Unobserved Heterogeneity in the Productivity Distribution and Gains From Trade

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Abstract

A correct parametric approximation of the productivity distribution is essential to calculate Gains From Trade (GFT) in heterogeneous firms models. This paper argues that heterogeneity in productivity is best captured by Finite Mixture Models (FMMs). FMMs build on the existence of unobserved subpopulations in the data. As such, they are generally consistent with models of firm dynamics differing between groups of firms and allow for a very flexible distribution fit. We find FMMs to increase this fit by more than 70% compared to currently considered distributions. A GFT exercise with Portuguese data reveals that only FMMs approximate the 'true gains' reasonably well.

Keywords: Finite Mixture Model, firm size distribution, productivity distribution, Gains From Trade

JEL Codes: L11, F11, F12

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1 Introduction

It is well-known that the choice of a parametric productivity distribution significantly affects Gains From Trade (GFT) estimates (Head et al., 2014; Nigai, 2017; Bee and Schiavo, 2018) and alters the channels through which trade affects welfare (Arkolakis et al., 2012; Bas et al., 2017; Melitz and Redding, 2015; Fernandes et al., 2018). To date, however, there is no consensus on what this parametric approximation should be. Some authors argue a single distributional form such as Pareto (Axtell, 2001), Lognormal (Head et al., 2014) or Weibull (Bee and Schiavo, 2018) suffices to define the productivity distribution. Others build on the idea that a single distribution can not adequately capture the heterogeneity in productivity. This results in combinations of distributions such as the Double-Pareto (Arkolakis, 2016), Double-Pareto Lognormal (Sager and Timoshenko, 2019) or Lognormal-Pareto (Nigai, 2017). Nevertheless, Dewitte (2020) demonstrates that none of these currently considered distributions are able to provide a sufficiently good fit to the data.

This paper argues that heterogeneity in the productivity distribution can be captured most adequately by Finite Mixture Models (FMMs). A FMM is essentially a weighted sum of an a priori unknown number of individual densities. As such, it is a semi-parametric approximation that allows for discrete subpopulations to define the overall distribution. The flexible, semi-parametric nature of FMMs renders them favorable both from a theoretical and empirical point of view.

From a theoretical point of view, the generative process of a FMM corresponds to a simple combination of the generative processes of the underlying individual densities. A FMM can therefore easily generalize, and is generally consistent with, existing models of firm dynamics. First, FMMs allows to combine a specified generative process of firm dynamics across groups of firms to capture additional, unspecified heterogeneity. Luttmer (2007), for instance, generalizes his single-sector model with a finite mixture specification to a multi-sector model. This in order to capture additional heterogeneity across industries and obtain a satisfactory fit to the data. Second, a finite mixture specification is generally consistent with the mechanisms considered to differentiate firm dynamics between groups of firms. The differences in growth rates between financially constrained and unconstrained firms by Cabral and Mata (2003), for instance, can be respecified into a finite mixture specification. FMMs provide an empirical tool that can account for dynamics to differ between groups of firms without having, but not excluding the possibility, to specify the mechanisms that drive these differences a priori. These mechanisms can be left 'unobserved'.

We illustrate the excellent empirical performance of FMMs using the domestic sales[1] of the *population* of active Portuguese firms in 2006. Our contributions to the literature are threefold. First, we have access to a representative dataset on the sales distribution. This allows us to evaluate the performance of parametric distributions on the complete productivity distribution as well as to focus on both the *left* and right tail. Moreover, it insulates us from erroneous conclusions due to truncated or unrepresentative data in the left tail of the distribution (Perline, 2005). Second, we introduce a multitude of new, economically relevant distributions to the productivity distribution

[1]We rely on the distributional relation between productivity and positive domestic sales, under specific model assumptions (Nigai, 2017; Dewitte, 2020), to evaluate parametric approximations of the productivity distribution.

literature. Fitting and comparing up to 52 different distributions helps to reveal features of the data that are of importance when deciding on a specific parametric distribution. Third, our analysis relies on a clear statistical framework to distinguish between distributional fits. Based on the Bayesian Information Criterion (BIC), the currently favored Double-Pareto Lognormal (Sager and Timoshenko, 2019) and Lognormal-Pareto (Nigai, 2017) come in ranked sixteenth and thirty-first out of 52 distributions respectively, while FMMs top the charts. Moreover, a Kolmogorov-Smirnov test reveals that only FMMs provide a distribution fit that is not rejected by the data. FMMs reduce the maximum deviation from the empirical Cumulative Distribution Function (the Kolmogorov-Smirnov test statistic) by more than 70% compared to the Double-Pareto Lognormal distribution and by more than 90% compared to the Lognormal-Pareto distribution. This performance is not surprising, as we show that the Double-Pareto Lognormal and Lognormal-Pareto distribution can be interpreted as constraints of the more general mixture specification.

A Gains From Trade application demonstrates the importance of correctly approximating the productivity distribution in heterogeneous firms models à la Melitz (2003), and underlines the straightforward implementation of FMMs into such models. We contribute to the literature providing quantitative expressions necessary to calibrate a heterogeneous firms model for all distributions considered. Our calibration exercise reveals that when reducing variable trade costs by two thirds, FMMs are able to track the 'true GFT' (obtained from the empirical distribution) closely, while a single Lognormal distribution underestimates these GFT by $\pm 11\%$ and a Lognormal-Pareto distribution overestimates them by $\pm 13\%$.

The paper is organized as follows. In the following section we start by linking the large literature on the parametric approximation of size distributions, spanning the fields of efficiency analysis, physics, regional and actuarial science, to the productivity distribution literature. From this overview, it becomes apparent that the literature on productivity distributions lacks a clear statistical framework that differentiates between a sufficiently large number of alternative distributions over a representative data range. We therefore establish a methodology that uniformly fits a large number of distributions both to complete and truncated datasets, and present evaluation methods to differentiate between these distribution in section 3. Our database on firm sales is discussed in section 4. We provide our empirical results in section 5 and discuss the implications of these results for GFT calculations in section 6. Section 7 concludes.

2 Literature Review

This section provides an overview of the literature related to firm size/productivity distributions. We discuss why the Pareto distribution can only match the tail of size distributions while single hump-shaped distributions such as the Lognormal or the Weibull distribution can not accurately match both the tail and the bulk of the distribution. Size distributions are therefore best approximated by a combination of distributions, of which we consider three types: mixture, piecewise composite and multiplicative distributions. We argue that finite mixtures are preferable both from an empirical and theoretical point of view because of their flexible, semi-parametric nature.

2.1 Single distributions

The *Pareto distribution* has been dominating heterogeneous firms models (Melitz, 2003). Even though the Melitz (2003)-model is not restricted to this distributional choice, its empirical performance (see for instance Axtell (2001); Gabaix (2009); Levy (2009); di Giovanni et al. (2011)) and convenience led to a widespread reliance on the Pareto distribution for social welfare and economic policy analysis.[2] The fit of a Pareto distribution is usually evaluated using its Cumulative Distribution Function (CDF), which follows a straight line on a log-log scale with the shape parameter (k) as slope:

$$G_P(x; x_{min}, k) = 1 - \left(\frac{x_{min}}{x}\right)^k, \qquad x \geq x_{min}. \tag{1}$$

Figure 1 compares a fitted Pareto survival function ($\text{CDF}^c = 1 - \text{CDF}$) with the empirical survival function of Portuguese firm-level sales in 2006 on a log-log scale for the complete dataset (upper panel). It is immediately clear that the Pareto distribution is not a good fit to the complete distribution due to the existence of a hump in the middle.[3]

The popularity of the Pareto distribution, however, rests on its ability to provide a close fit to lower-truncated[4] data with predominantly large observations.[5] Just as every curved line looks straight when one zooms in close enough, so too does the distribution of firm sales appear to be straight when truncated sufficiently. Both the left (lower left panel) and right tail (lower right panel) exhibit linearity of the CDF and survival functions respectively on a log-log scale, in line with Pareto behavior in the tails of the distribution.[6] The apparent straight line behavior of the tails can therefore just as well be approximated by a surprisingly large class of distributions including, but not restricted to, (finite mixtures of) the Exponential, Lognormal, Gamma and Weibull distributions.[7] Proof of which is the performance of the Lognormal distribution in the lower panels of Figure 1.[8]

[2]See Arkolakis et al. (2012) for an overview of work relying on the Melitz-Pareto combination.

[3]See also the Probability Density Function (PDF) in Appendix Figure 2.

[4]An *upper-truncated* version of the Pareto distribution has also been used to explain the existence of zero trade flows across country pairs (Helpman et al., 2008; Feenstra, 2018) and to demonstrate the relevance of heterogeneous firms models (Melitz and Redding, 2014). A discussion on the economic relevance of, and an extension of the analysis to, upper-truncated distributions falls outside the scope of this paper. The methodology set out in this paper allows to truncate any kind of distribution both from above and/or below (see section 3).

[5]Note that the influential paper of Axtell (2001) does not rely on truncated data but unintentionally favors the Pareto distribution due to data binning (Virkar and Clauset, 2014) and methodological choices (Clauset et al., 2009; Bottazzi et al., 2015) characteristic of that time.

[6]The Inverse Pareto distribution is specified as

$$G_{IP}(x; x_{max}, k) = 1 - \left(\frac{x_{max}}{x}\right)^{-k}, \qquad x \leq x_{max}.$$

[7]Perline (2005) defines this class of distributions within the Gumbel domain of attraction.

[8]Even though Pareto and Lognormal distributions exhibit qualitatively different behavior in their upper tails, their apparent quantitative similar behavior in the upper tail for Lognormals with large variance is well-documented (Malevergne et al., 2011).

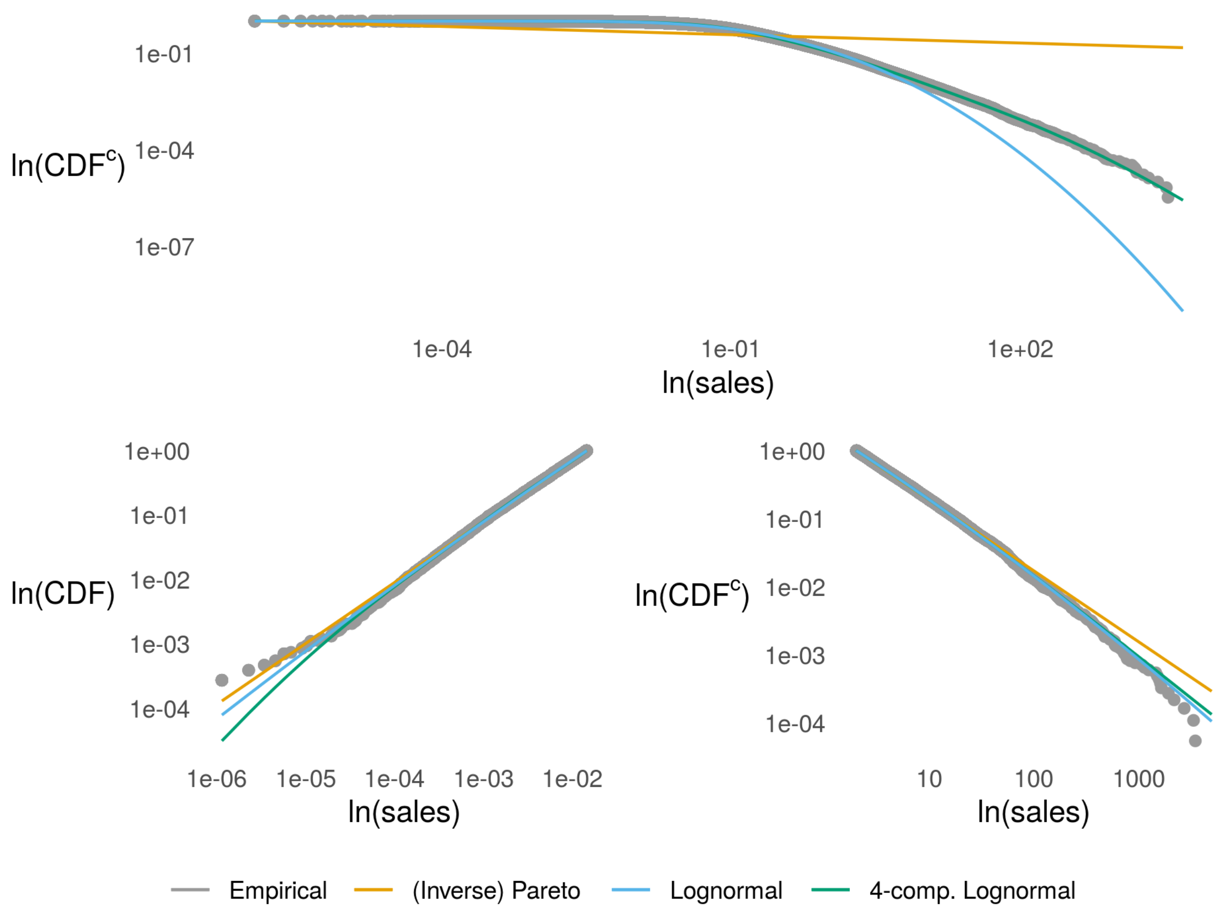

Figure 1: Empirical survival function of Portuguese domestic sales in 2006 (upper panel) on a log-log scale with fitted (Inverse) Pareto and (4-component mixture of) the Lognormal distributions. The lower left and right panels focus on distributions fitted solely to the left and right tail respectively.

Notes: (Truncated) Distributions are fitted using maximum likelihood methods (cf. infra) to the complete and truncated datasets independently. Tail truncation points are determined by the best-fitting (Inverse) Pareto distributions according to the Kolmogorov-Smirnov statistic.

These alternative *hump-shaped distributions* are claimed to provide a better fit to complete size distributions (see Bee and Schiavo (2018) for the Weibull and Eeckhout (2004, 2009); Head et al. (2014); Fernandes et al. (2018) for the Lognormal distribution). In the firm size literature, this claim is usually supported by comparing their performance with a limited number of alternative distributions, mostly Pareto, using the low-powered R-squared.[9] Even though homogeneous hump-shaped distributions such as the Lognormal can adequately fit the tail or the bulk of the empirical distribution, they cannot do both simultaneously. This is easily observable from the upper panel of Figure 1 where the single Lognormal distribution, when fitted to the complete size distribution, does not fit the right tail of the complete productivity distribution while matching the bulk rather satisfactorily.

2.2 Combined distributions

As single distributions are not capable of accurately matching both the bulk and the tail(s) of the productivity distribution, recent research focuses on combinations of distributions. We consider three types of combinations: mixture, piecewise composite and product distributions. To our knowledge, mixture distributions have not been fitted to the productivity distribution. Nevertheless, current applications of both the piecewise composite and product distributions can be interpreted as constraints of the more general mixture specification.

2.2.1 Mixture distributions

Finite Mixture Models (FMMs) are essentially a weighted sum of I individual densities $m_i(\cdot)$:

$$g(x|\boldsymbol{\Psi}) = \sum_{i=1}^{I} \pi_i m_i(x|\boldsymbol{\theta}_i), \qquad \pi_i \geq 0, \quad \sum_{i=1}^{I} \pi_i = 1 \tag{2}$$

where I represents the number of components or discrete subpopulations, π_i is the probability of belonging to component i, $\boldsymbol{\theta_i}$ the component-specific parameter vector of density $m_i(\cdot)$ and $\boldsymbol{\Psi} = (\pi_1, \dots, \pi_{I-1}, \boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_I)$ is the vector of all model parameters (McLachlan and Peel, 2000). They are also referred to as Latent Class Models (LCM) provided that the number of components, and thus also the mixing parameter itself, does not have to be specified a priori but is determined by the data. As such, a finite mixture model provides a semi-parametric approach ideal to fully capture the heterogeneity of size distributions.[10]

The aptitude of Finite Mixture models has already been explored in the context of efficiency analysis (see for instance Beard et al. (1997); Orea and Kumbhakar (2004); El-Gamal and Inanoglu (2005); Greene (2005)), city sizes (Kwong and Nadarajah, 2019) and actuarial losses (Miljkovic and Grün, 2016). It has, to our knowledge, not been applied to productivity distributions before.

[9] See Clauset et al. (2009) for an explanation as to why the R-squared has low power in a distributional context.

[10] A semi-parametric approach is to be favored over a nonparametric approach in the case of heavy-tailed distributions such as firm size. This is because the heavy tails renders nonparametric procedures less efficient (Clauset et al., 2009; Dewitte, 2020).

The generative process of a FMM corresponds to a simple combination of the generative processes of the underlying individual densities and can therefore easily generalize, and is generally consistent with, existing models of firm dynamics.[11] First, FMMs allows to combine a specified generative process of firm dynamics across groups of firms to capture additional, unspecified heterogeneity. Luttmer (2007), for instance, generalizes his single-sector model with a finite mixture specification to a multi-sector model. This allows to capture additional heterogeneity across industries and obtain a satisfactory fit to the data. Similarly, Rossi-Hansberg and Wright (2007) argue the need to account for cross-sectoral differences in their initial single-sector model specification to achieve an accurate description of the cross-sectional size distribution of US firms.

Second, a finite mixture specification is generally consistent with the mechanisms that differentiate firm dynamics between groups of firms. Firm dynamics are argued to differ between groups of firms depending on whether or not they are financially constrained (Cooley and Quadrini, 2001; Cabral and Mata, 2003; Desai et al., 2003; Albuquerque and Hopenhayn, 2004; Clementi and Hopenhayn, 2006; Angelini and Generale, 2008), innovate (Costantini and Melitz, 2008; Atkeson and Burstein, 2010), add or drop products (Klette and Kortum, 2004; Lentz and Mortensen, 2008), add or drop management layers (Caliendo and Rossi-Hansberg, 2012; Caliendo et al., 2020), incur specific market penetration costs (Arkolakis, 2016), et cetera. As (Rossi-Hansberg and Wright, 2007, p. 1641) paraphrase Jovanovic (1982): "many of the mechanisms in the literature undoubtedly contributed toward an explanation of establishment dynamics". To date, however, it remains unclear which mechanism, or mechanisms, dominate. There are "many sources of heterogeneity that support the idea of discrete subpopulations likely to differ in important characteristics" (Perline, 2005, p.80). Finite Mixture Models provide an empirical tool that can account for dynamics to differ between groups of firms as determined by the data. As such, they can account for most, or even a combination, of the proposed mechanisms without having to specify these mechanisms a priori. The mechanisms can be left 'unobserved'.

