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Abstract

The paper examines the utility function representability of the lexicographic preferences over multiple attributes. A sufficient and necessary condition is provided: a lexicographic preference is utility function representable if and only if there is at most one attribute with uncountable levels and when such an attribute exists, it is the least important one. An auxiliary result of independent interest for general rational preferences is proved stating that the class of utility function representable sets is closed in countable unions under some mild condition.

Keywords: Lexicographic Preference; Utility Function Representation; Representable Class; Pseudo-continuous Points.

JEL Classification Numbers: D01, D11.

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1 Introduction

A lexicographic rule orders alternatives over attributes in the same way as a dictionary orders words over letters. When using lexicographic rule to make decisions, an agent evaluates alternatives first on the most important attribute, and if there are ties, on the second-most important attribute, and so forth. There is good evidence that people use lexicographic rules. Tversky et al. (1988) examine the rules that consumers use for choice and matching tasks involving public policies, job applicants, and benefit plans. Yee et al. (2007) report that approximately two-thirds of their subjects use lexicographic rules for evaluating SmartPhones.

Even though lexicographic rules occur in many decision-making scenarios, lexicographic rules can not be represented by a function in general. A result known at least since Debreu (1954) says that it is impossible to construct a utility function representing lexicographic preferences over two or more real-valued attributes. However, lexicographic utility functions can exist over discrete attributes. Substantial theoretical research has also examined conditions under which continuous lexicographic utility functions can exist (e.g., Fishburn 1974, 1975), the possibility of representing such preferences by multiple functions (e.g., Bridges 1983, Chateauneuf 1987, Knoblauch 2000), and the formulation of models for probabilistic lexicographic preferences (e.g.,Tversky 1972).

In this paper, we focus on the study of the utility function representation about lexicographic preferences under different scenarios. We find that the cardinality of different attributes will determine whether the lexicographic preference can be represented by a utility function. It is shown that a lexicographic preference is representable by a utility function if and only if there is at most one attribute with uncountable levels and when such an attribute exists, it is the last one, i.e., the least important one.

The intuition behind the results is that, if the lexicographic preference can be represented by a utility function, the preference is determined by one-dimension information. When compressing the preference information from multiple dimensions (i.e., attributes) to one dimension (i.e., utility value) while preserving the order, if an attribute dominates some other attribute(s), it must allow some utility space to accommodate the preference information of the less important attributes. As will be shown in Proposition 2, if an attribute has uncountable levels, any candidate utility function for that attribute has points in its range that are not isolated, which means there is no place to embody preference information in the less important attributes.

In order to prove the sufficient and necessary conditions for representability of the lexicographic preferences, we develop an auxiliary lemma that also applies for general preferences. It describes a condition that guarantee a countable union of subsets on which a specific preference has utility function representations is also representable by a function.

The remainder of the paper is organized as follows. Section 2 lays down a general framework for studying the utility function representation problem of any rational preferences. Section 3 provides the sufficient and necessary condition for the utility function representability of the lexicographic preferences with at most one attribute with uncountable levels. Section 4 gives an auxiliary lemma that is useful for the proof of the main theorem and of independent interest itself. Section 5 proves the main theorem according to whether the number of attributes is finite or infinite. Section 6 concludes the paper.

2 Notations and Concepts

In this section, we give a general framework about the utility function representation. Consider a set $\mathbb{X}$ of possible objects (i.e., alternatives) that might be chosen by an agent who has a preference relation $\succcurlyeq$. Each alternative of $\mathbb{X}$ is characterized by n attributes, where n can be finite or infinite. Denote $N = \{1, \cdots, n\}$ when n is finite; $N = \mathbb{N}$ when n is infinite. Here, the attributes can be very general. For example, in the consumer theory, an attribute can be a good, a combination of goods, a combination of goods and their characteristics, etc. For any specific attribute $i \in N$, its domain is denoted by $\mathbb{X}_i$ which has at least two distinct elements. Therefore, the set of all alternatives is denoted as $\mathbb{X} = \times_{i \in N} \mathbb{X}_i$ and each alternative is characterized by a vector $(x_1, \cdots, x_n) \in \times_{i \in N} \mathbb{X}_i$.

For any nonempty $I \subseteq N$, we denote $X_I = \times_{i \in I} \mathbb{X}_i$. For any $x \in \mathbb{X}$ and $I \subseteq N$, we denote $x = (x_I, x_{N \setminus I})$, where $x_I \in \mathbb{X}_I$ and $x_{N \setminus I} \in \mathbb{X}_{N \setminus I}$. Furthermore, if I is a singleton, that is, $I = \{i\}$ for some i, we denote $\mathbb{X}_{-i} = \mathbb{X}_{N \setminus \{i\}}$ and $x = (x_i, x_{-i})$, where $x_i \in \mathbb{X}_i$ and $x_{-i} \in \mathbb{X}_{-i}$.

The agent's preference on $\mathbb{X}$ is represented by a binary relation $\succcurlyeq$, where $x \succcurlyeq y$ stands for "the agent prefers x to y". The corresponding strict preference is denoted as $x \succ y$, i.e., $x \succcurlyeq y$ and not $y \succcurlyeq x$. The indifference relation is defined as $x \sim y$ if and only if $x \succcurlyeq y$ and $y \succcurlyeq x$. Furthermore, for each attribute $i \in N$, the agent has a preference $\succcurlyeq_i$ on $\mathbb{X}_i$. All these preferences, $\succcurlyeq$ and $\succcurlyeq_i, i \in N$ are assumed to be rational, that is, they are complete and transitive.

