On the value of life

Colignatus, Thomas

Samuel van Houten Genootschap

20 August 2020

On the value of life

Evaluating the effect of public intervention by a measure that is more impartial and fair with respect to the age groups and the sexes and that balances the effect measures given by the number of lives saved and the amount of life-years gained

Thomas Colignatus, October 16 2003, August 20 2020

http://thomascool.eu

Summary

The national budget affects life and death via its allocations in areas such as traffic safety, flood control, public health and the like. When the cost-effectiveness of an intervention is evaluated, common effect measures are the number of lives extended (saved) and the expected life-years gained. The latter are usually adjusted for quality of life, giving QALYs, and discounted. In models that support decision making on the national aggregates, the subjects can be reduced to representative agents that are scored only on these dimensions. The lives extended measure is impartial to age and sex. The life-years measures however are biased in age and sex, since young people have a higher life expectancy than the old and women have a higher life expectancy than men, and policy advice might reflect that bias. It seems advisable to devise a measure that is more impartial and fair with respect to the age groups and the sexes. An alternative is to value a single life at 100%, and to measure the life-years gain with respect to that 100%. In addition, rather than fine-tune policy with interpersonal utility comparisons, one could choose a utility norm for the representative agent. A possible norm for time preference and diminishing marginal utility of life is the square root. The square root is easier to communicate than logarithmic utility or some rate of discount, but has comparable effect. A life of 100 years then has value 10, a life of 25 years has value 5, so that by age 25 half of life is passed. The considerations of both 100% range and square root utility lead to the following age & sex adjusted gain measure. When a person has age *a,* experiences an event (accident, disease) with a life expectancy of *d* years, but might have an intervention such that the life expectancy could become *e*, then the current effect measures are the single life saved and the absolute life-years gain $x = e - d$, but the proposed compromise gain measure is $g[x \mid a, d] = \sqrt{x} \,/\, \sqrt{a + d + x}$. The square root gives the utility of the representative agent, *g* gives the impact for interpersonal comparison, and aggregate utility is found by summing the g_i over the individuals *i*. For example, saving (from acute death, $d = 0$) a baby ($a = 0$) has the same value, namely 1, whether it is a boy (life expectancy at birth, $x = 75.94$) or girl ($x = 80.71$) (data Statistics Netherlands 2002). As another example, let the unit share $s = x / (a + e)$ be 25% for one person and 81% for another person so that the last person would weigh more than three times as much in this respect. For above gain measure, $g = \sqrt{s}$ and the weight ratio becomes 50% / 90%, so that the last person now weighs less than half so that there is more equality. The paper compares various gain measures within the context of social welfare maximization. The update in 2020 has a more explicit discussion of Fair Innings (FI) and Proportional Shortfall (PS), and it is shown in a better manner that the UnitSqrt can be an acceptable compromise.

Keywords: social welfare; decision making; risk; health; quality of life; cost-effectiveness; discounting; fair innings; proportional shortfall; unitsqrt

Note: This notebook is implemented in *Mathematica*, see Cool (2001a, 2020). The page numbers in the table of contents in the update are off.

Competing interests: The author commercially sells Cool (2001a, 2020). The author might also benefit when the proposed effect measure is adopted.

1. Introduction

1.1. National budget allocations and their impact on life & death

The national budget affects life and death via its allocations in areas such as traffic safety, flood control, public health and the like. To measure the impact of interventions, the literature recognises the following main measures for the value of life:

(1) The number of *lives extended* (saved), the NL measure, with the "rule of rescue" (ROR).

(2) The amount of *life-years gained*, the LY measure.

(3) The amount of *quality adjusted life-years gained*, the QALY measure.

(4) Discounted values of these, at different rates.

(5) Elaborations of above measures, which are the topic of this present paper. A derived measure is the average life-years gain per saved person; another derived measure is the average per participating person. The main criteria are in Section 3.1. This update of 2020 now also includes Appendix G with a more explicit discussion of the "*fair innings*" (FI) by Williams (1997) and the "*proportional shortfall*" (PS) by the Dutch national health insurance, see Van de Wetering et al. (2013) and in Dutch ZIN (2018ab). The discussion of these alternative measures was more implicit in 2003 and can better be made more explicit. The UnitSqrt measure combines the best features of FI and PS. Thus it is more useful to maintain the body of the text of 2003, that develops the notion of the UnitSqrt, and refer readers interested in the comparison with FI and PS to Appendix G.

The measures are distinguished by counting ("how many") and weighing ("how much"). Financial benefits and costs will be included to derive the (Incremental) Cost-Effectiveness Ratio's (ICER) of the (net) cost per life saved or cost per life year gained. Gold et al. (1996) discuss these measures from the viewpoint of public health. Jonkman et al. (2003) give an overview for e.g. environmental risks, industrial accidents and flood risks. The Dutch environmental planning bureau RIVM report (2003) recalls the 1989 Dutch official risk target of at most 1 death per million per year (apart of crude annual mortality of some 8800 per million in general) and it discusses ways to deal with rising risks.

Since the measures of *lives extended* and *life-years gained* are different, they can cause different conclusions on the priority of budget allocations. The two main dimensions where they differ are age and sex. The advantage of the lives extended measure is that it seems neutral on age and sex, while the life-years gained measure is not. For example, consider the situation of full recovery after the intervention, so that there is also no difference between LY and QALY. Since the young have a higher life expectancy than the old, the LY measure will cause a bias to the advantage of the young and the disadvantage of the old. Also, women live longer than men, on average, and thus the use of the LY measure will cause a bias to the advantage of women and the disadvantage of men. Full recovery is only an example, and the situation will differ per accident or disease, but it is clear that there could be a bias. Indeed, if one chooses for the life-years measure, this bias is apparently the preferred one.

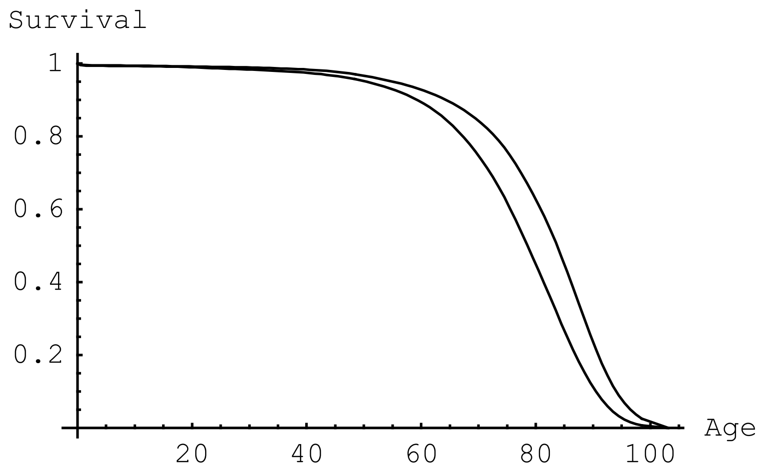
There is an argument that the life-years gained are the more basic measure and that the measure of the lives saved is only a proxy. In some cases only the number of lives saved are mentioned, but it is often rather clear what the implication would be for the number of life-years gained. For floods, for example, one would take the average life expectancy, and multiply this with the number of lives saved. In such cases, translations can be made, at least in principle. This would suggest that the life-years gained measure is the more relevant effect.

This argument that selects the life-years gained as the basic measure however runs counter to the implied bias for age and sex. It will be useful to give an example. In the Netherlands 2002, life expectancy at birth is 75.94 years for boys and 80.71 years for girls (Statistics Netherlands 2002). The difference between men and women is reduced when their life expectations are discounted, but still remains sizeable, especially at older age when there are less years for discounting. The following table gives some key data, where the "average person" is determined using population weights.

Life expectancy	At birth			At 65		
Sex	Male	Female	Average	Male	Female	Average
No discounting	75.94	80.71	78.35	15.78	19.3	17.61
At 3%	30.25	30.72	30.49	12.61	14.71	13.73

For the research on public interventions, the table has the implication that, at 3% discounting, a male baby is 30.25 / 12.61 = 2.4 times more important than a 65 old male. A programme of the same cost should save at least 2.4 male pensioners before it can outweigh a programme on saving male babies.

The following is the survival plot for Dutch men and women.



Consider a theoretical example that highlights the issue. Suppose that 1000 men and 1000 women are screened for some disease at age 65 with a test that costs $100 per person, and that 1 man and 1 woman are saved from nearly immediate death, with full life expectancy restored. The discounted Cost-Effectiveness Ratio's in terms of dollars per life year gained per man, woman and average person are:

```
100 * 1000 / {12.61, 14.71, 13.73}
```

{7930.21, 6798.1, 7283.32}

If the cut-off point is chosen between $6798 and $7930 per life year gained, then only women are treated. The health authority may try to impose equality for the sexes by various methods such as: (a) decide on the maximum, (b) use a different age cut-off point per sex, (c) compensate by tests on other diseases, (d) assign different budgets, (e) impose the use of the average life expectancy and forbid the distinction by sex for allocation purposes. Precisely that such rules could be used, suggests that the life-years gained measure is not sufficient.

From the viewpoint of life-years, having the male sex is a serious disease that might warrant prenatal screening. For biological reproduction only a small number of males needs to be born. For example, if 1 in a 1000 of births were male, mankind still is not in danger of extinction, and this would add approximately 5 * 499 life-years per 1000 persons, or 2.5 years per person, which would be a major achievement in public health (as measured by life-years gained). By comparison, Van den Akker-Van Marle et al. (2002) estimated that total elimination of cervical cancer would yield a gain in life expectancy of 46 days per person (no discounting).

Conversely, could we find a measure such that this dramatic difference between the ages and sexes disappears, so that neither age nor sex itself qualifies as a disease?

The QALY measure is intended for its own sake, namely quality adjustment of health (and not utility), but it is also thought sometimes to provide a compromise between lives extended and life-years gained. Indeed, having more measures adds to the richness of the decision situation, but may also create confusion when conflicting decisions can be supported. In this situation the QALY measure has gained some prominence as the single compromise measure to use. However, Gold et al. (1996:8) rightly remark: "To calculate the total health effect of an intervention, analysts sum all quality-adjusted life-years. This simple addition implies that QALY's are of equal value no matter who gains them or when they occur during the life span. Both intuition and research suggest that this is not the case and that deviations from this assumption are substantial and important". For the present discussion it is relevant that the QALY measure still appears to be biased on age and sex as well. The proposed QALY measures try to incorporate elements of age and sex, for example by scoring subjects "as appropriate for a person's age" (Gold et al. (1996:128)), but there still remains the overall bias. One wonders about a quality index that compares male and female life, and life at different ages.

It may be noted that biasing one group via one programme will again increase their bias for other programmes as well. This cumulating bias would only stop when the quality of life would diminish due to over-treatment. It may be doubted whether this is the proper direction for public policy making.

1.2. The objective of this analysis

This paper considers the question whether we could find a reasonable balance between these measures, such that we could use an age and sex adjusted measure of the (quality adjusted) life-years gain of the intervention. Such a measure would correct the biases of the two original measures, and would absolve us from choosing the one or the other. Clearly, such a measure involves moral choices, but it must be noted that such choices are made anyhow, while the use of an intersubjectively accepted measure could streamline these discussions and decisions.

The issue is considered here from the viewpoint of modeling and decision making at the population level. Thus we do not consider the alternative positions of the individual level (at bedside, or clinical) or for health institutions or for example insurance companies. Of course, there is the generally felt desire that the same measure of success is used at all levels, but it must also be observed that this goal has some limitations. At the individual level, there is a wealth of information about the individual person so that subtler moral considerations are possible. This wealth of information does not exist at the population level. In the future, with the advance of information and computer technology, and in a perhaps more ideal system of public decision making, such information could be accessible also for public planning. But in the current situation, planning at the population level will have to use models of representative agents that rely on only a few characteristics. Indeed, in current modeling, the representative agent is reduced to sex, age, life saved and life-years gained. With such limited information, care must still be taken to prevent biases. This paper thus must be seen in the modeling context, without implications for the individual level, and the proper question considered is whether aggregate validity can be enhanced by better handling of the characteristics of the representative agent.

Though much of the argumentation will be readily understood, such as the bias for age and sex, I have decided to still write out the various smaller steps, so that clarity should be enhanced.

1.3. The context of public decision making

1.3.1. A reference model for utility maximization

In this discussion, there is a clear distinction between the private and public point of view. The situation can be clarified by using the Lipscomb et al. (1996:236) two-period model, where an individual maximises utility U over health H and composite consumption X subject to a budget constraint of B with market rate of interest r:

$$\max \qquad U = U(H_1, X_1, H_2, X_2)$$

$$\text{subject to} \qquad p_1 H_1 + p'_1 X_1 + (p_2 H_2 + p'_2 X_2) / (1 + r) = B$$

For twice differentiable utility that is concave in each argument (see Lipscomb op. cit. for other caveats), a condition for the optimum is:

$$\left(\frac{\partial U}{\partial H_1} / \frac{\partial U}{\partial H_2}\right) = (1 + r)\ p_1 / \ p_2$$

Consider now the public health context, however. This context adds other constraints, for example to warrant that a minimal level of health is attained, using minimum standards $H_{t,\,\mathrm{MIN}}$ as targets and subsidies S as means:

$$\text{also subject to: } H_t \geq H_{t,\,\mathrm{MIN}} \;\; \& \;\; S_t = p_t\, H_{t,\,\mathrm{MIN}} \;\; \& \;\; \text{replace } B \rightarrow B' = B + S_1 + S_2 \,/\, (1+r)$$

This example is chosen such that the public health conditions can be eliminated by substitution and relabeling the variables into $h_t = (H_t - H_{t,\,\mathrm{MIN}})$, so that the problem takes public health efforts as given and concentrates on health 'outside the system of public health'.

With this extension, the model describes a game situation where both the individual and the public are deciding on the individual's health. Typical public health considerations can be recognised here. When health is provided within the system, some people may be induced to risk it by other activities outside of it. If the public health authority would only hand out money to the poor so that they can acquire their own services in the market, then there is the risk that the money is spent on other items than health, such as alcohol and drugs. Similarly, it might be cheaper to compensate people for a loss of health rather than to prevent that loss, though that is not accepted public health policy.

1.3.2. Public and private decision making

The measures of effect and their evaluation, our question on finding a compromise and the approach for answering it are best understood within the general context of public decision making. It is only this framework of public decision making that makes it sensible to try to find a balance between a life and a (quality adjusted) life year. In the other context of commercial decision making, this question loses most if not all of its relevance.

In the market perspective, the value of life is a residual that follows from other decisions. Each individual determines the value of his/her life him/herself, determines the willingness to pay for protective and medical services, and tries to find that service at that price. The market also provides that service as long as it is profitable to do so. Equilibrium is defined in terms of price and marginal cost of service. Thus, demand and supply concern health services, variable H above is well-defined in terms of health services rather than health itself, and it is dubious whether health can be measured intersubjectively. The aggregate relation to the lives extended or life-years gained is only a statistical result and not a primary objective. People are interested in their own particular combination of life saved and life-years gained, the market average is a mix of all these individual preferences, and this mix need not have a stable relation with the protective and medical services and their costs.

In the public context, the objective is different. It is to maximise an effect subject to a given budget. The marginal value of a life or life year follows from the budget, which might affect the decision to change the budget. The decisions involve interpersonal comparison of utility, and there could be norms that abstract from individual variation in order to arrive at impartiality. The objective could be for example that one maximizes the life-years of the most disadvantaged (which would follow Rawl's criterion). In the theory of public decision making (see e.g. Mueller (1989)), there are two main reasons for collective allocation of funds:

(a) the good is a public good, i.e. without the possibility to discriminate between people,

(b) there are merit and demerit goods, generally causing taxes for the rich and subsidies for the poor.

These two theoretical reasons are not without problems for practical application. For flood control, it seems clear that a flood would affect all people indiscriminately in the flooded area, but on second thought one realises that the rich may have boats or helicopters. For public health, an infectious disease affects all, but the rich might still pay for better care. What transpires from these theoretical and practical points is that one can always recognize two levels in an implementation: (i) there is a basic infrastructure that applies to all and (ii) there is a superstructure that allows for individual variability. Sometimes the infrastructure that applies to all has the form of a provision for the poor only, but then the assumption is that the better-off provide for their own anyway. This two-leveled structure is the main cause why QALY measurements enter the discussion: it is a method to handle the basic infrastructure that is not part of the market mechanism.

Addendum 2020: Within the public sphere, there still remains the problem of "the cricket and the ant". If there would be an ideal scoring measure for ants, then what to do with a cricket that has been living irresponsibly and that, facing disease or even death, wants to be treated like an ant ? The ideal measure would apply for a general class of deserving persons, consisting of generally responsible behaviour, perhaps in practice calibrated on the lives of doctors and health workers. The class of ants can be extended with youngsters likely with a grace period up to 25 years where youngsters might need to learn from accidents. Ants of all ages would also have an occasional glitch, that would be regarded as insurable. Concerning the crickets, there seem to be no developed ideas for chronic irresponsible behaviour of the 25+ ages who either burden the public health system or perhaps at times simply are neglected. It seems unavoidable that national insurance also develops a system that keeps track of irresponsible behaviour, with bonus / malus categories. However, the latter discussion should not be confused with the present topic of valuing deserving ant lives.

1.3.3. Social welfare theory

Any effort to find a proper effect measure runs into the issue in Social Welfare Theory that relates to the 'non-comparability of individual utilities' and the 'impossibility of aggregating preferences'. Gold et al. (1996:32) remark: "A substantial literature, spanning economics, philosophy, and political science, addresses the possible specifications of the social welfare function and the ways that such a distributive scheme might be elicited from the views of members of society. The literature suggests that there is no consensus on the specific form the social utility function should take; it appears to be impossible to select a specific weighing scheme from any universally accepted set of first principles (Sen, 1995). Consequently, much of the economic literature concerned with improvements in well-being avoids choosing weights to be attached to the utilities of different individuals."

It appears, however, that there exists a common misunderstanding about that 'impossibility theorem', and it appears that more is possible than commonly thought. This is discussed more extensively by Colignatus (2001b, 2014). In summary, a society will always make a decision anyhow, and hence the moral and social objective is to find a decent solution that can be substantiated. It is better to define 'the ideal' as something that can be realistically attained rather than see the ideal as a combination of axioms that cannot be combined anyway (as happens with the axioms of that 'impossibility theorem'). People may not agree on everything, but they could agree on what is workable. Hence, there is scope for an effort to find a measure for the value of life that would work in practice. When it can be shown that the current measures of *lives saved* and *life-years gained* have drawbacks that can be repaired, then there would be an advance in decision making.

1.3.4. The role of money

A key point is the role of money. It actually needs an explanation why money is not a generally accepted measure for the success of public health, traffic safety or flood control. For an economist, money is a medium of exchange and store of value, and in principle it might be used as a measure of success. Money only tells that so much of x trades for so much of y, and nothing else. When the payment for flood control or hospital bills or car safety features buys additional years of life, then an estimate of the consumers' surplus in terms of money is an implied measure of success, and it exists whether we think it morally suspect or not. Yet, money is not a commonly accepted measure of effect. Part of the explanation is 'paternalism' ('merit goods') where a majority decides what is good for the minority, for example that the poor should not simply get money but might be furnished with better health. Part of the explanation however also derives from the need to account for public funds. It requires a system of management to identify who are the true poor and who are only pretending to be poor, and it requires a similar system of accountability to increase the likelihood that the subsidies are allocated properly. This aspect of management and control is the main reason for the use of QALY's in the public domain.

Interestingly, the former Dean of the Yale University School of Medicine remarked in a speech for the Australian Society for Medical Research, Rosenberg (2002):

"First, increases in life expectancy in the United States between 1970 and 1990 were worth roughly US$2.8 trillion dollars a year. This huge sum represents a rate of return on the research investment of greater than a hundred to one! Second, reduced mortality from cardiovascular disease alone was estimated to be worth US$1.5 trillion a year. Third, improvements in life expectancy account for nearly half of the actual gain in US living standards during the past 50 years. Fourth, the likely returns from future medical research are so high that the pay-off for any plausible portfolio of investments will be enormous. For example, research that would lead to reducing cancer deaths by as little as 10% would be worth US$4 trillion. I was stunned by these results." And: "I was always taught to consider these outcomes as incalculable. To have an economic value put on our national investment and to find that it was so large was surprising and exhilarating."