2.2.2 Piecewise composite distributions

Piecewise composite distributions have a probability density specified as:

$$g(x|\boldsymbol{\theta}) = \begin{cases} \alpha_1 m_1^*(x|\boldsymbol{\theta}_1) & \text{if} \quad c_0 < x \leq c_1 \\ \alpha_2 m_2^*(x|\boldsymbol{\theta}_2) & \text{if} \quad c_1 < x \leq c_2 \\ \vdots & \qquad \vdots \\ \alpha_I m_I^*(x|\boldsymbol{\theta}_I) & \text{if} \quad c_{I-1} < x \leq c_I \end{cases} \tag{3}$$

where $\forall i \in I : m_i^*(x|\boldsymbol{\theta}_i) = \frac{m_i(x|\boldsymbol{\theta}_i)}{\int_{c_{i-1}}^{c_i} m_i(x|\boldsymbol{\theta}_i)dx}$ is the probability density function (PDF) of $m_i(x|\boldsymbol{\theta}_i)$

[11]Note that while this paper conceptualizes the generality of FMMs from a generative perspective, it is not able to provide evidence in favor of any specific generative process. See the methodology section (section 4), Appendix B and the conclusion (Section 7) for a more elaborate evaluation of current limitations regarding this paper's discussion of (the generative processes of) FMMs.

truncated at the cutoffs c_{i-1}, c_i. For this distribution to be well-behaved, additional differentiability and continuity conditions are imposed that determine the value of both component cutoffs (c_i) and probabilities (α_i) (Bakar et al., 2015), so that the vector of all model parameters reduces to the combination of the component-specific parameter vectors: $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_I)$.

While these composite distributions can be formed from many individual parametric distributions, applications mostly focus on Lognormal distributions with Pareto tails. The 'Inverse Pareto-Lognormal-Pareto' distribution has been applied in the city size literature (Ioannides and Skouras, 2013; Luckstead and Devadoss, 2017), while the 'Lognormal-Pareto' version was applied by Nigai (2017) to the Melitz (2003) model for GFT calculations. Dewitte (2020) generalizes the implementation of the piecewise composite distributions to allow for any underlying density in three-, and two- piecewise composite distributions, mainly focusing on Pareto-tailed piecewise composites.

From the distribution specification in equation 3, it can be observed that piecewise composite distributions can be interpreted as mixtures of truncated densities with component probabilities restricted to ensure continuity and differentiability (Scollnik, 2007).[12] This contrasts with the general mixture specification (eq. 2), where component probabilities can be interpreted as the probability that an individual observation belongs to a certain group of observations. Moreover, the generative process of piecewise distributions is rather ambiguous. It is for instance not clear yet which firm dynamics could explain the existence of hard cutoffs that separate the Lognormal from the Pareto distribution.

2.2.3 Product distributions

Alternatively, distributions can be combined into a product distribution: a probability distribution constructed as the distribution of the product of random variables having two other known distributions. The product distribution mainly used in the literature, the Double-Pareto Lognormal distribution, results from the product of a Lognormal with a (Double-)Pareto distributed random variable (Reed and Jorgensen, 2004). This distribution is found to approximate city size distributions well (Reed, 2002; Giesen et al., 2010), while Sager and Timoshenko (2019) applied the distribution to Brazilian export data.

A generative process for this Double-Pareto Lognormal distribution exists (Reed and Hughes, 2002; Reed, 2002; Reed and Jorgensen, 2004) and is applicable to heterogeneous firms models (Arkolakis, 2016). Interestingly, the Double-Pareto Lognormal distribution can be seen as a structured infinite mixture of Lognormal distributions (Reed, 2002, p.13).[13] The Double-Pareto Lognormal distribution can therefore be absorbed by the more flexible mixture distributions as specified

[12]This becomes even more clear when we rewrite the specification of the piecewise composite distribution (eq. 3) as the weighted sum of truncated densities: $g(x|\boldsymbol{\theta}) = \alpha_1 \mathbb{I}(c_0 < x \leq c_1) m_1^*(x|\boldsymbol{\theta}_1) + \alpha_2 \mathbb{I}(c_1 < x \leq c_2) m_2^*(x|\boldsymbol{\theta}_2) + \ldots + \alpha_I \mathbb{I}(c_{I-1} < x \leq c_I) m_I^*(x|\boldsymbol{\theta}_I)$.

[13]In the context of firm size, this could mean that each age (= time since entry in the market) group of firms is distributed Lognormally at a certain point in time. The reason the overall firm size distribution is not Lognormal is that these groups of firms have not all been evolving for the same length of time. The overall distribution of size will be a mixture of Lognormal distributions (across age groups) with time since entry as mixing parameter. When this mixing parameter is exponentially distributed, firm size will be Double-Pareto Lognormally distributed.

in equation 2. Whereas the Double-Pareto Lognormal may suffer from misspecification and/or oversimplification by imposing a structure on the mixture distribution, a FMM allows the data to determine the mixture structure needed to capture the heterogeneity that is present in the data.

3 Methodology

The literature review reveals the myriad of empirical evidence in favor of qualitatively very different distributions fits to productivity. This points at the lack of a clear statistical framework that differentiates between a sufficiently large number of distributions over a representative data range. In this section, we establish a methodology that uniformly fits the large, but relevant, range of single and combined distributions to both complete and truncated data. We then present statistical tests to differentiate between the fitted distributions.

3.1 Distribution fitting

We rely on Maximum Likelihood (ML)[14] over all firms $b \in B$ to fit all considered distributions to the data. We consider the (Inverse) Pareto, hump-shaped distributions (Lognormal, Weibull, Fréchet, Gamma, Exponential and Burr) and combinations of these distributions in the form of mixtures, piecewise composite or product distributions. We limit piecewise composite and product distributions to available Pareto-tailed extensions of the considered hump-shaped distributions.[15] In the case of FMMs, ML is wrapped in an Expectation-Maximization (EM) algorithm to estimate the component probabilities. The estimation methods allow to fit the distributions to both complete and truncated data. This will not only allow us to single out and focus on tail performance, but also to generalize the proposed distributional fits to unrepresentative and/or truncated data.

3.1.1 (Inverse) Pareto

Complete data The ML estimator for the shape parameter k over all firms $b \in B$ can easily be obtained as

[14] The choice for Maximum Likelihood contrasts with the productivity distribution literature, where popular fitting techniques rely on the minimization of squared errors between a log-linearization of the theoretical and empirical PDFs/CDFs (Axtell, 2001; di Giovanni and Levchenko, 2013; Head et al., 2014; Freund and Pierola, 2015; Bas et al., 2017; Nigai, 2017; Bee and Schiavo, 2018). Such methods, however, might not be apt to fit distribution functions. For instance, reported parameters in the literature are, to our knowledge, not obtained from a regression procedure restricted to estimate a properly normalized distribution function. Parameters obtained from an estimation procedure must result in a probability density function that integrates to 1 over the range from the lower bound up to the upper bound (due to its normalization properties) (Clauset et al., 2009). While it is possible to incorporate such constraints in the regression analysis, it has never been reported to our knowledge. Moreover, it is unclear to which extent the standard errors obtained from these methods are valid (Clauset et al., 2009; Bottazzi et al., 2015). Maximum likelihood methods do not suffer from such problems.

[15] See Appendix Tables 1, 2 and 3 for an overview of the specifications for all distributions considered. Considered distributions are chosen based on their occurrence in the economic literature.

$$k_{IP} = \left[\frac{1}{B}\sum_{b=1}^{B} ln\frac{x_{max}}{x_b}\right]^{-1}, \qquad k_P = \left[\frac{1}{B}\sum_{b=1}^{B} ln\frac{x_b}{x_{min}}\right]^{-1}. \tag{4}$$

The ML estimator of the scale parameters equals the maximum and minimum observation: $\hat{x}_{min} = \min(x)$, $\hat{x}_{max} = \max(x)$, as the likelihood function is monotonically increasing (decreasing) in x_{min} (x_{max}).

Truncated data The (Inverse) Pareto distribution is a special distribution, being truncated from (above) below by definition.[16] This means that the (upper) lower truncation point lies within the parameter space of the distribution, and distribution fits can be optimized accordingly. The ML estimator as specified above merely assumes the exogenously applied truncation points as scale parameter.

Obtaining an accurate estimate for the (upper) lower bound is, however, vital to the accuracy of the estimated shape parameter $\hat{k}$. Choosing a (maximum) minimum too (high) low results in a biased shape parameter, as one will be fitting a power-law to non-power-law data. Choosing a value too (low) high, on the other hand, increases the statistical error and bias from finite size effects on the shape parameter, as one discards legitimate data points. Moreover, it is widely documented that the minimum and shape parameter of the Pareto distribution exhibit a positive correlation (Eeckhout, 2004; di Giovanni and Levchenko, 2013; Head et al., 2014; Freund and Pierola, 2015; Bee and Schiavo, 2018).

Many practices therefore co-exist to determine the (upper) lower truncation point, without consensus on the best practice to determine this scale parameter of the (Inverse) Pareto-distribution. In the case of the Pareto distribution, some rely on visual techniques, looking for a 'kink' in the distribution above which the relationship between log rank and log size is approximately linear (di Giovanni and Levchenko, 2013; Bas et al., 2017). Some use export sales, and assume as such a truncation parameter equal to the minimum of sales, e.g. Freund and Pierola (2015). Others determine their minimum to ensure a Pareto parameter large enough to deliver finite moments when calibrating their theoretical models (Head et al., 2014; Bee and Schiavo, 2018). Still others estimate the minimum, assuming a mixed Lognormal-Pareto distribution (Malevergne et al., 2011; Bakar and Nadarajah, 2013; Nigai, 2017). Such methods are either subject to possibly large measurement errors and inconsistencies or restrictive in their need to assume a distributional relation between the bulk and the tail of the distribution.

In order to obtain an accurate estimate for the lower bound, we rely on a formal decision rule developed by Clauset et al. (2009). For the ordered productivity set $\{x_b; b = 1, \ldots, B\}$, we evaluate every x_b as a potential (x_{max}) x_{min}, estimating the ML estimate of the power-law exponent k. We

[16] Fully truncated (both from below and above) Pareto distributions can be deduced from a truncated probability density function (see eq. 6) and have been used in the economic literature (Helpman et al., 2008; Melitz and Redding, 2014; Feenstra, 2018).

then use the Kolmogorov-Smirnov statistic to select the optimum (x_{max}) x_{min}. It is defined as the cutoff which minimizes the maximum absolute deviation of the empirical from the theoretical CDF:

$$T_{KS,\hat{x}_{max}} = \sup_{x \leq \hat{x}_{max}} \left| \frac{1}{B} \sum_{b=1}^{B} \mathbb{I}(x_b \leq \hat{x}_{max}) - G_{IP}(x; \hat{k}, \hat{x}_{max}) \right|$$
$$T_{KS,\hat{x}_{min}} = \sup_{x \geq \hat{x}_{min}} \left| \frac{1}{B} \sum_{b=1}^{B} \mathbb{I}(x_b \geq \hat{x}_{min}) - G_{P}(x; \hat{k}, \hat{x}_{min}) \right|, \quad (5)$$

where $\mathbb{I}_A$ is the indicator of event A.

3.1.2 Hump-shaped, piecewise composite and product distributions

Complete data The maximum likelihood of the considered *hump-shaped distributions* (Lognormal, Weibull, Fréchet, Gamma, Exponential and Burr) is straightforward and estimation methods are widely available. We also consider *piecewise composite distributions* as Pareto-tailed extensions of these hump-shaped distributions. The ML estimator of these distributions has no closed form and needs to be approached numerically, see Dewitte (2020). Pareto-tailed extensions in the form of *product distributions*, on the other hand, are less generally available. We consider the Double-Pareto Lognormal distribution (Reed and Jorgensen, 2004). This distribution is the result of multiplying a Double Pareto, used by among others Arkolakis (2016), with a Lognormal distribution. Reducing the parameter space of the Double Pareto allows us to consider the Left- and Right-Pareto Lognormal distribution respectively. Also in this case, the ML estimator has no closed form solution and needs to be approached numerically (Reed and Jorgensen, 2004).

Truncated data Consisting of individual truncated densities, the estimation of piecewise composite distributions on truncated data is by its definition straightforward. Maximum likelihood methods for the remaining hump-shaped and product distributions can easily be adapted by truncating the distribution to be restricted within the domain of the data. The resulting truncated probability density function ($g^*(x)$) is then specified within the (exogenously determined) boundaries $x \in \left[c^l, c^u\right]$:

$$g^*(x) = \frac{g(x)}{G(c^u) - G(c^l)}. \quad (6)$$

3.1.3 FMM

Complete data Direct maximum likelihood estimation of a FMM (see eq. 2) is not straightforward, since the number of components I is a priori unknown. The log-likelihood function can be written as

$$logL(x|\mathbf{\Psi}) = \sum_{b=1}^{B}\sum_{i=1}^{I} z_{bi}\left[log(\pi_i) + log(m_i(x_b|\boldsymbol{\theta}_i))\right], \tag{7}$$

where z_{bi} is an unobserved component indicator equal to one if the observation x_b originates from subpopulation i and zero otherwise. Two steps need to be taken iteratively in order to be able to maximize this equation. The Expectation (E)-step of the s-th iteration consists of determining the conditional expectation of eq. 7 given the observed data and the current parameter estimates from iteration $s-1$:

$$\begin{aligned} Q(\mathbf{\Psi}|\mathbf{\Psi}^{(s-1)}) &= E\left[logL(x|\mathbf{\Psi})|x, \mathbf{\Psi}^{(s-1)}\right] \\ &= \sum_{b=1}^{B}\sum_{i=1}^{I} \pi_{bi}^{(s)}\left[log(\pi_i) + log(m_i(x_b|\boldsymbol{\theta}_i))\right], \end{aligned} \tag{8}$$

where the missing data z_{ni} is replaced by the posterior probability that x_b belongs to the ith mixture:

$$\pi_{bi}^{(s)} = E\left[z_{bi}|x_b, \mathbf{\Psi}^{(s-1)}\right] = \frac{\pi_i^{(s-1)} m_i(x_b|\boldsymbol{\theta}_i^{(s-1)})}{\sum_{i=1}^{I}\pi_i^{(s-1)} m_i(x_b|\boldsymbol{\theta}_i^{(s-1)})}. \tag{9}$$

The Maximization (M)-step then, consists of maximizing the Q-function over the parameter vector $\mathbf{\Psi}$:

$$\mathbf{\Psi}^{(s)} = \max_{\mathbf{\Psi}} Q(\mathbf{\Psi}|\mathbf{\Psi}^{(s-1)}). \tag{10}$$

Each iteration updates the E- and M-step until the algorithm converges (See Miljkovic and Grün (2016) and McLachlan and Peel (2000) for a more elaborate overview).

The validity of the proposed estimation technique does not depend on its ability to identify the unobserved component indicator z_{bi}. FMMs can be utilized in two ways. First, they can be used as a semi-parametric, flexible approximation of the overall distribution. Second, they are model-based clustering methods when a certain distribution is imposed (Fop et al., 2018; Grün, 2018). While both applications rely on the idea that discrete subpopulations define the overall distribution, the semi-parametric approximation does not claim to correctly identify these subpopulations (z_{bi}). This paper relies on FMMs as a semi-parametric approximation of the productivity distribution. See Appendix B for a more elaborate discussion on the difference between both applications and their relevance for the current analysis.

Truncated data The EM-algorithm can be adapted to fitting data only to truncated data within the (exogenously determined) boundaries $x \in \left[c^l, c^u\right]$. We specify the conditional densities

$$
\begin{aligned}
g(x|\mathbf{\Psi}, c^l \leq x \leq c^u) &= \frac{\sum_{i=1}^{I} \pi_i m_i(x|\boldsymbol{\theta_i})}{G(c^u|\mathbf{\Psi}) - G(c^l|\mathbf{\Psi})} \\
&= \sum_{i=1}^{I} \pi_i \frac{M_i(c^u|\boldsymbol{\theta_i}) - M_i(c^l|\boldsymbol{\theta_i})}{G(c^u|\mathbf{\Psi}) - G(c^l|\mathbf{\Psi})} \frac{m_i(x|\boldsymbol{\theta_i})}{M_i(c^u|\boldsymbol{\theta_i}) - M_i(c^l|\boldsymbol{\theta_i})} \\
&= \sum_{i=1}^{I} \eta_i m_i(x|\boldsymbol{\theta_i}, c^l \leq x \leq c^u), \qquad (11)
\end{aligned}
$$

with $\eta_i > 0$, $\sum_{i=1}^{I} \eta_i = 1$ and M_i the component-specific Cumulative Distribution Function. The Q-function becomes

$$
\begin{aligned}
Q(\mathbf{\Psi}|\mathbf{\Psi}^{(s-1)}) &= E\left[logL(x|\mathbf{\Psi})|x, \mathbf{\Psi}^{(s-1)}\right] \\
&= \sum_{b=1}^{B} \sum_{i=1}^{I} \pi_{bi}^{(s)} \left[log(\eta_i) + log(m_i(x_b|\boldsymbol{\theta}_i, c^l \leq x_b \leq c^u))\right], \qquad (12)
\end{aligned}
$$

where the posterior probability that x_b comes from the ith mixture is not affected by the truncation:

$$
\pi_{bi}^{(s)} = \frac{\eta_i^{(s-1)} m_i(x_b|\boldsymbol{\theta}_i^{(s-1)}, c^l \leq x_b \leq c^u))}{\sum_{i=1}^{I} \eta_i^{(s-1)} m_i(x_b|\boldsymbol{\theta}_i^{(s-1)}), c^l \leq x_b \leq c^u)} = \frac{\pi_i^{(s-1)} m_i(x_b|\boldsymbol{\theta}_i^{(s-1)})}{\sum_{i=1}^{I} \pi_i^{(s-1)} m_i(x_b|\boldsymbol{\theta}_i^{(s-1)})}. \qquad (13)
$$

The M-step then again consists of maximizing the Q-function over the parameters $\mathbf{\Psi}$. Iterating over the E- and M-step until the algorithm converges provides us with distributions fitted to the truncated data.

