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Abstract

Based on results discussed by Meinhardt (2013), which presents a dual characterization of the pre-kernel by a finite union of solution sets of a family of quadratic and convex objective functions, we could derive some results related to the single-valuedness of the pre-kernel. Rather than extending the knowledge of game classes for which the pre-kernel consists of a single point, we apply a different approach. We select a game from an arbitrary game class with a single pre-kernel element satisfying the non-empty interior condition of a payoff equivalence class, and then establish that the set of related and linear independent games which are derived from this pre-kernel point of the default game replicates this point also as its sole pre-kernel element. Hence, a bargaining outcome related to this pre-kernel element is stable. Furthermore, we establish that on the restricted subset on the game space that is constituted by the convex hull of the default and the set of related games, the pre-kernel correspondence is single-valued, and therefore continuous. In addition, we provide sufficient conditions that preserve the pre-nucleolus property for related games even when the default game has not a single pre-kernel point. Finally, we apply the same techniques to related solutions of the pre-kernel, namely the modiclus and anti-pre-kernel, to work out replication results for them.

Keywords: Transferable Utility Game, Pre-Kernel, Pre-Nucleolus, Anti-Pre-Nucleolus, Modiclus, Uniqueness of the Pre-Kernel, Convex Analysis, Fenchel-Moreau Conjugation, Indirect Function, Stability Analysis.

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1 INTRODUCTION

In bargaining situation people often have strong concern on the execution and stability of an agreed-upon contract. In particular, where multilateral agreements are non-binding, compliance becomes a crucial issue in order to avoid its obstruction. People may reduce fulfillment if they feel that they have been treated unfair. This may happen when agents are not acting self-constraint and are not refraining from using their powers to exploit one another. However, an agreement can be achieved when agents are following some fairness standards, then compliance is reality and obstruction is held to account. Apart from the Shapley value, a standard of fairness in cooperative game theory is the (pre-)kernel in accordance with its rich axiomatic foundation, i.e., principles of distributive arbitration. Moreover, this solution concept is also interesting from a non-cooperative point of view, since it results as a solution of a **Nash program**, which allows a reinterpretation of the kernel based on a non-cooperative bargaining game without making any use of interpersonal utility comparisons. Hence, a non-cooperative foundation of the kernel solution is established while formulating a bargaining process that will lead the players to the proposed solution whenever they follow the described rules (cf. Serrano (1997); Chang and Hu (2016)).

The kernel as well as the pre-kernel are in general set-valued solution concepts of cooperative game theory, which are only under very specific conditions single-valued. In this context, the coincidence of the kernel with the nucleolus – that is, the kernel consists of a single point – is only known for some classes of transferable utility games. In particular, it was established by Maschler et al. (1972) that for the class of convex games – introduced by Shapley (1971) – the kernel and the nucleolus coincide. Moreover, Arin and Feltkamp (1997) have established that for the class of veto-rich transferable utility games both solution concepts coalesce. Similar, Getán et al. (2012) were able to extend this result to the class of zero-monotonic almost-convex games. However, for the class of average-convex games, there is only some evidence that both solution concepts coalesce. For getting an overview of the recent developments in this field, we refer the inclined reader to Iñarra et al. (2020).

In order to advance our understanding about the stability of a bargaining outcome based on the principles of distributive justice related to the pre-kernel, we shall focus on its single-valuedness to abstract from the selection issue while identifying the conditions under those a variation within the game parameter space does not affect this bargaining agreement, and the fulfillment of the contract can be assured even under the new parameter setting (cf. Theorem. 4.4). However, such an analysis of stability requests a different approach as focusing, for instance, on the convexity property of a game. Rather than to investigate the game classes which possess a single pre-kernel element, we propose an alternative approach to investigate this issue while applying results and techniques provided in the book by Meinhardt (2013). There, it was shown that the pre-kernel of the grand coalition can be characterized by a finite union of solution sets of a family of quadratic and convex functions (Theorem 7.3.1). This dual representation of the pre-kernel is based on a Fenchel-Moreau generalized conjugation of the characteristic function. This generalized conjugation was introduced by Martinez-Legaz (1996), which he called the indirect function. Immediately thereafter, it was Meseguer-Artola (1997) who proved that the pre-kernel can be derived from an over-determined system of non-linear equations. This over-determined system of non-linear equations is equivalent to a minimization problem, whose set of global minima is equal to the pre-kernel set. Though an explicit structural form of the objective function that would allow a better and more comprehensive understanding of the pre-kernel set could not be performed.

The characterization of the pre-kernel set by a finite union of solution sets was possible due to a partition of the domain of the objective function into a finite number of payoff sets. From each payoff vector contained into a particular payoff set the same quadratic and convex function is induced. The collection of all these functions on the domain composes the objective function from which a pre-kernel element can be singled out. Moreover, each payoff set creates a linear mapping that maps payoff vectors into a vector subspace of balanced excesses. Equivalent payoff sets which reflects the same underlying bargaining situation produce the same vector subspace. The vector of balanced excesses generated by a pre-kernel point is contained into the vector subspace spanned by the basis vectors derived from the payoff set that contains this pre-kernel element. In contrast, the vectors of unbalanced excesses induced from the minima of a quadratic function do not belong to their proper vector subspace. An orthogonal projection maps these vectors on this vector subspace of the space of balanced excesses (cf. Meinhardt (2013, Chap. 5-7)).

From this structure a replication result of a pre-kernel point can be obtained. This is due that from the payoff set that contains the selected pre-kernel element, and which satisfies in addition the non-empty interior condition, a null space in the game space can be identified that allows a variation within the game parameter without affecting the pre-kernel properties of this payoff vector. Thus, a pre-kernel element of a TU game is replicable as a pre-kernel solution of a related game, whenever the pre-kernel element of the default game belongs to a payoff equivalence class, which satisfies the non-empty interior property. Then a full dimensional ellipsoid can be inscribed from which some parameter bounds can be specified within coalitional values can be varied without destroying the pre-kernel properties of the solution from the default game. These bounds specify a redistribution of the bargaining power among coalitions while supporting the selected pre-imputation still as a pre-kernel point. Even though the values of the maximum surpluses have been varied, the set of most effective coalitions remains unaltered by the parameter change. This indicates that a bargaining outcome related to this specific pre-kernel point remains stable against a variation in the game parameter space, and obstruction is held account. Hence, a set of related games can be determined, which are linear independent, and possess the selected pre-kernel element of the default game as well as a pre-kernel point (cf. Meinhardt (2013, Sect. 7.6)).

Applying this approach to the stability analysis of cartel agreements goes beyond the usual convexity investigation that is normally conducted in the literature, for instance, see Zhao (1999); Norde et al. (2002); Driessen and Meinhardt (2005, 2010). For an application of this approach while studying the stability of cartel agreements related to the pre-kernel, we refer the reader to Meinhardt (2018a).

To the best of our knowledge such kind of stability analysis of a bargaining outcome was up to now only conducted by scholars for linear solution concepts like the Shapley value[1], but not for non-linear solutions like the pre-kernel and pre-nucleolus. However, through the non-linearity of these solution concepts, a stability analysis necessitates a much broader and more sophisticated machinery of mathematical tools than under the Shapley value, for instance. For that reason, we shall repeat as well as generalize the requested preliminaries in the course of the analysis, which have been introduced in full scale by Meinhardt (2013). Otherwise, one will find our analysis as not comprehensible and as too complex due to the missing context. This is the price every open-minded game theorist has to pay to substantially advance the knowledge frontier while applying mathematical techniques to produce new answers to not well understood problems rather than looking on a deformation of a well known game theoretical problem with inadequate methods.

In the sequel of this paper, we will establish that the set of related games, which are derived from a default game exhibiting a singleton pre-kernel, must also possess this point as its sole pre-kernel element,

[1]For an overview of the most recent developments in this highly dynamic research field we refer the reader to Algaba et al. (2020, Chap. 6. & 7.). Even there, the application of the theory of linear algebraic groups reveals to us that the Borel-groups (minimal parabolic groups) are acting on the bases of TU games, and the Shapley value remains stable whenever the change of basis is located in the same orbit. In the same vein Hernández-Lamoneda et al. (2007) were able to compute a decomposition for the space of cooperative games under the action of the symmetric group $\mathcal{S}_n$ to identify all irreducible subspaces that are relevant to study symmetric linear solutions, this result was extended by Hernández-Lamoneda et al. (2009) for games in partition function form.

and therefore coincides with the pre-nucleolus. Notice that these games need not necessarily be convex, average-convex, totally balanced, or zero-monotonic. They could belong to different subclasses of games, however, they must satisfy the non-empty interior condition. Moreover, we show that the pre-kernel correspondence in the game space restricted to the convex hull that is constituted by the extreme points, which are specified by the default and related games, is single-valued, and therefore continuous.

In addition, we establish that this approach is also applicable to related solutions of the pre-kernel, namely the modiclus and anti-pre-kernel. The former solution concept was invented by Sudhölter (1993), whereas the latter was discussed by Funaki and Meinhardt (2006). The modiclus, even known under the name modified nucleolus, takes besides the primal power also the dual power of coalitions into account. In contrast, the anti-pre-kernel is pointing to the dual power of coalitions without explicitly introducing the dual game, that is, without changing the game context (cf. Meinhardt (2018b)). In particular, we show that the modiclus is identical to the pre-nucleolus or even to the anti-pre-nucleolus under regular conditions for a class of shifted games. To turn then in the next step – based on these results – to the aforementioned replication results w.r.t. the modiclus. Finally, we exhibit a replication result for an exposed element of the anti-pre-kernel for convex games. That is to say, we prove that the anti-pre-nucleolus for the class of convex games, which is also an element of the the anti-pre-kernel, remains at least an element of those under a very specific change in the game parameter, even though when the initial properties cannot be preserved. Hence, the object under consideration is in general not anymore the anti-pre-nucleolus of the induced game. Notice in this context that the anti-pre-kernel for convex games needs not to be a singleton – in contrast to the pre-kernel.

The structure of the paper is organized as follows: In the Section 2 we introduce some basic notations and definitions to investigate the coincidence of the pre-kernel with the pre-nucleolus. Section 3 provides the concept of the indirect function and gives a dual pre-kernel representation in terms of a solution set. In the next step, the notion of lexicographically smallest most effective coalitions is introduced in order to identify payoff equivalence classes on the domain of the objective function from which a pre-kernel element can be determined. Moreover, relevant concepts from Meinhardt (2013) are reconsidered. Section 4 studies the single-valuedness of the pre-kernel for related games. However, Section 5 investigates the continuity of the pre-kernel correspondence. In Section 6 some sufficient conditions are worked out under which the pre-nucleolus of a default game can preserve the pre-nucleolus property for related games. Whereas in Section 7 we introduce the notions of the anti-pre-nucleolus and anti-pre-kernel to finally extend certain attributes from the dual characterization of the pre-kernel to the anti-pre-kernel. In Section 8 we are going to discuss the definition of the modiclus, and some game properties which are useful in connection with the foregoing considerations to derive replication results within Section 9. There, we provide new replication results related to the modiclus and anti-pre-kernel. A few final remarks close the paper by Section 10.

2 Some Preliminaries

A cooperative game with transferable utility is a pair $\langle N, v\rangle$, where N is the non-empty finite player set $N := \{1, 2, \ldots, n\}$, and v is the characteristic function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) := 0$. A player i is an element of N, and a coalition S is an element of the power set of 2^N. The real number $v(S) \in \mathbb{R}$ is called the value or worth of a coalition $S \in 2^N$. Let S be a coalition, the number of members in S will be denoted by $s := |S|$. We assume throughout that $v(N) > 0$ and $n \geq 2$ is valid. In addition, we identify a cooperative game by the vector $v := (v(S))_{S\subseteq N} \in \mathcal{G}^n = \mathbb{R}^{2^n}$, if no confusion can arise. Finally, the relevant game space for our investigation is defined by $\mathcal{G}(N) := \{v \in \mathcal{G}^n \mid v(\emptyset) = 0 \wedge v(N) > 0\}$.

If $\mathbf{x} \in \mathbb{R}^n$, we apply $x(S) := \sum_{k \in S} x_k$ for every $S \in 2^N$ with $x(\emptyset) := 0$. The set of vectors $\mathbf{x} \in \mathbb{R}^n$ which satisfies the efficiency principle $v(N) = x(N)$ is called the **pre-imputation set** and it is defined by

$$\mathcal{I}^*(N, v) := \{\mathbf{x} \in \mathbb{R}^n \mid x(N) = v(N)\}, \tag{2.1}$$

or more concisely as $\mathcal{I}^*(v)$, where an element $\mathbf{x} \in \mathcal{I}^*(v)$ is called a pre-imputation. The set of pre-imputations which satisfies in addition the **individual rationality property** $x_k \geq v(\{k\})$ for all $k \in N$ is called the **imputation set** $\mathcal{I}(N, v)$.

A vector that results from a vector $\mathbf{x}$ by a **transfer** of size $\delta \geq 0$ between a pair of players $i, j \in N, i \neq j$, is referred to as $\mathbf{x}^{\,i,j,\delta} = (x_k^{i,j,\delta})_{k \in N}$, which is given by

$$\mathbf{x}^{i,j,\delta}_{N \setminus \{i,j\}} = \mathbf{x}_{N \setminus \{\, i,j\}}, \; x_i^{i,j,\delta} = x_i - \delta \quad \text{and} \quad x_j^{i,j,\delta} = x_j + \delta. \tag{2.2}$$

A **side-payment** for the players in N is a vector $\mathbf{z} \in \mathbb{R}^n$ such that $z(N) = 0$.

A **solution concept**, denoted as σ, on a non-empty set $\mathcal{G}$ of games is a correspondence on $\mathcal{G}$ that assigns to any game $v \in \mathcal{G}$ a subset $\sigma(N, v)$ of $\mathcal{I}^*(N, v)$. This set can be empty or just be single-valued, in the latter case, the solution σ is a function and is simply called a value.

The **core** of a game $\langle N, v\rangle$ is a set-valued solution that is constituted by the imputations satisfying besides the individual rationality property as well as the coalitional rationality property, i.e. the core of a game $v \in \mathcal{G}^n$ is given by

$$\mathcal{C}(N, v) := \{\mathbf{x} \in \mathcal{I}(N, v) \,|\, x(N) = v(N) \text{ and } x(S) \geq v(S) \; \forall \; S \subset N\}. \tag{2.3}$$

The core of a n-person game may be empty. Whenever it is non-empty we have some incentive for mutual cooperation in the grand coalition.

Given a vector $\mathbf{x} \in \mathcal{I}^*(v)$, we define the **excess** of coalition S with respect to the pre-imputation $\mathbf{x}$ in the game $\langle N, v\rangle$ by

$$e^v(S, \mathbf{x}) := v(S) - x(S). \tag{2.4}$$

Take a game $v \in \mathcal{G}^n$. For any pair of players $i, j \in N, i \neq j$, the **maximum surplus** of player i over player j with respect to any pre-imputation $\mathbf{x} \in \mathcal{I}^*(v)$ is given by the maximum excess at $\mathbf{x}$ over the set of coalitions containing player i but not player j, thus

$$s_{ij}(\mathbf{x}, v) := \max_{S \in \mathcal{G}_{ij}} e^v(S, \mathbf{x}) \qquad \text{where } \mathcal{G}_{ij} := \{S \mid i \in S \text{ and } j \notin S\}. \tag{2.5}$$

The set of all pre-imputations $\mathbf{x} \in \mathcal{I}^*(v)$ that balances the maximum surpluses for each distinct pair of players $i, j \in N, i \neq j$ is called the **pre-kernel** of the game v, and is defined by

$$\mathcal{PK}(v) := \{\mathbf{x} \in \mathcal{I}^*(v) \mid s_{ij}(\mathbf{x}, v) = s_{ji}(\mathbf{x}, v) \quad \text{for all } i, j \in N, i \neq j\}. \tag{2.6}$$

The pre-kernel has the advantage of addressing a stylized bargaining process, in which the figure of argumentation is a **pairwise equilibrium procedure** of claims while relying on best arguments, that is, the coalitions that will best support the claim. The pre-kernel solution characterizes all those imputations in which all pairs of players $i, j \in N, i \neq j$ are in equilibrium with respect to their claims.

In order to define the pre-nucleolus of a game $v \in \mathcal{G}^n$, take any $\mathbf{x} \in \mathbb{R}^n$ to define a 2^n-tuple vector $\theta(\mathbf{x})$ whose components are the excesses $e^v(S, \mathbf{x})$ of the 2^n coalitions $S \subseteq N$, arranged in decreasing order, that is,

$$\theta_i(\mathbf{x}) := e^v(S_i, \mathbf{x}) \geq e^v(S_j, \mathbf{x}) =: \theta_j(\mathbf{x}) \qquad \text{if} \qquad 1 \leq i \leq j \leq 2^n. \tag{2.7}$$

Ordering the so-called complaint or dissatisfaction vectors $\theta(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ by the lexicographic order $\leq_L$ on $\mathbb{R}^n$, we shall write

$$\theta(\mathbf{x}) <_L \theta(\mathbf{y}) \qquad \text{if } \exists \text{ an integer } 1 \leq k \leq 2^n, \tag{2.8}$$

such that $\theta_i(\mathbf{x}) = \theta_i(\mathbf{y})$ for $1 \leq i < k$ and $\theta_k(\mathbf{x}) < \theta_k(\mathbf{y})$. Furthermore, we write $\theta(\mathbf{x}) \leq_L \theta(\mathbf{y})$ if either $\theta(\mathbf{x}) <_L \theta(\mathbf{y})$ or $\theta(\mathbf{x}) = \theta(\mathbf{y})$. Now the pre-nucleolus $\mathcal{PN}(N, v)$ over the pre-imputations set $\mathcal{I}^*(v)$ is defined by

$$\mathcal{PN}(N, v) = \{\mathbf{x} \in \mathcal{I}^*(N, v) \mid \theta(\mathbf{x}) \leq_L \theta(\mathbf{y}) \; \forall \; \mathbf{y} \in \mathcal{I}^*(N, v)\}\,. \tag{2.9}$$

The **pre-nucleolus** of any game $v \in \mathcal{G}^n$ is non-empty as well as unique, and it is referred to as $\nu(v)$ if the game context is clear from the contents or $\nu(N, v)$ otherwise.

Moreover, both solutions can be uniquely characterized by a set of axioms. In order to formalize such an axiomatization, let $\langle N, v \rangle \in \mathcal{G}$ be a game s.t. $\emptyset \neq S \subseteq N$ and let $\vec{x} \in \mathcal{I}^*(N, v)$. The **Davis/Maschler reduced game** w.r.t. S and $\vec{x}$ is the game $\langle S, v_{S,\vec{x}} \rangle$ as given by

$$v_{S,\vec{x}}(T) := \begin{cases} 0 & \text{if } T = \emptyset \\ v(N) - x(N \backslash S) & \text{if } T = S \\ \max_{Q \subseteq N \backslash S} \; (v(T \cup Q) - x(Q)) & \text{otherwise.} \end{cases} \tag{2.10}$$

This game type has been introduced by Davis and Maschler (1965) to study the kernel.

Definition 2.1 (DM-RGP). *A solution σ on $\mathcal{G}$ satisfies the reduced game property (RGP), if for $\langle N, v\rangle \in \mathcal{G}, \emptyset \neq S \subseteq N$ and $\mathbf{x} \in \sigma(N, v)$, then $\langle S, v_{S,\mathbf{x}} \rangle \in \mathcal{G}$ and $\mathbf{x}_S \in \sigma(S, v_{S,\mathbf{x}})$.*

Let σ be a solution concept on the set $\mathcal{G}$, and $\mathcal{U}$ the universe of players. In addition, define the permutation group by $\text{Sym}(N) := \{\vartheta : N \rightarrow N \,|\, \vartheta \text{ is bijective}\}$ acting on the game space $\mathcal{G}$ by linear transformations. Hence, each bijection $\vartheta \in \text{Sym}(N)$ corresponds to a linear and invertible transformation of an element of the vector space $\mathcal{G}$ by defining a permuted game $\vartheta\, v(S) := v(\vartheta^{-1} S)$ for every $\vartheta \in \text{Sym}(N), v \in \mathcal{G}$ and $S \subseteq N$, whereas $\vartheta\, S := \{\vartheta(i) \,|\, i \in S\}$ and $\vartheta^{-1}\, S := \{i \,|\, \vartheta(i) \in S\}$. Hence, the games $\langle N, v \rangle$ and $\langle \vartheta N, \vartheta v \rangle$ are equivalent, where we have written the group operations for the sake of convenience as junction.

1. A solution σ on $\mathcal{G}$ satisfying **non-emptiness (NE)**, if $\sigma(N, v) \neq \emptyset$ for every $\langle N, v \rangle \in \mathcal{G}$.
2. A solution σ on $\mathcal{G}$ is **single-valued (SIVA)**, if $|\,\sigma(N, v)\,| = 1$ for every $\langle N, v \rangle \in \mathcal{G}$.
3. A solution σ on $\mathcal{G}$ is **Pareto optimal (PO)**, if $\sum_{k \in N} \sigma(N, v)_k = v(N)$ for every $\langle N, v \rangle \in \mathcal{G}$.
4. A solution σ on $\mathcal{G}$ satisfies the **equal treatment property (ETP)**, if $\langle N, v \rangle \in \mathcal{G}$, $\vec{x} \in \sigma(N, v)$ and if $k, l \in N$ s.t. $k \sim_v l$, then $x_k = x_l$.
5. A solution σ on $\mathcal{G}$ satisfies **anonymity (AN)**, if for $\langle N, v \rangle \in \mathcal{G}$, for a bijection $\vartheta \in \text{Sym}(N)$ and for $\langle \vartheta N, \vartheta v \rangle \in \mathcal{G}$ implying $\sigma(\vartheta N, \vartheta v) = \vartheta(\sigma(N, v))$.
6. A solution σ on $\mathcal{G}$ fulfills the **Covariance under Strategic Equivalence (COV)** property, if for $\langle N, v_1 \rangle, \langle N, v_2 \rangle \in \mathcal{G}$, with $v_2 = t \cdot v_1 + \mathfrak{m}$ for some $t \in \mathbb{R} \backslash \{0\}, \mathfrak{m} \in \mathbb{R}^{2^n}$, then $\sigma(N, v_2) = t \cdot \sigma(N, v_1) + \mathbf{m}$, whereas $\mathbf{m} \in \mathbb{R}^n$ and $\mathfrak{m}$ is the vector of measures obtained from $\mathbf{m}$.
7. A solution σ on $\mathcal{G}$ possesses the **converse reduced game property (CRG)** property, if for $\langle N, v \rangle \in \mathcal{G}$ with $|N| \geq 2$, $\vec{x} \in \mathcal{I}^*(N, v)$, $\langle S, v_{S,\vec{x}} \rangle \in \mathcal{G}$ and $\vec{x}_S \in \sigma(S, v_{S,\vec{x}})$ for every $S \in \{T \subseteq N \,|\, |T| = 2\}$, then $\vec{x} \in \sigma(N, v)$.

Theorem 2.1 (Sobolev (1975)). *If $\mathcal{U}$ is an infinite player set, then there exists a unique solution σ on $\mathcal{G}_{\mathcal{U}}$ satisfying single-valuedness (SIVA), anonymity (AN), covariance under strategic equivalence (COV), and reduced game property (RGP), which is the pre-nucleolus.*

In addition, we want to discuss some important game properties. A game $v \in \mathcal{G}^n$ is said to be **monotonic** if

$$v(S) \leq v(T) \qquad \forall \emptyset \neq S \subseteq T. \tag{2.11}$$

Thus, whenever a game is monotonic, a coalition T can guarantee to its member a value at least as high as any sub-coalition S can do. This subclass of games is referred to as $\mathcal{MN}^n$. A game $v \in \mathcal{G}^n$ satisfying the condition

$$v(S) + v(T) \leq v(S \cup T) \qquad \forall S, T \subseteq N, \text{ with } S \cap T = \emptyset, \tag{2.12}$$

is called **superadditive**. This means, that two disjoint coalitions have some incentive to join into a mutual coalition. This can be regarded as an incentive of merging economic activities into larger units. We denote this subclass of games by $\mathcal{SA}^n$. However, if a game $v \in \mathcal{G}^n$ satisfies

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \qquad \forall S, T \subseteq N, \tag{2.13}$$

or equivalently

$$v(S \cup \{i\}) - v(S) \leq v(S \cup \{i, j\}) - v(S \cup \{j\}) \qquad \text{if } S \subseteq N \backslash \{i, j\}, \tag{2.14}$$

then it is called **convex**. In this case, we will observe a strong incentive for a mutual cooperation in the grand coalition, due to its achievable over proportionate surpluses while increasing the scale of cooperation. This subclass of games has been introduced by Shapley (1971), and we denote it by $\mathcal{CV}^n$. Convex games having a non-empty core and the Shapley value is the center of gravity of the extreme point of the core (cf. Shapley (1971)), that is, a convex combination of the vectors of marginal contributions, which are core imputations for convex games. It should be evident that $\mathcal{CV}^n \subset \mathcal{SA}^n$ is satisfied.

3 A Dual Pre-Kernel Representation

The concept of a Fenchel-Moreau generalized conjugation – also known as the indirect function of a characteristic function game – was introduced by Martinez-Legaz (1996), and provides the same information as the n-person cooperative game with transferable utility under consideration. This approach was successfully applied in Meinhardt (2013) to give a dual representation of the pre-kernel solution of TU games by means of solution sets of a family of quadratic objective functions. In this section, we review some crucial results extensively studied in Meinhardt (2013, Chap. 5 & 6) as the building blocks to investigate the single-valuedness of the pre-kernel correspondence and its replication property.

The **convex conjugate** or **Fenchel transform** $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$ (where $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\, \infty\}$) of a convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ (cf. Rockafellar (1970, Section 12)) is defined by

$$f^*(\mathbf{x}^*) = \sup_{\mathbf{x} \in \mathbb{R}^n} \{\langle\, \mathbf{x}^*, \mathbf{x}\,\rangle - f(\mathbf{x})\} \qquad \forall \mathbf{x}^* \in \mathbb{R}^n.$$

Observe that the Fenchel transform f^* is the point-wise supremum of affine functions $p(\mathbf{x}^*) = \langle\, \mathbf{x}, \mathbf{x}^*\,\rangle - \mu$ such that $(\mathbf{x}, \mu) \in (\mathcal{C} \times \mathbb{R}) \subseteq (\mathbb{R}^n \times \mathbb{R})$, whereas $\mathcal{C}$ is a convex set. Thus, the Fenchel transform f^* is again a convex function.

We can generalize the definition of a Fenchel transform (cf. Martinez-Legaz (1996)) by introducing a fixed non-empty subset $\mathcal{K}$ of $\mathbb{R}^n$, then the conjugate of a function $f : \mathcal{K} \to \overline{\mathbb{R}}$ is $f^c : \mathbb{R}^n \to \overline{\mathbb{R}}$, given by

$$f^c(\mathbf{x}^*) = \sup_{\mathbf{x} \in \mathcal{K}} \{ \langle\, \mathbf{x}^*, \mathbf{x} \,\rangle - f(\mathbf{x}) \} \qquad \forall \mathbf{x}^* \in \mathbb{R}^n,$$

which is also known as the **Fenchel-Moreau conjugation**.

A vector $\mathbf{x}^*$ is said to be a subgradient of a convex function f at a point $\mathbf{x}$, if

$$f(\mathbf{z}) \geq f(\mathbf{x}) + \langle\, \mathbf{x}^*, \mathbf{z} - \mathbf{x} \,\rangle \qquad \forall \mathbf{z} \in \mathbb{R}^n.$$

The set of all subgradients of f at $\mathbf{x}$ is called the subdifferentiable of f at $\mathbf{x}$ and it is defined by

$$\partial f(\mathbf{x}) := \{ \mathbf{x}^* \in \mathbb{R}^n \mid f(\mathbf{z}) \geq f(\mathbf{x}) + \langle\, \mathbf{x}^*, \mathbf{z} - \mathbf{x} \,\rangle \quad (\forall \mathbf{z} \in \mathbb{R}^n) \}.$$

The set of all subgradients $\partial f(\mathbf{x})$ is a closed convex set, which could be empty or may consist of just one point. The multivalued mapping $\mathbf{x} \mapsto \partial f(\mathbf{x})$ is called the subdifferential of f.

Theorem 3.1 (Martinez-Legaz (1996))**.** *The indirect function $\pi : \mathbb{R}^n \to \mathbb{R}$ of any n-person TU game is a non-increasing polyhedral convex function such that*

(i) $\partial \pi(\mathbf{x}) \cap \{-1, 0\}^n \neq \emptyset \qquad \forall \mathbf{x} \in \mathbb{R}^n$,
(ii) $\{-1, 0\}^n \subset \bigcup_{\mathbf{x} \in \mathbb{R}^n} \partial \pi(\mathbf{x})$, *and*
(iii) $\min_{\mathbf{x} \in \mathbb{R}^n} \ \pi(\mathbf{x}) = 0$.

Conversely, if $\pi : \mathbb{R}^n \to \mathbb{R}$ satisfies (i)-(iii) then there exists a unique n-person TU game $\langle N, v \rangle$ having π as its indirect function, its characteristic function is given by

$$v(S) = \min_{\mathbf{x} \in \mathbb{R}^n} \left\{ \pi(\mathbf{x}) + \sum_{k \in S} x_k \right\} \qquad \forall\ S \subset N. \tag{3.1}$$

According to the above result, the associated **indirect function** $\pi : \mathbb{R}^n \to \mathbb{R}_+$ is given by:

$$\pi(\mathbf{x}) = \max_{S \subseteq N} \left\{ v(S) - \sum_{k \in S} x_k \right\},$$

for all $\mathbf{x} \in \mathbb{R}^n$. A characterization of the pre-kernel in terms of the indirect function is due to Meseguer-Artola (1997). In the course of our discussion we present this representation in its most general form, although we restrict ourselves to the trivial coalition structure $\mathcal{B} = \{N\}$.

The pre-imputation that comprises the possibility of compensation between a pair of players $i, j \in N, i \neq j$, is denoted as $\mathbf{x}^{\,i,j,\delta} = (x_k^{\,i,j,\delta})_{k \in N} \in \mathcal{I}^*(v)$, with $\delta \geq 0$, which is given by

$$\mathbf{x}^{\,i,j,\delta}_{N \backslash \{i,j\}} = \mathbf{x}_{N \backslash \{\, i,j\}}, \; x_i^{\,i,j,\delta} = x_i - \delta \quad \text{and} \quad x_j^{\,i,j,\delta} = x_j + \delta.$$

By the next Lemma we shall establish that the indirect function π of game v can be related to the maximum surpluses.

Lemma 3.1 (Meseguer-Artola (1997))**.** *Let $\langle N, v \rangle$ be a n-person cooperative game with side payments. Let π and s_{ij} be the associated indirect function and the maximum surplus of player i against player j, respectively. If $\mathbf{x} \in \mathcal{I}^*(N, v)$, then the equality:*

$$s_{ij}(\mathbf{x}, v) = \pi(\mathbf{x}^{\,i,j,\delta}) - \delta$$

holds for every $i, j \in N$, $i \neq j$, and for every $\delta \geq \delta_1(\mathbf{x}, v)$, where:

$$\delta_1(\mathbf{x}, v) := \max_{k \in N, S \subset N \backslash \{k\}} |v(S \cup \{k\}) - v(S) - x_k|.$$

Proof. For a proof consult Meseguer-Artola (1997); Meinhardt (2013). □

A characterization of the pre-kernel in terms of the indirect function is due to Meseguer-Artola (1997). As aforementioned, we focus on the trivial coalition structure $\mathcal{B} = \{N\}$ only, since our algorithm evaluates a pre-kernel element for the grand coalition. However, this approach can also be applied for more general coalition structures (cf. Meinhardt (2018d, Section 5.2.2)). Though we concentrate on the trivial one $\mathcal{B} = \{N\}$, we nevertheless restate this result through its most general form.

Proposition 3.1 (Meseguer-Artola (1997); Meinhardt (2013)). *For a TU game with indirect function π, a pre-imputation $\mathbf{x} \in \mathcal{I}^*(N, v)$ is in the pre-kernel of $\langle N, v\rangle$ for the coalition structure $\mathcal{B} = \{B_1, \ldots, B_l\}$, $\mathbf{x} \in \mathcal{PK}(v, \mathcal{B})$, if, and only if, for every $k \in \{1, 2, \ldots, l\}$, every $i, j \in B_k,\ i < j$, and some $\delta \geq \delta_1(v, \mathbf{x})$, one gets*

$$\pi(\mathbf{x}^{\,i,j,\delta}) = \pi(\mathbf{x}^{\,j,i,\delta}).$$

Meseguer-Artola (1997) was the first who recognized that based on the result of Proposition 3.1 a pre-kernel element can be derived as a solution of an over-determined system of non-linear equations. For the trivial coalition structure $\mathcal{B} = \{N\}$ the over-determined system of non-linear equations is given by

$$\begin{cases} f_{ij}(\mathbf{x}) = 0 & \forall i, j \in N, i < j \\ f_0(\mathbf{x}) = 0 \end{cases} \tag{3.2}$$

where, for some $\delta \geq \delta_1(\mathbf{x}, v)$,

$$f_{ij}(\mathbf{x}) := \pi(\mathbf{x}^{\,i,j,\delta}) - \pi(\mathbf{x}^{\,j,i,\delta}) \qquad \forall i, j \in N, i < j, \tag{3.2-a}$$

and

$$f_0(\mathbf{x}) := \sum_{k \in N} x_k - v(N). \tag{3.2-b}$$

To any over-determined system an equivalent minimization problem is associated such that the set of global minima coincides with the solution set of the system (cf. Meinhardt (2013, Sec. 5.3)). The solution set of such a minimization problem is the set of values for $\mathbf{x}$ which minimizes the following function

$$h(\mathbf{x}) := \sum_{\substack{i,j \in N \\ i<j}} (f_{ij}(\mathbf{x}))^2 + (f_0(\mathbf{x}))^2 \geq 0 \qquad \forall\, \mathbf{x} \in \mathbb{R}^n. \tag{3.3}$$

As we will notice in the sequel, this optimization problem is equivalent to a least squares adjustment. For further details see Meinhardt (2013, Chap. 6). From the existence of the pre-kernel and objective function h of type (3.3), we get the following relation:

Corollary 3.1 (Meinhardt (2013)). *For a TU game $\langle N, v\rangle$ with indirect function π, it holds that*

$$h(\mathbf{x}) = \sum_{\substack{i,j \in N \\ i<j}} (f_{ij}(\mathbf{x}))^2 + (f_0(\mathbf{x}))^2 = \min_{\mathbf{y} \in \mathcal{I}^0(v)} h(\mathbf{y}) = 0, \tag{3.4}$$

if, and only if, $\mathbf{x} \in \mathcal{PK}(v)$.

