilon_{it}$, $x_{\ell it} = \mu_{\ell i} + \sum_{s=1}^{2} \gamma_{\ell si}^0 f_{st}^0 + v_{\ell it}$ $\ell = 1, 2$; $i = 1, .., N$; $t = -50, .., T$ and the first 50 observations are discarded; $f_{st}^0 = \rho_{fs} f_{st-1}^0 + (1 - \rho_{fs}^2)^{1/2} \zeta_{st}$, $\zeta_{st} \sim i.i.d.N(0, 1)$, $\gamma_{si}^0 \sim i.i.d.N(0, 1)$ for $s = 1, 2, 3$, $\varepsilon_{it} = \varsigma_\varepsilon \sigma_{it} (\epsilon_{it} - 1)/\sqrt{2}$, $\epsilon_{it} \sim i.i.d.\chi_1^2$ with $\sigma_{it}^2 = \eta_i \varphi_t$, $\eta_i \sim i.i.d.\chi_2^2/2$, and $\varphi_t = t/T$ for $t = 0, 1, ..., T$ and unity otherwise; $\gamma_{\ell si}^0 = \rho_{\gamma,1s} \gamma_{3i}^0 + (1 - \rho_{\gamma,1s}^2)^{1/2} \xi_{1si}$, $\gamma_{2si}^0 = \rho_{\gamma,2s} \gamma_{si}^0 + (1 - \rho_{\gamma,2s}^2)^{1/2} \xi_{2si}$, $\xi_{\ell si} \sim i.i.d.N(0, 1)$, $v_{\ell it} = \rho_{v,\ell} v_{\ell it-1} + (1 - \rho_{v,\ell}^2)^{1/2} \varpi_{\ell it}$, $\varpi_{\ell it} \sim i.i.d.N(0, \varsigma_v^2 \sigma_{\varpi_{\ell,i}}^2)$, $\sigma_{\varpi_{\ell,i}}^2 \sim i.i.d.U[0.5, 1.5]$ for $\ell = 1, 2$, $s = 1, 2$. We set $\{\rho, \psi, \psi_1, \beta_1, \beta_2\} = \{0.4, 0.25, 0.20, 3, 1\}$. In addition, we specify $\rho_{fs} = \rho_{\gamma,2s} = \rho_{v,\ell} = 0.5$ and $\rho_{\gamma,1s} = 0.0$ for all ℓ, s. Case I specifies $N = 100\tau$ and $T = 25\tau$ for $\tau = 1, 2, 4$. Case II specifies $N = 25\tau$ and $T = 100\tau$ for $\tau = 1, 2, 4$. Case III specifies $N = T = 50\tau$ for $\tau = 1, 2, 4$.

Table C.6: DGP with spatial-time lag. Simulation results for $\psi = 0.25$

Panel A	IV					QMLE				
$\pi_u = 1/4$	**Case I:** $N = 100\tau$, $T = 25\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.250	.017	.070	.047	.998	.250	.011	.087	.105	1.00
2	.250	.009	.046	.058	1.00	.250	.005	.015	.064	1.00
4	.250	.004	.052	.058	1.00	.250	.002	.046	.067	1.00
	Case II: $N = 25\tau$, $T = 100\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.251	.017	.219	.083	.998	.250	.012	.045	.177	1.00
2	.250	.008	.041	.063	1.00	.250	.005	.008	.120	1.00
4	.250	.004	.071	.061	1.00	.250	.002	.011	.077	1.00
	Case III: $N = 50\tau$, $T = 50\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.251	.017	.264	.064	.999	.250	.011	.010	.114	1.00
2	.250	.011	.104	.058	1.00	.250	.005	.016	.088	1.00
4	.250	.004	.013	.063	1.00	.250	.002	.004	.052	1.00

Panel B	IV					QMLE				
$\pi_u = 3/4$	**Case I:** $N = 100\tau$, $T = 25\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.250	.024	.158	.055	.984	.249	.022	.341	.145	1.00
2	.250	.012	.046	.065	1.00	.250	.009	.094	.085	1.00
4	.250	.006	.074	.049	1.00	.250	.004	.109	.073	1.00
	Case II: $N = 25\tau$, $T = 100\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.251	.023	.248	.089	.975	.250	.031	.131	.382	1.00
2	.250	.011	.034	.062	1.00	.250	.013	.162	.233	1.00
4	.250	.006	.093	.060	1.00	.250	.005	.001	.132	1.00
	Case III: $N = 50\tau$, $T = 50\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.251	.023	.394	.076	.983	.250	.023	.018	.196	1.00
2	.250	.011	.009	.063	1.00	.250	.010	.051	.127	1.00
4	.250	.006	.040	.061	1.00	.250	.005	.009	.070	1.00