3.2 Distribution evaluation

We rely on multiple distinct criteria to differentiate between the distributions. First, we consider whether the proposed parametric distribution is a sufficiently good fit to the data. We then differentiate between distributions using information criteria.

Goodness of fit We follow Dewitte (2020) in evaluating the parametric distributions by summarizing the distance between the empirical and parametric rth moment of the distribution by the 1- and ∞-norm:

$$
S^r = \sum_y \Delta^r(y), \qquad T^r = \sup_y \Delta^r(y), \qquad (14)
$$

where $\Delta^r(y)$ is the normalized absolute deviation:

$$\Delta^r(y) = \frac{\left|\frac{1}{B}\sum_{b=1}^{B}\mathbb{I}(x_b \geq y)x_b^r - \int_y^\infty x^r g(x|\mathbf{\Psi})dx\right|}{\frac{1}{B}\sum_{b=1}^{B} x_b^r}. \tag{15}$$

$\mathbb{I}(A)$ is the indicator of event A and $\mu_y^r = \int_y^\infty x^r g(x|\mathbf{\Psi})dx$ is the y-bounded, rth-moment of the parametric distribution, with r taking positive values. Evaluated at the 0th-moment of the distribution, the test statistic T^0 corresponds with the Kolmogorov-Smirnov (KS) test statistic, quantifying the largest distance between the empirical and parametric CDF. This is the sole specification of the statistic specified on which we can rely to provide statistically underpinned claims regarding the accuracy of the distributional assumption with respect to its empirical counterpart. Nevertheless, Dewitte (2020) argues that evaluating these test statistics at higher moments of the distribution ($r > 0$) can be informative on the distributional fit, especially relating to their use in heterogeneous firms models (see also section 6).[17] Whereas the ∞-norm contains only information on the largest distance, the 1-norm provides information on the distance between both distributions over the complete distributional space, weighting all distances equally. The normalization factor allows us to interpret the distances on a scale of zero to one for all moments, similar to the interpretation of the standard KS test statistic.

As we rely on estimated parameters, asymptotic distributions are not available for the test statistics. We therefore rely on a parametric bootstrap:

1. Assume B i.i.d. random variables with distribution $G(\cdot|\mathbf{\Psi})$;
2. Estimate the parameters $\mathbf{\Psi}$ of the distribution using MLE and calculate the rth moment implied by the parametric distribution: $\hat{\mu}^r$;
3. $H_0 : \mu^r = \hat{\mu}^r$ with test statistic $t \in \{S^r, T^r\}$;
4. Draw N bootstrap samples of size B from $G(\cdot|\hat{\mathbf{\Psi}})$;
5. For each sample of the parametric distribution, calculate the bootstrapped test statistics $t^* \in \left\{(S^{\tilde{r}})^*, (T^{\tilde{r}})^*\right\}$;[18]
6. The p-value is then defined as

$$\hat{p} = \frac{1}{N+1}\left[\sum_{n=1}^{N}\mathbb{I}\left(t_n^* \geq t\right) + 1\right]. \tag{16}$$

The bootstrap exercise should therefore be interpreted as 'the likelihood of observing a deviation between the moments of the empirical and parametric distribution as large as t under the null

[17] We have no knowledge of statistical tests that evaluate distributional fits based on bounded higher moments of the distribution.

[18] Note that we do not re-fit the parametric distribution to the bootstrap sample. The vastness of the dataset at our availability in the empirical section results both in a large computational burden but also a very precise estimation of the distribution parameters. The influence of not refitting the parametric distribution to the bootstrap sample is therefore negligent.

hypothesis', allowing us to evaluate whether the distributional assumption provides a good fit to the evaluated moments of the distribution.

Information Criteria We differentiate between distributions based on the log-likelihood, the Aikaike or Bayesian Information Criteria. When possible, we can differentiate between two distributions based on the ratio of their likelihoods:

$$LR = \sum_{b=1}^{B} ln \frac{g_1(x_b; \cdot)}{g_2(x_b, \cdot)} \tag{17}$$

with $g_{1,2}$ the probability densities of the respective distributions. If these distributions are non-nested (Vuong, 1989), the test statistic amounts to the sample average of this ratio, standardized by a consistent estimate of its standard deviation. The null hypothesis states that both classes of distributions are equally far (in the Kullback and Leibler (1951) divergence/relative entropy sense) from the true distribution. If this is true, our test statistic will follow (asymptotically) a Gaussian distribution with mean zero. If the null is false, and $g_1(\cdot)$ is closer to the truth, the test statistic diverges to $+\infty$ with probability one. If $g_2(\cdot)$ fits the data better, it diverges to $-\infty$ (Vuong, 1989).

The Aikaike Information criterion penalizes the log-likelihood information for the number of parameters (to avoid overfitting) and is defined as $AIC = 2np - 2ln(L)$ with np the number of parameters and $ln(L)$ the log-likelihood. Similarly, the Bayesian Information criterion corrects for the number of parameters as $BIC = npln(B) - 2ln(L)$. Differentiation between distributions relies then on te relative distance of the BICs: $\Delta BIC = BIC_1 - BIC_2$. The value of ΔBIC implies strong evidence in favor of distribution 1 if $B > 10$, moderate evidence if $6 < B \leq 10$ and weak evidence if $2 < B \leq 6$ (Kass and Raftery, 1995). AIC and BIC statistics are considered adequate when choosing the number of components for a suitable FMM (McLachlan and Peel, 2000).

4 Data

We use firm-level data from Portugal to evaluate the empirical performance of FMMs compared to "traditional" distributions such as the Log-normal or Pareto distribution. The main source of information is Sistema de Contas Integradas das Empresas (SCIE, Enterprise Integrated Accounts System) in the year 2006, a dataset covering the universe of active Portuguese firms that has been used already by, among others: (Carreira and Teixeira, 2016; Dias et al., 2016; Fernandes and Ferreira, 2017; Bastos et al., 2018; Fonseca et al., 2018).[19] It contains data both on firm-level sales and number of employees. Moreover, each firm has a unique identification number that allows us to link this dataset with a dataset on international trade.

The firm size distribution of Portugal was earlier the object of study by Cabral and Mata (2003), who relied on a longitudinal matched employer-employee dataset covering all business units with

[19]A comparison between SCIE and the OECD SBDS database proves the full coverage of firms in our dataset for the Portuguese economy (see Table 6).

at least one wage earner in the Portuguese economy (Quadros de Pessoal). They provide evidence that the firm size distribution of Portugal is not very different from other countries such as France, the United States, Germany, Japan and the United Kingdom.

We rely on the distributional relation between productivity and positive domestic sales, under specific model assumptions (Nigai, 2017; Dewitte, 2020), to evaluate parametric approximations of the productivity distribution. Relying on domestic rather than total sales corrects for the impact of international trade on the firm size distribution (di Giovanni et al., 2011). We reduce our dataset discarding self-employed companies[20], resulting in a dataset covering the positive domestic sales of 299,935 Portuguese firms in 2006.

5 Results

We fit the distributions to Portuguese domestic sales in the year 2006. We initially focus on fitting the Pareto, Lognormal, combinations of Pareto and Lognormal and up to 5-component mixtures of Lognormals to the complete data. This proves to be sufficient for our main message. We show that our results hold when focusing on the tails of the data, can be extended to other economically relevant distributions, are robust to sample selection and outliers and can be externally validated on city size data.

5.1 Complete data

Single distributions can not sufficiently capture the heterogeneity of the productivity distribution. Table 1 displays the selected distribution fits, ordered according to their log-likelihood. One immediately observe that single parametric distributions provide the worst fits. This demonstrates the need, as the evolution of the literature indicates (Nigai, 2017; Sager and Timoshenko, 2019), to combine distributions in order to adequately capture heterogeneity in productivity. The Pareto distribution, for instance, provides a really bad fit to the distribution with a Goodness of fit statistic of up to 267 times bigger than the best fitting mixture of Lognormals.[21].

Finite mixture models greatly improve the distributional fit, without over-fitting the data. According to the log-likelihood, distributions with a larger number of parameters provide a better fit to the data, even when parameter correction ($R_{AIC,BIC}$) is applied. The BIC values indicate that the 4-component Lognormal provides the best fit to the data. This demonstrates that the performance of FMMs is not the result of over-fitting, but of FMMs being able to capture heterogeneity of which other distributional forms are not capable. The currently favored Double-Pareto Lognormal (Sager and Timoshenko, 2019) and Lognormal-Pareto (Nigai, 2017) distribution are ranked fourth and eighth respectively. The structure imposed on a general mixture specification in order to attain these specific piecewise composite or product distributions (see section 2.2) is therefore not

[20]Disregarding individual companies renders our dataset more comparable with earlier datasets used to evaluate productivity distributions such as the ORBIS database used by Nigai (2017).

[21]The higher the Goodness of fit statistic, the larger the deviation between the empirical and parametric distribution (see eq. 15)

Table 1: Selected distribution fits to Portuguese domestic sales in 2006.

		Goodness of fit		Information Criteria		
Distribution	Parms.	T_a^0	S_b^0	Loglike	R_{AIC}	R_{BIC}
5-comp. Lognormal	14	0.18	0.11	12,776	1	2^{+++}
		(0.10;0.25)	(0.08;0.32)			
4-comp. Lognormal	11	0.19	0.11	12,770	2	1
		(0.09;0.25)	(0.08;0.32)			
3-comp. Lognormal	8	0.29	0.34	12,723	3	3^{+++}
		(0.10;0.24)**	(0.09;0.32)**			
Double-Pareto Lognormal	4	0.66	0.80	12,429	4	4^{+++}
		(0.09;0.25)***	(0.08;0.33)***			
2-comp. Lognormal	5	0.53	0.71	12,401	5	5^{+++}
		(0.10;0.24)***	(0.09;0.32)***			
Inv. Pareto-Lognormal-Pareto	4	0.81	1.01	12,231	6	6^{+++}
		(0.09;0.26)***	(0.08;0.34)***			
Inv. Pareto-Lognormal	3	3.02	4.26	9,198	7	7^{+++}
		(0.09;0.24)***	(0.08;0.31)***			
Lognormal-Pareto	3	2.56	3.78	8,721	8	8^{+++}
		(0.09;0.25)***	(0.08;0.32)***			
Left-Pareto Lognormal	3	3.23	4.91	8,059	9	9^{+++}
		(0.10;0.25)***	(0.09;0.32)***			
Right-Pareto Lognormal	3	2.82	4.38	8,028	10	10^{+++}
		(0.09;0.25)***	(0.08;0.32)***			
Lognormal	2	2.93	5.03	7,372	11	11^{+++}
		(0.10;0.25)***	(0.08;0.33)***			
Pareto	2	48.34	68.18	-436,227	12	12^{+++}
		(0.09;0.25)***	(0.08;0.33)***			

Notes: All distributions fitted using Maximum Likelihood.
Values between parentheses report the 5th and 95th quantile of the parametric bootstrapped test statistic with 999 replications. ***, **, * indicate significance of this test at 1%, 5% and 10% respectively.
$^{+++}$, $^{++}$, $^{+}$ indicates the difference between this distribution's BIC and the first-ranked distribution in terms of BIC (ΔBIC) providing strong evidence in favour of the first-ranked distribution ($\Delta BIC > 10$), moderate evidence ($6 < \Delta BIC \leq 10$) and weak evidence ($2 < \Delta BIC \leq 6$) respectively.
$_a$ Values multiplied by 100 for expositional purpose, $_b$ Values divided by 1,000 for expositional purpose.

warranted.

Finite mixture models are the sole parametric specifications that are not rejected by the data. Focusing on goodness-of-fit criteria around the 0th moment, we observe that these follow the log-likelihood ranking closely. The 4- and 5-component Lognormal distributions reduce the largest deviation from the empirical CDF (T^0) by more than 70% ($\frac{0.66-0.19}{0.66} \times 100$) compared to the Double-Pareto Lognormal distribution and by more than 90% compared to the Lognormal-Pareto distribution. This pattern is consistent over the complete range of the data, as is apparent from the cumulative error of the CDF fit (S^0). Moreover, none of the currently favored parametric distributions provide a good fit to the data. Only for the 4- and 5-component Lognormal distributions the null hypothesis that the data originates from the proposed parametric distribution can not be rejected.

Figure 2[22] provides a visual insight into the numerical results of Table 1. It plots the normalized absolute deviation between the empirical and parametric CDF. The figure shows the large errors related to the Lognormal distribution. Augmenting the Lognormal distribution with a Pareto right-tail as in Nigai (2017) improves the fit only marginally. While it does provide a slightly better fit in the right tail of the distribution, this comes at the cost of a worse fit to the left-tail of the distribution and an almost equally bad fit to the bulk of the distribution as the Lognormal distribution. The best-fitting Pareto-tailed Lognormal, the Double-Pareto Lognormal, does a better job at fitting the distribution. However, it clearly lags behind in comparison with the 4-component Lognormal, which only displays marginal errors both in the bulk and the tails of the data. This tail performance becomes even more apparent when considering the Quantile-Quantile plot in Figure 3.

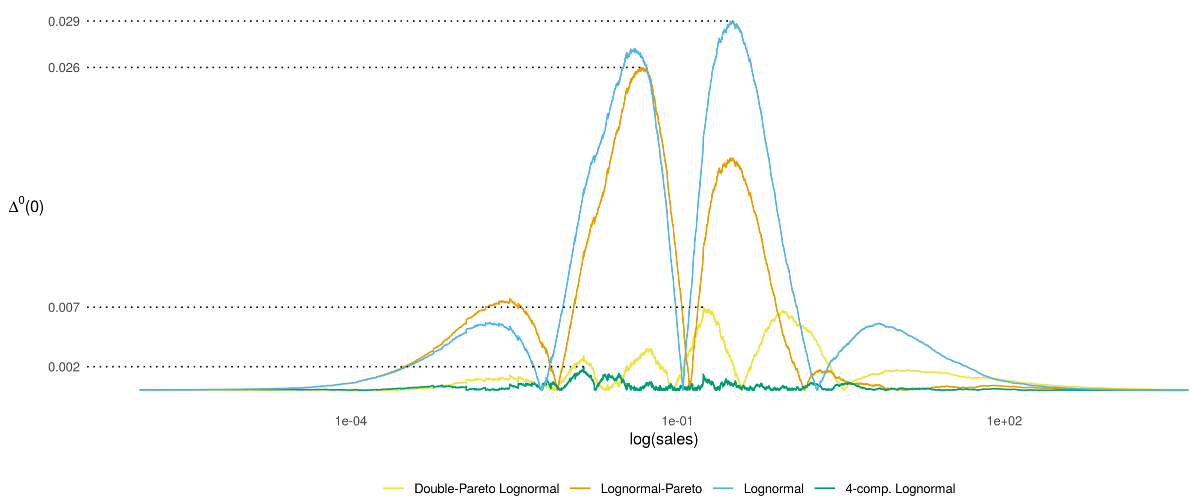

Figure 2: Normalized Absolute Deviation between the empirical and Double-Pareto Lognormal, Lognormal-Pareto, Lognormal and 4-component Lognormal CDFs over the complete range of domestic sales in Portugal, 2006.

[22]This representation of the results is essentially a visually more interpretable version of the Probability-Probability plot (see Appendix Figure 3).

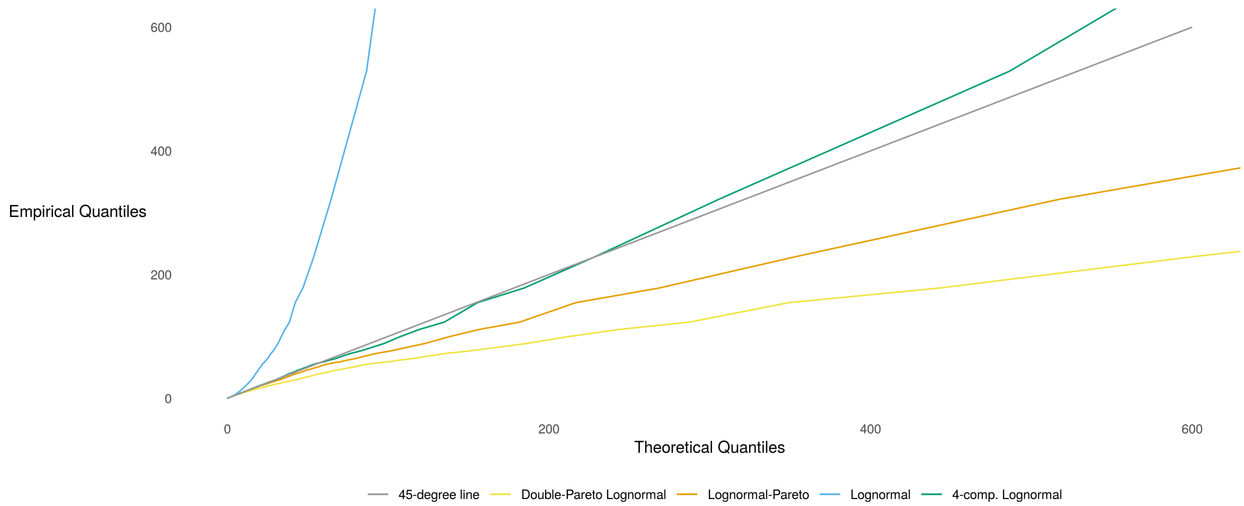

Figure 3: Quantile-Quantile plot for the Double-Pareto Lognormal, Lognormal-Pareto, Lognormal and 4-component Lognormal over approximately 99.99% of domestic sales in Portugal, 2006.
Note: Quantiles are capped at 600 for expositional purposes, leaving out approximately the upper 0.01% of the data.

5.2 Truncated data

Allowing for heterogeneity in distributions clearly provides a better fit when fitting the complete distribution, but what about when we fit the tails only? This is mostly interesting from the Pareto point of view, which is often claimed to be a good fit to the right tail of the productivity distribution.[23]

Table 2 displays the results of fitting the (Inverse) Pareto to the (left) right tail of the distribution using the methods described in section 3. We recovered the best-fitting truncation point for the (Inverse) Pareto distribution, assigning 8.53% and 6.07% of the data to the left and right tail respectively. We reduced our dataset according to these truncation parameters and fitted truncated mixtures of Lognormals to both tails of the distribution for comparison. This approach puts the Pareto distribution twice in the advantage. First, it is free from a parametric specification for the bulk of the distribution. Second, the truncation parameter is chosen in function of the best-fitting (Inverse) Pareto distribution. As a result, the (Inverse) Pareto, as well as (mixtures of) the Lognormal, provide a good fit to the tails according to the Kolmogorov-Smirnov test.