1.3.5. Discounting

Since we deal with life, we also deal with time, time preference and discounting. Discounting will feature strongly in our discussion. Cohen (2003), whose paper is by itself advisable, rightly calls attention to the fact that the reference model for utility maximization, given above and taken from Gold et al. (1996), has been formulated such that it tends to imply discounting. Above, that reference model was used primarily to distinguish the private from the public context. It is not necessarily the right model to deal with time in that context. In the discussion below, the suggestion arises that social discounting (e.g. at the long run market rate of interest) only applies to costs and that decisions on life could be done differently.

1.3.6. Solution approach

From these considerations it would seem to follow that the effect measures have only limited meaning at the aggregate level. This insight has the consequence that a model with a representative agent would seem to be acceptable. For the present purposes, it could suffice to simply impose a norm for that representative agent rather than develop a complete model that would capture all individual variability.

1.4. The structure of the argument

1.4.1. The proposed solution

The structure of the argument below can best be understood when also the structure of the proposed solution is recognised. Let us first define the key parameters in the problem. Let a person have age a with a remaining life expectancy of $e[a]$ that is also sex dependent (but not put into formulas). At birth $e^* = e[0]$. We can use symbol y to stand for the life-years, either expected or including past life. When the person becomes ill or has an accident with a life expectancy of d years, there might be a treatment or intervention such that the life expectancy could become $e = d + x$, possibly different from full recovery $e[a]$, so that the LY effect measure is the (expected) absolute gain $x = e - d$. The variable x can be seen as added to d or as taken away from e because of the accident or disease.

These variables can also be used to represent a *burden of disease* calculation. Namely, the burden of a particular disease is the loss in life expectancy $burden = e[a] - d = x$, where d is the remaining expectancy after incurring the disease. Prevention of the accident or disease can be seen as an intervention that "restores" the original life expectancy that otherwise would have been lost. The present format is more general than for only $e[a]$, since it also allows for interventions that result in e rather than restoring $e[a]$.

At issue is now to find a transform that would adjust for age and sex. The solution approach recognises the following steps:

(1) There is a measure for (quality adjusted) life-years y. (Life itself is implied by the existence of the person.)

(2) The representative agent has utility function $u[y]$ that applies to those (quality adjusted) life-years y. The model of such a representative agent is imposed by the public authority. The proposal is to use $u[y] = \sqrt{y}$.

(3) There is a social gain function $g[u, y]$ that applies to that utility (taking account of both life and life-years). Society then maximises the sum of all individual gains, $\sum_i g_i$. The proposed gain function is $g[x \mid a, d] = u[x] / u[a + d + x]$. With the square root as utility, the gain function will be called the "unit square root" (UnitSqrt).

(4) When there are a-1 quality adjusted years with scores $0 \le q_i \le 1$, then total quality of life enjoyed at the beginning of calendar age a is $aq = \Sigma\, q_i$. A quality adjusted gain has $g[xq \mid aq, dq]$, with $eq = dq + xq$ the quality adjusted expectation for the particular intervention. Observe that aq can be found by simply adding the q's for the relevant calendar age, while the expectation eq tends to require the cohort life table (for the population average).

When the agents would be altruistic then the gain could be seen as full part of individual utility. However, it suffices here to distinguish between individual utility and its impact on social welfare via the gain function and the summation of all gains. This formulation neglects interaction, say between mother and child, and assumes that all gains can be assigned to individuals. The weight of an individual in social welfare depends upon the shape of the gain function.

In the following, we will discuss some utility functions, notably discounting, logarithmic utility and the square root, and see what gain functions could be used. The combined criteria will be illustrated by graphs and numerical examples.

1.4.2. Individual discounting versus social discounting

Discounting will feature strongly in this discussion. The common assumption of diminishing marginal utility of any consumption good also seems to apply to the length of life, and causes here the proposal of the square root utility. This is similar to discounting of life-years gains, but this similarity needs to be discussed.

There is a difference between discounting for an individual and discounting over various individuals (over time). Apparently, already A.C. Pigou warned that discounting could define away our descendants. F.P. Ramsey, the early theorist of dynamic consumption theory, was a strong proponent for a zero discount rate. Hueting (1991, 1992) restates that discounting implies a zero preference for environmental sustainability since all benefits for future generations are basically discounted to zero. Hueting's solution is to impose a constraint so that all variables evolve subject to that constraint. Only the existing population is relevant, since our preferences for the future generations enter our utility functions, and cause us to impose that constraint (or not), see Hueting & De Boer (2019).

The proposed UnitSqrt measure has an individual and aggregate aspect. The square root has the same effect as discounting the life expectancy. Discounting for an individual seems allowable, since it applies to only one individual. Interpersonal comparison of utility of different individuals is achieved by valuing each individual total life at 100% each, and to value the extension of life by the UnitSqrt contribution to that 100%. Thus, time preference and diminishing marginal utility are associated for an individual, but there is no such consideration for a group of individuals. The idea then is that discounting for groups could be a category mistake.

1.4.3. Discounting and quality of life

The issues of discounting and quality of life adjustment complicate the discussion. Note how time preference and diminishing marginal utility enter the discussion:

(i) diminishing marginal utility applies to the sum of life-years,

(ii) quality of life scores of the separate years could be discounted separately.

When there are n quality adjusted years with scores $0 \leq q_i \leq 1$, then total quality of life is $y = \Sigma\, q_i$, the average quality is y / n, and then the utility of the representative individual would at least be $u[y, n, y / n]$. Here y can be both the expectation and realised life. Under the time trade-off interpretation of quality of life, y is a sufficient statistic that contains all relevant information on the other two variables, and then $u[y, n, y / n]$ collapses to $u[y]$ again. However, there arises a difference between utility and discounting, for example with $\sqrt{q_1 + ... + q_n}$ versus PV[$\{q_1, ..., q_n\}$, r] for rate of discount r. For example, the square root utility of two periods $\{1, q\}$ and $\{q, 1\}$ has $\sqrt{1 + q} = \sqrt{q + 1}$ for some $0 < q < 1$, but this will not hold for discounting. Diminishing marginal utility on how much the agent lives now differs from time preference on when the agent lives. When there is no quality adjustment, i.e. when $y = n$, then a utility function with diminishing marginal utility and discounting are comparable, but otherwise they differ.

The market and the public authority have different objectives here.

(a) For the market, each decision would be taken with more knowledge about the individual characteristics than would be available by other methods, at least in principle. Consumers would willingly pay a price for services that they want, to avoid undesirable states of health. What is required is a clear description of the health states and information about the treatment and its likelihood of success. Quality of life measurement as the valuation of the effect on a [0, 1] scale does not add information on the state itself. The price that equilibrates the market concerns the service that might cause that effect, but this is not the price of the effect itself.

(b) For public decision making, the allocation of resources concerns public goods and (de-) merit goods. Something like a quality of life measurement might sometimes be feasible as a tool to decide on such allocation. This would apply in particular to the infrastructure provided for the poor (since the richer have their own additional market). For the public authority who uses a representative agent, diminishing marginal utility (criterion (i) above) might suffice.

It follows that criterion (ii) seems to have less relevance.

As Quality of Life might be based upon the Time Trade-Off approach, that issue is discussed in Appendix C. A key point may however be mentioned here. A time trade-off question for a questionnaire may run like: "How many calendar years of your disability would you be willing to sacrifice, if you could trade with one year without disability again ?" Such a question however is fraught with problems, since it implies the whole decision problem that includes not only a "pure time trade-off" but also utility, risk, individual discounting, calendar time issues, and personal circumstances such as family and income. MacKeigan et al. (2002) for example discuss the possibility of double discounting in such QALY measurements. For this reason, Appendix C will employ a theoretical "Degree of Living" that is "pure time trade-off" though not operationally defined. The use of the Degree of Living concept should clarify the point that when Quality of Life is discounted, at the public health programming level, then this would likely not be based upon the Degree of Living content or "pure time trade-off" in that Quality of Life but rather on additional utility considerations.

1.4.4. The appendices

The appendices discuss aspects that are relevant for this discussion but that stray from the main argument.

Appendix A discusses some general notions on social welfare.

Appendix B considers discounting when life is continuous and costs are discrete.

Appendix C considers discounting of degrees of living, i.e. the pure Time Trade-Off interpretation of Quality of Life that is free from any consideration of utility.

Appendix D considers numerical examples. Our discussion is targeted at application at the population level, but stylized examples of individual cases highlight some properties of the proposed effect measure.

Appendix E projects these numerical examples over time and age.

Appendix F discusses conditions for consistency.

Appendix G discusses the measures of Fair Innings (FI) and Proportional Shortfall (PS) more explicitly. Since the UnitSqrt measure combines the best of the features of these measures, the body of this paper provides the development of the main proposal of the UnitSqrt measure, and comparison with these alternative measures, as already indicated in Section 3.1, can thus be developed more in this Appendix.

PM1. For the graphs, *a, d* and *x* generally have been taken as variables with the origin at 0, which eases the interpretation. The life extension functions have been defined in terms of *x,* but the numerical examples in Appendix D have been defined in terms of *e.*

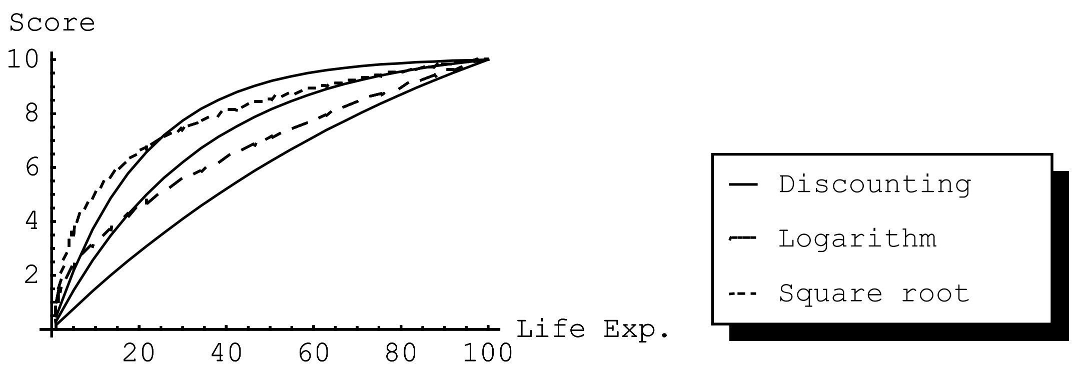
PM2. Before we consider the examples below, it is necessary to mention a point of psychology, in line with Kahneman et al. (1982). For a patient who has 5 years to live, it (likely) matters psychologically whether we propose (i) to double the life expectancy, (ii) to add 5 years, (iii) to aspire at a total life expectancy of 10 years. All these proposals come down numerically to the same thing, but they will convey different psychological messages, and can cause different responses. The first conveys a sense of richness, the second might be most neutral, the third conveys a sense of poverty ("Ten year *max*, that's it."). Clearly, these connotations require some care when interpreting the examples. Again, the discussion in this paper properly concerns a representative agent for aggregate modeling, and not a real individual person at the bedside.

2. Utility functions for the representative agent

The representative agent has time preference and diminishing marginal utility of living. What would be a good function to express these? It is not trivial either what life-years y to take. The utility function of the representative agent might be taken as $u[a + e]$ or $u[e]$ or even $u[x]$, since there is a difference between expectation and memory from the past. However, part of that issue can also be tackled by the social gain function, as for example the proposed gain function $g[x \mid a, d] = u[x] / u[a + d + x]$ imposes a reference point in total life expectancy. For this section it then suffices to select $u[.]$.

A common approach in cost-effectiveness analysis for public health is to discount the life-years. Alternatively, Luenberger (1998), chapter 15, shows the relevance of logarithmic utility for investment theory. The logarithmic scale however is less communicative, see also the Richter scale for earthquakes that takes some exercise to get used to. Lack of communication may cost lives when people have difficulty to judge what risk they will be exposed to. A more straightforward function is the square root, since the value of $\sqrt{100}$ is easier to communicate than Log[100]. The following plot compares these functions. The unit Present Value $\frac{1-(r+1)^{-n}}{r}$ for a discrete constant stream of one year of living per calendar year is given as a drawn line. The plot considers three rates of discount, 1% (lowest drawn line), 3% (middle) and 5% (highest drawn line). Log[n] is the fine dashed line, and $\sqrt{n}$ the coarse dashed line. All functions are scaled such that they give 10 at 100, so that we actually plot A PV[n, r] and B Log[n]. The scaling factors are irrelevant for applications at the public level since they apply to all individuals.

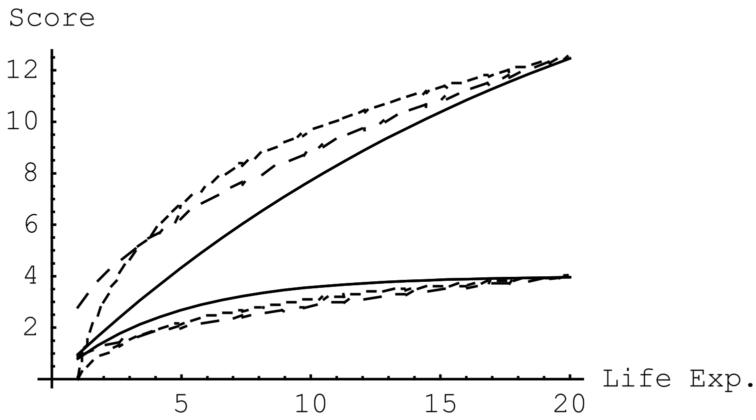
```
PVComparePlot[Sqrt, Log, {.01, .03, .05}, AxesLabel → {"Life Exp.", "Score"}]
```



The conclusions from these graphs are (a) that the shapes are similar, (b) that the Log looks like 5% discounting (an estimate that minimizes the SSE gives $r = 7\%$) and (c) that the square root is between 1% and 3% discounting.

The PV, with different rates of discount, seems to allow for more variation in (individual) time preference. However, this variability can be reproduced by the logarithm and square root by including a scaling factor. We can test this property by looking at some examples. Consider two persons, one with a rate of discount of 5% and the other with a rate of 25%. Their valuation of living an additional 20 years ranges from 12.5 (when $r = 5\%$) to 4 (when $r = 25\%$). The following graph compares the PV with the Log and the square root, now scaling the latter to the associated PV.

```
PVCompare2Plot[.05, .25, 20, AxesLabel → {"Life Exp.", "Score"}]
```



The general shape of the PV can thus be reproduced by both the Log and square root if we apply an adequate scaling factor. Where we said that the two persons had different discount rates, we could also have said that they had different scaling factors. Indeed, the values differ, since the graphs don't overlap, but, given the uncertainties involved, the differences are not that large.

What can we conclude from this comparison ? The point is that when a representative agent is modeled, then we have to select one rate of discount or one scaling factor. Given the uncertainties involved, it would seem advisable to select the utility function that can best be communicated, the square root.

3. Possible criteria for the social gain function

3.1. Formula's for the criteria

What will be a fair social gain function $g[x \mid a, d]$? Some formats are:

Absolute = $x = e - d$

Discounted = PV[x, r] / $(1+r)$^d

Relative = $x / (a+d)$

Unit = $x / (a+d+x)$

Valued = $u[x] / u[a+d+x]$

Marginal = $(u[e] - u[d]) / u[e]$

The absolute measure gives the level of life-years gained $x = e - d$. For the example agents of age 16 and 60, with their addition of 6 years, there is an equal score, namely 6. The absolute measure does not explicitly refer to the age of the person. In practice it relies on it, since often $e = e[a]$.

The discounted value of x is the present value of the constant stream of x years, PV[x, r], subsequently discounted for the period d that precedes it.

The relative measure is $x / (a + d)$. The relative measure has the interpretation that a person's life expectancy after the accident or disease would be $a + d$ if nature had its course, and the relative increase of life with x towards $a + e$ is taken with respect to that origin. For the relative measure with $d = 0$, an increase of 6 years involves a 37.5% increase for a 16-years old but only a 10% increase for a 60-years old. One 16-years old then is about as much as four 60-years olds. The relative measure also implies a dependence upon sex, since women will tend to have higher percentage increases.

An approach is to take each life as a whole, and to allocate 100% to each life. The relative addition to each life, or the share of the gain in total life, then is the unit measure x / (a + e), where a + e gives the individual life expectancy. For example with d = 0, for the 16-years old the score is 6 / 22 = 27% and for the 60-years old the score is 6 / 66 = 9%.

The different denominators (a + d) and (a + e) depend upon age and are not fixed at birth. This is especially relevant since life expectancy tends to differ with age. One might want to impose a norm that the relative impact of an intervention is related to the life expectation at birth e^*, giving x / e^*. This however has the consequence that older people receive preferential treatment, since $e^* < (a + e[a])$. Making the denominator dependent upon the individual life, has the benefit of imposing more equality amongst the different ages.

Comparing these measures, one cannot escape the impression that the unit measure still is too sensitive to the young. Adding 6 years to a 60-year old or to a 16-year old is neutral when considering the absolute measure, but is highly favourable to the 16-year old when considered relatively, even when unitized. The absolute and relative measures do not yet account for the utility function u[.].

The valued measure $u[x]$ / $u[a+d+x]$ expresses the idea that one refocuses on x. The numerator values the addition to life in a way that is independent of age, while it takes account of diminishing marginal utility. The denominator values the total duration of life. This approach seems theoretically most satisfying. The value is always in the range [0, 1]. With the square root, the measure becomes $\sqrt{x / (a + d + x)}$ which is the square root of the unit measure. This expression will be called the UnitSqrt measure. The UnitSqrt measure orders events in the same manner as the Unit measure, but the relative differences are smaller. For x = 6 and d = 0, for the 60-years old the score is 30% and for the 16-years old the score is 52%. The age groups are not equal, as in the absolute measure, but they are not so unequal as in the other measures.

The marginal addition to utility that is age-sensitive is ($u[a+d+x]$ - $u[a+d]$), and this measure is biased against old age due to diminishing marginal utility. An application is in "*fair innings*", see Appendix G. The unitised measure, i.e. the addition as a share of the total, is (1 - $u[a+d]$ / $u[a+d+x]$), and this enhances the bias since the old age has a higher denominator. In case of the square root, the unitised measure is (1 - $\sqrt{(a + d) / (a + d + x)}$) and it seems much like the percentage increase.

A final measure of success to mention here considers only the remaining life-years. Any memory of age is forgotten here, except for the lingering impact when e = $e[a]$. Rather than taking the increase e / d, it is useful to be more sex-neutral and normalise to 100%, which gives 1 - d / e = x / (d + x), which measure is known as "*proportional shortfall*", see Appendix G. It is more general to incorporate the notion of marginal utility and time preference, which gives (1- $u[d]$ / $u[e]$). The measure gives the utility share of x in the total remaining utility (assuming lack of memory of age).

The table below collects some results for our examples. The main result are the ratio's of the scores of the 16-years old and the 60-years old, for d = 0 and a fixed gain of 6 years and the restoration to full life expectancy $e[a]$ (for an 'average person'). We can note that the different measures maintain the priority order (all ratio's are greater or equal to 1), but they have a different impact in the number of 60-years olds that go into a 16-years old (the A16/A60 column). The example of a fixed gain, our example of 6 years in the first three columns, can be misleading, since restoration to normal life expectancy in the last three columns favours the 16-years old with 63 years and the 60-years old with 22 years, meaning that the young person weighs as much as 3 senior citizens, at least according to the absolute gain measure. If we adopt the UnitSqrt measure, however, then it does not matter much whether the gain is 6 years or life restoring, since in both cases one needs 1.73 numbers of 60-years olds to compensate for a 16-years old. (That this outcome is exactly the same, is a matter of coincidence with respect to the life expectancies in the Netherlands. Different outcomes arise for age 50 or age 70.)

