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A NOTE ON R^2 IN THE INSTRUMENTAL VARIABLES MODEL[1]

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The properties of two goodness-of-fit measures are analyzed for the Instrumental Variables model. One measure is based on the two step approach to IV estimation and is equal to the coefficient of determination in the second stage equation. This measure therefore measures the fit of a transformation of the original model. The other measure is the one proposed by Barten (1987) for the linear model without a constant term. In comparison with the other R^2-measure, Bartens measure has as an advantage that it measures the fit of the original model. As a disadvantage, the measure does not have a relation with the chisquared statistic, testing whether all slope coefficients are equal to zero. (JEL : C30)

Two R^2-measures of goodness of fit are analyzed for the Instrumental Variables (IV) model. One measure is based on the two step approach to IV estimation and is equal to the coefficient of determination in the second stage model. This measure generalizes to the R^2 proposed by Carter and Nagar (1977) for a single structural equation when estimated by 2SLS. The other measure is the one proposed for the linear model without a constant term by Barten (1987). This measure applies to the original model, but does not exhibit a relationship with the standard chi-squared test. Consider the model

$$y = Z\delta + u . \tag{1}$$

The dependent variable y is a T vector ; Z is a Txk_z matrix of regressor variables ; and u is a T vector of random errors. The $\{u_i\}$ are i.i.d. with $E(u) = 0$ and $Var(u) = \sigma^2 I_T$. The matrix Z contains the unity vector, s, and so the familiar partition $Z = [sZ^*]$ and $\delta = (\alpha\ \delta^{*\prime})'$ can be made, with α the constant and δ^* the slope parameters. Furthermore, plim $(T^{-1}Z'Z) = Q_{zz}$ and Z is correlated

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with the errors, $\text{plim}(T^{-1} Z' u) \neq 0$, and so ordinary least squares estimation is not consistent. There exists a $T \times K_w$ matrix of instruments, W, with $k_w \geq k_z$, for which $\text{plim}(T^{-1} W' W) = Q_{ww}$; $\text{plim}(T^{-1} W' u) = 0$; and the instruments are correlated with the variables in Z, with $\text{plim}(T^{-1} W' Z) = Q_{wz} \neq 0$. Q_{zz} and Q_{wz} are finite and so nonstationary variables are ruled out. The consistent instrumental variables (IV) estimator is then defined as

$$\hat{\delta}_{IV} = (Z' P_w Z)^{-1} Z' P_w y = (\hat{\alpha}_{IV} \; \hat{\delta}^{*\prime}_{IV})',$$

where, for general matrix A, P_A is the projection matrix $A(A'A)^{-1} A'$ The IV estimator, $\hat{\delta}_{IV}$, is the same as the ordinary least squares estimator of δ in the model

$$y = \hat{Z}\delta + u*, \tag{2}$$

with $\hat{Z} = P_w Z$, the OLS prediction of Z after regression of Z on W. This is a two step (2S) approach to IV estimation. Note that variables in Z that are not asymptotically correlated with u, like the vector of units, are apart of W, and so part of $\hat{Z}$: $\hat{Z} = P_w [s \; Z^*] = [s \; \hat{Z}^*]$, and hence, $\delta' \hat{Z}' N \hat{Z} \delta = \delta^{*\prime} \hat{Z}^{*\prime} N \hat{Z}^* \delta$, where N is the matrix that transforms variables into deviations from the mean : $N = I - T^{-1} ss'$.