Notes: The data generating process is $y_{it} = \psi \sum_{j=1}^{N} w_{ij} y_{jt} + \psi_1 \sum_{j=1}^{N} w_{ij} y_{jt-1} + \alpha_i + \rho y_{it-1} + \sum_{\ell=1}^{2} \beta_\ell x_{\ell it} + u_{it}$, $u_{it} = \sum_{s=1}^{3} \gamma_{si}^0 f_{st}^0 + \varepsilon_{it}$, $x_{\ell it} = \mu_{\ell i} + \sum_{s=1}^{2} \gamma_{\ell si}^0 f_{st}^0 + v_{\ell it}$ $\ell = 1, 2$; $i = 1, .., N; t = -50, .., T$ and the first 50 observations are discarded; $f_{st}^0 = \rho_{fs} f_{st-1}^0 + (1-\rho_{fs}^2)^{1/2} \zeta_{st}$, $\zeta_{st} \sim i.i.d.N(0,1)$, $\gamma_{si}^0 \sim i.i.d.N(0,1)$ for $s = 1, 2, 3$, $\varepsilon_{it} = \varsigma_\varepsilon \sigma_{it} (\epsilon_{it} - 1)/\sqrt{2}$, $\epsilon_{it} \sim i.i.d.\chi_1^2$ with $\sigma_{it}^2 = \eta_i \varphi_t$, $\eta_i \sim i.i.d.\chi_2^2/2$, and $\varphi_t = t/T$ for $t = 0, 1, ..., T$ and unity otherwise; $\gamma_{\ell si}^0 = \rho_{\gamma,1s} \gamma_{3i}^0 + (1 - \rho_{\gamma,1s}^2)^{1/2} \xi_{1si}$, $\gamma_{2si}^0 = \rho_{\gamma,2s} \gamma_{si}^0 + (1 - \rho_{\gamma,2s}^2)^{1/2} \xi_{2si}$, $\xi_{\ell si} \sim i.i.d.N(0,1)$, $v_{\ell it} = \rho_{v,\ell} v_{\ell it-1} + (1-\rho_{v,\ell}^2)^{1/2} \varpi_{\ell it}$, $\varpi_{\ell it} \sim i.i.d.N(0, \varsigma_v^2 \sigma_{\varpi_{\ell,i}}^2)$, $\sigma_{\varpi_{\ell,i}}^2 \sim i.i.d.U[0.5, 1.5]$ for $\ell = 1, 2$, $s = 1, 2$. We set $\{\rho, \psi, \psi_1, \beta_1, \beta_2\} = \{0.4, 0.25, 0.20, 3, 1\}$. In addition, we specify $\rho_{fs} = \rho_{\gamma,2s} = \rho_{v,\ell} = 0.5$ and $\rho_{\gamma,1s} = 0.0$ for all ℓ, s. Case I specifies $N = 100\tau$ and $T = 25\tau$ for $\tau = 1, 2, 4$. Case II specifies $N = 25\tau$ and $T = 100\tau$ for $\tau = 1, 2, 4$. Case III specifies $N = T = 50\tau$ for $\tau = 1, 2, 4$.

Table C.7: **DGP with spatial-time lag. Simulation results for** $\psi_1 = 0.20$, $\pi_u = 1/4$

	IV					QMLE				
	Case I: $N = 100\tau$, $T = 25\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.200	.023	.223	.056	.988	.201	.008	.421	.106	1.00
2	.200	.011	.179	.053	1.00	.200	.003	.139	.007	1.00
4	.200	.006	.043	.046	1.00	.200	.002	.145	.072	1.00
	Case II: $N = 25\tau$, $T = 100\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.196	.021	.264	.073	.986	.200	.008	.045	.143	1.00
2	.200	.011	.055	.063	1.00	.200	.004	.015	.107	1.00
4	.200	.006	.123	.062	1.00	.200	.002	.020	.061	1.00
	Case III: $N = 50\tau$, $T = 50\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.196	.021	.257	.061	.989	.200	.007	.138	.111	1.00
2	.200	.011	.094	.057	1.00	.200	.003	.108	.079	1.00
4	.200	.006	.041	.061	1.00	.200	.002	.025	.055	1.00

Notes: The data generating process is $y_{it} = \psi \sum_{j=1}^{N} w_{ij} y_{jt} + \psi_1 \sum_{j=1}^{N} w_{ij} y_{jt-1} + \alpha_i + \rho y_{it-1} + \sum_{\ell=1}^{2} \beta_\ell x_{\ell it} + u_{it}$, $u_{it} = \sum_{s=1}^{3} \gamma_{si}^0 f_{st}^0 + \varepsilon_{it}$, $x_{\ell it} = \mu_{\ell i} + \sum_{s=1}^{2} \gamma_{\ell si}^0 f_{st}^0 + v_{\ell it}$ $\ell = 1, 2$; $i = 1, .., N; t = -50, .., T$ and the first 50 observations are discarded; $f_{st}^0 = \rho_{fs} f_{st-1}^0 + (1-\rho_{fs}^2)^{1/2} \zeta_{st}$, $\zeta_{st} \sim i.i.d.N(0,1)$, $\gamma_{si}^0 \sim i.i.d.N(0,1)$ for $s = 1, 2, 3$, $\varepsilon_{it} = \varsigma_\varepsilon \sigma_{it}(\epsilon_{it} - 1)/\sqrt{2}$, $\epsilon_{it} \sim i.i.d.\chi_1^2$ with $\sigma_{it}^2 = \eta_i \varphi_t$, $\eta_i \sim i.i.d.\chi_2^2/2$, and $\varphi_t = t/T$ for $t = 0, 1, ..., T$ and unity otherwise; $\gamma_{\ell si}^0 = \rho_{\gamma,1s} \gamma_{3i}^0 + (1 - \rho_{\gamma,1s}^2)^{1/2} \xi_{1si}$, $\gamma_{2si}^0 = \rho_{\gamma,2s} \gamma_{si}^0 + (1 - \rho_{\gamma,2s}^2)^{1/2} \xi_{2si}$, $\xi_{\ell si} \sim i.i.d.N(0,1)$, $v_{\ell it} = \rho_{\upsilon,\ell} v_{\ell it-1} + (1-\rho_{\upsilon,\ell}^2)^{1/2} \varpi_{\ell it}$, $\varpi_{\ell it} \sim i.i.d.N(0, \varsigma_\upsilon^2 \sigma_{\varpi_{\ell,i}}^2)$, $\sigma_{\varpi_{\ell,i}}^2 \sim i.i.d.U[0.5, 1.5]$ for $\ell = 1, 2$, $s = 1, 2$. We set $\{\rho, \psi, \psi_1, \beta_1, \beta_2\} = \{0.4, 0.25, 0.20, 3, 1\}$. In addition, we specify $\rho_{fs} = \rho_{\gamma,2s} = \rho_{\upsilon,\ell} = 0.5$ and $\rho_{\gamma,1s} = 0.0$ for all ℓ, s. Case I specifies $N = 100\tau$ and $T = 25\tau$ for $\tau = 1, 2, 4$. Case II specifies $N = 25\tau$ and $T = 100\tau$ for $\tau = 1, 2, 4$. Case III specifies $N = T = 50\tau$ for $\tau = 1, 2, 4$.