	Age 16	Age 60	A16/A60	Age 16	Age 60	A16/A60
Fixed gain or e[a]	6	6	1	62.97	21.62	2.91
Relative	0.38	0.1	3.75	3.94	0.36	10.92
Unit	0.27	0.09	3.	0.8	0.26	3.01
UnitSqrt	0.52	0.3	1.73	0.89	0.51	1.73

From the evaluation above, it appears that the "valued" approach with the square root seems theoretically most satisfying. Below, the various measures will be investigated with both plots (main body of the text) and more elaborate numerical examples (Appendix D). These sections all use the following function:

```
?LifeGain
```

```
LifeGain[a, d, x (, u) (, r) (, f)] for
   a = age
   d = the number of years to death due to a disease
   x = the added life expectancy due to treatment
   e = d + x = the life expectancy after treatment
   u = utility (with default option Utility -> "Sqrt")
   r = rate of interest (default taken from Options[LifeGain])
   f = an age-dependent Life Expectancy function, such that e = f[a]
gives the treatment score according to the utility
   "Absolute"   =  e - d  = x
   "Discounted" =  PV[x, r] / (1+r)^d
   "Relative"   =  (e-d )/ (a+d) = x / (a+d)
   "Unit"       =  (e-d) / (a+e) = x / (a+d+x)
   "Sqrt"       =  Sqrt[e-d] / Sqrt[a+e] = Sqrt[x / (a+d+x)]
   "UnitLog"    =  Log[1+e-d] / Log[1+a+e] = Log[1+x] / Log[1+a+d+x]
   "Marginal"  =  (u[e] - u[d]) / u[e] for default Utility
   other "u"    =  u[x] / u[a+d+x] for u = ToExpression["u"]

When f is specified, then x is recalculated as Max[f[a] - d, x']
  where x' is the original input (when left out, it will get default 0)
```

Note that, while these measures are defined even for a situation of death from accident or disease, they also allow the determination of the value of life itself, when we substitute the age at birth ($a = 0$), death at birth ($d = 0$), and the treatment "helping at birth", all with the (arbitrary) life expectancy. For the Valued measures, the value of life is 1, which corresponds with 'one life' in general.

```
LifeGain[0, 0, any e, "Sqrt"]
```

1

A possible drawback of this method of valuing life is that advances in life expectancy seem not appreciated. For example, when a future generation would live longer, then they still get a life value 1. The suggestion is that the rise in life expectancy derives from the national outlays on protection of the currently living, rather than that it is an explicit objective for future generations.

3.2. Graphical examples of the criteria

It will be instructive to consider the following situations:

(1) Fixed values for d and x, x small, for various ages

(2) Fixed values for d and x, x large, for various ages

(3) Fixed value for d, x dependent upon full recovery, $x = e[a] - d$, for various ages

(4) Fixed value for a and e, x dependent as $x = e - d$, for various years of death from disease

(5) Fixed value for *a* and *d,* for various values of *x.*

The following graphs have this legend:

the Unit Sqrt gain measure: drawn (non-dashed) (purple) line

the Unit gain measure: fine dashed (red) line

the Relative gain measure: coarse dashed (blue) line

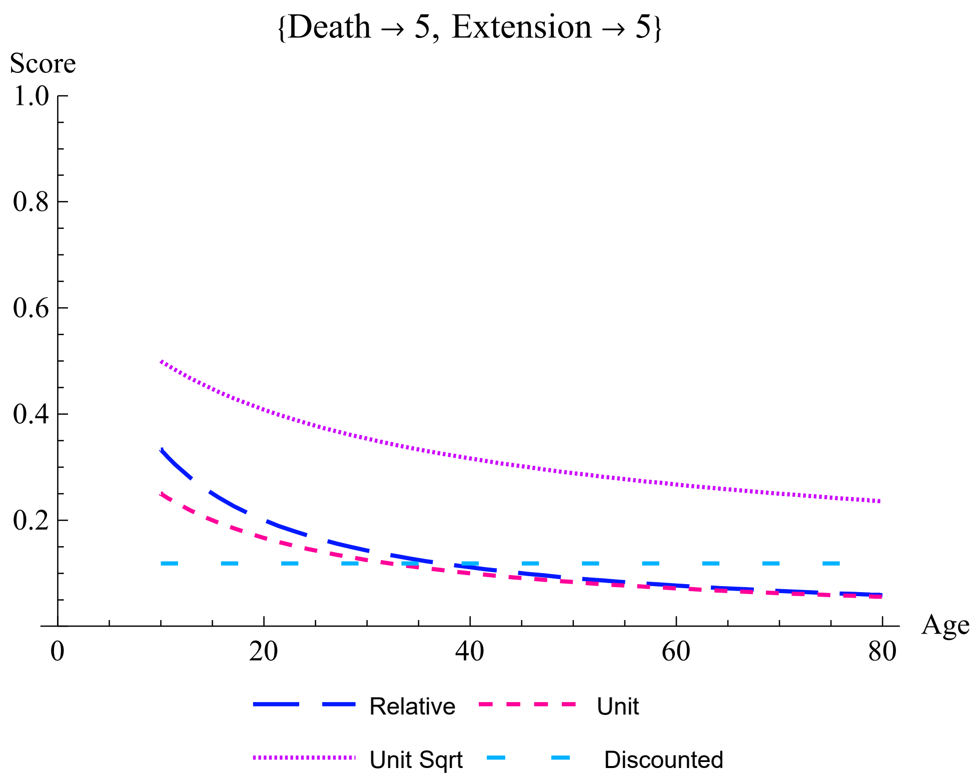
the Discounted gain measure: broken line, light blue

The Absolute measure needs no plot, since the plots depend upon x as input. The behaviour of the Absolute measure is similar to that of the Discounted one, and has a similar scale. The Discounted measure has been scaled by the rate of discount (default $r = 3\%$) to put the result in the same range as the other [0, 1] measures.

As said, the way to read these graphs is that a flat profile implies equality amongst the categories considered, while different ratio's would also be relevant if they would be the moral objective.

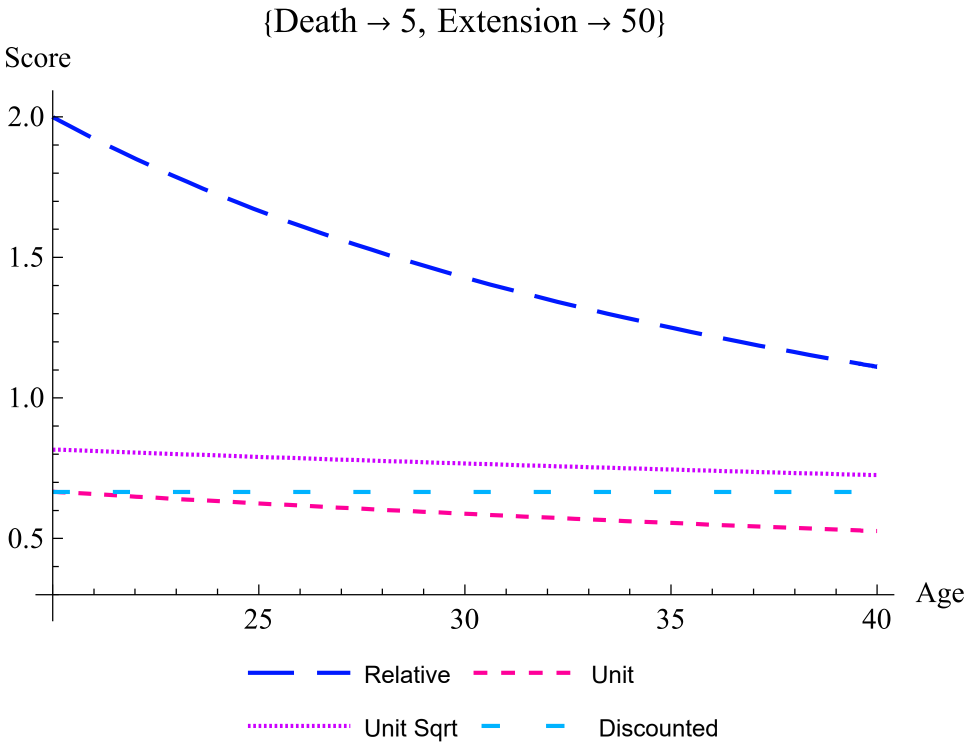
(1) For example, consider the person with $d = 5$, $x = 5$, and $e = 10$. Let us consider the measures for age 10 till 80. The relative measure gives most variation in the impact per age, the unit measure gives a medium amount of variation, and the UnitSqrt measure gives less variation, and of course the constant x and its discounted value show no variation.

```
LifeGainPlot[a, 5, 5, {10, 80}]
```



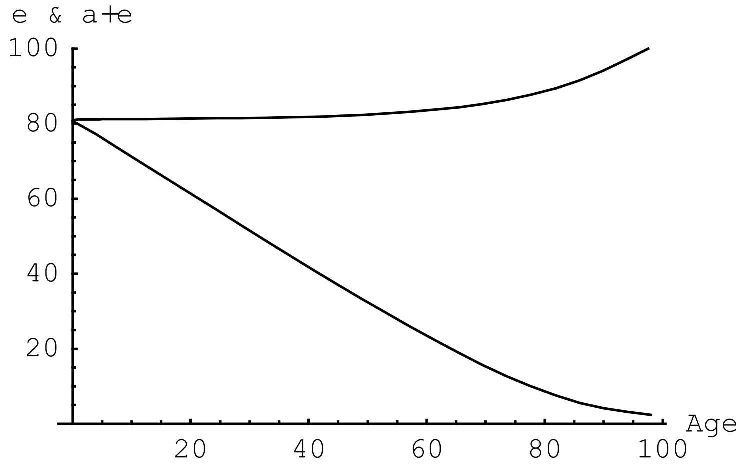
(2) The smoothing effect is rather dramatic when we consider the case of a death in 5 years that can be prevented with an additional extension of 50 years, for a starting age range of 20 till 40 years (living up to at most 95). While the absolute gain measure $x = 50$ is constant, the other measures show an effect of the age.

```
LifeGainPlot[a, 5, 50, {20, 40}, PlotRange → All, AxesOrigin → {20, 0.3}]
```



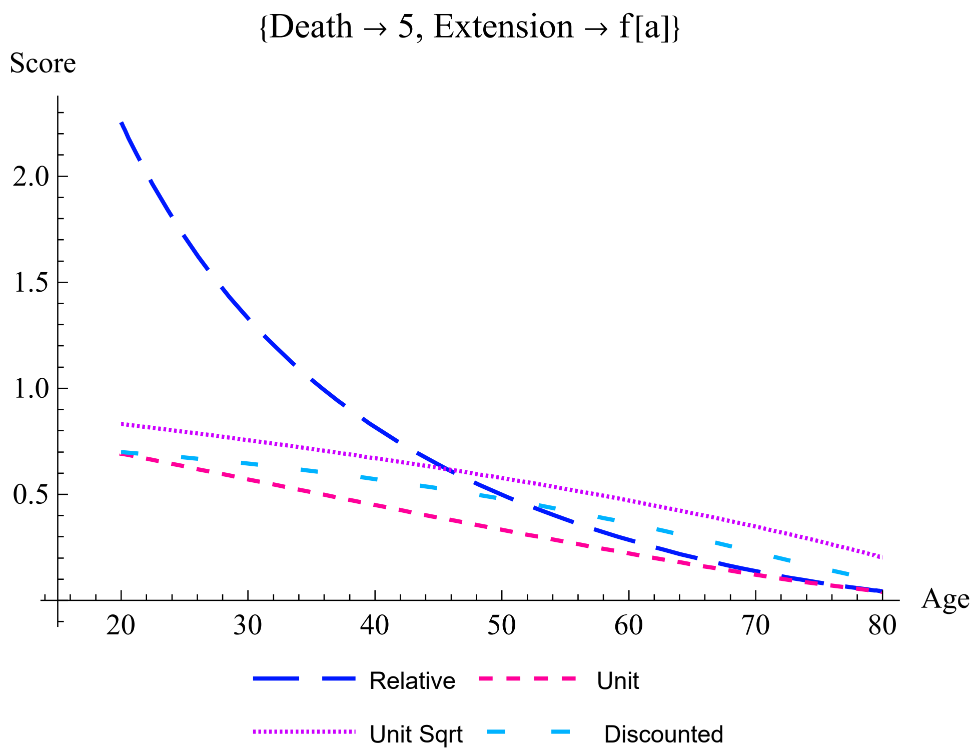
(3) For fully cured persons, the life expectancy will be that of normal healthy people again, $e = e[a]$. The next plot gives the example for Dutch women, showing both e and $a + e$.

```
Plot[{fem[a], a+fem[a]}, {a, 0, 98}, PlotRange → {0, 100}, AxesLabel →
{"Age", "e & a+e"}];
```



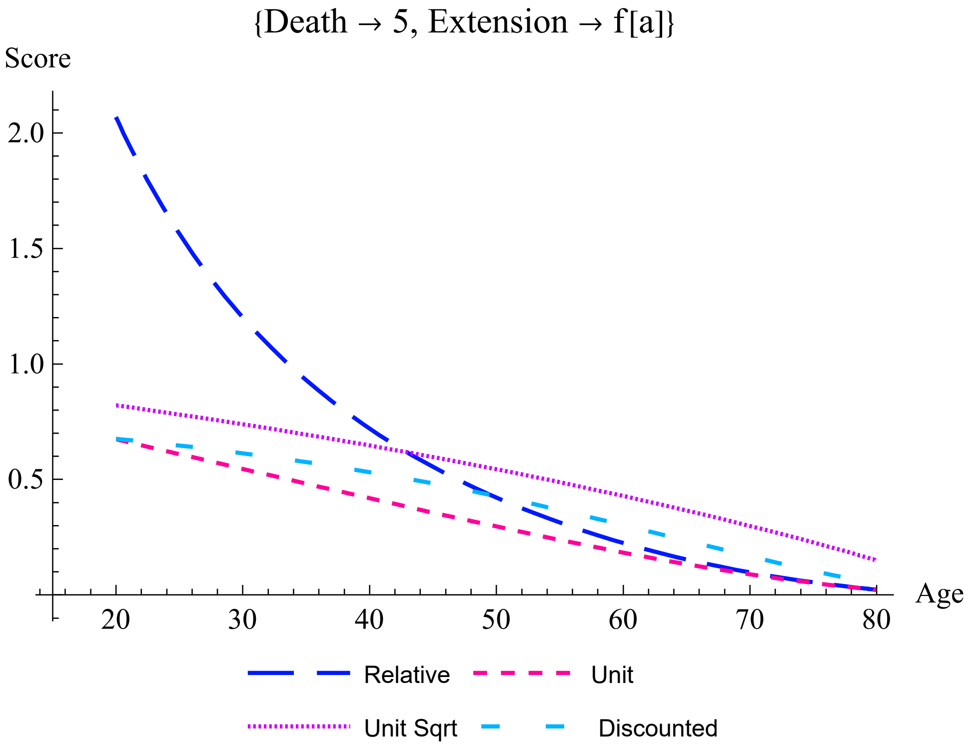
The subsequent plot shows the impact when we assume death from disease in 5 years while full recovery is possible. For example, when a precursor of cancer is diagnosed at age a, with a prognosis of death in 5 years, a treatment could restore original life expectancy at age a. The relative gain measure is high for young ages, both the other measures have a smaller range, and the UnitSqrt measure shows least variation.

```
LifeGainPlot[a, 5, fem[a]-5, {20, 80}, PlotLabel → "{Death → 5, Extension →
f[a]}",
PlotRange → All, AxesOrigin -> {15, 0}]
```



The same plot for the Dutch males.

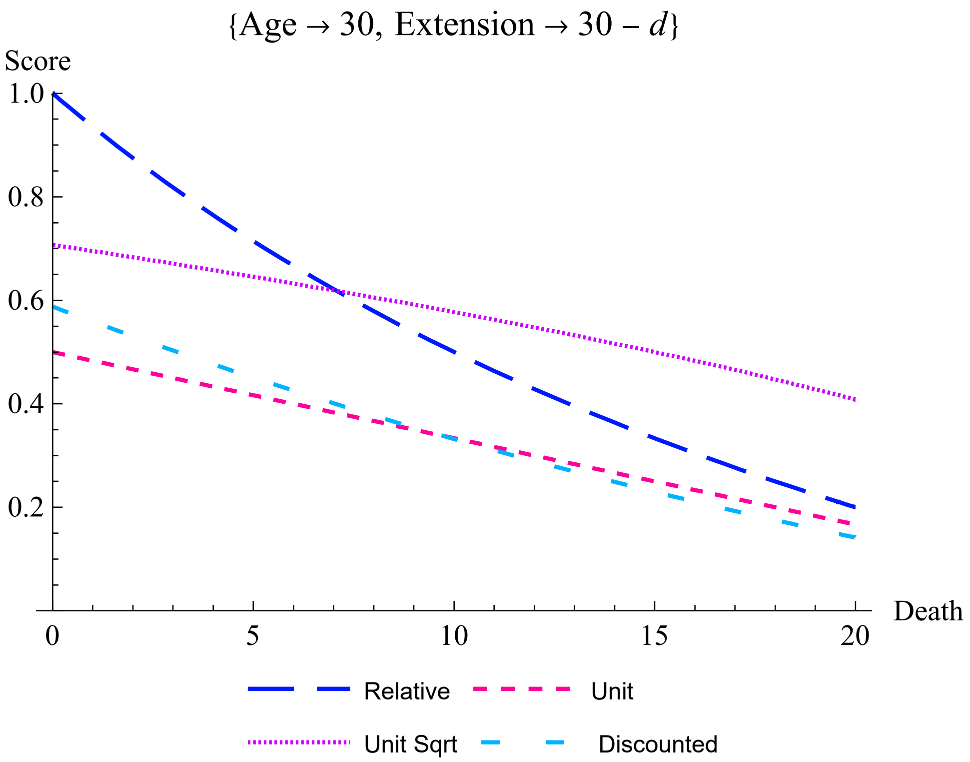
```
LifeGainPlot[a, 5, mal[a]-5, {20, 80}, PlotLabel → "{Death → 5, Extension →
f[a]}", PlotRange → All, AxesOrigin -> {15, 0}]
```



The plots for males and females are rather alike. The argument that there would be a large difference was relevant for the absolute life-years gain measure and not for the unitized measures.