Proof. To establish the equivalence between the pre-kernel set and the set of global minima, we have to notice that in view of Theorem 3.1 $\min_{\mathbf{y}} h = 0$ is in force. Now, we prove necessity while taking a pre-kernel element, i.e. $\mathbf{x} \in \mathcal{PK}(v)$, then the efficiency property is satisfied with $f_0(\mathbf{x}) = 0$ and the maximum surpluses $s_{ij}(\mathbf{x}, v)$ must be balanced for each distinct pair of players i, j, implying that $f_{ij}(\mathbf{x}) = 0$ for all $i, j \in N, i < j$ and therefore $h(\mathbf{x}) = 0$. Thus, we are getting $\mathbf{x} \in M(h)$. To prove sufficiency, assume that $\mathbf{x} \in M(h)$, then $h(\mathbf{x}) = 0$ with the implication that the efficiency property $f_0(\mathbf{x}) = 0$ and $f_{ij}(\mathbf{x}) = 0$ must be valid for all $i, j \in N, i < j$. This means that the difference $f_{ij}(\mathbf{x}) = (\pi(\mathbf{x}^{i,j,\delta}) - \pi(\mathbf{x}^{j,i,\delta}))$ is equalized for each distinct pair of indices $i, j \in N, i < j$. Thus, $\mathbf{x} \in \mathcal{PK}(v)$. It turns out that the minimum set coincides with the pre-kernel, i.e., we have:

$$M(h) = \{\mathbf{x} \in \mathcal{I}^*(v) \mid h(\mathbf{x}) = 0\} = \mathcal{PK}(v), \tag{3.5}$$

with this argument we are done. □

Corollary 3.1 gives an alternative characterization of the pre-kernel set in terms of a solution set. Singling out a pre-kernel element by solving the above minimization problem is, for instance, possible by a modified *Steepest Descent Method*. However, a direct method is not applicable. This is due to fact that the objective function h is the difference of two convex functions and that due to Theorem 3.1 the indirect function π is a non-increasing polyhedral convex function. This implies that function h is not continuous differentiable everywhere and that its structural form is ambiguous. Nevertheless, Proposition 6.2.2 (cf. Meinhardt (2013)) characterizes the objective function h as the composite of a finite family of quadratic functions. For brevity, we do not discuss the whole details which would go beyond the scope of the paper, here we focus only on the aspect that the domain of function h can be partitioned into payoff equivalence classes. On each payoff equivalence class a quadratic and convex function can be identified. Pasting the finite collection of quadratic and convex functions together reproduces function h. For a thorough and more detailed discussion of this topic, we refer the reader to Section 5.4 and 6.2 in Meinhardt (2013).

To understand the structural form of the objective function h, we will first identify equivalence relations on its domain. To start with, we define the set of **most effective** or **significant coalitions** for each pair of players $i, j \in N, i \neq j$ at the payoff vector $\mathbf{x}$ by

$$\mathcal{C}_{ij}(\mathbf{x}) := \{S \in \mathcal{G}_{ij} \mid s_{ij}(\mathbf{x}, v) = e^v(S, \mathbf{x})\}. \tag{3.6}$$

When we gather for all pair of players $i, j \in N, i \neq j$ all these coalitions that support the claim of a specific player over some other players, we have to consider the concept of the collection of most effective or significant coalitions w.r.t. $\mathbf{x}$, which we define as in Maschler et al. (1979, p. 315) by

$$\mathcal{C}(\mathbf{x}) := \bigcup_{\substack{i,j \in N \\ i \neq j}} \mathcal{C}_{ij}(\mathbf{x}). \tag{3.7}$$

Observe that the set $\mathcal{C}_{ij}(\mathbf{x})$ for all $i, j \in N, i \neq j$ does not have cardinality one, which is required to identify a partition on the domain of function h. Now let us choose for each pair $i, j \in N, i \neq j$ a descending ordering on the set of most effective coalitions in accordance with their size, and within such a collection of most effective coalitions having smallest size, the lexicographical minimum is singled out, then we obtain the required uniqueness to partition the domain of h. To see this, notice that from the set of most effective or significant coalitions of a pair of players $i, j \in N, i \neq j$ at the payoff vector $\mathbf{x}$ the smallest cardinality over the set of most effective coalitions is defined as

$$\Phi_{ij}(\mathbf{x}) := \min \left\{ |S| \;\middle|\; S \in \mathcal{C}_{ij}(\mathbf{x}) \right\}. \tag{3.8}$$

Gathering all these sets having smallest cardinality for all pairs of players $i, j \in N, i \neq j$, we end up with

$$\Psi_{ij}(\mathbf{x}) := \left\{ S \in \mathcal{C}_{ij}(\mathbf{x}) \;|\, \Phi_{ij}(\mathbf{x}) = |S| \right\}. \tag{3.9}$$

With respect to an arbitrary payoff vector $\mathbf{x}$, the set of coalitions of smallest cardinality $\Psi_{ij}(\mathbf{x})$ which is minimized w.r.t. the lexicographically order $<_L$ is determined by

$$\mathcal{S}_{ij}(\mathbf{x}) := \left\{ S \in \Psi_{ij}(\mathbf{x}) \;\middle|\; S <_L T \text{ for all } S \neq T \in \Psi_{ij}(\mathbf{x}) \right\} \quad \forall i, j \in N, i \neq j. \tag{3.10}$$

We denote this set as the **lexicographically smallest most effective coalitions** w.r.t. $\mathbf{x}$. Gathering all these collections we are able to specify the set of lexicographically smallest most effective coalitions w.r.t. $\mathbf{x}$ through

$$\mathcal{S}(\mathbf{x}) := \bigcup_{\substack{i,j \in N \\ i \neq j}} \mathcal{S}_{ij}(\mathbf{x}). \tag{3.11}$$

This set will be indicated in short as the set of **lexicographically smallest coalitions** or just more succinctly **most effective coalitions** whenever no confusion can arise. Notice that this set is never empty and can uniquely be identified. This implies that the cardinality of this set is equal to $n \cdot (n-1)$. In the following we will observe that from these type of sets equivalence relations on the domain $dom\, h$ can be identified.

To see this, consider the correspondence $\mathcal{S}$ on $dom\, h$ and two different vectors, say $\mathbf{x}$ and $\vec{\gamma}$, then both vectors are said to be equivalent w.r.t. the binary relation $\sim$ if, and only if, they induce the same set of lexicographically smallest coalitions, that is, $\mathbf{x} \sim \vec{\gamma}$ if, and only if, $\mathcal{S}(\mathbf{x}) = \mathcal{S}(\vec{\gamma})$. In case that the binary relation $\sim$ is reflexive, symmetric and transitive, then it is an **equivalence relation** and it induces **equivalence classes** $[\vec{\gamma}]$ on $dom\, h$ which we define through $[\vec{\gamma}] := \{\mathbf{x} \in dom\ h\ | \mathbf{x} \sim \vec{\gamma}\}$. Thus, if $\mathbf{x} \sim \vec{\gamma}$, then $[\mathbf{x}] = [\vec{\gamma}]$, and if $\mathbf{x} \nsim \vec{\gamma}$, then $[\mathbf{x}] \cap [\vec{\gamma}] = \emptyset$. This implies that whenever the binary relation $\sim$ induces equivalence classes $[\vec{\gamma}]$ on $dom\, h$, then it partitions the domain $dom\, h$ of the function h. The resulting collection of equivalence classes $[\vec{\gamma}]$ on $dom\, h$ is called the quotient of $dom\, h$ modulo $\sim$, and we denote this collection by $dom\, h/\sim$. We indicate this set as an equivalence class whenever the context is clear, otherwise we apply the term payoff set or payoff equivalence class.

Proposition 3.2 (Meinhardt (2013))**.** *The binary relation $\sim$ on the set $dom\, h$ defined by $\mathbf{x} \sim \vec{\gamma} \iff \mathcal{S}(\mathbf{x}) = \mathcal{S}(\vec{\gamma})$ is an equivalence relation, which forms a partition of the set $dom\, h$ by the collection of equivalence classes $\{[\vec{\gamma}_k]\}_{k \in J}$, where J is an arbitrary index set. Furthermore, for all $k \in J$, the induced equivalence class $[\vec{\gamma}_k]$ is a convex set.*

Proof. For a proof see Meinhardt (2013, p. 59). □

By Proposition 3.2, we observe that a payoff equivalence class can alternatively be specified through

$$[\vec{\gamma}] := \left\{ \mathbf{x} \in \mathbb{R}^N \;| x(N) = v(N) \text{ and } \mathcal{S}(\mathbf{x}) = \mathcal{S}(\vec{\gamma}) \right\}.$$

The cardinality of the collection of the payoff equivalence classes induced by a TU game is finite (cf. Meinhardt (2013, Proposition 5.4.2.)). Furthermore, on each payoff equivalence class $[\vec{\gamma}]$ from the $dom\, h$ a unique quadratic and convex function can be identified. Therefore, there must be a finite composite of these functions that constitutes the objective function h. In order to construct such a quadratic and convex function suppose that $\vec{\gamma} \in [\vec{\gamma}]$. From this vector we obtain the collection of most effective coalitions $\mathcal{S}(\vec{\gamma})$ in accordance with Proposition 3.2. Then observe that the differences in the values between a pair $\{i, j\}$ of players are defined by $\alpha_{ij} := (v(S_{ij}) - v(S_{ji})) \in \mathbb{R}$ for all $i, j \in N,\ i < j$, and $\alpha_0 := v(N) > 0$ w.r.t. $\mathcal{S}(\vec{\gamma})$. All of these q-components compose the q-coordinates of a payoff independent vector $\vec{\alpha}$, with $q = \binom{n}{2} + 1$. A vector that reflects the degree of unbalancedness of excesses for all pair of players, is denoted by $\vec{\xi} \in \mathbb{R}^q$, that is a q-column vector, which is given by

$$\begin{aligned} \xi_{ij} &:= e^v(S_{ij}, \vec{\gamma}) - e^v(S_{ji}, \vec{\gamma}) = v(S_{ij}) - \gamma(S_{ij}) - v(S_{ji}) + \gamma(S_{ji}) \quad \forall\, i, j \in N,\ i < j, \\ &= v(S_{ij}) - v(S_{ji}) + \gamma(S_{ji}) - \gamma(S_{ij}) = \alpha_{ij} + \gamma(S_{ji}) - \gamma(S_{ij}) \quad \forall\, i, j \in N,\ i < j, \\ \xi_0 &:= v(N) - \gamma(N) = \alpha_0 - \gamma(N). \end{aligned} \tag{3.12}$$

In view of Proposition 3.2, all vectors contained in the equivalence class $[\vec{\gamma}]$ induce the same set $\mathcal{S}(\vec{\gamma})$, and it holds

$$\xi_{ij} := e^v(S_{ij}, \vec{\gamma}) - e^v(S_{ji}, \vec{\gamma}) = s_{ij}(\vec{\gamma}, v) - s_{ji}(\vec{\gamma}, v) =: \zeta_{ij} \quad \forall\, i, j \in N,\ i < j. \tag{3.13}$$

The payoff dependent configurations $\vec{\xi}$ and $\vec{\zeta}$ having the following interrelationship outside its equivalence class: $\vec{\xi} \neq \vec{\zeta}$ for all $\mathbf{y} \in [\vec{\gamma}]^c$. Moreover, equation (3.13) does not necessarily mean that for $\vec{\gamma}', \vec{\gamma}^* \in [\vec{\gamma}],\ \vec{\gamma}' \neq \vec{\gamma}^*$, it holds $\vec{\xi}' = \vec{\xi}^*$. Hence, the vector of unbalanced excesses $\vec{\xi}$ is only equal with the vector of unbalanced maximum surpluses $\vec{\zeta}$ if the corresponding pre-imputation $\vec{\gamma}$ is drawn from its proper equivalence class $[\vec{\gamma}]$.

In addition, we write for sake of simplicity that $\mathbf{E}_{ij} := (\mathbf{1}_{S_{ji}} - \mathbf{1}_{S_{ij}}) \in \mathbb{R}^n,\ \forall i, j \in N, i < j$, and $\mathbf{E}_0 := -\mathbf{1}_N \in \mathbb{R}^n$. Notice that $\mathbf{1}_S$ is the **indicator function** or **characteristic vector** $\mathbf{1}_S : N \mapsto \{0, 1\}$ given by $\mathbf{1}_S(k) := 1$ if $k \in S$, otherwise $\mathbf{1}_S(k) := 0$. Combining these q-column vectors, we can construct an $(n \times q)$-matrix in $\mathbb{R}^{n \times q}$ referred to as $\mathbf{E}$, and which is given by

$$\mathbf{E} := [\mathbf{E}_{1,2}, \ldots, \mathbf{E}_{n-1,n}, \mathbf{E}_0] \in \mathbb{R}^{n \times q}. \tag{3.14}$$

Proposition 3.3 (Quadratic Function). *Let $\langle N, v \rangle$ be a TU game with indirect function π, then an arbitrary vector $\vec{\gamma}$ in the domain of h, i.e. $\vec{\gamma} \in dom\, h$, induces a quadratic function:*

$$h_\gamma(\mathbf{x}) = (1/2) \cdot \langle\, \mathbf{x}, \mathbf{Q}\, \mathbf{x}\, \rangle + \langle\, \mathbf{x}, \mathbf{a}\, \rangle + \alpha \qquad \mathbf{x} \in dom\, h, \tag{3.15}$$

where $\mathbf{a}$ is a column vector of coefficients, α is a scalar and $\mathbf{Q}$ is a symmetric $(n \times n)$-matrix with integer coefficients taken from the interval $[-n \cdot (n-1), n \cdot (n-1)]$.

Proof. The proof is given in Meinhardt (2013, pp. 66-68). □

By the above discussion, the objective function h and the quadratic as well as convex function h_γ of type (3.15) coincide on the payoff set $[\vec{\gamma}]$ (cf. Meinhardt (2013, Lemma 6.2.2)). However, on the complement $[\vec{\gamma}]^c$ it holds $h \neq h_\gamma$. Moreover, in view of Meinhardt (2013, Proposition 6.2.2) function h is composed of a finite family of quadratic and convex functions of type (3.15).

Proposition 3.4 (Least Squares). *A quadratic function h_γ given by equation (3.15) is equivalent to*

$$\langle\, \vec{\alpha} + \mathbf{E}^\top\, \mathbf{x}, \vec{\alpha} + \mathbf{E}^\top\, \mathbf{x}\, \rangle = \|\, \vec{\alpha} + \mathbf{E}^\top\, \mathbf{x}\, \|^2. \tag{3.16}$$

Therefore, the matrix $\mathbf{Q} \in \mathbb{R}^{n^2}$ *can also be expressed as* $\mathbf{Q} = 2 \cdot \mathbf{E}\;\mathbf{E}^\top$*, and the column vector* $\mathbf{a}$ *as* $2 \cdot \mathbf{E}\;\vec{\alpha} \in \mathbb{R}^n$*. Finally, the scalar* α *is given by* $\|\vec{\alpha}\|^2$*, where* $\mathbf{E} \in \mathbb{R}^{n \times q}, \mathbf{E}^\top \in \mathbb{R}^{q \times n}$ *and* $\vec{\alpha} \in \mathbb{R}^q$*.*

Proof. The proof can be found in Meinhardt (2013, pp. 70-71). □

Realize that the transpose of a vector or a matrix is denoted by the symbols $\mathbf{x}^\top$, and $\mathbf{Q}^\top$ respectively.

Lemma 3.2 (Meinhardt (2013))**.** *Let* $\mathbf{x}, \vec{\gamma} \in \operatorname{dom} h, \mathbf{x} = \vec{\gamma} + \mathbf{z}$ *and let* $\vec{\gamma}$ *induces the matrices* $\mathbf{E} \in \mathbb{R}^{n \times q}, \mathbf{E}^\top \in \mathbb{R}^{q \times n}$ *determined by formula* (3.14)*, and* $\vec{\alpha}, \vec{\xi} \in \mathbb{R}^q$ *as in equation* (3.12)*. If* $\mathbf{x} \in M(h_\gamma)$*, then*

1. $-\mathbf{E}^\top\,\mathbf{x} = \mathbf{P}\,\vec{\alpha}$*.*
2. $\mathbf{E}^\top\,\vec{\gamma} = \mathbf{P}\,(\vec{\xi} - \vec{\alpha}) = (\vec{\xi} - \vec{\alpha})$*.*
3. $-\mathbf{E}^\top\,\mathbf{z} = \mathbf{P}\,\vec{\xi}$*.*

In addition, let $q := \binom{n}{2} + 1$*. The matrix* $\mathbf{P} \in \mathbb{R}^{q^2}$ *is either equal to* $2 \cdot \mathbf{E}^\top\,\mathbf{Q}^{-1}\mathbf{E}$*, if the matrix* $\mathbf{Q} \in \mathbb{R}^{n^2}$ *is non-singular, or it is equal to* $2 \cdot \mathbf{E}^\top\,\mathbf{Q}^\dagger\mathbf{E}$*, if the matrix* $\mathbf{Q}$ *is singular. Furthermore, it holds for the matrix* $\mathbf{P}$ *that* $\mathbf{P} \neq \mathbf{I}_q$ *and rank* $\mathbf{P} \leq n$*.*

Proof. The proof is given in Meinhardt (2013, pp. 80-81). □

Notice that $\mathbf{Q}^\dagger$ is the **Moore-Penrose** or **pseudo-inverse** matrix of matrix $\mathbf{Q}$, if matrix $\mathbf{Q}$ is singular. This matrix is unique according to the following properties: (1) general condition, i.e. $\mathbf{Q}\,\mathbf{Q}^\dagger\,\mathbf{Q} = \mathbf{Q}$, (2) reflexive, i.e. $\mathbf{Q}^\dagger\,\mathbf{Q}\,\mathbf{Q}^\dagger = \mathbf{Q}^\dagger$, (3) normalized, i.e. $(\mathbf{Q}\,\mathbf{Q}^\dagger)^\top = \mathbf{Q}^\dagger\,\mathbf{Q}$, and finally (4) reversed normalized, i.e. $(\mathbf{Q}^\dagger\,\mathbf{Q})^\top = \mathbf{Q}\,\mathbf{Q}^\dagger$.

Proposition 3.5 (Orthogonal Projection Operator)**.** *Matrix* $\mathbf{P}$ *is idempotent and self-adjoint, i.e.* $\mathbf{P}$ *is an orthogonal projection operator.*

Proof. The proof can be found in Meinhardt (2013, p. 86). □

Lemma 3.3 (Meinhardt (2013).)**.** *Let* $\mathcal{E}$ *be a subspace of* $\mathbb{R}^q$ *with basis* $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ *derived from the linear independent vectors of matrix* $\mathbf{E}^\top$ *having rank* m*, with* $m \leq n$*, and let* $\{\mathbf{w}_1, \ldots, \mathbf{w}_{q-m}\}$ *be a basis of* $\mathcal{W} := \mathcal{E}^\perp$*. In addition, define matrix* $E^\top := [\mathbf{e}_1, \ldots, \mathbf{e}_m] \in \mathbb{R}^{q \times m}$*, and matrix* $W^\top := [\mathbf{w}_1, \ldots, \mathbf{w}_{q-m}] \in \mathbb{R}^{q \times (q-m)}$*, then for any* $\vec{\beta} \in \mathbb{R}^q$ *it holds*

1. $\vec{\beta} = [E^\top\;W^\top] \cdot \mathbf{c}$ *where* $\mathbf{c} \in \mathbb{R}^q$ *is a coefficient vector, and*
2. *the matrix* $[E^\top\;W^\top] \in \mathbb{R}^{q \times q}$ *is invertible, that is, we have*

$$[E^\top\;W^\top]^{-1} = \begin{bmatrix}(E\,E^\top)^{-1}\,E \\ (W\,W^\top)^{-1}\,W\end{bmatrix}.$$

Proof. For a proof see Meinhardt (2013, pp. 90-91). □

Notice that $\mathcal{E}$ can be interpreted as indicating a vector subspace of balanced excesses. A pre-imputation will be mapped into its proper vector subspace of balanced excesses $\mathcal{E}$, i.e. the vector subspace induced by the pre-imputation. However, the corresponding vector of unbalanced excesses generated by this pre-imputation is an element of this vector subspace of balanced excesses, if the pre-imputation is also a pre-kernel point. Hence, the vector of balanced excesses coincides with the vector of balanced maximum surpluses. This is a consequence of Lemma 3.2 or see Proposition 8.4.1 in Meinhardt (2013). Otherwise, this vector of unbalanced excesses will be mapped by the orthogonal projection $\mathbf{P}$ on $\mathcal{E}$. More information about the properties of this kind of vector subspace can be found in Meinhardt (2013, pp. 87-113 and 138-168).

Proposition 3.6 (Positive General Linear Group). *Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ as well as $\{\mathbf{e}_1^1, \ldots, \mathbf{e}_m^1\}$ be two ordered bases of the subspace $\mathcal{E}$ derived from the payoff sets $[\vec{\gamma}]$ and $[\vec{\gamma}_1]$, respectively. In addition, define the associated basis matrices $E^\top, E_1^\top \in \mathbb{R}^{q \times m}$ as in Lemma 3.3, then the unique transition matrix $X \in \mathbb{R}^{m^2}$ such that $E_1^\top = E^\top X$ is given, is an element of the positive general linear group, that is $X \in GL^+(m)$.*

Proof. The proof can be found in Meinhardt (2013, p. 101). □

Proposition 3.6 denotes two payoff sets $[\vec{\gamma}]$ and $[\vec{\gamma}_1]$ as equivalent, if there exists a transition matrix X from the positive general linear group, that is $X \in \mathrm{GL}^+(m)$, such that $E_1^\top = E^\top X$ is in force. Notice that the transition matrix X must be unique (cf. Meinhardt (2013, p. 102)). The underlying group action (cf. Meinhardt (2013, Corollary 6.6.1)) can be interpreted that a bargaining situation is transformed into an equivalent bargaining situation. For a thorough discussion of a group action onto the set of all ordered bases, the interested reader should consult Meinhardt (2013, Sect. 6.6).

The vector space $\mathbb{R}^q$ is an orthogonal decomposition by the subspaces $\mathcal{E}$ and $\mathcal{N}_{\mathbf{E}}$. We denote in the sequel a basis of the orthogonal complement of space $\mathcal{E}$ by $\{\mathbf{w}_1, \ldots, \mathbf{w}_{q-m}\}$. This subspace of $\mathbb{R}^q$ is identified by $\mathcal{W} := \mathcal{N}_{\mathbf{E}} = \mathcal{E}^\perp$. In addition, we have $\mathbf{P}\,\mathbf{w}_k = \mathbf{0}$ for all $k \in \{1, \ldots, q-m\}$. Thus, we can obtain the following corollary

Corollary 3.2 (Meinhardt (2013)). *If $\vec{\gamma}$ induces the matrices $\mathbf{E} \in \mathbb{R}^{n \times q}, \mathbf{E}^\top \in \mathbb{R}^{q \times n}$ determined by formula (3.14), then with respect to the Euclidean inner product, getting*

1. $\mathbb{R}^q = \mathcal{E} \oplus \mathcal{W} = \mathcal{E} \oplus \mathcal{E}^\perp$.

A consequence of the orthogonal projection method presented by the next theorem and corollary is that every payoff vector belonging to the intersection of the minimum set of function h_γ and its payoff equivalence class $[\vec{\gamma}]$ is a pre-kernel element. This due to $h_\gamma = h$ on $[\vec{\gamma}]$.

Theorem 3.2 (Orthogonal Projection Method). *Let $\vec{\gamma}_k \in [\vec{\gamma}]$ for $k = 1, 2, 3$. If $\vec{\gamma}_2 \in M(h_\gamma)$ and $\vec{\gamma}_k \notin M(h_\gamma)$ for $k = 1, 3$, then $\vec{\zeta}_2 = \vec{\xi}_2 = \mathbf{0}$, and consequently $\vec{\gamma}_2 \in \mathcal{PK}(v)$.*

Proof. For a proof see Meinhardt (2013, pp. 109-111). □

Corollary 3.3 (Meinhardt (2013)). *Let be $[\vec{\gamma}]$ an equivalence class of dimension $3 \leq m \leq n$, and $\mathbf{x} \in M(h_\gamma) \cap [\vec{\gamma}]$, then $\vec{\alpha} = \mathbf{P}\,\vec{\alpha}$, and consequently $\mathbf{x} \in \mathcal{PK}(v)$.*

$$\mathcal{PK}(N, v) = \bigcup_{k \in \mathcal{J}'} M(h_{\gamma_k}, \overline{[\vec{\gamma}_k]}). \tag{3.17}$$

where $\mathcal{J}'$ is a finite index set such that $\mathcal{J}' := \{k \in \mathcal{J} \,|\, g(\vec{\gamma}_k) = 0\}$. In addition, $g(\vec{\gamma}_k) = 0$ is the minimum value of a minimization problem under constraints of function h_{γ_k} over the closed payoff set $\overline{[\vec{\gamma}_k]}$. The solution sets $M(h_{\gamma_k}, \overline{[\vec{\gamma}_k]})$ are convex. Taking the finite union of convex sets gives us a non-convex set if $|\,\mathcal{J}'\,| \geq 2$. Hence, the pre-kernel set is generically a non-convex set. For the class of convex games and three person games we have $|\,\mathcal{J}'\,| = 1$, which implies that the per-kernel must be a singleton. From these ingredients we are able to design a method to compute an element of the pre-kernel.

Now we are in the position to provide an algorithm to single out a pre-kernel point. That is, given a TU game $\langle N, v\rangle$ with indirect function π and objective function h of type (3.3) on the domain $dom\, h$, the method described below generates a sequence of payoff vectors on $dom\, h$ that converges under regime of orthogonal projection to a pre-kernel point.

To this end, we consider a mapping that sends a point $\vec{\gamma}$ to a point $\vec{\gamma}_{\circ} \in M(h_{\vec{\gamma}})$ through

$$\Gamma(\vec{\gamma}) := -\Big(\mathbf{Q}^{\dagger}\;\mathbf{a}\Big)(\vec{\gamma}) = -\Big(\mathbf{Q}^{\dagger}_{\vec{\gamma}}\;\mathbf{a}_{\vec{\gamma}}\Big) = \vec{\gamma}_{\circ} \in M(h_{\vec{\gamma}}) \qquad \forall \vec{\gamma} \in \mathbb{R}^n, \tag{3.18}$$

where $\mathbf{Q}_{\vec{\gamma}}$ and $\mathbf{a}_{\vec{\gamma}}$ are the matrix and the column vector induced by vector $\vec{\gamma}$, respectively. Notice that matrix $\mathbf{Q}^{\dagger}_{\vec{\gamma}}$ is the pseudo-inverse of matrix $\mathbf{Q}_{\vec{\gamma}}$. In addition, the set $M(h_{\vec{\gamma}})$ is the solution set of function $h_{\vec{\gamma}}$. Under a regime of orthogonal projection this mapping induces a cycle free method to evaluate a pre-kernel point for any class of TU games. We restate here Algorithm 8.1.1 of Meinhardt (2013) in a more succinctly written form through Method 3.1.

Algorithm 3.1: Procedure to seek for a Pre-Kernel Element

Data: Arbitrary TU Game $\langle\, N, v\,\rangle$, and a payoff vector $\vec{\gamma}_0 \in \mathbb{R}^n$.
Result: A payoff vector s.t. $\vec{\gamma}_{k+1} \in \mathcal{PK}(N, v)$.
begin
  $k \longleftarrow 0, \quad \mathcal{S}(\vec{\gamma}_{-1}) \longleftarrow \emptyset$
  Select an arbitrary starting point $\vec{\gamma}_0$
  if $\vec{\gamma}_0 \notin \mathcal{PK}(N, v)$ **then** Continue
  else Stop
  Determine $\mathcal{S}(\vec{\gamma}_0)$
  if $\mathcal{S}(\vec{\gamma}_0) \neq \mathcal{S}(\vec{\gamma}_{-1})$ **then** Continue
  else Stop
  repeat
    if $\mathcal{S}(\vec{\gamma}_k) \neq \emptyset$ **then** Continue
    else Stop
    Compute $\mathbf{E}_k$ and $\vec{\alpha}_k$ from $\mathcal{S}(\vec{\gamma}_k)$ and v
    Determine $\mathbf{Q}_k$ and $\mathbf{a}_k$ from $\mathbf{E}_k$ and $\vec{\alpha}_k$
    Calculate by Formula (3.18) $\mathbf{x}$
    $k \longleftarrow k + 1$
    $\vec{\gamma}_{k+1} \longleftarrow \mathbf{x}$
    Determine $\mathcal{S}(\vec{\gamma}_{k+1})$
  until $\mathcal{S}(\vec{\gamma}_{k+1}) = \mathcal{S}(\vec{\gamma}_k)$
end

Meinhardt (2013, Theorem 8.1.2) establishes that this iterative procedure converges toward a pre-kernel point. In view of Meinhardt (2013, Theorem 9.1.2) we even know that at most $\binom{n}{2} - 1$-iteration steps are sufficient to successfully terminate the search process. However, we have some empirical evidence that generically at most $n + 1$-iteration steps are needed to determine an element from the pre-kernel set (cf. Meinhardt (2013, Appendix A)). This method has also been proven to be useful in finding a N-shaped pre-kernel (cf. Meinhardt (2014)).

Example 3.1. To illustrate how the algorithm works, we introduce a minimum cost spanning tree game. The player set given by $N = \{1, 2, 3\}$ represents the users of a common good provided by a common supplier 0. Then the distribution system consists of links among members $N_0 = \{0\} \cup N$. The costs associated to buildup the links is given by the following cost matrix

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & 0 & 1 & 2 \\ 3 & 1 & 0 & 3 \\ 5 & 2 & 3 & 0 \end{bmatrix}, \tag{3.19}$$

where each entry denotes the cost of constructing the link $\{i, j\}$. For a more thorough investigation of minimum cost spanning tree games we refer to Curiel (1997). In the next step, let us define a savings game by

$$v(S) := \sum_{k \in S} c(\{k\}) - c(S) \qquad \forall\, S \subseteq N. \tag{3.20}$$

From the cost matrix (3.19), we derive a minimum cost spanning tree game from which a savings game is obtained through formula (3.20). The derived minimum cost spanning tree and savings game are given by Table 3.1.

Table 3.1: MCST and Savings Game

Game	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
c	1	3	5	2	3	6	4
v[a,b,c]	0	0	0	2	3	2	5

[a] Kernel: $(2, 1, 2)$
[b] Nucleolus: $(2, 1, 2)$
[c] Shapley Value: $(11/6, 4/3, 11/6)$

As a starting point we focus on the pre-selected efficient payoff vector $\mathbf{y}_0 = (-1, 2, 3)^\top$ to see of how we can determine a per-kernel point by means of Algorithm 3.1 for our specific savings game example. From the vector $\mathbf{y}_0$, we get the excess vector $exc(\mathbf{y}_0) = (0, 1, -2, -3, 1, 1, -3, 1)$.

In the next step, we look on the maximum surpluses for all pair of players. Recall that for any pair of players $i, j \in N, i \neq j$, the maximum surplus of player i over player j with respect to any pre-imputation $\mathbf{x}$ is given by the maximum excess at $\mathbf{x}$ over the set of coalitions containing player i but not player j, thus

$$s_{ij}(\mathbf{x}, v) := \max_{S \in \mathcal{G}_{ij}} e^v(S, \mathbf{x}) \qquad \text{where } \mathcal{G}_{ij} := \{S \mid i \in S \text{ and } j \notin S\}.$$

The expression $s_{ij}(\mathbf{x}, v)$ describes the maximum amount at the pre-imputation $\mathbf{x}$ that player i can gain without the cooperation of player j.

From this excess vector $exc(\mathbf{y}_0)$ we get now the subsequent set of lexicographically smallest coalitions for each pair of players:

$$\mathcal{S}(\mathbf{y}_0) = \{\{1\}, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}\}$$

whereas the order of the pairs of players in $\mathcal{S}(\mathbf{y}_0)$ is given by

$$\{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}.$$

For instance, for the pair of players $(1, 2)$, we find out these coalitions that support the claim of player 1 without counting on the cooperation of player 2; these are the coalitions $\{\{1\}, \{1,3\}\}$ having excess

$(1, 1)$. We see here that both coalitions have maximum surplus. This set is not unique, we determine the coalitions that have smallest cardinality, which is $\{1\}$. This set is unique and through this set we have also determined the coalition that has lexicographical minimum. To observe how we have to proceed if this set is not unique, let us assume that $n = 4$, then the set of coalitions supporting player 1 without counting on the cooperation of player 2 is $\{\{1\}, \{1, 3\}, \{1, 4\}, \{1, 3, 4\}\}$. Moreover, let us assume that the coalitions $\{\{1, 3\}, \{1, 4\}, \{1, 3, 4\}\}$ have maximum surpluses, then the smallest cardinality is 2 and we single out the coalitions $\{\{1, 3\}, \{1, 4\}$ and taking finally the lexicographical minimum, which is $\{1, 3\}$.

For the reverse pair $(2, 1)$ we find out that coalition $\{2\}$ supports best the claim of player 2 without taking into account the cooperation of player 1. Proceeding in the same way for the remaining pairs, then we derive a matrix $\mathbf{E}$ by $\mathbf{E}_{ij} = \mathbf{1}_{S_{ji}} - \mathbf{1}_{S_{ij}}$ for each $i, j \in N, i < j$, and $\mathbf{E}_0 = \mathbf{1}_N$. Notice that $\mathbf{1}_S : N \mapsto \{0, 1\}$ is the characteristic vector given by $\mathbf{1}_S(k) := 1$ if $k \in S$, otherwise $\mathbf{1}_S(k) := 0$. Then matrix $\mathbf{E}$ is defined by

$$\mathbf{E} := [\mathbf{E}_{1,2}, \ldots, \mathbf{E}_{2,3}, \mathbf{E}_0] \in \mathbb{R}^{3\times4}.$$

We realize that vector $\mathbf{E}_{1,2}$ is given by $(0, 1, 0)^\top - (1, 0, 0)^\top = (-1, 1, 0)^\top$ and $\mathbf{E}_0 = (1, 1, 1)^\top$. Proceeding in an analogous way for the remaining pair of players $(1, 3)$ and $(2, 3)$, matrix $\mathbf{E}$ is quantified by

$$\mathbf{E} = \begin{bmatrix} -1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

A column vector $\mathbf{a}$ can be obtained by $2 \cdot \mathbf{E}\ \vec{\alpha} \in \mathbb{R}^n$ whereas the vector $\vec{\alpha}$ is given by $\alpha_{ij} := (v(S_{ji}) - v(S_{ij})) \in \mathbb{R}$ for all $i, j \in N, i < j$, and $\alpha_0 := v(N)$. Therefore, vector $\vec{\alpha}$ is given by $(0, 0, 1, 5)^\top$.

From this matrix, we construct matrix $\mathbf{Q}$ by $2 \cdot \mathbf{E}\ \mathbf{E}^\top$, inserting its numbers, matrix $\mathbf{Q}$ is specified by

$$\mathbf{Q} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

The column vector $\mathbf{a}$ is given by $\mathbf{a} = (10, 8, 12)^\top$.

Solving this system of linear equations $\mathbf{Q}\ \mathbf{x} - \mathbf{a} = \mathbf{0}$, or alternatively $\mathbf{E}^\top\ \mathbf{x} - \vec{\alpha} = \mathbf{0}$, we get as a solution $\mathbf{y}_1 = (5/3, 4/3, 2)^\top$. The corresponding excess vector is given through

$$exc(\mathbf{y}_1) = (0, -5/3, -4/3, -2, -1, -2/3, -4/3, 0).$$

We observe that the maximum surpluses are not balanced. Hence, we need at least an additional iteration step to complete.