Nevertheless, despite the advantage for the (Inverse) Pareto distribution, it seems that (mixtures of) the Lognormal distribution provide a significantly better fit to the tails of the data. (Mixtures of) the Lognormal distribution have a higher log-likelihood and lower deviation from the empirical CDF than the (Inverse) Pareto distribution. This results in the likelihood ratio test significantly rejecting Pareto in favor of (mixtures of) the Lognormal distribution, which is in line with earlier results reported in related literature (Clauset et al., 2009). When correcting for the number of parameters, the BIC reveals that the single Lognormal distribution is sufficient to fit the tail only.

[23]Note that this argument carries the normative value that obtaining a good fit for larger firms is absolute, regardless of the implications for the fit to smaller firms.

A mixture of Lognormals insufficiently improves the fit in order to justify the corresponding increase in number of parameters.

Table 2: Selected distribution fits to the tails of Portuguese domestic sales in 2006.

		Goodness of fit		Information Criteria		
Distribution	Parms.	T_a^0	S_b^0	Loglike	R_{AIC}	R_{BIC}
		Left tail (N=25,588, 8.53% of the data)				
5-comp. Trunc. Lognormal	14	0.63	0.04	108,196.19***	5	6^{+++}
		(0.32;0.85)	(0.02;0.10)			
4-comp. Trunc. Lognormal	11	0.61	0.04	108,195.05***	4	5^{+++}
		(0.33;0.85)	(0.02;0.09)			
3-comp. Trunc. Lognormal	8	0.58	0.04	108,194.44***	1	4^{+++}
		(0.33;0.86)	(0.02;0.10)			
2-comp. Trunc. Lognormal	5	0.77	0.06	108,189.93***	3	3^{+++}
		(0.32;0.84)*	(0.02;0.09)			
Trunc. Lognormal	2	1.02	0.10	108,186.99***	2	1
		(0.32;0.85)**	(0.02;0.09)**			
Inv. Pareto	2	0.80	0.10	108,183.90	6	2^{++}
		(0.33;0.84)*	(0.02;0.09)**			
		Right tail (N=18,217, 6.07% of the data)				
5-comp. Trunc. Lognormal	14	0.62	0.03	-47,896.59***	5	6^{+++}
		(0.39;1.00)	(0.02;0.07)			
Trunc. Lognormal	2	0.70	0.04	-47,897.86***	1	1
		(0.38;0.97)	(0.02;0.08)			
2-comp. Trunc. Lognormal	5	0.71	0.04	-47,897.99***	2	3^{+++}
		(0.38;1.01)	(0.02;0.08)			
3-comp. Trunc. Lognormal	8	0.68	0.04	-47,898.60***	3	4^{+++}
		(0.38;0.99)	(0.02;0.08)			
4-comp. Trunc. Lognormal	11	0.68	0.04	-47,898.62***	4	5^{+++}
		(0.39;1.00)	(0.02;0.08)			
Pareto	2	0.86	0.08	-47,910.44	6	2^{+++}
		(0.38;0.99)	(0.02;0.08)*			

Notes: All distributions fitted using Maximum Likelihood.
Values between parentheses report the 5th and 95th quantile of the parametric bootstrapped test statistic with 999 replications. ***, **, * indicate significance of this test at 1%, 5% and 10% respectively.
Similarly, ***, **, * indicate significance at 1%, 5% and 10% respectively for the likelihood ratio test between (Inverse) Pareto and (mixtures of) the Lognormal distribution.
$^{+++}$, $^{++}$, $^{+}$ indicates the difference between this distribution's BIC and the first-ranked distribution in terms of BIC (ΔBIC) providing strong evidence in favour of the first-ranked distribution ($\Delta BIC > 10$), moderate evidence ($6 < \Delta BIC \leq 10$) and weak evidence ($2 < \Delta BIC \leq 6$) respectively.
$_a$ Values multiplied by 100 for expositional purpose, $_b$ Values divided by 1,000 for expositional purpose.

5.3 Extension to other distributions

The superior performance of FMMs is not limited to the Lognormal distribution. Appendix Table 7 displays the results of fits to the complete data expanding to FMMs of distributions often used in the economic literature such as the Exponential, Gamma, Weibull, Burr and Fréchet distribution. Most of these mixtures are not able to match the performance of the Lognormal. Only the Burr distribution provides an equivalent fit to the PDF and CDF.[24] Compared to Pareto-tailed combinations of distributions, we find that also mixtures of Weibull and Gamma are able to provide an improved distribution fit. Overall, the currently favored Double-Pareto Lognormal (Sager and Timoshenko, 2019) and Lognormal-Pareto (Nigai, 2017) distribution are ranked sixteenth and thirty-first respectively according to BIC, out of 52 considered distributions.

The consistent excellent performance of the Lognormal distribution can be motivated from two perspectives. From the perspective of overall fit, a mixture of (log-) normal distributions with sufficient components is assumed to be able to approach all distributions (McLachlan and Peel, 2000). From a generative perspective for individual components, the Lognormal distribution is the realization of applying the Central Limit Theorem (CLT) in the log domain: firm heterogeneity will approximately be Lognormal if it is the multiplicative product of many independent random variables. This corresponds with extensions of heterogeneous firms models à la Melitz (2003) that consider multi-dimensional firm heterogeneity, taking into consideration the product dimension (Bernard et al., 2009) or uncertainty in demand and/or supply (see for instance De Loecker (2011); Bas et al. (2017); Sager and Timoshenko (2019); Gandhi et al. (0)).

5.4 Robustness

We scrutinize the robustness of our results with a number of additional analyses. First, we examine whether our results are not caused by sample selection. We therefore restrict our dataset to the manufacturing sector only (see Appendix Table 8) and find the performance of FMMs to improve relative to Pareto-tailed distributions. Second, we inspect whether our results are not due to outliers in the tails of the distribution. We discard the first and last 1,000 observations of the dataset. Results in Appendix Table 9 again confirm the superiority of FMMs.

We validate our approach externally fitting the considered distributions to the U.S. Census 2000 city size distribution data. This dataset has been subject to an extensive debate in the city size literature, including the discussion between Eeckhout (2004, 2009) and Levy (2009).[25] Appendix Table 10 provides the test results, demonstrating that the city size distribution is neither Lognormal, Pareto nor Pareto-tailed Lognormal. It is best approximated by a 2-component Lognormal distribution (according to the BIC). These results provide an overview of the city size literature up till now and are in line with the findings of Kwong and Nadarajah (2019).

6 Gains From Trade implications

In this section, we integrate the distribution fits from the previous section into a heterogeneous firms framework à la Melitz (2003). This allows us to perform a GFT exercise along the lines of

[24]The Burr distribution fails to match higher moments of the data, however. See also section 6.

[25]The dataset is available at https://www.aeaweb.org/aer/data/sept09/20071478_data.zip.

(Melitz and Redding, 2015; Bee and Schiavo, 2018) and investigate the importance of providing a good fit to the productivity distribution for GFT calculations.

Our setup is a two-country symmetric heterogeneous firms model with a finite number of firms.[26] The parameterization of our model is standard (Head et al., 2014; Melitz and Redding, 2015; Bee and Schiavo, 2018). We work with two symmetric countries i and j and choose labor in one country as the numeraire, so that $W^i = W^j = 1$. We choose fixed entry costs $f^e = 0.545$ and set fixed costs equal to one ($f^{ii} = f^{ij} = 1$). The elasticity of substitution is set to four.

Finally, we need to capture the heterogeneity distribution. Assuming a parametric distribution and under the assumption of an *infinite* number of firms, we can calculate the necessary analytical expressions using the distributional parameters from our empirical analysis to capture heterogeneity. Following Nigai (2017), we can also capture heterogeneity directly from the empirical, *finite*, data. To allow comparison between GFT obtained assuming a parametric distribution and GFT obtained from the finite data, we perform a parametric bootstrap. This parametric bootstrap generates a range of finite sample estimates under the hypothesis that the observed data is generated by a certain parametric distribution, which can be compared with the observed finite data (Dewitte, 2020).

We calculate the changes in welfare due to a trade shock (Gains From Trade), which can be written as log changes in real per-capita income due to an exogenous increase in variable trade costs τ_{ij} to τ'_{ij}. This can be further decomposed into the channels through which trade affects welfare: trade costs (τ^{ij}), the number of firms (M^i), the probality of successful entry into the domestic market ($m^0_{\omega^{ii*}}$), the average productivity of firms exporting from i to j ($m^{\sigma-1}_{\omega^{ij*}}$)[27] and the bilateral trade share (λ^{ij}):

$$\begin{aligned} 100 \times ln\frac{(\mathbb{W}^i)'}{\mathbb{W}^i} &= 100 \times -ln\frac{(P^i)'}{P^i} \qquad (18) \\ &= 100 \times -\left[ln\frac{(\tau^{ij})'}{(\tau^{ij})} - \frac{1}{\sigma-1}\left(ln\frac{(M^i)'}{M^i} - ln\frac{(m^0_{\omega^{ii*}})'}{m^0_{\omega^{ii*}}} + ln\frac{(m^{\sigma-1}_{\omega^{ij*}})'}{m^{\sigma-1}_{\omega^{ij*}}} - ln\frac{(\lambda^{ij})'}{\lambda^{ij}}\right)\right]. \end{aligned}$$

Our exercise reduces the variable trade costs from $\tau^{ij} = 3$ to $(\tau^{ij})' = 1$. The obtained GFT are displayed in Figure 4. This figure presents the parametric bootstrapped distribution of GFT by means of box-plots delineating the 5th, 25th, 50th, 75th and 95th quantile. Empirical GFT are indicated by the vertical blue line. Green circles are the average parametric finite sample GFT and the parametric plug-in population estimates of GFT are shown by yellow diamonds.

We observe that heavy-(Pareto-) tailed distributions significantly overestimate GFT, while relatively light-tailed distributions underestimate GFT. Mixture models are the only distributions that provide an approximation of GFT that is not rejected by the data. The distributions in Figure 4 are ordered according to their distance from the empirical GFT. As such, we can interpret the

[26] See Appendix C for a full workout of the model.

[27] We define average productivity here as average productivity unconditional on successful entry, in contrast to the definition conditional on successful entry in (Melitz, 2003, p.1702).

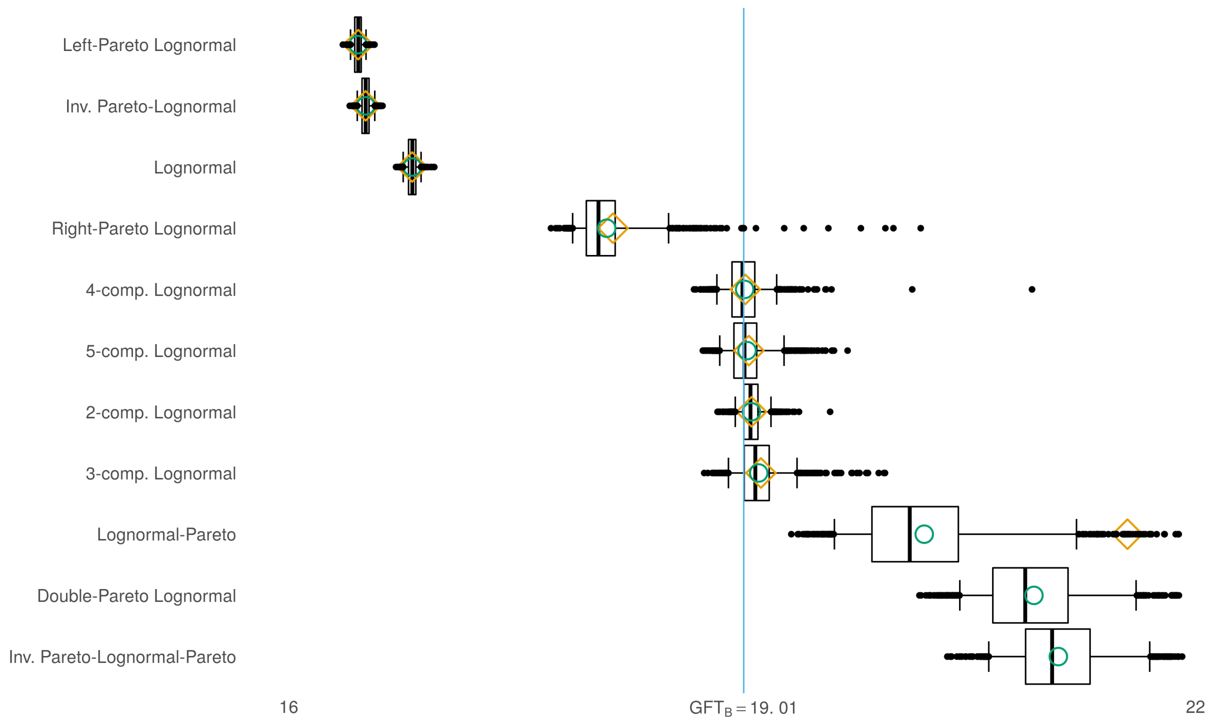

Figure 4: Gains from a reduction in variable trade costs $\tau^{ij} = 3$ to $(\tau^{ij})' = 1$.

Notes: Box-plots display the 5th, 25th, 50th, 75th and 95th quantile of the asymptotic distribution of parametric finite sample GFT obtained from a bootstrap with 999 replications. Yellow diamonds represent the parametric plug-in (population) estimates of GFT. Green circles are the average parametric bootstrapped finite sample GFT and the empirical sample GFT are indicated by the vertical blue line. All sample values obtained from a sample of 299,935 firms.

4-component Lognormal distribution as providing the closest fit to the GFT obtained from the empirical distribution. Where the empirical values imply an increase in real income per capita of 19.01% when reducing variable trade costs from 3 to 1, the 4-component Lognormal distribution closely predicts this to be 19.02%, as can be deduced from the parametric plug-in population estimates (yellow diamonds). Moreover, the close fit results in a very good approximation of the empirical GFT, as can be deduced from the parametric bootstrapped finite sample GFT being at least as small as the empirical GFT in more than 5% of the cases (the box-plot overlaps with the vertical blue line). This contrasts with the simple Lognormal distribution underestimating the empirical GFT by about 11% with 16.8% predicted GFT, and with the Lognormal-Pareto distribution overestimating the empirical GFT by approximately 13%, with 21.55% predicted GFT.

Deviations from GFT calculations can be mainly attributed to errors in capturing the evolution of average productivity of exporting firms and bilateral trade shares. Table 3 reports the weighted components of welfare gains (see eq. 18) for all considered distributional forms, allowing us to evaluate the channels trough which the differences in GFT between distributions come about. We observe that the deviation of the parametric results compared to the empirical distribution are relatively small for the changes in number of firms and in the probability of successful entry into the domestic market. The largest differences can be found for the changes in average productivity of exporting firms and in the trade shares. Heavy-tailed distributions largely underestimate the positive effect of the increase in average productivity of exporting firms and the negative effect of the increase in the bilateral trade shares compared to the empirical distribution, while the reverse is true for lighter-tailed distributions.

Table 3: Decomposition of procentual welfare gains from a reduction in variable trade costs $\tau^{ij} = 3 \rightarrow (\tau^{ij})' = 1$.

Distribution	Parms.	$ln\frac{(\mathbb{W}^i)'}{\mathbb{W}^i}$	$-\ln\frac{(\tau^{ij})'}{(\tau^{ij})}$	$\frac{1}{\sigma-1}ln\frac{(M^i)'}{M^i}$	$\frac{1}{\sigma-1}ln\frac{(m^0_{\omega ii*})'}{m^0_{\omega ii*}}$	$\frac{1}{\sigma-1}ln\frac{(m^{\sigma-1}_{\omega ij*})'}{m^{\sigma-1}_{\omega ij*}}$	$-\frac{1}{\sigma-1}ln\frac{(\lambda^{ij})'}{\lambda^{ij}}$
Pareto	2	-	1.10	-	-	-	-
		(-0.00;0.00)***	(1.10;1.10)	(-0.22;-0.22)***	(-0.00;0.00)***	(0.00;0.00)***	(-0.88;-0.88)***
Left-Pareto Lognormal	3	0.16	1.10	-0.17	0.15	0.60	-1.51
		(0.16;0.17)***	(1.10;1.10)	(-0.17;-0.17)***	(0.15;0.15)***	(0.58;0.62)***	(-1.53;-1.49)***
Inv. Pareto-Lognormal	3	0.17	1.10	-0.17	0.15	0.58	-1.49
		(0.16;0.17)***	(1.10;1.10)	(-0.17;-0.17)***	(0.15;0.15)***	(0.56;0.60)***	(-1.51;-1.47)***
Lognormal	2	0.17	1.10	-0.17	0.15	0.53	-1.44
		(0.17;0.17)***	(1.10;1.10)	(-0.17;-0.17)***	(0.15;0.15)***	(0.51;0.55)***	(-1.46;-1.42)***
Right-Pareto Lognormal	3	0.18	1.10	-0.18	0.17	0.28	-1.19
		(0.18;0.19)**	(1.10;1.10)	(-0.19;-0.18)	(0.17;0.18)	(0.23;0.33)**	(-1.24;-1.13)**
Empirical	0	0.19	1.10	-0.18	0.18	0.20	-1.10
4-comp. Lognormal	11	0.19	1.10	-0.18	0.18	0.20	-1.10
		(0.19;0.19)	(1.10;1.10)	(-0.19;-0.18)	(0.17;0.18)	(0.18;0.22)	(-1.13;-1.08)
5-comp. Lognormal	14	0.19	1.10	-0.19	0.18	0.20	-1.10
		(0.19;0.19)	(1.10;1.10)	(-0.19;-0.18)	(0.17;0.19)	(0.17;0.22)	(-1.12;-1.07)
2-comp. Lognormal	5	0.19	1.10	-0.17	0.17	0.23	-1.13
		(0.19;0.19)	(1.10;1.10)	(-0.18;-0.17)***	(0.16;0.17)***	(0.22;0.25)***	(-1.15;-1.12)***
3-comp. Lognormal	8	0.19	1.10	-0.18	0.18	0.19	-1.09
		(0.19;0.19)	(1.10;1.10)	(-0.19;-0.18)	(0.17;0.18)	(0.16;0.22)	(-1.12;-1.06)
Lognormal-Pareto	3	0.22	1.10	-0.22	0.22	0.02	-0.90
		(0.20;0.21)***	(1.10;1.10)	(-0.22;-0.20)***	(0.20;0.22)***	(0.04;0.14)***	(-1.04;-0.93)***
Double-Pareto Lognormal	4	-	1.10	-	-	-	-
		(0.20;0.22)***	(1.10;1.10)	(-0.20;-0.19)***	(0.19;0.20)***	(0.02;0.09)***	(-0.98;-0.90)***
Inv. Pareto-Lognormal-Pareto	4	-	1.10	-	-	-	-
		(0.21;0.22)***	(1.10;1.10)	(-0.20;-0.18)*	(0.18;0.20)***	(0.01;0.08)***	(-0.97;-0.89)***

Notes: Values between parentheses report the 5th and 95th quantile of the parametric bootstrapped statistics with 999 replications. ***, **, * indicate the rejection of a signifcant overlap of the parametric bootstrapped statistic with the empirical statistic at 1%, 5% and 10% respectively.