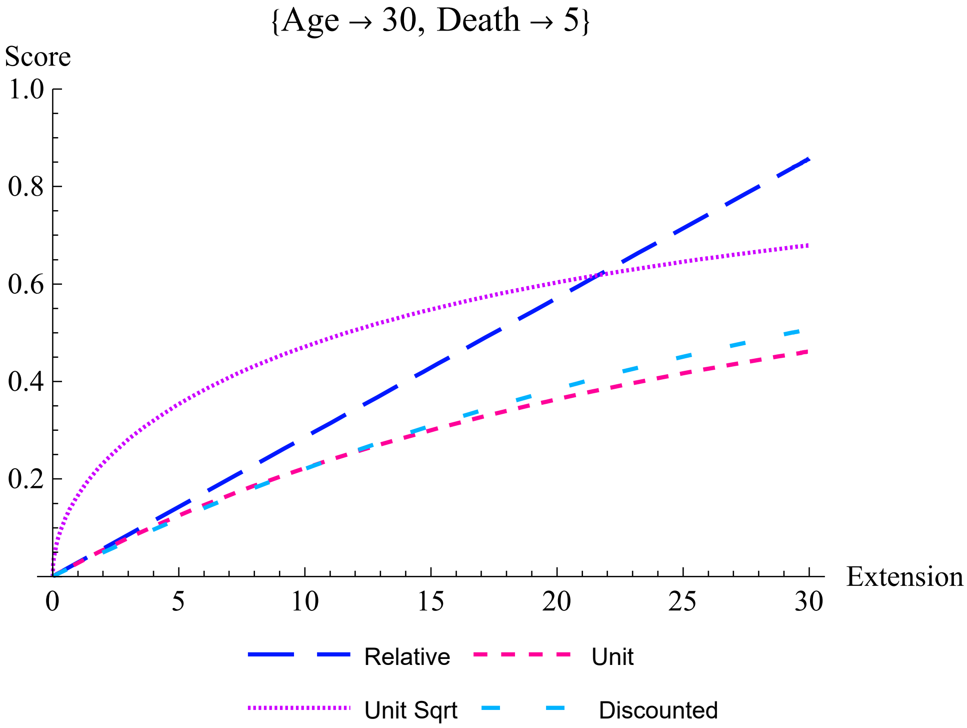
(4) Consider persons of age a = 30 with a given life expectation of an additional e = 30 years after treatment. Suppose that these persons have different prognoses of death without treatment. The following graph gives the measures for d in the range [0, 20]. Though the absolute difference $x = e - d$ thus varies in the range [10, 30], and though the relative effect ranges from 33% till 100%, the Unit and UnitSqrt measures are more stable, and the UnitSqrt is most stable.

```
LifeGainPlot[30, d, 30 - d, {0, 20}]
```



(5) The UnitSqrt measure tends to show least variation in the graphs above, but can also show quite some variation. Consider persons of age 30 with an expectancy of 5 years from disease, but with varying life extensions when subjected to treatment or intervention. The relative measure is proportional to the extension, the Unit and UnitSqrt gain measures show the decreasing marginal utility, but the UnitSqrt measure shows a strong sensitivity to the lower values in the range. This strong effect in the lower range also means less effect for the higher range.

```
LifeGainPlot[30, 5, x, {0, 30}]
```



In summary, these graphs confirm the theoretical argument made before. Given the current practice of discounting, let us focus on comparing the performance of the UnitSqrt measure with discounting. In the examples (2) and (3) above, both functions track each other, but they differ in the other graphs. The following table reviews the plots.

	Discounting	*UnitSqrt*
(1) Fixed d and x, x small	Constant	Higher value for young age
(2) Fixed d and x, x large	Constant	Limited variation
(3) Fixed d, $x = e[a] - d$	Like UnitSqrt	Like discounting
(4) Fixed a and e, $x = e - d$	Value drops with d	Value drops less strongly
(5) Fixed a and d, various x	Slowly rising value	Fast increase for low values

Where the UnitSqrt deviates from discounting, it deviates in a manner that is attractive from the point of fairness. The UnitSqrt gain measure tends to smooth the impacts of age and life expectancy, and to emphasize the life-years gain in the near future compared to the distant future. As such, it satisfies important considerations for a compromise between the two original measures of 'lives saved' and 'life-years gained'.

3.3. Numerical examples of the criteria

3.3.1. Two kinds of examples

The various measures can also be clarified by considering some more numerical examples. This paper provides two kinds of examples.

3.3.2. Reminiscent of clinical application

The measure proposed here is intended for application at the population level. However, situations that remind one of clinical application can clarify some of the effects. It will be useful to include also the 'cost of treatment' and the 'annual income after treatment' as variables, to provide a cost-effectiveness evaluation context. Since this exercise does not add to the conclusions and only provides more clarification, it is put in appendix D.

3.3.3. A population in the steady state

The following calculations apply at the population level. The calculations use a survival function that generates a population in the steady state. When a population is saved from a flood, the lives saved have an average life expectancy that differs from the one at birth. The life expectancy at an age can be weighted by the numbers surviving at that age according to the survival function. The calculation can be done separately for men and women.

For example, the life expectancy at birth for Dutch women is:

```
sex = Female;

lifeExpAtBirth = LExample[sex, 0]
```

80.71

This life expectancy is also found by numerical integration as the area under the survival curve for women.

```
NIntegrate[survival[sex, age], {age, 0, 103}]
```

80.7096

To calculate the average life expectancy, we thus use the weights = survival / (life expectancy at birth):

```
AverageLE[sex_] := NIntegrate[survival[sex, age]*
            LExample[sex, age],
            {age, 0, 98.5}] / LExample[sex, 0]

AverageLE[sex]
```

41.5777

The representative agent has a square root utility function, so that expected utility is:

```
AvSqrtLE[sex_] := NIntegrate[survival[sex, age]*
           Sqrt[LExample[sex, age]],
           {age, 0, 98.5}] /  LExample[sex, 0]

AvSqrtLE[sex]
```

6.15443

If the population is saved, for example from a flood that would have meant instant death, the life gain measure would give:

```
AverageGain[sex_] := NIntegrate[survival[sex, age]*
              LifeGain[age, 0, LExample[sex, age], "Sqrt"],
              {age, 0, 98.5}] / LExample[sex, 0]

AverageGain[sex]
```

0.678302

When we collect the results for men and women, and add a column for the percentage difference:

Sex	Male	Female	$\frac{200\,(\text{Female}-\text{Male})}{\text{Female}+\text{Male}}$
Life expectancy at birth	75.94	80.71	6.09001
Average life expectancy	39.365	41.5777	5.46736
Average $\sqrt{\text{LE}}$	5.99993	6.15443	2.54223
Average Life Gain	0.680196	0.678302	−0.278868

Men and women differ 6% in life expectancy at birth and 5.5% in average life expectancy, but the average UnitSqrt life gain from saving them differs only a quarter of a percentage, namely 0.28%, and the difference has a reversed sign, so that it is more valuable to save the Dutch male population than the Dutch female population.

4. Quality of life and discounting

Above discussion is relatively simple when the life-years are not adjusted for quality of life. The problem becomes more complex when there is adjustment for quality of life. The crux of the problem has already been outlined in section 1.4.3.

The idea of the quality of life correction is that it only affects the measure of health. It thus should not be confused with utility considerations. However, in combination with discounting, there could be moral consequences that make the QALY adjustment non-neutral, so that the original suggestion of only correcting for health might not be maintained.

Consider two agents who live one year of full quality 1 and another year of diminished quality $q < 1$. They only experience this in different order, so that there are time series $\{1, q\}$ and $\{q, 1\}$. Clearly, for all positive rates of discount:

$$1 + \frac{q}{1+r} > q + \frac{1}{1+r}$$

An individual with a positive rate of discount will prefer $\{1, q\}$ above $\{q, 1\}$. However, a public authority who has to choose between these two persons, also must make an interpersonal comparison of utility, and would rather be indifferent, since both persons score equally, at $1 + q = q + 1$.

This is the way of discounting that is neutral to the order of events:

```
LifeGain[a, 0, 1 + q, "Discounted", r]
```

$$\frac{1 - (r+1)^{-q-1}}{r}$$

This method of discounting compares to the UnitSqrt criterion. That has the advantage of including the age.

```
LifeGain[a, 0, 1 + q]
```

$$\sqrt{\frac{q+1}{a+q+1}}$$

There is a difference between an individual and the public authority. An individual is involved in time preference, but the public authority is also involved in interpersonal comparison of utility. Admittedly, a person involved in time preference might also be seen as comparing two ego's along two different time paths, but comparing two different ego's is different from comparing two persons. Above example shows that a particular method of discounting runs against equity, and it is an important counterexample against discounting in a blind manner. For a public authority, discounting annual costs separately is relevant for the financial aspect of cost-effectiveness, but it is dubious with respect to the separate years of a life gain.

This example can be made stronger by including costs. When both treatments are cost-effective, then the morality of an equitable choice is underlined. Let the costs for the first agent be $\{c_1, c_2\}$ and for the second agent be $\{c_3, c_4\}$, let f be the discount factor $1 / (1 + r)$, and let the discounted costs be equal $\bar{c} = c_1 + c_2 f = c_3 + c_4 f$. We can also impose unit cost equality in both years, to make a strong case. Then $c_1 / 1 = c_3 / q$ for the first year and $c_2 / q = c_4 / 1$ for the second year. Cost can be normalised by selecting $c_1 = 1$. These equations solve into $c_3 = q$, $c_2 = q\, c_4$, and $c_4 = 1 / f$. Thus, for every combination of q and r we can find a theoretical case with equal unit costs per period such that the discounted costs are equal. When the discounted costs are equal and the effect measures are equal at $\sqrt{1 + q}$ then the two agents are basically the same. This enhances the point that it is morally decent not to discount the outcomes separately.

We thus have to make a distinction between diminishing marginal utility of life and time preference, in relation to the Time Trade-Off interpretation of quality of life. This suggests the conclusion that u[y] and $g[u, y]$ are preferably formulated for the sum y, regardless of whether the separate years are adjusted for quality or not.

This discussion on discounting causes a heightened awareness of problems with adjustment for quality anyhow.

Consider the representative agent's utility $u[y, n, y / n]$ again. The time trade-off interpretation of quality of life adjustment has some drawbacks. The calendar time in the period $(y, n]$ can contain events that are unique and that cannot take place in the period before y. The usual examples are milestone events such as anniversaries and the like. Perhaps more importantly, there is always the probability that a treatment would be found such that a longer life would be possible. This aspect is necessarily linked to calendar time, and it is difficult to see how it could be substituted for quality of life. Similarly for mental processes, where for example a (longer) sickbed might help to accept death. The agent may well prefer to live longer in calendar time, even though the quality adjusted amount of living does not change. Finally, there seems always to be some dependency on money. When a person with state q_1 is offered some millions of dollars then the state may jump to q_2. Is this sum of money only the cost of treatment or is there a wealth effect? Hence, the time trade-off interpretation of the quality of life meets with doubt.

This paper will not try to solve this question of the adequacy of the time trade-off interpretation of the QALY measure. There however is a solution approach. (a) When the time trade-off interpretation is considered acceptable, then use u[y] and $g[u, y]$, with y taken as quality adjusted life-years. (b) When the time trade-off interpretation is considered inacceptable, then the calendar years are important, and then use $u[n]$ and $g[u, n]$ with calendar life-years n rather than y. (c) When there is a proportion π in the population for which the time trade-off interpretation is acceptable, apply that interpretation for them, and subsequently don't apply that approach for $(1 - \pi)$.

This argument is embedded in some general notions. Appendix A collects the issues in social welfare, Appendices B and C collect issues on discounting.

5. Additional considerations

5.1. Other aspects to consider

The issue discussed here is rather complex, though the proposed solution seems rather simple. It will be useful to indicate some other aspects to consider. Some of these aspects can be discussed at more length in the appendices.

5.2. Allocation over time

The individual scores can be cumulated over the periods of extended living $(d, e]$, and this gives the total distribution over time. Similarly, the individual scores can be cumulated over the ages of extended living $(a+d, a+d+x]$, and the sums per age give the age distribution. How this is done may be clear for discounting, as this is the more conventional method, but it may be less clear for the UnitSqrt measure. The results are shown and discussed in Appendix *E*.

Since discounting is piecewise linear, the method of 'first discounting per individual and then summing' will give the same result as 'first summing per year and then discounting'. For the UnitSqrt, the total gain is the sum over the individuals. A gain can be spread over the period x that is relevant for the individual. This may be stated such that the first year has contribution $\sqrt{1}$, the second year contribution $\sqrt{2} - \sqrt{1}$, the third year contribution $\sqrt{3} - \sqrt{2}$ and so on up to $\sqrt{x} - \sqrt{x-1}$ (or its integer floor). Perhaps we should think of that in this manner, perhaps we should drop the notion of social discounting.

5.3. Conditions for consistency

The measures should be consistent under certain kinds of transformations. What happens when the person ages to $a + \delta$, and the moment of death becomes $d - \delta$, assuming that x remains the same ? What happens when the treatment is split up into two steps, first with extension ξ and subsequently at age $a + \xi$ with extension $x - \xi$? Appendix F discusses these questions, and shows that consistency requires proper accounting per person.

A person can have various treatments over his or her life, and some have more treatments than others. These treatments can be for flood control, heart attacks, etcetera. For single treatments, the life extension measures serve well to judge the issue. Adding up the separate scores for an individual however is another issue. The absolute measure of the total life-years gain for the individual is by itself a meaningful figure. For the other gain measures, the calculation of the total requires a choice on what would be the best approach, also in the light that treatments can be multiplied by arbitrary division.

5.4. Stochastics

The prognoses d and x are only expectations, while properly there would be a distribution with a confidence interval. Developing this angle leads too far now. Note that for example Gold et al. (1996) Chapter 8 contains techniques to approximate confidence intervals. Presently, we may at least consider minimum and maximum values of a range, with intervals $[d^-, d^+]$ and $[x^-, x^+]$ giving minimal and maximal gain measures $g[d^+, x^-]$ and $g[d^-, x^+]$. For example, for a person at age 30, when the interval for d is [3, 7] and for x is [8, 12], then the various measures give these maximal and minimal ranges:

```
ForSet[{"Absolute", "Discounted", "Relative", "Unit", "Sqrt"}, c,
 N[{c, {LifeGain[a, d⁺, x⁻, c], LifeGain[a, d⁻, x⁺, c]}, {LifeGain[a, d⁻, x⁻, c],
    LifeGain[a, d⁺, x⁺, c]}} /.{a → 30, d⁻ → 3, d⁺ → 7, x⁻ → 8, x⁺ → 12}]]
```

Absolute	{8., 12.}	{8., 12.}
Discounted	{5.70765, 9.10932}	{6.42401, 8.09352}
Relative	{0.216216, 0.363636}	{0.242424, 0.324324}
Unit	{0.177778, 0.266667}	{0.195122, 0.244898}
Sqrt	{0.421637, 0.516398}	{0.441726, 0.494872}

Conclusion

We started out with the two main measures for the success of an intervention, namely *lives extended* and *life-years gained*. It appears possible to impose more impartiality with respect to the impact of age and sex differences. This can be done by valuing a life at 100% and by adjusting the life extension for the diminishing marginal utility of living. The theoretical argument, the graphs and the numerical examples corroborate the impression that the particular choice of the UnitSqrt measure defined above is an appealing compromise. To be sure, a balanced decision will always take all measures into account.

Some next steps for further research would be:

(1) Determine what the cost-effectiveness ratio's and frontiers would be in terms of this measure in comparison with the existing measures, for some well-known interventions.

(2) Show how decisions would be affected, depending upon the different effect measures.

(3) Inform other researchers about these results, and then do a systematic review of their views on the approach.

(4) See how the theoretical measure compares to preferences as shown by the general public, see Reckers-Droog et al. (2019).

Appendix A: Some general notions in social welfare

Introduction

The following collects some angles from the viewpoint of social welfare that would be relevant for the discussion but that distract from the main line of argumentation above.

Equity and equality

The question of finding a balance between the two measures of lives saved and life-years gained touches some deep roots. There are parallels in general notions of *equality* and *equity*.

The measure of lives saved can be associated with the notion of equality. An example of equality is that all people have a vote. In the same manner, each life could be treated with the same sense of urgency. The fundamental idea is that society makes a reasonable effort to prevent death and then it is up to nature how long one survives. The second measure can be associated with the notion of equity or fairness - where people have a share in the proceeds, and are treated according to that share. For example, for a commercial company, the ordinary stocks and shares that don't bear interest are straightforwardly called 'equity'.

Cool (2001b:185-191) discusses some equity rules on fairly dividing a cake. When the minimum is also the proportional share, then all should get that proportional share, so that all are at the minimum. When the minimum is less than the proportional share for the reason that some agents are willing to settle for less than the proportional share, then the difference is immediately claimed by those who want more than the minimum.

The equality versus equity problem has roots in the distinction between merely counting ('how many') and weighing ('how much'). What is important to realise, in this discussion, is that there will always be a measure that pertains to individuals, cq. the number of cases. The individuals namely provide the life-years. Not recording the number of cases would be destruction of information. Similarly, whenever a medical docter has a patient to treat, it is this patient that is in focus, whatever balance he or she tries to strike with the life-years gain measure in general.

Public Choice theory

Classic economic theory has only a limited scope. Its argument of diminishing marginal utility and time preference is rather theoretical, applies only to a single person and does not allow interpersonal comparison of utility. There are some arguments from interpersonal comparison that are disregarded but that might be relevant. An older person with much personal investment in knowledge and experience might be considered (with a 'sunk cost' argument) as having a higher priority than a youngster without such investments, even though the life-years gained would favour the youngster. A pensioner might be able to pay for an intervention and make this investment worthwhile, but from the viewpoint of the pension fund any life saved only adds to the costs. In cases like these, standard economics only helps to describe the situation, but little else.

Public Choice theory may have a wider scope, see Mueller (1989). In Public Choice, the hypothesis of individual utility maximization still is relevant. It is not sufficient to simply assume that government officials or even medical doctors are benevolent. What counts are the facts, that might be explained by individual profit seeking. By describing how society arrives at interpersonal comparisons of utility, one can make forecasts of what will happen.

With respect to the market for risk control (floods, traffic safety, and so on), its demand and supply and their regulation, it is necessary to be aware of all kinds of processes that deviate from naive assumptions. The principal (the national state) may formulate the objectives, but the agents that should implement these will have their own goals. Some groups may have more power or money than others and may increase their priority. At the smallest level of detail, for example concerning a slot in an operation schedule, decisions may be made that may be dubious from another point of view.

It is important to be aware of these notions. In the mean time, it is important to note that the present discussion concerns a measure for modeling at the population level, and that it is far removed from application in other domains.

On Nord (1992) on QALY and SAVE

Nord (1992) provides some strong points of critique on the QALY measure that are relevant for this discussion. A point however is that Nord's paper seems to be targeted at the clinical level, while the present paper concerns the aggregate modeling level. Let us consider some of the points however since they clarify our discussion:

(a) "(.. A) life in a wheelchair is considered not only less healthy than a life without disability but also of less value"

My impression is that the point is different. People who are in a wheelchair just now, are certainly as valuable as any other person. Given the choice, many might want to be out of the wheelchair, however. So, if we can prevent that new people will get into the wheelchair, then this is an advance. Treatments that end up with more people in a wheelchair then should be weighted down compared to treatments that keep them up and walking. This 'weighting down' does not necessarily have to be called 'quality of life'. The penalty might also be expressed as the amount of money required to compensate for the loss (which amount could be different per person).

(b) "(the QALY) disregards the starting point and end point. It disregards the fact that a small but significant improvement for a person in a bad state may be preferred by society to a more substantial improvement for a person in a less severe state (...)"

The proposed UnitSqrt life gain measure reflects part of this phenomenon. Consider a child of 6 that might die in 3 months, but that might live another 5 years with treatment. The child gets a higher score than a person of 30 who is prognosed to die in 10 years, and who might live another 15 years with treatment. However, a person of 30 who gains another 40 years, wins out again.

```
{LifeGain[6, 0.25, 5., "Sqrt"],
 LifeGain[30, 10, 15.,  "Sqrt"],
 LifeGain[30, 10, 40.,  "Sqrt"]}
```

{0.666667, 0.522233, 0.707107}

The point of this paper is that this outcome, though arbitrary, may still be acceptable for a modeling exercise, since such an exercise deals with averages only and it does not deal with intangibles that cannot be resolved at a high modeling level anyway.

For example, one could say that above numerical result conforms with Nord's proposition, but it also might be said that it shows both to be dubious. Life for the child might be psychologically bad, even though it would be in excellent health for five additional years, and dying at age 11 might be a horrible prospect. The person of age 30 might be a young mother with dependent children, for whom the added 15 years would be crucial. Then again, are children better off losing their mother directly or knowing that she will die in 15 years ?

One returns to the general point that the proposed life gain measure is not intended for clinical application, but it derives from some mathematical and ethical considerations on representative agents, and it could be useful for general decision making at the population level.

(c) "(...) also disregards the fact that if two patients are in the same state of dysfunction but differ with respect to potential for improvement society may wish to give them the same priority, on the ground that they are equally entitled to treatment"

In that case, apparently, 'satisfied entitlements' is the measure of success. This measure can be implemented for cost-effectiveness studies. However, the point remains, to what extent does society provide entitlements ? Is that not linked to an expectation of lives saved or life-years gained ? (Note that this argument is different in the market sector of insurance.)

(d) "The health services - as well as politicians and the general public - are concerned with providing care for living, breathing, feeling and thinking individuals, not maximising numbers of abstract time entities."