In this paper, the utility function representation problem of $\{\mathbb{X}, \succcurlyeq\}$ with respect to $\{\mathbb{X}_i, \succcurlyeq_i \}_{i \in N}$ will be studied. The situation faced by the agent is summarized as $\{\{\mathbb{X}_i, \succcurlyeq_i\}_{i \in N}, \{\mathbb{X}, \succcurlyeq\}\}$. For any preference relation $\{\mathbb{X}, \succcurlyeq\}$, it is utility function representable, if there exists a function, which assigns each alternative $x \in \mathbb{X}$ a real number and keeps the order of the preference.

Definition 1. *A preference relation $\{\mathbb{X}, \succcurlyeq\}$ is utility function representable if there exists a function $u : \mathbb{X} \to \mathbb{R}$, such that for all $x', x'' \in \mathbb{X}$,*

$$x' \succcurlyeq x'' \iff u(x') \geq u(x'').$$

When the above condition holds, we say that the function $u(\cdot)$ represents $\succcurlyeq$ over $\mathbb{X}$.

Generally, there is no direct relationship between the utility function representability of $\{\mathbb{X}_i, \succcurlyeq_i\}$ and $\{\mathbb{X}, \succcurlyeq\}$. Firstly, as shown by Debreu (1954) and/or Mas-Colell et al.(1995), it is impossible to construct a utility function representing lexicographic preferences over two or more real-valued attributes. Therefore the utility function representability of $\{\mathbb{X}_i, \succcurlyeq_i\}$ for all i does not guarantee the utility function representability of $\{\mathbb{X}, \succcurlyeq\}$. On the other hand, if there is no restriction on $\{\mathbb{X}, \succcurlyeq\}$, it can induce a preference on each component $\mathbb{X}_i$ other than $\succcurlyeq_i$ in an arbitrary way. In particular, even though for some i $\{\mathbb{X}_i, \succcurlyeq_i\}$ can not be represented by a utility function, $\{\mathbb{X}, \succcurlyeq\}$ may be representable by some utility functions as shown in the following example.

Example 1. *The set $\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2$ consists of possible objects that might be chosen, with each object having 2 attributes. It is assumed that $\mathbb{X}_1 = \mathbb{Y}_1 \times \mathbb{Y}_2$, $\mathbb{X}_2 = \mathbb{Y}_3$, where $\mathbb{Y}_1 = \mathbb{Y}_2 = \mathbb{Y}_3 = \mathbb{R}_+$. Furthermore, the preference relation $\succcurlyeq_1$ over $\mathbb{X}_1$ is a lexicographic preference and $\succcurlyeq_2$ over $\mathbb{X}_2$ is the ordinary ordering $\geq$ on $\mathbb{R}_+$. As for $\succcurlyeq$ over $\mathbb{X}$, it is defined as $(x'_1, x'_2) \succcurlyeq (x''_1, x''_2)$ if and only if $y'_1 + y'_2 + y'_3 \geq y''_1 + y''_2 + y''_3$.*

In Example 1, $\{\mathbb{X}_1, \succcurlyeq_1\}$ can not be represented by any function, however, $\{\mathbb{X}, \succcurlyeq\}$ can be represented simply by $u(\cdot) = y_1 + y_2 + y_3$. Removing $\mathbb{X}_2$ will see the inconsistency of $\succcurlyeq$ and $\succcurlyeq_1$. In order to make a meaningful connection between $\{\mathbb{X}, \succcurlyeq\}$ and its components $\{\mathbb{X}_i, \succcurlyeq_i\}_{i=1}^n$, a basic condition is required.

Definition 2. *$\{\{\mathbb{X}_i, \succcurlyeq_i\}_{i \in N}, \{\mathbb{X}, \succcurlyeq\}\}$ is said to be consistent, equivalently, $\{\mathbb{X}, \succcurlyeq\}$ is consistent with $\{\mathbb{X}_i, \succcurlyeq_i\}_{i \in N}$, if for every $i \in N$ and every $x_{-i} \in \mathbb{X}_{-i}$,*

$$x'_i \succcurlyeq_i x''_i \iff (x'_i, x_{-i}) \succcurlyeq (x''_i, x_{-i}).$$

The consistency condition says that the preference relation remains the same over any specific attribute, whenever the other attributes are the same.

Since we aim to study the lexicographic preference on $\mathbb{X}$ and its utility function representation problem, let us define the lexicographic preference over $\{\mathbb{X}_i, \succcurlyeq_i\}_{i \in N}$.

Definition 3. *Given* $\{\mathbb{X}_i, \succcurlyeq_i\}_{i\in N}$, $\{\mathbb{X}, \succcurlyeq\}$, *where* $\mathbb{X} = \times_{i\in N}\mathbb{X}_i$, *is a lexicographic preference on* $\mathbb{X}$ *if for any* $x' \neq x'' \in \mathbb{X}$,

$$x' \succ x'' \Leftrightarrow x'_j \succ_j x''_j$$

here, $j = \min\{i : i \in N \text{ and } x'_i \nsim_i x''_i\}$. *Furthermore, if* $x'_i \sim_i x''_i, \forall i \in N$, *then* $x' \sim x''$.

As is easily seen, a lexicographic preference $\succcurlyeq$ on $\mathbb{X}$ satisfies the consistency property. Thus for $\{\mathbb{X}, \succcurlyeq\}$ to be representable by a utility function, it is necessary that the following condition holds. This condition is assumed throughout the rest of this paper.