For the second iteration step we use the vector $\mathbf{y}_1 = (5/3, 4/3, 2)^\top$ while applying the procedure from above to get matrix $\mathbf{E}$ by

$$\mathbf{E} = \begin{bmatrix} -1 & -1 & 0 & 1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

with $\vec{\alpha}$ is given by $(-3, 0, 1, 5)^\top$. Constructing again matrix $\mathbf{Q}$ through

$$\mathbf{Q} = \begin{bmatrix} 6 & 0 & 2 \\ 0 & 6 & -2 \\ 2 & -2 & 8 \end{bmatrix}.$$

The column vector $\mathbf{a}$ is given by $\mathbf{a} = (16, 2, 18)^{\top}$. Solving this system of linear equations $\mathbf{Q}\,\mathbf{x} - \mathbf{a} = \mathbf{0}$, we get as a solution $\mathbf{y}_2 = (2, 1, 2)^{\top}$. The corresponding excess vector is given through

$$exc(\mathbf{y}_2) = (0, -2, -1, -2, -1, -1, -1, 0).$$

We can check out that the maximum surpluses are balanced, hence the vector $\mathbf{y}_2 = (2, 1, 2)^{\top}$ is a pre-kernel element of the game. Notice that in this specific case, we needed only two iteration steps to complete. This is the theoretical expected upper bound of iteration steps, since by Theorem 9.2.1 of Meinhardt (2013, p. 222), we have $\binom{3}{2} - 1 = 3 - 1 = 2$. #

4 The Single-Valuedness of the Pre-Kernel

In this section we apply results and techniques employed in the work of Meinhardt (2013). Namely, we prove in a first step that the linear mapping of a pre-kernel element into a specific vector subspace of balanced excesses is a singleton. Secondly, that there cannot exist a different and non-transversal vector subspace of balanced excesses in which a linear transformation of a pre-kernel element can be mapped. This enables us to study the single-valuedness of the pre-kernel solution of a related TU game derived from a pre-kernel element of a default game.

For conducting this line of investigation some additional concepts are needed. In a first step we introduce the definition of a **unanimity game**, which is indicated as: $\mathbf{u}_T(S) := 1$, if $T \subseteq S$, otherwise $\mathbf{u}_T(S) := 0$, whereas $T \subseteq N, T \neq \emptyset$. The collection of all unanimity games forms a **unanimity/game basis**. A formula to express the coordinates of this basis is given by

$$v = \sum_{\substack{T \subset N,\\ T \neq \emptyset}} \lambda^v_T\, \mathbf{u}_T \iff \lambda^v_T = \sum_{\substack{S \subset T,\\ S \neq \emptyset}} (-1)^{t-s} \cdot v(S),$$

if $\langle N, v\rangle$, where $|S| = s$, and $|T| = t$. A coordinate λ^v_T is said to be an unanimity coordinate of game $\langle N, v\rangle$, and vector λ^v is called the unanimity coordinates of game $\langle N, v\rangle$. Notice that we assume here that the game is defined in $\mathbb{R}^{2^n-1}$ rather than $\mathbb{R}^{2^n}$, since we want to write for sake of convenience the **game basis** in matrix form without a column and row of zeros. Thus we write

$$v = \sum_{\substack{T \subset N,\\ T \neq \emptyset}} \lambda^v_T\, \mathbf{u}_T = [\mathbf{u}_{\{1\}}, \ldots, \mathbf{u}_{\{N\}}]\, \lambda^v = \mathfrak{U}\, \lambda^v$$

where the unanimity basis $\mathfrak{U}$ is in $\mathbb{R}^{p' \times p'}$ with $p' = 2^n - 1$. In addition, define the **unity games (Dirac games)** $\mathbf{1}^T$ for all $T \subseteq N$ as: $\mathbf{1}^T(S) := 1$, if $T = S$, otherwise $\mathbf{1}^T(S) := 0$.

In the next step, we select a payoff vector $\vec{\gamma}$, which also determines its payoff set as a representative by $[\vec{\gamma}]$. With regard to Proposition 3.2, this vector induces in addition a set of lexicographically smallest most effective coalitions indicated by $\mathcal{S}(\vec{\gamma})$. Implying that we get the configuration $\vec{\alpha}$ by the q-coordinates $\alpha_{ij} := (v(S_{ij}) - v(S_{ji})) \in \mathbb{R}$ for all $i, j \in N, i < j$, and $\alpha_0 := v(N)$. Furthermore, we can also define a set of vectors as the differences of unity games w.r.t. the set of lexicographically smallest most effective coalitions, which is given by

$$\mathbf{v}_{ij} := \mathbf{1}^{S_{ij}} - \mathbf{1}^{S_{ji}} \quad \text{for } S_{ij}, S_{ji} \in \mathcal{S}(\vec{\gamma}) \quad \text{and} \quad \mathbf{v}_0 := \mathbf{1}^N, \tag{4.1}$$

whereas $\mathbf{v}_{ij}, \mathbf{v}_0 \in \mathbb{R}^{p'}$ for all $i, j \in N, i < j$. With these column vectors, we can identify matrix $\boldsymbol{\mathcal{V}} := [\mathbf{v}_{1,2}, \ldots, \mathbf{v}_{n-1,n}, \mathbf{v}_0] \in \mathbb{R}^{p' \times q}$. Then we obtain $\vec{\alpha} = \boldsymbol{\mathcal{V}}^\top v$ with $v \in \mathbb{R}^{p'}$ due to the removed empty set. Moreover, by the measure $y(S) := \sum_{k \in S} y_k$ for all $\emptyset \neq S \subseteq N$, we extend every payoff vector $\mathbf{y}$ to a vector $\overline{\mathbf{y}} \in \mathbb{R}^{p'}$, and define the excess vector at $\mathbf{y}$ by $\overline{e}_{\mathbf{y}} := v - \overline{\mathbf{y}} \in \mathbb{R}^{p'}$, then we get $\vec{\xi}_{\mathbf{y}} = \boldsymbol{\mathcal{V}}^\top \overline{e}_{\mathbf{y}}$. From matrix $\boldsymbol{\mathcal{V}}^\top$, we can also derive an orthogonal projection $\mathbf{P}_{\mathcal{V}}$ specified by $\boldsymbol{\mathcal{V}}^\top (\boldsymbol{\mathcal{V}}^\top)^\dagger \in \mathbb{R}^{q \times q}$ such that $\mathbb{R}^q = \mathcal{V} \oplus \mathcal{V}^\perp$ is valid, i.e. the rows of matrix $\boldsymbol{\mathcal{V}}^\top$ are a spanning system of the vector subspace $\mathcal{V} \subseteq \mathbb{R}^{q \times q}$, thus $\mathcal{V} := span\{\mathbf{v}_{1,2}^\top, \ldots, \mathbf{v}_{n-1,n}^\top, \mathbf{v}_0^\top\}$. Vector subspace $\mathcal{V}$ reflects the power of the set of lexicographically smallest most effective coalitions. In contrast, vector subspace $\mathcal{E}$ reflects the ascribed unbalancedness in the coalition power w.r.t. the bilateral bargaining situation obtained at $\vec{\gamma}$ through $\mathcal{S}(\vec{\gamma})$. The next results show how these vector subspaces are intertwined.

Lemma 4.1 (Meinhardt (2013)). *Let $\mathbf{E}^\top \in \mathbb{R}^{q \times n}$ be defined as in Equation (3.14), $\boldsymbol{\mathcal{V}}^\top \in \mathbb{R}^{q \times p'}$ as by Equation (4.1), then there exists a matrix $\mathbf{Z}^\top \in \mathbb{R}^{p' \times n}$ such that $\mathbf{E}^\top = \boldsymbol{\mathcal{V}}^\top \mathbf{Z}^\top$ if, and only if, $\mathcal{R}_{\mathbf{E}^\top} \subseteq \mathcal{R}_{\boldsymbol{\mathcal{V}}^\top}$, that is, $\mathcal{E} \subseteq \mathcal{V}$.*

Proof. The proof is given in Meinhardt (2013, p. 141). □

Notice that the minimal rank of matrix $\boldsymbol{\mathcal{V}}^\top$ is bounded by $\mathbf{E}^\top$ which is equal to $m < n$ with the consequence that we get in this case $\mathcal{V} = \mathcal{E}$. However, the maximal rank is equal to q, and then $\mathcal{V} = \mathbb{R}^q$ (cf. Meinhardt (2013, Corollary 7.4.1)).

Lemma 4.2 (Meinhardt (2013)). *Let $\vec{\alpha}, \vec{\xi} \in \mathbb{R}^q$ as in Equation (3.12), then the following relations are satisfied on the vector space $\mathcal{V}$:*

1. $\mathbf{P}_{\mathcal{V}}\, \vec{\alpha} = \vec{\alpha} \in \mathcal{V}$
2. $\mathbf{P}_{\mathcal{V}}\, \vec{\xi} = \vec{\xi} \in \mathcal{V}$
3. $\mathbf{P}_{\mathcal{V}}\, (\vec{\xi} - \vec{\alpha}) = (\vec{\xi} - \vec{\alpha}) \in \mathcal{V}$
4. $\mathbf{P}_{\mathcal{V}}\, \mathbf{E}^\top = \mathbf{P}\, \mathbf{E}^\top = \mathbf{E}^\top$, *hence* $\mathcal{E} \subseteq \mathcal{V}$
5. $\mathbf{E}\, \mathbf{P}_{\mathcal{V}} = \mathbf{E}\, \mathbf{P} = \mathbf{E}$, *hence* $\mathcal{R}_{\mathbf{E}} \subseteq \mathcal{V}$.

Proof. For a proof see Meinhardt (2013, p. 142). □

It was worked out by Meinhardt (2013, Sect. 7.6) that a pre-kernel element of a specific game can be replicated as a pre-kernel element of a related game whenever the non-empty interior property of the payoff set, in which the pre-kernel element of default game is located, is satisfied. In this case, a full dimensional ellipsoid can be inscribed from which some bounds can be specified within the game parameter can be varied without destroying the pre-kernel properties of the payoff vector of the default game. These bounds specify a redistribution of the bargaining power among coalitions while supporting the selected pre-imputation still as a pre-kernel point. Although the values of the maximum excesses have been changed by the parameter variation, the set of lexicographically smallest most significant coalitions remains unaffected.

Lemma 4.3 (Meinhardt (2013)). *If $\mathbf{x} \in M(h_\gamma^v)$, then $\mathbf{x} \in M(h_\gamma^{v^\mu})$ for all $\mu \in \mathbb{R}$, where $v^\mu := \boldsymbol{\mathcal{U}}(\lambda^v + \mu \Delta)$ and $\mathbf{0} \neq \Delta \in \mathcal{N}_{\boldsymbol{\mathcal{W}}} = \{\Delta \in \mathbb{R}^{p'} \mid \boldsymbol{\mathcal{W}} \Delta = \mathbf{0}\}$, where $\boldsymbol{\mathcal{W}} := \boldsymbol{\mathcal{V}}^\top \boldsymbol{\mathcal{U}} \in \mathbb{R}^{q \times p'}$.*

Proof. Let $\mathbf{x}$ be a minimizer of function h^v_γ under game v, then $\mathbf{x}$ remains a minimizer for a function $h^{v^\mu}_\gamma$ induced by game v^μ whenever $\mathbf{Q}\,\mathbf{x} = -2\,\mathbf{E}\,\vec{\alpha} = -\mathbf{a}$ remains valid. Since the payoff vector has induced the matrices $\mathbf{Q}, \mathbf{E}$ and matrix $\mathcal{V}$ defined by $[\mathbf{v}_{1,2}, \ldots, \mathbf{v}_{n-1,n}, \mathbf{v}_0]$, where the vectors are defined as by formula (4.1). We simply have to prove that the configuration $\vec{\alpha}$ remains invariant against an appropriate change in the game parameter. Observing that matrix $\mathcal{W}$ has a rank equal to or smaller than $q = \binom{n}{2} + 1$, say $m \leq q$, then the null space of matrix $\mathcal{W}$ has rank of $p' - m$, thus $\mathcal{N}_{\mathcal{W}} \neq \{\emptyset\}$. But then exists some $\mathbf{0} \neq \Delta \in \mathbb{R}^{p'}$ s.t. $\Delta \in \mathcal{N}_{\mathcal{W}}$ and $v^\mu = \mathcal{U}(\lambda^v + \mu\Delta)$ for $\mu \in \mathbb{R}\backslash\{\mathbf{0}\}$, getting

$$\mathcal{W}\;\lambda^{v^\mu} = \mathcal{W}\,(\lambda^v + \mu\Delta) = \mathcal{V}^\top\,(v + \mu v^\Delta) = \mathcal{V}^\top\,v = \vec{\alpha},$$

whereas $\mathcal{W}\,\Delta = \mathcal{V}^\top\,v^\Delta = \mathbf{0}$ with $v^\Delta := \mathcal{U}\,\Delta$. This argument proves that the configuration $\vec{\alpha}$ remains invariant against a change in the game parameter space by $v^\Delta \neq \mathbf{0}$. This implies that the payoff vector $\mathbf{x}$ is also a minimizer for function $h^{v^\mu}_\gamma$ under game v^μ. □

Lemma 4.4 (Meinhardt (2013))**.** *If $[\vec{\gamma}]$ has non-empty interior and $\mathbf{x} \in \mathcal{PK}(v) \subset [\vec{\gamma}]$, then there exists some critical bounds given by*

$$\delta^\varepsilon_{ij}(\mathbf{x}) = \frac{\pm\sqrt{\bar{c}}}{\|\mathbf{E}^\top(\mathbf{1}_j - \mathbf{1}_i)\|} \neq 0 \qquad \forall i,j \in N, i \neq j, \tag{4.2}$$

with $\bar{c} > 0$ and $\|\mathbf{E}^\top(\mathbf{1}_j - \mathbf{1}_i)\| > 0$.

Proof. Define a set $\varepsilon := \{\mathbf{y}\,|h^v_\gamma(\mathbf{y}) \leq \bar{c}\} \subset [\vec{\gamma}]$, whereas $h^v_\gamma(\mathbf{y}) = (1/2) \cdot \langle\, \mathbf{y}, \mathbf{Q}\,\mathbf{y}\,\rangle + \langle\, \mathbf{y}, \mathbf{a}\,\rangle + \alpha$. By assumption the payoff set $[\vec{\gamma}]$ has non-empty interior, we can say that ε is the ellipsoid of maximum volume obtained by Formula (3.15) that lies inside of the convex payoff set $[\vec{\gamma}]$. This ellipsoid must have a strictly positive volume, since the payoff equivalence class $[\vec{\gamma}]$ has non-empty interior, hence we conclude that $\bar{c} > 0$. Of course, the set ε is a convex subset of the convex set $[\vec{\gamma}]$, therefore $h^v = h^v_\gamma$ on ε. Moreover, the solution set $M(h^v_\gamma)$ is a subset of the ellipsoid ε, which is the smallest non-empty ellipsoid of the form (3.15), i.e., it is its center in view of Theorem 3.2. By our supposition $\mathcal{PK}(v) \subset [\vec{\gamma}]$, we conclude that $M(h^v) = M(h^v_\gamma) = \mathcal{PK}(v)$ must be satisfied. In the next step similar to Maschler et al. (1979), we define some critical numbers $\delta^\varepsilon_{ij}(\mathbf{x}) \in \mathbb{R}$ s.t.

$$\delta^\varepsilon_{ij}(\mathbf{x}) := \max\{\delta \in \mathbb{R}\,|\,\mathbf{x}^{\,i,j,\delta} = \mathbf{x} - \delta\,\mathbf{1}_i + \delta\,\mathbf{1}_j \in \varepsilon\} \qquad \forall\, i,j \in N, i \neq j. \tag{4.3}$$

That is, the number $\delta^\varepsilon_{ij}(\mathbf{x})$ is the maximum amount that can be transferred from i to j while remaining in the ellipsoid ε. This number is well defined for convex sets having non-empty interior.

In addition, observe that $\mathbf{x}^{\,i,j,\delta^\varepsilon} = \mathbf{x} - \delta^\varepsilon_{ij}(\mathbf{x})\,\mathbf{1}_i + \delta^\varepsilon_{ij}(\mathbf{x})\,\mathbf{1}_j$ is a unique boundary point of the ellipsoid ε of type (3.15) with maximum volume. Having specified by the point $\mathbf{x}^{\,i,j,\delta^\varepsilon}$ a boundary point, getting

$$\begin{aligned}
&h^v(\mathbf{x}^{\,i,j,\delta^\varepsilon}) = h^v_\gamma(\mathbf{x}^{\,i,j,\delta^\varepsilon}) = \bar{c} > 0 \iff \\
&\|\mathbf{E}^\top\,\mathbf{x}^{\,i,j,\delta^\varepsilon} + \vec{\alpha}\|^2 = \bar{c} \iff \|\mathbf{E}^\top\,\mathbf{x} + \vec{\alpha} + \delta^\varepsilon_{ij}(\mathbf{x})\,\mathbf{E}^\top(\mathbf{1}_j - \mathbf{1}_i)\|^2 = \bar{c} \iff \\
&\|\mathbf{E}^\top\,\mathbf{x} + \vec{\alpha}\|^2 + 2 \cdot \delta^\varepsilon_{ij}(\mathbf{x})\,\langle\,\mathbf{E}^\top\,\mathbf{x} + \vec{\alpha}, \mathbf{E}^\top(\mathbf{1}_j - \mathbf{1}_i)\,\rangle + (\delta^\varepsilon_{ij}(\mathbf{x}))^2\,\|\mathbf{E}^\top(\mathbf{1}_j - \mathbf{1}_i)\|^2 = \bar{c} \iff \\
&(\delta^\varepsilon_{ij}(\mathbf{x}))^2\,\|\mathbf{E}^\top(\mathbf{1}_j - \mathbf{1}_i)\|^2 = \bar{c} \qquad \forall\, i,j \in N, i \neq j.
\end{aligned}$$

The last conclusion follows, since by assumption we have $\mathbf{x} \in \mathcal{PK}(v)$, which is equivalent to $h^v(\mathbf{x}) = h^v_\gamma(\mathbf{x}) = 0$, and therefore we obtain $\mathbf{E}^\top\,\mathbf{x} + \vec{\alpha} = \mathbf{0}$. In addition, the volume of the ellipsoid ε is strictly positive such that $\bar{c} > 0$, this result implies that $(\delta^\varepsilon_{ij}(\mathbf{x}))^2$ as well as $\|\mathbf{E}^\top(\mathbf{1}_j - \mathbf{1}_i)\|^2$ must also be strictly positive. Therefore, we get finally (4.2). □

Theorem 4.1 (Meinhardt (2013)). *If $[\vec{\gamma}]$ has non-empty interior and $\mathbf{x} \in \mathcal{PK}(v) \subset [\vec{\gamma}]$, then $\mathbf{x} \in \mathcal{PK}(v^{\mu})$ for all $\mu \cdot v^{\Delta} \in [-\mathsf{C}, \mathsf{C}]^{p'}$, where $v^{\mu} = v + \mu \cdot v^{\Delta} \in \mathbb{R}^{p'}$, $\mu \in \mathbb{R}$*

$$\mathsf{C} := \min_{i,j \in N, i \neq j} \left\{ \left| \frac{\pm\sqrt{\bar{c}}}{\|\mathbf{E}^{\top}(\mathbf{1}_j - \mathbf{1}_i)\|} \right| \right\}, \tag{4.4}$$

and $\mathbf{0} \neq \Delta \in \mathcal{N}_{\mathcal{W}} = \{\Delta \in \mathbb{R}^{p'} \mid \mathcal{W}\Delta = \mathbf{0}\}$ with matrix $\mathcal{W} := \mathcal{V}^{\top}\,\mathcal{U}$.

Proof. By Lemma 4.4 $\mathbf{x}^{\,i,j,\delta^{\varepsilon}} \in \varepsilon \subset [\vec{\gamma}]$, is a unique boundary point of the ellipsoid ε of type (3.15) with maximum volume. We conclude that either (1) $s_{ij}(\mathbf{x}^{\,i,j,\delta^{\varepsilon}}) = s_{ij}(\mathbf{x}) + \delta^{\varepsilon}_{ij}(\mathbf{x})$ if $S \in \mathcal{G}_{ij}$, or (2) $s_{ji}(\mathbf{x}^{\,i,j,\delta^{\varepsilon}}) = s_{ji}(\mathbf{x}) - \delta^{\varepsilon}_{ij}(\mathbf{x})$ if $S \in \mathcal{G}_{ji}$, or otherwise (3) $s_{ij}(\mathbf{x}^{\,i,j,\delta^{\varepsilon}}) = s_{ij}(\mathbf{x})$ is satisfied. Moreover, let $v, v^{\mu}, v^{\Delta} \in \mathbb{R}^{p'}$ and recall that $v^{\mu} = \mathcal{U}(\lambda^{v} + \mu\Delta)$ with $\mathbf{0} \neq \Delta \in \mathcal{N}_{\mathcal{W}}$. Then it holds $v^{\mu}(S) = v(S) + \mu \cdot v^{\Delta}(S)$ for all $S \in 2^{n}\backslash\{\emptyset\}$. In the next step, extend the pre-kernel element $\mathbf{x}$ to a vector $\overline{\mathbf{x}}$ by the measure $x(S) := \sum_{k \in S} x_k$ for all $S \in 2^{n}\backslash\{\emptyset\}$, then define the excess vector by $\overline{e} := v - \overline{\mathbf{x}}$. Due to these definitions, we obtain for $\vec{\xi}^{\,v^{\mu}}$ at $\mathbf{x}$:

$$\vec{\xi}^{\,v^{\mu}} = \mathcal{V}^{\top}\,\overline{e}^{\mu} = \mathcal{V}^{\top}\,(v^{\mu} - \overline{\mathbf{x}}) = \mathcal{V}^{\top}\,(v - \overline{\mathbf{x}} + \mu \cdot v^{\Delta}) = \mathcal{V}^{\top}\,(v - \overline{\mathbf{x}}) = \mathcal{V}^{\top}\,\overline{e} = \vec{\xi} = \mathbf{0}.$$

By Lemma 4.3, the system of excesses remains balanced for all $\mu \in \mathbb{R}$. However, the system of maximum surpluses remains invariant on a hypercube specified by the critical values of the ellipsoid ε. Thus, for appropriate values of μ the expression $\mu \cdot v^{\Delta}(S)$ belongs to the non-empty interval $[-\mathsf{C}, \mathsf{C}]$ for $S \in 2^{n}\backslash\{\emptyset\}$. This interval specifies the range in which the game parameter can vary without having any impact on the set of most effective coalition given by $\mathcal{S}(\mathbf{x})$. Thus, the coalitions $\mathcal{S}(\mathbf{x})$ still have maximum surpluses for games defined by $v^{\mu} = \mathcal{U}(\lambda^{v} + \mu\Delta)$ for all $\mu\mathcal{U}\,\Delta = \mu \cdot v^{\Delta} \in [-\mathsf{C}, \mathsf{C}]^{p'}$. Hence the pre-kernel solution $\mathbf{x}$ is invariant against a change in the hypercube $[-\mathsf{C}, \mathsf{C}]^{p'}$. The conclusion follows. □

Meinhardt (2013, Sec. 7.6) has shown by some examples that the specified bounds by Theorem 4.1 are not tight, in the sense that pre-kernel points belonging to the relative interior of a payoff set can also be the object of a replication. However, pre-kernel elements which are located on the relative boundary of a payoff set are probably not replicable. Therefore, there must exist a more general rule to reproduce a pre-kernel element for a related game v^{μ}.

In the course of our discussion, we establish that the single pre-kernel element of a default game which is an interior point of a payoff set is also the singleton pre-kernel of the derived related games. In a first step, we show that there exists a unique linear transformation of the pre-kernel point of a related game into the vector subspace of balanced excesses $\mathcal{E}$. This means, there is no other pre-kernel element in a payoff equivalence class that belongs to the same set of ordered bases, i.e. reflecting an equivalent bargaining situation with a division of the proceeds of mutual cooperation in accordance with the pre-kernel solution. Secondly, we prove that there can not exist any other vector subspace of balanced excesses $\mathcal{E}_1$ non-transversal to $\mathcal{E}$ in which a pre-kernel vector can be mapped by a linear transformation. That is, there exists no other non-equivalent payoff set in which an other pre-kernel point can be located.

Lemma 4.5 (Meinhardt (2013)). *Let $\vec{\gamma}$ induces matrix $\mathbf{E}$, then*

$$(\mathbf{E}^{\top})^{\dagger} = 2 \cdot \mathbf{Q}^{\dagger}\mathbf{E} \in \mathbb{R}^{n \times q}.$$

Proof. Remind from Lemma 3.2 that $\mathbf{P} = 2 \cdot \mathbf{E}^{\top}\,\mathbf{Q}^{\dagger}\mathbf{E}$ holds. In addition, note that we have the following relation $\mathbf{Q}^{\dagger}\mathbf{Q} = (\mathbf{E}^{\top})^{\dagger}\,\mathbf{E}^{\top}$ which is an orthogonal projection onto $\mathcal{R}_{\mathbf{E}}$. Then obtaining

$$\begin{aligned} 2 \cdot \mathbf{Q}^{\dagger}\mathbf{E} &= 2 \cdot \mathbf{Q}^{\dagger}\mathbf{Q}\mathbf{Q}^{\dagger}\mathbf{E} = 2 \cdot (\mathbf{E}^{\top})^{\dagger}\mathbf{E}^{\top}\mathbf{Q}^{\dagger}\mathbf{E} \\ &= (\mathbf{E}^{\top})^{\dagger}(2 \cdot \mathbf{E}^{\top}\mathbf{Q}^{\dagger}\mathbf{E}) = (\mathbf{E}^{\top})^{\dagger}\mathbf{P} = (\mathbf{E}^{\top})^{\dagger}. \end{aligned}$$

The last equality follows from Lemma 4.2. This argument terminates the proof. □

Notice that in the sequel SO(n) denotes the special orthogonal group, whereas GL$^+(n)$ denotes the positive general linear group (cf. Meinhardt (2013, pp. 99-109)).

Proposition 4.1. *Let $E_1^\top = E^\top X$ with $X \in SO(n)$, that is $[\vec{\gamma}] \sim [\vec{\gamma}_1]$. In addition, assume that the payoff equivalence class $[\vec{\gamma}]$ induced from TU game $\langle N, v \rangle$ has non-empty interior such that $\{\mathbf{x}\} = \mathcal{PK}(v) \subset [\vec{\gamma}]$ is satisfied, then there exists no other pre-kernel element in payoff equivalence class $[\vec{\gamma}_1]$ for a related TU game $\langle N, v^\mu \rangle$, where $v^\mu = v + \mu \cdot v^\Delta \in \mathbb{R}^{p'}$, as defined by Lemma 4.3.*

Proof. By the way of contradiction suppose that $\mathbf{x}, \mathbf{y} \in \mathcal{PK}(v^\mu)$ with $\mathbf{y} \in [\vec{\gamma}_1]$ is valid. Then we get

$$h^{v^\mu}(\mathbf{x}) = h_\gamma^{v^\mu}(\mathbf{x}) = \|\mathbf{E}^\top \mathbf{x} + \vec{\alpha}\|^2 = 0 \quad \text{and} \quad h^{v^\mu}(\mathbf{y}) = h_{\gamma_1}^{v^\mu}(\mathbf{y}) = \|\mathbf{E}_1^\top \mathbf{y} + \vec{\alpha}_1\|^2 = 0,$$

implying that

$$\mathbf{P}\,\vec{\alpha} = \vec{\alpha} \in \mathcal{E} \qquad \text{and} \qquad \mathbf{P}\,\vec{\alpha}_1 = \vec{\alpha}_1 \in \mathcal{E}. \tag{4.5}$$

Moreover, we have $E_1^\top = E^\top X$ with $X \in \mathrm{SO}(n)$, then $\mathcal{E} \subseteq \mathcal{V} \cap \mathcal{V}_1$ in accordance with Lemma 7.4.1 by Meinhardt (2013). Now assume that $\vec{\alpha}_1 = \boldsymbol{\mathcal{V}}_1^\top v^\mu$ holds with $\mathcal{V}_1 \subseteq \mathcal{V}$. The latter supposition implies $\boldsymbol{\mathcal{V}}_1^\top = \mathbf{P}_\mathcal{V}\,\boldsymbol{\mathcal{V}}_1^\top$, since for every $\vec{\beta} \in \mathcal{V}$ we get $\vec{\beta} = \mathbf{P}_\mathcal{V}\,\vec{\beta}$ (cf. Remark 6.5.1 Meinhardt (2013)). According to $\mathcal{V}_1 \subseteq \mathcal{V}$ it also holds $\mathcal{N}_{\mathcal{W}_1} \supseteq \mathcal{N}_\mathcal{W}$. Our hypothesis $\mathbf{y} \in \mathcal{PK}(v^\mu)$ implies

$$\mathbf{0} = \mathbf{E}_1^\top \mathbf{y} + \vec{\alpha}_1 = \boldsymbol{\mathcal{V}}_1^\top \mathbf{Z}^\top \mathbf{y} + \boldsymbol{\mathcal{V}}_1^\top v^\mu = \boldsymbol{\mathcal{V}}_1^\top \mathbf{Z}^\top \mathbf{y} + \boldsymbol{\mathcal{V}}_1^\top (v + \mu \cdot v^\Delta) = \boldsymbol{\mathcal{V}}_1^\top (v - \overline{\mathbf{y}}),$$

whereas the vector of measures $\overline{\mathbf{y}}$ is expressed by $\overline{\mathbf{y}} = -\mathbf{Z}^\top \mathbf{y}$ (cf. Meinhardt (2013, p. 141)). The result $\boldsymbol{\mathcal{V}}_1^\top (v - \overline{\mathbf{y}}) = \mathbf{0}$ yields to $\mathbf{y} \in \mathcal{PK}(v)$, which is a contradiction. Therefore, we conclude that $\mathcal{V} \subset \mathcal{V}_1$ must be satisfied.

In addition, from $\vec{\alpha}_1 = \boldsymbol{\mathcal{V}}_1^\top v^\mu$ we obtain $\mathbf{P}_\mathcal{V}\,\vec{\alpha}_1 = \boldsymbol{\mathcal{V}}^\top (\boldsymbol{\mathcal{V}}^\top)^\dagger \boldsymbol{\mathcal{V}}_1^\top v^\mu \neq \boldsymbol{\mathcal{V}}_1^\top v^\mu = \vec{\alpha}_1$ in accordance with $\mathbf{P}_\mathcal{V}\,\boldsymbol{\mathcal{V}}_1^\top \neq \boldsymbol{\mathcal{V}}_1^\top$, in fact, it holds $\mathcal{V} \subset \mathcal{V}_1$. Thus, we have $\mathbf{P}_\mathcal{V}\,\vec{\alpha}_1 \notin \mathcal{V}$ contradicting that $\mathbf{P}_\mathcal{V}\,\vec{\alpha}_1 = \vec{\alpha}_1 \in \mathcal{E} \subseteq \mathcal{V} \subset \mathcal{V}_1$ holds true. From this, we conclude that $\vec{\alpha}_1 = \boldsymbol{\mathcal{V}}^\top v^\mu$ must be in force.

Furthermore, from (4.5) we have

$$\mathbf{P}\,\vec{\alpha} - \vec{\alpha} = \mathbf{P}\,\vec{\alpha}_1 - \vec{\alpha}_1 = \mathbf{0} \in \mathcal{E} \Longleftrightarrow \mathbf{P}\,(\vec{\alpha} - \vec{\alpha}_1) = (\vec{\alpha} - \vec{\alpha}_1) \in \mathcal{E}.$$

Therefore, obtaining the equivalent expression

$$\mathbf{E}^\top (X\,\mathbf{y} - \mathbf{x}) = (\vec{\alpha} - \vec{\alpha}_1) = \boldsymbol{\mathcal{V}}^\top v - \boldsymbol{\mathcal{V}}^\top (v + \mu \cdot v^\Delta) = \mathbf{0},$$

then $\mathbf{x} = X\,\mathbf{y}$, since matrix $\mathbf{E}^\top$ has full rank due to $\{\mathbf{x}\} = \mathcal{PK}(v)$. Furthermore, notice that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle (\mathbf{E}^\top)^\dagger \vec{\alpha}, (\mathbf{E}_1^\top)^\dagger \vec{\alpha}_1 \rangle = \langle (\mathbf{E}^\top)^\dagger \vec{\alpha}, X^{-1} (\mathbf{E}^\top)^\dagger \vec{\alpha} \rangle = \langle 2\,\mathbf{Q}^\dagger\,\mathbf{E}\,\vec{\alpha}, 2\,X^{-1}\,\mathbf{Q}^\dagger\,\mathbf{E}\,\vec{\alpha} \rangle \neq \mathbf{0}$$

Matrix $\mathbf{E}^\top$ has full rank, and $\mathbf{Q}$ is symmetric and positive definite, hence $\mathbf{Q}^\dagger = \mathbf{Q}^{-1}$, and the above expression can equivalently be written as

$$\begin{aligned} \langle \mathbf{Q}^\dagger\,\mathbf{a}, X^{-1}\,\mathbf{Q}^\dagger\,\mathbf{a} \rangle &= \langle \mathbf{Q}^{-1}\,\mathbf{a}, X^{-1}\,\mathbf{Q}^{-1}\,\mathbf{a} \rangle = \langle \mathbf{a}, \mathbf{Q}\,X^{-1}\mathbf{Q}^{-1}\,\mathbf{a} \rangle \\ &= \langle \mathbf{a}, X_1\mathbf{a} \rangle = \langle \mathbf{a}, \mathbf{a}_1 \rangle \neq \mathbf{0}, \end{aligned} \tag{4.6}$$

while using $\mathbf{a} = 2\,\mathbf{E}\,\vec{\alpha}$ from Proposition 3.4, and with similar matrix $X_1 = \mathbf{Q}\,X^{-1}\mathbf{Q}^{-1}$ as well as $\mathbf{a}_1 = X_1\,\mathbf{a}$. According to $\mathbf{E}_1^\top = \mathbf{E}^\top X$ with $X \in \mathrm{SO}(n)$, we can write $X = \mathbf{Q}^{-1}(2\,\mathbf{E}\,\mathbf{E}_1^\top)$. But then

$$X_1 = \mathbf{Q}\,X^{-1}\mathbf{Q}^{-1} = \mathbf{Q}\,(2\,\mathbf{E}\,\mathbf{E}_1^\top)^{-1}.$$

Since we have $X \in \text{SO}(n)$, it holds $X^{-1} = X^{\top}$ implying that

$$X_1^{\top} = X^{-1} = (2\,\mathbf{E}\,\mathbf{E}_1^{\top})^{-1}\,\mathbf{Q} = (2\,\mathbf{E}\,\mathbf{E}_1^{\top})\,\mathbf{Q}^{-1} = X^{\top} = X_1^{-1},$$

which induces $X = \mathbf{Q}^{-1}\,(2\,\mathbf{E}\,\mathbf{E}_1^{\top}) = \mathbf{Q}\,(2\,\mathbf{E}\,\mathbf{E}_1^{\top})^{-1} = X_1$. Now, observe

$$\begin{aligned} X_1 &= \mathbf{Q}\,X^{-1}\mathbf{Q}^{-1} = \mathbf{Q}\,X^{\top}\mathbf{Q}^{-1} = \mathbf{Q}\,(2\,\mathbf{E}\,\mathbf{E}_1^{\top})\,\mathbf{Q}^{-1}\,\mathbf{Q}^{-1} \\ &= \mathbf{Q}\,(2\,\mathbf{E}\,\mathbf{E}^{\top}\,X)\,\mathbf{Q}^{-2} = \mathbf{Q}^2\,X\,\mathbf{Q}^{-2}, \end{aligned}$$

hence, we can conclude that $X = \mathbf{I}$ implying $X_1 = \mathbf{I}$ as well. We infer that $\mathbf{x} = \mathbf{y}$ contradicting the assumption $\mathbf{x} \neq \mathbf{y}$ due to $\mathbf{x} \in [\vec{\gamma}]$, and $\mathbf{y} \in [\vec{\gamma}_1]$. With this argument we are done. □

Proposition 4.2. *Impose the same conditions as under Proposition 4.1 with the exception that $X \in GL^{+}(n)$, then there exists no other pre-kernel element in payoff equivalence class $[\vec{\gamma}_1]$ for a related TU game $\langle\, N, v^{\mu}\,\rangle$.*

Proof. By the proof of Proposition 4.1 the system of linear equations $\mathbf{E}^{\top}\,(X\,\mathbf{y} - \mathbf{x}) = \mathbf{0}$ is consistent, then we get $\mathbf{x} = X\,\mathbf{y}$ by the full rank of matrix $\mathbf{E}^{\top}$. By Equation 4.6 we obtain similar matrix $X_1 = \mathbf{Q}\,X^{-1}\mathbf{Q}^{-1}$, hence the matrix X_1 is in the same orbit (conjugacy class) as matrix X^{-1}, this implies that $E^{\top} = E_1^{\top}\,X^{-1} = E_1^{\top}\,X_1$ must be in force. But then $E^{\top} = E^{\top}\,X\,X_1$, which requires that $X\,X_1 = \mathbf{I}$ must be satisfied in view of the uniqueness of the transition matrix $X \in \text{GL}^{+}(m)$ (cf. Meinhardt (2013, p. 102)). In addition, we have $\mathbf{a}_1 = X_1\,\mathbf{a}$ as well as $\mathbf{a}_1 = 2\,\mathbf{E}_1\,\vec{\alpha} = X\,\mathbf{a}$. Therefore, we obtain $X\,\mathbf{a}_1 = \mathbf{a} = X^2\,\mathbf{a}$. From this we draw the conclusion in connection with the uniqueness of the transition matrix X that $X = \mathbf{I}$ is valid. Hence, $\mathbf{x} = \mathbf{y}$ as required. □

Proposition 4.3. *Assume $[\vec{\gamma}] \sim [\vec{\gamma}_1]$, and that the payoff equivalence class $[\vec{\gamma}]$ induced from TU game $\langle\, N, v\,\rangle$ has non-empty interior such that $\{\mathbf{x}\} = \mathcal{PK}(v) \subset [\vec{\gamma}]$ is satisfied, then there exists no other pre-kernel element in payoff equivalence class $[\vec{\gamma}_1]$ for a related TU game $\langle\, N, v^{\mu}\,\rangle$, where $v^{\mu} = v + \mu \cdot v^{\Delta} \in \mathbb{R}^{p'}$, as defined by Lemma 4.3.*

Proof. We have to establish that there is no other element $\mathbf{y} \in \mathcal{PK}(v^{\mu})$ such that $\mathbf{y} \in [\vec{\gamma}_1]$ is valid, whereas $\mathbf{y} \notin \mathcal{PK}(v)$ in accordance with the single-valuedness of the pre-kernel for game v. In view of Theorem 4.1 the pre-kernel $\{\mathbf{x}\} = \mathcal{PK}(v)$ of game $\langle\, N, v\,\rangle$ is also a pre-kernel element of the related game $\langle\, N, v^{\mu}\,\rangle$, i.e. $\mathbf{x} \in \mathcal{PK}(v^{\mu})$ with $\mathbf{x} \in [\vec{\gamma}]$ due to Corollary 3.2.