Our results confirm the findings of Dewitte (2020) that a good fit to truncated average sales proves to be a critical predictor of the performance of GFT calculations. A ranking of the distributions according to GFT performance does not closely follow the ranking of the fit to the 0th moment of the distribution (the CDF). The Double-Pareto Lognormal, for instance, provides a closer fit to the empirical CDF than the Right-Pareto Lognormal, but provides worse GFT approximations. This can be attributed to the relatively heavy tail of the Double-Pareto Lognormal, resulting in a large error when calculating higher moments of the distribution. As such, a ranking of distributions based on the fit to average sales proves to be a better indicator of GFT performance, as can be deduced from the statistics T^1 in Appendix Table 7.

These findings are not the result of a specific parametrization of the model. Figure 4 displays the percentage errors in parametric GFT calculations relative to the empirical benchmark for different parametrization scenarios. Our findings are robust for different values of the elasticity of substitution (left upper panel) and fixed entry costs (left bottom panel), as well as for different starting values for the iceberg trade costs (right upper panel) and for a reduction in fixed rather than variable trade costs (right bottom panel).

7 Conclusion

This paper provides evidence that heterogeneity in the productivity distribution can be captured most adequately by Finite Mixture Models. A clear statistical framework differentiates between the fit of 52 distributions to domestic sales of the population of active Portuguese firms in 2006. The flexible, semi-parametric nature of FMMs results in a substantial empirical performance improvement compared to currently favored distributions in the firm size literature. Moreover, FMMs are the only distributions providing an approximation of Gains From Trade that is not rejected by the data.

Even though our results provide strong evidence in favor of FMMs, we take no stance on distribution type nor on the mixing parameter (or mechanism) that defines the underlying discrete subpopulations. It is clear that the two are closely interconnected, and therefore not easily identifiable. Further research is necessary to be able to define which mechanisms result in multiple individual densities defining the overall productivity distribution.

The idea of FMMs also opens many new venues for ongoing research. For instance, the mechanisms driving firm-level dynamics in aggregate growth models are determined by the parametric approximation of the productivity distribution (see for instance Luttmer (2007); Arkolakis (2016)). A correct parametric approximation is then essential to motivate the determinants of a firm's productivity growth. In relation to this, the estimation of productivity usually relies on a first-order Markov process that is identical for the complete population. Concurrently, however, it is recognized that productivity dynamics are endogenous to exporting (De Loecker, 2013), importing (Kasahara and Rodrigue, 2008), innovation (Aw et al., 2011), management practices (Bloom and Reenen, 2011; Caliendo et al., 2020), et cetera. Introducing Finite Mixture Modeling into the estimation procedures would allow, semi-parametrically, to control for such discrete subpopulations without the risk of model misspecification. Moreover, the potential identification of these subpopulations provides the opportunity to discriminate between the many different mechanisms (see for instance Cabral and Mata (2003); Klette and Kortum (2004); Rossi-Hansberg and Wright (2007); Atkeson and Burstein (2010); Caliendo et al. (2020)) that drive the existence of such subpopulations. Also, the propagation of firm-level volatility to the aggregate level mainly relies on a Pareto specification for the right tail of the productivity distribution (Gabaix, 2011; di Giovanni and Levchenko, 2012; Carvalho and Grassi, 2019). FMMs as well are sufficiently heavy-tailed to motivate granularity.

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Online Appendix to "Unobserved Heterogeneity in the Productivity Distribution and Gains From Trade"

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8th July 2020

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Appendix A Additional Figures and table

A.1 Figures

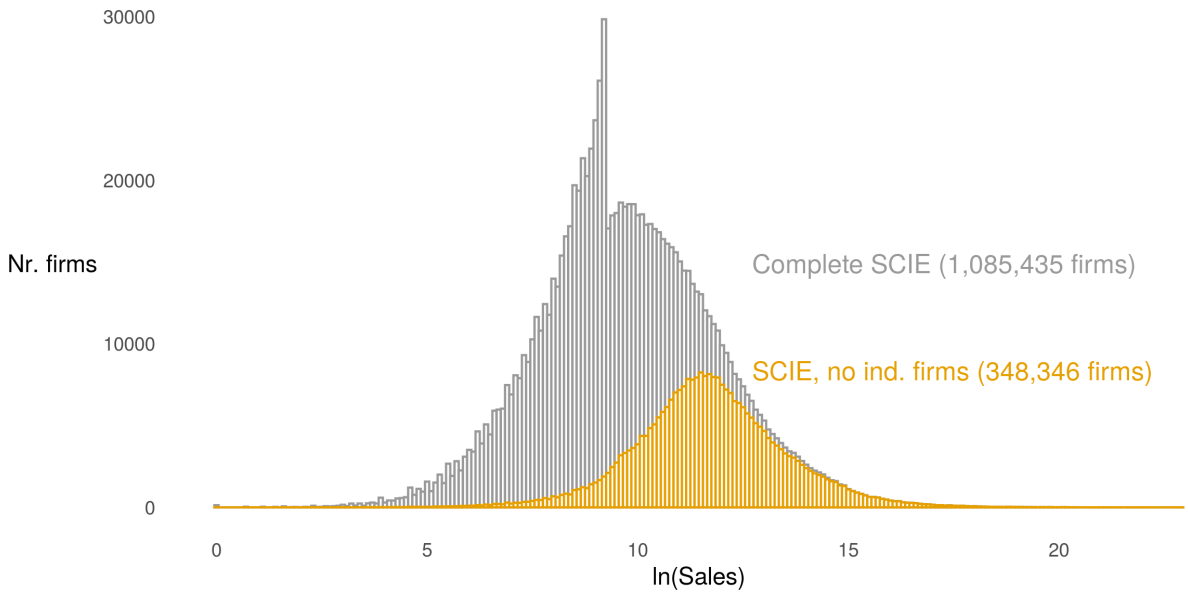

Figure 1: Density comparison of the SCIE dataset with and without individual companies.

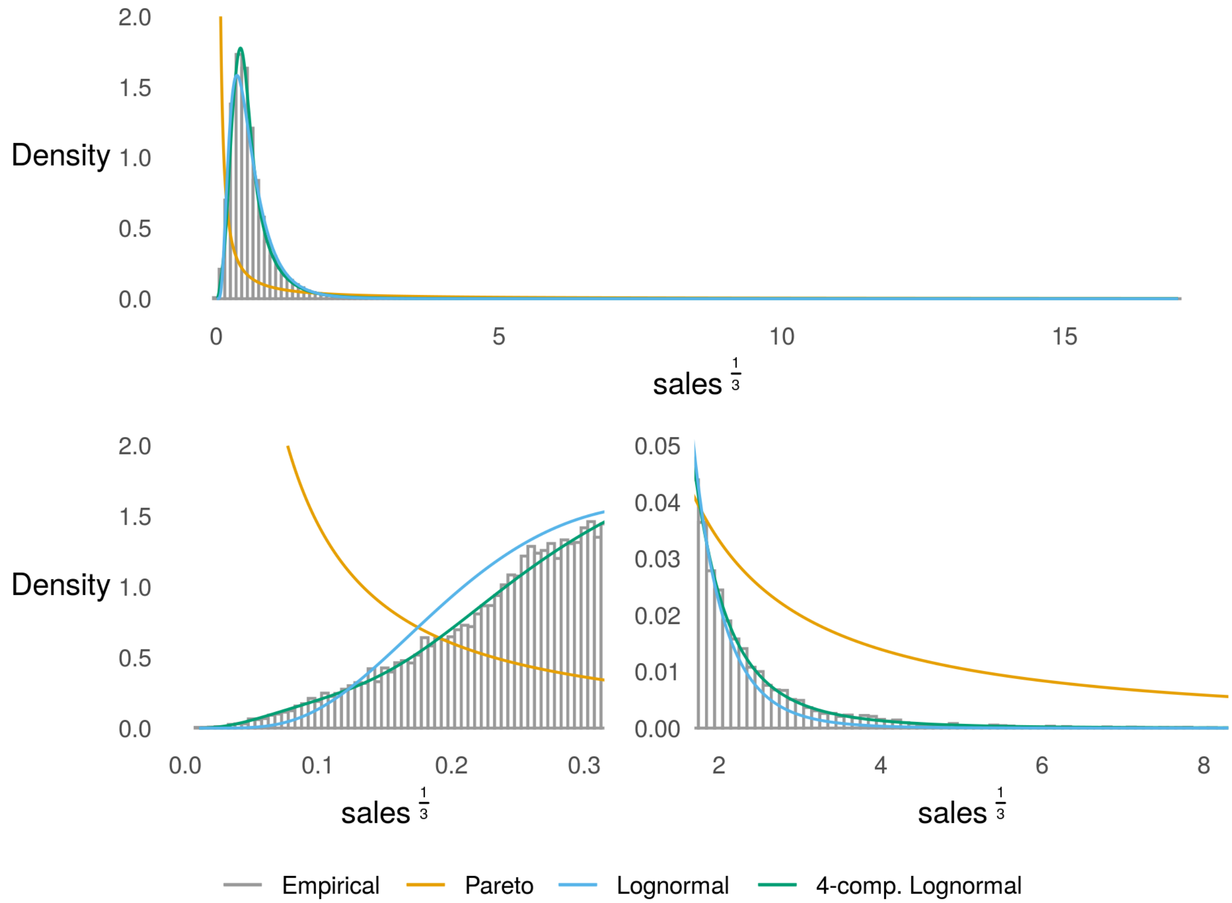

Figure 2: Empirical probability density function of Portuguese firm productivity in 2006 (upper panel) with fitted Pareto and (4-component) Lognormal densities. The lower left and right panels focus in on the left and right tail respectively.

Notes: Productivity is measured as domestic sales (relative to the mean) to the power of $1/(\sigma - 1)$ with σ, the elasticity of substitution between varieties, set to four. Distributions are fitted using maximum likelihood methods (cf. infra) to the complete dataset.

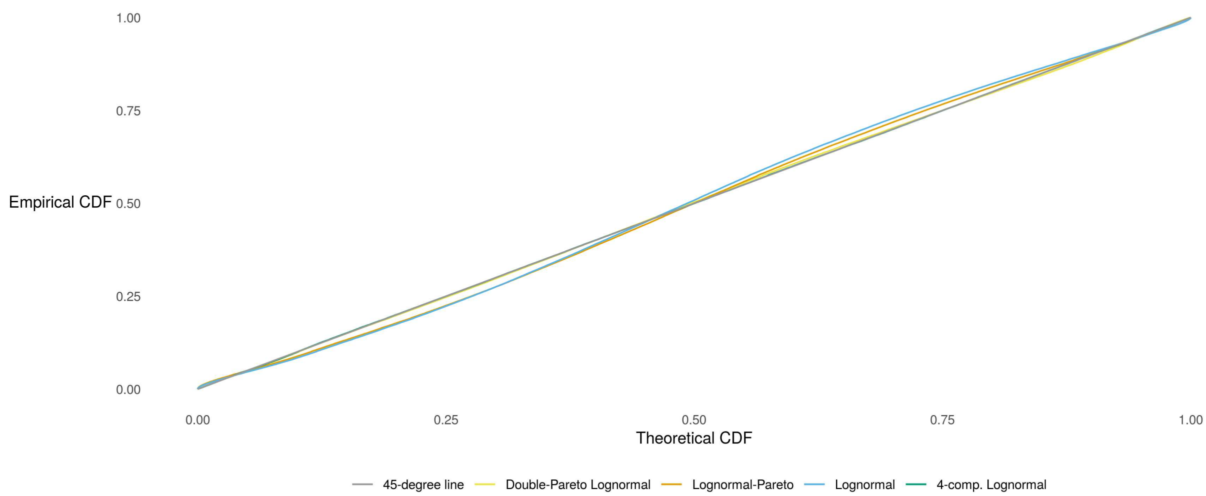

Figure 3: Probability-Probability plot for the Double-Pareto Lognormal, Lognormal-Pareto, Lognormal and 4-component Lognormal over the complete range of domestic sales in Portugal, 2006.

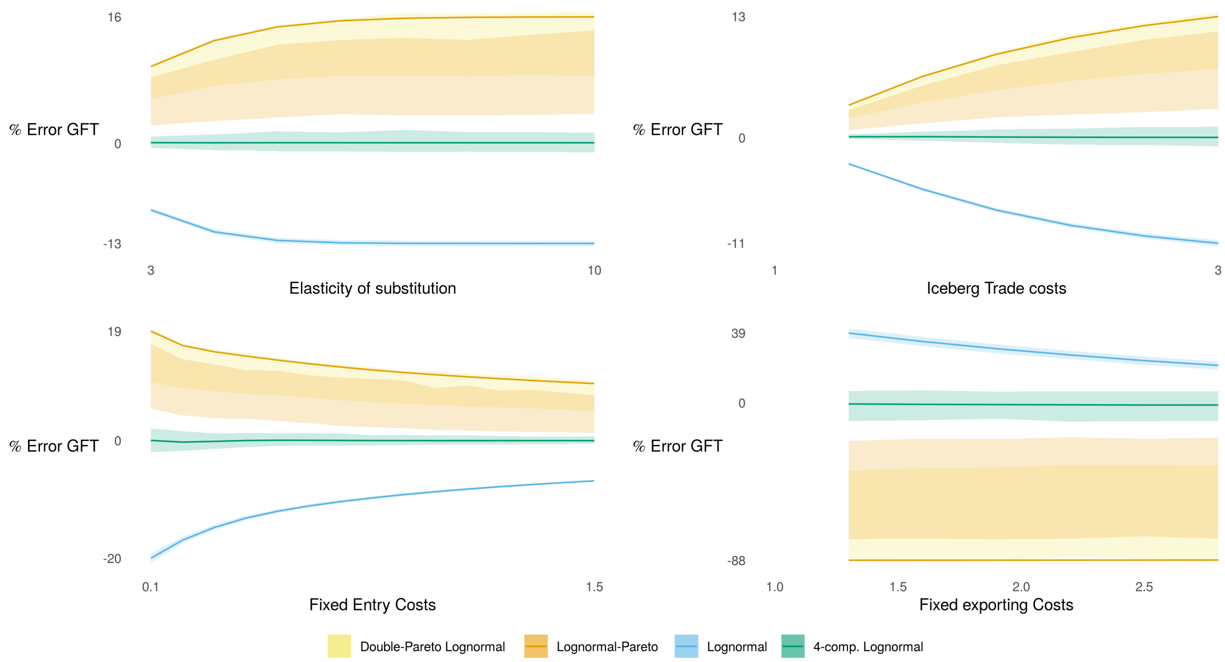

Figure 4: Percentage errors in parametric GFT calculations relative to the empirical benchmark for different values of the elasticity of substitution (left upper panel) and different fixed entry costs (left bottom panel) for a reduction in variable trade costs $\left(\tau^{ij} = 3 \rightarrow (\tau^{ij})' = 1\right)$. The right upper panel displays percentage errors in parametric GFT for different starting values of the iceberg trade costs $\left(\tau^{ij} \in [1;3] \rightarrow (\tau^{ij})' = 1\right)$. The bottom left panel showcases the error in parametric GFT for a reduction in fixed exporting costs with different starting values $\left(f^{ij} \in [1;3] \rightarrow (f^{ij})' = 1\right)$.

Notes: Full lines represent the parametric population GFT, while shaded areas delineate the 5th and 95th quantile of the parametric bootstrapped (999 replications) finite sample GFT. The Double-Pareto Lognormal has no finite population GFT value.

A.2 Tables

Table 1: Overview of all distributions considered.

Distribution	**Abbreviation**	**Support**	**Parameters**	**Change in parameters from power transformation** ax^b
Pareto	P	$[x_{min}, \infty[$	k, x_{min}	$kb, \left(\frac{x_{min}}{a}\right)^{\frac{1}{b}}$
Inverse Pareto	IP	$[0, x_{max}]$	k, x_{max}	$kb, \left(\frac{x_{max}}{a}\right)^{\frac{1}{b}}$
Lognormal	LN	$[0, \infty[$	μ, Var	$\frac{\mu - lna}{b}, \ \frac{Var}{b}$
Weibull	W	$[0, \infty[$	k, s	$bk, \left(\frac{s}{a}\right)^{\frac{1}{b}}$
Exponential	Exp	$[0, \infty[$	s	$W\left(b, \left(\frac{s}{a}\right)^{\frac{1}{b}}\right)$
Burr	B	$[0, \infty[$	k, c, s	$k, bc, \left(\frac{s}{a}\right)^{\frac{1}{b}}$
Fréchet	F	$[0, \infty]$	k, s	$bk, \left(\frac{s}{a}\right)^{\frac{1}{b}}$
Generalized Gamma	GG	$[0, \infty[$	k, c, s	$bk, bc, \left(\frac{s}{a}\right)^{\frac{1}{b}}$
Gamma	G	$[0, \infty[$	k, s	$GG\left(bk, b, \left(\frac{s}{a}\right)^{\frac{1}{b}}\right)$
Finite Mixture Model	FMM	See ind. comp.	$\boldsymbol{\Psi}$	See ind. comp.
Piecewise composite	PC	See ind. comp.	$\boldsymbol{\theta}$	See ind. comp.
Double-Pareto Lognormal	DPLN	$[0, \infty[$	k_1, μ, Var, k_2	$\frac{k_1}{b}, b\mu + log(a), Var, \frac{k_2}{b}$
Left-Pareto Lognormal	LPLN	$[0, \infty[$	k_1, μ, Var	$\frac{k_1}{b}, b\mu + log(a), Var$
Right-Pareto Lognormal	RPLN	$[0, \infty[$	μ, Var, k_2	$b\mu + log(a), Var, \frac{k_2}{b}$

Table 2: Overview of the probability and cumulative density functions of single distributions considered.