The answer is the same general point, that these life gain measures are not intended for clinical application.

Nord then suggests the following denominator of value: "saving the life of a young person, and restoring him or her to full health. This particular outcome is suggested as the unit of measurement on the ground that most people will probably regard it as the maximum benefit that a single individual can obtain."

However, Nord himself gives the examples of entitlement or of 'significant improvement' that could well deserve priority above this 'maximum benefit'.

In summary, Nord correctly points to weaknesses in the QALY measure, and some aspects find an answer in the proposed UnitSqrt life gain measure. Differences seems mainly caused by the difference in application, either the clinical level (Nord) or the population level (this paper).

On Bleichrodt and Johannesson (1997) on experimental results

Bleichrodt and Johannesson (1997) give an excellent theoretical review and application of the standard gamble, time trade-off and rating scale interpretations, and subsequently the ranking properties of QALYs. It is very nice to see how all these angles come together.

My own earlier work, that has different roots, allows some comments that seem useful.

Cool (2001a:327-351) develops a general concept of risk when there are more than one possible negative outcomes, ρ = -E[x; x < 0]. That discussion shows that the Von Neumann - Morgenstern approach of expected utility doesn't apply for practice. The assumption of independence is killed both by risk asymmetry and by the dependence upon the level of income (which also can be an explanation for the Allais paradox). The Arrow-Pratt measure of risk aversion is rather a measure of diminishing marginal utility and less a measure of risk aversion. An alternative approach is to include risk (as defined) within utility as a separate dimension. My impression at this stage is that there is room for improvement in the standard gamble estimates of QALYs (if QALYs are still seen as a proper approach).

Colignatus (2001b:220-245, 2014) discusses the Elo-Rasch model for ratings. It is likely one of the better ways of rating. Aggregating individual rating scores by using the implied rankings will run into the problem known from Kenneth Arrow's Theorem on voting and aggregating preferences. Bleichrodt and Johannesson (1997) indeed apply both Borda and Pairwise voting methods. Colignatus (2001b, 2014) proposes a "Fixed Point Borda" as a compromise with attractive features.

On Ferrer Carbonell (2003)

The interesting thesis by Ferrer Carbonell (2003) investigates, amongst others, also the rating system (grades on a report card) pioneered by Kapteyn and Van Praag. It leads too far to go into that now, but it is useful to mention it.

On the selection of the rate of discount

Under some conditions, the growth of capital is also the rate of return to capital, and this can be the market rate of interest. Let K be the stock of capital and let output Y depend upon capital by means of a production function $f[K]$, and in particular (inversely) $K = \kappa\ Y$, with κ the capital coefficient that states how much capital is required per unit of output. Output is equal to income and distributed between consumption and investment, $Y = C + I$. The new stock of capital is $K_{+1} = K + I$. The rate of growth is $g = K_{+1} / K - 1 = I / K$. Output is also cost $Y = W + P$, with wage costs W and profits $P = r\ K$ with r the rate of return on capital. If we assume for the steady state that $W = C$, then $P = I$ and $r = g$. The rate of interest (on money) would equal the rate of return on capital, and then be equal to economic growth.

A reasonable value for the rate of discount could be average economic growth, about 2.5%.

It is useful to recall Luenberger (1998), Chapter 15, and in particular page 421. In a single period investment decision, expected value maximisation tends to cause the decision to invest all capital. In a multi-period investment problem, the expected value is a less adequate guide, and it is better to maximise capital growth, causing the decision not to invest all resources but to save some in order to be able to start again when the investment is lost. The latter approach can be stated as expected utility maximisation with logarithmic utility.

Via the link between logarithmic utility and growth and via the link between growth and the rate of interest, we find a link between logarithmic utility and the rate of discount. This though differs from the graph that shows that logarithmic utility and discounting have the same overall shape.

The point remains that the square root is easier to communicate than the logarithm.

Appendix B: Discounting when life is continuous

Summary

Background. In cost-effectiveness analysis (CEA) of public health interventions, a frequently used Cost-Effectiveness Ratio (CER) gives the costs per life-year gained. When calculating the ratio, care must be taken that discrete values are discounted discretely and continuous variables are discounted in continuous manner. **Method.** Theoretical exposition with graphical and numerical examples. **Results**. Costs can be continuous or discrete, but collected cost data normally come in discrete form and thus can be discounted discretely. Life is a continuous flow and hence life-years must be discounted continuously. The difference between discrete and continuous discounting is given by a continuity correction factor. This factor is a linear term and the size of the correction is about half the rate of discount. **Conclusion.** With separable utility, the priority order in terms of the CERs for the different alternative interventions is not affected. When utility is not separable, then the priority order is affected.

Introduction

The difference between discrete and continuous variables causes the difference between counting and measuring or between *'how many'* and *'how much'*. Continuous variables can be made discrete, the simplest example is that of a flow of water from a faucet measured by a bucket each five minutes. This method of quantification is alright, as long as that bucket is not perceived as being full during the whole five minutes.

In cost-effectiveness analysis (CEA) of public health interventions, the Cost-Effectiveness Ratio (CER) is often chosen as the cost per life year gained, see for example Gold et al. (1996). Since life is a continuous flow, the basic modeling strategy would be to model cost as a flow too, see for example the model by Manton & Stallard (1988:36). However, in practice, costs are observed as discrete sums, for example as budget expenditures, and the discrete approach seems to be the more common modeling method anyhow, even for human capital modeling, see Grossman (2000). The difference between continuous and discrete models would seem to vanish if the period is made small enough, but in practice the period is long, often a year, and then some continuity correction would be required. The objective of the present discussion is to clarify the reason and size of that continuity correction.

The issue of continuity correction is particularly relevant for discounting. Costs and life-years gained usually are discounted, and common discount rates are 0, 3 and 5%. Our objective then is also to clarify the distinction between (a) discounting of values at specific moments, such as costs or bills that have to be paid at the end of a year, and (b) discounting of continuous streams, such as the stream of life that occurs continuously.

There are two main findings:

(a) The continuity correction has the size of about half of the rate of discount.

(b) The continuity correction factor only depends upon the discount rate, so that it is independent of the quality of life that can vary over life.

The practical impact of our finding thus is modest. Other authors such as Johannesson (1992), Gyrd-Hansen & Sogaard (1998) and Cairns & Van der Pol (1997) discuss problems in discounting that cause differences of much larger magnitude. Yet it is useful to have conceptual clarity.

There are two other residual findings that concern possible sources of confusion:

(i) The discount factor can formulated as $(1+r)^{-t}$ or as $e^{-i\,t}$, for $i = \text{Log}[1 + r]$ and time variable t and the natural number $e = 2.71828...$ One possible source for confusion can be clarified immediately. The rates of discount of 0, 3 and 5% concern r and one must apply the transformation to find i. The use of $e^{-i\,t}$only derives from mathematical convenience or convention as distinguished from computational convenience.

(ii) There is a particular method in finance and banking, namely *compounding continuously*, that is a more serious possible cause for confusion. The terms *discounting continuously* and *compounding continuously* seem to convey the same idea, but it is important to avoid that confusion, as explained below.

Below, the concepts of discounting and quality adjustment are accepted without discussion. It is useful, however, to note that discussion still is possible, see e.g. Chapman (2002).

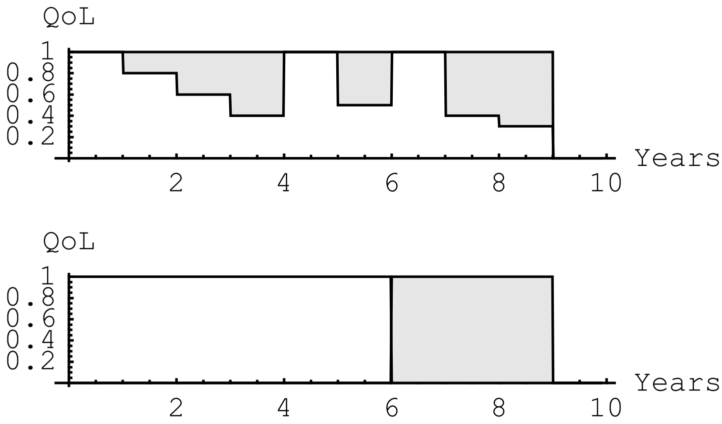
Definition of terms

Let y measure quality adjusted life-years, and let n be the associated calendar years (though alternatively months or seconds - in general 'period'). With qualities of life $q_j \in [0, 1]$ in a calendar-sequence $\{q_1, ..., q_n\}$, we have $y = \sum_{j=1}^{n} q_j$. For full quality life, we have $q_j = 1$, so that $y = \sum_{j=1}^{n} 1 = n$. The size of the period can be adjusted to get a natural number for the total life. For a less than full quality life, we get $y < n$, and sometimes this is expressed by saying that the 'time trade-off' (q) is less than calendar time (n). See e.g. Johannesson et al. (1994), Lipscomb et al. (1996) and Bleichrodt & Johannesson (1997) for good discussions of these concepts. Figure 1 is a typical example of an not-discounted 'time trade-off plot' for a person with a sequence of 9 calendar years of varying quality so that the time trade-off 'normal time' is only 6 years. In graph (i) we see how the quality of life is distributed over time, and in graph (ii) we see more clearly, by collecting the two categories, that the proportion of 'lost time' is 3 out of 9 years.

Figure 1: A typical example of an not-discounted time trade-off plot,

(i) over time, (ii) aggregation of normal time and lost time

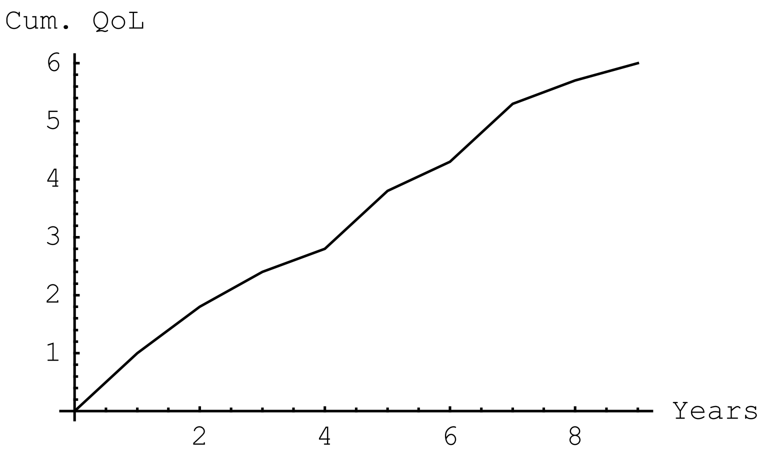
```
TimeTradeOffPlot[{1, .8, .6, .4, 1, .5, 1, .4, .3}, AxesLabel → {"Years",
"QoL"}];
```



The q_j will generally be seen as flows. For example for the first year, the subject has lived zero years at $t = 0$, has lived half a year at $t = 1/2$, and has lived one year at $t = 1$. Taking the integral of the flow of life will give the completed life. The cumulation of Figure 1 is done in Figure 2.

Figure 2: Cumulated life-years

```
PlotLine[CumulateDuration[{1, .8, .6, .4, 1, .5, 1, .4, .3}],
AxesLabel → {"Years", "Cum. QoL"}];
```



Costs can also be continuous, such as electricity. The question arises how continuous variables can be translated to discrete values, in the context of discounting.

Discrete versus continuous life

Let the discount rate be r. If life were discrete, the q_j values are taken at specific moments. In the financial practice of taking values at the end of the year, the present value is $\text{PV}[\{q_1, \ldots, q_n\}, r] = \sum_{j=1}^{n} q_j \big/ (1+r)^j$. In the special case of a constant value of $q_j = x$ for n calendar years then this expression simplifies to PV[n, r, x] = $\frac{1-(1+r)^{-n}}{r}\,x$. The q_j values can also be booked at the beginning of the year as expected life e_i for the next year (note the difference between e and $\mathbb{e}$). Then each discount factor is $(1 + r)$ higher, giving PV* = $(1 + r)$ PV. For a constant periodic value of $q_j = x$ for n calendar years we also may write PV* = $x + \frac{1-(1+r)^{-n+1}}{r}\,x = x +$ PV[n-1, r, x].

The model changes when life is considered to come to us in a continuous stream. This experience of continuity is consistent with the phenomenon that Figure 1 presents surfaces. We collected surfaces to identify the total time trade-off. The figure does not give a line graph with values at specific moments in time. Note that the continuous model is consistent with discrete values for integrals over subperiods, see the continuous formulation of life and the calculation of the life expectation by Manton & Stallard (1988:20). At issue is however how this is affected by discounting.

When there is a constant flow with the rate of $q[t] = x$ per time unit, then after T time units (e.g. $T = n$), the total flow that has passed is $X = \int_0^T x\, dt = x\, T$. With r the discount rate, then the discount factor $\frac{1}{1+r}$ can be applied between any two moments that are at a distance of one time unit. For moments at a distance of t time units, the discount factor $\left(\frac{1}{1+r}\right)^t$ applies. For example, multiplying the factors of subperiods $t_1 = 1/4$ and $t_2 = 3/4$, gives the factor for the total period again. Using $i = \mathrm{Log}[1 + r]$, the discount factor is also $\mathbb{e}^{-i\,t}$. Each instance of flow $q[t]$ is at distance t from 0, so that its discounted value at 0 is $q[t]\left(\frac{1}{1+r}\right)^t$. The Continuous Present Value of all values between 0 and T is CPV = $\int_0^T x \left(\frac{1}{1+r}\right)^t dt = \frac{1-(1+r)^{-T}}{\mathrm{Log}(1+r)}\,x = \frac{1-\mathbb{e}^{-i\,T}}{i}\,x$. The denominator of the CPV can be written as i, suggesting a discrete PV format, but the numerator also gets an $\mathbb{e}$-term, destroying that formal identity. The CPV differs from the discrete PV by a constant factor f = CPV / PV = $\frac{r}{\mathrm{Log}(1+r)} = r / i$. Note that $i = \mathrm{Log}[1+r] < r$ for normal discount rates $0 < r < 1$, so that $f > 1$. An important conclusion is that this factor is independent from the level of the (constant) quality of life and the number of periods. When discrete life expectancy is booked at the beginning of the period, then the factor is f^* = CPV / PV* = $f / (1+r)$, and $f^* < 1$.

For values of q_j that differ per period, we find the same factors f and f^*. The reason is that the factors hold for each period and can be isolated from the summation. Alternatively, we can discount each period to its beginning, and thereafter discount discretely. The factors thus hold in general, independent from the quality of life and its variation and the number of periods. It essentially concerns a correction for continuity in the single period. Thus in general, when a discrete present value has been given, then the proper continuous value can be found by multiplying with correction factors f or f^*. Table 1 gives values for the common discount rates, using six significant digits to allow for easier verification of the results.

Table 1: Correction factors for multiplication of

discrete present values, per discount rate

Rate	Beginning	End
3%	0.985365	1.01493
5%	0.975997	1.0248
10%	0.953824	1.04921

Conversely, when the true value is the continuous value, the discrete value gives an error. The error of booking life at the end and then using the discrete PV must be measured with respect to CPV, and thus is $(1/f - 1) * 100$ percent. Since continuous life starts immediately, booking life only at the end gives an underestimation of discounted life. When life is booked discretely at the beginning, then the error factor is $((1+r)/f - 1) * 100$, and this overestimates life. Table 2 gives outcomes for common discount rates, and for those values the error is about the size of 1/2 of the discount rate.

Table 2: Error percentage compared to the true value of discounted life-years gain

when discounting life discretely at either beginning or end of period, per discount rate

Rate	Beginning	End
3%	1.49	−1.47
5%	2.46	−2.42
10%	4.84	−4.69

Discrete versus continuous costs

We can repeat the same analysis to costs. The general approach is to make costs continuous by describing them as services. Though capital outlays such as electricity plants are discrete, their services are continuous. When a customer buys a car or a can of biscuits, this can be described as buying the services of those products.

When both costs and life-years are continuous, but are modeled as discrete, then there would be no need for the continuity correction, since both would be corrected in the same manner and the ratio would remain the same.

In practice, however, costs are generally taken discretely. For example, the various costs per month are added to give an annual total, or an average cost per patient is multiplied with the number of patients to arrive at the total. It could already be an advance when costs would be discounted at the market rate of interest to arrive at the annual total. In all practicality, costs are discrete.

With costs discrete such that the discrete PV applies and with life continuous such that the CPV applies, the continuity correction factor would need to apply too.

Consequences for CEA

A central question is whether the priority order of the projects is affected, i.e. for the projects that are considered in a cost-effectiveness analysis. In CEA, the ordering criterion is the CER, and then one considers the "incremental cost" / "incremental life year gain". Assume a CEA project that discounted costs and life-years in a discrete manner. The assumption of discrete costs remains the same, but let us then apply the continuity correction to the life-years. Assume that the same discount rate holds for all individuals. From the linearity of the transformation of the denominator, it would follow that the order is not affected.

The curvature that expresses the dependence upon the discount rate is always affected, since the different discount rates have different correction factors. This curvature however is not a separate decision criterion.

When the health-CER is affected but the non-health interventions are not affected (such as concerning consumption and education) then the priority order with respect to those other interventions need not be affected when the allocation is based upon other criteria.

Cost-effectiveness conclusions however are affected when preferences are not separable. Chapman (2002:411) uses the term "domain independence" for the possible disagreement between health and money discount rates. The more general notion is separability, as e.g. defined by Barten (1977:31): "(...) the preference ordering is separable into mutually exclusive groups of goods if the preference ordering of a certain group is independent of what one consumes of the goods outside the group". For example, the consumer choice of the vacation country (Greece or Mexico) can be separable from the choice of the new kind of car (Mercedes or Toyota). For medical decision making, the point is that when decisions on 'quality of life' cannot be separated from other utility aspects, such as the level of income and consumption, then discounting (or the time preference that it stands for) could well affect results. This holds *a fortiori* when the aggregate choice is not modeled for a representative agent only but is modeled by heterogeneous individual utility. When individuals have different discount rates so that their curvature is affected, then their individual orders need not be affected, and if the aggregate priority depends upon individual order only, then the aggregate need not be affected either. However, when the aggregate depends upon more than just order, then the aggregate priority is affected.

Compounding continuously

The discussion about *continuous discounting* contains a possible point of confusion, and the above derivation of the present value of a continuous flow has been stated more explicitly to prevent this confusion.

The point of possible confusion concerns the issue of *continuous compounding* This is a specific technique in finance and banking, and a discussion of it can be found in basic textbooks in finance, such as Luenberger (1998:21) and Bodie & Merton (2000:108). The above discussion concerns (i) a continuous flow and it uses (ii) compound interest (i.e. interest on interest), but it is *not necessarily* a case of continuous compounding.

The explanation of continuous compounding starts with the distinction between the nominal rate and the effective rate. Let a bank advertise with a nominal discount rate of R per year while it actually compounds over m subperiods. That is, the bank applies rate $\frac{R}{m}$ in each subperiod (for example $m = 12$ months), so that the effective rate per year is $r = \left(1 + \frac{R}{m}\right)^m - 1$. Some customers only look at the advertised nominal rate, and are not fully aware whether the bank compounds per quarter or per month or even more. By taking more subperiods, the effective rate over the whole period rises. Taking an infinite amount of subperiods, $m \to \infty$, then the situation is called "continuous compounding", and the effective annual rate in this limit is $r = e^R - 1$.

Note that $R = \text{Log}[1 + r]$, so that the term $\text{Log}[1 + r]$ reappears, and thus $R = i$ when there is continuous compounding. Note however the difference between the contexts of discussion.

For each rate of discount r (our discussion) there is an $i = \text{Log}[1 + r]$, but this does not imply that actual banks and other financial institutions indeed advertise with this $R = i$ and apply continuous compounding with it. This may be the case, but it may also not be the case.

When there is a nominal rate R (the finance discussion) but no continuous compounding, then the effective rate follows as $r = \left(1 + \frac{R}{m}\right)^m - 1$ and then there is an $i = \text{Log}[1 + r]$ that would differ from R.