Table C.8: DGP with spatial-time lag. Simulation results for $\beta_2 = 1$

Panel A			**IV**					**QMLE**		
$\pi_u = 1/4$					**Case I:** $N = 100\tau$, $T = 25\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	1.00	.040	.169	.052	.738	1.00	.032	.202	.117	.912
2	1.00	.018	.033	.048	.997	1.00	.014	.064	.082	1.00
4	1.00	.009	.027	.054	1.00	1.00	.007	.038	.060	1.00
					Case II: $N = 25\tau$, $T = 100\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	1.00	.022	.050	.073	.757	1.00	.034	.312	.181	.850
2	1.00	.011	.034	.063	.998	1.00	.015	.061	.117	1.00
4	1.00	.006	.003	.062	1.00	1.00	.007	.007	.071	1.00
					Case III: $N = 50\tau$, $T = 50\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	1.00	.036	.102	.058	.780	1.00	.030	.230	.115	.933
2	1.00	.018	.022	.054	.998	1.00	.013	.080	.073	1.00
4	1.00	.009	.022	.052	1.00	1.00	.007	.003	.069	1.00

Panel B			**IV**					**QMLE**		
$\pi_u = 3/4$					**Case I:** $N = 100\tau$, $T = 25\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	1.01	.069	.950	.110	.339	1.06	.093	6.40	.406	.352
2	1.00	.025	.083	.051	.982	1.01	.029	1.34	.141	.981
4	1.00	.012	.037	.054	1.00	1.00	.012	.374	.079	1.00
					Case II: $N = 25\tau$, $T = 100\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	1.00	.049	.008	.083	.531	1.07	.118	7.07	.524	.176
2	1.00	.025	.051	.072	.968	1.02	.040	1.81	.303	.876
4	1.00	.012	.004	.063	1.00	1.00	.014	.359	.134	1.00
					Case III: $N = 50\tau$, $T = 50\tau$					
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	1.00	.050	.210	.063	.528	1.05	.086	5.16	.371	.313
2	1.00	.024	.013	.055	.985	1.01	.028	1.15	.157	.987
4	1.00	.012	.027	.050	1.00	1.00	.012	.225	.092	1.00

Notes: The data generating process is $y_{it} = \psi \sum_{j=1}^{N} w_{ij} y_{jt} + \psi_1 \sum_{j=1}^{N} w_{ij} y_{jt-1} + \alpha_i + \rho y_{it-1} + \sum_{\ell=1}^{2} \beta_\ell x_{\ell it} + u_{it}$, $u_{it} = \sum_{s=1}^{3} \gamma_{si}^0 f_{st}^0 + \varepsilon_{it}$, $x_{\ell it} = \mu_{\ell i} + \sum_{s=1}^{2} \gamma_{\ell si}^0 f_{st}^0 + v_{\ell it}$ $\ell = 1,2$; $i = 1,..,N; t = -50,..,T$ and the first 50 observations are discarded; $f_{st}^0 = \rho_{fs} f_{st-1}^0 + (1-\rho_{fs}^2)^{1/2} \zeta_{st}$, $\zeta_{st} \sim i.i.d.N(0,1)$, $\gamma_{si}^0 \sim i.i.d.N(0,1)$ for $s = 1,2,3$, $\varepsilon_{it} = \varsigma_\varepsilon \sigma_{it}(\epsilon_{it} - 1)/\sqrt{2}$, $\epsilon_{it} \sim i.i.d.\chi_1^2$ with $\sigma_{it}^2 = \eta_i \varphi_t$, $\eta_i \sim i.i.d.\chi_2^2/2$, and $\varphi_t = t/T$ for $t = 0,1,...,T$ and unity otherwise; $\gamma_{\ell si}^0 = \rho_{\gamma,1s} \gamma_{3i}^0 + (1 - \rho_{\gamma,1s}^2)^{1/2} \xi_{1si}$, $\gamma_{2si}^0 = \rho_{\gamma,2s} \gamma_{si}^0 + (1 - \rho_{\gamma,2s}^2)^{1/2} \xi_{2si}$, $\xi_{\ell si} \sim i.i.d.N(0,1)$, $v_{\ell it} = \rho_{v,\ell} v_{\ell it-1} + (1-\rho_{v,\ell}^2)^{1/2} \varpi_{\ell it}$, $\varpi_{\ell it} \sim i.i.d.N(0, \varsigma_v^2 \sigma_{\varpi_{\ell,i}}^2)$, $\sigma_{\varpi_{\ell,i}}^2 \sim i.i.d.U[0.5, 1.5]$ for $\ell = 1,2$, $s = 1,2$. We set $\{\rho, \psi, \psi_1, \beta_1, \beta_2\} = \{0.4, 0.25, 0.20, 3, 1\}$. In addition, we specify $\rho_{fs} = \rho_{\gamma,2s} = \rho_{v,\ell} = 0.5$ and $\rho_{\gamma,1s} = 0.0$ for all ℓ, s. Case I specifies $N = 100\tau$ and $T = 25\tau$ for $\tau = 1,2,4$. Case II specifies $N = 25\tau$ and $T = 100\tau$ for $\tau = 1,2,4$. Case III specifies $N = T = 50\tau$ for $\tau = 1,2,4$.