The two estimation approaches, although resulting in the same estimator for δ, give rise to two different definitions of predicted values and residuals. The direct IV estimation gives

$$\hat{y}_{IV} = Z\hat{\delta}_{IV} \; ; \; \hat{u}_{IV} = y - \hat{y}_{IV},$$

whereas the prediction of y using the second stage fit from 2S estimation gives.

$$\hat{y}_{2S} = \hat{Z}\hat{\delta}_{IV} \; ; \; \hat{u}_{2S} = y - \hat{y}_{2S}.$$

This difference, of course, has an impact on R^2 : The goodness of fit in the 2S case will be based on the (dis)similarity between y and $\hat{y}_{2s}$, in the IV case on that between y and $\hat{y}_{IV}$. Obviously, one would ideally consider the IV results alone, and not the results based on a transformation of the original variables. The problem however is that, in the case of direct IV application, the total variation $y' Ny$ cannot be decomposed into explained variation and residual variation, whereas it can in the 2S case, thus

$$y'Ny = \hat{y}_{IV}'\hat{N}y_{IV} + 2\hat{y}_{IV}'\hat{N}u_{IV} + \hat{u}_{IV}'N\hat{u}_{IV},$$

and

$$y'Ny = \hat{y}_{2s}'\hat{N}y_{2s} + \hat{u}_{2s}'\hat{u}_{2s}.$$

The problem with defining an R^2 directly on IV results is therefore equivalent to the problem encountered in the linear model without a constant term ; one could therefore consider a measure put forward by Barten (1987) for this model.

First, the 2S case is discussed. Estimation of (2) is simply OLS estimation and so the coefficient of determination is[2] :

$$R^2_{2s} = \frac{\hat{y}'_{2s}N\hat{y}_{2s}}{y'Ny} = \frac{\hat{\delta}_{IV}'\hat{Z}'N\hat{Z}\hat{\delta}_{IV}}{\hat{\delta}_{IV}'\hat{Z}'N\hat{Z}\hat{\delta}_{IV} + \hat{u}_{2s}'\hat{u}_{2s}} = \frac{\hat{\delta}^{*\prime}_{IV}\hat{Z}^{*\prime}N\hat{Z}^{*}\hat{\delta}^{*}_{IV}}{\hat{\delta}^{*\prime}_{IV}\hat{Z}^{*\prime}N\hat{Z}^{*}\hat{\delta}^{*}_{IV} + \hat{u}_{2s}'\hat{u}_{2s}}$$

R^2_{2s} has the following properties :

(*i*) $0 \le R^2_{2s} \le 1$; $R^2_{2s} = 0$ if $\hat{\delta}^*_{IV} = 0$, $R^2_{2s} = 1$ iff $\hat{u}_{2s} = 0$.

(*ii*) R^2_{2s} is the squared sample correlation coefficient of y and $\hat{y}_{2s}$.

(*iii*) Under $H_0 : \delta^* = 0$, the test statistic $TR^2_{2s}/(1 - R^2_{2s})$ converges in distribution to a chi-squared variable with $K_z - 1$ degrees of freedom (see Carter and Nagar (1977) and Knight (1980)).

(*iv*) Define
$M_w = I - P_w$; $H_{\hat{Z}} = \text{plim}(T^{-1}\hat{Z}'N\hat{Z})$; $G_z = \text{plim}(T^{-1}Z'M_wZ)$. The probability limit of R^2_{2s} is then given by

$$Plim\ R^2_{2s} = \frac{\delta'H_{\hat{Z}}\delta}{\delta'H_{\hat{Z}}\delta + \sigma^2 + \delta'G_z\delta + 2plim(T^{-1}\delta'Z'u)},$$

so plim $R^2_{2s} = 0$ if $\delta^* = 0$; plim $R^2_{2s} = 1$ iff $\sigma^2 + \delta'G_z\delta + 2\text{plim}(T^{-1}\delta'Z'u) = 0$. Note that plim $R^2_{2s} < 1$ even if $\sigma^2 = 0$, reflecting the fact that there are right hand endogenious variables.