Assumption 1. *For each* $i \in N$, $\{\mathbb{X}_i, \succcurlyeq_i\}$ *is utility function representable.*

3 The main results

For the convenience of the statement and the proof of our main results, we assume that all the involved preferences are antisymmetric. In other words, each element is equivalent solely to itself. Recall that $N = \{1, \dots, n\}$ for some $n \in \mathbb{N}$ or $N = \mathbb{N}$ (then we say $n = \infty$). Only the former case admits the last element which corresponds to the least important attribute. Now we present the main theorem of this paper as follows.

Main Theorem. *Given consistent* $\{\{\mathbb{X}_i, \succcurlyeq_i\}_{i\in N}, \{\mathbb{X}, \succcurlyeq\}\}$, *where* $\succcurlyeq$ *is the antisymmetric lexicographic preference on* $\mathbb{X}$, *then* $\{\mathbb{X}, \succcurlyeq\}$ *is utility function representable if and only if there is at most one attribute with uncountable levels and when such an attribute exists, it is the least important one.*

Remark 1. *The antisymmetric condition on* $\succcurlyeq$ *can be relaxed. In fact, let* $\sim$ *be the equivalent relation related to* $\succcurlyeq$, *i.e.,* $x \sim y$ *if and only if* x *and* y *are indifferent. Without the antisymmetric condition, we can consider the quotient space* $\tilde{\mathbb{X}} := \mathbb{X}/\sim$. *Then the preference on* $\tilde{\mathbb{X}}$ *induced by* $\succcurlyeq$ *(still denoted by* $\succcurlyeq$*) is antisymmetric. The representability of the original* $\succcurlyeq$ *on* $\mathbb{X}$ *is the same as the representability of* $\succcurlyeq$ *on* $\tilde{\mathbb{X}}$, *with each equivalent class sharing a common utility value. Note that without the antisymmetric condition, for each* i, $\mathbb{X}_i$ *can be uncountable as long as* $\mathbb{X}_i/\sim_i$ *is countable when* $i < n$.

As an example, let $\mathbb{X}_i = \mathbb{R}$ *for all* $i = 1, \dots, n$. *For* $i < n$, $x_i \succcurlyeq_i y_i$ *if and only if* $[x_i] \geq [y_i]$. *Here* $[a]$ *stands for the largest integer not exceeding* $a \in \mathbb{R}$. *For* $i = n$, *let* $x_n \succcurlyeq_n y_n$ *if and only if* $x_n \geq y_n$ *in* $\mathbb{R}$. *Note that for* $i < n$, $x_i \sim_i y_i$ *if and only if* $[x_i] = [y_i]$. *Thus* $\mathbb{X}_i/\sim_i$ *amounts to* $\mathbb{Z}$, *the set of all integers with the ordinary order. Now that the lexicographic* *preference (with respect to the ordinary orders) on* $(\times_{i=1}^{n-1}\mathbb{Z}) \times \mathbb{R}$ *is representable, so is the non-antisymmetric lexicographic preference on* $\times_{i=1}^{n}\mathbb{R}$ *(with respect to* $\succcurlyeq_i, i = 1, \ldots n$*).*

As a corollary to the main theorem, if we focus on the situation where $\mathbb{X}_i = \mathbb{Q}$ or $\mathbb{R}$, we have the following.

Corollary 1. *Given consistent* $\{\{\mathbb{X}_i, \succcurlyeq_i\}_{i\in N}, \{\mathbb{X}, \succcurlyeq\}\}$*, where* $\mathbb{X}_i$ *is either* $\mathbb{Q}$ *or* $\mathbb{R}$*, with the ordinary order in* $\mathbb{R}$*, and* $\succcurlyeq$ *is the lexicographic preference on* $\mathbb{X} = \times_{i\in N}\mathbb{X}_i$*, then* $\{\mathbb{X}, \succcurlyeq\}$ *is utility function representable if and only if* $\mathbb{X}_i = \mathbb{Q}$ *for all* $i < n$*.*

We remark that in the above corollary $n = \infty$ is allowed, which corresponds to $N = \mathbb{N}$.

The proof of the main theorem will immediately follow from the two theorems in section 5. For that purpose we introduce an auxiliary result on the class of subsets on which the preference can be represented by utility functions.

4 An auxiliary lemma on representable class

To facilitate the further analysis, we introduce a class $\mathscr{S}$ of subsets of $\mathbb{X}$ such that restricted to each set in the class the preference $\succcurlyeq$ can be represented by some utility function. Then if $\{\mathbb{X}, \succcurlyeq\}$ can be represented by some utility function, $\mathbb{X}$ belongs to the class. Thus whether $\{\mathbb{X}, \succcurlyeq\}$ can be represented by a utility function is reduced to check whether $\mathbb{X} \in \mathscr{S}$ or not. Such a class of sets is defined as the representable class of $\{\mathbb{X}, \succcurlyeq\}$. Before defining the representable class, we need to define the restricted preference as follows.

Definition 4. *For the preference relation* $\succcurlyeq$ *on* $\mathbb{X}$ *and a nonempty subset* $S \subset \mathbb{X}$*, the restricted preference* $\succcurlyeq_S$ *on* $S \subseteq \mathbb{X}$ *is defined as, for any* $x', x'' \in S$

$$x' \succcurlyeq_S x'' \Longleftrightarrow x' \succcurlyeq x''.$$

The restricted preference $\succcurlyeq_S$ preserves the preference order when $\succcurlyeq$ is restricted on S. So, if a function represents $\{\mathbb{X}, \succcurlyeq\}$, then the function restricted to S represents the corresponding $\{S, \succcurlyeq_S\}$.