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Table C.9: **DGP with endogenous covariate. Simulation results for** $\beta_1 = 3$, $\pi_u = 1/4$

	IV					QMLE				
	Case I: $N = 100\tau$, $T = 25\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	3.00	.055	.089	.058	.514	3.63	.634	21.0	1.00	.724
2	3.00	.025	.025	.046	.968	3.60	.602	20.0	1.00	.990
4	3.00	.012	.012	.048	1.00	3.59	.590	19.6	1.00	1.00
	Case II: $N = 25\tau$, $T = 100\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	3.00	.049	.020	.077	.564	3.54	.545	17.9	1.00	.448
2	3.00	.025	.015	.073	.970	3.54	.546	18.1	1.00	.812
4	3.00	.012	.020	.063	1.00	3.56	.566	18.8	1.00	.972
	Case III: $N = 50\tau$, $T = 50\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	3.00	.050	.032	.064	.780	3.58	.578	19.1	1.00	.518
2	3.00	.026	.001	.068	.998	3.58	.576	19.2	1.00	.952
4	3.00	.013	.018	.055	1.00	3.57	.571	19.0	1.00	1.00

Notes: The data generating process is $y_{it} = \psi \sum_{j=1}^{N} w_{ij} y_{jt} + \psi_1 \sum_{j=1}^{N} w_{ij} y_{jt-1} + \alpha_i + \rho y_{it-1} + \sum_{\ell=1}^{2} \beta_\ell x_{\ell it} + u_{it}$, $u_{it} = \sum_{s=1}^{3} \gamma_{si}^0 f_{st}^0 + \varepsilon_{it}$, $x_{\ell it} = \mu_{\ell i} + \sum_{s=1}^{2} \gamma_{\ell si}^0 f_{st}^0 + v_{\ell it}$ $\ell = 1, 2$; $i = 1, .., N; t = -50, .., T$ and the first 50 observations are discarded; $f_{st}^0 = \rho_{fs} f_{st-1}^0 + (1 - \rho_{fs}^2)^{1/2} \zeta_{st}$, $\zeta_{st} \sim i.i.d.N(0, 1)$, $\gamma_{si}^0 \sim i.i.d.N(0, 1)$ for $s = 1, 2, 3$, $\varepsilon_{it} = \varsigma_\varepsilon \sigma_{it} (\epsilon_{it} - 1)/\sqrt{2}$, $\epsilon_{it} \sim i.i.d.\chi_1^2$ with $\sigma_{it}^2 = \eta_i \varphi_t$, $\eta_i \sim i.i.d.\chi_2^2/2$, and $\varphi_t = t/T$ for $t = 0, 1, ..., T$ and unity otherwise; $\gamma_{\ell si}^0 = \rho_{\gamma,1s} \gamma_{3i}^0 + (1 - \rho_{\gamma,1s}^2)^{1/2} \xi_{1si}$, $\gamma_{2si}^0 = \rho_{\gamma,2s} \gamma_{si}^0 + (1 - \rho_{\gamma,2s}^2)^{1/2} \xi_{2si}$, $\xi_{\ell si} \sim i.i.d.N(0, 1)$, $v_{1it} = \rho_{\upsilon,1} v_{1it-1} + (1 - \rho_{\upsilon,1}^2)^{1/2} \varpi_{1it} + 0.5\varepsilon_{it}$, $v_{2it} = \rho_{\upsilon,2} v_{2it-1} + (1 - \rho_{\upsilon,2}^2)^{1/2} \varpi_{2it}$ and $v_{3it} = \rho_{\upsilon,3} v_{3it-1} + (1 - \rho_{\upsilon,3}^2)^{1/2} \varpi_{3it}$, with $\varpi_{\ell it} \sim i.i.d.N(0, \varsigma_\upsilon^2 \sigma_{\varpi_{\ell,i}}^2)$, $\sigma_{\varpi_{\ell,i}}^2 \sim i.i.d.U\,[0.5, 1.5]$ for $\ell = 1, 2$, $s = 1, 2, 3$. $x_{3it} = \mu_{3i} + v_{1it} + \varphi v_{3it}$, where φ is set such that the correlation between $(v_{1it} + 0.5\varepsilon_{it})$ and $(v_{1it} + \varphi v_{3it})$ equals 0.5. We specify $\rho_{fs} = \rho_{\gamma,2s} = \rho_{\upsilon,\ell} = 0.5$ and $\rho_{\gamma,1s} = 0.0$ for all ℓ, s.