Extend the payoff element $\mathbf{y}$ to a vector $\overline{\mathbf{y}}$ by the measure $y(S) := \sum_{k \in S} y_k$ for all $S \in 2^n \backslash \{\emptyset\}$, then define the excess vector by $\overline{e}^{\mu} := v^{\mu} - \overline{\mathbf{y}}$. Moreover, compute the vector of unbalanced excesses $\vec{\xi}^{\,v^{\mu}}$ at $\mathbf{y}$ for game v^{μ} by $\mathcal{V}_1^{\top}\,\overline{e}^{\mu}$. This vector is also the vector of unbalanced maximum surpluses, since $\mathbf{y} \in [\vec{\gamma}_1]$, and therefore $h^{v^{\mu}} = h^{v^{\mu}}_{\gamma_1}$ on $[\vec{\gamma}_1]$ in view of Lemma 6.2.2 by Meinhardt (2013). Notice that in order to have a pre-kernel element at $\mathbf{y}$ for the related game v^{μ} it must hold $\vec{\xi}^{\,v^{\mu}} = \mathbf{0}$. In addition, by hypothesis $[\vec{\gamma}] \sim [\vec{\gamma}_1]$, it must hold $\mathbf{E}^{\top} = \mathcal{V}^{\top}\,\mathbf{Z}^{\top}$ and $\mathbf{E}_1^{\top} = \mathcal{V}_1^{\top}\,\mathbf{Z}^{\top}$ in view of Lemma 4.1, thus $E_1^{\top} \neq E^{\top}\,X$ for all $X \in \text{GL}^{+}(n)$. This implies that we derive the corresponding matrices $\mathcal{W} := \mathcal{V}^{\top}\,\mathcal{U}$ and $\mathcal{W}_1 := \mathcal{V}_1^{\top}\,\mathcal{U}$, respectively.

We have to consider two cases, namely $\Delta \in \mathcal{N}_{\mathcal{W}} \cap \mathcal{N}_{\mathcal{W}_1}$ and $\Delta \in \mathcal{N}_{\mathcal{W}} \backslash \mathcal{N}_{\mathcal{W}_1}$.

1. Suppose $\Delta \in \mathcal{N}_{\mathcal{W}} \cap \mathcal{N}_{\mathcal{W}_1}$, then we get

$$\vec{\xi}^{\,v^{\mu}} = \mathcal{V}_1^{\top}\,\overline{e}^{\mu} = \mathcal{V}_1^{\top}\,(v^{\mu} - \overline{\mathbf{y}}) = \mathcal{V}_1^{\top}\,(v - \overline{\mathbf{y}} + \mu \cdot v^{\Delta}) = \mathcal{V}_1^{\top}\,(v - \overline{\mathbf{y}}) = \mathcal{V}_1^{\top}\,\overline{e} = \vec{\xi}^{\,v} \neq \mathbf{0}.$$

Observe that $\vec{\xi}^{\,v} = \mathcal{V}_1^{\top}\,(v - \overline{\mathbf{y}}) \neq \mathbf{0}$, since vector $\mathbf{y} \in [\vec{\gamma}_1]$ is not a pre-kernel element of game v.

2. Now suppose $\Delta \in \mathcal{N}_{\mathcal{W}} \backslash \mathcal{N}_{\mathcal{W}_1}$, then

$$\vec{\xi}^{v^{\mu}} = \mathcal{V}_1^{\top}\,\overline{e}^{\mu} = \mathcal{V}_1^{\top}\,(v^{\mu} - \overline{\mathbf{y}}) = \mathcal{V}_1^{\top}\,(v - \overline{\mathbf{y}} + \mu \cdot v^{\Delta}) = \mathcal{V}_1^{\top}\,\overline{e} + \mu \cdot \mathcal{V}_1^{\top}\,v^{\Delta} = \vec{\xi}^{v} + \mu \cdot \mathcal{V}_1^{\top}\,v^{\Delta} \neq \mathbf{0}.$$

Since, we have $\mathcal{V}_1^{\top}\,(v - \overline{\mathbf{y}}) \neq \mathbf{0}$ as well as $\mathcal{V}_1^{\top}\,v^{\Delta} \neq \mathbf{0}$, and $\mathcal{V}_1^{\top}\,v^{\Delta}$ can not be expressed by $-\mathcal{V}_1^{\top}\,(v - \overline{\mathbf{y}})$ in accordance with our hypothesis. To see this, suppose that the vector Δ is expressible in this way, then it must hold $\Delta = -\frac{1}{\mu}\,(\mathcal{W}_1)^{\dagger}\,\vec{\xi}^{v}$. However, this implies

$$\mathcal{W}\,\Delta = -\frac{1}{\mu}\,\mathcal{W}\,(\mathcal{W}_1)^{\dagger}\,\vec{\xi}^{v} = -\frac{1}{\mu}\,(\mathcal{V}^{\top}\,\mathcal{U})\,(\mathcal{V}_1^{\top}\,\mathcal{U})^{\dagger}\,\vec{\xi}^{v} = -\frac{1}{\mu}\,\mathcal{V}^{\top}\,(\mathcal{V}_1^{\top})^{\dagger}\,\vec{\xi}^{v} \neq \mathbf{0}.$$

This argument terminates the proof. □

To complete the single-valuedness investigation, we need to establish that the single pre-kernel element of the default game also preserves the pre-nucleolus property for the related games, otherwise we can be sure that there must exist at least a second pre-kernel point for the related game different from the first one. For doing so, we introduce the following set:

Definition 4.1. *For every $\mathbf{x} \in \mathbb{R}^n$, and $\psi \in \mathbb{R}$ define the set*

$$\mathcal{D}^{v}(\psi, \mathbf{x}) := \{S \subseteq N \,|\, e^{v}(S, \mathbf{x}) \geq \psi\}\,, \tag{4.7}$$

and let $\mathcal{B} = \{S_1, \ldots, S_m\}$ be a collection of non-empty sets of N. We denote the collection $\mathcal{B}$ as balanced whenever there exist positive numbers w_S for all $S \in \mathcal{B}$ such that we have $\sum_{S \in \mathcal{B}} w_S \mathbf{1}_S = 1_N$. The numbers w_S are called weights for the balanced collection $\mathcal{B}$. Be reminded that $\mathbf{1}_S$ is the indicator function or characteristic vector $\mathbf{1}_S : N \mapsto \{0, 1\}$ given by $\mathbf{1}_S(k) := 1$ if $k \in S$, otherwise $\mathbf{1}_S(k) := 0$.

A characterization of the pre-nucleolus in terms of balanced collections is due to Kohlberg (1971).

Theorem 4.2. *Let $\langle\, N, v\,\rangle$ be a TU game and let be $\mathbf{x} \in \mathcal{I}^{*}(v)$. Then $\mathbf{x} = \nu(N, v)$ if, and only if, for every $\psi \in \mathbb{R}$, $\mathcal{D}^{v}(\psi, \mathbf{x}) \neq \emptyset$ implies that $\mathcal{D}^{v}(\psi, \mathbf{x})$ is a balanced collection over N.*

Theorem 4.3. *Let $\langle\, N, v\,\rangle$ be a TU game that has a singleton pre-kernel such that $\{\mathbf{x}\} = \mathcal{PK}(v) \subset [\vec{\gamma}]$, and let $\langle\, N, v^{\mu}\,\rangle$ be a related game of v derived from $\mathbf{x}$, then $\mathbf{x} = \nu^{*}(N, v^{\mu})$, whereas the payoff equivalence class $[\vec{\gamma}]$ has non-empty interior.*

Proof. By our hypothesis, $\mathbf{x} = \nu(N, v)$ is an interior point of an inscribed ellipsoid with maximum volume $\varepsilon := \{\mathbf{y}' \,|\, h_{\gamma}^{v}(\mathbf{y}') \leq \bar{c}\} \subset [\vec{\gamma}]$, whereas h_{γ}^{v} is of type (3.15) and $\bar{c} > 0$ (cf. Lemma 4.4). This implies by Theorem 4.1 that this point is also a pre-kernel point of game v^{μ}, there is no change in set of lexicographically smallest most effective coalitions $\mathcal{S}(\mathbf{x})$ under v^{μ}. The min-max excess value ψ^{*} obtained by iteratively solving the LP (6.4-6.7) of Maschler et al. (1979, p. 332) for game v is smaller than the maximum surpluses derived from $\mathcal{S}(\mathbf{x})$, this implies that there exists a $\bar{\psi} \geq \psi^{*}$ s.t. $\mathcal{S}(\mathbf{x}) \subseteq \mathcal{D}^{v}(\bar{\psi}, \mathbf{x})$, that is, it satisfies Property I of Kohlberg (1971). Moreover, matrix $\mathbf{E}^{\top}$ induced from $\mathcal{S}(\mathbf{x})$ has full rank, therefore, the column vectors of matrix $\mathbf{E}^{\top}$ are a spanning system of $\mathbb{R}^n$. Hence, we get *span* $\{\mathbf{1}_S \,|\, S \in \mathcal{S}(\mathbf{x})\} = \mathbb{R}^n$, which implies that the corresponding matrix $[\mathbf{1}_S]_{S \in \mathcal{S}(\mathbf{x})}$ must have rank n, therefore collection $\mathcal{S}(\mathbf{x})$ is balanced. In addition, we can choose the largest $\psi \in \mathbb{R}$ s.t. $\emptyset \neq \mathcal{D}^{v}(\psi, \mathbf{x}) \subseteq \mathcal{S}(\mathbf{x})$ is valid, which is a balanced set. Furthermore, we have $\mu \cdot v^{\Delta} \in [-\mathsf{C}, \mathsf{C}]^{p'}$. Since $\mathsf{C} > 0$, the set $\mathcal{D}^{v}(\psi - 2\,\mathsf{C}, \mathbf{x}) \neq \emptyset$ is balanced as well. Now observe that $e^{v}(S, \mathbf{x}) - \mathsf{C} \leq e^{v}(S, \mathbf{x}) + \mu \cdot v^{\Delta}(S) \leq e^{v}(S, \mathbf{x}) + \mathsf{C}$ for all $S \subseteq N$. This implies $\mathcal{D}^{v}(\psi, \mathbf{x}) \subseteq \mathcal{S}(\mathbf{x}) \subseteq \mathcal{D}^{v^{\mu}}(\psi - \mathsf{C}, \mathbf{x}) \subseteq \mathcal{D}^{v}(\psi - 2\,\mathsf{C}, \mathbf{x})$, hence, $\mathcal{D}^{v^{\mu}}(\psi - \mathsf{C}, \mathbf{x})$ is balanced. Let $c \in [-\mathsf{C}, \mathsf{C}]$, and from the observation $\lim_{c \uparrow 0} \mathcal{D}^{v^{\mu}}(\psi + c, \mathbf{x}) = \mathcal{D}^{v^{\mu}}(\psi, \mathbf{x}) \supseteq \mathcal{D}^{v}(\psi, \mathbf{x})$, we draw the conclusion $\mathbf{x} = \nu(N, v^{\mu})$. □

Theorem 4.4. *Assume that the payoff equivalence class $[\vec{\gamma}]$ induced from TU game $\langle N, v \rangle$ has non-empty interior. In addition, assume that game $\langle N, v \rangle$ has a singleton pre-kernel such that $\{\mathbf{x}\} = \mathcal{PK}(v) \subset [\vec{\gamma}]$ is satisfied, then the pre-kernel $\mathcal{PK}(v^{\mu})$ of a related TU game $\langle N, v^{\mu} \rangle$, as defined by Lemma 4.3, consists of a single point, which is given by $\{\mathbf{x}\} = \mathcal{PK}(v^{\mu})$.*

Proof. This result follows from Theorems 4.1, 4.3, and Propositions 4.2, 4.3. □

REMARK 4.1.
Theorem 4.4 states that an agreement point based on the norms of distributive arbitration of the pre-kernel remains stable, whenever the parameter change within the characteristic function does not exceed a certain bound, i.e., the absolute redistribution of bargaining power among the coalitions that materializes by game v^{μ} is not too large to disturb the relative order of bargaining power between the coalitions. Then the original contract based on the pre-kernel element $\mathbf{x}$ remains stable, and cannot be obstructed while referring to the principles of distributive justice related to those. That is to say that the underlying bargaining agenda must remain stable. This implicates that in case that subjects have agreed upon that their standard of fairness is best reflected by the pre-kernel solution, they have to single out a new bargaining agreement under this understanding when a variation within the game parameter space changes the underlying negotiation structure. Changing in this context to another solution like the Shapley value implies that the preferences over the set of norms of distributive arbitration are not stable. But then any kind of stability analysis must be obsolete, which must lie at the heart of any economical study (cf., for instance, with Meinhardt (2002, Chap. 2), Meinhardt (2017)). Furthermore, we observe by this kind of stability analysis that it makes only sense for solution concepts which exist on every game class. Narrowing the set of solution concepts of any game theoretical and practical relevance considerably. For a practical application, we refer to Meinhardt (2018a). ◇

To summarize: by the above consideration we have established a single-valuedness result of the pre-kernel by conducting the following steps: (1) The linear mapping of a pre-kernel element into a specific vector subspace of balanced excesses $\mathcal{E}$ consists of a single point. (2) There can not exist any other non-transversal vector subspace of balanced excesses $\mathcal{E}_1$ in which a linear transformation of pre-kernel element can be mapped. (3) The pre-kernel coincides with the pre-nucleolus of the set of related games. This excludes the possibility that there must exist at least a second pre-kernel point, namely the pre-nucleolus.

Example 4.1. In order to illuminate the foregoing discussion of replicating a pre-kernel element consider a four person average-convex but non-convex game that is specified by

$$v(N) = 16, v(\{1,2,3\}) = v(\{1,2,4\}) = v(\{1,3,4\}) = 8,$$
$$v(\{1,3\}) = 4, v(\{1,4\}) = 1, v(\{1,2\}) = 16/3, v(S) = 0 \text{ otherwise},$$

with $N = \{1,2,3,4\}$. Further inspection reveals that this game has a veto-player, who is player one. Hence, apart from average-convexity, the game is even a veto-rich game, but then the pre-kernel must coalesce with the pre-nucleolus (cf. Arin and Feltkamp (1997)), whereas its outcome is given by the point: $\nu(v) = \mathcal{PK}(v) = (44/9, 4, 32/9, 32/9)$. Obviously, the set $\mathcal{S}(\nu(v)) = \{\{2\},\{3\},\{4\},\{1,2\},\{1,3,4\}\}$ is balanced, from this set a boundary vector $\vec{b} = (4, 32/9, 32/9, 80/9, 12)$ is obtained by $\nu(v)(S)$ for $S \in \mathcal{S}(\nu(v))$. Define matrix $\mathbf{A}$ by $[\mathbf{1}_S]_{S \in \mathcal{S}(\nu(v))}$, then the solution of the system $\mathbf{A}\,\mathbf{x} = \vec{b}$ reproduces the pre-nucleolus. Moreover, this imputation is even an interior point. To see this, select an $\epsilon > 0$ sufficiently small and let $\mathbf{z} \in \mathbb{R}^4$ s.t. $z(N) = 0$, then one can establish that $\mathcal{S}(\nu(v) + \epsilon\,\mathbf{z}) = \mathcal{S}(\nu(v))$ holds. Thus, the non-empty interior condition is valid. Hence, by Theorem 4.1 a redistribution of the bargaining power among coalitions can be obtained while supporting the imputation $(44/9, 4, 32/9, 32/9)$ still as a pre-kernel element for a set of related games. In order to get a null space $\mathcal{N}_{\mathcal{W}}$ with maximum dimension we set the parameter μ to 0.9. In this case, the rank of matrix $\mathcal{W}$ must be equal to 4, and we could derive at most 11-linear independent games which replicate the point $(44/9, 4, 32/9, 32/9)$ as a pre-kernel element. Theorem 4.4 even states that this point is also the sole pre-kernel point, hence the pre-kernel coincide with the pre-nucleolus for these games (see Table 4.1).

Furthermore, as it will be established in the forthcoming section, the pre-kernel is stable within a fixed game setting. However, it is not robust against a change in the bargaining agenda, i.e., against a change of the figure of argumentation during a bargaining process. That is, shifting the exchange of proposals from the α-argumentation[2] to the γ-argumentation[3], for instance, might make a pre-kernel agreement vulnerable (cf. Meinhardt (2018a)). To observe its stability, we just mention here that the convex hull of the collection of games $\{v, \ldots, v_{11}\}$ form a set in the game space – the game setting has not changed and with it not the figure of argumentation during the bargaining process, only the bargaining power of the coalitions has been changed. Within this convex set, each TU game has exactly the above element as its sole pre-kernel point. This implies that we have identified a stable bargaining scenario, where a settlement of an agreement is not problematic while referring to the principles of distributive justice related to the pre-kernel. The agreement based on these principles remains stable after having varied the parameter set within the specified range. Therefore, the selected agreement point cannot be obstructed by the principles of distributive justice related to the pre-kernel within this subset of the game space. The incentive to cooperate remains valid.

In addition note that none of these 11-linear independent related games is average-convex. Only two games, namely v_1 and v_3 are zero-monotonic and super-additive. Nevertheless, all games have a non-empty core and are semi-convex. The cores of the games have between 16 and 24-vertices, and have volumes that range from approximately 80 to 127 percent of the default core. TU game v_2 has the smallest and v_3 the largest core.[4]

Table 4.1: List of Games[f] which possess the same unique Pre-Kernel[a] as v

				$\mu = 0.9$				
Game	$\{1\}$	$\{2\}$	$\{1,2\}$	$\{3\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$	$\{4\}$
v	0	0	16/3	0	4	0	8	0
v_1	18/49	32/95	127/24	-1/24	256/59	4/13	175/22	-1/24
v_2	-9/25	21/38	89/16	11/48	231/58	42/71	385/47	11/48
v_3	-14/45	-1/40	201/41	-28/65	39/11	-19/44	142/19	-28/65
v_4	0	0	16/3	0	159/47	16/33	107/14	0
v_5	0	0	16/3	0	149/40	-37/102	497/66	0
v_6	0	0	16/3	0	4	-5/47	143/19	0
v_7	0	0	16/3	0	4	-5/47	143/19	0
v_8	0	0	16/3	0	149/40	-37/102	497/66	0
v_9	0	0	16/3	0	149/40	-37/102	497/66	0
v_{10}	0	0	16/3	0	4	-5/47	143/19	0
v_{11}	0	0	16/3	0	4	-5/47	143/19	0

Continued on next page

[2]In the sense of Aumann (1961).

[3]In the sense of Hart and Kurz (1983).

[4]The example can be reproduced while using our MATLAB toolbox *MatTuGames* 2020b. The results can also be verified with our Mathematica package *TuGames* 2020a.

Table 4.1 – continued from previous page

Game	$\{1,4\}$	$\{2,4\}$	$\{1,2,4\}$	$\mu = 0.9$ $\{3,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	N	ACV [d]	ZM [e]
v	1	0	8	0	8	0	16	Y	Y
v_1	79/59	4/13	175/22	-4/57	792/95	10/33	16	N	Y
v_2	57/58	42/71	385/47	4/7	325/38	31/56	16	N	N
v_3	6/11	-19/44	142/19	-27/47	319/40	-29/55	16	N	Y
v_4	41/34	-3/46	428/53	7/34	8	14/25	16	N	N
v_5	203/120	2/41	167/19	-5/24	8	-9/19	16	N	N
v_6	1	23/29	139/16	0	8	18/31	16	N	N
v_7	1	-5/47	139/16	0	8	-8/25	16	N	N
v_8	19/24	2/41	71/9	83/120	8	26/61	16	N	N
v_9	19/24	2/41	71/9	-5/24	8	-9/19	16	N	N
v_{10}	1	-5/47	475/61	0	8	18/31	16	N	N
v_{11}	1	-5/47	475/61	0	8	-8/25	16	N	N

[a] Pre-Kernel and Pre-Nucleolus: $(44/9, 4, 32/9, 32/9)$
[b] Modiclus: $\varsigma^*(N, v) = (8, 53/12, 37/12, 1/2)$
[c] Shapley Value: $\phi(N, v) = (247/36, 121/36, 113/36, 95/36)$
[d] ACV: Average-Convex Game
[e] ZM: Zero-Monotonic Game
[f] Note: Computation performed with MatTuGames (cf. Meinhardt (2020b)).

To visualize some results, we have plotted the core of game v as a yellow polytope in connection with the imputation set (skeleton-like triangle), the Shapley value (blue enlarged dot), modiclus (green enlarged dot; forthcoming in Section 8), and pre-nucleolus (red enlarged dot), which is identical to the nucleolus. By the figure, we observe that all mentioned point solutions are belonging to the core.[5]. #

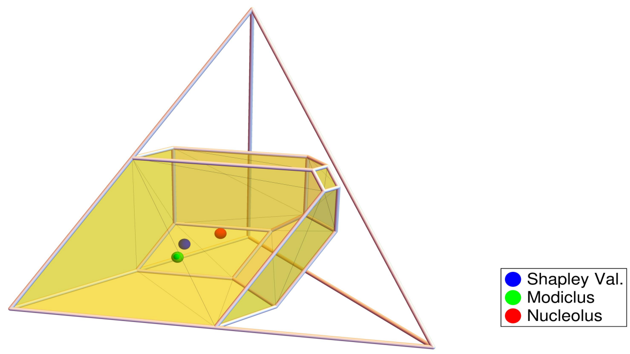

Figure 1: The Imputation Set, Core, Modiclus, Nucleolus, and Shapley value of Game v

[5]The figures have been generated with the graphical extensions of the Mathematica Package TuGames implemented within Meinhardt (2020a).

Different applications of the theorem can be found in Meinhardt (2013, Sec. 7.6) and in Meinhardt (2018a). The latter reference sets this finding in the context of industrial cooperation (cf. Zhao (2018)), i.e., cooperative oligopoly games with transferable technologies. In particular, to investigate the stability of a cartel agreement against a redistribution of the bargaining power among coalitions or the figure of argumentation.

5 On the Continuity of the Pre-Kernel

In the previous section, we have established single-valuedness of the pre-kernel on the set of related games. Here, we generalize these results while showing that even on the convex hull comprising the default and related games in the game space, the pre-kernel must be a single point and is identical with the element specified by the default game. Furthermore, the pre-kernel correspondence restricted on this convex subset in the game space must be single-valued, and therefore continuous.

Recall that the relevant game space is defined through $\mathcal{G}(N) := \{v \in \mathcal{G}^n \,|\, v(\emptyset) = 0 \wedge v(N) > 0\}$, and

$$\mathcal{G}^n_{\mu,v} := \left\{ v^{\mu} \in \mathcal{G}(N) \,|\, \mu \cdot v^{\Delta} \in [-\mathsf{C}, \mathsf{C}]^{p'} \right\}.$$

This set is the translate of a convex set by v, which is also convex and non-empty with dimension $p' - m'$, if matrix $\mathcal{W}$ has rank $m' \leq q < p'$. Then we can construct a convex set in the game space $\mathcal{G}(N)$ by taking the convex hull of game v and the convex set $\mathcal{G}^n_{\mu,v}$, thus

$$\mathcal{G}^n_c := conv\ \{v, \mathcal{G}^n_{\mu,v}\}.$$

Theorem 5.1. *The pre-kernel $\mathcal{PK}(v^{\mu^*})$ of game v^{μ^*} belonging to $\mathcal{G}^n_c$ is a singleton, and is equal to $\{\mathbf{x}\} = \mathcal{PK}(v)$.*

Proof. Let be $\{\mathbf{x}\} = \mathcal{PK}(v)$ for game v. Take a convex combination of games in $\mathcal{G}^n_c$, hence

$$v^{\mu^*} = \sum_{k=1}^{m} t_k \cdot v^{\mu}_k + t_{m+1} \cdot v = \sum_{k=1}^{m} t_k \cdot (v + \mu \cdot v^{\Delta}_k) + t_{m+1} \cdot v = v + \mu \sum_{k=1}^{m} t_k \cdot v^{\Delta}_k + \mu\, t_{m+1} \cdot \mathbf{0} = v + \mu \cdot v^{\Delta^*},$$

with $v^{\Delta^*} := \sum_{k=1}^{m} t_k \cdot v^{\Delta}_k + t_{m+1} \cdot \mathbf{0}$, where $0 \leq t_k \leq 1, \forall k \in \{1, 2, \ldots, m+1\}$, and $\sum_{k=1}^{m+1} t_k = 1$. Then $\mu\, v^{\Delta^*} \in [-\mathsf{C}, \mathsf{C}]^{p'}$, thus the set of lexicographically smallest coalitions $\mathcal{S}(\mathbf{x})$ does not change. By Theorem 4.1 the vector $\{\mathbf{x}\} = \mathcal{PK}(v)$ is also a pre-kernel element of game v^{μ^*}. But then by Theorem 4.4 the pre-kernel of game v^{μ^*} consists of a single point, therefore $\{\mathbf{x}\} = \mathcal{PK}(v^{\mu^*})$. □

Example 5.1. To see that even on the convex hull $\mathcal{G}^4_c$, which is constituted by the default and related games of Table 4.1, a particular TU game has the same singleton pre-kernel, we choose the following vector of scalars $\vec{t} = (1, 3, 8, 1, 2, 4, 3, 5, 7, 9, 2, 3)/48$ such that $\sum_{k=1}^{12} t_k = 1$ is given to construct by the convex combination of games presented by Table 4.1 a TU game v^{μ^*} that reproduces the imputation $(44/9, 4, 32/9, 32/9)$ as its sole pre-kernel. The TU game v^{μ^*} on this convex hull in the game space that replicates this pre-kernel is listed through Table 5.1:

Table 5.1: A TU Game v^{μ^*} on $\mathcal{G}_c^4$ with the same singleton Pre-Kernel as v [a,b]

S	$v^{\mu^*}(S)$	S	$v^{\mu^*}(S)$	S	$v^{\mu^*}(S)$	S	$v^{\mu^*}(S)$
$\{1\}$	$-1/23$	$\{1,2\}$	$134/25$	$\{2,4\}$	$173/1125$	$\{1,3,4\}$	$576/71$
$\{2\}$	$8/71$	$\{1,3\}$	$530/137$	$\{3,4\}$	$19/144$	$\{2,3,4\}$	$15/232$
$\{3\}$	$2/75$	$\{1,4\}$	$179/178$	$\{1,2,3\}$	$1436/187$	N	16
$\{4\}$	$2/75$	$\{2,3\}$	$-8/157$	$\{1,2,4\}$	$1946/239$		

[a] Pre-Kernel and Pre-Nucleolus: $(44/9, 4, 32/9, 32/9)$
[b] Note: Computation performed with MatTuGames (cf. Meinhardt (2020b)).

This game is neither average-convex nor zero-monotonic, however, it is again semi-convex and has a rather large core with a core volume of 97 percent w.r.t. the core of the average-convex game, and 20 vertices in contrast to 16 vertices respectively. #

Let $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces. A set-valued function or correspondence σ of $\mathcal{X}$ into $\mathcal{Y}$ is a rule that assigns to every element $x \in \mathcal{X}$ a non-empty subset $\sigma(x) \subset \mathcal{Y}$. Given a correspondence $\sigma : \mathcal{X} \twoheadrightarrow \mathcal{Y}$, the corresponding graph of σ is defined by

$$Gr(\sigma) := \{(x, y) \in \mathcal{X} \times \mathcal{Y} \mid y \in \sigma(x)\}\,. \tag{5.1}$$

Definition 5.1. *A set-valued function $\sigma : \mathcal{X} \twoheadrightarrow \mathcal{Y}$ is closed, if $Gr(\sigma)$ is a closed subset of $\mathcal{X} \times \mathcal{Y}$*

The graph of the pre-kernel correspondence $\mathcal{PK}$ is given by

$$Gr(\mathcal{PK}) := \left\{(v, \mathbf{x}) \mid v \in \mathcal{G}^n, \mathbf{x} \in \mathcal{I}^0(v),\;\; s_{ij}(\mathbf{x}, v) = s_{ji}(\mathbf{x}, v) \quad \text{for all } i, j \in N, i \neq j \right\}.$$

Similar, the graph of the solution set of function h^v of type (3.3) is specified by

$$\begin{aligned} Gr(M(h^v)) &:= \left\{(v, \mathbf{x}) \mid v \in \mathcal{G}^n, \mathbf{x} \in \mathcal{I}^0(v),\;\; h^v(\mathbf{x}) = 0 \right\} \\ &= \bigcup_{k \in \mathcal{J}'} \left\{(v, \mathbf{x}) \mid v \in \mathcal{G}^n, \mathbf{x} \in \overline{[\vec{\gamma}_k]},\;\; h^v_{\gamma_k}(\mathbf{x}) = 0 \right\} = \bigcup_{k \in \mathcal{J}'} Gr(M(h^v_{\gamma_k}, \overline{[\vec{\gamma}_k]})), \end{aligned}$$

with $\mathcal{J}' := \{k \in \mathcal{J} \mid g(\vec{\gamma}_k) = 0\}$. This graph is equal to the finite union of graphs of the restricted solution sets of quadratic and convex functions $h^v_{\gamma_k}$ of type (3.15). The restriction of each solution set of function $h^v_{\gamma_k}$ to $\overline{[\vec{\gamma}_k]}$ is bounded, closed, and convex (cf. Meinhardt (2013, Lemmata 7.1.3, 7.3.1)), hence each graph $Gr(M(h^v_{\gamma_k}, \overline{[\vec{\gamma}_k]}))$ from the finite index set $\mathcal{J}'$ is bounded, closed and convex.

Proposition 5.1. *The following relations are satisfied between the above graphs:*

$$Gr(\mathcal{PK}) = Gr(M(h^v)) = \bigcup_{k \in \mathcal{J}'} Gr(M(h^v_{\gamma_k}, \overline{[\vec{\gamma}_k]})). \tag{5.2}$$

Hence, the pre-kernel correspondence $\mathcal{PK} : \mathcal{G}(N) \twoheadrightarrow \mathbb{R}^n$ is closed and bounded.