Definition 5. *The representable class* $\mathscr{S}$ *of* $\{\mathbb{X}, \succcurlyeq\}$ *is the set of all* $S \subseteq \mathbb{X}$ *such that* $\{S, \succcurlyeq_S\}$ *can be represented by some utility function. That is,*

$$\mathscr{S} = \{S \subset \mathbb{X} : S \neq \emptyset, \text{ and } \exists u : S \to R, \text{ such that } u(\cdot) \text{ represents } \succcurlyeq_S \}.$$

Obviously the representable class $\mathscr{S}$ contains every singleton of $\mathbb{X}$. A natural question is whether it is closed under countable unions. We will see in Example 3 that this is not true in general. But under some extra condition the answer is yes. The following definition is related to the extra condition.

Definition 6. *For a subset S of $\mathbb{X}$ and the restricted preference $\succcurlyeq_S$ over S that is representable by a function $u : S \to \mathbb{R}$, we say $x \in S$ is pseudo-continuous with respect to u if*

$$\sup\{u(s) : s \in S, s \prec x\} = u(x), \ \ \inf\{u(s) : s \in S, s \succ x\} = u(x),$$

and either one of the following conditions holds:

(i) there exists some $x' \in \mathbb{X} \setminus S$ such that $x' \prec x$ and $\{s \in S : s \prec x\} \subset \{y \in \mathbb{X} : y \prec x'\}$;

(ii) there exists some $x' \in \mathbb{X} \setminus S$ such that $x' \succ x$ and $\{s \in S : s \succ x\} \subset \{y \in \mathbb{X} : y \succ x'\}$.

Here we notice that the concept of pseudo-continuity depends on both u and $\mathbb{X}$. Function u takes charge of the "continuous" part.[1] Replacing u with some other u' that also represents S may be not "continuous" at the point studied. $\mathbb{X}$ corresponds to the "pseudo" part in "pseudo-continuous", because it has "gap" around the studied point from the perspective of $\mathbb{X}$. We can see this more clearly in the following example.

Example 2. *Let $\mathbb{X} = \mathbb{R}$ with the ordinary order $\succcurlyeq$. Let $S_1 = (-\infty, \sqrt{2}) \cup \{2\} \cup (\sqrt{2}+2, \infty)$ and $S_2 = (-\infty, \sqrt{2}) \cup \{3\} \cup (\sqrt{2}+2, \infty)$. Define utility functions over them, respectively:*

$$u_1(x) = \begin{cases} x & \text{if } x < \sqrt{2}, \\ \sqrt{2} & \text{if } x = 2, \\ x - 2 & \text{if } x > \sqrt{2}+2; \end{cases} \qquad u_2(x) = \begin{cases} x & \text{if } x < \sqrt{2}, \\ \sqrt{2} & \text{if } x = 3, \\ x - 2 & \text{if } x > \sqrt{2}+2. \end{cases}$$

Obviously u_i represents the restricted preference $\succcurlyeq_{S_i}$ for both i in $\{1, 2\}$. It is easy to see that $x_1 = 2$ and $x_2 = 3$ are pseudo-continuous points for S_1 and S_2, with respect to u_1 and u_2, respectively.

Now we are ready to state the following lemma.

Lemma 1. *Let $\mathscr{S}$ be the representable class of $\{\mathbb{X}, \succcurlyeq\}$, and for each $k \in \mathbb{N}$, $S_k \in \mathscr{S}$ and $\{S_k, \succcurlyeq_{S_k}\}$ can be representable by $u_k : S_k \to \mathbb{R}$ with at most countably many pseudo-continuous points. Then $\cup_{k\in\mathbb{N}} S_k \in \mathscr{S}$.*

[1]If $\mathbb{X}$ is endowed with the order topology induced by $\succcurlyeq$, S with the induced topology, and $\mathbb{R}$ with the usual Eaclidean topology, then $u : S \to \mathbb{R}$ is indeed continuous at any pseudo-continuous point of S. Recall that the order topology has a subbase consisting of the sets of the form $\{y \in \mathbb{X} : y \succ x\}$ or $\{y \in \mathbb{X} : y \prec x\}$ for some $x \in \mathbb{X}$.

Proof. For each $k \in \mathbb{N}$, $S_k \in \mathscr{S}$ implies that there exists a function $u_k : S_k \to \mathbb{R}$ that represents $\succcurlyeq$ restricted to S_k. Let $Q_k \subset S_k$ be the countable set (maybe empty) of pseudo-continuous points of the representing function u_k on S_k.

Let $M_{1,k} = \{z \in u_k(S_k) : \exists a_0 > 0, \forall a \in (0, a_0), z - a \notin u_k(S_k)\}$, the set of all "locally minimal" values of u_k. And similarly, $M_{2,k} = \{z \in u_k(S_k) : \exists a_0 > 0, \forall a \in (0, a_0), z + a \notin u_k(S_k)\}$, the set of all "locally maximal" values of u_k. Both $M_{1,k}$ and $M_{2,k}$ are countable. Let D_k be an arbitrary countable and dense subset of $u_k(S_k)$, and $J_k = D_k \cup M_{1,k} \cup M_{2,k}$. J_k is also a countable set. Let $C_k = u_k^{-1}(J) \subseteq S_k$ be the preimage of J_k under u_k. C_k also contains all "locally maximal" and "locally minimal" (with respect to $\succcurlyeq_{S_k}$) points in S_k. Take $C = \cup_{k=1}^{\infty}(C_k \cup Q_k)$. It is a countable set since each Q_k is countable by assumption. So we can choose a positive valued real function $g : C \to (0, \infty)$ with the property that $\sum_{x \in C} g(x) < 1$. Define $u : \cup_{k=1}^{\infty} S_k \to (0, 1)$ by

$$u(x) = \sum_{c \in C, c \preccurlyeq x} g(c).$$

Now if $x \succcurlyeq y$ in $\cup_{k=1}^{\infty} S_k$, then $\{c \in C, c \preccurlyeq x\} \supseteq \{c \in C, c \preccurlyeq y\}$. Consequently, $u(x) = \sum_{c \in C, c \preccurlyeq x} g(c) \geq \sum_{c \in C, c \preccurlyeq y} g(c) = u(y)$.