Table C.10: Size and power performance for the J test statistic, $\pi_u = 1/4$

	Panel A			Panel B			Panel C		
	size			power			size		
τ	I	II	III	I	II	III	I	II	III
1	.056	.098	.079	.264	.351	.309	.055	.091	.067
2	.057	.065	.067	.821	.897	.889	.051	.066	.057
4	.052	.048	.051	1.00	1.00	1.00	.050	.052	.050

Notes: The data generating process is $y_{it} = \psi \sum_{j=1}^{N} w_{ij} y_{jt} + \psi_1 \sum_{j=1}^{N} w_{ij} y_{jt-1} + \alpha_i + \rho y_{it-1} + \sum_{\ell=1}^{2} \beta_\ell x_{\ell it} + u_{it}$, $u_{it} = \sum_{s=1}^{3} \gamma_{si}^0 f_{st}^0 + \varepsilon_{it}$, $x_{\ell it} = \mu_{\ell i} + \sum_{s=1}^{2} \gamma_{\ell si}^0 f_{st}^0 + v_{\ell it}$ $\ell = 1, 2$; $i = 1, .., N; t = -50, .., T$ and the first 50 observations are discarded; $f_{st}^0 = \rho_{fs} f_{st-1}^0 + (1 - \rho_{fs}^2)^{1/2} \zeta_{st}$, $\zeta_{st} \sim i.i.d.N(0, 1)$, $\gamma_{si}^0 \sim i.i.d.N(0, 1)$ for $s = 1, 2, 3$, $\varepsilon_{it} = \varsigma_\varepsilon \sigma_{it} (\epsilon_{it} - 1)/\sqrt{2}$, $\epsilon_{it} \sim i.i.d.\chi_1^2$ with $\sigma_{it}^2 = \eta_i \varphi_t$, $\eta_i \sim i.i.d.\chi_2^2/2$, and $\varphi_t = t/T$ for $t = 0, 1, ..., T$ and unity otherwise; $\gamma_{\ell si}^0 = \rho_{\gamma,1s} \gamma_{3i}^0 + (1 - \rho_{\gamma,1s}^2)^{1/2} \xi_{1si}$, $\gamma_{2si}^0 = \rho_{\gamma,2s} \gamma_{si}^0 + (1 - \rho_{\gamma,2s}^2)^{1/2} \xi_{2si}$, $\xi_{\ell si} \sim i.i.d.N(0, 1)$, $v_{1it} = \rho_{\upsilon,1} v_{1it-1} + (1 - \rho_{\upsilon,1}^2)^{1/2} \varpi_{1it} + 0.5\varepsilon_{it}$, $v_{2it} = \rho_{\upsilon,2} v_{2it-1} + (1 - \rho_{\upsilon,2}^2)^{1/2} \varpi_{2it}$ and $v_{3it} = \rho_{\upsilon,3} v_{3it-1} + (1 - \rho_{\upsilon,3}^2)^{1/2} \varpi_{3it}$, with $\varpi_{\ell it} \sim i.i.d.N(0, \varsigma_\upsilon^2 \sigma_{\varpi_{\ell,i}}^2)$, $\sigma_{\varpi_{\ell,i}}^2 \sim i.i.d.U\,[0.5, 1.5]$ for $\ell = 1, 2$, $s = 1, 2, 3$. $x_{3it} = \mu_{3i} + v_{1it} + \varphi v_{3it}$, where φ is set such that the correlation between $(v_{1it} + 0.5\varepsilon_{it})$ and $(v_{1it} + \varphi v_{3it})$ equals 0.5. We specify $\rho_{fs} = \rho_{\gamma,2s} = \rho_{\upsilon,\ell} = 0.5$ and $\rho_{\gamma,1s} = 0.0$ for all ℓ, s. Case I specifies $N = 100\tau$ and $T = 25\tau$ for $\tau = 1, 2, 4$. Case II specifies $T = 100\tau$ and $N = 25\tau$ for $\tau = 1, 2, 4$. Case III specifies $N = T = 50\tau$ for $\tau = 1, 2, 4$.

Table C.11: **DGP with normal and homoskedastic errors.** $\rho = 0.4$, $\pi_u = 3/4$

	IV					QMLE				
	Case I: $N = 100\tau$, $T = 25\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.400	.022	.265	.064	.994	.373	.031	6.65	.694	.886
2	.400	.010	.049	.049	1.00	.391	.011	2.32	.520	1.00
4	.400	.005	.037	.055	1.00	.396	.005	1.02	.438	1.00
	Case II: $N = 25\tau$, $T = 100\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.400	.020	.036	.090	1.00	.396	.012	1.08	.152	1.00
2	.400	.010	.067	.077	1.00	.398	.005	.443	.074	1.00
4	.400	.005	.047	.056	1.00	.399	.003	.252	.096	1.00
	Case III: $N = 50\tau$, $T = 50\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.399	.020	.140	.062	.998	.392	.015	2.05	.228	1.00
2	.400	.010	.080	.062	1.00	.396	.006	1.03	.154	1.00
4	.400	.006	.067	.052	1.00	.398	.003	.484	.150	1.00