Proof. The equality of the graph of the pre-kernel and the solution set of function h^v follows in accordance with Corollary 3.1. Finally, the last equality is a consequence of Theorem 7.3.1 by Meinhardt (2013). From this argument boundedness and closedness follows. □

Definition 5.2. *The correspondence $\sigma : \mathcal{X} \twoheadrightarrow \mathcal{Y}$ is said to be upper hemi-continuous (**uhc**) at x if for every open set $\mathcal{O}$ containing $\sigma(x) \subseteq \mathcal{O}$ it exists an open set $\mathcal{Q} \subseteq \mathcal{Y}$ of x such that $\sigma(x') \subseteq \mathcal{O}$ for every $x' \in \mathcal{Q}$. The correspondence σ is **uhc**, if it is **uhc** for each $x \in \mathcal{X}$.*

Definition 5.3. *The correspondence $\sigma : \mathcal{X} \twoheadrightarrow \mathcal{Y}$ is said to be lower hemi-continuous (**lhc**) at x if for every open set $\mathcal{O}$ in $\mathcal{Y}$ with $\sigma(x) \cap \mathcal{O} \neq \emptyset$ it exists an open set $\mathcal{Q} \subseteq \mathcal{Y}$ of x such that $\sigma(x') \cap \mathcal{O} \neq \emptyset$ for every $x' \in \mathcal{Q}$. The correspondence σ is **lhc**, if it is **lhc** for each $x \in \mathcal{X}$.*

Lemma 5.1. *Let $\mathcal{X}$ be a non-empty and convex polyhedral subset of $\mathbb{R}^{\tilde{p}}$, and $\mathcal{Y} \subseteq \mathbb{R}^{\tilde{n}}$. If $\sigma : \mathcal{X} \twoheadrightarrow \mathcal{Y}$ is a bounded correspondence with a convex graph, then σ is lower hemi-continuous.*

Proof. For a proof see Peleg and Sudhölter (2007, pp. 185-186). □

Theorem 5.2. *The pre-kernel correspondence $\mathcal{PK} : \mathcal{G}(N) \twoheadrightarrow \mathbb{R}^n$ is on $\mathcal{G}_c^n$ upper hemi-continuous as well as lower hemi-continuous, that is, continuous.*

Proof. The non-empty set $\mathcal{G}_c^n$ is a bounded polyhedral set, which is convex by construction. We draw from Proposition 5.1 the conclusion that the graph of the pre-kernel correspondence is bounded and closed. Form Theorem 5.1 it follows $|\,\mathcal{J}'\,| = 1$ on $\mathcal{G}_c^n$, this implies that the graph of the pre-kernel correspondence is also convex on $\mathcal{G}_c^n$. The sufficient conditions of Lemma 5.1 are satisfied, hence $\mathcal{PK}$ is lower hemi-continuous on $\mathcal{G}_c^n$. It is known from Theorem 9.1.7. by Peleg and Sudhölter (2007) that $\mathcal{PK}$ is upper hemi-continuous on $\mathcal{G}(N)$. Hence, on the restricted set $\mathcal{G}_c^n$, the set-valued function $\mathcal{PK}$ is upper and lower hemi-continuous, and therefore continuous. Actual, it is a continuous function on $\mathcal{G}_c^n$ in view of $|\,\mathcal{J}'\,| = 1$. □

Corollary 5.1. *The pre-kernel correspondence $\mathcal{PK} : \mathcal{G}(N) \twoheadrightarrow \mathbb{R}^n$ is on $\mathcal{G}_c^n$ single-valued and constant.*

Example 5.2. To observe that on the restricted set $\mathcal{G}_c^4$ the pre-kernel correspondence $\mathcal{PK} : \mathcal{G}(N) \twoheadrightarrow \mathbb{R}^n$ is single-valued and continuous, we exemplarily select a line segment in $\mathcal{G}_c^4$ to establish that all games on this segment have the same singleton pre-kernel. For this purpose, we resume Example 4.1 and 5.1. Then we choose a vector of scalars $\vec{t}^{\epsilon} := (1,3,8,1,2,4+\epsilon,3,5,7,9,2-\epsilon,3)/48$ with $t_k^{\epsilon} \geq 0$ for each k such that $\sum_{k=0}^{11} t_k^{\epsilon} = 1$ and $\epsilon \in [-2,2]$. Thus, we define the line segment in $\mathcal{G}_c^4$ through TU game v^{μ^*} from Example 5.1 by

$$\mathcal{G}_c^{4,l} := \left\{ \sum_{k=0}^{11} t_k^{\epsilon} \cdot v_k^{\mu} \;\middle|\; v_k^{\mu} \in \mathcal{G}_c^4, \epsilon \in [-2,2] \right\}.$$

Therefore, for each game in the line segment $\mathcal{G}_c^{4,l}$, we can write

$$\begin{aligned} v^{\epsilon} &:= \sum_{k=1}^{11} t_k^{\epsilon} \cdot v_k^{\mu} + t_0^{\epsilon} \cdot v = \sum_{k=1}^{11} t_k \cdot v_k^{\mu} + t_0 \cdot v + \frac{\epsilon}{48}\,(v_6^{\mu} - v_{11}^{\mu}) = v^{\mu^*} + \frac{\epsilon}{48}\,(v_6^{\mu} - v_{11}^{\mu}) \\ &= v + \mu \cdot v^{\Delta^*} + \frac{\epsilon\,\mu}{48}\,(v_6^{\Delta} - v_{11}^{\Delta}). \end{aligned}$$

We extend the pre-kernel element $\mathbf{x} = (44/9, 4, 32/9, 32/9)$ to a vector $\overline{\mathbf{x}}$ in order to define the excess vector under game v as $\overline{e} := v - \overline{\mathbf{x}}$, and for game v^{ϵ} as $\overline{e}^{\,v^{\epsilon}} := v^{\epsilon} - \overline{\mathbf{x}}$, respectively. According to these definitions, we get for $\vec{\zeta}^{v^{\epsilon}} = \vec{\xi}^{v^{\epsilon}}$ at $\mathbf{x}$ the following chain of equalities:

$$\vec{\xi}^{v^{\epsilon}} = \boldsymbol{\mathcal{V}}^{\top}\,\overline{e}^{\,v^{\epsilon}} = \boldsymbol{\mathcal{V}}^{\top}\left(v - \overline{\mathbf{x}} + \mu \cdot v^{\Delta^*} + \frac{\epsilon\,\mu}{48}\,(v_6^{\Delta} - v_{11}^{\Delta})\right) = \boldsymbol{\mathcal{V}}^{\top}\,(v - \overline{\mathbf{x}}) = \boldsymbol{\mathcal{V}}^{\top}\,\overline{e} = \vec{\xi} = \vec{\zeta} = \mathbf{0},$$

The last equality is satisfied, since $\mathbf{x}$ is the pre-kernel of game v. Recall that it holds $\mu\, v^{\Delta^*}, \mu\, v_6^{\Delta}, \mu\, v_{11}^{\Delta} \in [-\mathsf{C}, \mathsf{C}]^{15}$, whereas $\boldsymbol{\mathcal{V}}^\top v^{\Delta^*} = \boldsymbol{\mathcal{V}}^\top v_6^{\Delta} = \boldsymbol{\mathcal{V}}^\top v_{11}^{\Delta} = \mathbf{0}$ is in force. Therefore, for each TU game $v^\epsilon \in \mathcal{G}_c^{4,l}$ we obtain

$$\mathcal{PK}(v^\epsilon) = (44/9, 4, 32/9, 32/9).$$

The pre-kernel correspondence $\mathcal{PK}$ is a single-valued and constant mapping on $\mathcal{G}_c^{4,l}$. Hence its is continuous on the restriction $\mathcal{G}_c^{4,l}$, and due to Theorem 5.2 a fortiori on $\mathcal{G}_c^{4}$. #

6 Preserving the Pre-Nucleolus Property

In this section we study some conditions under which a pre-nucleolus of a default game can preserve the pre-nucleolus property in order to generalize the above results in the sense to identify related games with a sole pre-kernel point even when the default game has a set-valued pre-kernel point. This question can only be addressed with limitation, since we are not able to make it explicit while giving only sufficient conditions under which the pre-kernel point must be at least disconnected, otherwise it must be a singleton. However, a great deal of our investigation is devoted to work out explicit conditions under which the pre-nucleolus of a default game will loose this property under a related game.

For the next result remember that a balanced collection $\mathcal{B}$ is called minimal balanced, if it does not contain a proper balanced sub-collection.

Theorem 6.1. *Let $\langle\, N, v\,\rangle$ be a TU game that has a set-valued pre-kernel such that $\mathbf{x} \in \mathcal{PK}(v)$, $\mathbf{y} = \nu(v)$ with $\mathbf{x}, \mathbf{y} \in [\vec{\gamma}]_v$, and $\mathbf{x} \neq \mathbf{y}$ is satisfied. In addition, let $\langle\, N, v^\mu\,\rangle$ be a related game of v with $\mu \neq 0$ derived from $\mathbf{x}$ such that $\mathbf{x} \in \mathcal{PK}(v^\mu) \cap [\vec{\gamma}]_{v^\mu}$, and $\mathbf{y} \notin [\vec{\gamma}]_{v^\mu}$ holds. If the collection $\mathcal{S}^v(\mathbf{x})$ as well as its sub-collections are not balanced,*

1. *then $\mathbf{y} \notin \mathcal{PN}(v^\mu)$.*
2. *Moreover, if in addition $\mathbf{x} = \mathbf{y} \notin [\vec{\gamma}]_{v^\mu}$, then $\mathbf{x} \notin \mathcal{PN}(v^\mu)$.*

Proof. The proof starts with the first assertion.

1. By our hypothesis, $\mathbf{x}$ is a pre-kernel element of game v and a related game v^μ that is derived from $\mathbf{x}$. There is no change in set of lexicographically smallest most effective coalitions $\mathcal{S}^v(\mathbf{x})$ under v^μ due to $\mathbf{x} \in [\vec{\gamma}]_{v^\mu}$, hence $\mathcal{S}^v(\mathbf{x}) = \mathcal{S}^{v^\mu}(\mathbf{x})$. Moreover, we have $\mu \cdot v^\Delta \in \mathbb{R}^{p'}$. Furthermore, it holds $\mathbf{y} = \nu(v)$ by our assumption. Choose a balanced collection $\mathcal{B}$ that contains $\mathcal{S}^v(\mathbf{x})$ such that $\mathcal{B}$ is minimal. Then single out any $\psi \in \mathbb{R}$ such that the balanced set $\mathcal{D}^v(\psi, \mathbf{y})$ satisfies $\mathcal{S}^v(\mathbf{x}) \subseteq \mathcal{B} \subseteq \mathcal{D}^v(\psi, \mathbf{y}) \neq \emptyset$. Now choose $\epsilon > 0$ such that $\mathcal{D}^v(\psi, \mathbf{y}) = \mathcal{D}^v(\psi - 2\,\epsilon, \mathbf{y})$ is given. The set $\mathcal{D}^v(\psi - 2\,\epsilon, \mathbf{y})$ is balanced as well. Observe that due to $\mathbf{x} \in [\vec{\gamma}]_{v^\mu}$ we get $\mu \cdot v^\Delta(S) \leq \epsilon$ for all $S \subset N$. However, it exists some coalitions $S \in \mathcal{S}^v(\mathbf{x})$ such that $e^v(S, \mathbf{y}) - \epsilon \not\leq e^v(S, \mathbf{y}) + \mu \cdot v^\Delta(S)$ holds. Let $c \in [-\epsilon, \epsilon]$, now as $\lim_{c \uparrow 0} \mathcal{D}^{v^\mu}(\psi + c, \mathbf{y}) = \mathcal{D}^{v^\mu}(\psi, \mathbf{y})$ we have $\mathcal{D}^{v^\mu}(\psi, \mathbf{y}) \subseteq \mathcal{D}^v(\psi, \mathbf{y})$. Furthermore, we draw the conclusion that $\mathcal{S}^v(\mathbf{x}) \not\subseteq \mathcal{D}^{v^\mu}(\psi, \mathbf{y})$ is given due to $\mathcal{S}^v(\mathbf{x}) = \mathcal{S}^v(\mathbf{y}) \neq \mathcal{S}^{v^\mu}(\mathbf{y})$. Therefore, we obtain $\mathcal{D}^{v^\mu}(\psi, \mathbf{y}) \subset \mathcal{B} \subseteq \mathcal{D}^v(\psi - 2\,\epsilon, \mathbf{y})$. To see this, assume that $\mathcal{D}^{v^\mu}(\psi, \mathbf{y})$ is balanced, then we get $\mathcal{B} \subseteq \mathcal{D}^{v^\mu}(\psi, \mathbf{y})$, since $\mathcal{B}$ is minimal balanced. This implies $\mathcal{S}^v(\mathbf{x}) \subseteq \mathcal{D}^{v^\mu}(\psi, \mathbf{y})$. However, this contradicts $\mathcal{S}^v(\mathbf{x}) \not\subseteq \mathcal{D}^{v^\mu}(\psi, \mathbf{y})$. We conclude that $\mathcal{D}^{v^\mu}(\psi, \mathbf{y}) \subset \mathcal{B}$ must hold, but then the set $\mathcal{D}^{v^\mu}(\psi, \mathbf{y})$ can not be balanced. Hence, $\mathbf{y} \notin \mathcal{PN}(v^\mu)$.
2. Finally, if $\mathbf{x} = \mathbf{y}$, then $\mathbf{x}$ is the pre-nucleolus of game v, but it does not belong anymore to payoff equivalence class $[\vec{\gamma}]$ under v^μ, that is, $[\vec{\gamma}]$ has shrunk. Therefore, $\mathcal{S}^v(\mathbf{x}) \neq \mathcal{S}^{v^\mu}(\mathbf{x})$. Define from the set $\mathcal{S}^v(\mathbf{x})$ a minimal balanced collection $\mathcal{B}$ that contains $\mathcal{S}^v(\mathbf{x})$. In the next step, we can single out any $\psi \in \mathbb{R}$ such that the balanced set $\mathcal{D}^v(\psi, \mathbf{x})$ satisfies $\mathcal{S}^v(\mathbf{x}) \subseteq \mathcal{B} \subseteq \mathcal{D}^v(\psi, \mathbf{x}) \neq \emptyset$. In view of

$\mathbf{x} \in \mathcal{PK}(v^{\mu})$, it must exist an $\epsilon > 0$ within the maximum surpluses can be varied without effecting the pre-kernel property of $\mathbf{x}$ even when $\mathbf{x} \notin [\vec{\gamma}]_{v^{\mu}}$, thus we have $\mu \cdot v^{\Delta}(S) \leq \epsilon$ for all $S \subset N$. This implies that $\mathcal{D}^{v}(\psi, \mathbf{x}) \subseteq \mathcal{D}^{v}(\psi - 2\,\epsilon, \mathbf{x})$ is in force. The set $\mathcal{D}^{v}(\psi - 2\,\epsilon, \mathbf{x})$ is balanced as well. However, it exists some coalitions $S \in \mathcal{S}^{v}(\mathbf{x})$ such that $e^{v}(S, \mathbf{x}) - \epsilon \not\leq e^{v}(S, \mathbf{x}) + \mu \cdot v^{\Delta}(S)$ is valid. Let $c \in [-\epsilon, \epsilon]$, now as $\lim_{c \uparrow 0} \mathcal{D}^{v^{\mu}}(\psi + c, \mathbf{x}) = \mathcal{D}^{v^{\mu}}(\psi, \mathbf{x})$ we have $\mathcal{D}^{v^{\mu}}(\psi, \mathbf{x}) \subseteq \mathcal{D}^{v}(\psi, \mathbf{x})$. Furthermore, we draw the conclusion that $\mathcal{S}^{v}(\mathbf{x}) \not\subseteq \mathcal{D}^{v^{\mu}}(\psi, \mathbf{x})$ is given due to $\mathcal{S}^{v}(\mathbf{x}) \neq \mathcal{S}^{v^{\mu}}(\mathbf{x})$. Therefore, we obtain $\mathcal{D}^{v^{\mu}}(\psi, \mathbf{x}) \subset \mathcal{B} \subseteq \mathcal{D}^{v}(\psi - 2\,\epsilon, \mathbf{x})$ by the same reasoning as under (1). Then the set $\mathcal{D}^{v^{\mu}}(\psi, \mathbf{x})$ can not be balanced. Hence, $\mathbf{x} \notin \mathcal{PN}(v^{\mu})$.

□

Theorem 6.2. *Let $\langle\, N, v \,\rangle$ be a TU game that has a set-valued pre-kernel such that $\mathbf{x} \in \mathcal{PK}(v) \cap [\vec{\gamma}]$, $\{\mathbf{y}\} = \mathcal{PN}(v) \cap [\vec{\gamma}_1]$ is satisfied, and let $\langle\, N, v^{\mu} \,\rangle$ be a related game of v with $\mu \neq 0$ derived from $\mathbf{x}$ such that $\mathbf{x} \in \mathcal{PK}(v^{\mu}) \cap [\vec{\gamma}]$ holds. If $\Delta \in \mathcal{N}_{\mathcal{W}} \backslash \mathcal{N}_{\mathcal{W}_1}$, then $\mathbf{y} \notin \mathcal{PK}(v^{\mu})$ and a fortiori $\mathbf{y} \notin \mathcal{PN}(v^{\mu})$.*

Proof. From the payoff equivalence classes $[\vec{\gamma}]$ and $[\vec{\gamma}_1]$ we derive the corresponding matrices $\mathcal{W} := \mathcal{V}^{\top}\,\mathcal{U}$ and $\mathcal{W}_1 := \mathcal{V}_1^{\top}\,\mathcal{U}$, respectively. By assumption, it is $\Delta \in \mathcal{N}_{\mathcal{W}} \backslash \mathcal{N}_{\mathcal{W}_1}$ satisfied. From this argument, we can express the vector of unbalanced excesses $\vec{\xi}^{\,v^{\mu}}$ at $\mathbf{y}$ by

$$\vec{\xi}^{\,v^{\mu}} = \mathcal{V}_1^{\top}\,\overline{e}^{\mu} = \mathcal{V}_1^{\top}\,(v^{\mu} - \overline{\mathbf{y}}) = \mathcal{V}_1^{\top}\,(v - \overline{\mathbf{y}} + \mu \cdot v^{\Delta}) = \vec{\xi}^{\,v} + \mu \cdot \mathcal{V}_1^{\top}\,v^{\Delta} = \mu \cdot \mathcal{V}_1^{\top}\,v^{\Delta} \neq \mathbf{0}.$$

Observe that $\vec{\xi}^{\,v} = \mathcal{V}_1^{\top}\,(v - \overline{\mathbf{y}}) = \mathbf{0}$, since vector $\mathbf{y} \in [\vec{\gamma}_1]$ is a pre-kernel element of game v. However, due to $\Delta \in \mathcal{N}_{\mathcal{W}} \backslash \mathcal{N}_{\mathcal{W}_1}$, we obtain $\mathcal{V}_1^{\top}\,v^{\Delta} \neq \mathbf{0}$, it follows that $\mathbf{y} \notin \mathcal{PK}(v^{\mu})$. The conclusion follows that $\mathbf{y} \notin \mathcal{PN}(v^{\mu})$ must hold. □

Theorem 6.3. *Let $\langle\, N, v \,\rangle$ be a TU game that has a set-valued pre-kernel such that $\mathbf{x} \in \mathcal{PK}(v) \backslash \mathcal{PN}(v)$ and $\mathbf{x} \in [\vec{\gamma}]$. If $\langle\, N, v^{\mu} \,\rangle$ is a related game of v with $\mu \neq 0$ derived from $\mathbf{x}$ such that $\mathbf{x} \in \mathcal{PK}(v^{\mu}) \cap [\vec{\gamma}]$ holds, then $\mathbf{x} \notin \mathcal{PN}(v^{\mu})$.*

Proof. According to our assumption $\mathbf{x}$ is not the pre-nucleolus of game v, this implies that there exists some $\psi \in \mathbb{R}$ such that $\mathcal{D}^{v}(\psi, \mathbf{x}) \neq \emptyset$ is not balanced. Recall that the set of lexicographically smallest most effective coalitions $\mathcal{S}^{v}(\mathbf{x})$ has not changed under v^{μ}, since $\mathbf{x}$ is a pre-kernel element of game v^{μ} which still belongs to the payoff equivalence class $[\vec{\gamma}]$. Then exists a bound $\epsilon > 0$ within the maximum surpluses can be varied without effecting the pre-kernel property of $\mathbf{x}$. Thus, we get $\mathcal{D}^{v}(\psi, \mathbf{x}) = \mathcal{D}^{v}(\psi - 2\,\epsilon, \mathbf{x}) \neq \emptyset$ is satisfied. Then $e^{v}(S, \mathbf{x}) - \epsilon \leq e^{v}(S, \mathbf{x}) + \mu \cdot v^{\Delta}(S) \leq e^{v}(S, \mathbf{x}) + \epsilon$ for all $S \subseteq N$, therefore, this implies $\mathcal{D}^{v^{\mu}}(\psi - \epsilon, \mathbf{x}) = \mathcal{D}^{v}(\psi, \mathbf{x})$. The set $\mathcal{D}^{v^{\mu}}(\psi - \epsilon, \mathbf{x})$ is not balanced, we conclude that $\mathbf{x} \notin \mathcal{PN}(v^{\mu})$. □

Theorem 6.4. *Assume that the payoff equivalence class $[\vec{\gamma}]$ induced from TU game $\langle\, N, v \,\rangle$ has non-empty interior. In addition, assume that the pre-kernel of game $\langle\, N, v \,\rangle$ constitutes a line segment such that $\mathbf{x} \in \mathcal{PN}(v) \cap \partial\overline{[\vec{\gamma}]}$, $\mathcal{PK}(v) \cap \overline{[\vec{\gamma}_1]}$, and $\mathbf{x} \in \mathcal{PK}(v^{\mu}) \cap [\vec{\gamma}]$ is satisfied, then the pre-kernel $\mathcal{PK}(v^{\mu})$ of a related TU game $\langle\, N, v^{\mu} \,\rangle$ with $\mu \neq 0$ derived from $\mathbf{x}$ is at least disconnected, otherwise unique.*

Proof. In the fist step, we have simply to establish that for game v^{μ} the pre-imputations lying on the part of the line segment included in payoff equivalence class $[\vec{\gamma}_1]$ under game v will loose their pre-kernel properties due to the change in the game parameter. In the second step, we have to show that the pre-nucleolus $\mathbf{x}$ under game v is also the pre-nucleolus of the related game v^{μ}.

1. First notice that the payoff equivalence class $[\vec{\gamma}]$ has full dimension in accordance with its non-empty interior condition. This implies that the vector $\mathbf{x}$ must be the sole pre-kernel element in $\overline{[\vec{\gamma}]}$ (cf. with the proof of Theorem 7.8.1 in Meinhardt (2013)). By our hypothesis, it is even a boundary point

of the payoff equivalence class under game v. Moreover, it must hold $[\vec{\gamma}] \nsim [\vec{\gamma}_1]$, since the rank of the induced matrix $\mathbf{E}^\top$ is n, and that of $\mathbf{E}_1^\top$ is $n-1$, therefore, we have $E_1^\top \neq E^\top X$ for all $X \in \mathrm{GL}^+(n)$.

In the next step, we select an arbitrary pre-kernel element from $\mathcal{PK}(v) \cap \overline{[\vec{\gamma}_1]}$, say $\mathbf{y}$. By hypothesis, there exists a related game v^μ of v such that $\mathbf{x} \in \mathcal{PK}(v^\mu) \cap [\vec{\gamma}]$ holds, that is, there is no change in matrix $\mathbf{E}$ and vector $\vec{\alpha}$ implying $h^{v^\mu}(\mathbf{x}) = h^{v^\mu}_\gamma(\mathbf{x}) = 0$. This implies that for game v^μ the payoff equivalence class $[\vec{\gamma}]$ has been enlarged in such a way that we can inscribe an ellipsoid with maximum volume $\varepsilon := \{\mathbf{y}' \,|h^{v^\mu}_\gamma(\mathbf{y}') \leq \bar{c}\}$, whereas $h^{v^\mu}_\gamma$ is of type (3.15) and $\bar{c} > 0$ (cf. Lemma 4.4.). It should be obvious that element $\mathbf{x}$ is an interior point of ε, since $\mathbf{x} = M(h^{v^\mu}_\gamma) \subset \varepsilon \subset [\vec{\gamma}]$. We single out a boundary point $\mathbf{x}'$ in $\partial\overline{[\vec{\gamma}]}$ under game v^μ which was a pre-kernel element under game v, and satisfying after the parameter change the following properties: $\mathbf{x}' \in \partial\overline{[\vec{\gamma}]} \cap \overline{[\vec{\gamma}_1]}$ with $\mathbf{x}' = \mathbf{x} + \mathbf{z}$, and $\mathbf{z} \neq \mathbf{0}$. This is possible due to the fact that the equivalence class $[\vec{\gamma}]$ has been enlarged at the expense of equivalence class $[\vec{\gamma}_1]$, which has shrunk or shifted by the change in the game parameter. Observe now that two cases may happen, that is, either $\mathbf{x}' \in \varepsilon$ or $\mathbf{x}' \notin \varepsilon$. In the former case, we have $h^{v^\mu}_\gamma(\mathbf{x}') = h^{v^\mu}(\mathbf{x}') = h^{v^\mu}_{\gamma_1}(\mathbf{x}') = \bar{c} > 0$, and in the latter case, we have $h^{v^\mu}_\gamma(\mathbf{x}') = h^{v^\mu}(\mathbf{x}') = h^{v^\mu}_{\gamma_1}(\mathbf{x}') > \bar{c} > 0 = h^{v}(\mathbf{x}') = h^{v}_{\gamma_1}(\mathbf{x}')$.

From $h^{v^\mu}_{\gamma_1}(\mathbf{x}') > 0$, and notice that the vector of unbalanced excesses at $\mathbf{x}'$ is denoted as $\vec{\xi}^{\,v^\mu}$, we derive the following relationship

$$h^{v^\mu}_{\gamma_1}(\mathbf{x}') = \|\,\vec{\xi}^{\,v^\mu}\,\|^2 = \|\,\vec{\xi}^{\,v} + \mu \cdot \mathcal{V}_1^\top\, v^\Delta\,\|^2 = \|\,\mu \cdot \mathcal{V}_1^\top\, v^\Delta\,\|^2 = \mu^2 \cdot \|\,\mathcal{V}_1^\top\, v^\Delta\,\|^2 > 0,$$

with $\mu \neq 0$. Thus, we have $\mathcal{V}_1^\top\, v^\Delta \neq \mathbf{0}$, and therefore $\Delta \in \mathcal{N}_{\mathcal{W}} \backslash \mathcal{N}_{\mathcal{W}_1}$. Observe that $\vec{\xi}^{\,v} = \mathcal{V}_1^\top\,(v - \overline{\mathbf{x}'}) = \mathbf{0}$, since vector $\mathbf{x}' \in \overline{[\vec{\gamma}_1]}$ is a pre-kernel element of game v. Take the vector $\mathbf{y} \in [\vec{\gamma}_1]$ from above that was on the line segment as vector $\mathbf{x}'$ under game v which constituted a part of the pre-kernel of game v, we conclude that $\mathbf{y} \notin \mathcal{PK}(v^\mu)$ in accordance with $\mathcal{V}_1^\top\, v^\Delta \neq \mathbf{0}$.

2. By our hypothesis, $\mathbf{x}$ is the pre-nucleolus of game v, and an interior point of equivalence class $[\vec{\gamma}]$ of the related game v^μ. Using a similar argument as under (1) we can inscribe an ellipsoid with maximum volume ε, whereas $h^{v^\mu}_\gamma$ is of type (3.15) and $\bar{c} > 0$. In view of the assumption that $\mathbf{x}$ is also pre-kernel element of game v^μ, we can draw the conclusion that the set of lexicographically smallest most effective coalitions $\mathcal{S}(\mathbf{x})$ has not changed under v^μ. But then, we have $\mu \cdot v^\Delta \in [-\mathsf{C}, \mathsf{C}]^{p'}$. In addition, there exists a $\bar{\psi} \geq \psi^*$ s.t. $\mathcal{S}(\mathbf{x}) \subseteq \mathcal{D}^v(\bar{\psi}, \mathbf{x})$, that is, it satisfies Property I of Kohlberg (1971). Moreover, matrix $\mathbf{E}^\top$ induced from $\mathcal{S}(\mathbf{x})$ has full rank, therefore, the column vectors of matrix $\mathbf{E}^\top$ are a spanning system of $\mathbb{R}^n$. Hence, we get *span* $\{\mathbf{1}_S \,|\, S \in \mathcal{S}(\mathbf{x})\} = \mathbb{R}^n$ as well, which implies that matrix $[\mathbf{1}_S]_{S \in \mathcal{S}(\mathbf{x})}$ has rank n, the collection $\mathcal{S}(\mathbf{x})$ must be balanced. In accordance with vector $\mathbf{x}$ as the pre-nucleolus of game v, we can choose the largest $\psi \in \mathbb{R}$ s.t. $\emptyset \neq \mathcal{D}^v(\psi, \mathbf{x}) \subseteq \mathcal{S}(\mathbf{x})$ is valid, which is a balanced set. Since $\mathsf{C} > 0$, the set $\mathcal{D}^v(\psi - 2\,\mathsf{C}, \mathbf{x}) \neq \emptyset$ is balanced as well. Now observe that $e^v(S, \mathbf{x}) - \mathsf{C} \leq e^v(S, \mathbf{x}) + \mu \cdot v^\Delta(S) \leq e^v(S, \mathbf{x}) + \mathsf{C}$ for all $S \subseteq N$. This implies $\mathcal{D}^v(\psi, \mathbf{x}) \subseteq \mathcal{S}(\mathbf{x}) \subseteq \mathcal{D}^{v^\mu}(\psi - \mathsf{C}, \mathbf{x}) \subseteq \mathcal{D}^v(\psi - 2\,\mathsf{C}, \mathbf{x})$, hence, $\mathcal{D}^{v^\mu}(\psi - \mathsf{C}, \mathbf{x})$ is balanced. To conclude, let $c \in [-\mathsf{C}, \mathsf{C}]$, and from the observation $\lim_{c \uparrow 0} \mathcal{D}^{v^\mu}(\psi + c, \mathbf{x}) = \mathcal{D}^{v^\mu}(\psi, \mathbf{x}) \supseteq \mathcal{D}^v(\psi, \mathbf{x})$, we draw the implication $\mathbf{x} = \nu(N, v^\mu)$.

Finally, recall that the vector $\mathbf{x}$ is also the unique minimizer of function $h^{v^\mu}_\gamma$, which is an interior point of payoff equivalence class $[\vec{\gamma}]$, therefore the pre-kernel of the related game v^μ can not be connected. Otherwise the pre-kernel of the game consists of a single point. □

Corollary 6.1. *Let $\langle\, N, v\,\rangle$ be a TU game that has a non single-valued pre-kernel such that $\mathbf{x} \in \mathcal{PN}(v) \cap \partial\overline{[\vec{\gamma}]}$ and let $\langle\, N, v^\mu\,\rangle$ be a related game of v derived from $\mathbf{x}$, whereas $\mathbf{x} \in$ int $[\vec{\gamma}]_{v^\mu}$, then $\mathbf{x} = \nu(N, v^\mu)$.*

7 THE ANTI-PRE-KERNEL AND ANTI-PRE-NUCLEOLUS

Generically, cooperative game theory studies the circumstances under which mutual cooperation becomes attractive to establish an agreement. The purpose of this subsection is now the reverse study, that is, to focus on the conditions under which it becomes unattractive within the game context, that is, under which conditions a possible outcome becomes unstable within a bargaining situation. For seeing its relevance, one has to take into account that a transferable utility game represents a virtual bargaining situation where arguments as claims or proposals can be exchanged through communication to reach an agreement. Although the communicational aspect is not visible by the characteristic function, it is, nevertheless, of fundamental importance of how we have to read and understand a TU game. Under this consideration of a TU game, it is a crucial aspect to identify the outcomes which are not supportable as agreements to motivate a partner to move in order to finally reach an agreement point (see for more details Meinhardt (2018a)).

To this end, let us notice that the **anti-imputation set** $\mathcal{I}^{\#}(N,v)$ is specified by

$$\mathcal{I}^{\#}(N,v) := \left\{ \vec{x} \in \mathcal{I}^{*}(N,v) \;\middle|\; \sum_{k \in N\setminus\{i\}} x_k \leq v(N \setminus \{i\}) \quad \text{for all } i \in N \right\}, \tag{7.1}$$

whereas $\mathcal{I}^{*}(N,v)$ is the pre-imputation set. Obviously, we get $\mathcal{I}^{\#}(N,v) = \mathcal{I}(N,v^{*})$ or $\mathcal{I}^{\#}(N,v^{*}) = \mathcal{I}(N,v)$, since it holds for all $i \in N$ the subsequent inequality

$$x(N\backslash\{i\}) \leq v^{*}(N\backslash\{i\}) \iff x(N\backslash\{i\}) \leq v(N) - v(\{i\}) \iff x_i \geq v(\{i\}),$$

whenever $x(N) = v(N)$ is given.

The **anti-core** of a game $\mathcal{C}^{\#}(N,v)$ is the set of pre-imputations satisfying besides the anti-individual rationality property also the anti-coalitional rationality property, i.e. the anti-core of a game $v \in \mathcal{G}^n$ is given by

$$\mathcal{C}^{\#}(N,v) := \{\mathbf{x} \in \mathcal{I}^{*}(N,v) \,|\, x(N) = v(N) \text{ and } x(S) \leq v(S)\; \forall\, S \subset N\}. \tag{7.2}$$

The anti-core of a n-person game may be empty. Whenever it is non-empty, this set specifies the imputations that makes mutual cooperation in the grand coalition unattractive. An anti-core selection is a blocking outcome in the sense that such a payoff distribution relies in the feasible set of all coalitions. Thus, a payoff distribution located in the anti-core can be blocked by any coalition. Therefore, each coalition can formulate an objection against this allocation. As a consequence, the anti-core is the set of all pre-imputation that prevents mutual cooperation in the grand coalition. Or to put it differently, it describes the set of all allocations that are vulnerable by preventive arguments. For convex games the set of vulnerable allocations is empty, but non-empty for concave games. Moreover, the grand coalition might not distribute to its members a value that exceeds the value that the intermediate coalitions can produce to their members. Hence, the formation of a larger coalition is not rewarding. By this interpretation, we realize that it is a conceptual misunderstanding of mutual cooperation if one speaks in the context of the anti-core from a dual core.

A particular example of a blocking outcome under a non-empty anti-core is the anti-pre-nucleolus. Thus, if the anti-core exists, then the anti-pre-nucleolus under which the smallest excesses of all coalitions are maximized must belong to the anti-core, that is, all excesses must be greater than or equal to zero. In this specific case, the anti-pre-nucleolus is a payoff distribution that prevents mutual cooperation in the grand coalition.

Even though it is easily seen that the anti-core of a dual game $\langle N, v^* \rangle$ coincides with the core $\mathcal{C}(N,v)$ of game $\langle N, v \rangle$, we clarify the relationship first. Starting with the **anti-core** $\mathcal{C}^{\#}(N,v^*)$ of a dual game $\langle N, v^* \rangle$, we get

$$\begin{aligned} &\mathcal{C}^{\#}(N,v^*) := \{\vec{x} \in \mathcal{I}^*(N,v) |\, x(S) \leq v^*(S) \quad \text{for all } S \subseteq N\} \\ &\iff \{\vec{x} \in \mathcal{I}^*(N,v) |\, x(S) + x(N \backslash S) - x(N \backslash S) \leq v(N) - v(N \backslash S) \quad \text{for all } S \subseteq N\} \\ &\iff \{\vec{x} \in \mathcal{I}^*(N,v) |\, x(N \backslash S) \geq v(N \backslash S) \quad \text{for all } S \subseteq N\} = \mathcal{C}(N,v). \end{aligned} \tag{7.3}$$

Resume now that a solution σ is called **self dual**, if $\sigma(N,v) = \sigma(N,v^*)$ for all $\langle N, v \rangle \in \mathcal{G}_{\mathcal{U}}$. Notice that for additive games the core is self dual, that is, the core coincides with the core of the dual game. This is due to $v = v^*$. However, the core of the game is also identical to the anti-core of the game, i.e., $\mathcal{C}(N,v) = \mathcal{C}^{\#}(N,v)$. Implying that both the incentives and disincentives of cooperating in the grand coalition are weak.

It was a commonly held belief of game theorists that the pre-nucleolus of the modest bankruptcy game is identical to the pre-nucleolus of its dual game, the greedy bankruptcy game. Following Funaki and Meinhardt (2006), we provide the framework to illustrate that this belief is false.

In this respect, remember that the **dual** $v^* : 2^N \to \mathbb{R}$ of the TU game $\langle N, v \rangle$ is defined by $v^*(S) = v(N) - v(N \backslash S)$ for all $S \subseteq N$. The worth $v^*(S)$ is the amount from which coalition S cannot be prevented from by the opponents whenever they receive $v(N \backslash S)$. The preventive power of the opponents prevent coalition S to get more than $v^*(S)$. Or to put it differently, $v^*(S)$ is the amount coalition S prevents access from outsiders. The pre-kernel of the dual game v^*, known as the **dual pre-kernel**, is denoted by $\mathcal{PK}^d(N,v)$, and it takes the preventive power of coalitions during a negotiation into account. Notice that the subsequent equality holds $\mathcal{PK}^d(N,v) = \mathcal{PK}(N,v^*)$ for any $v \in \mathcal{G}^n$. Observe that by this relation it is just enough to investigate the preventive power of coalitions within the initial game setting rather than deriving the dual.