Thus there is a distinction between continuous discounting and continuous compounding. Some research papers in finance and economics assume a nominal discount rate R that is compounded continuously, and thus they may try to avoid the confusion by making sure that $i = R$. However, this may also cause the confusion that it would always be the case that $i = R$. It is more useful to work directly with the effective discount rate r and disregard how banks arrive at it.

Conclusion

It makes some difference whether life is regarded as a continuous stream or as a sequence of discrete events. With linearity and separable utility, the priority order of (quality of) life interventions is not affected. When utility is not separable, then the priority order would be affected.

Appendix C: Discounting and degree of living

Introduction

We will use the term "degree of living" to express a pure time equivalent approach or "pure Time Trade-Off" that abstracts from utility considerations and such. The "degree of living" will be a purely theoretical term without any operational measurement, but it has the advantage for now that at least our concepts can be clear. Note that "quality of life" has met with various interpretations and operational measurements, such as Time Trade-Off, Standard Gamble and Rating Scale as used in the literature, so that there is some danger of confusion with other elements.

Discounting of life is problematic and taking the square root seems acceptable, but the latter is similar, in some ways, to discounting. This situation is confusing. The discussion should give clarity however how the one differs from the other.

Recall the definition that $y = \Sigma\, q_i$, where y are life-years that can be realised or expected. If only years of living fully are considered, $q_i = 1$ except that the last q_n can still be a fractional year. (Alternatively, the period is taken so small that the life duration can be approximated by an integer value.) Consider a utility function $U[y]$ with diminishing marginal utility, such that $\partial_{y,y} U[y] < 0$, . Consider then the following utility functions:

(1) $\overline{y}$ = Sqrt[n] or $\overline{y}$ = CPV[n, r] = $\frac{1-(1+r)^{-n}}{\text{Log}[1+r]}$ using continuous discounting for the calendar years (or fully living e.g. when $y = n$). The present value assumes values at the end of the period and discounts to the beginning of the first period, following financial convention.

(2) $\hat{y}$ = Sqrt[y] and $\hat{y}$ = CPV[y, r] for the sum y, the life-years equivalent.

(3) The degrees of living can be discounted separately, giving $\tilde{y} = U[q_1, ..., q_n, r]$ = CPV[$\{q_1, ..., q_n\}$, r] $\neq$ CPV[y, r] (though there is equality when $y = n$). Here is no analogue for the Sqrt.

(4) There is a social welfare function with a correction for age and sex such as the UnitSqrt transformation.

These possibilities cause various indicators for time-equivalence. The time T of a (discounted) full life can be found from solving T from $\tilde{y}$ = CPV[$\tilde{T}$, r]. It is also possible to express the time effect by a ratio, notably to calendar time, thus y / n, $\hat{y} / \overline{y}$ or $\tilde{y} / \overline{y}$.

In (1) and (2) only diminishing marginal utility is used, and in that sense, the Sqrt can be compared to discounting. The discounting formula may also be more flexible since it contains the additional parameter r, but the square root can also be extended with a scale factor. While the discounting formula is used, the philosophy of discounting however is not fully applied, since agents with profiles $\{1, q\}$ and $\{q, 1\}$ are treated equally, with outcome $U[1 + q]$.

In (3), the discounting philosophy is also applied, creating a difference between agents $\{1, q\}$ and $\{q, 1\}$.

There thus is a distinction between preference for 'living' and preference for 'living when'. At the public decision making level, the representative agent can be assumed to prefer a longer life, with a diminishing marginal utility of living longer. But it is also useful to impose distributive justice, and to let society be indifferent as to when this living occurs. Thus (3) is rejected, and (4) is applied to (1) or (2).

Degree of Living

The definition of a calendar sequence of degrees of living $\{q_1, ..., q_n\}$ is that the scores only have meaning when they allow us to say that a person will live less even though the calendar time runs faster. It is not quite a matter of wanting to live shorter (that could involve a utility comparison), but of actually living less.

Consider a person with a sequence of 9 calendar years with various degrees of living.

```
degrees = {1, 1, .6, .4, 1, .5, 1, .4, .3};
```

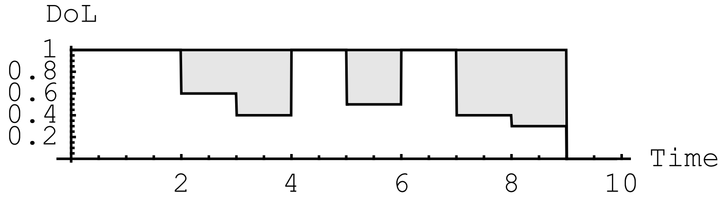
The statistical accounting without discounting is straightforward.

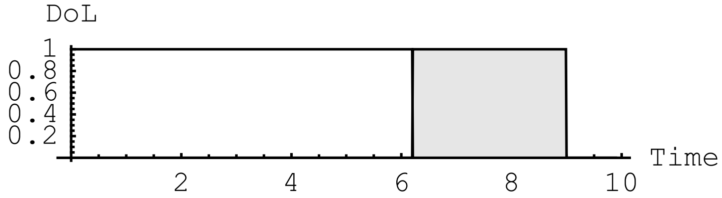
```
TimeTradeOff[degrees]
```

{Time → 9, WeightedSum → 6.2, Time(Loss) → 2.8, DegreeOfLiving → 0.688889}

We can plot the situation in two ways: (a) the distribution over time, (b) the collection of all the 'time proper', as opposed to the 'lost time'. In plot (b) we see the proportions more clearly.

```
TimeTradeOffPlot[degrees];
```





The summary statement for this agent is: "Given the current parameters, the best prognosis now is that you will live 9 calendar years, but some of these at a lower rate, so that you will actually live only 6.2 full years. If there would be a medical solution to let you live fully during those 6.2 years followed by sudden death, then this would be equivalent in terms of living as such."

Note that this only defines the concept of a degree of living but it does not suggest that such a switch actually can be made or be made. That would be a decision based upon utility, that also includes other considerations. Some might accept the time trade-off, but others would prefer calendar time because some events are causally linked to calendar time. A state worse than death would not be indicated by a negative health value but by the utility function too.

Discounting

There are two views on discounting:

(i) The degrees of living are merely counting, like calendar years are counting, and then there is no discounting of separate values.

(ii) The degrees of living are years that are consumed at different moments in time, and then discounting of separate values might be warranted.

Consider how life-years are added to a life: this happens incrementally and without discounting. Marginal utility of the life expectation applies to the sum, but this sum contains a memory how much has already been included. Adding life-years is ageing, but differs from ageing in the sense of losing quality. This loss of quality due to ageing is modeled not necessarily by discounting.

Though the technique of discounting would be feasible, viewpoint (i) above seems more appropriate theoretically. However, for Quality of Life measures that are not "pure Time Trade-Off" and that contain utility considerations, discounting might be appropriate.

Approach (ii) leads to the following:

```
PV[degrees, r]
```

$$\frac{1}{r+1}+\frac{1}{(r+1)^2}+\frac{0.6}{(r+1)^3}+\frac{0.4}{(r+1)^4}+\frac{1}{(r+1)^5}+\frac{0.5}{(r+1)^6}+\frac{1}{(r+1)^7}+\frac{0.4}{(r+1)^8}+\frac{0.3}{(r+1)^9}$$

For example with a rate of discount of 3%, we can find the following values. The label "Discounting" indicates the discounted value $\tilde{y}$, the label "DegreeOfLiving" indicates the ratio of this to the PV of the calendar years, thus $\tilde{y}$ / $\bar{y}$, and the label "Time[Full]" indicates the solution of $\tilde{y}$ = PV[$\tilde{T}$, r].

```
TimeTradeOff[PV, degrees, .03]
```

{Discounting → 5.45808, Time → 9, PV[9, 0.03] → 7.78611,
DegreeOfLiving → 0.701002, Time(Loss) → 2.95041, Time(Full) → 6.04959}

In this example, the discounted degree of living (0.7) is larger than the not-discounted value (0.689) since the disease years are at the end and weigh less.

Since living is continuous, we actually should use the continuous present value. The solution of the full time is not different since the continuity correction applies to both sides of the equation.

```
TimeTradeOff[degrees, .03]
```

{Discounting → 5.53955, Time → 9, ContinuousPV[9, 0.03] → 7.90233,
DegreeOfLiving → 0.701002, Time(Loss) → 2.95041, Time(Full) → 6.04959}

Though method (ii) is feasible, it is not by itself convincing.

Using very short periods

The result of the continuous PV can be reproduced by the discrete PV when we take very short periods. In that case, the rate of discount per period becomes very small so that the continuity correction approaches unity.

Take the example of a person who starts out with 30 years of living well, and who is struck by some disease with various states. The various summary statistics will be a weighted sum based upon the {duration(i), degree(i)} states.

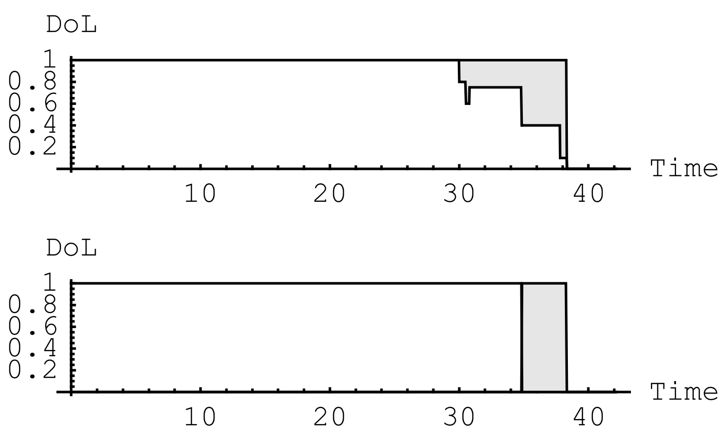
```
dur = {{30, 1}, {0.5, 0.8}, {0.3, 0.6}, {4, 0.75}, {3, 0.4}, {0.5, 0.1}};
```

These scores disregard discounting yet.

```
TimeTradeOff[dur]
```

{Time → 38.3, WeightedSum → 34.83, Time(Loss) → 3.47, DegreeOfLiving → 0.909399}

```
TimeTradeOffPlot[dur];
```



Discounting can be done by the continuous formula or by distinguishing subperiods using the Greatest Common Divisor (GCD) and then apply the discrete model (with subperiod rate of discount). The discounted degree of living appears to be higher than the undiscounted value, since the disease state consumes a shorter period at the end.

```
TimeTradeOff[dur, .03]
```

{Discounting → 21.6945, Time → 38.3, ContinuousPV[38.3, 0.03] → 22.9253, DegreeOfLiving → 0.946313, Time(Loss) → 3.61762, Time(Full) → 34.6824}

```
TimeTradeOff[PV, dur, .03]
```

$\left\{\text{Period} \to \left\{\text{GCD} \to \frac{1}{10}, \text{Time} \to 383, \text{PV}[383, 0.00296025] \to 228.915, \text{Time(Loss)} \to 36.1762, \text{Time(Full)} \to 346.824\right\}, \text{Discounting} \to 216.625, \text{Time} \to 38.3, \text{PV}[38.3, 0.03] \to 228.915, \text{DegreeOfLiving} \to 0.946313, \text{Time(Loss)} \to 3.61762, \text{Time(Full)} \to 34.6824\right\}$

Appendix D: Numerical examples of the criteria

Some random cases

The various measures can also be clarified by considering some numerical examples. The proposed effect measure of the UnitSqrt is intended for application at the level of national budget allocation, but some stylized clinical examples will help to clarify the meaning. It will be useful to include also the 'cost of treatment' and the 'annual income after treatment' as variables, to provide a cost-effectiveness evaluation context, so that one can get an impression of the functioning of the proposed index. The following five cases are basically random but have been selected from some random runs for their contrast. Time is measured in years; and costs and income are measured in thousands of dollars (of a base year).

```
rp = RandomPatient["Example"];
```

```
RandomPatient[Table, rp]
```

	Sex	Age	Years to Death	Life expectancy after treatment	Treatment Cost	Annual income after treatment
1	Male	16.3	3.6	56.8	5.2	88
2	Female	12.6	6.	69.5	5.7	17
3	Male	26.2	3.	12.	3.7	34
4	Female	52.5	0.75	2.	8.4	50
5	Female	64.	9.9	20.9	2.5	53

Evaluation in terms of the criteria

The data on the cases can be evaluated in terms of the various criteria. The annual income after treatment can be discounted to a present value (PV), and a measure of the return on investment (ROI) is given by the present value divided by the cost of treatment. For the PV, the life expectancy after treatment can be used, counting only the additional years $x = e\text{-}d$ of income, but also discounting the d years, and allowing for a pension of 70% at age 65. The default rate of discount is 3%.

```
ap = AnalysePatient[rp];
```

```
AnalysePatient[Table, ap]
```

	Absolute	Discounted	Relative	Unit	UnitSqrt	Marginal	PV/Cost	PV
1	53.2	23.749	2.673	0.728	0.853	0.748	393.372	2045.535
2	63.5	23.644	3.414	0.773	0.879	0.706	68.001	387.608
3	9.	7.125	0.308	0.236	0.485	0.5	65.477	242.263
4	1.25	1.183	0.023	0.023	0.151	0.388	7.039	59.132
5	11.	6.905	0.149	0.13	0.36	0.312	102.473	256.183

Note that the coefficient of variation of the UnitSqrt is amongst the smallest, confirming that it shows less dispersion in general.

```
AnalysePatient["Statistics", ap]
```

	Absolute	Discounted	Relative	Unit	UnitSqrt	Marginal	PV/Cost	PV
Mean	27.59	12.521	1.314	0.378	0.546	0.531	127.273	598.144
S	28.548	10.476	1.604	0.349	0.316	0.192	152.653	817.525
S/Mean	1.035	0.837	1.221	0.923	0.579	0.362	1.199	1.367

One possible approach is to treat the cases in order of their rank.

```
AnalysePatient[Table, RankOrder, ap]
```

	Absolute	Discounted	Relative	Unit	UnitSqrt	Marginal	PV/Cost	PV
1	4	5	4	4	4	5	5	5
2	5	4	5	5	5	4	3	4
3	2	3	3	3	3	3	2	2
4	1	1	1	1	1	2	1	1
5	3	2	2	2	2	1	4	3

Some observations are:

(1) Case 1 (a teenage boy) has a higher score on the economic criteria compared to case 2 (a teenage girl, likely to be poor) but she has a higher score on all medical criteria except for discounted gain. Though the teenage girl will likely be poor, she also will live longer, which improves her economic value.

It is actually clarifying to closer look into values for the discounted life-years gains. The girl is expected to live at least 10 years longer, but this has little effect, giving the small values of the discount factors so far in the future. There is a stronger immediate effect from the fact that the boy is expected to die within 3.6 years, while she will only die in 6 years. When the discount rate would be 2.90594%, then they would both have an equal discounted life-years gain value, he wins for higher values, she wins for lower values.

```
LifeGain[16.3, 3.6, 56.8 - 3.6, "Discounted", r] ==
LifeGain[12.6, 6,   69.5 - 6,   "Discounted", r]
```

$$\frac{1-\frac{1}{(r+1)^{53.2}}}{r\,(r+1)^{3.6}} == \frac{1-\frac{1}{(r+1)^{63.5}}}{r\,(r+1)^{6}}$$

```
FindRoot @@ {%, {r, .03}}
```

$\{r \to 0.0290594\}$

(2) Case 4 (a middle aged rich woman) has the lowest score on all criteria (except the marginal increment), but the intervention still is profitable from an economic point of view (PV / Cost > 1).

(3) Case 5 (a retiring rich lady) scores low on all criteria, but her economic value is high. Of course, if her pension would be counted as a cost, then she would score low on all counts.

(4) Case 3 (a young man in his twenties, with a middle income) has overall middle scores, and notably scores less than case 2.

(5) For these cases, all relative measures give the same rank order, and they differ slightly from the absolute rank order. In absolute terms, case 5 would get priority over case 3, but in relative terms case 3 would get priority over case 5.

Cost-effectiveness ratio's

Though all interventions are cost-effective from a financial point of view, since the PV of each future income outweighs the cost of each intervention, let us now disregard those PVs. Dividing the cost of the intervention by the various life extension measures, gives the Cost-Effectiveness Ratio or the unit cost per life extension measure. The costs per UnitSqrt appear to be less dispersed. This also means that decisions might be less easier to make if one wants to base decisions upon large differences.

```
apc = AnalysePatientCost[ap];
```

```
AnalysePatientCost[Table, apc]
```

	Cost	Absolute	Discounted	Relative	Unit	UnitSqrt	Marginal
1	5.2	0.098	0.219	1.945	7.145	6.095	6.95
2	5.7	0.09	0.241	1.67	7.37	6.481	8.072
3	3.7	0.411	0.519	12.004	15.704	7.623	7.4
4	8.4	6.72	7.103	357.84	366.24	55.465	21.67
5	2.5	0.227	0.362	16.795	19.295	6.945	8.019

Remember that costs are in $1000. For example for case 1:

(a) the treatment costs $98 per additional life year

(b) the treatment costs $219 per additional discounted life year

(c) the treatment costs $19.45 for each % increase in life duration (the base taken at original moment of death)

(d) the treatment costs $71.45 for each % point increase in life duration (the base taken in whole life)

(e) the treatment costs $60.95 for each % point increase in Sqrt life duration (the base taken in Sqrt of whole life)

(f) the treatment costs $69.50 for each % point increase in Sqrt life addition (the base at the current moment).

We can determine the rank orders again.

```
AnalysePatientCost[Table, RankOrder, apc]
```

	Cost	Absolute	Discounted	Relative	Unit	UnitSqrt	Marginal
1	3	2	1	2	1	1	1
2	4	1	2	1	2	2	4
3	2	4	4	3	3	4	2
4	5	5	5	5	5	5	5
5	1	3	3	4	4	3	3

For example, case 4 is the most expensive on all scores and there is not really a cheapest one. Case 1 is cheapest in terms of the price per UnitSqrt. Given that the UnitSqrt smoothes, it helps when the absolute costs are low as well.

Indeed, if we would order the cases by minimal unit cost, and take a cut-off point at a budget of 15 thousand dollars, then cases 1, 2 and 5 would be selected, with an average cost of $64.04 per UnitSqrt % point, and a budget slack of still $1.6 thousand.

```
PatientBudgetAllocation[UnitCost, 15, apc, 6]
```

{UnitCost(List) → {6.09546, 6.48126, 7.62276, 55.4654, 6.9454}, Order → {1, 2, 5, 3, 4}, Take → {1, 2, 5}, Not → {3, 4}, BudgetUsed → 13.4, Rest → 1.6, Sum → 2.0925, UnitCost(Average) → 6.40381, MarginalPatients → 4, MarginalCost → 8.4, PotentialBudgetOverrun → −6.8}

Note, however, that when we would allocate the whole budget and maximise the sum in UnitSqrts subject to that budget constraint, then the optimal solution appears to consist of cases 1, 2, and 3, with a total cost of 14.6 thousand dollar. Perhaps surprisingly, case 5 drops out, even though she is cheapest, both absolutely and in terms of unit cost. Her bad luck is, that maximising the sum of UnitSqrts subject to the budget constraint is something else than requiring efficiency in unit cost, since the budget slack can be used to introduce some inefficiency.

```
PatientBudgetAllocation[Budget, 15, apc, 6]
```

{UnitCost → {6.09546, 6.48126, 7.62276, 55.4654, 6.9454}, Take → {1, 2, 3}, Not → {4, 5}, BudgetUsed → 14.6, Rest → 0.4, Sum → 2.21794, UnitCost(Average) → 6.58268, MarginalPatients → {1, 4, 5}, MarginalCost → 1.5, PotentialBudgetOverrun → −1.1}

Cases versus values

At some basic level, our discussion refers to the difference between counting cases and weighing values. These two approaches are linked by taking an average value. Let n be the number of cases and v be the sum of all values (scores) of these cases, then $m = v / n$ is the average or mean value. Since Log[v] = Log[n] + Log[m], we can find a straight line contour in the Log[n] × Log[m] space that connects all points with equal total value. When we have subgroups, e.g. men versus women, then we can determine different contour lines, and the absolute differences in logs allow a percentage difference interpretation of the levels.