On the other hand, if $x \prec y$ in $\cup_{k=1}^{\infty} S_k$, we need show $u(x) < u(y)$. Without loss of generality assume $x \in S_j$ and $y \in S_k$. From the definition of function u it suffices to show that there exists some $c \in C$ satisfying $x \prec c \preccurlyeq y$. This is trivially true if y is "locally maximal" or "locally minimal", for we can take $c = y$.

If y is neither "locally maximal" nor "locally minimal" in S_k, there exists some sequence $\{y_{k,\ell}, \ell = 1, 2, \dots\} \subset S_k$ such that $\{u_k(y_{k,\ell})\}_{\ell=1}^{\infty}$ is a strictly increasing sequence converging to $u_k(y)$ as ℓ tends to infinity. If $x \prec \hat{y}$, for some $\hat{y} \in S_k$ with $\hat{y} \prec y$, by the construction of C_k, we can find some $z \in C_k$ and $\ell \in \mathbb{N}$ such that $\hat{y} \preccurlyeq z \prec y_{k,\ell} \prec y$. In such a case we can take $c = z$. If no such $\hat{y}$ exists, then y is pseudo-continuous with respect to u_k. We can take $c = y$.

Thus u represents $\succcurlyeq_{\cup_{k=1}^{\infty} S_k}$. In other words, $\cup_{k=1}^{\infty} S_k \in \mathscr{S}$. □

Lemma 1 tells us that when the utility function representability of the restricted preferences is closed under countable unions. However, such a property can not extend to uncountable unions. Otherwise, every rational preference would be represented by some utility function, because each singleton can be represented by some utility function.

The condition of at most countably many pseudo-continuous points is crucial for Lemma 1. Without this condition the conclusion of Lemma 1 may be no longer true, as is shown by the following example. Let $\mathbb{R}$ and $\mathbb{Q}$ be endowed with the ordinary order $\geq$.

Example 3. *The lexicographic preference on $\mathbb{R} \times \mathbb{Q}$ is not representable by any utility function. However, $\mathbb{R} \times \mathbb{Q} = \cup_{q \in \mathbb{Q}}(\mathbb{R} \times \{q\})$ is a countable union of representable subsets, with a common representing function $u(r, q) = r$ on those subsets. The problem lies in that each of these subsets, $\mathbb{R} \times \{q\}$, admits a continuum of pseudo-continuous points. In fact, every point is pseudo-continuous.*

Note that $\mathscr{S}$ is partially ordered with inclusion relation $\subseteq$. Furthermore, $S \in \mathscr{S}$ is a maximal element if there is no other $S' \in \mathscr{S}$ such that $S \subseteq S'$.

Theorem 1. *Let $\mathscr{S}$ be the representable class of $\{\mathbb{X}, \succcurlyeq\}$, then $\{\mathbb{X}, \succcurlyeq\}$ can be represented by a utility function if and only if $\mathscr{S}$ has a maximal element.*

Proof. "Only if" part is obvious. As for the "if" part, suppose not. Then there exists some maximal element M of $\mathscr{S}$ that is different from $\mathbb{X}$. First of all, because M is an element of $\mathscr{S}$, there exists $u : M \to \mathbb{R}$ that represents preference $\succcurlyeq$ restricted to M. Since a strictly monotone transform preserves the preference, we can assume without loss of generality that u is valued in $(0, 1)$. Next choose an arbitrary element $\bar{x} \in \mathbb{X} \setminus M$. Define $L_M(\bar{x}) = \{x : x \in M, x \prec \bar{x}\}$, $U_M(\bar{x}) = \{x : x \in M, x \succcurlyeq \bar{x}\}$, and $a = \frac{1}{2}\left(\sup\{u(x) : x \in L_M(\bar{x})\} + \inf\{u(x) : x \in U_M(\bar{x})\} + 1\right)$. Here by convention $\sup \emptyset = 0$ and $\inf \emptyset = 1$. Now we define a function $\bar{u}$ on $M \cup \{\bar{x}\}$ by

$$\bar{u}(x) = \begin{cases} u(x) & \text{if } x \in L_M(\bar{x}) \\ a & \text{if } x = \bar{x} \\ u(x) + 1 & \text{if } x \in U_M(\bar{x}). \end{cases}$$

It is straightforward to show that $\bar{u}$ represents $\succcurlyeq$ on $M \cup \{\bar{x}\}$, which contradicts M being a maximal element of $\mathscr{S}$. Therefore $\mathbb{X} \in \mathscr{S}$, which concludes the theorem. □

We remark that the results in Lemma 1 and Theorem 1 in fact hold for general rational preferences on a general space $\mathbb{X}$.

5 Proof of the main results

The main theorem will immediately follow from Theorems 2 and 3 below.