Distribution	**PDF**	**CDF**
P	$\frac{kx_{min}^{k}}{x^{k+1}}$	$1-\left(\frac{x_{min}}{x}\right)^{k}$
IP	$\frac{kx_{max}^{-k}}{x^{-k+1}}$	$1-\left(\frac{x_{max}}{x}\right)^{-k}$
LN	$\frac{1}{xVar\sqrt{2\pi}}e^{-(lnx-\mu)^2/2Var^2}$	$\Phi\left(\frac{lnx-\mu}{Var}\right)$
W	$\frac{k}{s}\left(\frac{x}{s}\right)^{k-1}e^{-\left(\frac{x}{s}\right)^{k}}$	$1-e^{-\left(\frac{x}{s}\right)^{k}}$
Exp	$\frac{1}{s}e^{-\frac{x}{s}}$	$1-e^{-\frac{x}{s}}$
B	$\frac{\frac{kc}{s}\left(\frac{x}{s}\right)^{c-1}}{\left(1+\left(\frac{x}{s}\right)^{c}\right)^{k+1}}$	$1-\frac{1}{\left(1+\left(\frac{x}{s}\right)^{c}\right)^{k}}$
F	$\frac{k}{s}\left(\frac{x}{s}\right)^{-1-k}e^{-\left(\frac{x}{s}\right)^{-k}}$	$e^{-\left(\frac{x}{s}\right)^{-k}}$
GG[a]	$\frac{c}{s^{k}\Gamma(\frac{k}{c})}x^{k-1}e^{-\left(\frac{x}{s}\right)^{c}}$	$\frac{1}{\Gamma(\frac{k}{c})}\gamma\left(\frac{k}{c},\left(\frac{x}{s}\right)^{c}\right)$
G[a]	$\frac{1}{s^{k}\Gamma(k)}x^{k-1}e^{-\frac{x}{s}}$	$\frac{1}{\Gamma(k)}\gamma\left(k,\frac{x}{s}\right)$

Notes: [a]$\Gamma(x)$ stands for the Gamma function, while $\gamma(s,x)$ stands for the lower incomplete Gamma function with upper bound x.

Table 3: Overview of the probability and cumulative density functions of combined distributions considered.

Distribution	**PDF**	**CDF**
FMM	$\sum_{i=1}^{I} \pi_i m_i(x\mid\boldsymbol{\theta}_i)$	$\sum_{i=1}^{I} \pi_i M(x\mid\boldsymbol{\theta}_i)$
PC[a]	$\begin{cases} \frac{\alpha_1}{1+\alpha_1+\alpha_2} m_1^*(x\mid\boldsymbol{\theta}_1) & \text{if} \quad 0 < x \leq c_1 \\ \frac{1}{1+\alpha_1+\alpha_2} m_2^*(x\mid\boldsymbol{\theta}_2) & \text{if} \quad c_1 < x \leq c_2 \\ \frac{\alpha_2}{1+\alpha_1+\alpha_2} m_3^*(x\mid\boldsymbol{\theta}_3) & \text{if} \quad c_2 < x < \infty \end{cases}$	$\begin{cases} \frac{\alpha_1}{1+\alpha_1+\alpha_2} \frac{M_1(x\mid\boldsymbol{\theta}_1)}{M_1(c_1\mid\boldsymbol{\theta}_1)} & \text{if} \quad 0 < x \leq c_1 \\ \frac{\alpha_1}{1+\alpha_1+\alpha_2} + \frac{1}{1+\alpha_1+\alpha_2} \frac{M_2(x\mid\boldsymbol{\theta}_2)-M_2(c_1\mid\boldsymbol{\theta}_2)}{M_2(c_2\mid\boldsymbol{\theta}_2)-M_2(c_1\mid\boldsymbol{\theta}_2)} & \text{if} \quad c_1 < x \leq c_2 \\ \frac{1+\alpha_1}{1+\alpha_1+\alpha_2} + \frac{\alpha_2}{1+\alpha_1+\alpha_2} \frac{M_3(x\mid\boldsymbol{\theta}_3)-M_3(c_2\mid\boldsymbol{\theta}_3)}{1-M_3(c_2\mid\boldsymbol{\theta}_3)} & \text{if} \quad c_2 < x < \infty \end{cases}$
DPLN[b]	$\frac{k_2 k_1}{k_2+k_1}\left[x^{-k_2-1} e^{k_2\mu+\frac{k_2^2 Var^2}{2}} \Phi\left(\frac{lnx-\mu-k_2 Var^2}{Var}\right) + x^{k_1-1} e^{-k_1\mu+\frac{k_1^2 Var^2}{2}} \Phi^c\left(\frac{lnx-\mu+k_1 Var^2}{Var}\right)\right]$	$\Phi\left(\frac{lnx-\mu}{Var}\right) - \frac{1}{k_2+k_1}\left[k_1 x^{-k_2} e^{k_2\mu+\frac{k_2^2 Var^2}{2}} \Phi\left(\frac{lnx-\mu-k_2 Var^2}{Var}\right) - k_2 x^{k_1} e^{-k_1\mu+\frac{k_1^2 Var^2}{2}} \Phi^c\left(\frac{lnx-\mu+k_1 Var^2}{Var}\right)\right]$
LPLN[b]	$k_1 x^{k_1-1} e^{-k_1\mu+\frac{k_1^2 Var^2}{2}} \Phi^c\left(\frac{lnx-\mu+k_1 Var^2}{Var}\right)$	$\Phi\left(\frac{lnx-\mu}{Var}\right) - x^{k_1} e^{-k_1\mu+\frac{k_1^2 Var^2}{2}} \Phi^c\left(\frac{lnx-\mu+k_1 Var^2}{Var}\right)$
RPLN[b]	$k_2 x^{-k_2-1} e^{k_2\mu+\frac{k_2^2 Var^2}{2}} \Phi\left(\frac{lnx-\mu-k_2 Var^2}{Var}\right)$	$\Phi\left(\frac{lnx-\mu}{Var}\right) - x^{-k_2} e^{k_2\mu+\frac{k_2^2 Var^2}{2}} \Phi\left(\frac{lnx-\mu-k_2 Var^2}{Var}\right)$

Notes: [a] $\forall i \in I : m_i^*(x) = \frac{m_i(x)}{\int_{c_{i-1}}^{c_i} m_i(x)dx}$, [b] Φ and Φ^c stand for the standard normal and complementary standard normal cdfs.

Table 4: Expression of the y-bounded rth moment (μ_y^r) for the single distributions considered.

Distribution	μ_y^r	**Additional parameter restrictions**[a]
P	$-\left(y\right)^{r-k}\,\frac{k\omega_{min}^k}{r-k}$	$k>r$
IP	$k\omega_{max}^{-k}\frac{\left(\omega_{max}\right)^{r+k}-\left(y\right)^{r+k}}{r+k}$	-
LN	$e^{\frac{r\left(rVar^2+2\mu\right)}{2}}\left[1-\Phi\left(\frac{lny-\left(rVar^2+\mu\right)}{Var}\right)\right]$	-
W[c]	$s^{\sigma_s-1}\Gamma\left(\frac{\sigma_s-1}{k}+1,\left(\frac{y}{s}\right)^k\right)$	-
Exp[c]	$s^{\sigma_s-1}\Gamma\left(\sigma_s+1,\frac{y}{s}\right)$	-
B[b]	$s^r k\left[\boldsymbol{B}\left(\frac{r}{c}+1,k-\frac{r}{c}\right)-\boldsymbol{B}\left(\frac{\left(\frac{y}{s}\right)^c}{1+\left(\frac{y}{s}\right)^c};\frac{r}{c}+1,k-\frac{r}{c}\right)\right]$	$c>r,\ kc>r$
F[c]	$s^{\sigma_s-1}\left[1-\Gamma\left(1-\frac{\sigma_s-1}{k},\left(\frac{y}{s}\right)^{-k}\right)\right]$	$k>r$
GG[c]	$\frac{s^{\sigma_s-1}}{\Gamma\left(\frac{k}{c}\right)}\Gamma\left(\frac{\sigma_s-1+k}{c},\left(\frac{y}{s}\right)^c\right)$	-
G[c]	$\frac{s^{\sigma_s-1}}{\Gamma(k)}\Gamma\left(\sigma_s-1+k,\frac{y}{s}\right)$	-

Notes: [a] Additional parameter restrictions represent parameter restrictions needed to keep the statistic finite. [b] $\boldsymbol{B}(a,b)$ stands for the beta function, while $\boldsymbol{B}(x,a,b)$ stands for the lower incomplete beta function with upper bound x. [c] $\Gamma(x)$ stands for the Gamma function, while $\Gamma(s,x)$ stands for the upper incomplete Gamma function with lower bound x.

Table 5: Expression of the y-bounded rth moment (μ_y^r) for the combined considered.

Distribution	μ_y^r	**Additional parameter restrictions**[a]
FMM	$\sum_{i=1}^{I} \pi_i (\mu_i)_y^r$	See ind. comp.
PC	$\begin{cases} \frac{\alpha_1}{1+\alpha_1+\alpha_2} \frac{(\mu_1)_y^r - (\mu_1)_{c_1}^r}{M_1(c_1)} + \frac{1}{1+\alpha_1+\alpha_2} \frac{(\mu_2)_{c_1}^r - (\mu_2)_{c_2}^r}{M_2(c_2)-M_2(c_1)} + \frac{\alpha_2}{1+\alpha_1+\alpha_2} \frac{(\mu_3)_y^r}{1-M_3(c_2)} & \text{if} \quad 0 < y \leq c_2 \\ \frac{1}{1+\alpha_1+\alpha_2} \frac{(\mu_2)_y^r - (\mu_2)_{c_2}^r}{M_2(c_2)-M_2(c_1)} + \frac{\alpha_2}{1+\alpha_1+\alpha_2} \frac{(\mu_3)_{c_2}^r}{1-M_3(c_2)} & \text{if} \quad c_1 < y \leq c_2 \\ \frac{\alpha_2}{1+\alpha_1+\alpha_2} \frac{(\mu_3)_y^r}{1-M_3(c_2)} & \text{if} \quad c_2 < y < \infty \end{cases}$	See ind. comp.
DPLN	$-\frac{k_2 k_1}{k_2+k_1} e^{k_2\mu + \frac{k_2^2 Var^2}{2}} \frac{y^{\sigma_s - k_2 - 1}}{\sigma_s - k_2 - 1} \Phi\left(\frac{lny - \mu - k_2 Var^2}{Var}\right) - \frac{k_2 k_1}{k_2+k_1} \frac{1}{r-k_2} e^{\frac{r^2 Var^2 + 2\mu r}{2}} \Phi^c\left(\frac{lny - rVar^2 - \mu}{Var}\right) - \frac{k_2 k_1}{k_2+k_1} e^{-k_1\mu + \frac{k_1^2 Var^2}{2}} \frac{y^{\sigma_s + k_1 - 1}}{\sigma_s + k_1 - 1} \Phi^c\left(\frac{lny - \mu + k_1 Var^2}{Var}\right) + \frac{k_2 k_1}{k_2+k_1} \frac{1}{r+k_1} e^{\frac{r^2 Var^2 + 2\mu r}{2}} \Phi^c\left(\frac{lny - rVar^2 - \mu}{Var}\right)$	$k_2 > r$
LPLN	$-k_1 e^{-k_1\mu + \frac{k_1^2 Var^2}{2}} \frac{y^{\sigma_s + k_1 - 1}}{\sigma_s + k_1 - 1} \Phi^c\left(\frac{lny - \mu + k_1 Var^2}{Var}\right) + \frac{k_1}{r+k_1} e^{\frac{r^2 Var^2 + 2\mu r}{2}} \Phi^c\left(\frac{lny - rVar^2 + \mu}{Var}\right)$	-
RPLN	$-k_2 e^{k_2\mu + \frac{k_2^2 Var^2}{2}} \frac{y^{\sigma_s - k_2 - 1}}{\sigma_s - k_2 - 1} \Phi\left(\frac{lny - \mu - k_2 Var^2}{Var}\right) - \frac{k_2}{r-k_2} e^{\frac{r^2 Var^2 + 2\mu r}{2}} \Phi^c\left(\frac{lny - rVar^2 + \mu}{Var}\right)$	$k_2 > r$

Notes: [a] Additional parameter restrictions represent parameter restrictions needed to keep the statistic finite.

Table 6: Coverage ratio of SCIE vs OECD SDBS database.

	Number of Enterprises						Total Employment						Turnover					
NACE Rev.2	1-9	10-19	20-49	50-249	> 250	Total	1-9	10-19	20-49	50-249	> 250	Total	1-9	10-19	20-49	50-249	> 250	Total
13	100				100	100												
14	100	100	100	100		100		100	100					100	100			
15	100	100	100	100	100	100	100	100	100			100	100	100	100			100
16				100	100	100						100						100
17	100	100	100	100	100	100				100	100	100				100	100	100
18	100	100	100	100	100	100				100	100	100				100	100	100
19	100	100	100	100	100	100		100	100	100				100	100	100		
20	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
21	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
22	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
23					100	100												
24	100	100	100	100	100	100				100	100					100	100	
25	100	100	100	100	100	100				100	100	100				100	100	100
26	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
27	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
28	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
29	100	100	100	100	100	100	100			100	100	100	100			100	100	100
30	100	100	100	100	100	100												
31	100	100	100	100	100	100			100	100	100	100			100	100	100	100
32	100	100	100	100	100	100		100	100	100	100			100	100	100	100	
33	100	100	100	100	100	100	100	100	100				100	100	100			
34	100	100	100	100	100	100			100	100	100				100	100	100	
35	100	100	100	100	100	100			100	100	100				100	100	100	
36	100	100	100	100	100	100		100		100	100	100		100		100	100	100
37	100	100	100	100		100				100		100				100		100
40	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
41	100	100	100	100	100	100	100	100	100			100	100	100	100			100
45	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
50	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
51	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
52	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
55	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
60	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
61	100	100	100	100	100	100		100		100		100		100		100		100
62	100	100	100	100	100	100			100	100		100			100	100		100
63	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
64	100	100	100	100	100	100	100			100	100	100	100			100	100	100
70	100	100	100	100	100	100	100	100	100			100	100	100	100			100
71	100	100	100	100	100	100			100			100			100			100
72	100	100	100	100	100	100	100			100	100	100	100			100	100	100
73	100	100	100	100		100						100						100
74	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100

Notes: Each cell corresponds to the ratio of our dataset compared to the data from the OECD structural SDBS database for the year 2006. Size classes are based on total employment. Empty cells and absent industries are due to missing information from SBDS, even though the data is available in our SCIE database.

Table 7: Distribution fits to Portuguese domestic sales in 2006.

		Goodness of fit				Information Criteria		
Distribution	Parms.	T_a^0	S_b^0	T_a^1	S_b^1	Loglike	R_{AIC}	R_{BIC}
5-comp. Lognormal	14	0.18	0.11	3.08	2.37	12,776	1	3^{+++}
		(0.10;0.25)	(0.08;0.32)	(2.03;9.73)	(0.82;26.24)			
4-comp. Lognormal	11	0.19	0.11	2.78	0.13	12,770	2	2
		(0.09;0.25)	(0.08;0.32)	(2.07;9.23)	(0.83;24.67)			
5-comp. Burr	19	0.19	0.12	-	-	12,767	3	7^{+++}
		(0.10;0.25)	(0.08;0.32)	(-;-)	(-;-)			
4-comp. Burr	15	0.24	0.14	-	-	12,754	4	6^{+++}
		(0.10;0.25)*	(0.08;0.32)	(-;-)	(-;-)			
3-comp. Burr	11	0.25	0.17	-	-	12,748	6	4^{+++}
		(0.09;0.25)*	(0.08;0.30)	(-;-)	(-;-)			
2-comp. Burr	7	0.20	0.20	-	-	12,745	5	1
		(0.09;0.25)	(0.08;0.32)	(-;-)	(-;-)			
5-comp. Weibull	14	0.25	0.14	6.96	11.95	12,731	7	8^{+++}
		(0.10;0.25)**	(0.08;0.31)	(1.29;5.00)***	(0.59;13.45)*			
3-comp. Lognormal	8	0.29	0.34	4.39	9.91	12,723	8	5^{+++}
		(0.10;0.24)**	(0.09;0.32)**	(2.34;11.34)	(0.93;30.68)			
5-comp. Gamma	14	0.26	0.16	7.27	0.09	12,639	9	9^{+++}
		(0.10;0.26)**	(0.09;0.33)	(1.29;5.11)***	(0.44;14.23)			
Inv. Pareto-Burr	4	0.51	0.61	-	-	12,561	10	10^{+++}
		(0.09;0.24)***	(0.08;0.33)***	(-;-)	(-;-)			
Inv. Pareto-Burr-Pareto	5	0.51	0.61	-	-	12,561	11	11^{+++}
		(0.09;0.25)***	(0.08;0.33)***	(-;-)	(-;-)			
5-comp. Exponential	9	0.32	0.23	7.96	0.15	12,548	12	12^{+++}
		(0.09;0.26)***	(0.09;0.31)	(1.31;4.78)***	(0.40;12.83)			
4-comp. Weibull	11	0.31	0.25	14.75	27.04	12,543	13	13^{+++}
		(0.09;0.25)***	(0.08;0.31)	(0.87;3.44)***	(0.29;8.78)***			
Burr-Pareto	4	0.73	0.95	-	-	12,451	15	15^{+++}
		(0.09;0.25)***	(0.08;0.33)***	(-;-)	(-;-)			
Burr	3	0.73	0.95	-	-	12,451	14	14^{+++}
		(0.10;0.24)***	(0.08;0.31)***	(-;-)	(-;-)			
Double-Pareto Lognormal	4	0.66	0.80	-	-	12,429	16	16^{+++}
		(0.09;0.25)***	(0.08;0.33)***	(-;-)	(-;-)			