Considering above set of 5 random cases, we find that 2 are men and 3 are women, and we can determine the subgroup total scores and averages.

```
sel[s_] := Select[rp , (First[#] === s) &]
sel[f_, s_] := Map[f, Transpose[AnalysePatient[sel[s]]]]
men = sel[Add, Male];
women = sel[Add, Female];
```

These give the total scores per subgroup:

```
AnalysePatient[Table, {men, women}]
```

	Absolute	Discounted	Relative	Unit	UnitSqrt	Marginal	PV/Cost	PV
1	62.2	30.875	2.982	0.963	1.338	1.248	458.849	2287.798
2	75.75	31.731	3.586	0.926	1.391	1.406	177.514	702.923

These are the average values:

```
AnalysePatient[Table,{men/2, women/3}]
```

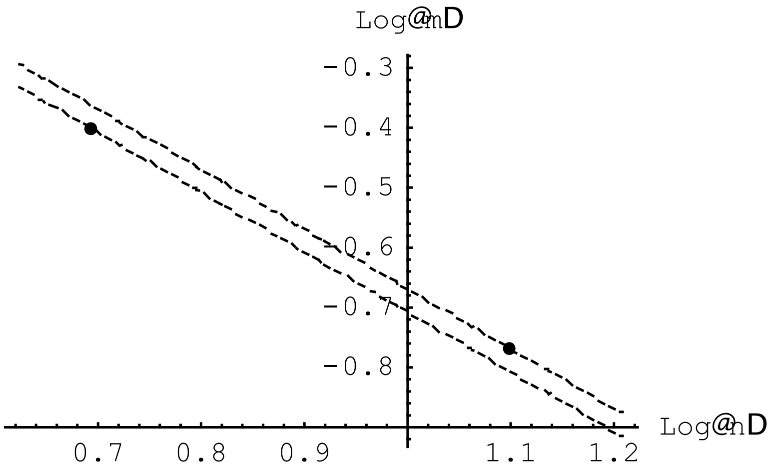
	Absolute	Discounted	Relative	Unit	UnitSqrt	Marginal	PV/Cost	PV
1	31.1	15.437	1.491	0.482	0.669	0.624	229.424	1143.899
2	25.25	10.577	1.195	0.309	0.464	0.469	59.171	234.308

Let us take the UnitSqrt measure and determine the {Log[n], Log[m]} points of the two groups, and then show the equal sum contours.

```
points = Log[{{2, men[[5]] / 2}, {3, women[[5]] / 3}}]
```

$$\begin{pmatrix} \log(2) & -0.40161 \\ \log(3) & -0.768694 \end{pmatrix}$$

```
EqualSumContour[points, AxesLabel → {Log[n], Log[m]}];
```



The vertical or horizontal distance between the lines is about 0.04, which means a 4% difference in sum total values. If the averages remain constant, the number of men in the sample would need to rise with 4% to get the same total value as the women. The random list of cases that we used actually seems typical, since the average UnitSqrt life-years gain of men is larger than for women.

When evaluating various treatments or programmes of intervention, then the total value v would be the main criterion for success. But as said, one would never lose sight of the number of lives saved, n, since these enter via the average score per person saved, v / n. When the measure that is used is not adjusted for age and sex, then the average becomes a more important factor in decision making. For example, for the absolute life-years gain, a high overall score can be caused by a huge value for a few individuals or by a meagre average for a large number, and likely the latter would be more interesting (unless the average becomes imperceptible). When the measure of success however has been adjusted for age and sex, such as is the case for the UnitSqrt measure, then one would be less sensitive to this composition effect, since there already has been a correction. However, the number of cases does not disappear even then, and still can be included as a separate factor in decision making.

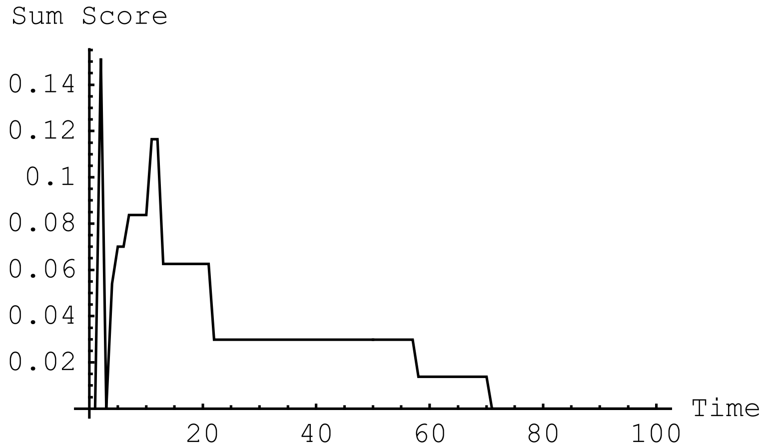
Appendix E: Distributions over time and age

Consider the numerical examples of the cases discussed in Appendix D. Their life gain scores can be assigned to different moments in time and age. Collecting these scores then gives distributions over time and age. The distribution over time is related to the discussion about discounting.

When we take the various UnitSqrt scores of the cases and allocate them evenly over their individual periods of extended living [d+1, e], then we can sum these scores per year, which gives the distribution over time.

```
res = PatientToTimeAxis[rp, "Sqrt"];
```

```
PlotLine[res, AxesLabel → {"Time", "Sum Score"}];
```

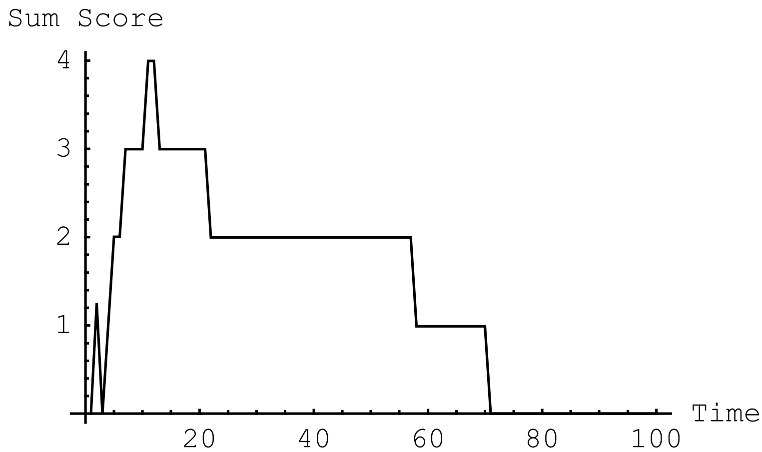


The scores can also be added over the whole period without additional discounting, since the UnitSqrt values already include time preference. An alternative plot, not shown here, would be to take the values $\sqrt{1}$, ($\sqrt{2}$ - $\sqrt{1}$), ($\sqrt{3}$ - $\sqrt{2}$), ..., all divided by the same $\sqrt{a + e}$.

We can also plot the absolute gain. We recognise the same features in the graph, but the proportions are different.

```
resAbs = PatientToTimeAxis[rp, "Absolute"];
```

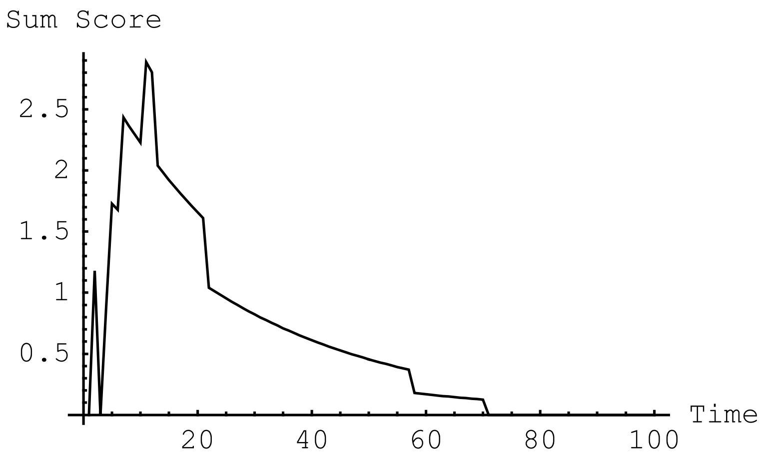
```
PlotLine[resAbs, AxesLabel → {"Time", "Sum Score"}];
```



The absolute scores neglect time preference. With social discounting using the discrete method, we get:

```
disAbs = Discounting[resAbs, 0.03];
```

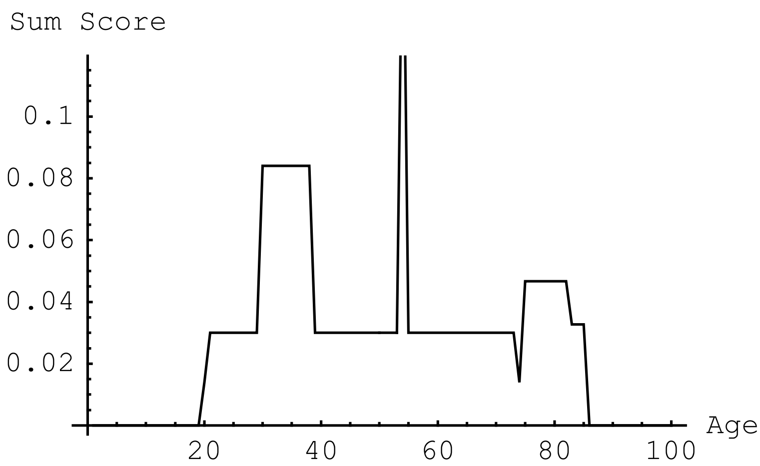
```
PlotLine[disAbs, AxesLabel → {"Time", "Sum Score"}];
```



When we allocate the scores evenly over their individual ages of extended living [a+d+1, a+e], then this gives the age distribution. We again compare the two main contenders. Again the features of the graphs are similar, but the relative heights differ. The age distribution created in this manner is flatter for the absolute measure and more pronounced for the UnitSqrt measure.

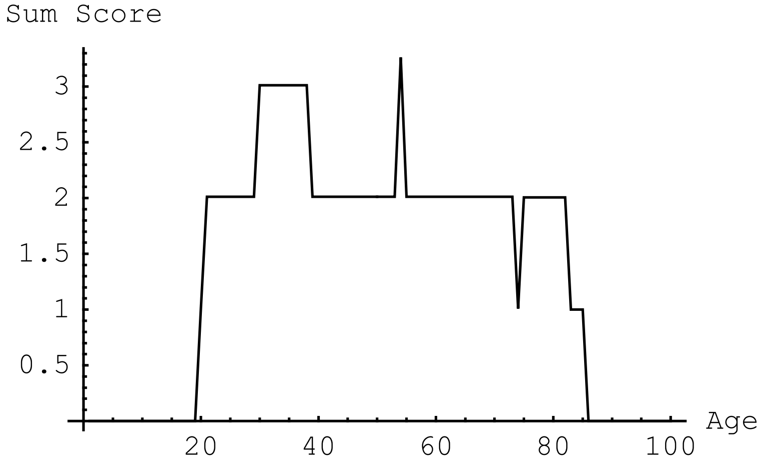
```
resAge = PatientToTimeAxis["Age", rp, "Sqrt"];
```

```
PlotLine[resAge, AxesLabel → {"Age", "Sum Score"}];
```




```
absAge = PatientToTimeAxis["Age", rp, "Absolute"];
```

```
PlotLine[absAge, AxesLabel → {"Age", "Sum Score"}];
```



Appendix F: Conditions for consistency

Suppose that a subject is given two interventions, one for flood control and one for a heart attack. Part of the effects will overlap, since both interventions are required to keep the subject alive. Some will not overlap, for example when flood control comes first, adds 10 years of life expectancy at that age, and when the other is later, say after five years, and secures more life expectancy at that age.

In the context of Public Choice, it is also possible that a treatment is split up in two separate treatments, supposedly with the argument of specialisation, but perhaps with the objective to generate more income or UnitSqrt score-points for the services provided.

Should we collect all interventions per person, or can we trust that interventions are defined by themselves or that the market works properly ?

Two conditions for consistency can be mentioned.

The first is whether the same results arises when the age rises to $a + \delta$, and the moment of death then becomes $d - \delta$, assuming that x remains the same.

```
ForSet[{"Absolute", "Relative", "Unit", "Sqrt"}, c,
LifeGain[a, d, x, c] == LifeGain[a + δ, d - δ, x, c]]
```

{True, True, True, True}

Another issue is what happens when the treatment with effect x, say $x = 10$, is split up in separate steps. Consider a patient of age 30 who is likely to die in 5 years, and split the treatment into two steps, first with extension ξ and subsequently, after 5 years, with extension $x - \xi$. There now are two separate UnitSqrt scores, and adding them could increase the total score for the patient. This does not happen for the absolute gain measure, that is neutral to such splitting. But the other measures are sensitive to it. We can investigate this split-up effect without worrying about the age effect, since there is no separate influence here as shown above.

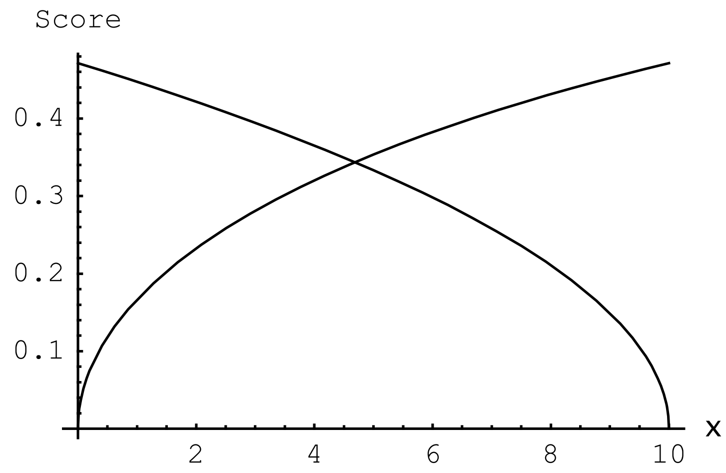
```
split[x_, ξ_, c___] := {LifeGain[30, 5, ξ, c], LifeGain[35, ξ, x-ξ, c]}
```

The UnitSqrt scores at the borders are 0.47, showing that there are two ways to reach age 45 in maximal manner. Inbetween values of ξ show a reduction of the score, when these cases would be considered as separate patients.

```
split[10, ξ, "Sqrt"]
```

$$\left\{\sqrt{\frac{\xi}{\xi+35}}, \frac{\sqrt{10-\xi}}{3\sqrt{5}}\right\}$$

```
Plot @@ {split[10, ξ, "Sqrt"], {ξ, 0, 10}, AxesLabel → {ξ, "Score"}};
```



When these scores are added, we see a top at $\xi = 5$. Thus, when the patient is treated at 30 with a life extension of 5, and is treated again at 35 with another life extension of 5, then the sum of the separate life gain measures is highest, and higher than 0.47. The average effect (dividing by 2), though, is lower than 0.47.

```
Plot @@ {{tot = Add[split[10, ξ, "Sqrt"]], tot/2}, {ξ, 0, 10}, AxesLabel →
{ξ, "Score (Sum)"}};
```

This property that the total can be so large seems contradictory to requirements of consistency, but closer reflection shows that this need not be the case.

(a) If we assume that the gain measure is applied at the clinical level, which is not intended but might help to clarify the implications, and if the score on each single event would be relevant for the priority of a treatment, then there would be a danger of arbitrarily splitting up of treatments. However, if the arbitrary splitting up of the treatment would also increase the total cost of the treatment, then there would be some check on this.

(b) If we return to the domain of national policy making, for which the UnitSqrt measure is intended, there is the design of intervention programmes, and there the scores indeed might be split up and summed. Using the UnitSqrt score, and allowing the multiple entry of individuals, an intervention programme with intermediate treatments would then indeed look more effective than a programme with only one treatment. Again, fixed costs per treatment would penalise this effect.

It follows that the application of the UnitSqrt measure requires careful definition of the treatment, and records must be kept per individual. For the other life gain measures the same arguments for consistency hold. The main criterion for consistency appears to be that one tries to avoid multiple entries per individual. However, when there are really different treatments, such as one for flood control and one for treatment of heart disease, then the decision on the separate issues would be based upon the separate UnitSqrts.

Appendix G: Comparing Fair Innings, Proportional Shortfall, UnitSqrt

LifeGain function

The LifeGain function in *The Economics Pack* collects the various measures in a uniform format. This has the advantage that outcomes for particular cases can be compared without uncertainty about the input and the calculation.

The LifeGain function has been formulated in terms of an intervention that causes recovery. With age a and remaining life expectancy d because of an accident or disease, then the intervention would recover life expectancy with an additional x to the value $e = d + x$. The variable x can be seen as added to d or as taken away from e because of the accident or disease. We can distinguish the average age-specific expectation $e[a]$ and some other value e due to treatment.

The LifeGain format thus can also represent a *burden of disease calculation*. Namely, the burden of a particular disease is the loss in life expectancy $burden = e[a] - d = x$, where d is the remaining expectancy after incurring the disease. Prevention of the accident or disease can be seen as an intervention that “restores” the original life expectancy that otherwise would have been lost. The LifeGain format is more general than for only $e[a]$, since it also allows for interventions that have an effect $x = e - d$, that differs from $e[a] - d$, namely with a new life expectancy $e = d + x$.

Criteria considered in this Appendix

This Appendix discusses the following criteria for allocation of resources for treatment.

- The “Rule of Rescue” (ROR) or the “lives saved” (extended) measure. Patients are ordered on values of d, and those with the lowest values of d are selected. These cases can be described as “most urgent” but it is rather tautological that the urgency follows the selection rule.

- The “*prospective health*” criterion gives priority to patients for who the potential improvement x is highest. This value of x is called “absolute shortfall”.

- The “*proportional shortfall*” (PS) criterion, which is the prospective health but relative to what is possible in the future, giving $x / (d + x)$. This neglects the age of the patients (i.e. what already has been achieved in the past).

- The “*fair innings*” (FI) that uses a norm for what a decent duration of life would be. Williams (1997) selects 61 years of age, adjusted for quality of life. In its discussion of “fair innings”, ZIN (2018a) selects 85 years of age, presumably also with quality adjustment.

- The Unit(Sqrt) measure takes each life as 100%, no matter its duration.

Sometimes it is stated that the patients can be helped with “the worst health”. This term however requires interpretation. In the perspective of prospective health a case might be seen as “worst” when the shortfall is largest, but in the perspective of fair innings a lower life expectancy would be seen as worst, which might come with lower values of the absolute shortfall.

The purpose of this Appendix is to clarify the relations between these measures. This paper "*On the value of life*" and its *UnitSqrt* measure originated as Colignatus (2003), independently from the "*fair innings*" (FI) argument by Williams (1997), and independently from the "*proportional shortfall*" (PS) measure, as chosen by ZIN (2018ab) for national health insurance in Holland. The 2003 version already included or indicated these alternative measures by their structural form but not by these particular names and uses. It is useful now to discuss them explicitly. The PS measure was already developed in the 1990s and has been in use since then, when ZIN was called CZV, see Van de Wetering et al. (2013). The ZIN (2018a) report can be much appreciated. Its appendix (2018b) allows for a perspective on such thinking in Holland since it collects responses by Dutch health research institutes. Since the measure of proportional shortfall has been in use for such a long time, those researchers have been acquainted with the measure, and their commentary might not be as critical as might be possible from a fresh consideration. Reckers-Droog et al. (2019) compare the official criterion of proportional shortfall with preferences amongst the general public, and they conclude:

"Our results indicate that the public prefers prioritising relatively more severely ill patients when patients' ages are equal and younger patients when patients' disease severity is equal. The latter preferences were found to be irrespective of patients' severity levels in the current design. Our results suggest that the public considers patients' age to be highly important in setting priorities. Current decision-making frameworks do not reflect these preferences."