Notes: The data generating process is $y_{it} = \psi \sum_{j=1}^{N} w_{ij} y_{jt} + \alpha_i + \rho y_{it-1} + \sum_{\ell=1}^{2} \beta_\ell x_{\ell it} + u_{it}$, $u_{it} = \sum_{s=1}^{3} \gamma_{si}^0 f_{st}^0 + \varepsilon_{it}$, $x_{\ell it} = \mu_{\ell i} + \sum_{s=1}^{2} \gamma_{\ell si}^0 f_{st}^0 + v_{\ell it}$ $\ell = 1,2$; $i = 1,..,N; t = -50,..,T$ and the first 50 observations are discarded; $f_{st}^0 = \rho_{fs} f_{st-1}^0 + (1-\rho_{fs}^2)^{1/2} \zeta_{st}$, $\zeta_{st} \sim i.i.d.N(0,1)$, $\gamma_{si}^0 \sim i.i.d.N(0,1)$ for $s = 1,2,3$, $\varepsilon_{it} = \varsigma_\varepsilon \epsilon_{it}$, $\epsilon_{it} \sim i.i.d.N(0,1)$; $\gamma_{\ell si}^0 = \rho_{\gamma,1s} \gamma_{3i}^0 + (1-\rho_{\gamma,1s}^2)^{1/2} \xi_{1si}$, $\gamma_{2si}^0 = \rho_{\gamma,2s} \gamma_{si}^0 + (1-\rho_{\gamma,2s}^2)^{1/2} \xi_{2si}$, $\xi_{\ell si} \sim i.i.d.N(0,1)$, $v_{\ell it} = \rho_{v,\ell} v_{\ell it-1} + (1-\rho_{v,\ell}^2)^{1/2} \varpi_{\ell it}$, $\varpi_{\ell it} \sim i.i.d.N(0, \varsigma_v^2 \sigma_{\varpi_{\ell,i}}^2)$, $\sigma_{\varpi_{\ell,i}}^2 \sim i.i.d.U[0.5,1.5]$ for $\ell = 1,2$, $s = 1,2$. We set $\{\rho, \psi, \beta_1, \beta_2\} = \{0.4, 0.25, 3, 1\}$. In addition, we specify $\rho_{fs} = \rho_{\gamma,2s} = \rho_{v,\ell} = 0.5$ and $\rho_{\gamma,1s} = 0.0$ for all ℓ, s. Case I specifies $N = 100\tau$ and $T = 25\tau$ for $\tau = 1,2,4$. Case II specifies $N = 25\tau$ and $T = 100\tau$ for $\tau = 1,2,4$. Case III specifies $N = T = 50\tau$ for $\tau = 1,2,4$.

Table C.12: **DGP with normal and homoskedastic errors.** $\psi = 0.25$, $\pi_u = 3/4$

	IV					QMLE				
	Case I: $N = 100\tau$, $T = 25\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.250	.026	.134	.064	.964	.247	.034	1.19	.172	.792
2	.250	.012	.048	.062	1.00	.250	.015	.010	.104	1.00
4	.250	.005	.045	.055	1.00	.250	.007	.079	.084	1.00
	Case II: $N = 25\tau$, $T = 100\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.250	.024	.398	.098	.978	.248	.039	2.11	.264	.710
2	.250	.011	.048	.076	1.00	.249	.016	.540	.174	1.00
4	.250	.005	.036	.056	1.00	.251	.008	.203	.106	1.00
	Case III: $N = 50\tau$, $T = 50\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	.249	.024	.354	.067	.984	.249	.034	.303	.218	.816
2	.251	.011	.192	.058	1.00	.250	.015	.140	.100	1.00
4	.250	.005	.016	.052	1.00	.250	.007	.019	.108	1.00

Notes: See notes of Table C.11.

Table C.13: DGP with normal and homoskedastic errors. $\beta_1 = 3$, $\pi_u = 3/4$

	IV					QMLE				
	Case I: $N = 100\tau$, $T = 25\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	3.00	.080	.006	.071	.218	3.01	.089	.455	.200	.232
2	3.00	.038	.006	.060	.753	3.01	.040	.476	.132	.794
4	3.00	.018	.008	.056	1.00	3.00	.018	.187	.088	1.00
	Case II: $N = 25\tau$, $T = 100\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	3.00	.077	.024	.096	.223	3.00	.098	.122	.286	.182
2	3.00	.037	.008	.070	.864	3.00	.040	.067	.170	.736
4	3.00	.018	.019	.063	1.00	3.00	.018	.045	.104	1.00
	Case III: $N = 50\tau$, $T = 50\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	3.00	.075	.037	.074	.263	3.01	.093	.161	.240	.917
2	3.00	.037	.004	.059	.773	3.01	.037	.175	.114	1.00
4	3.00	.017	.004	.049	1.00	3.00	.017	.094	.082	1.00