By the work of Funaki and Meinhardt (2006), we know that the pre-nucleolus of the dual game v^* (dual pre-nucleolus) coincides with the anti-pre-nucleolus of the game v. Here, we restate the theorem

Theorem 7.1 (Funaki and Meinhardt (2006))**.** *Let $\langle N, v \rangle \in \mathcal{G}$ be a transferable utility game. Then the pre-nucleolus of the dual game v^* coincides with the anti-pre-nucleolus of the game v, i.e.;*

$$\mathcal{PN}^d(N,v) = \mathcal{PN}(N,v^*) = \mathcal{PN}^{\#}(N,v). \tag{7.4}$$

Proof. See, for instance, Funaki and Meinhardt (2006) or Meinhardt (2018b). □

The same we can illustrate for the dual pre-kernel and the anti-pre-kernel. For doing so, consider that for a game v, such that for any pair of players $i,j \in N, i \neq j$, the minimum surplus (maximum loss) of player i over player j with respect to the pre-imputation $\mathbf{x} \in \mathcal{I}^*(N,v)$, is given by the minimum excess at $\mathbf{x}$ over the set of coalitions containing player i but not player j. The **anti-surplus** (minimum surplus) is defined to be

$$s^{\#}_{ij}(\mathbf{x},v) := \min_{S \in \mathcal{G}_{ij}} e^v(S,\mathbf{x}) \qquad \text{where } \mathcal{G}_{ij} := \{S \mid i \in S \text{ and } j \notin S\}. \tag{7.5}$$

The expression $s^{\#}_{ij}(\mathbf{x})$ describes the minimal amount at the pre-imputation $\mathbf{x}$ that player i can gain without the cooperation of player j. The set of all pre-imputations $\mathbf{x} \in \mathcal{I}^*(N,v)$ that balances the minimum surpluses for each distinct pair of player $i,j \in N, i \neq j$, is called the **anti-pre-kernel** of the game v, and is defined by

$$\mathcal{PK}^{\#}(N,v) := \{\mathbf{x} \in \mathcal{I}^*(N,v) \mid s^{\#}_{ij}(\mathbf{x},v) = s^{\#}_{ji}(\mathbf{x},v) \quad \text{for all } i,j \in N, i \neq j\}. \tag{7.6}$$

Similar, to the solution concept of the dual pre-kernel, we introduce the notion of an anti-pre-kernel of the dual game v^*, that will be called the **dual anti-pre-kernel**, which is denoted by $(\mathcal{PK}^{\#})^d(N,v)$.

The next theorem asserts that the pre-kernel of the dual game v^* (dual pre-kernel) coincides with the anti-pre-kernel of v. We restate this result in connection with its proof.

Theorem 7.2 (Funaki and Meinhardt (2006))**.** *Let v be a transferable utility game. Then the pre-kernel of the dual game v^* (dual pre-kernel) coincides with the anti-pre-kernel of v, that is*

$$\mathcal{PK}^d(N,v) = \mathcal{PK}(N,v^*) = \mathcal{PK}^{\#}(N,v). \tag{7.7}$$

Proof. Using the definition of the dual game v^* and the definition of the pre-kernel (2.6), then we obtain

$$\begin{aligned} s_{ij}(\vec{x},v^*) &= \max_{S\in\mathcal{G}_{ij}}[v^*(S)-x(S)] = \max_{S\in\mathcal{G}_{ij}}[v(N)-v(N\backslash S)-x(S)] \\ &= \max_{N\backslash S\in\mathcal{G}_{ji}}[-v(N\backslash S)+x(N\backslash S)] = -\min_{T\in\mathcal{G}_{ji}}[v(T)-x(T)] = -s^{\#}_{ji}(\vec{x},v). \end{aligned}$$

Thus, the maximum surplus of player i against player j in the dual game v^* is equal to the negative of the anti-surplus of player j against player i in the game v. Applying the definition of the pre-kernel, and we get

$$s_{ij}(\vec{x},v^*) = s_{ji}(\vec{x},v^*) \Longleftrightarrow s^{\#}_{ji}(\vec{x},v) = s^{\#}_{ij}(\vec{x},v).$$

Thus, the dual pre-kernel coincides with the anti-pre-kernel. □

REMARK 7.1 (Anti-Pre-Kernel).
It should be obvious that due to $v^{**}=v$, we have $\mathcal{PK}(N,v^{**}) = \mathcal{PK}(N,v)$, and the pre-kernel of the game v is identical to the anti-pre-kernel of the dual game v^*, hence it holds that $\mathcal{PK}(N,v) = \mathcal{PK}^{\#}(N,v^*)$. ◇

Notice that in this respect we can apply the indirect function approach. For seeing this let us recall Lemma 3.1 that states that $s_{ij}(\mathbf{x},v) = \pi(\mathbf{x}^{i,j,\delta}) - \delta$ holds true for every $i,j\in N$, $i\neq j$, and for every $\delta \geq \delta_1(\mathbf{x},v)$. Thus, in case of the game v, we get for any $\mathbf{x}$

$$\pi(\mathbf{x}^{i,j,\delta}) = s_{ij}(\mathbf{x},v)+\delta = -s^{\#}_{ji}(\vec{x},v^*)+\delta, \tag{7.8}$$

for every $i,j\in N$, $i\neq j$. This reveals that the indirect function of game v can also be expressed in terms of the minimum surpluses of its dual. Similar, the indirect function of the dual game v^* can be characterized in terms of the minimum surpluses of its primal v. Thus, if we denote the indirect function of the dual game as π^*, we can introduce a new non-negative valued objective function $h^{\#}$ on $\mathcal{I}^*(N,v)$ through

$$\begin{aligned} h^{\#}(\mathbf{x}) &= \sum_{\substack{i,j\in N\\ i<j}} (\pi^*(\mathbf{x}^{i,j,\delta}) - \pi^*(\mathbf{x}^{j,i,\delta}))^2 + (v^*(N)-x(N))^2 \\ &= \sum_{\substack{i,j\in N\\ i<j}} (s^{\#}_{ij}(\vec{x},v) - s^{\#}_{ji}(\vec{x},v))^2 + (v(N)-x(N))^2 \qquad \mathbf{x}\in\mathcal{I}^*(N,v). \end{aligned} \tag{7.9}$$

Alike to function h of type 3.3, it is neither quadratic nor convex. By this arguments we observe that the whole machinery of the indirect function is applicable in case of the anti-pre-kernel of game v.

In contrast to the pre-kernel of a game which balances all pairs of players $i,j\in N, i\neq j$ of maximum surpluses we observe by Definition 7.6 that the anti-pre-kernel balances all pairs $i,j\in N, i\neq j$

of minimum surpluses instead. Thus with some appropriate modifications for Algorithm 3.1, we can describe a method of evaluating an element of the anti-pre-kernel. As a starting point we have to rewrite Definition 3.6 of the set of most effective coalitions in such a way that we select out those coalitions having minimum surpluses, that is, we want single out the less effective coalitions for all pairs of players $i, j \in N, i \neq j$. For doing so, we formally define the set of **less effective** or **insignificant coalitions** for each pair of players $i, j \in N, i \neq j$ at the payoff vector $\mathbf{x}$ by

$$\mathcal{C}_{ij}^{\#}(\mathbf{x}) := \left\{ S \in \mathcal{G}_{ij} \;\middle|\; s_{ij}^{\#}(\mathbf{x}, v) = e^{v}(S, \mathbf{x}) \right\}. \tag{7.10}$$

Substituting in the Definition (3.7) up to (3.10) the set of most effective coalitions by the set of less effective coalitions let us end up with the set of **lexicographically smallest less effective coalitions** w.r.t. $\mathbf{x}$ which is defined by

$$\mathcal{S}^{\#}(\mathbf{x}) := \left\{ \mathcal{S}_{ij}^{\#}(\mathbf{x}) \;\middle|\; i, j \in N, i \neq j \right\}. \tag{7.11}$$

Similar, we can construct from the set $\mathcal{S}^{\#}(\mathbf{x})$ a rectangular matrix denoted as $\mathbf{E}^{\#} \in \mathbb{R}^{n \times q}$. From this matrix and its transpose a matrix $\mathbf{Q}^{\#} \in \mathbb{R}^{n^2}$ is determined by $\mathbf{Q}^{\#} = 2 \cdot (\mathbf{E})^{\#} \, (\mathbf{E}^{\#})^{\top}$, and a column vector $\mathbf{a}^{\#}$ by $2 \cdot \mathbf{E}^{\#} \, \vec{\alpha}^{\#} \in \mathbb{R}^{n}$. Defining each component of the vector as $\alpha_{ij}^{\#} := (v(S_{ij}^{\#}) - v(S_{ji}^{\#})) \in \mathbb{R} \;\; \forall i, j \in N, i < j$ and $\alpha_{0}^{\#} := v(N)$. Finally, the scalar $\alpha^{\#}$ is given by $\|\vec{\alpha}^{\#}\|^2$, whereas $\mathbf{E}^{\#} \in \mathbb{R}^{n \times q}$, $(\mathbf{E}^{\#})^{\top} \in \mathbb{R}^{q \times n}$ and $\vec{\alpha}^{\#} \in \mathbb{R}^{q}$.

From vector $\vec{\gamma}$ the matrix $\mathbf{Q}^{\#}$, column vector $\mathbf{a}^{\#}$, and scalar $\alpha^{\#}$ are induced from which a modified quadratic and convex function can be specified through

$$h_{\gamma}^{\#}(\mathbf{x}) = (1/2) \cdot \langle\, \mathbf{x}, \mathbf{Q}^{\#} \mathbf{x} \,\rangle + \langle\, \mathbf{x}, \mathbf{a}^{\#} \,\rangle + \alpha^{\#} \qquad \mathbf{x} \in \mathbb{R}^{n}, \tag{7.12}$$

which reflects the underlying minimum surpluses w.r.t. payoff equivalence class $[\vec{\gamma}]^{\#}$. Now, observe that we can replicate from the finite set of equivalence classes the function $h^{\#}$ s.t. it is composed of a finite family of quadratic functions of type (7.12) (analogously to Meinhardt (2013, Proposition 6.2.2)). Provided that the game context is clear, we refer to $h^{\#}$, otherwise to $h^{v\,\#}$. Similar, for function $h_{\gamma}^{\#}$ with $h_{\gamma}^{v\,\#}$ in case of an ambiguous context.

In the next step, we select a payoff vector $\vec{\gamma}$, which also determines its payoff set $[\vec{\gamma}]^{\#}$. With regard to the binary relation $\sim$ on the set $dom\, h^{\#}$ defined by $\mathbf{x} \sim \vec{\gamma} \iff \mathcal{S}^{\#}(\mathbf{x}) = \mathcal{S}^{\#}(\vec{\gamma})$ is an equivalence relation, which forms a partition of the set $dom\, h^{\#}$ by the collection of equivalence classes $\{[\vec{\gamma}_k]^{\#}\}_{k \in J}$, where J is an arbitrary index set. This vector induces in addition a set of lexicographically smallest less effective coalitions indicated by $\mathcal{S}^{\#}(\vec{\gamma})$. Furthermore, we can also define a set of vectors as the differences of unity games w.r.t. the set of lexicographically smallest less effective coalitions, which is given by

$$\mathbf{v}_{ij}^{\#} := \mathbf{1}^{S_{ij}} - \mathbf{1}^{S_{ji}} \quad \text{for } S_{ij}, S_{ji} \in \mathcal{S}^{\#}(\vec{\gamma}) \quad \text{and} \quad \mathbf{v}_0 := \mathbf{1}^{N}, \tag{7.13}$$

whereas $\mathbf{v}_{ij}^{\#}, \mathbf{v}_0 \in \mathbb{R}^{p'}$ for all $i, j \in N, i < j$. With these column vectors, we can identify matrix $\boldsymbol{\mathcal{V}}^{\#} := [\mathbf{v}_{1,2}^{\#}, \ldots, \mathbf{v}_{n-1,n}^{\#}, \mathbf{v}_0] \in \mathbb{R}^{p' \times q}$. Notice that we get $(\boldsymbol{\mathcal{V}}^{\#})^{\top} v = \vec{\alpha}^{\#}$ in this context.

Redefining the mapping (3.18) by replacing the corresponding matrix, vector and scalar by their anti-counterparts, we obtain a mapping that sends a point $\vec{\gamma}$ to a point $\vec{\gamma}_{\circ} \in M(h_{\gamma}^{\#})$. This mapping is identified through

$$\Gamma^{\#}(\vec{\gamma}) := -\Big((\mathbf{Q}^{\#})^{\dagger} \, \mathbf{a}^{\#} \Big)(\vec{\gamma}) = -\Big((\mathbf{Q}^{\#})_{\vec{\gamma}}^{\dagger} \, \mathbf{a}_{\vec{\gamma}}^{\#} \Big) = \vec{\gamma}_{\circ} \in M(h_{\gamma}^{\#}) \qquad \forall \vec{\gamma} \in \mathbb{R}^{n}, \tag{7.14}$$

where $\mathbf{Q}^{\#}_{\vec{\gamma}}$ and $\mathbf{a}^{\#}_{\vec{\gamma}}$ are the matrix and the column vector induced by vector $\vec{\gamma}$, respectively. The solution set of function $h^{\#}_{\gamma}$ is denoted by $M(h^{\#}_{\gamma})$.

By the above procedure we provided a dual characterization of the anti-pre-kernel – a solution concept from transferable utility games that incorporates the preventive power of coalitions and which gives the same solution as the dual pre-kernel – without explicitly introducing the dual game. That is to say, we have preserved the original game context. Consequently, we do not have changed the theoretical framework in which subjects are involved while considering the preventive power effect. Thus, the reference and comparability to the original game setting is not lost by this approach (cf. Meinhardt (2018b,c)). In contrast, to the modiclus, which we are going to discuss in the subsequent section.

After having observed that the pre-kernel and the anti-pre-kernel can be both characterized by the indirect function approach, we introduce now a method similar to those of Algorithm 3.1 for computing an anti-pre-kernel element of a TU game $\langle N, v \rangle$, which we have formalized by means of pseudo-code through Algorithm 7.1:

Algorithm 7.1: Procedure to seek for an Anti-Pre-Kernel Element

Data: Arbitrary TU Game $\langle N, v \rangle$, and a payoff vector $\vec{\gamma}_0 \in \mathbb{R}^n$.
Result: A payoff vector s.t. $\vec{\gamma}_{k+1} \in \mathcal{PK}^{\#}(v)$.
begin
  $k \longleftarrow 0, \quad \mathcal{S}^{\#}(\vec{\gamma}_{-1}) \longleftarrow \emptyset$
  Select an arbitrary starting point $\vec{\gamma}_0$
  if $\vec{\gamma}_0 \notin \mathcal{PK}^{\#}(v)$ **then** Continue
  else Stop
  Determine $\mathcal{S}^{\#}(\vec{\gamma}_0)$
  if $\mathcal{S}^{\#}(\vec{\gamma}_0) \neq \mathcal{S}^{\#}(\vec{\gamma}_{-1})$ **then** Continue
  else Stop
  repeat
    if $\mathcal{S}^{\#}(\vec{\gamma}_k) \neq \emptyset$ **then** Continue
    else Stop
    Compute $\mathbf{E}^{\#}_k$ and $\vec{\alpha}^{\#}_k$ from $\mathcal{S}^{\#}(\vec{\gamma}_k)$ and v
    Determine $\mathbf{Q}^{\#}_k$ and $\mathbf{a}^{\#}_k$ from $\mathbf{E}^{\#}_k$ and $\vec{\alpha}^{\#}_k$
    Calculate by Formula (7.14) $\mathbf{x}$
    $k \longleftarrow k+1$
    $\vec{\gamma}^{\#}_{k+1} \longleftarrow \mathbf{x}$
    Determine $\mathcal{S}^{\#}(\vec{\gamma}_{k+1})$
  until $\mathcal{S}^{\#}(\vec{\gamma}_{k+1}) = \mathcal{S}^{\#}(\vec{\gamma}_k)$
end

Example 7.1. Let us resume Example 4.1. By the foregoing discussion we became aware that game v is average-convex that possesses a sole pre-kernel point. Applying the computation procedure from Algorithm 7.1 we get an element of the anti-pre-kernel that is quantified through $(8, 53/12, 37/12, 1/2)$. Conducting an analogous analysis in lieu thereof for the anti-pre-kernel as outlined below for the pre-kernel – which we formalize in the forthcoming Section 9 –, we infer that this point is also the sole anti-pre-kernel element that coincides with the anti-pre-nucleolus. In addition, we even realize that this point is also identical to the modiclus. From Meinhardt (2018b, Section 10) it is known that that the modiclus coincides with the anti-pre-nucleolus for the class of PS, weighted graph, and modest bankruptcy games. Though we have evidence from the literature that coincidence is not necessarily guaranteed for convex games (cf. Sudhölter (1997a, Section 3)). #

8 The Modiclus

The modiclus is a solution concept from cooperative game theory, which was invented by Sudhölter (1993), and introduced into the literature by Sudhölter (1996); Sudhölter (1997a); Rosenmüller and Sudhölter (2004); Raghavan and Sudhölter (2005) or Peleg and Sudhölter (2007) just to mention the best-known of those. Sometimes this solution concept is called the modified nucleolus to stress its close relationship to the (pre-)nucleolus, which was introduced by Schmeidler (1969). However, as it was worked out by Meinhardt (2018b), there is also a close relationship to the anti-pre-nucleolus (cf. Funaki and Meinhardt (2006)). This is caused by the fact that the modiclus takes the primal as well as the preventive power (dual power) of coalitions during a stylized bargaining scenario into account.

Most solution concepts incorporate the exercised or potential power of a coalition (primal power) to enforce their claims of its members. However, as it was discussed in the previous Section 7, each coalition has also a preventive power (dual power) that can be captured by the dual game of v, which is defined by $v^*(S) = v(N) - v(N\backslash S)$ for all $S \subseteq N$. Resume that $v^*(S)$ is the amount that coalition S cannot be prevented from when the complement $N\backslash S$ receives the payoff $v(N\backslash S)$ or in other words, the amount coalition S prevents access from outsiders. Thus, if the opponents $N\backslash S$ of a coalition S are powerful, the larger is the amount coalition S must renounce. Implying that the potential bargaining power of S must be weak, and vice versa. Solution concepts which incorporate the preventive power of coalitions are, for instance, the aforementioned anti-(pre-)nucleolus or anti-(pre)-kernel.

However, the modiclus simultaneously consider besides the primal power also the preventive power of the game. Thus, the modiclus addresses in a stylized bargaining process not only to the exercised power that a coalition can carry out, but also to its potential power to weakening the bargaining situation of opponents. Consequently, its figure of argumentation is pointing to a different stylized bargaining scenario. This means that subjects who are trying to obtain an agreement based on the principles of distributive justice related to the modiclus are pointing even to the dual game to enforce it within the original bargaining context, namely the dual game, which is not strategical equivalent to the former. From this point of view, it reveals arbitrariness in the imposed argumentation, and can therefore only be considered as a relative obscure and artificial perception of modeling a stylized bargaining process due to its changes in the bargaining agenda. As a consequence the theoretical framework in which subjects are involved changes, with the result that the reference and comparability to the original setting is lost. By this argument, we do not think so that this conceptual defect is healed due to the self-duality of the modiclus (cf. Meinhardt (2018b,c)).

Despite this obvious criticism, we nevertheless suppose that it is worthwhile to investigate this solution concept further with a greater accurateness as it was done in the prevailing literature. We are convinced that this solution concept merits more than an opaque and erroneous presentation that has been provided in the past by his inventor (cf. Meinhardt (2019)). In particular, we observe its full potential in advancing our understanding of compliance w.r.t. a bargaining agreement (cf. Meinhardt (2018a)). Especially, its close relationship to the pre-nucleolus and pre-kernel will be quite helpful to establish new insights in this direction. For doing so, we size – similar to the pre-kernel – on replication results. However, before we can focus on this issue, we have to invoke a short detour to discuss the modiclus in some details.

In order to define the modiclus, denoted as $\varsigma^*(N, v)$, of a game $v \in \mathcal{G}^n$, take any $\mathbf{x} \in \mathcal{I}^*(N, v)$ to define a $(2^{2\cdot n} - 1)$-tuple vector $\tilde{\theta}(\mathbf{x}, v)$ whose components are the bi-excesses $e^v(S, \mathbf{x}) - e^v(T, \mathbf{x})$ of coalitions $S, T \subseteq N$, arranged in decreasing order, that is,

$$\begin{aligned} \tilde{\theta}_i(\mathbf{x}, v) := e^v(S_k, \mathbf{x}) - e^v(T_l, \mathbf{x}) \geq e^v(S_o, \mathbf{x}) - e^v(T_p, \mathbf{x}) =: \tilde{\theta}_j(\mathbf{x}, v) \\ \text{if} \qquad 1 \leq i = k + l \leq j = o + p \leq 2^{2\cdot n} - 1. \end{aligned} \tag{8.1}$$

Arranging the so-called bi-complaint or bi-dissatisfaction vectors $\tilde{\theta}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{I}^*(N,v)$ by the lexicographic order $\leq_L$ on $\mathbb{R}^{2^{2\cdot n}-1}$, we shall write

$$\tilde{\theta}(\mathbf{x},v) <_L \tilde{\theta}(\mathbf{z},v) \qquad \text{if } \exists \text{ an integer } 1 \leq k \leq 2^{2\cdot n}-1, \tag{8.2}$$

such that $\tilde{\theta}_m(\mathbf{x},v) = \tilde{\theta}_m(\mathbf{z},v)$ for $1 \leq m < k$ and $\tilde{\theta}_k(\mathbf{x},v) < \tilde{\theta}_k(\mathbf{z},v)$. Furthermore, we write $\tilde{\theta}(\mathbf{x},v) \leq_L \tilde{\theta}(\mathbf{z},v)$ if either $\tilde{\theta}(\mathbf{x},v) <_L \tilde{\theta}(\mathbf{z},v)$ or $\tilde{\theta}(\mathbf{x},v) = \tilde{\theta}(\mathbf{z},v)$. Notice that we omit in the sequel the game in the term $\tilde{\theta}$ provided that the context is clear.

Then the modiclus $\mathcal{MD}(N,v)$ over the pre-imputations set $\mathcal{I}^*(N,v)$ is defined by

$$\mathcal{MD}(N,v) = \left\{\mathbf{x} \in \mathcal{I}^*(N,v) \mid \tilde{\theta}(\mathbf{x}) \leq_L \tilde{\theta}(\mathbf{z}) \; \forall\, \mathbf{z} \in \mathcal{I}^*(N,v)\right\}. \tag{8.3}$$

The **modiclus** of any game $v \in \mathcal{G}^n$ is non-empty as well as unique, and it is referred to as $\varsigma^*(N,v)$. At this set the total bi-complaint $\tilde{\theta}(\mathbf{x})$ is lexicographically minimized over the non-empty, compact and convex imputation set $\mathcal{I}^*(N,v)$. Notice that by the construction of the vector $\tilde{\theta}(\mathbf{x})$, it inherits the same structure as $\theta(\mathbf{x})$, which exhibits its relation to the (pre-)nucleolus (cf. Peleg and Sudhölter (2007)).

Analogously to the (pre-)nucleolus, the modiclus $\varsigma^*(N,v)$ can be determined while solving recursively a sequence of linear programs. For describing a method, we suppose that $n \geq 2$. First we define for $k = -1$ the set $\mathcal{Q}^{-1} := \emptyset$, and for $k = 0$ we define $\tilde{\mathcal{P}}^0 := (2^N \times 2^N)\backslash(\{\emptyset\} \times \{\emptyset\})$. Then we consider for each $k \in \mathbb{N}_0$ the subsequent linear problem

$$\begin{aligned} \tau^k &= \min \; t \\ &\text{s.t. } e^v(S,\mathbf{y}) - e^v(T,\mathbf{y}) \leq t \qquad \forall (S,T) \in \tilde{\mathcal{P}}^k \text{ and } \mathbf{y} \in \mathcal{X}^k \\ &\quad \left\{e^v(S,\mathbf{y}) - e^v(T,\mathbf{y}) = \tau^{i-1} \qquad \forall (S,T) \in \mathcal{Q}^i\right\}_{i \in \{-1,0,1,\ldots,k-1\}}, \end{aligned} \tag{8.4}$$

whereas $\mathcal{Q}^k := \{(S,T) \in \tilde{\mathcal{P}}^{k-1} \mid e^v(S,\mathbf{y}) - e^v(T,\mathbf{y}) = \tau^{k-1}\}$, and $\tilde{\mathcal{P}}^k := \tilde{\mathcal{P}}^{k-1}\backslash\mathcal{Q}^{k-1}$. Finally, we define $\mathcal{X}^0 := \mathcal{I}^*(N,v)$ and

$$\mathcal{X}^k := \{\mathbf{y} \in \mathcal{X}^{k-1} \mid e^v(S,\mathbf{y}) - e^v(T,\mathbf{y}) \leq \tau^{k-1} \; \forall (S,T) \in \tilde{\mathcal{P}}^{k-1}\},$$

for $k \in \mathbb{N}_+$.

Algorithm 8.1: Method for Computing the Modiclus

Data: Arbitrary TU Game $\langle N, v \rangle$.
Result: A payoff vector s.t. $\mathbf{y}_{k+1} = \varsigma^*(N,v)$.

```
begin
    k ⟵ 0, P̃^0 := (2^N × 2^N)\({∅} × {∅}), Q^{-1} := ∅
    repeat
        if P̃^k ≠ ∅ then Continue
        else Stop
        Determine a pair (τ^k, y_k) s.t. solves LP (8.4)
        Determine X^{k+1} := {y ∈ X^k | e^v(S,y) − e^v(T,y) ≤ τ^k ∀(S,T) ∈ P̃^k}
        if y_k ∈ X^{k+1} then Continue
        else Stop
        Determine Q^k := {(S,T) ∈ P̃^{k−1} | e^v(S,y) − e^v(T,y) = τ^{k−1}}
        Determine P^{k̃+1} := P̃^k\Q^k
        k ⟵ k + 1
    until P̃^{k+1} = P̃^k
end
```

For each $k \in \mathbb{N}_+$ the feasible set of linear programs (8.4) constitutes a non-empty convex polytope s.t. a pair $(\tau^k, \mathbf{y}_k)$ exists which minimizes LP (8.4).

Resume that if $\mathbf{x} \in \mathcal{I}^*(N, v)$, then

$$\begin{aligned} e^v(N\backslash S, \mathbf{x}) &= v(N\backslash S) - x(N\backslash S) = -v(N) + v(N\backslash S) - x(N\backslash S) + x(N) \\ &= -v^*(S) + x(S) = e^{-v^*}(S, -\mathbf{x}) = -(v^*(S) - x(S)) = -e^{v^*}(S, \mathbf{x}), \end{aligned}$$

for all $S \subseteq N$. Then we redefine the bi-complaint vector by

$$\begin{aligned} \bar{\theta}_i(\mathbf{x}, v) &:= e^v(S_k, \mathbf{x}) + e^{v^*}(T_l, \mathbf{x}) \geq e^v(S_o, \mathbf{x}) + e^{v^*}(T_p, \mathbf{x}) =: \bar{\theta}_j(\mathbf{x}, v) \\ &\text{if} \qquad 1 \leq i = k + l \leq j = o + p \leq 2^{2 \cdot n} - 1, \end{aligned} \tag{8.5}$$

to collocate it in a non increasing order by $\bar{\theta}(\mathbf{x}, v)$ for all $\mathbf{x} \in \mathcal{I}^*(N, v)$, and we can rewrite the modiclus $\mathcal{MD}(N, v)$ over the pre-imputations set $\mathcal{I}^*(N, v)$ as

$$\mathcal{MD}(N, v) = \left\{ \mathbf{x} \in \mathcal{I}^*(N, v) \mid \bar{\theta}(\mathbf{x}) \leq_L \bar{\theta}(\mathbf{z}) \; \forall \, \mathbf{z} \in \mathcal{I}^*(N, v) \right\}. \tag{8.6}$$

due to $\tilde{\theta}(\mathbf{y}, v) = \bar{\theta}(\mathbf{y}, v)$ for all $\mathbf{y} \in \mathcal{I}^*(N, v)$. From this result we can even deduce that the modiclus must be self-dual. Hence, $\mathcal{MD}(N, v) = \mathcal{MD}(N, v^*)$.

In order to derive a characterization of the modiclus that is based on the preventive power of coalitions, we need to introduce and discuss some dual game properties, which are the reflections of the concepts introduced in Subsection 8.2.

8.1 Some Primal Game Properties to Characterize the Modiclus

In the course of this Section, we introduce and discuss some game properties, which are indispensable to get a characterization of the modiclus and its related solutions. For a more thoroughly discussion of this topic we refer the reader to Meinhardt (2018b).

To start with, we define the maximal excess of game v w.r.t. the distribution $\mathbf{x}$ by

$$\varkappa(\mathbf{x}, v) := \max_{S \subseteq N} e^v(S, \mathbf{x}).$$

From this and $e^v(S, \mathbf{x}) = -e^{v^*}(N\backslash S, \mathbf{x})$ for all $S \subseteq N$ and $\mathbf{x} \in \mathcal{I}^*(N, v)$, we obviously get

$$\varkappa(\mathbf{x}, v) = -\min_{S \subseteq N} e^{v^*}(N\backslash S, \mathbf{x}).$$

Furthermore, by the definition we observe that $\varkappa(\mathbf{x}, v) = \pi(\mathbf{x})$ must hold. In order to avoid any confusion related to the modiclus or related solution concept, we rely on the notation that was introduced/applied in the literature. We do discuss the link to the indirect function π only when it is necessary.

In addition, define the set $\bar{N} = N \times \{0, 1\}$ s.t. $N \times \{0\} = N$ and $N \times \{1\} = N^*$, which forms a partition through $\bar{N} = N \uplus N^*$. Consider the mapping $\vartheta : N \to N^*$ that is bijective s.t. $\vartheta(i) = i^*$ is given for each $i \in N$ in order to notice that each coalition $S \subseteq N$ can be mapped one-to-one and onto to a coalition $\vartheta(S) = S^*$ in N^* s.t. $|S| = |S^*|$ holds.

For studying the axiomatization of the modiclus, we define the **dual extension** of a game v by

$$\bar{v}(S \uplus T^*) := v(S) + v^*(T) \qquad \text{for all } S, T \subseteq N, \tag{8.7}$$

whereas the **primal extension** of the dual game v^* is specified by

$$\bar{v}^*(S \uplus T^*) := v^*(S) + v(T) \qquad \text{for all } S, T \subseteq N, \tag{8.8}$$

with the relation

$$\begin{aligned}
\bar{v}^*(S \uplus T^*) &= \bar{v}(N \uplus N^*) - \bar{v}((N \uplus N^*)\backslash(S \uplus T^*)) \\
&= \bar{v}(N^* \uplus N) - \bar{v}(N\backslash S \uplus N^*\backslash T^*) \\
&= \bar{v}(N^* \uplus N) - v(N\backslash S) - v^*(N\backslash T) \\
&= 2 \cdot v(N) - v(N\backslash S) - v(N) + v(T) \\
&= v(N) - v(N\backslash S) + v(T) = v^*(S) + v(T) = v(T) + v^*(S) \\
&= \bar{v}(T \uplus S^*),
\end{aligned} \tag{8.9}$$

for all $S, T \subseteq N$.

Definition 8.1 (Dual Cover)**.** *Let $\langle N, v\rangle$ be a TU game and let $\vartheta : N \to N^*$ be a bijective map with $i \mapsto \vartheta(i)$ such that $\bar{N} = N \uplus N^*$. Define the* ***dual cover game****, denoted as $\langle \bar{N}, \tilde{v}\rangle$, by*

$$\tilde{v}(S \uplus T^*) := \max\{v(S) + v^*(T), v^*(S) + v(T)\} \qquad \forall\, S, T \subseteq N, \tag{8.10}$$

which is the ***dual cover*** *of game v.*

This game accounts for an optimistic assessment of the combined influence of the exercised and preventive power of a pair of coalitions. That means, a pair of coalitions pointing during a negotiation on the highest amount of its combined power to assure for its members the largest possible share.

Definition 8.2 (LED and Diverse Game Properties)**.** *Let $\langle N, v\rangle$ be a TU game.*

1. *We define the upper marginal contribution respectively lower marginal contribution of a player $k \in N$ to a coalition w.r.t. game $\langle N, v\rangle$ by*

$$\begin{aligned}
r_k(N, v) &:= \max_{S \subseteq N\backslash\{i\}} \big(v(S \cup \{k\}) - v(S)\big) \\
l_k(N, v) &= \min_{S \subseteq N\backslash\{i\}} \big(v(S \cup \{k\}) - v(S)\big).
\end{aligned}$$

2. *The TU game $\langle N, w\rangle$ is a* ***shift game*** *of v, if there exists a real number $t \in \mathbb{R}$ s.t.*

$$w(S) := \begin{cases} v(S) + t & \text{if } \emptyset \neq S \neq N \\ v(S) & \text{otherwise.} \end{cases} \tag{8.11}$$

*This game is denoted as the t-**shift** game of v, defined as $v^{\wedge t}$.*

3. *Let $\mathbf{x} \in \mathcal{I}^*(N, v)$ be a pre-imputation, and define the* ***excess comparability cover*** *game $\langle N, v^x\rangle$ by*

$$v^x(S) := \begin{cases} v(S) & \text{if } S \in \{\emptyset, N\}, \\ \max\Big\{v(S) + \varkappa + 2\varkappa^*, v^*(S) + \varkappa^* + 2\varkappa\Big\} & \text{otherwise,} \end{cases} \tag{8.12}$$

for all $S \subseteq N$ with $\varkappa := \varkappa(\mathbf{x}, v)$ and $\varkappa^* := \varkappa(\mathbf{x}, v^*,)$. More concisely, we denote the excess comparability cover game as ECC game.

4. *For each* $\mathbf{x} \in \mathcal{I}^*(N, v)$*, we define the* ***large excess difference*** *by*

$$\boldsymbol{\Lambda}(\mathbf{x}, v) := \min_{\emptyset \neq T \subset N} \big(v(T) - v^*(T)\big) - \overline{\varkappa}, \tag{8.13}$$

whereas $\overline{\varkappa} := \max\{e^v(S, \mathbf{x}) \,|\, \emptyset \neq S \subset N\}$ *indicates the maximal nontrivial excess at the pre-imputation* $\mathbf{x}$*. It is applied the convention that* $\min \emptyset = \infty$ *and* $\max \emptyset = -\infty$ *holds, in addition, with* $\boldsymbol{\Lambda}(\mathbf{x}, v) = 0$*, if* $|N| = 1$*.*

5. *The TU game* $\langle N, v\rangle$ *possesses the* ***large excess difference property (LED)*** *w.r.t.* $\mathbf{x} \in \mathcal{I}^*(N, v)$*, if* $\boldsymbol{\Lambda}(\mathbf{x}, v) \geq 0$ *holds.*

6. *A vector* $\mathbf{d}^v \in \mathbf{R}^N$ *is denoted as the* ***difference vector of maximal and minimal marginal contributions*** *w.r.t.* v *and* $\mathbf{x}$*, and is defined by*

$$\begin{aligned} d_i^v &:= r_i(N, v) - l_i(N, v) \\ &= \max_{S \subseteq N \setminus \{i\}} \big(v(S \cup \{i\}) - v(S)\big) - \min_{S \subseteq N \setminus \{i\}} \big(v(S \cup \{i\}) - v(S)\big), \end{aligned}$$

for each $i \in N$ *with the convention that* $d_i^v = 0$*, if* $|N| = 1$*.*

Notice that the difference vector $\mathbf{d}^v \in \mathbf{R}^N$ of maximal and minimal marginal contributions w.r.t. v and $\mathbf{x}$ specifies the reasonableness range in which a solution vector $\mathbf{x}$ ought to be located. It ought to be bounded from above by the maximal amount of marginal contribution w.r.t. a particular coalition, and to be bounded from below by the minimal marginal contribution of a player to a coalition. Thus, it is reasonable that a player should not receive an amount that is beyond this upper bound, and an amount that is not below the lower bound in order to get a fair bargaining outcome. Now, the difference vector $\mathbf{d}^v \in \mathbf{R}^N$ measures the fairness range in which an outcome can be vary.