Firstly, let us reflect on the situation where each attribute has at most countable levels. As is well known, if a set is at most countable, there exists a one-to-one function from it to integers. And, after the transformation, each point has a distance of at least 1 from the other points. So, for each point, it has enough space to include more points, even an interval. However, such transformations do not preserve the ordering, which is required by the utility function representation. If the attributes are at most countable, does there exist an order-preserving function which allows adding more points with each point? The answer is affirmative as is shown by the following proposition.

Proposition 1. *For any $\{\mathbb{X}, \succcurlyeq\}$, suppose $\mathbb{X}$ is at most countable. There exists a order-preserving function $u : \mathbb{X} \to [0, 1]$, such that $\forall x \in \mathbb{X}$, $d(u(x), u(\mathbb{X} \setminus \{x\})) := \inf\{|u(x) - u(y)| : y \in \mathbb{X} \setminus \{x\}\} > 0$.*

Proof. Because $\mathbb{X}$ is at most countable, we can list all elements in $\mathbb{X}$ as $\{x_1, \cdots, x_k, \cdots\}$ and find a sequence of positive numbers $\mu_k > 0, k = 1, 2, \cdots$, with $\sum_{k=1}^{\infty} \mu_k < 1$. Now we define $u(\cdot)$ by

$$u(x_k) = \sum_{x_{k'} \prec x_k} \mu_{k'} + \frac{1}{2}\mu_k,\ x_k \in \mathbb{X}.$$

The function $u(\cdot)$ defined above preserves the order on $\mathbb{X}$ and its range is a subset of $(0, 1)$. Furthermore, the distance from μ_k to $\{u(x) : x \in \mathbb{X} \setminus \{x_k\}\}$ is at least $\frac{\mu_k}{2} > 0$, which concludes the proposition. □

As is shown by the proposition above, if $\mathbb{X}_1$ is countable, we have an order-preserving function from it to $[0, 1]$ and each point has space to embrace an interval with it. To make the thing clearer, we consider a simple situation where $n = 2$ and $\mathbb{X}_1 = \mathbb{Q}, \mathbb{X}_2 = \mathbb{R}$ in the following example.

Example 4. *Take $n = 2$. Consider $\mathbb{X}_1 = \mathbb{Q}$ and $\mathbb{X}_2 = \mathbb{R}$, with the ordinary order for real numbers. The lexicographic preference defined on $\mathbb{X}_1 \times \mathbb{X}_2$ is representable by a utility function.*

Proof. Analogously as in the proof of Proposition 1, we define $u_1(\cdot)$ as

$$u_1(x_k) = \sum_{x_{k'} \prec x_k} \mu_{k'} + \frac{1}{2}\mu_k, \forall x_k \in \mathbb{X}_1$$

Now, for each x_k, there is a "gap" of size μ_k from other point. These gaps allow us to insert some compressed utility representation of $\{(x_k, x) : x \in \mathbb{X}_2\}$ for each k. That is, for $\mathbb{X}_2$, we define function $u_2 = \frac{e^x}{1+e^x} - \frac{1}{2}$ on it. Then we can define function

$$u(x', x'') = u_1(x') + \mu_{\iota(x')} u_2(x'')$$

Here, $\iota : \mathbb{X}_1 \to \mathbb{N}$ is an arbitrary bijection that assigns each $x' \in \mathbb{X}_1$ its location in the list $\{x_1, \cdots, x_k, \cdots\}$. It is easy to show that such a function can represent the lexicographic preference defined with $n = 2, \mathbb{X}_1 = \mathbb{Q}, \mathbb{X}_2 = \mathbb{R}$. □

We can extend the Example 4 to more general cases. However, Proposition 1 does not hold when there are uncountable levels in $\mathbb{X}$, which means that for every order-preserving function from $\mathbb{X}$ to $\mathbb{R}$, there always exists at least one element in $\mathbb{X}$ which has no "gap", so there is no space to insert an interval the same as Example 2. We describe this in the following proposition.

Proposition 2. *For any $\{\mathbb{X}, \succcurlyeq\}$, suppose $\mathbb{X}$ is uncountable. For any order-preserving function $u : \mathbb{X} \to [0, 1]$, there exists at least one point $x \in \mathbb{X}$ such that $d(u(x), u(\mathbb{X} \setminus \{x\})) = 0$.*

Proof: The proof follows from the proof of Theorem 2. □

By Proposition 2, if for some attribute, it has uncountable levels, there is no "gap" to include an interval around it. So, if this is the unique attribute with uncountable levels and it is listed in the last location, then it is not a problem. However, when it is not listed in the last location, the problem arises. We summarize this in the following theorem.

Theorem 2. *For consistent $\{\{\mathbb{X}_i, \succcurlyeq_i\}_{i\in N}, \{\mathbb{X}, \succcurlyeq\}\}$, suppose $N = \{1, \cdots, n\}$ is finite. Then $\{\mathbb{X}, \succcurlyeq\}$ is representable by a utility function if and only if all but the last attributes admit countable levels.*

Proof. "If" part is easy to prove by Lemma 1. Let u_n be a utility function representing $\succcurlyeq_n$ on $\mathbb{X}_n$. Given $x_{-n} \in \mathbb{X}_{-n}$, consider the set $S_{-n}(x_{-n}) = \{(x_{-n}, x_n) : x_n \in \mathbb{X}_n\}$, by the property of consistency, the reduced preference on $S_{-n}(x_{-n})$ can be represented by function $\hat{u}_n : S_{-n}(x_{-n}) \to \mathbb{R}$ defined by $\hat{u}_n(x_{-n}, x_n) = u_n(x_n)$. So $S_{-n}(x_{-n}) \in \mathscr{S}$. Furthermore, $S_{-n}(x_{-n})$ contains no pseudo-continuous point with respect to $\hat{u}_n$. Also, $\mathbb{X}_{-n}$ is at most countable by assumption. Thus, by Lemma 1, $\cup_{x_{-n}\in\mathbb{X}_{-n}} S_{-n}(x_{-n}) = \mathbb{X} \in \mathscr{S}$. This concludes that $\{\mathbb{X}, \succcurlyeq\}$ is representable by some utility function.