Notes: The data generating process is $y_{it} = \psi \sum_{j=1}^{N} w_{ij} y_{jt} + \alpha_i + \rho y_{it-1} + \sum_{\ell=1}^{2} \beta_\ell x_{\ell it} + u_{it}$, $u_{it} = \sum_{s=1}^{3} \gamma_{si}^0 f_{st}^0 + \varepsilon_{it}$, $x_{\ell it} = \mu_{\ell i} + \sum_{s=1}^{2} \gamma_{\ell si}^0 f_{st}^0 + v_{\ell it}$ $\ell = 1, 2$; $i = 1, .., N; t = -50, .., T$ and the first 50 observations are discarded; $f_{st}^0 = \rho_{fs} f_{st-1}^0 + (1 - \rho_{fs}^2)^{1/2} \zeta_{st}$, $\zeta_{st} \sim i.i.d.N(0, 1)$, $\gamma_{si}^0 \sim i.i.d.N(0, 1)$ for $s = 1, 2, 3$, $\varepsilon_{it} = \varsigma_\varepsilon \epsilon_{it}$, $\epsilon_{it} \sim i.i.d.N(0, 1)$; $\gamma_{\ell si}^0 = \rho_{\gamma,1s} \gamma_{3i}^0 + (1 - \rho_{\gamma,1s}^2)^{1/2} \xi_{1si}$, $\gamma_{2si}^0 = \rho_{\gamma,2s} \gamma_{si}^0 + (1 - \rho_{\gamma,2s}^2)^{1/2} \xi_{2si}$, $\xi_{\ell si} \sim i.i.d.N(0, 1)$, $v_{\ell it} = \rho_{v,\ell} v_{\ell it-1} + (1 - \rho_{v,\ell}^2)^{1/2} \varpi_{\ell it}$, $\varpi_{\ell it} \sim i.i.d.N(0, \varsigma_v^2 \sigma_{\varpi_{\ell,i}}^2)$, $\sigma_{\varpi_{\ell,i}}^2 \sim i.i.d.U\,[0.5, 1.5]$ for $\ell = 1, 2$, $s = 1, 2$. We set $\{\rho, \psi, \beta_1, \beta_2\} = \{0.4, 0.25, 3, 1\}$. In addition, we specify $\rho_{fs} = \rho_{\gamma,2s} = \rho_{v,\ell} = 0.5$ and $\rho_{\gamma,1s} = 0.0$ for all ℓ, s. Case I specifies $N = 100\tau$ and $T = 25\tau$ for $\tau = 1, 2, 4$. Case II specifies $N = 25\tau$ and $T = 100\tau$ for $\tau = 1, 2, 4$. Case III specifies $N = T = 50\tau$ for $\tau = 1, 2, 4$.

Table C.14: DGP with normal and homoskedastic errors.$\beta_2 = 1$, $\pi_u = 3/4$

	IV					QMLE				
	Case I: $N = 100\tau$, $T = 25\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	1.01	.078	1.13	.076	.263	1.05	.104	5.14	.312	.318
2	1.00	.036	.031	.063	.801	1.02	.041	1.62	.150	.810
4	1.00	.016	.019	.054	1.00	1.00	.017	.397	.086	1.00
	Case II: $N = 25\tau$, $T = 100\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	1.00	.069	.102	.091	.271	1.05	.109	5.07	.368	.212
2	1.00	.033	.008	.058	.864	1.01	.040	1.07	.168	.808
4	1.00	.016	.006	.053	1.00	1.00	.017	.370	.082	1.00
	Case III: $N = 50\tau$, $T = 50\tau$									
τ	Mean	RMSE	ARB	Size	Power	Mean	RMSE	ARB	Size	Power
1	1.00	.072	.115	.085	.928	1.05	.100	5.02	.296	.258
2	1.00	.033	.002	.058	1.00	1.02	.041	1.52	.160	.820
4	1.00	.016	.017	.058	1.00	1.00	.016	.298	.082	1.00

Notes: See notes of Table C.13.

Online Appendix D: Additional results for the application in Section 5

Table D.1 reports results in terms of different specifications and/or different estimation approaches. In particular, Column (1) reproduces Column "Full" in Table 5.1 and is included for ease of comparison. Column (2) treats $INEFF$ as strictly exogenous with respect to the idiosyncratic error. Therefore, $INEFF$ is instrumented by the defactored value of the same variable. It is clear from the p-value of the J-test that this particular identification strategy is not valid, an outcome which confirms that $INEFF$ is subject to reverse causality. Hence, an external instrument is required for consistent parameter estimation. In comparison to Column (1), major discrepancies in Column (2) include: (i) the estimated autoregressive coefficient appears to be biased upwards, with the difference being statistically significant; (ii) the coefficient of operational inefficiency is not statistically significant, which is counterintuitive. This finding shows the importance of the ability of our approach to potentially allow for general forms of endogeneity.

Column (3) reports results from running an IV regression using the same instruments as in Column (1) but without controlling for a common factor component. That is, essentially in this case $\mathbf{M}_{\widehat{\mathbf{F}}} = \mathbf{M}_{\widehat{\mathbf{F}}_{-1}} = \mathbf{M}_{\widehat{\mathbf{H}}} = \mathbf{I}_T$. As expected, the model is rejected based on the J-test. Inconsistency of parameter estimation manifests mainly via the estimated autoregressive and spatial lag coefficients, both of which appear to be biased in opposite directions. This bears important implications for long-run direct and indirect estimated effects. Thus, allowing for unobserved common factors appears to be crucial in this application.

Column (4) reports results from a "naive" model based on two-way fixed effects estimation without spatial effects and common factors. In this case, the magnitude of the estimated autoregressive coefficient is more than twice as much as that in Column (1), which implies that bias exceeds 100%. Moreover, the estimated coefficient of operational inefficiency appears to have a negative sign, which is also indicative of large bias due to reverse causality.