Next, we define the set of symmetric pre-imputation by

$$\mathcal{SI}^*(\bar{N}, \tilde{v}) := \big\{\mathbf{z} \in \mathcal{I}^*(\bar{N}, \tilde{v}) \;\big|\; z_i = z_{i^*} \quad \text{for all } i \in N\big\}.$$

Lemma 8.1 (Rosenmüller and Sudhölter (2004))**.** *Let* $\langle N, v\rangle$ *be a TU game and let* $\langle \bar{N}, \tilde{v}\rangle$ *its associated dual cover as given by Definition 8.1 and take* $\mathbf{x} := (\vec{x}, \vec{x}^*) \in \mathcal{SI}^*(\bar{N}, \tilde{v})$*, then the reduced game* $\tilde{v}_{N,\mathbf{x}}$ *is specified by*

$$\tilde{v}_{N,\mathbf{x}}(S) := \begin{cases} 0 & \text{if } S = \emptyset \\ v(N) & \text{if } S = N \\ \max\big\{v(S) + \varkappa(\vec{x}, v^*), v^*(S) + \varkappa(\vec{x}, v)\big\} & \text{otherwise.} \end{cases} \tag{8.14}$$

In addition, the maximal excesses of game $\bar{v}, \bar{v}^*, \tilde{v}$ *and* $w := \tilde{v}_{N,\mathbf{x}}$ *w.r.t.* $\mathbf{x}$ *are given by* $\varkappa(\mathbf{x}, w) = \varkappa(\mathbf{x}, \tilde{v}) = \varkappa(\mathbf{x}, \bar{v}) = \varkappa(\mathbf{x}, \bar{v}^*) = \varkappa^* + \varkappa$*. Notice that* w *is the reduced game of* $\tilde{v}$ *w.r.t. coalition* N *and* $\mathbf{x}$*.*

Proof. See Meinhardt (2018b). □

Lemma 8.2 (Sudhölter (1997a).)**.** *Let* $\langle N, v\rangle$ *be a TU game s.t.* $|N| \geq 2$*, and choose* $\mathbf{x} \in \mathcal{I}^*(N, v)$*. Then the following relations are satisfied*

1.
$$\boldsymbol{\Lambda}(\mathbf{x}, v) = \min_{\emptyset \neq S, T \subset N} \Big(\min\Big\{e^v(S, \mathbf{x}), e^v(T, \mathbf{x})\Big\} - e^v(S, \mathbf{x}) - e^{v^*}(T, \mathbf{x})\Big). \tag{8.15}$$

2. $\boldsymbol{\Lambda}(\mathbf{x}, v^{\wedge t}) = \boldsymbol{\Lambda}(\mathbf{x}, v) + t$ *for* $t \in \mathbb{R}$*.*

3. *The shift game $v^{\wedge t}$ of v meets LED w.r.t $\mathbf{x}$ if, and only if, $t \geq -\mathbf{\Lambda}(\mathbf{x}, v)$.*

Proof. See Meinhardt (2018b). □

Theorem 8.1 (Sudhölter (1997a); Peleg and Sudhölter (2007))**.** *Let $\langle N, v\rangle$ be a TU game and let $\langle \bar{N}, \tilde{v}\rangle$ its associated dual cover as given by Definition 8.1, then $\varsigma^{*}(N, v) = \nu^{*}(\bar{N}, \tilde{v})_{N}$, that is, the modiclus of game $\langle N, v\rangle$ coincides with the restriction of the pre-nucleolus of the dual cover game $\langle \bar{N}, \tilde{v}\rangle$ to N.*

Proof. See Meinhardt (2018b). □

Lemma 8.3 (Sudhölter (1997a))**.** *Let $\langle N, v\rangle$ be a TU game. The pre-nucleolus of each shift game of v is equal to the pre-nucleolus of v, i.e., $\nu^{*}(N, v) = \nu^{*}(N, v^{\wedge t})$ for all $t \in \mathbb{R}$.*

Lemma 8.4 (Sudhölter (1997a,b))**.** *Let $\langle N, v\rangle$ be a TU game. If $\mathbf{x}$ is Pareto optimal, then the ECC game v^{x} satisfies LED w.r.t. $\mathbf{x}$. Moreover, it holds $v^{x}(S) = \tilde{v}_{N,\mathbf{x}}(S) + t$ for all $S \subset N$ and $v^{x}(N) = v(N)$ with $t := (\varkappa + \varkappa^{*})$. Whereas the game $\tilde{v}_{N,\mathbf{x}}$ is specified by Lemma 8.1.*

Proof. See Meinhardt (2018b). □

Corollary 8.1 (Meinhardt (2018b))**.** *Let $\langle N, v\rangle$ fulfills LED w.r.t. $\mathbf{x}$, then*

1. *the ECC game is given by $v^{\mathbf{x}}(S) = v(S) + \varkappa$ for all $S \subseteq N$ and $v^{\mathbf{x}}(N) = v(N)$;*
2. *the reduced game $\tilde{v}_{N,\mathbf{x}}$ coincides with v.*

Whereas the game $\tilde{v}_{N,\mathbf{x}}$ is specified by Lemma 8.1.

REMARK 8.1.
In accordance with the second assertion of Corollary 8.1 and in connection with Theorem 8.1, the following relationship between the modiclus and pre-nucleolus is immediately revealed

$$\varsigma^{*}(N, v) = \nu^{*}(\bar{N}, \tilde{v})_{N} = \nu^{*}(N, \tilde{v}_{N,\mathbf{x}}) = \nu^{*}(N, v), \tag{8.16}$$

whenever $\langle N, v\rangle$ fulfills LED w.r.t. $\mathbf{x}$. ◇

Definition 8.3 (Diverse Solution Properties)**.** *Let σ be a solution concept on the set $\mathcal{G}$, and $\mathcal{U}$ the universe of players.*

1. *A solution σ on $\mathcal{G}$ is called **reasonable from above (REAB)**, if $\langle N, v\rangle \in \mathcal{G}, \mathbf{x} \in \sigma(N, v)$, then $\mathbf{x} \leq \mathbf{r}$, whereas $\mathbf{r}$ is defined by 8.2 (1a).*
2. *A solution σ on $\mathcal{G}$ is called **reasonable from below (REBE)**, if $\langle N, v\rangle \in \mathcal{G}, \mathbf{x} \in \sigma(N, v)$, then $\mathbf{x} \geq \mathbf{l}$, whereas $\mathbf{l}$ is defined by 8.2 (1b).*
3. *A solution σ on $\mathcal{G}$ is called **reasonable (RE)** from both sides, if it meets REBE as well as REAB.*
4. *A solution σ on $\mathcal{G}$ fulfills **excess comparability (EC)**, if $\langle N, v\rangle \in \mathcal{G}, \mathbf{x} \in \sigma(N, v)$ and $\langle N, v^{x}\rangle \in \mathcal{G}$, then $\mathbf{x} \in \sigma(N, v^{x})$, whereas $\langle N, v^{x}\rangle$ is defined by 8.2 (3).*
5. *A solution σ satisfies the **large excess difference consistency (LEDCONS)**, if for $\langle N, v\rangle \in \mathcal{G}, \mathbf{x} \in \sigma(N, v)$ and v satisfies LED w.r.t. $\mathbf{x}$, then $\langle S, v_{S,\mathbf{x}}\rangle \in \mathcal{G}$ and $\mathbf{x}_{S} \in \sigma(S, v_{S,\mathbf{x}})$.*
6. *A solution σ meets the **dual replication property (DRP)**, if for $\langle N, v\rangle \in \mathcal{G}$, for a bijection $\vartheta \in Sym(\bar{N})$ with $\bar{N} = N \uplus N^{*}$ s.t. $\langle \vartheta\bar{N}, \bar{w}\rangle \in \mathcal{G}$, where $\bar{w} := \vartheta\bar{v}$ (cf. Equation (8.7)), and if $\mathbf{x} \in \sigma(N, v)$, then $\vartheta(\mathbf{x}, \mathbf{x}^{*}) \in \sigma(\vartheta\bar{N}, \bar{w})$.*

7. *A solution σ meets the **dual cover property (DCP)**, if for $\langle N, v\rangle \in \mathcal{G}$, for a bijection $\vartheta \in Sym(\bar{N})$ with $\bar{N} = N \uplus N^{*}$ s.t. $\langle \vartheta\bar{N}, \tilde{w}\rangle \in \mathcal{G}$, where $\tilde{w} := (\vartheta\tilde{v})^{\wedge t}$ with $t = 6d^{v}(N)$ (cf. Equation (8.10)), and if $\mathbf{x} \in \sigma(N, v)$, then $\vartheta(\mathbf{x}, \mathbf{x}^{*}) \in \sigma(\vartheta\bar{N}, \tilde{w})$.*

Lemma 8.5 (Sudhölter (1997a))**.** *Let $\langle N, v\rangle$ be a TU game.*

1. *Let $\vec{y} := \nu^{*}(N, v)$ be the pre-nucleolus of v. If v fulfills LED w.r.t. $\vec{y}$, then $\vec{y} = \varsigma^{*}(N, v)$.*
2. *Consider a pre-imputation $\mathbf{x} \in \mathcal{I}^{*}(N, v)$ and a proper coalition $\emptyset \neq S \subseteq N$. If v satisfies LED w.r.t. $\mathbf{x}$, then $\langle S, v_{S,\mathbf{x}}\rangle$ satisfies LED w.r.t. $\mathbf{x}_{S}$.*
3. *Let $t := 6 \cdot \mathbf{d}^{v}(N)$, and assume that the pre-imputation $\vec{x} \in \mathcal{I}^{*}(N, v)$ is reasonable on both sides w.r.t. v, then the shifted dual cover game $\tilde{v}^{\wedge t}$ fulfills LED w.r.t. the symmetric pre-imputation $\mathbf{x} := (\vec{x}, \vec{x}^{*}) \in \mathcal{SI}^{*}(\bar{N}, \tilde{v})$.*
4. *Let v satisfies LED w.r.t. $\vec{x} \in \mathcal{I}^{*}(N, v)$, then v fulfills LED w.r.t. $\vec{y} := \nu^{*}(N, v)$.*

Proof. See Meinhardt (2018b) □

A characterization of the modiclus in terms of the above properties can be given by:

Theorem 8.2 (Sudhölter (1997a))**.** *Let $\mathcal{U}$ be an infinite player set. Then the modiclus is the unique solution concept on $\mathcal{G}_{\mathcal{U}}$ satisfying **SIVA, COV, EC, LEDCONS**, and **DCP**, whereas **DCP** can also be replaced by **DRP**.*

Proof. See Meinhardt (2018b). □

Alternatively, the axiomatization of the modiclus can be characterized by SIVA, COV, DCP, and derived game property (DGP), which is a modification of the reduced game property w.r.t. the modiclus. This characterization comes very close to those of the pre-nucleolus, which is provided by SIVA, anonymity (AN), COV, and the reduced game property (RGP). This reveals that the modiclus is a derivative of the pre-nucleolus (cf. Meinhardt (2018b)).

8.2 Some Dual Game Properties to Characterize the Modiclus

In order to derive a characterization of the modiclus that is based on the preventive power of coalitions, we need to introduce and discuss some dual game properties, which we shall provide here within the original game context. Hence, we referring to the preventive power of a coalition without leaving the original game and introducing the dual game. We avoid by this construction that we are pointing to a different stylized bargaining scenario with the consequence that we are therefore changing the theoretical framework in which subjects are involved.

The minimal excess of game v w.r.t. the distribution $\mathbf{x}$ is given by

$$\underline{\varkappa}(\mathbf{x}, v) := \min_{S \subseteq N} e^{v}(S, \mathbf{x}).$$

It is evident that from this and $e^{v}(S, \mathbf{x}) = -e^{v^{*}}(N\backslash S, \mathbf{x})$ for all $S \subseteq N$ and $\mathbf{x} \in \mathcal{I}^{*}(N, v)$, we get

$$\underline{\varkappa}(\mathbf{x}, v) = -\max_{S \subseteq N} e^{v^{*}}(N\backslash S, \mathbf{x}) = -\varkappa(\mathbf{x}, v^{*}).$$

Definition 8.4 (SED and Diverse Game Properties)**.** *Let $\langle N, v\rangle$ be a TU game.*

1. *Let* $\mathbf{x} \in \mathcal{I}^*(N,v)$ *be a pre-imputation, and define the* ***excess comparability floor*** *game* $\langle N, v_x^{\#}\rangle$ *by*

$$v_x^{\#}(S) := \begin{cases} v(S) & \text{if } S \in \{\emptyset, N\}, \\ \min\left\{v(S) + \underline{\varkappa} + 2\underline{\varkappa}^*, v^*(S) + \underline{\varkappa}^* + 2\underline{\varkappa}\right\} & \text{otherwise}, \end{cases} \tag{8.17}$$

   for all $S \subseteq N$ *with* $\underline{\varkappa} := \underline{\varkappa}(\mathbf{x}, v)$ *and* $\underline{\varkappa}^* := \underline{\varkappa}(\mathbf{x}, v^*)$*. More concisely, we denote the excess comparability floor game as ECF game.*

2. *For each* $\mathbf{x} \in \mathcal{I}^*(N,v)$*, we define the* ***small excess difference*** *by*

$$\underline{\mathbf{\Lambda}}(\mathbf{x}, v) := \max_{\emptyset \neq T \subset N} \big(v(T) - v^*(T)\big) - \underline{\varkappa}_m, \tag{8.18}$$

   whereas $\underline{\varkappa}_m := \min\{e^v(S,\mathbf{x}) \,|\, \emptyset \neq S \subset N\}$ *indicates the minimal nontrivial excess at the pre-imputation* $\mathbf{x}$*. It is applied the convention that* $\min \emptyset = \infty$ *and* $\max \emptyset = -\infty$ *holds, in addition, with* $\underline{\mathbf{\Lambda}}(\mathbf{x}, v) = 0$*, if* $|N| = 1$*.*

3. *The TU game* $\langle N, v\rangle$ *possesses the* ***small excess difference property (SED)*** *w.r.t.* $\mathbf{x} \in \mathcal{I}^*(N,v)$*, if* $\underline{\mathbf{\Lambda}}(\mathbf{x}, v) \leq 0$ *holds.*

REMARK 8.2 (Meinhardt (2018b)).
Notice that a TU game $\langle N, v\rangle$ possesses the **small excess difference property (SED)** w.r.t. $\mathbf{x}$ if, and only if, the dual game $\langle N, v^*\rangle$ of v possesses the **large excess difference property (LED)** w.r.t. $\mathbf{x}$ (cf. Formula (8.13)). To observe this, assume that the dual game $\langle N, v^*\rangle$ fulfills LED w.r.t. $\mathbf{x}$, hence $\mathbf{\Lambda}(\mathbf{x}, v^*) \geq 0$, from which we get the subsequent chain of equivalence relations:

$$\begin{aligned}
\mathbf{\Lambda}(\mathbf{x}, v^*) &:= \min_{\emptyset \neq T \subset N} \big(v^*(N\backslash T) - v(N\backslash T)\big) - \overline{\varkappa} \geq 0 \\
&\iff \min_{\emptyset \neq T \subset N} \big(v(N) - v(T) - v(N\backslash T)\big) - \max_{\emptyset \neq S \subset N} \big(e^{v^*}(N\backslash S, \mathbf{x})\big) \geq 0 \\
&\iff -\max_{\emptyset \neq T \subset N} \big(v(T) + v(N\backslash T) - v(N)\big) - \max_{\emptyset \neq S \subset N} \big(-e^v(S, \mathbf{x})\big) \geq 0 \\
&\iff -\max_{\emptyset \neq T \subset N} \big(v(T) - v^*(T)\big) + \min_{\emptyset \neq S \subset N} \big(e^v(S, \mathbf{x})\big) \geq 0 \\
&\iff \max_{\emptyset \neq T \subset N} \big(v(T) - v^*(T)\big) - \min_{\emptyset \neq S \subset N} \big(e^v(S, \mathbf{x})\big) \leq 0 \\
&\iff \max_{\emptyset \neq T \subset N} \big(v(T) - v^*(T)\big) - \underline{\varkappa}_m =: \underline{\mathbf{\Lambda}}(\mathbf{x}, v) \leq 0
\end{aligned}$$

Conversely, a TU game $\langle N, v\rangle$ satisfies the LED property w.r.t. $\mathbf{x}$ if, and only if, its dual $\langle N, v^*\rangle$ satisfies the SED property w.r.t. $\mathbf{x}$. Thus, we can refer to an induced property of the preventive power within the original game context (bargaining context) without explicitly introducing the dual game. Finally, recognize that a game might satisfy both, i.e., LED and SED, or at least one of those, or none of those w.r.t. $\mathbf{x}$. ◇

Definition 8.5 (Dual Floor)**.** *Let* $\langle N, v\rangle$ *be a TU game and let* $\vartheta : N \to N^*$ *be a bijective map with* $i \mapsto \vartheta(i)$ *such that* $\bar{N} = N \uplus N^*$*. Define the* ***dual floor game****, denoted as* $\langle \bar{N}, \tilde{\bar{v}}\rangle$*, by*

$$\tilde{\bar{v}}(S \uplus T^*) := \min\{v(S) + v^*(T), v^*(S) + v(T)\} \qquad \forall\, S, T \subseteq N \tag{8.19}$$

which is called the ***dual floor*** *of game* v*.*

The game $\langle \tilde{N}, \tilde{\tilde{v}} \rangle$ as defined by Equation (8.19) is the floor of the dual extension of $\langle N, v \rangle$ and the primal extension of the dual game $\langle N, v^* \rangle$. This game reflects a prudence assessment of the combined influence of the potential (primal) and preventive (dual) power of a pair of coalitions in a stylized bargaining scenario. The game is the reverse counterpart to the dual cover game that accounts for an optimistic assessment of the combined influence of the primal and dual extension.

Consider a game $\langle N, v \rangle \in \mathcal{G}, \emptyset \neq S \subseteq N$ and let $\mathbf{x} \in \mathcal{I}^*(N, v)$. The **anti-reduced game** w.r.t. S and $\mathbf{x}$ is the game $\langle S, v^{\#}_{S,\mathbf{x}} \rangle$ as given by

$$v^{\#}_{S,\mathbf{x}}(T) := \begin{cases} 0 & \text{if } T = \emptyset \\ v(N) - x(N \backslash S) & \text{if } T = S \\ \min_{Q \subseteq N \backslash S} \left(v(T \cup Q) - x(Q) \right) & \text{otherwise.} \end{cases} \tag{8.20}$$

The definition leads to the idea to impose a particular consistency requirement on a solution concept – in order to qualify as a desirable outcome of a game – while incorporating the preventive power of outsiders in the sense that partners can only referring to their smallest claim. This is a weaker requirement on a solution concept that ought to be satisfied as under the Davis/Maschler reduced game. Under this scenario partners of a subgroup S focus on a pessimistic view that the preventive power of outsiders allows them only to capture the smallest amount that would be obtainable while entering into a joint cooperation with outsiders to enforce higher shares for them in a negotiation. As a consequence, partners can only refer to weaker claims to get higher shares w.r.t. a proposal $\mathbf{x}$, which was offered during a negotiation. This means that a solution can be classified as stable or consistent provided that the preventive power of outsiders discourage any subgroup of players to deviate from the proposal while leaving the grand coalition in order to play their own game with the help of some outsiders. In this context, the imposed norm on the solution, which we denote anti-reduced game property (ARGP), can also be seen as subgame perfectness or a stability requirement of a cooperative solution concept. As worked out by Meinhardt (2018b), the anti-pre-nucleolus fulfills ARGP. This means roughly stated that all possible projections of the anti-pre-nucleolus of the default game onto the restricted sets equal the anti-pre-nucleolus of the associated anti-reduced games. We formalize this notion by incorporating the subsequent definition:

Definition 8.6 (Anti-RGP). *A solution σ on $\mathcal{G}$ satisfies the anti-reduced game property (ARGP), if for $\langle N, v \rangle \in \mathcal{G}, \emptyset \neq S \subseteq N$ and $\mathbf{x} \in \sigma(N, v)$, then $\langle S, v^{\#}_{S,\mathbf{x}} \rangle \in \mathcal{G}$ and $\mathbf{x}_S \in \sigma(S, v^{\#}_{S,\mathbf{x}})$.*

Lemma 8.6. *Let $\langle N, v \rangle$ be a TU game and $\mathbf{y} := \nu^{\#}(N, v)$ be the anti-pre-nucleolus of v. In addition, consider a pre-imputation $\mathbf{x} \in \mathcal{I}^*(N, v)$ and a proper coalition $\emptyset \neq S \subseteq N$.*

1. *If v fulfills SED w.r.t. $\mathbf{y}$, then $\mathbf{y} = \varsigma^*(N, v)$.*
2. *If v satisfies SED w.r.t. $\mathbf{x}$, then $\langle S, v^{\#}_{S,\mathbf{x}} \rangle$ satisfies SED w.r.t. $\mathbf{x}_S$ for all $\emptyset \neq S \subseteq N$.*
3. *Let $t := -6 \cdot \mathbf{d}^v(N)$, and assume that the pre-imputation $\vec{x} \in \mathcal{I}^*(N, v)$ is reasonable on both sides w.r.t. v, then the shifted dual floor game $\tilde{\tilde{v}}^{\wedge t}$ fulfills SED w.r.t. the symmetric pre-imputation $\mathbf{x} := (\vec{x}, \vec{x}^*) \in \mathcal{SI}^*(\bar{N}, \tilde{\tilde{v}})$.*
4. *Let v satisfies SED w.r.t. $\vec{x} \in \mathcal{I}^*(N, v)$, then v fulfills SED w.r.t. $\vec{y} := \nu^{\#}(N, v)$.*

Proof. See Meinhardt (2018b) □

Example 8.1. Again resume Example 4.1. We have worked out by Example 7.1 that the modiclus is identical to the anti-pre-nucleolus of game v, which was given by the pre-imputation $\mathbf{x} := (8, 53/12, 37/12, 1/2)$. In order to observe whether the game satisfies either LED or SED w.r.t. the vector $\mathbf{x}$, we compute the large excess difference and the small excess difference, which are given by $\mathbf{\Lambda}(\mathbf{x}, v) = -15.5 < 0$ and $\underline{\mathbf{\Lambda}}(\mathbf{x}, v) = 0$ respectively. From the latter result we realize that the game must fulfill SED w.r.t. the payoff vector $\mathbf{x}$. Hence, the pre-imputation $\mathbf{x}$ must be an element of the core due to Meinhardt (2018b, Lemma 7.2). In view of Lemma 8.6 we notice that the game satisfies SED w.r.t. the anti-pre-nucleolus, from which it immediately follows that it must coincide with the modiclus of the game. Thus, the modiclus belongs to the core of game v.

For a more thorough discussion of this topic, we refer the reader to Meinhardt (2018b).

Definition 8.7 (SEDCONS and REC)**.** *Let σ be a solution concept on the set $\mathcal{G}$, and $\mathcal{U}$ the universe of players.*

1. *A solution σ satisfies the **small excess difference consistency (SEDCONS)**, if for $\langle N, v\rangle \in \mathcal{G}, \mathbf{x} \in \sigma(N, v)$ and v satisfies SED w.r.t. $\mathbf{x}$, then $\langle\, S, v^{\#}_{S,\mathbf{x}}\,\rangle \in \mathcal{G}$ and $\mathbf{x}_S \in \sigma(S, v^{\#}_{S,\mathbf{x}})$.*
2. *A solution σ on $\mathcal{G}$ fulfills **reverse excess comparability (REC)**, if $\langle N, v\rangle \in \mathcal{G}, \mathbf{x} \in \sigma(N, v)$ and $\langle N, v^{\#}_{x}\rangle \in \mathcal{G}$, then $\mathbf{x} \in \sigma(N, v^{\#}_{x})$.*
3. *A solution σ meets the **primal replication property (PRP)**, if for $\langle N, v\rangle \in \mathcal{G}$, for a bijection $\vartheta \in Sym(\bar{N})$ with $\bar{N} = N \uplus N^*$ s.t. $\langle \vartheta\bar{N}, \bar{w}^*\rangle \in \mathcal{G}$, where $\bar{w}^* := \vartheta\bar{v}^*$ (cf. Equation (8.8)), and if $\mathbf{x} \in \sigma(N, v)$, then $\vartheta(\mathbf{x}, \mathbf{x}^*) \in \sigma(\vartheta\bar{N}, \bar{w}^*)$.*
4. *A solution σ meets the **dual floor property (DFP)**, if for $\langle N, v\rangle \in \mathcal{G}$, for a bijection $\vartheta \in Sym(\bar{N})$ with $\bar{N} = N \uplus N^*$ s.t. $\langle \vartheta\bar{N}, \tilde{\bar{w}}\rangle \in \mathcal{G}$, where $\tilde{\bar{w}} := (\vartheta\tilde{\bar{v}})^{\wedge t}$ with $t = -6d^v(N)$ (cf. Equation (8.19)), and if $\mathbf{x} \in \sigma(N, v)$, then $\vartheta(\mathbf{x}, \mathbf{x}^*) \in \sigma(\vartheta\bar{N}, \tilde{\bar{w}})$.*

REMARK 8.3.
The idea behind the excess comparability floor game $\langle N, v^{\#}_{x}\rangle$ is analogous to that of the ECC game v^x, and can be summarized as follows: Consider a Pareto optimal offer $\mathbf{x}$ from the set $\mathcal{I}^*(N, v)$ to divide the proceeds of $v(N)$. In order to obtain such a division, the partners involved of the offer to split the revenue have agreed upon that the figure of argumentation must consider the potential as well as the preventive power of coalitions, but by a reflective consideration. As a consequence, the resultant dissatisfaction alluded to the proposal $\mathbf{x}$ incorporates the combined smallest degree of dissatisfaction that arises with the proposal $\mathbf{x}$, that is, the combined smallest losses of the potential and preventive power that a proper subgroup of partners must bear while either pointing to the exercised or preventive power. These losses have taken into account by a comparable and purely symmetrical consideration w.r.t. the offer $\mathbf{x}$. This is accomplished that the modified minimal excesses w.r.t. v and v^* coincide. For a proper coalition these amounts were corrected either by its worth $v(S)$ or by its preventive value $v^*(S)$. These values give pairs of comparable modified excesses w.r.t. $\mathbf{x}$, where the smallest level of dissatisfaction is selected. #

Having discussed these properties, we can provide an axiomatization of the modiclus that is based on the preventive power of coalitions without explicitly pointing to the dual game.

Theorem 8.3 (Meinhardt (2018b))**.** *Let $\mathcal{U}$ be an infinite player set. Then the modiclus is the unique solution concept on $\mathcal{G}_{\mathcal{U}}$ satisfying **SIVA, COV, SEDCONS, REC**, and **DFP**, whereas **DFP** can also be replaced by **PRP**.*

Proof. Meinhardt (2018b) □

Nevertheless, in view of this axiomatization of the modiclus is still pointing to different game as well as to different solution contexts via SEDCONS, REC and DFP or PRP (see also Meinhardt (2018b)).

9 Replication of Related Solutions

This Section is devoted to study replication results of related solutions of the pre-kernel with the help of indirect function approach. This shall allow us to derive replication results of the modiclus. By an initial step, we establish that the modiclus is identical to the pre-nucleolus or even to the anti-pre-nucleolus under regular conditions for a class of shifted games. In a next step, we extend these results while applying the indirect function approach. In this context, the game properties of the preceding sections are quite helpful to derive these replication results w.r.t. the modiclus. The main results are referring to the ECC (cf. Theorem 9.1) and to the ECF game (cf. Corollary 9.9) respectively. Moreover, in accordance with the preventive power of coalitions, there is also close relationship to the anti-pre-nucleolus with the consequence that we can apply these properties to get even for an exposed element of the anti-pre-kernel a replication result (cf. Proposition 9.2). Though this relationship meets our expectation, when remember us that for the class of PS, weighted graph, and modest bankruptcy games the modiclus coincides with the anti-pre-nucleolus (cf. Meinhardt (2018b, Section 10)).

In order to start with our discussion, we provide a simplification of the anti-reduced game $\tilde{\tilde{v}}^{\#}_{N,\mathbf{x}}$ w.r.t. the grand coalition N and a pre-imputation $\mathbf{x}$ (cf. Definition (8.20)), which we restate here without proof through:

Lemma 9.1 (Meinhardt (2018b)). *Let $\langle N, v\rangle$ be a TU game and let $\langle \bar{N}, \tilde{\tilde{v}}\rangle$ its associated dual floor as given by Definition 8.5 and take $\mathbf{x} := (\vec{x}, \vec{x}^{*}) \in \mathcal{SI}^{*}(\bar{N}, \tilde{\tilde{v}})$, then the anti-reduced game $\tilde{\tilde{v}}^{\#}_{N,\mathbf{x}}$ is specified by*

$$\tilde{\tilde{v}}^{\#}_{N,\mathbf{x}}(S) := \begin{cases} 0 & \text{if } S = \emptyset \\ v(N) & \text{if } S = N \\ \min\left\{v(S) + \underline{\varkappa}(\vec{x}, v^{*}), v^{*}(S) + \underline{\varkappa}(\vec{x}, v)\right\} & \text{otherwise.} \end{cases} \tag{9.1}$$

In addition, the minimal excesses of game $\bar{v}^{*}, \bar{v}, \tilde{\tilde{v}}$ and $w := \tilde{\tilde{v}}^{\#}_{N,\mathbf{x}}$ w.r.t. $\mathbf{x}$ are given by $\underline{\varkappa}(\mathbf{x}, w) = \underline{\varkappa}(\mathbf{x}, \tilde{\tilde{v}}) = \underline{\varkappa}(\mathbf{x}, \bar{v}) = \underline{\varkappa}(\mathbf{x}, \bar{v}^{*}) = \underline{\varkappa}^{*} + \underline{\varkappa}$. Notice that w is the anti-reduced game of $\tilde{\tilde{v}}$ w.r.t. coalition N and $\mathbf{x}$.

Proof. See Meinhardt (2018b). □

Proposition 9.1 (Meinhardt (2018b)). *Let $\langle N, v\rangle$ be a TU game, set $\vec{y} := \varsigma^{*}(N, v)$, and let $t_0 \in \mathbb{R}$ be a critical number s.t. $\mathbf{\Lambda}(\vec{y}, v^{\wedge t_0}) \geq 0$ is satisfied. If $\nu^{*}(N, v) = \nu^{*}(N, v^{\vec{y}})$, then $\varsigma^{*}(N, v) = \varsigma^{*}(N, v^{\wedge t_0})$.*

Proof. Set $u := v^{\wedge t_0}$ and by the supposition $\mathbf{\Lambda}(\vec{y}, u) \geq 0$, the game u fulfills LED w.r.t. $\vec{y}$. Notice that due to Lemma 8.2 (2) such a critical number $t_0 \in \mathbb{R}$ can be assured. Recall now the Theorems 2.1, 8.1, and the Lemmata 8.3, 8.4, then

$$y := \varsigma^{*}(N, v) = \nu^{*}(\bar{N}, \tilde{v})_N = \nu^{*}(N, \tilde{v}_{N,\mathbf{y}}) = \nu^{*}(N, v^{\vec{y}}).$$

In the next step define $w := u^{\vec{y}}$. In this respect, remind Corollary 8.1 and Remark 8.1, from which we deduce in connection with LED

$$\begin{aligned} \varsigma^{*}(N, u) &= \nu^{*}(\bar{N}, \tilde{u})_N = \nu^{*}(N, \tilde{u}_{N,\mathbf{y}}) = \nu^{*}(N, u) \\ &= \nu^{*}(N, w) = \nu^{*}(N, v^{\wedge t}) = \nu^{*}(N, v), \end{aligned}$$

whereas $t := 2 \cdot t_0 + \varkappa(\mathbf{x}, v)$. Notice that from Corollary 8.1 (2), we get $u = \tilde{u}_{N,\mathbf{y}}$, and from Corollary 8.1 (1), we obtain

$$w = u + \vec{\varkappa}^{u} = v + \bar{\mathbf{t}}_0 + \vec{\varkappa}^{u} = v + 2 \cdot \bar{\mathbf{t}}_0 + \vec{\varkappa}^{v} = v + \bar{\mathbf{t}},$$

with $\bar{\mathbf{t}}_0 := (\mathbf{t}_0, 0)$ for $\mathbf{t}_0 \in \mathbb{R}^{2^n-1}$; $\bar{\mathbf{t}} := (\mathbf{t}, 0)$ for $\mathbf{t} \in \mathbb{R}^{2^n-1}$; $\vec{\varkappa}^u := (\vec{\varkappa}(\mathbf{x}, u), 0) \in \mathbb{R}^{2^n-1}$ as well as $\vec{\varkappa}^v := (\vec{\varkappa}(\mathbf{x}, v), 0) \in \mathbb{R}^{2^n-1}$. Hence, we can apply Lemma 8.3.