2. R^2_{2s} is the measure also advocated by Pesaran and Smith (1994) for the IV model.

Next, Barten's measure will be considered for the results directly based on IV estimation. The measure is defined as (see Barten (1987) :

$$R_B^2 = \frac{\hat{y}_{IV}' N \hat{y}_{IV}}{\hat{y}_{IV}' N \hat{y}_{IV} + \hat{u}_{IV}' \hat{u}_{IV}} = \frac{\hat{\delta}_{IV}' Z' NZ \hat{\delta}_{IV}}{\hat{\delta}_{IV}' Z' Z \hat{\delta}_{IV} + \hat{u}_{IV}' \hat{u}_{IV}}$$

Properties of R_B^2 are :

(*i*) $0 \leq R_B^2 \leq 1$; $R_B^2 = 0$ if $\hat{\delta}_{IV}^* = 0$, $R_B^2 = 1$ if $\hat{u}_{IV} = 0$.

(*ii*) R_B^2 is *not* the squared sample correlation coefficient of y and $\hat{y}_{IV}$.

(*iii*) Because the denominator of R_B^2 will not stay the same if the right-hand side of (1) is changed, the measure is *not* related to the chi-squared statistic for testing the hypothesis that all coefficients except the constant are zero.

(*iv*) The probability limit of R_B^2 is given by

$$plim\ R_B^2 = \frac{\delta' H_z \delta}{\delta' H_z \delta + \sigma^2} = \frac{\delta^{*\prime} H_z . \delta^*}{\delta^{*\prime} H_{z*} \delta^* + \sigma^2},$$

with $H_z = plim\,(T^{-1} Z' N Z)$; $H_{z*} = \text{plim}\,(T^{-1} Z^{*\prime} N Z^*)$, and so plim $R_B^2 = 0$ if $\delta^* = 0$; plim $R_B^2 = 1$ iff $\sigma^2 = 0$.

The results are clear. The problem that $y' N y$ cannot be decomposed into an explained variation and a residual variation on the basis of IV results, can be overcome either at the expense of using 2S results, and so transforming the results of interest, or by using Barten's R_B^2. R_B^2 can be considered to be an estimator of a kind of squared population correlation coefficient in model (1), *viz.*

$$P_2 = \frac{E(\delta' Z' NZ\delta)}{E(\delta' Z' NZ\delta + u'u)},$$

with E the expectation operator, whereas R_{2s}^2 is an estimator of the equivalent squared population correlation coefficient in the second stage equation (2). This fact seems to make R_B^2 superior to R_{2S}^2. The argument against using R_B^2, however, is the problem that its denominator changes if the right-hand side of (1) changes, thus losing the link with the chi-squared statistic.

If (1) is a single structural equation of a simultaneous equations system, including the (over) identifying restrictions, the R^2 measure as proposed by Carter and Nagar (1977, p.42), R^2_{CN}, is identical to R^2_{2s} when the estimation method is 2SLS. Hence, R^2_{CN} can be considered to estimate the squared population correlation coefficient in the estimated second stage model, and, as Knight (1980, p.268) noted, the probability limit of R^2_{CN} equals the probability limit of the OLS-coefficient of determination in the reduced form equation. Also, when the partially restricted reduced from (PRRF) estimation method is used, it is easily seen that R^2_{CN} is a goodness-of-fit measure merely for the partially restricted reduced form equation (Knight, 1980, p.267).

Carter and Nagar (1977, p.45) also proposed an R^2-measure for a complete simultaneous equations system, which is equivalent to the goodness-of-fit measure of McElroy (1977) for seemingly unrelated regression equations. Again, this measure does not apply to the structural system, but to the (restricted) reduced form of it. A generalization of Barten's measure for the structural system, that is invariant to changes in scale of the dependent variable (see Neudecker and Windmeijer (1991)) is

$$R^{2*}_B = \frac{\hat{y}'_{IV}(\Sigma \otimes N)\hat{y}_{IV}}{\hat{y}'_{IV}(\Sigma \otimes N)\hat{y}_{IV} + \hat{u}'_{IV}\hat{u}_{IV}}$$

with Σ the variance matrix of the rows of the $T \times M$ matrix of disturbances of the system. As before, the major problem with R^{2*}_B is that the denominator of the measure is not constant for different linear specifications for the same dependent variables.

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