For the complexity of evaluation of national resource allocation see e.g. also Paulden & Culyer (2010).

Quality of Life adjusted variables

An improvement upon the 2003 version of this paper is that this present version better clarifies how the UnitSqrt[*a, d, x*] measure must be applied to *quality of life adjusted variables*: see Section 1.4.1 for UnitSqrt[*aq, dq, xq*]. (PM. The 2003 version was at risk of using UnitSqrt[*a, dq, xq*].)

See Gold et al. (2002) for the comparison of HALYs, QALYs and DALYs, and see Murray (1994) for the adjustment for disability. DALYs are determined by health experts and are related to ICD-10 categories, while QALYs tend to be based upon patient questionnaires and allow for a population average. The latter is the preferred approach here. (PM. The ZIN (2018a:7) characterisation of DALY and QALY is not convincing.) The LifeTable routine in *The Economics Pack,* see Cool (2001a, 2020), now includes UnitSqrtLE and UnitSqrtQALE for the scenario of an accident or disease with instantaneous death and an intervention with recovery to the (quality adjusted) age-specific life expectancy. Thus UnitSqrtLE[] = UnitSqrt[*a*, 0, *e*[*a*]] and UnitSqrtQALE[] = UnitSqrt[*aq*[*a*], 0, *eq*[*a*]] indexed by calendar age *a*.

The functions have the same structure for both adjusted and unadjusted variables. While it is understood that input variables best be adjusted for quality of life, we may still use unadjusted variables in the following, namely for ease of writing and reading single-character variables.

Inclusion of FI and PS in the LifeGain format

For reference, it is useful to restate the UnitSqrt measure of performance:

```
LifeGain[a, d, x, "Sqrt"]
```

$$\sqrt{\frac{x}{a+d+x}}$$

When the intervention restores the original age-specific life expectancy *e*[*a*] then *x* = *e*[*a*] - *d,* and the measure of performance is:

```
LifeGain[a, d, e[a] - d, "Sqrt"]
```

$$\sqrt{\frac{e(a)-d}{e(a)+a}}$$

In the WHO Global Burden of Disease study, a disability factor *dis* is determined per disease and age group with life years LY, so that the burden = *dis* LY, and then the disease- and age-specific DALY follows as DALY = (1 - *dis*) LY. When this is related to the life expectancy then the *burden* = *e*[*a*] - *d*. When this is related to quality of life and the QALE then the *burden* = *eq*[*a*] - *dq*, in which the calendar age *a* has the role of an index. ZIN (2018ab) uses the term “burden” (Dutch “ziektelast”) but also uses QALE rather than DALE.

The LifeGain function has now been extended with the "fair innings" (FI) and "proportional shortfall" (PS) measures, see their definitions below. Fair Innings requires an age norm which can be entered via Options[Life-Gain], so that the overall input structure is not affected.

Level and proportional shortfall

The level shortfall was already given in 2003 as follows.

```
LifeGain[a, d, x, "Absolute"]
```

$$x$$

The proportional shortfall (PS) was already identified in the 2003 version of this paper as the “marginal” approach. It doesn’t include the age *a* or *aq* in the denominator. ZIN (2018a:12): “However a point of discussion about this method is that there is no distinction in prioritising different age-groups.”

PS thus is calculated as burden / expectation. For unadjusted variables it is (*e*[*a*] - *d*) / *e*[*a*] = *x* / (*d* + *x*). For quality of life adjusted variables it is (*eq*[*a*] - *dq*) / *eq*[*a*] = *xq* / (*dq* + *xq*), in which calendar age *a* functions as an index.

We have the following input formats that actually do the same.

```
LifeGain[a, d, x, "PrShortfall"]
```

$$\frac{x}{d+x}$$

```
LifeGain[a, d, x, "Marginal", Utility → Identity]
```

$$1-\frac{d}{d+x}$$

```
LifeGain[a, d, e[a] - d, "Marginal", Utility → Identity]
```

$$1-\frac{d}{e(a)}$$

```
PrShortfall[a, d, e - d]
```

$$1-\frac{d}{e}$$

ZIN (2018a:10) gives this example of proportional shortfall: “A loss of 5 QALYs given an expectation of 10 QALYs gives the same proportional shortfall as a loss of 20 QALYs given an expectation of 40 QALYs.”

Given the distinction between $x = e[a] - d$ and $x' = e - d$, or that a treatment might not succeed in fully recovering the average life expectancy, we can also imagine a “shortfall” measure $x' / e[a]$. However, there is no discussion of this by ZIN (2018a) and we leave it be.

```
? PrShortfall
```

Symbol

PrShortfall[a, d, x] is LifeGain[a, d, x, "Marginal", Utility -> Identity]

Fair innings

Williams himself refers to John Harris (1985:91-94) "*The Value of Life*" (Routledge and Kegan Paul). This reference indicates that there is a general reservoir for notions of fairness, so that the reader might understand that I arrived at the UnitSqrt measure independently from these authors. In 2020, the SARS-CoV-2 pandemic caused me to return to the issue, see Colignatus (2020ab). The ZIN (2018ab) report on allocation within Dutch national health insurance alerted me to their "proportional shortfall" approach and the "fair innings" by Williams (1997). This Appendix has been written to include these measures and to make comparisons.

A remarkable point in Williams (1997) is his focus on differences in life expectancy due to the social and economic status (SES) of cases. It makes sense to calculate values conditional to SES, namely for the evaluation of how society is performing. It is another issue to use these outcomes for the allocation of resources. On equity considerations, one would tend to allocate national resources for the group of all men and all women separately, not distinguished by SES. At the individual bedside level, the SES would help estimate the life expectancy, but then there are more criteria to allow for a more informed decision. To allocate national resources using the SES conditionality would require additional arguments, that likely go beyond the use of the current variables a, d and x.

Williams (1997) observes that the UK Social Classes SC1&2 have a life expectancy at birth of 72, and SC4&5 of 67 years. The QALE values are 66 and 57. Subsequently, Williams finds: “Even a man in SC4 and 5 will eventually achieve the ‘fair innings’, here supposed to be 61 QALYs, [ftnt] by the time he is 64 years old.” While it is dubious to make national allocation conditional on SES, it can be observed that the distinction allows Williams to find an age norm for the notion of fair innings. However, such a norm might also be found by taking population weights and then find the average life expectancy at birth for all men. ZIN (2018ab) raises the norm to 85 years, presumably with quality adjustment. However, life expectancy at birth for Dutch males however is around 80 and quality adjusted around 65 . We take FI as indifferent to sex, so these norms may also be taken for females. Thus we have the option FairInnings → {80, 65}.

With a^* the age norm for fair innings, then $x' = \text{Min}[a^*, a + d + x] - \text{Min}[a^*, a + d]$ is the scope for a *FI-acceptable* gain in life expectancy, which can be labeled as “FI Level”. ZIN (2018b:8) confirms: "After 85 the burden of disease becomes zero. This is the consequence of fair innings, and also its limitation that does not conform to social preferences. This is also the reason to advance PS [Proportional Shortfall]."

A relative format may be "FI Ratio" = x' / a^*. Relevant is also "FI Unit" = $x' / \text{Min}[a^*, a + d + x]$. For some cases the latter has a lower denominator and thus a higher outcome, which seems more appropriate for the philosophy of Fair Innings. QALE formats require input of aq, dq, xq and aq^*.

```
LifeGain[a, d, x, "FI Unit"]
```

$$\frac{\min(80,\ a + d + x) - \min(80,\ a + d)}{\min(80,\ a + d + x)}$$

The drawback of the FI exposition by Williams (1997) is that it requires the additional parameters a^* and aq^* which are also dependent upon time and location. He might have been open to the notion that the Unit(Sqrt) measure takes each life as 100%, so that all are treated equally in this respect, and so that there is no need for an additional parameter. However, Williams (1997) explicitly argued that "fair innings" "(...) reflects the feeling that everyone is entitled to some 'normal' span of health (usually expressed in life years, e.g. 'three score years and ten') and anyone failing to achieve this has been cheated, whilst anyone getting more than this is 'living on borrowed time'." Thus, a proper presentation of "fair innings" (as seen by Williams) cannot avoid such a norm for what would be a "normal span". While the Unit(Sqrt) measure conveys many aspects of a notion of fair innings, we still must adopt the formulation by Williams (1997) as a codification of his proposal of "fair innings". A rejection of Williams' implementation is not a rejection of a notion of fair innings (100% could be fair too, as it allows each life to run its course) but only a rejection of Williams's particular implementation. It may well be that the Unit(Sqrt) measure better captures what Williams (1997) was after, and that he merely did not develop this at the time. It is a pity that the 2003 paper did not reach him (Alan Williams (1927 - 2005)) and that I was unaware at the time of his work.

```
? FairInnings
```

Symbol

FairInnings –> {norm LE, norm QALE}, in Options[LifeGain],

gives the norm of what would be a fair duration of life,

taken as the (quality adjusted) life expectancy at birth.

Special formats for LifeGain[a, d, x, string] are, and write FI or FI QALE:

"FI (QALE) Level" gives Min[a+d+x, norm] – Min[a+d, norm]

"FI (QALE) Ratio" gives (Min[a+d+x, norm] – Min[a+d, norm]) / norm

"FI (QALE) Unit" gives 1 – Min[a + d, norm] / Min[a + d + x, norm]

Cases mentioned by Van de Wetering et al. (2013) and ZIN (2018a:13)

Van de Wetering et al. (2013) compare a 70-year-old, who loses 1 of remaining 5 years, with a 30-year-old, who loses 1 of 40 years due to lack of treatment. (i) Williams's implementation of fair innings has 1 / $a*$ for both, with both ages below $a*$ = 80. (ii) Proportional shortfall favours the elderly person, who gets to live to 75. (iii) UnitSqrt favours the younger person who gets to live to (only) 70. The FI Unit measure also indicates that the youngster makes a better contribution to fair innings (because the final age of 70 is below the other's 75).

```
lis = {"Sqrt", "PrShortfall",
   "Absolute", "FI Level", "FI Ratio", "FI Unit", "Rescue"};
TableForm[{
    LifeGain[70, 4, 1, #] & /@ lis,
    LifeGain[30, 39, 1, #] & /@ lis} // N // Transpose,
 TableHeadings → {lis, {"Old", "Young"}}]
```

	Old	Young
Sqrt	0.11547	0.119523
PrShortfall	0.2	0.025
Absolute	1.	1.
FI Level	1.	1.
FI Ratio	0.0125	0.0125
FI Unit	0.0133333	0.0142857
Rescue	4.	39.

ZIN (2018a:13) mentions two example cases that can be reproduced here as far as possible. The ZIN[2018a) example of Fair Innings takes the norm as 85, which doesn't fit the Dutch situation, so that we adapt to our choice of parameters.

(i) An old person, with a loss of 1 QALY, given an average expectation of 2 QALYs.

(ii) A young person, with a loss of 30 QALYs, given an average expectation of 60 QALYs.

Unfortunately these values don't allow a numerical distinction between d and x. We have set the FI norm at 80 years of age, and take 75 as the old age (since any value ≥ 80 gives x' = 0). Let the youngster be 20 years of age.

```
lis = {"Sqrt", "PrShortfall",
   "Absolute", "FI Level", "FI Ratio", "FI Unit", "Rescue"};
TableForm[{
    LifeGain[75, 1, 1, #] & /@ lis,
    LifeGain[20, 30, 30, #] & /@ lis} // N // Transpose,
 TableHeadings → {lis, {"Old", "Young"}}]
```

	Old	Young
Sqrt	0.113961	0.612372
PrShortfall	0.5	0.5
Absolute	1.	30.
FI Level	1.	30.
FI Ratio	0.0125	0.375
FI Unit	0.012987	0.375
Rescue	1.	30.

The ZIN (2018a:13) example with a^* = 85 presumably has their "old person" also at that age $a = a^*$. Subsequently they have the formula (85 - (85 - 1)) / 85. Apparently they subtract the burden of disease from a^*, giving $a^* - d$, and then they apparently determine the gain as $x = a^* - (a^* - d)$. It is hard to say since they took $x = d$. If their token age really is $a = a^*$ then their formula conflicts with the earlier statement that there would be no advancement beyond a^*. I do not understand what formula they apply for this particular example. The formula implemented in *The Economics Pack* uses their formula in their Table 2 in ZIN (2018a:11) while also including the maximum condition on a^*, which gives x' = Min[a^*, $a + d + x$] - Min[a^*, $a + d$].

References

Colignatus is the name in science for Thomas Cool, econometrician (Groningen 1982) and teacher of mathematics (Leiden 2008)

M.E. van den Akker-Van Marle, M. van Ballegooijen, G.J. van Oortmarssen, R. Boer, J.D.F. Habbema (2002), "Cost-effectiveness of cervical cancer screening: comparison of screening policies", Journal of the National Cancer Institute, 94:193-204

Barten, A. (1977), "The systems of consumer demand functions approach: A review", Econometrica, Vol 45, No 1

Bleichrodt, H., and M. Johannesson (1997), "Standard gamble, time trade-off and rating scale: experimental results on the ranking properties of QALYs", J. of Health Economics 16:155-175

Bodie, Z. and R. Merton (2000), "Finance", Prentice Hall

Cairns, J.A. and M.J. van der Pol (1997), "Saving future lives. A comparison of three discounting models", Health Economics 6:341-350

Chapman, G.B. (2002), "Your money or your health: Time preference and trading money for health", Health Economics 401-416

Cohen B. J. (2003) , "Discounting in cost-utility analysis of healthcare interventions: reassessing current practice", Pharmacoeconomics. 2003;21(2):75-87

Cool, Th. (2001a, 2020), "The Economics Pack. Applications of *Mathematica*", Thomas Cool *Consultancy & Econometrics,* http://thomascool.eu/TheEconomicsPack/index.html. The software has been updated but the 2001a User Guide is "as is".

Colignatus, Th. (2001b, 2014), "Voting Theory for Democracy. Using The Economics Pack *Applications of Mathematica* for Direct Single Seat Elections", Thomas Cool *Consultancy & Econometrics,* http://thomascool.eu/Papers/VTFD/Index.html

Colignatus, Th. (2003), "On the value of life", working paper, https://ideas.repec.org/p/wpa/wuwppe/0310003.html

Colignatus, Th. (2020a), "Covid-19 and the value of life", https://boycottholland.wordpress.com/2020/03/31/-covid-19-and-the-value-of-life/

Colignatus, Th. (2020b), "Redesign of the didactics of S(E)IR(D) -> SI(EY)A(CD) models of infectious epidemics", https://zenodo.org/record/3894161

Ferrer Carbonell, A. (2003), "Quantitative analysis of well-being with economic applications", (17/01/03), Thesis 295, Tinbergen Institute, www.tinbergen.nl

Gold, M., J. Siegel, L. Russell, M. Weinstein (1996), "Cost-effectiveness in health and medicine", Oxford

Gold, Marthe, and David Stevenson, Dennis Fryback (2002), "HALYs and QALYs and DALYs, Oh My: Similarities and Differences in Summary Measures of Population Health", Annu. Rev. Public Health 2002. 23:115–34

Grossman, M. (2000), "The human capital model", in A.J. Culyer and J.P. Newhouse, "Handbook of health economics", Volume 1A:347-408

Gyrd-Hansen, D. and J. Sogaard (1998), "Discounting life-years: Whither time preference ?", Health Economics 7:121-127

Hueting, R. (1991), "The use of the discount rate in cost-benefit analysis for different uses of a humid tropical forest area", Ecological Economics, 3, p43-57

Hueting, R. (1992), contribution in the discussion in "Committee of international development institutions on the environment (cidie): workshop on environmental and natural resource accounting, Summary Record, Nairobi, 24-26 February 1992", Environmental Economics Series Paper No. 3, United Nations Environment Programme (UNEP), see http://www.unep.org/unep/products/eeu/ecoserie/ecos3/ecos38.htm

Hueting, R. and B. de Boer (2019), "National Accounts and environmentally Sustainable National Income", Eburon Academic Publishers and University of Chicago Press, http://www.sni-hueting.info/EN/NA-eSNI/index.html

Johannesson, M. (1992), "On the discounting of gained life-years in cost-effectiveness analysis", International Journal of Technology Asessment in Health Care, *:2, 359-364

Johannesson, M., J. Pliskin, M. Weinstein (1994), "A note on QALY's, time tradeoff, and discounting", Med. Decision Making 14:188-193

Jonkman, S.N., P.H.A.J.M. van Gelder, J.K. Vrijling (2003), "An overview of quantitative risk measures for loss of life and economic damage", Journal of Hazardous Materials A99, 1-30

Kahneman D., A.Tversky, P. Slovic (eds) (1982), "Judgment under uncertainty: Heuristics and biases", CUP

Lipscomb, J., M.C. Weinstein, G.W. Torrance (1996), "Time preference", pp 214-246, in M.R. Gold, J.E. Siegel, L.B. Russell, M.C. Weinstein (eds) (1996), "Cost-effectiveness in health and medicine", Oxford

Luenberger, D. (1998), "Investment Science", Oxford

MacKeigan L.D., A. Gafni, B.J. O'Brien (2002), "Double discounting of QALYs", Health Econ. 2003 Feb;12(2):165-9.

Manton, K, and E. Stallard (1988), "Chronic disease modelling", Charles Griffin & Co (UK), Oxford (USA)

Mueller, D. (1989), "Public Choice II", Cambridge

Murray CJ (1994), "Quantifying the burden of disease: the technical basis for disability-adjusted life years", Bull World Health Organ. 1994;72(3):429-445.

Nicolet, Anna, and Antoinette D I van Asselt, Karin M Vermeulen, Paul F M Krabbe (2020), "Value judgment of new medical treatments: Societal and patient perspectives to inform priority setting in The Netherlands", PLoS One, 2020 Jul 9;15(7):e0235666. doi: 10.1371/journal.pone.0235666.

Nord, E. (1992), "An alternative to QALYs: the saved young life equivalent (SAVE)", BMJ 305:875-7

Paulden, M. and Culyer, A.J. (2010), "Does cost-effectiveness analysis discriminate against patients with short life expectancy? : Matters of logic and matters of context. Working Paper", CHE Research Paper . Centre for Health Economics, University of York , York, UK.

Reckers-Droog, Vivian, and Job van Exel, Werner Brouwer (2019), "Equity Weights for Priority Setting in Healthcare: Severity, Age, or Both?", Value in Health, Volume 22, Issue 12, P1441-1449, December 01, 2019, https://doi.org/10.1016/j.jval.2019.07.012

RIVM (2003), "Nuchter omgaan met risico's", RIVM rapport 251701047, Bilthoven, The Netherlands

Rosenberg, L. (2002), "Exceptional economic returns on investments in medical research", speech for the Australian Society for Medical Research, at http://www.mja.com.au/public/issues/177_07_071002/ros10377_fm.html

Rutten-van Molken, M., J. van Busschbach, F. Rutten (eds) (+/- 1999), "Van kosten tot effecten. Een handleiding vor evaluatiestudies in de gezondheidszorg", Elsevier Gezondheidszorg

Wetering, EJ van de, and Stolk EA, van Exel NJ, Brouwer WB (2013), "Balancing equity and efficiency in the Dutch basic benefits package using the principle of proportional shortfall", Eur J Health Econ. 2013;14(1):107-115. doi:10.1007/s10198-011-0346-7

Williams, A. (1997), "Intergenerational equity: an exploration of the 'Fair Innings' argument", Health Economics 6, 117–32

Zorginstituut Nederland (ZIN) (2018a), "Ziektelast in de praktijk. De theorie en praktijk van het berekenen van ziektelast bij pakketbeoordelingen, https://www.zorginstituutnederland.nl/publicaties/rapport/2018/05/07/ziektelast-in-de-praktijk

Zorginstituut Nederland (ZIN) (2018b), "Bijlage bij rapport 'Ziektelast in de praktijk': Samenvatting reacties externe partijen op concept rapport", https://www.zorginstituutnederland.nl/publicaties/rapport/2018/05/07/ziektelast-in-de-praktijk