2-comp. Lognormal	5	0.53	0.71	8.70	10.15	12,401	17	17^{+++}
		(0.10;0.24)***	(0.09;0.32)***	(1.32;5.87)**	(0.54;16.11)			
Inv. Pareto-Lognormal-Pareto	4	0.81	1.01	-	-	12,231	18	18^{+++}
		(0.09;0.26)***	(0.08;0.34)***	(-;-)	(-;-)			
4-comp. Gamma	11	0.40	0.63	11.95	0.26	12,173	19	19^{+++}
		(0.10;0.25)***	(0.08;0.32)***	(1.00;3.92)***	(0.26;10.38)			
Inv. Pareto-Fréchet-Pareto	4	1.11	1.48	-	-	11,953	20	20^{+++}
		(0.09;0.25)***	(0.08;0.33)***	(-;-)	(-;-)			
3-comp. Weibull	8	0.69	0.92	20.31	39.45	11,855	21	21^{+++}
		(0.10;0.25)***	(0.09;0.31)***	(0.73;2.60)***	(0.23;6.78)***			
4-comp. Exponential	7	0.57	0.89	13.91	0.36	11,801	22	22^{+++}
		(0.10;0.25)***	(0.09;0.32)***	(0.95;3.61)***	(0.34;9.44)			
Inv. Pareto-Weibull-Pareto	4	1.60	2.00	-	-	11,338	24	24^{+++}
		(0.09;0.25)***	(0.08;0.32)***	(-;-)	(-;-)			
Weibull-Pareto	3	1.60	2.00	-	-	11,338	23	23^{+++}
		(0.09;0.25)***	(0.08;0.32)***	(-;-)	(-;-)			
Inv. Pareto-Gamma-Pareto	4	1.70	2.17	-	-	11,249	26	26^{+++}
		(0.10;0.26)***	(0.08;0.35)***	(-;-)	(-;-)			
Gamma-Pareto	3	1.70	2.17	-	-	11,249	25	25^{+++}
		(0.09;0.25)***	(0.08;0.32)***	(-;-)	(-;-)			
Inv. Pareto-Exponential-Pareto	3	1.97	2.71	-	-	11,044	27	27^{+++}
		(0.10;0.25)***	(0.09;0.33)***	(-;-)	(-;-)			
Exponential-Pareto	2	2.00	2.83	-	-	11,012	28	28^{+++}
		(0.09;0.25)***	(0.08;0.32)***	(-;-)	(-;-)			
3-comp. Gamma	8	1.00	1.56	19.47	0.62	10,288	29	29^{+++}
		(0.10;0.25)***	(0.09;0.32)***	(0.73;2.75)***	(0.30;7.18)			
Inv. Pareto-Lognormal	3	3.02	4.26	45.72	127.97	9,198	30	30^{+++}
		(0.09;0.24)***	(0.08;0.31)***	(0.43;1.67)***	(0.16;4.36)***			
Lognormal-Pareto	3	2.56	3.78	562.07	1683.18	8,721	31	31^{+++}
		(0.09;0.25)***	(0.08;0.32)***	(169.39;451.06)***	(371.67;1342.23)***			
3-comp. Exponential	5	1.64	2.60	22.73	0.87	8,387	32	32^{+++}
		(0.09;0.25)***	(0.08;0.32)***	(0.67;2.36)***	(0.21;6.27)			
Left-Pareto Lognormal	3	3.23	4.91	46.00	127.70	8,059	33	33^{+++}
		(0.10;0.25)***	(0.09;0.32)***	(0.41;1.58)***	(0.18;4.05)***			
Right-Pareto Lognormal	3	2.82	4.38	19.27	49.88	8,028	34	34^{+++}
		(0.09;0.25)***	(0.08;0.32)***	(3.10;11.90)**	(1.23;32.23)**			
Lognormal	2	2.93	5.03	41.38	113.04	7,372	35	35^{+++}

		(0.10;0.25)***	(0.08;0.33)***	(0.47;1.84)***	(0.17;4.76)***			
2-comp. Weibull	5	2.10	3.19	35.20	72.85	6,442	36	36^{+++}
		(0.09;0.24)***	(0.08;0.31)***	(0.42;1.50)***	(0.16;3.80)***			
2-comp. Fréchet	5	6.92	10.64	-	-	-3,041	37	37^{+++}
		(0.09;0.25)***	(0.08;0.32)***	(-;-)	(-;-)			
5-comp. Fréchet	14	6.96	10.63	-	-	-3,045	40	40^{+++}
		(0.10;0.25)***	(0.09;0.31)***	(-;-)	(-;-)			
3-comp. Fréchet	8	6.96	10.63	-	-	-3,046	38	38^{+++}
		(0.09;0.25)***	(0.08;0.32)***	(-;-)	(-;-)			
4-comp. Fréchet	11	6.98	10.63	-	-	-3,047	39	39^{+++}
		(0.10;0.25)***	(0.09;0.32)***	(-;-)	(-;-)			
2-comp. Gamma	5	4.00	5.93	31.79	2.24	-3,381	41	41^{+++}
		(0.09;0.25)***	(0.08;0.32)***	(0.45;1.67)***	(0.14;4.45)			
2-comp. Exponential	3	7.06	11.51	37.63	3.23	-18,112	42	42^{+++}
		(0.10;0.25)***	(0.08;0.33)***	(0.38;1.40)***	(0.14;3.52)*			
Inv. Pareto-Weibull	3	9.18	16.52	54.06	123.38	-29,711	43	44^{+++}
		(0.10;0.25)***	(0.08;0.31)***	(0.26;0.92)***	(0.11;2.22)***			
Weibull	2	9.18	16.51	54.06	123.40	-29,713	44	43^{+++}
		(0.09;0.25)***	(0.09;0.32)***	(0.25;0.90)***	(0.15;2.20)***			
Fréchet	2	8.91	16.72	-	-	-32,908	45	45^{+++}
		(0.10;0.26)***	(0.08;0.33)***	(-;-)	(-;-)			
Fréchet-Pareto	3	8.91	16.72	-	-	-32,908	46	46.5^{+++}
		(0.10;0.25)***	(0.08;0.31)***	(-;-)	(-;-)			
Inv. Pareto-Fréchet	3	8.91	16.72	-	-	-32,908	46	46.5^{+++}
		(0.09;0.25)***	(0.08;0.33)***	(-;-)	(-;-)			
Inv. Pareto-Gamma	3	20.93	32.98	50.26	9.56	-104,785	48	48^{+++}
		(0.10;0.25)***	(0.08;0.33)***	(0.22;0.76)***	(0.13;1.81)***			
Gamma	2	20.98	33.03	50.29	9.58	-104,878	49	49^{+++}
		(0.10;0.25)***	(0.08;0.32)***	(0.22;0.76)***	(0.11;1.71)***			
Exponential	1	44.64	79.71	60.73	16.76	-299,935	50	50^{+++}
		(0.10;0.25)***	(0.09;0.33)***	(0.15;0.49)***	(0.11;1.08)***			
Inv. Pareto-Exponential	2	44.64	79.71	60.73	16.76	-299,935	51	51^{+++}
		(0.09;0.24)***	(0.08;0.32)***	(0.15;0.51)***	(0.11;1.12)***			
Pareto	2	48.34	68.18	-	-	-436,227	52	52^{+++}
		(0.09;0.25)***	(0.08;0.33)***	(-;-)	(-;-)			

Notes: All distributions fitted using Maximum Likelihood.

Values between parentheses report the 5th and 95th quantile of the parametric bootstrapped test statistic with 999 replications. ***, **, * indicate significance of this test at 1%, 5% and 10% respectively.

+++, ++, + indicates the difference between this distribution's BIC and the first-ranked distribution in terms of BIC (ΔBIC) providing strong evidence in favour of the first-ranked distribution ($\Delta BIC > 10$), moderate evidence ($6 < \Delta BIC \leq 10$) and weak evidence ($2 < \Delta BIC \leq 6$) respectively.

[a] Values multiplied by 100 for expositional purpose, [b] Values divided by 1,000 for expositional purpose.

Table 8: Distribution fits to domestic sales of the Portuguese manufacturing sector in 2006.

		Goodness of fit		Information Criteria		
Distribution	Parms.	T_a^0	S_b^0	Loglike	R_{AIC}	R_{BIC}
5-comp. Burr	19	0.24	0.02	-2,095	7	10^{+++}
		(0.25;0.66)	(0.03;0.12)			
5-comp. Lognormal	14	0.28	0.03	-2,095	3	5^{+++}
		(0.25;0.67)	(0.03;0.12)			
4-comp. Burr	15	0.24	0.02	-2,096	5	6^{+++}
		(0.25;0.67)	(0.03;0.12)			
3-comp. Burr	11	0.23	0.03	-2,099	4	3^{+++}
		(0.25;0.67)	(0.03;0.12)			
2-comp. Burr	7	0.22	0.02	-2,099	1	1
		(0.25;0.66)	(0.03;0.12)			
3-comp. Lognormal	8	0.28	0.02	-2,101	2	2^{+++}
		(0.25;0.69)	(0.03;0.12)			
4-comp. Lognormal	11	0.27	0.02	-2,101	6	4^{+++}
		(0.25;0.67)	(0.03;0.12)			
5-comp. Weibull	14	0.34	0.03	-2,104	8	7^{+++}
		(0.26;0.65)	(0.03;0.11)			
5-comp. Gamma	14	0.31	0.04	-2,114	9	8^{+++}
		(0.26;0.66)	(0.03;0.12)			
4-comp. Weibull	11	0.40	0.04	-2,131	10	9^{+++}
		(0.26;0.65)	(0.03;0.11)			
5-comp. Exponential	9	1.29	0.15	-2,171	11	13^{+++}
		(0.25;0.66)***	(0.03;0.12)***			
4-comp. Gamma	11	0.50	0.09	-2,178	12	15^{+++}
		(0.26;0.65)	(0.03;0.12)			
Inv. Pareto-Fréchet-Pareto	4	0.65	0.09	-2,187	13	11^{+++}
		(0.25;0.65)*	(0.03;0.12)			
Inv. Pareto-Burr	4	0.88	0.13	-2,197	14	12^{+++}
		(0.26;0.65)***	(0.03;0.12)**			
Inv. Pareto-Burr-Pareto	5	0.88	0.13	-2,197	15	14^{+++}
		(0.25;0.69)***	(0.03;0.12)**			
3-comp. Weibull	8	0.83	0.14	-2,222	16	17^{+++}
		(0.25;0.66)***	(0.03;0.11)**			
2-comp. Lognormal	5	0.73	0.11	-2,232	17	16^{+++}
		(0.26;0.67)**	(0.03;0.12)*			
Double-Pareto Lognormal	4	1.08	0.17	-2,245	18	18^{+++}
		(0.26;0.66)***	(0.03;0.12)***			
4-comp. Exponential	7	1.18	0.16	-2,251	19	20^{+++}
		(0.26;0.66)***	(0.03;0.12)***			
Inv. Pareto-Lognormal-Pareto	4	1.27	0.18	-2,263	20	19^{+++}
		(0.25;0.66)***	(0.03;0.11)***			
Burr-Pareto	4	1.18	0.25	-2,284	22	22^{+++}
		(0.26;0.67)***	(0.03;0.12)***			
Burr	3	1.18	0.25	-2,284	21	21^{+++}
		(0.26;0.67)***	(0.03;0.12)***			
Inv. Pareto-Gamma-Pareto	4	1.65	0.28	-2,346	23	24^{+++}
		(0.25;0.65)***	(0.03;0.12)***			

Inv. Pareto-Weibull-Pareto	4	1.62	0.28	-2,348	24	25^{+++}
		(0.25;0.68)***	(0.03;0.12)***			
Gamma-Pareto	3	1.58	0.27	-2,355	26	26^{+++}
		(0.25;0.68)***	(0.03;0.12)***			
Weibull-Pareto	3	1.58	0.27	-2,355	27	27^{+++}
		(0.27;0.67)***	(0.03;0.11)***			
Exponential-Pareto	2	1.57	0.27	-2,355	25	23^{+++}
		(0.26;0.68)***	(0.03;0.12)***			
Inv. Pareto-Exponential-Pareto	3	1.57	0.27	-2,355	28	28^{+++}
		(0.26;0.67)***	(0.03;0.12)***			
3-comp. Gamma	8	1.01	0.20	-2,408	29	29^{+++}
		(0.26;0.68)***	(0.03;0.12)***			
3-comp. Exponential	5	1.44	0.26	-2,608	30	30^{+++}
		(0.25;0.65)***	(0.03;0.12)***			
Inv. Pareto-Lognormal	3	4.18	0.81	-2,875	31	31^{+++}
		(0.26;0.65)***	(0.03;0.12)***			
2-comp. Weibull	5	2.19	0.43	-2,918	32	32^{+++}
		(0.25;0.66)***	(0.03;0.12)***			
Lognormal-Pareto	3	3.25	0.64	-3,051	33	33^{+++}
		(0.25;0.67)***	(0.03;0.12)***			
Left-Pareto Lognormal	3	4.39	0.89	-3,103	34	34^{+++}
		(0.25;0.65)***	(0.03;0.11)***			
Right-Pareto Lognormal	3	3.51	0.73	-3,143	35	35^{+++}
		(0.26;0.66)***	(0.03;0.12)***			
Lognormal	2	3.96	0.88	-3,250	36	36^{+++}
		(0.25;0.65)***	(0.03;0.11)***			
2-comp. Gamma	5	3.35	0.71	-4,108	37	37^{+++}
		(0.25;0.65)***	(0.03;0.12)***			
5-comp. Fréchet	14	8.11	1.79	-4,863	38	40^{+++}
		(0.26;0.67)***	(0.03;0.12)***			
4-comp. Fréchet	11	8.35	1.80	-4,870	39	39^{+++}
		(0.25;0.66)***	(0.03;0.12)***			
3-comp. Fréchet	8	8.59	1.82	-4,881	40	38^{+++}
		(0.26;0.67)***	(0.03;0.12)***			
2-comp. Fréchet	5	9.55	1.92	-4,955	41	41^{+++}
		(0.25;0.67)***	(0.03;0.12)***			
2-comp. Exponential	3	5.75	1.20	-5,550	42	42^{+++}
		(0.25;0.67)***	(0.03;0.12)***			
Inv. Pareto-Weibull	3	9.91	2.42	-8,321	44	44^{+++}
		(0.26;0.65)***	(0.03;0.12)***			
Weibull	2	9.91	2.42	-8,321	43	43^{+++}
		(0.26;0.67)***	(0.03;0.11)***			
Inv. Pareto-Fréchet	3	10.04	2.61	-9,885	46	46^{+++}
		(0.26;0.68)***	(0.03;0.12)***			
Fréchet-Pareto	3	10.04	2.61	-9,885	47	47^{+++}
		(0.25;0.69)***	(0.03;0.13)***			
Fréchet	2	10.04	2.61	-9,885	45	45^{+++}
		(0.25;0.65)***	(0.03;0.11)***			
Inv. Pareto-Gamma	3	20.68	4.43	-17,309	48	48^{+++}
		(0.26;0.67)***	(0.03;0.11)***			
Gamma	2	20.72	4.44	-17,318	49	49^{+++}

		(0.25;0.67)***	(0.03;0.12)***			
Exponential	1	43.48	10.42	-41,128	50	50+++
		(0.27;0.66)***	(0.03;0.12)***			
Inv. Pareto-Exponential	2	43.48	10.42	-41,128	51	51+++
		(0.26;0.65)***	(0.03;0.11)***			
Pareto	2	49.14	9.43	-66,043	52	52+++
		(0.26;0.65)***	(0.03;0.11)***			

Notes: All distributions fitted using Maximum Likelihood.
Values between parentheses report the 5th and 95th quantile of the parametric bootstrapped test statistic with 999 replications. ***, **, * indicate significance of this test at 1%, 5% and 10% respectively.
+++, ++, + indicates the difference between this distribution's BIC and the first-ranked distribution in terms of BIC (ΔBIC) providing strong evidence in favour of the first-ranked distribution ($\Delta BIC > 10$), moderate evidence ($6 < \Delta BIC \leq 10$) and weak evidence ($2 < \Delta BIC \leq 6$) respectively.
$_a$ Values multiplied by 100 for expositional purpose, $_b$ Values divided by 1,000 for expositional purpose.

Table 9: Distribution fits to Portuguese domestic sales leaving out the first and last 1,000 observations in 2006.