Table D.1: **Additional results on bank risk-taking model (full sample)**

	(1)	**(2)**	**(3)**	**(4)**
$\widehat{\rho}$ (AR parameter)	0.405***	0.541***	0.654***	0.829***
	(0.060)	(0.067)	(0.045)	(0.004)
$\widehat{\psi}$ (spatial parameter)	0.449***	0.382***	0.301***	–
	(0.104)	(0.089)	(0.044)	
$\widehat{\beta}_1$ (inefficiency)	0.331***	0.486	0.352***	-0.095***
	(0.086)	(0.331)	(0.106)	(0.027)
$\widehat{\beta}_2$ (CAR)	0.011**	0.011**	0.010**	-0.122***
	(0.005)	(0.006)	(0.005)	(0.021)
$\widehat{\beta}_3$ (size)	0.031	0.021	0.027	-0.011
	(0.072)	(0.048)	(0.062)	(0.019)
$\widehat{\beta}_4$ (buffer)	-0.033**	-0.016**	-0.021**	0.124***
	(0.015)	(0.012)	(0.011)	(0.021)
$\widehat{\beta}_5$ (profitability)	-0.002	-0.004**	-0.006	0.010
	(0.002)	(0.002)	(0.002)	(0.011)
$\widehat{\beta}_6$ (quality)	0.224***	0.226***	0.237***	0.213***
	(0.035)	(0.045)	(0.032)	(0.015)
$\widehat{\beta}_7$ (liquidity)	1.438***	0.838***	0.671***	.481***
	(0.213)	(0.188)	(0.157)	(0.061)
$\widehat{\beta}_8$ (inst. pressure)	-0.022	0.044	0.033	0.071***
	(0.041)	(0.041)	(0.041)	(0.024)
$\widehat{r}_y$	1	1	0	–
$\widehat{r}_x$	2	2	0	–
J-test	28.649	44.401	58.677	–
	[0.156]	[0.003]	[0.000]	

Notes: Column (1) presents results on the full model, as documented in Section 5.1. In column (2) $INEFF$ is treated as strictly exogenous with respect to the idiosyncratic error and therefore it is instrumented by the defactored value of $INEFF$. Column (3) reports results obtained by running IV estimation using the same instruments as in (1) but without allowing for a common factor component. That is, in this case $\mathbf{M}_{\widehat{\mathbf{F}}} = \mathbf{M}_{\widehat{\mathbf{F}}_{-1}} = \mathbf{M}_{\widehat{\mathbf{H}}} = \mathbf{I}_T$. *Finally, Column (4) reports results from standard fixed effects estimation without spatial effects and common factors. Standard errors in parentheses.* *** $p < 0.01$, ** $p < 0.05$, * $p < 0.10$. *p-values in square brackets.*

Table D.2 below reports further results on the bank risk-taking model. In particular, Column (5) corresponds to the same model as that in Column (1) except that the spatial weighting matrix is computed based on dividend yield as opposed to debt ratio. As we can see, the results are very similar across all coefficients, which indicates that the choice of the spatial weighting matrix is not crucial in this application. Clearly, this is a desirable outcome.

Column (6) adds a spatial-time lag to the specification of the model. The corresponding estimated coefficient, $\hat{\psi}_1$, is not statistically significant at the 10% level. Moreover, the sum of $\hat{\psi}$ and $\hat{\psi}_1$ approximately equals 0.351, which is not statistically different from the spatial coefficient reported in Column (1). This indicates that, conditional on a spatial lag, there is no evidence for a spatial-time lag effect (diffusion).

Finally, Column (7) proxies profitability using the return on assets (ROA), which is defined as income after taxes and extraordinary items (annualized), expressed as a percentage of average total assets. The results between Columns (1) and (7) are almost identical, except this time the effect of profitability appears to be statistically significant at the 5% level.

Table D.2: Further results on bank risk-taking model (full sample)

	(1)	**(5)**	**(6)**	**(7)**
$\widehat{\rho}$ (AR parameter)	0.405*** (0.060)	0.403*** (0.059)	0.413*** (0.062)	0.399*** (0.057)
$\widehat{\psi}$ (spatial parameter)	0.449*** (0.104)	0.446*** (0.104)	0.667*** (0.247)	0.450*** (0.102)
$\widehat{\beta}_1$ (inefficiency)	0.331*** (0.086)	0.330*** (0.086)	0.217 (0.138)	0.322*** (0.083)
$\widehat{\beta}_2$ (CAR)	0.011** (0.005)	0.011** (0.004)	0.010** (0.004)	0.011** (0.004)
$\widehat{\beta}_3$ (size)	0.031 (0.072)	0.033 (0.073)	0.056 (0.074)	0.038 (0.072)
$\widehat{\beta}_4$ (buffer)	-0.033** (0.015)	-0.033** (0.015)	-0.031** (0.014)	-0.031** (0.015)
$\widehat{\beta}_5$ (profitability)	-0.002 (0.002)	-0.002 (0.002)	-0.002 (0.002)	-0.056** (0.025)
$\widehat{\beta}_6$ (quality)	0.224*** (0.035)	0.224*** (0.035)	0.221*** (0.035)	0.194*** (0.038)
$\widehat{\beta}_7$ (liquidity)	1.438*** (0.213)	1.440*** (0.214)	1.430*** (0.211)	1.455*** (0.214)
$\widehat{\beta}_8$ (inst. pressure)	-0.022 (0.041)	-0.022 (0.041)	-0.021 (0.040)	-0.025 (0.041)
$\widehat{\psi}_1$ (spatial-time lag)	–	–	-0.316 (0.304)	
$\widehat{r}_y$	1	1	1	1
$\widehat{r}_x$	2	2	2	2
J-test	28.649 [0.156]	28.051 [0.174]	30.290 [0.035]	28.937 [0.147]

Notes: Column (1) reports results on the full model, as documented in Section 5.1. Column (5) corresponds to the same model as that in Column (1) except that the spatial weighting matrix is computed based on dividend yield as opposed to debt ratio. In Column (6) we added a spatial-time lag into the baseline model. Finally, Column (7) proxies the capital adequacy ratio with another variable, defined as the total risk based capital expressed as a percent of risk-weighted assets. Standard errors in parentheses. *** $p < 0.01$, ** $p < 0.05$, * $p < 0.10$. *p-values in square brackets.*