By hypothesis, we have $\nu^*(N, v) = \nu^*(N, v^{\vec{y}})$, but then $\varsigma^*(N, v) = \varsigma^*(N, u)$ immediately follows. □

Corollary 9.1 (Meinhardt (2018b))**.** *Let $\langle N, v\rangle$ be a TU game, put $\vec{y} := \varsigma^*(N, v)$, and choose $t_0 \in \mathbb{R}$ as a critical number s.t. $\underline{\mathbf{\Lambda}}(\vec{y}, v^{\wedge t_0}) \leq 0$ is satisfied. If $\nu^{\#}(N, v) = \nu^{\#}(N, v^{\#}_{\vec{y}})$, then $\varsigma^*(N, v) = \varsigma^*(N, v^{\wedge t_0})$.*

Proof. For a proof see Meinhardt (2018b). □

Corollary 9.2 (Meinhardt (2018b))**.** *Let $\langle N, v\rangle$ be a TU game, put $\vec{y} := \varsigma^*(N, v)$, and choose $t_0 \in \mathbb{R}$ as a critical number s.t. $\mathbf{\Lambda}(\vec{y}, v^{\wedge t_0}) \geq 0$ is satisfied. If $\nu^*(N, v^{\wedge t_1}) = \nu^*(N, (v^{\wedge t_1})^{\vec{y}})$ for a number $t_1 < t_0$, then $\varsigma^*(N, v^{\wedge t}) = \varsigma^*(N, v^{\wedge t_0})$ for all $t \geq t_1$.*

Corollary 9.3 (Meinhardt (2018b))**.** *Let $\langle N, v\rangle$ be a TU game, put $\vec{y} := \varsigma^*(N, v)$, and choose $t_0 \in \mathbb{R}$ as a critical number s.t. $\underline{\mathbf{\Lambda}}(\vec{y}, v^{\wedge t_0}) \leq 0$ is satisfied. If $\nu^{\#}(N, v^{\wedge t_1}) = \nu^{\#}(N, (v^{\wedge t_1})^{\#}_{\vec{y}})$ for a number $t_1 > t_0$, then $\varsigma^*(N, v^{\wedge t}) = \varsigma^*(N, v^{\wedge t_0})$ for all $t \leq t_1$.*

REMARK 9.1 (Meinhardt (2018b)).
From the preceding results we infer that $\nu^{\#}(N, v^{\#}_{\vec{y}})$ is determined by a shifted game $v^{\wedge t}$, and the irrelevant game $\tilde{\tilde{v}}^{\#}_{N,\mathbf{y}} - v$ in accordance with $v^{\#}_{\vec{y}} = v + \bar{\mathbf{t}} + \tilde{\tilde{v}}^{\#}_{N,\mathbf{y}} - v$. This holds true whenever no coalition that determines $\nu^{\#}(N, \tilde{\tilde{v}}^{\#}_{N,\mathbf{y}} - v)$ appears in the determination set of $\nu^{\#}(N, v^{\wedge t})$. This allows a decomposition of $v^{\#}_{\vec{y}}$ in the sense of Hernández-Lamoneda et al. (2007) by the games $v^{\wedge t}$, and $\tilde{\tilde{v}}^{\#}_{N,\mathbf{y}} - v$ s.t. $\nu^{\#}(N, v) = \nu^{\#}(N, v^{\#}_{\vec{y}})$ is given. This means that the anti-pre-nucleolus of game $v^{\#}_{\vec{y}}$ can be dissected in two parts, since we have $\nu^{\#}(N, v^{\#}_{\vec{y}}) = \nu^{\#}(N, \tilde{\tilde{v}}^{\#}_{N,\mathbf{y}} - v) + \nu^{\#}(N, v^{\wedge t}) = \nu^{\#}(N, v^{\wedge t})$. Notice in this respect that it is not enough here that we get $\nu^{\#}(N, \tilde{\tilde{v}}^{\#}_{N,\mathbf{y}} - v) = \mathbf{0}$. If there is a non-empty intersection of the determination sets, we obtain nevertheless then $\nu^{\#}(N, v^{\#}_{\vec{y}}) \neq \nu^{\#}(N, \tilde{\tilde{v}}^{\#}_{N,\mathbf{y}} - v) + \nu^{\#}(N, v^{\wedge t}) = \nu^{\#}(N, v^{\wedge t})$. Of course, analogous arguments can be applied for the pre-nucleolus of the ECC game $v^{\vec{y}}$.
◇

Proposition 9.1 imposes very crude conditions on the primal power of a game in order to provide a coincidence results of modicli. However, the proposition neither tells us where the crucial condition $\nu^*(N, v) = \nu^*(N, v^{\vec{y}})$ is coming from nor of how we can assure those. To shed more light on this issue, we have to rely on some techniques and results introduced in Sections 3 & 4 (see also the the work of Meinhardt (2013)). Proceeding in this direction, we need to recall some definitions and notations as used for the discussion on Section 3 and Theorem 4.1. In particular, we have recall the critical numbers $\delta^{\varepsilon}_{ij}(\mathbf{x}) \in \mathbb{R}$ from Section 4, which we restate here for convenience sake

$$\delta^{\varepsilon}_{ij}(\mathbf{x}) := \max\left\{\delta \in \mathbb{R} \,|\, \mathbf{x}^{\,i,j,\delta} = \mathbf{x} - \delta\,\mathbf{1}_i + \delta\,\mathbf{1}_j \in \varepsilon\right\} \qquad \forall\, i, j \in N, i \neq j. \tag{4.3}$$

That is, the number $\delta^{\varepsilon}_{ij}(\mathbf{x})$ is the maximum amount that can be transferred from i to j while remaining in the ellipsoid ε, which is well defined number for convex sets having non-empty interior.

Theorem 9.1 (ECC Game and Replication of the Modiclus)**.** *Let $v^{\Delta} := \tilde{v}_{N,\mathbf{y}} - v$ and put $\vec{y} := \nu^*(N, v^{\vec{y}})$. If the induced payoff equivalence class $[\vec{\gamma}]$ of the ECC game $\langle N, v^{\vec{y}}\rangle$ has non-empty interior s.t. $\{\vec{y}\} =$*

$\mathcal{PK}(N, v^{\vec{y}}) \subset [\vec{\gamma}]$ and $v^{\Delta} \in [-\mathsf{C}, \mathsf{C}]^{p'}$ with

$$\mathsf{C} := \min_{i,j \in N, i \neq j} \left\{ \left| \frac{\pm\sqrt{\bar{c}}}{\|\mathbf{E}^{\top}(\mathbf{1}_j - \mathbf{1}_i)\|} \right| \right\}, \tag{9.2}$$

as well as $\mathbf{0} \neq \Delta \in \mathcal{N}_{\mathcal{W}} = \{\Delta \in \mathbb{R}^{p'} \mid \mathcal{W}\Delta = \mathbf{0}\}$ with matrix $\mathcal{W} := \mathcal{V}^{\top}\mathcal{U}$, then $\{\vec{y}\} = \{\nu^{*}(N, v)\} = \mathcal{PK}(N, v)$. Furthermore, we get $\varsigma^{*}(N, v) = \varsigma^{*}(N, v^{\wedge t_0})$ for t_0 as given by Proposition 9.1.

Proof. By Lemma 4.4 $\vec{y}^{\,i,j,\delta^{\varepsilon}} \in \varepsilon \subset [\vec{\gamma}]$, is a unique boundary point of the ellipsoid ε of type (3.15) with maximum volume. We conclude that either (1) $s_{ij}(\vec{y}^{\,i,j,\delta^{\varepsilon}}, v^{\vec{y}}) = s_{ij}(\vec{y}, v^{\vec{y}}) + \delta^{\varepsilon}_{ij}(\vec{y})$ if $S \in \mathcal{G}_{ij}$, or (2) $s_{ji}(\vec{y}^{\,i,j,\delta^{\varepsilon}}, v^{\vec{y}}) = s_{ji}(\vec{y}, v^{\vec{y}}) - \delta^{\varepsilon}_{ij}(\vec{y})$ if $S \in \mathcal{G}_{ji}$, or otherwise (3) $s_{ij}(\vec{y}^{\,i,j,\delta^{\varepsilon}}, v^{\vec{y}}) = s_{ij}(\vec{y}, v^{\vec{y}})$ is satisfied. Moreover, let $v, v^{\vec{y}}, v^{\Delta} \in \mathbb{R}^{p'}$ and notice that the ECC game $v^{\vec{y}}$ can be decomposed in accordance with Lemma 8.4 s.t. $v^{\vec{y}} = v^{\wedge t} + v^{\Delta} \in \mathbb{R}^{p'}$ is given. For convenience sake, we can rewrite the ECC game as

$$v^{\vec{y}} = v^{\wedge t} + v^{\Delta} = \mathcal{U}(\lambda^{v^{\wedge t}} + \Delta) = \mathcal{U}(\lambda^{v} + \lambda^{t} + \Delta)$$

with $\mathbf{0} \neq \Delta \in \mathcal{N}_{\mathcal{W}}$ and $v^{\Delta} = \mathcal{U}\,\Delta$. Then it holds $v^{\vec{y}}(S) = v(S) + t + v^{\Delta}(S)$ for all $S \in 2^{n}\backslash\{N, \emptyset\}$ and $v^{\vec{y}}(N) = v(N)$. In the next step, extend the pre-kernel element $\vec{y}$ to a vector $\overline{\mathbf{y}}$ by the measure $y(S) := \sum_{k \in S} y_k$ for all $S \in 2^{n}\backslash\{\emptyset\}$, then define the excess vector by $\overline{e}^{\vec{y}} := v^{\vec{y}} - \overline{\mathbf{y}}$. Due to these definitions, we obtain the following chain of equalities

$$\begin{aligned} \mathbf{0} = \vec{\xi}^{v^{\vec{y}}} = \mathcal{V}^{\top}\,\overline{e}^{\vec{y}} = \mathcal{V}^{\top}\,(v^{\vec{y}} - \overline{\mathbf{y}}) = \mathcal{V}^{\top}\,(v - \overline{\mathbf{y}} + \overline{\mathbf{t}} + v^{\Delta}) \\ \iff \mathbf{0} = \mathcal{V}^{\top}\,(v - \overline{\mathbf{y}}) + \mathcal{V}^{\top}\,\overline{\mathbf{t}} + \mathcal{V}^{\top}\,v^{\Delta} = \mathcal{V}^{\top}\,(v - \overline{\mathbf{y}}) = \mathcal{V}^{\top}\,\overline{e} = \vec{\xi}, \end{aligned}$$

for $\vec{\xi}^{v^{\vec{y}}}$ at $\vec{y}$ under the assumption that $\vec{y} := \nu^{*}(N, v^{\vec{y}})$ holds true from which $\vec{\xi}^{v^{\vec{y}}} = \mathbf{0}$ must be deduced (cf. Meinhardt (2013)). Observe that in this context we have $\mathcal{V}^{\top}\,\overline{\mathbf{t}} = \mathbf{0}$ as well as

$$\mathcal{V}^{\top}\,v^{\Delta} = \mathcal{V}^{\top}\,\mathcal{U}\Delta = \mathcal{W}\Delta = \mathbf{0}$$

in accordance with the imposed assumptions.

In terms of Lemma 4.3 the system of excesses remains balanced w.r.t. the game v, since $\vec{\xi} = \mathbf{0}$. Implying that the system of maximum surpluses remains invariant on a hypercube specified by the critical values of the ellipsoid ε, since by hypothesis the expression $v^{\Delta}(S)$ belongs to the non-empty interval $[-\mathsf{C}, \mathsf{C}]$ for $S \in 2^{n}\backslash\{\emptyset\}$. This interval specifies the range in which the game parameter can vary without having any impact on the set of most effective coalitions given by $\mathcal{S}(\vec{y})$. Thus, the coalitions $\mathcal{S}(\vec{y})$ still have maximum surpluses for the game defined by $v = \mathcal{U}\,\lambda^{v}$ due to the imposed condition $\mathcal{U}\,\Delta = v^{\Delta} \in [-\mathsf{C}, \mathsf{C}]^{p'}$. Hence the pre-kernel solution $\vec{y}$ of game $v^{\vec{y}}$ is invariant against a change in the hypercube $[-\mathsf{C}, \mathsf{C}]^{p'}$. In view of the Propositions 4.1 to 4.3 and Theorem 4.3, we draw the conclusion that $\nu^{*}(N, v) = \nu^{*}(N, v^{\vec{y}})$ must be in force. By the first part, the sufficient condition of Proposition 9.1 is satisfied, from which we obtain $\varsigma^{*}(N, v) = \varsigma^{*}(N, v^{\wedge t_0})$. This arguments closes the second part. □

REMARK 9.2 (Replication of the Modiclus).
Theorem 9.1 states that the modicli of the original game and an appropriate t-shift of that game coincide provided that the pre-nucleolus as an interior solution within a payoff equivalence induced by the corresponding ECC game remains invariant against a change in the game parameter. This means, that the pre-nucleolus of the default game belongs to the same equivalence class as under the ECC game implying that the set of most effective coalitions remains unaffected by this change in the parameter space. As a

consequence, the pre-kernel properties prevail and due to the validness of Theorem 4.3 even those of the pre-nucleolus. This allows us to employ Proposition 9.1 from which the replication result of modiclus can be finally deduced. Hence, the Theorem provides a rule of how one has to construct a game in order to replicate its modiclus. Obviously, reflected arguments can be applied for the anti-pre-nucleolus and modiclus #

In the course, we replicate this result for the anti-pre-nucleolus. For doing so, we get a sequence of results, which are the counterparts to those discussed in Section 4.

Corollary 9.4. *If $[\vec{\gamma}^{\#}]$ has non-empty interior and $\mathbf{x} \in \mathcal{PK}^{\#}(v) \subset [\vec{\gamma}^{\#}]$, then there exists some critical bounds given by*

$$\delta_{ij}^{\varepsilon^{\#}}(\mathbf{x}) = \frac{\pm\sqrt{\bar{c}}}{\|(\mathbf{E}^{\#})^{\top}(\mathbf{1}_j - \mathbf{1}_i)\|} \neq 0 \qquad \forall i, j \in N, i \neq j, \tag{9.3}$$

with $\bar{c} > 0$ and $\|(\mathbf{E}^{\#})^{\top}(\mathbf{1}_j - \mathbf{1}_i)\| > 0$.

Lemma 9.2. *If $\mathbf{x} \in M(h_{\gamma}^{v\,\#})$, then $\mathbf{x} \in M(h_{\gamma}^{v^{\mu}\,\#})$ for all $\mu \in \mathbb{R}$, where $v^{\mu\,\#} := \mathcal{U}(\lambda^{v} + \mu\Delta)$ and $\mathbf{0} \neq \Delta \in \mathcal{N}_{\mathcal{W}^{\#}} = \{\Delta \in \mathbb{R}^{p'} \mid \mathcal{W}^{\#}\Delta = \mathbf{0}\}$, where $\mathcal{W}^{\#} := (\mathcal{V}^{\#})^{\top}\,\mathcal{U} \in \mathbb{R}^{q \times p'}$.*

Proof. Let $\mathbf{x}$ be a minimizer of function $h_{\gamma}^{v\,\#}$ under game v, then $\mathbf{x}$ remains a minimizer for a function $h_{\gamma}^{v^{\mu}\,\#}$ induced by game v^{μ} whenever $\mathbf{Q}^{\#}\,\mathbf{x} = -2\,\mathbf{E}^{\#}\,\vec{\alpha}^{\#} = -\mathbf{a}^{\#}$ remains valid. Since the payoff vector has induced the matrices $\mathbf{Q}^{\#}, \mathbf{E}^{\#}$ and matrix $\mathcal{V}^{\#}$ defined by

$$[\mathbf{v}_{1,2}^{\#}, \ldots, \mathbf{v}_{n-1,n}^{\#}, \mathbf{v}_0],$$

where the vectors are defined as by formula (7.13). We simply have to prove that the configuration $\vec{\alpha}^{\#}$ remains invariant against an appropriate change in the game parameter. Observing that matrix $\mathcal{W}^{\#}$ has a rank equal to or smaller than $q = \binom{n}{2} + 1$, say $m \leq q$, then the null space of matrix $\mathcal{W}^{\#}$ has rank of $p' - m$, thus $\mathcal{N}_{\mathcal{W}^{\#}} \neq \{\emptyset\}$. But then exists some $\mathbf{0} \neq \Delta \in \mathbb{R}^{p'}$ s.t. $\Delta \in \mathcal{N}_{\mathcal{W}^{\#}}$ and $v^{\mu\,\#} = \mathcal{U}(\lambda^{v} + \mu\Delta)$ for $\mu \in \mathbb{R}\backslash\{\mathbf{0}\}$, getting

$$\mathcal{W}^{\#}\,\lambda^{v^{\mu}} = \mathcal{W}^{\#}\,(\lambda^{v} + \mu\Delta) = (\mathcal{V}^{\#})^{\top}\,(v + \mu v^{\Delta}) = (\mathcal{V}^{\#})^{\top}\,v = \vec{\alpha}^{\#},$$

whereas $\mathcal{W}^{\#}\,\Delta = (\mathcal{V}^{\#})^{\top}\,v^{\Delta} = \mathbf{0}$ with $v^{\Delta} := \mathcal{U}\,\Delta$. This argument proves that the configuration $\vec{\alpha}^{\#}$ remains invariant against a change in the game parameter space by $v^{\Delta} \neq \mathbf{0}$. This implies that the payoff vector $\mathbf{x}$ is also a minimizer for function $h_{\gamma}^{v^{\mu}\,\#}$ under game v^{μ}. □

To conclude this discussion, we introduce below the counterparts of the Propositions 4.1 to 4.3, Theorem 4.3 as well as Theorem 9.1.

Corollary 9.5. *Let $(E^{\#})_1^{\top} = (E^{\#})^{\top}\,X$ with $X \in SO(n)$, that is $[\vec{\gamma}^{\#}] \sim [\vec{\gamma}_1^{\#}]$. In addition, assume that the payoff equivalence class $[\vec{\gamma}^{\#}]$ induced from TU game $\langle\, N, v\,\rangle$ has non-empty interior such that $\{\mathbf{x}\} = \mathcal{PK}^{\#}(v) \subset [\vec{\gamma}^{\#}]$ is satisfied, then there exists no other anti-pre-kernel element in payoff equivalence class $[\vec{\gamma}_1^{\#}]$ for a related TU game $\langle\, N, v^{\mu}\,\rangle$, where $v^{\mu} = v + \mu \cdot v^{\Delta} \in \mathbb{R}^{p'}$, as defined by Lemma 9.2.*

Corollary 9.6. *Impose the same conditions as under Corollary 9.5 with the exception that $X \in GL^{+}(n)$, then there exists no other anti-pre-kernel element in payoff equivalence class $[\vec{\gamma}_1^{\#}]$ for a related TU game $\langle\, N, v^{\mu}\,\rangle$.*

Corollary 9.7. *Assume $[\vec{\gamma}^{\#}] \not\sim [\vec{\gamma}_1^{\#}]$, and that the payoff equivalence class $[\vec{\gamma}^{\#}]$ induced from TU game $\langle N, v \rangle$ has non-empty interior such that $\{\mathbf{x}\} = \mathcal{PK}^{\#}(v) \subset [\vec{\gamma}^{\#}]$ is satisfied, then there exists no other anti-pre-kernel element in payoff equivalence class $[\vec{\gamma}_1^{\#}]$ for a related TU game $\langle N, v^{\mu} \rangle$, where $v^{\mu} = v + \mu \cdot v^{\Delta} \in \mathbb{R}^{p'}$, as defined by Lemma 9.2.*

So far our investigation focused on an isolated anti-pre-kernel element. However, in order be sure that a single anti-pre-kernel element of the default game is also the anti-pre-nucleolus of a related game, it must satisfies Anti-Property I. Its definition reverses the arguments of Property I from Kohlberg (1971). For introducing this reversed concept, we have to impose first the set

Definition 9.1. *For every $\mathbf{x} \in \mathbb{R}^n$, and $\psi \in \mathbb{R}$ define the set*

$$\mathcal{D}^{\#}(N, v; \psi, \mathbf{x}) := \{S \subseteq N \mid e^{v}(S, \mathbf{x}) \leq -\psi\}. \tag{9.4}$$

Then we can state the definition of Anti-Property I through

Definition 9.2 (Anti-Property I). *A vector $\mathbf{x} \in \mathcal{I}^{*}(N, v)$ has Anti-Property I w.r.t. TU game $\langle N, v \rangle$, if for all $\psi \in \mathbb{R}$ s.t. $\mathcal{D}^{\#}(N, v; \psi, \mathbf{x}) \neq \emptyset$,*

$$Z(\mathcal{D}^{\#}(N, v; \psi, \mathbf{x})) = \{\mathbf{z} \in \mathbb{R}^n \mid z(S) \geq 0 \; \forall S \in \mathcal{D}^{\#}(N, v; \psi, \mathbf{x}), z(N) = 0\},$$

implies $z(S) = 0, \forall S \in \mathcal{D}^{\#}(N, v; \psi, \mathbf{x})$.

Thus, if the anti-pre-nucleolus does not satisfy Anti-Property I, we can be sure that there must exist at least a second anti-pre-kernel point for the related game different from the first one. Fortunately, this issue can be affirmed due to the next result

Corollary 9.8. *Let $\langle N, v \rangle$ be a TU game that has a singleton anti-pre-kernel such that $\{\mathbf{x}\} = \mathcal{PK}^{\#}(v) \subset [\vec{\gamma}^{\#}]$, and let $\langle N, v^{\mu} \rangle$ be a related game of v derived from $\mathbf{x}$, then $\mathbf{x} = \nu^{\#}(N, v^{\mu})$, whereas the payoff equivalence class $[\vec{\gamma}^{\#}]$ has non-empty interior.*

Lemma 9.3 (Meinhardt (2018b)). *Let $\langle N, v \rangle$ be a TU game. If $\mathbf{x}$ is Pareto optimal, then the ECF game $v_{\mathbf{x}}^{\#}$ satisfies SED w.r.t. $\mathbf{x}$. Moreover, it holds $v_{\mathbf{x}}^{\#}(S) = \tilde{v}_{N,\mathbf{x}}^{\#}(S) + t$ for all $S \subset N$ and $v_{\mathbf{x}}^{\#}(N) = v(N)$ with $t := (\underline{\varkappa} + \underline{\varkappa}^{*})$. Whereas the game $\tilde{v}_{N,\mathbf{x}}^{\#}$ is specified by Lemma 9.1.*

Proof. See Meinhardt (2018b). □

Corollary 9.9 (ECF Game and Replication of the Modiclus). *Let $v^{\Delta} := \tilde{\tilde{v}}_{N,\mathbf{y}} - v$ and put $\vec{y} := \nu^{\#}(N, v_{\vec{y}}^{\#})$. If the induced payoff equivalence class $[\vec{\gamma}^{\#}]$ of the ECF game $\langle N, v_{\vec{y}}^{\#} \rangle$ has non-empty interior s.t. $\{\vec{y}\} = \mathcal{PK}^{\#}(N, v_{\vec{y}}^{\#}) \subset [\vec{\gamma}^{\#}]$ and $v^{\Delta} \in [-\mathsf{C}^{\#}, \mathsf{C}^{\#}]^{p'}$ with*

$$\mathsf{C}^{\#} := \min_{i,j \in N, i \neq j} \left\{ \left| \frac{\pm\sqrt{\bar{c}}}{\|(\mathbf{E}^{\#})^{\top}(\mathbf{1}_j - \mathbf{1}_i)\|} \right| \right\}, \tag{9.5}$$

as well as $\mathbf{0} \neq \Delta \in \mathcal{N}_{\mathcal{W}^{\#}} = \{\Delta \in \mathbb{R}^{p'} \mid \mathcal{W}^{\#}\Delta = \mathbf{0}\}$ with matrix $\mathcal{W}^{\#} := (\mathcal{V}^{\#})^{\top}\mathcal{U}$, and matrix $\mathcal{V}^{\#}$ given by Definition 7.13, then $\{\vec{y}\} = \{\nu^{\#}(N, v)\} = \mathcal{PK}^{\#}(N, v)$. Furthermore, we get $\varsigma^{*}(N, v) = \varsigma^{*}(N, v^{\wedge t_0})$ for t_0 as given by Corollary 9.1.

Proof. For convenience sake define $u := v^{\#}_{\vec{y}}$ as well as $\mathcal{V}^{\top} := (\mathcal{V}^{\#})^{\top}$, and put $\vec{y} := \nu^{\#}(N,u)$. By Lemma 9.4 $\vec{y}^{\,i,j,\delta^{\varepsilon\,\#}} \in \varepsilon^{\#} \subset [\vec{\gamma}^{\#}]$, is a unique boundary point of the ellipsoid $\varepsilon^{\#}$ of type (7.12) with maximum volume. We conclude that either (1) $s^{\#}_{ij}(\vec{y}^{\,i,j,\delta^{\varepsilon\,\#}},u) = s^{\#}_{ij}(\vec{y},u) + \delta^{\varepsilon\,\#}_{ij}(\vec{y})$ if $S \in \mathcal{G}_{ij}$, or (2) $s^{\#}_{ji}(\vec{y}^{\,i,j,\delta^{\varepsilon\,\#}},u) = s^{\#}_{ji}(\vec{y},u) - \delta^{\varepsilon\,\#}_{ij}(\vec{y})$ if $S \in \mathcal{G}_{ji}$, or otherwise (3) $s^{\#}_{ij}(\vec{y}^{\,i,j,\delta^{\varepsilon\,\#}},u) = s^{\#}_{ij}(\vec{y},u)$ is satisfied. Moreover, let $v, u, v^{\Delta} \in \mathbb{R}^{p'}$ and notice that the ECF game u can be decomposed in accordance with Lemma 9.3 s.t. $u = v^{\wedge t} + v^{\Delta} \in \mathbb{R}^{p'}$ is given. For convenience sake, we can rewrite the ECF game as

$$u = v^{\wedge t} + v^{\Delta} = \mathfrak{U}(\lambda^{v^{\wedge t}} + \Delta) = \mathfrak{U}(\lambda^{v} + \lambda^{t} + \Delta)$$

with $\mathbf{0} \neq \Delta \in \mathcal{N}_{\mathcal{W}^{\#}}$ and $v^{\Delta} = \mathfrak{U}\,\Delta$. Then it holds $u(S) = v(S) + t + v^{\Delta}(S)$ for all $S \in 2^{n}\backslash\{N,\emptyset\}$ and $u(N) = v(N)$. In the next step, extend the anti-pre-kernel element $\vec{y}$ to a vector $\mathbf{y}$ by the measure $y(S) := \sum_{k\in S} y_k$ for all $S \in 2^{n}\backslash\{\emptyset\}$, then define the excess vector by $\vec{e}^{\,\vec{y}} := v^{\vec{y}} - \mathbf{y}$. Due to these definitions, we obtain the following chain of equalities

$$\begin{aligned} &\mathbf{0} = \vec{\xi}^{\,v^{\vec{y}}} = \mathcal{V}^{\top}\,\vec{e}^{\,\vec{y}} = \mathcal{V}^{\top}\,(v^{\vec{y}} - \mathbf{y}) = \mathcal{V}^{\top}\,(v - \mathbf{y} + \bar{\mathbf{t}} + v^{\Delta}) \\ \iff\; &\mathbf{0} = \mathcal{V}^{\top}\,(v - \mathbf{y}) + \mathcal{V}^{\top}\,\bar{\mathbf{t}} + \mathcal{V}^{\top}\,v^{\Delta} = \mathcal{V}^{\top}\,(v - \mathbf{y}) = \mathcal{V}^{\top}\,\vec{e} = \vec{\xi}, \end{aligned}$$

for $\vec{\xi}^{\,v^{\vec{y}}}$ at $\vec{y}$ under the assumption that $\vec{y} := \nu^{\#}(N,u)$ holds true from which $\vec{\xi}^{\,v^{\vec{y}}} = \mathbf{0}$ must be deduced. Observe that in this context we have $\mathcal{V}^{\top}\,\bar{\mathbf{t}} = \mathbf{0}$ as well as

$$\mathcal{V}^{\top}\,v^{\Delta} = \mathcal{V}^{\top}\,\mathfrak{U}\Delta = \mathcal{W}\Delta = \mathbf{0}$$

in accordance with the imposed assumptions.

Applying the arguments of Lemma 9.2, we observe that the system of excesses remains balanced w.r.t. the game v, since $\vec{\xi} = \mathbf{0}$. Implying that the system of minimum surpluses remains invariant on a hypercube specified by the critical values of the ellipsoid $\varepsilon^{\#}$, since by hypothesis the expression $v^{\Delta}(S)$ belongs to the non-empty interval $[-\mathsf{C}^{\#},\mathsf{C}^{\#}]$ for $S \in 2^{n}\backslash\{\emptyset\}$. This interval specifies the range in which the game parameter can vary without having any impact on the set of less effective coalitions given by $\mathcal{S}^{\#}(\vec{y})$. Thus, the coalitions $\mathcal{S}^{\#}(\vec{y})$ still have minimum surpluses for the game defined by $v = \mathfrak{U}\,\lambda^{v}$ due to the imposed condition $\mathfrak{U}\,\Delta = v^{\Delta} \in [-\mathsf{C}^{\#},\mathsf{C}^{\#}]^{p'}$. Hence the anti-pre-kernel point $\vec{y}$ of game $v^{\vec{y}}$ is invariant against a change in the hypercube $[-\mathsf{C}^{\#},\mathsf{C}^{\#}]^{p'}$. In view of the arguments from Corollaries 9.5 to 9.8, we draw the conclusion that $\nu^{\#}(N,v) = \nu^{\#}(N,u)$ must be in force. By the first part, the sufficient condition of Proposition 9.1 is satisfied, from which we obtain $\varsigma^{*}(N,v) = \varsigma^{*}(N,v^{\wedge t_0})$. This arguments closes the second part. □

Proposition 9.2 (Replication of an Anti-Pre-Kernel Element). *Let* $\langle N,v\rangle \in \mathcal{G}$ *be a convex game. Set* $\mathbf{x} := \nu^{\#}(N,v)$ *and assume that* $\mathbf{x} \in \mathcal{C}(N,v)$ *holds; put* $v^{\Delta} := \tilde{\tilde{v}}^{\#}_{N,\mathbf{x}} - v$*, then it holds* $(\mathcal{V}^{\#})^{\top}\,v^{\Delta} = \mathbf{0}$*, that is,* $\mathbf{x} \in \mathcal{PK}^{\#}(N,v^{\#}_{\mathbf{x}})$*. Whereas the game* $v^{\#}_{\mathbf{x}}$ *is given by Definition 8.17, game* $\tilde{\tilde{v}}^{\#}_{N,\mathbf{x}}$ *is specified by Lemma 9.1, and matrix* $\mathcal{V}^{\#}$ *is imposed by Definition 7.13.*

Proof. Set $\mathbf{x} := \nu^{\#}(N,v)$ as well as $u := \tilde{\tilde{v}}^{\#}_{N,\mathbf{x}}$, and notice that in view of the prerequisite, it holds $\mathbf{x} \in \mathcal{C}(N,v)$, but then the subsequent inequality is satisfied

$$\begin{aligned} &u(S) - v(S) = \min\left\{v(S) + \underline{\varkappa}(\mathbf{x},v^{*}), v^{*}(S) + \underline{\varkappa}(\mathbf{x},v)\right\} - v(S) \\ &\min\left\{\underline{\varkappa}(\mathbf{x},v^{*}), v^{*}(S) - v(S) + \underline{\varkappa}(\mathbf{x},v)\right\} \leq 0, \end{aligned}$$

due to convexity of v for all $S \subsetneq N$. Hence, $u \leq v$. Applying similar arguments as in Meinhardt (2013, Section 7.4 to 7.8) this means that the induced payoff equivalence class $[\vec{\gamma}]_v^{\#}$ of the anti-reduced game $\langle N, v \rangle$, which contains the anti-pre-nucleolus of game v, cannot shrink. Hence, we have $[\vec{\gamma}]_v^{\#} \subseteq [\vec{\gamma}]_u^{\#}$. Since, we have $\mathbf{x} \in [\vec{\gamma}]_u^{\#}$ in connection with $h_\gamma^{v\,\#} = h_\gamma^{u\,\#} = h^{u\,\#}$ on the restriction of $[\vec{\gamma}]_v^{\#}$, this implies $-\mathbf{E}^{\#}\mathbf{z} = \vec{\alpha}^{\#}$ for all $\mathbf{y} \in [\vec{\gamma}]_v$, since all payoffs within an equivalence class induce the same set $\mathcal{S}^{\#}(\vec{\gamma}, v)$ of smallest less effective coalitions. Obviously, the equation $-\mathbf{E}^{\#}\mathbf{z} = \vec{\alpha}^{\#}$ also holds on the enlarged convex set $[\vec{\gamma}]_u^{\#}$ due to $\mathcal{S}^{\#}(\mathbf{y}, v) = \mathcal{S}^{\#}(\mathbf{y}, u)$ for all $\mathbf{y} \in [\vec{\gamma}]_u^{\#}$ implying $\mathbf{z} \in \mathcal{PK}^{\#}(N, u)$. Therefore, we have $\alpha_{ij}^{\#} := (u(S_{ij}^{\#}) - u(S_{ji}^{\#})) = (v(S_{ij}^{\#}) - v(S_{ji}^{\#}))$ with $\{S_{ij}^{\#}\} = \mathcal{S}_{ij}^{\#}(\vec{\gamma}), \{S_{ji}^{\#}\} = \mathcal{S}_{ji}^{\#}(\vec{\gamma})$ for all $i, j \in N, i < j$, and $\alpha_0^{\#} := v(N)$ (cf. Section 7). In accordance with Remark 9.1 the ECF game $w := v_{\mathbf{x}}^{\#}$ can be decomposed to $v + \bar{\mathbf{t}} + \tilde{\tilde{v}}_{N,\mathbf{x}}^{\#} - v = v + \bar{\mathbf{t}} + v^{\Delta}$. For convenience sake define $\mathcal{V}^{\top} := (\mathcal{V}^{\#})^{\top}$ and extend the vector $\mathbf{x}$ to its measure $\overline{\mathbf{x}}$, but then

$$
\begin{aligned}
\vec{\xi}^{\,w} &= \mathcal{V}^{\top}\, \overline{e}^{w\,\mathbf{x}} = \mathcal{V}^{\top}\,(w - \overline{\mathbf{x}}) = \mathcal{V}^{\top}\,(v - \overline{\mathbf{x}} + \bar{\mathbf{t}} + v^{\Delta}) \\
&= \mathcal{V}^{\top}\,(v - \overline{\mathbf{x}}) + \mathcal{V}^{\top}\,\bar{\mathbf{t}} + \mathcal{V}^{\top}\,v^{\Delta} = \mathcal{V}^{\top}\,v^{\Delta} = \mathbf{0},
\end{aligned}
$$

for $\vec{\xi}^{\,w}$ at $\mathbf{x}$ under the assumption that $\mathbf{x} := \nu^{\#}(N, v)$ holds true from which $\vec{\xi}^{\,v} = \mathbf{0}$ must be drawn. Observe that in this context we have $\mathcal{V}^{\top}\,\bar{\mathbf{t}} = \mathbf{0}$, whereas the last equality arrives from

$$
\mathcal{V}^{\top}\,u = \mathcal{V}^{\top}\,v = \vec{\alpha}^{\#},
$$

in accordance with $\mathcal{S}^{\#}(\mathbf{y}, v) = \mathcal{S}^{\#}(\mathbf{y}, u)$ for all $\mathbf{y} \in [\vec{\gamma}]_u^{\#}$. Implying $\mathbf{x} \in \mathcal{PK}^{\#}(N, v_{\mathbf{x}}^{\#})$ as requested. □

REMARK 9.3.
Notice that we can only assure in the above proof that the anti-pre-nucleolus of a convex game v is an element of the anti-pre-kernel of game $v_{\mathbf{x}}^{\#}$. However, we cannot assure that the anti-pre-nucleolus properties are preserved under this very specific change in the game parameter. Hence, we can only say that a payoff vector, which is the anti-pre-nucleolus of game v and must be therefore located inside of the anti-pre-kernel, remains at least an element of those within a very specific derived game. ◇

10 Concluding Remarks

In this paper we have established that the set of related games derived from a default game with a single pre-kernel point must also possess this element as its single pre-kernel point. Moreover, we have shown that the pre-kernel correspondence in the game space restricted to the convex hull comprising the default and related games is single-valued and constant, and therefore continuous. Although we could provide some sufficient conditions under which the pre-nucleolus of a default game – whereas the pre-kernel constitutes a line segment – induces at least a disconnected pre-kernel for the set of related games, it is, however, still an open question if it is possible to obtain from a game with a set-valued pre-kernel some related games that have a singleton pre-kernel. In this respect, the knowledge of firmer conditions that preserve the pre-nucleolus property is of particular interest.

Furthermore, we have provided some replication results related to the modiclus and anti-pre-kernel while applying the same techniques as for the pre-kernel. This demonstrates the versatility of the introduced approach with that the stability issue of some particular bargaining outcomes can be investigated.

Even though we have not provided a new set of game classes with a sole pre-kernel element, we nevertheless think that the presented approach is also very useful to bring forward our knowledge about the classes of transferable utility games where the pre-kernel coalesces with the pre-nucleolus. To answer this question, one need just to select boundary points of the convex cone of the class of convex games to enlarge the convex cone within the game space to identify game classes that allow for a singleton pre-kernel.

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