"Only if" part: Since strictly increasing transforms preserve preferences, without loss of generality, we assume that for each i, $u_i : \mathbb{X}_i \to (0, 1)$ represents $\succcurlyeq_i$ on $\mathbb{X}_i$. If for some $i < n$ $\mathbb{X}_i$ is uncountable and there exists some function $\psi : \mathbb{X} \to (0, 1)$ representing $\succcurlyeq$. Let $\succcurlyeq_{i,i+1}$ be the reduced lexicographic preference on $\mathbb{X}_i \times \mathbb{X}_{i+1}$. Then by the property of consistency $\{\mathbb{X}_i \times \mathbb{X}_{i+1}, \succcurlyeq_{i,i+1}\}$ can be represented by function $\psi(\cdot, z) : \mathbb{X}_i \times \mathbb{X}_{i+1} \to (0, 1)$ for any fixed $z \in \mathbb{X}_{N\setminus\{i,i+1\}}$.

Take any two elements x'_{i+1} and x''_{i+1} from $\mathbb{X}_{i+1}$ with $x'_{i+1} \prec_{i+1} x''_{i+1}$. Let $T = u_i(\mathbb{X}_i)$. Consider the following two functions related to x'_{i+1} and x''_{i+1} respectively,

$$h' : T \to \mathbb{R}, \quad h'(t) = \psi(u_i^{-1}(t), x'_{i+1}, z); \qquad h'' : T \to \mathbb{R}, \quad h''(t) = \psi(u_i^{-1}(t), x''_{i+1}, z).$$

Obviously both h' and h'' are strictly increasing. Let D'_1 and D''_1 be subsets of T that consist of all discontinuous points of h' and h'' respectively (under ordinary topology reduced from the Euclidean topology on $\mathbb{R}$). Let $D_2 = \{t \in T : \exists a_0 > 0, \forall a \in (0, a_0), t + a \notin T\}$ be the set of all locally maximal points of T. Then $D'_1 \cup D''_1 \cup D_2$ is a countable subset of T. Take an arbitrary $t^* \in T \setminus (D'_1 \cup D''_1 \cup D_2)$. Then both h' and h'' are continuous at t^* and there exists a sequence $t_k \in T$ that is strictly decreasing and converges to t^* as $k \to \infty$. Let $x^* = u_i^{-1}(t^*)$ and $x_k = u_i^{-1}(t_k)$ for each k. Since ψ represents the lexicographic preference, we have

$$\psi(x_k, x'_{i+1}, z) < \psi(x_k, x''_{i+1}, z) < \psi(x_{k+1}, x'_{i+1}, z).$$

That is

$$h'(t_k) < h''(t_k) < h'(t_{k+1}).$$

Let k tend to infinity, we get $h'(t^*) = h''(t^*)$. In other words, $\psi(x^*, x'_{i+1}, z) = \psi(x^*, x''_{i+1}, z)$.

On the other hand, ψ representing the lexicographic preference implies $\psi(x^*, x'_{i+1}, z) < \psi(x^*, x''_{i+1}, z)$. The contradiction means that if there exists an attribute with uncountable levels, it is only possible to occur at the last (i.e., the nth) attribute. □

When there are infinite attributes, i.e., $N = \mathbb{N}$, we have the similar theorem with Theorem 3, whose proof is similar to that of Lemma 1.

Theorem 3. *Assume $\{\mathbb{X}, \succcurlyeq\}$ is consistent with $\{\mathbb{X}_i, \succcurlyeq_i\}_{i\in\mathbb{N}}$. Then $\{\mathbb{X}, \succcurlyeq\}$ is representable by a utility function if and only if $\mathbb{X}_i$ is at most countable for every $i \in \mathbb{N}$.*

Proof. The same proof for the necessary part of Theorem 2 works to imply that if $\mathbb{X}_i$ is uncountable for some i, $\mathbb{X}$ can not be represented by any utility function.

On the other hand, if $\mathbb{X}_i$ is countable for all $i \in \mathbb{N}$, we can construct a utility function to represent the lexicographic preference on $\mathbb{X} = \times_{i\in\mathbb{N}}\mathbb{X}_i$.

Fix some $\theta \notin \cup_{i\in\mathbb{N}}\mathbb{X}_i$. For each i, define $\bar{\mathbb{X}}_i = \mathbb{X}_i \cup \{\theta\}$. We extend $\succcurlyeq_i$ to $\bar{\mathbb{X}}_i$ by letting $\theta \prec_i x$ for every $x \in \mathbb{X}_i$. In this manner we also extend the lexicographic preference $\succcurlyeq$ on $\mathbb{X}$ to $\times_{i\in\mathbb{N}}\bar{\mathbb{X}}_i$.

Denote

$$S = \{(x_1, x_2, \dots) \in \times_{i\in\mathbb{N}}\bar{\mathbb{X}}_i : \exists k \text{ such that } x_i \neq \theta \text{ for all } i < k \text{ and } x_i = \theta \text{ for all } i \geq k\}$$

be the set of sequences ending with theta's. It is easy to see that S is a countable set. Therefore we can assign positive values to the points in S so that the total sum is finite. In other words, we have a function $v : S \to (0, \infty)$ satisfying $\sum_{s\in S} v(s) < \infty$.

Now define $u : \mathbb{X} \to \mathbb{R}$ by

$$u(x) = \sum_{s\in S, s\prec x} v(s).$$

Now if $x \succcurlyeq y$, then $\{s \in S, s \prec x\} \supseteq \{s \in S, s \prec y\}$, hence $u(x) \geq u(y)$ since v take strictly positive values.

To complete proving that u represents $\succcurlyeq$ on $\mathbb{X}$ we need only show for any $x \prec y$ in $\times_{i\in\mathbb{N}}\mathbb{X}_i$, there exists some $s \in S$ such that $x \prec s \prec y$.

Write $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$. Note that $x \prec y$ means that $x \neq y$. So there is some $k \in \mathbb{N}$ such that $x_k \neq y_k$ and $x_i = y_i$ for all $i < k$. Also $x_k \prec_k y_k$ in $\mathbb{X}_k$.

Take $s = (s_1, s_2, \dots) \in S$ as follows. For $i \leq k$, let $s_i = y_i$; for $i > k$ let $s_i = \theta$. Then $x \prec s \prec y$ in $\bar{\mathbb{X}}$.

Thus we know from the definition of u that $u(x) < u(y)$ if $x \prec y$ in $\mathbb{X}$. □

Remark 2. *The proof of Theorem 3 suggests that the results of Theorem 2 can be proved intuitively as follows. To each $\mathbb{X}_i$ we add an outside element θ and let it be ranked strictly lower to every element in $\mathbb{X}_i$. Embed partial product spaces $\times_{i=1}^{k}\mathbb{X}_i$, $k = 1, 2, \dots, n-1$, to $\bar{\mathbb{X}}$ by supplementing θ to the tails. Now analyze by induction on k. Since each ($\mathbb{X}_i, \succcurlyeq_i$) is representable (by u_i),the case for $k = 1$ is trivial. Suppose for general $k < n$, the lexicographic preference on $\times_{i=1}^{k}\mathbb{X}_i$ is representable, say by a function $u_{1,\dots,k}$. To accommodate other attributes with at least two different elements in an extended utility representation, it is necessary that every point in $u_{1,\dots,k}(\times_{i=1}^{k}\mathbb{X}_i)$ is "locally maximal", in the sense that there exists an interval to its right that includes no points of the set. This is also sufficient if we need accommodate only one further attribute (with at least two different levels), for the additional attribute can be distinguished by the interval to the right of each $u_{1,\dots,k}$-value point. As long as $\mathbb{X}_{k+1}$ is countable, we can continue this way until $k + 1 = n$, where $\mathbb{X}_n$ can be uncountable.*

On the other hand, if $\mathbb{X}_k$ is uncountable for some $k < n$, it is impossible for every point in $u_{1,\dots,k}(\times_{i=1}^{k}\mathbb{X}_i)$ to be "locally maximal" in $u_{1,\dots,k}(\times_{i=1}^{k}\mathbb{X}_i)$. The "locally maximal" points *are at most countably many, because each such point is "one arm's away" from other points on its right.*

6 Conclusion

In this paper, we mainly focus on the study of the utility function representation of lexicographic preferences. We find that the cardinality of different attributes will determine whether the lexicographic preference can be represented by a utility function. It is shown that lexicographic preference is representable by a utility function if and only if there is at most one attribute with uncountable levels and when such an attribute exists, it must be the least important one. An auxiliary result that is of independent interest is also established on how the union of countable "representable" subsets with respect to a preference is also "representable". The main results are expressed for antisymmetric lexicographic preferences. We also note that non-antisymmetric cases can be reduced to antisymmetric cases by considering the induced quotient spaces. Some examples are provided to help understand the related concepts and results.

References

[1] Bridges, D. S. 1983. A Numerical Representation of Intransitive Preferences on A Countable Set. *Journal of Economic Theory* 30, 213-217.

[2] Chateauneuf, A. 1987. Continuous Representation of A Preference Relation on A Connected Topological Space. *Journal of Mathematical Economics* 16, 139-146.

[3] Debreu, G. 1954. Representation of A Preference Ordering by A Numerical Function. R. M. Thrall, C. H. Coombs, R. L. Davis, eds. *Decision Processes.* John Wiley, New York, 159-165.

[4] Fishburn, P. C. 1974. Lexicographic Orders, Utilities and Decision Rules: A Survey. *Management Science* 20 (11) 1442-1471.

[5] Fishburn, P. C. 1975. Axioms for Lexicographic Preferences. *Review of Economic Studies.* 42(3) 415-419.

[6] Knoblauch, V. 2000. Lexicographic Orders and Preference Representation. *Journal of Mathematical Economics.* 34(2) 255-267.

[7] Mas-Colell, Andreu, Michael Dennis Whinston, and Jerry R. Green, 1995. Microeconomic Theory. New York: *Oxford University Press*.

[8] Royden, H.L. and Fitzpatrick, P., 2010. Real Analysis. *Prentice Hall*.

[9] Tversky, A. 1972. Elimination by Aspects: A Theory of Choice. *Psychological Review*. 79, 281-299.

[10] Tversky, A., S. Sattah, P. Slovic., 1988. Contingent Weighting in Judgment and Choice. *Psychological Review*. 95, 371-384.

[11] Yee, M., Dahan, E., Hauser, J. R., & Orlin, J., 2007. Greedoid-based Noncompensatory Inference. *Marketing Science*, 26(4), 532-549.