		Goodness of fit		Information Criteria		
Distribution	Parms.	T_a^0	S_b^0	Loglike	R_{AIC}	R_{BIC}
4-comp. Lognormal	11	0.18	0.18	23,100	1	1
		(0.09;0.24)	(0.09;0.32)			
5-comp. Lognormal	14	0.21	0.20	23,093	2	2^{+++}
		(0.09;0.24)	(0.08;0.30)			
3-comp. Lognormal	8	0.25	0.28	22,844	3	3^{+++}
		(0.10;0.25)*	(0.09;0.31)*			
5-comp. Weibull	14	0.33	0.27	22,764	4	4^{+++}
		(0.09;0.25)***	(0.08;0.32)			
5-comp. Gamma	14	0.36	0.29	22,758	5	5^{+++}
		(0.09;0.24)***	(0.08;0.31)*			
4-comp. Gamma	11	0.40	0.30	22,724	6	7^{+++}
		(0.09;0.24)***	(0.08;0.31)*			
2-comp. Lognormal	5	0.29	0.26	22,695	7	6^{+++}
		(0.10;0.25)**	(0.09;0.31)			
4-comp. Weibull	11	0.40	0.29	22,691	8	8^{+++}
		(0.10;0.25)***	(0.08;0.31)*			
4-comp. Exponential	7	0.60	0.34	22,544	9	9^{+++}
		(0.10;0.25)***	(0.08;0.33)**			
5-comp. Exponential	9	0.59	0.36	22,541	10	10^{+++}
		(0.09;0.25)***	(0.08;0.32)**			
3-comp. Burr	11	0.30	0.38	22,477	11	11^{+++}
		(0.09;0.26)***	(0.08;0.32)**			
3-comp. Weibull	8	0.40	0.66	22,247	12	12^{+++}
		(0.09;0.24)***	(0.08;0.31)***			
5-comp. Fréchet	14	0.67	0.56	22,240	13	13^{+++}
		(0.09;0.25)***	(0.08;0.32)***			
3-comp. Gamma	8	0.56	0.88	22,132	14	14^{+++}
		(0.10;0.25)***	(0.09;0.32)***			
4-comp. Fréchet	11	0.68	0.66	22,056	15	16^{+++}
		(0.09;0.26)***	(0.08;0.32)***			
3-comp. Exponential	5	0.64	1.00	22,025	16	15^{+++}
		(0.10;0.25)***	(0.08;0.33)***			
3-comp. Fréchet	8	0.67	0.65	21,911	17	17^{+++}
		(0.09;0.25)***	(0.09;0.33)***			
2-comp. Burr	7	0.80	0.76	21,614	22	22^{+++}
		(0.09;0.25)***	(0.08;0.30)***			
Inv. Pareto-Burr-Pareto	5	0.80	0.76	21,614	21	21^{+++}
		(0.09;0.24)***	(0.08;0.30)***			
Inv. Pareto-Burr	4	0.80	0.76	21,614	19	19^{+++}
		(0.09;0.25)***	(0.08;0.32)***			
Burr	3	0.80	0.76	21,614	18	18^{+++}
		(0.10;0.25)***	(0.08;0.32)***			
Burr-Pareto	4	0.80	0.76	21,614	20	20^{+++}
		(0.09;0.24)***	(0.08;0.31)***			
5-comp. Burr	19	0.80	0.76	21,614	23	24^{+++}

		(0.09;0.25)***	(0.08;0.31)***			
Double-Pareto Lognormal	4	1.04	1.36	21,592	24	23^{+++}
		(0.10;0.25)***	(0.09;0.33)***			
Inv. Pareto-Lognormal-Pareto	4	1.18	1.51	21,179	25	25^{+++}
		(0.10;0.25)***	(0.09;0.33)***			
Inv. Pareto-Lognormal	3	2.48	3.35	20,614	26	26^{+++}
		(0.10;0.25)***	(0.09;0.33)***			
Right-Pareto Lognormal	3	2.02	3.25	20,585	27	27^{+++}
		(0.10;0.25)***	(0.08;0.33)***			
Lognormal-Pareto	3	1.85	3.01	20,494	28	28^{+++}
		(0.10;0.25)***	(0.09;0.32)***			
Left-Pareto Lognormal	3	2.49	3.70	20,423	29	29^{+++}
		(0.09;0.25)***	(0.08;0.33)***			
Lognormal	2	2.35	3.68	20,407	30	30^{+++}
		(0.10;0.25)***	(0.09;0.33)***			
Inv. Pareto-Fréchet-Pareto	4	1.46	2.06	20,193	31	31^{+++}
		(0.10;0.25)***	(0.08;0.32)***			
Inv. Pareto-Weibull-Pareto	4	1.95	2.55	19,520	33	33^{+++}
		(0.10;0.25)***	(0.08;0.32)***			
Weibull-Pareto	3	1.95	2.55	19,520	32	32^{+++}
		(0.09;0.25)***	(0.08;0.33)***			
Inv. Pareto-Gamma-Pareto	4	2.06	2.75	19,441	35	35^{+++}
		(0.10;0.25)***	(0.09;0.32)***			
Gamma-Pareto	3	2.06	2.75	19,441	34	34^{+++}
		(0.09;0.25)***	(0.09;0.34)***			
2-comp. Weibull	5	1.31	2.24	19,404	36	38^{+++}
		(0.09;0.25)***	(0.08;0.32)***			
Inv. Pareto-Exponential-Pareto	3	2.20	3.02	19,394	38	37^{+++}
		(0.09;0.24)***	(0.08;0.31)***			
Exponential-Pareto	2	2.20	3.02	19,394	37	36^{+++}
		(0.09;0.24)***	(0.08;0.30)***			
2-comp. Fréchet	5	1.49	2.25	19,272	39	39^{+++}
		(0.10;0.25)***	(0.08;0.32)***			
2-comp. Gamma	5	2.30	3.50	16,675	40	40^{+++}
		(0.09;0.25)***	(0.08;0.32)***			
2-comp. Exponential	3	3.73	5.90	13,268	41	41^{+++}
		(0.10;0.26)***	(0.08;0.33)***			
Fréchet	2	7.68	12.96	-6,681	42	42^{+++}
		(0.09;0.26)***	(0.08;0.33)***			
Fréchet-Pareto	3	7.68	12.96	-6,681	44	43.5^{+++}
		(0.10;0.25)***	(0.08;0.32)***			
Inv. Pareto-Fréchet	3	7.68	12.96	-6,681	44	43.5^{+++}
		(0.10;0.25)***	(0.09;0.33)***			
Inv. Pareto-Weibull	3	8.40	14.21	-8,737	46	46^{+++}
		(0.10;0.26)***	(0.09;0.33)***			
Weibull	2	8.40	14.21	-8,738	45	45^{+++}
		(0.10;0.24)***	(0.09;0.31)***			
Inv. Pareto-Gamma	3	15.94	24.26	-43,526	47	47^{+++}
		(0.09;0.26)***	(0.08;0.34)***			
Gamma	2	15.94	24.27	-43,533	48	48^{+++}
		(0.09;0.24)***	(0.08;0.30)***			

Exponential	1	32.58	56.49	-139,654	49	49+++
		(0.10;0.25)***	(0.08;0.32)***			
Inv. Pareto-Exponential	2	32.58	56.49	-139,654	50	50+++
		(0.10;0.25)***	(0.09;0.33)***			
Pareto	2	37.07	55.02	-214,535	51	51+++
		(0.09;0.25)***	(0.08;0.32)***			

Notes: All distributions fitted using Maximum Likelihood.
Values between parentheses report the 5th and 95th quantile of the parametric bootstrapped test statistic with 999 replications. ***, **, * indicate significance of this test at 1%, 5% and 10% respectively.
+++, ++, + indicates the difference between this distribution's BIC and the first-ranked distribution in terms of BIC (ΔBIC) providing strong evidence in favour of the first-ranked distribution ($\Delta BIC > 10$), moderate evidence ($6 < \Delta BIC \leq 10$) and weak evidence ($2 < \Delta BIC \leq 6$) respectively.
$_a$ Values multiplied by 100 for expositional purpose, $_b$ Values divided by 1,000 for expositional purpose.

Table 10: Distribution fits to the U.S. Census 2000 city size distribution.

		Goodness of fit		Information Criteria		
Distribution	Parms.	T_a^0	S_b^0	Loglike	R_{AIC}	R_{BIC}
5-comp. Burr	19	0.22	0.02	-6,004	3	9^{+++}
		(0.33;0.87)	(0.02;0.09)			
3-comp. Burr	11	0.25	0.02	-6,006	1	2^{++}
		(0.33;0.85)	(0.02;0.10)			
4-comp. Burr	15	0.32	0.02	-6,008	2	5^{+++}
		(0.33;0.83)	(0.02;0.09)			
5-comp. Lognormal	14	0.58	0.05	-6,016	6	6^{+++}
		(0.32;0.87)	(0.02;0.10)			
4-comp. Lognormal	11	0.60	0.05	-6,016	5	4^{+++}
		(0.32;0.82)	(0.02;0.09)			
3-comp. Lognormal	8	0.62	0.05	-6,017	4	1
		(0.32;0.86)	(0.02;0.09)			
5-comp. Gamma	14	0.29	0.03	-6,033	7	10^{+++}
		(0.32;0.84)	(0.02;0.09)			
5-comp. Weibull	14	0.38	0.04	-6,037	9	12^{+++}
		(0.33;0.88)	(0.03;0.10)			
2-comp. Lognormal	5	0.71	0.05	-6,044	8	3^{+++}
		(0.33;0.82)	(0.02;0.09)			
2-comp. Burr	7	0.87	0.09	-6,056	10	7^{+++}
		(0.32;0.85)**	(0.02;0.09)*			
Right-Pareto Lognormal	3	1.33	0.17	-6,085	11	8^{+++}
		(0.31;0.84)***	(0.02;0.09)***			
Double-Pareto Lognormal	4	1.39	0.17	-6,085	12	11^{+++}
		(0.32;0.85)***	(0.02;0.09)***			
Inv. Pareto-Lognormal-Pareto	4	1.76	0.25	-6,135	14	14^{+++}
		(0.33;0.87)***	(0.02;0.09)***			
Lognormal-Pareto	3	1.75	0.25	-6,135	13	13^{+++}
		(0.33;0.86)***	(0.02;0.10)***			
4-comp. Weibull	11	0.69	0.08	-6,144	16	18^{+++}
		(0.32;0.82)	(0.02;0.09)*			
Inv. Pareto-Lognormal	3	1.90	0.27	-6,152	17	16^{+++}
		(0.33;0.84)***	(0.02;0.09)***			
Lognormal	2	1.89	0.27	-6,152	15	15^{+++}
		(0.32;0.85)***	(0.02;0.09)***			
Left-Pareto Lognormal	3	3.12	0.42	-6,152	18	17^{+++}
		(0.98;1.93)***	(0.14;0.29)***			
4-comp. Gamma	11	0.99	0.11	-6,163	19	19^{+++}
		(0.33;0.84)**	(0.02;0.09)**			
5-comp. Fréchet	14	1.73	0.15	-6,172	21	21^{+++}
		(0.33;0.85)***	(0.02;0.09)***			
4-comp. Fréchet	11	1.57	0.14	-6,174	20	20^{+++}
		(0.33;0.85)***	(0.02;0.09)***			
5-comp. Exponential	9	1.94	0.11	-6,260	22	23^{+++}
		(0.32;0.86)***	(0.02;0.10)**			
3-comp. Fréchet	8	1.61	0.17	-6,261	23	22^{+++}
		(0.32;0.83)***	(0.02;0.09)***			

4-comp. Exponential	7	1.79	0.14	-6,298	24	24^{+++}
		(0.32;0.84)***	(0.02;0.09)***			
Inv. Pareto-Burr	4	2.17	0.30	-6,370	26	26^{+++}
		(0.33;0.85)***	(0.03;0.09)***			
Inv. Pareto-Burr-Pareto	5	2.17	0.30	-6,370	28	28^{+++}
		(0.32;0.85)***	(0.02;0.10)***			
Burr-Pareto	4	2.17	0.30	-6,370	27	27^{+++}
		(0.32;0.85)***	(0.02;0.09)***			
Burr	3	2.17	0.30	-6,370	25	25^{+++}
		(0.32;0.84)***	(0.02;0.09)***			
3-comp. Weibull	8	1.71	0.18	-6,393	29	29^{+++}
		(0.32;0.84)***	(0.02;0.10)***			
Inv. Pareto-Fréchet-Pareto	4	3.05	0.40	-6,530	30	30^{+++}
		(0.33;0.83)***	(0.02;0.09)***			
3-comp. Gamma	8	2.37	0.25	-6,532	31	32^{+++}
		(0.33;0.85)***	(0.02;0.09)***			
2-comp. Fréchet	5	2.55	0.32	-6,538	32	31^{+++}
		(0.32;0.84)***	(0.02;0.09)***			
3-comp. Exponential	5	2.76	0.28	-6,633	33	33^{+++}
		(0.32;0.85)***	(0.02;0.09)***			
Inv. Pareto-Weibull-Pareto	4	3.60	0.48	-6,829	35	35^{+++}
		(0.32;0.86)***	(0.02;0.09)***			
Weibull-Pareto	3	3.60	0.48	-6,829	34	34^{+++}
		(0.32;0.88)***	(0.02;0.10)***			
Inv. Pareto-Gamma-Pareto	4	3.87	0.52	-6,848	37	39^{+++}
		(0.31;0.87)***	(0.02;0.09)***			
Gamma-Pareto	3	3.87	0.52	-6,848	36	37^{+++}
		(0.32;0.84)***	(0.02;0.10)***			
Inv. Pareto-Exponential-Pareto	3	3.96	0.54	-6,851	39	38^{+++}
		(0.32;0.82)***	(0.02;0.09)***			
Exponential-Pareto	2	3.96	0.54	-6,851	38	36^{+++}
		(0.32;0.85)***	(0.03;0.09)***			
2-comp. Weibull	5	2.96	0.32	-6,920	40	40^{+++}
		(0.32;0.87)***	(0.03;0.09)***			
Fréchet-Pareto	3	4.60	0.64	-7,404	42	42.5^{+++}
		(0.32;0.85)***	(0.02;0.09)***			
Inv. Pareto-Fréchet	3	4.60	0.64	-7,404	42	42.5^{+++}
		(0.33;0.85)***	(0.02;0.09)***			
Fréchet	2	4.60	0.64	-7,404	41	41^{+++}
		(0.32;0.84)***	(0.02;0.09)***			
2-comp. Gamma	5	4.23	0.54	-7,694	44	44^{+++}
		(0.32;0.86)***	(0.02;0.10)***			
2-comp. Exponential	3	7.16	0.84	-8,488	45	45^{+++}
		(0.33;0.84)***	(0.03;0.09)***			
Inv. Pareto-Weibull	3	8.31	1.13	-9,030	47	47^{+++}
		(0.32;0.85)***	(0.02;0.09)***			
Weibull	2	8.31	1.13	-9,030	46	46^{+++}
		(0.32;0.82)***	(0.02;0.09)***			
Inv. Pareto-Gamma	3	16.40	2.26	-13,169	48	49^{+++}
		(0.33;0.84)***	(0.02;0.09)***			
Gamma	2	16.42	2.26	-13,171	49	48^{+++}

		(0.32;0.87)***	(0.02;0.10)***			
Exponential	1	37.71	5.58	-25,359	50	50^{+++}
		(0.32;0.87)***	(0.02;0.09)***			
Inv. Pareto-Exponential	2	37.71	5.58	-25,359	51	51^{+++}
		(0.32;0.85)***	(0.02;0.09)***			
Pareto	2	41.69	5.06	-31,612	52	52^{+++}
		(0.33;0.83)***	(0.02;0.09)***			

Notes: All distributions fitted using Maximum Likelihood.

Values between parentheses report the 5th and 95th quantile of the parametric bootstrapped test statistic with 999 replications. ***, **, * indicate significance of this test at 1%, 5% and 10% respectively.

$^{+++}$, $^{++}$, $^{+}$ indicates the difference between this distribution's BIC and the first-ranked distribution in terms of BIC (ΔBIC) providing strong evidence in favour of the first-ranked distribution ($\Delta BIC > 10$), moderate evidence ($6 < \Delta BIC \leq 10$) and weak evidence ($2 < \Delta BIC \leq 6$) respectively.

$_a$ Values multiplied by 100 for expositional purpose, $_b$ Values divided by 1,000 for expositional purpose.

Table 11: Decomposition of procentual welfare gains from a reduction in variable trade costs $\tau^{ij} = 3 \rightarrow (\tau^{ij})' = 1$.

Distribution	Parms.	$ln\frac{U_i'}{U_i}$	$ln\frac{\tau_{ij}'}{\tau_{ij}}$	$ln\frac{M_i'}{M_i}$	$ln\frac{1-G(\omega_{ij}^*)'}{1-G(\omega_{ij}^*)}$	$ln\frac{\tilde{\omega}(\omega_{ij}^*)'}{\tilde{\omega}(\omega_{ij}^*)}$	$ln\frac{\lambda_{ij}'}{\lambda_{ij}}$
Pareto	2	-	1.10	-	-	-	-
		(-0.00;0.00)***	(1.10;1.10)	(-0.22;-0.22)***	(-0.00;0.00)***	(0.00;0.00)***	(-0.88;-0.88)***
Weibull	2	0.15	1.10	-0.16	0.12	1.35	-2.26
		(0.15;0.15)***	(1.10;1.10)	(-0.16;-0.16)***	(0.12;0.13)***	(1.30;1.41)***	(-2.32;-2.21)***
Inv. Pareto-Weibull	3	0.15	1.10	-0.16	0.12	1.35	-2.26
		(0.15;0.15)***	(1.10;1.10)	(-0.16;-0.16)***	(0.12;0.13)***	(1.30;1.41)***	(-2.32;-2.21)***
Left-Pareto Lognormal	3	0.16	1.10	-0.17	0.15	0.60	-1.51
		(0.16;0.17)***	(1.10;1.10)	(-0.17;-0.17)***	(0.15;0.15)***	(0.58;0.62)***	(-1.53;-1.49)***
Inv. Pareto-Lognormal	3	0.17	1.10	-0.17	0.15	0.58	-1.49
		(0.16;0.17)***	(1.10;1.10)	(-0.17;-0.17)***	(0.15;0.15)***	(0.56;0.60)***	(-1.51;-1.47)***
Lognormal	2	0.17	1.10	-0.17	0.15	0.53	-1.44
		(0.17;0.17)***	(1.10;1.10)	(-0.17;-0.17)***	(0.15;0.15)***	(0.51;0.55)***	(-1.46;-1.42)***
Right-Pareto Lognormal	3	0.18	1.10	-0.18	0.17	0.28	-1.19
		(0.18;0.19)**	(1.10;1.10)	(-0.19;-0.18)	(0.17;0.18)	(0.23;0.33)**	(-1.24;-1.13)**
2-comp. Weibull	5	0.18	1.10	-0.13	0.11	0.57	-1.46
		(0.18;0.18)***	(1.10;1.10)	(-0.14;-0.13)***	(0.11;0.11)***	(0.56;0.59)***	(-1.48;-1.45)***
3-comp. Weibull	8	0.19	1.10	-0.19	0.18	0.25	-1.16
		(0.18;0.19)***	(1.10;1.10)	(-0.19;-0.19)***	(0.18;0.19)***	(0.25;0.26)***	(-1.17;-1.15)***
4-comp. Weibull	11	0.19	1.10	-0.18	0.17	0.22	-1.12
		(0.19;0.19)***	(1.10;1.10)	(-0.18;-0.17)***	(0.16;0.17)***	(0.21;0.23)***	(-1.13;-1.11)***
5-comp. Weibull	14	0.19	1.10	-0.18	0.18	0.22	-1.12
		(0.19;0.19)**	(1.10;1.10)	(-0.19;-0.18)	(0.17;0.18)	(0.21;0.23)***	(-1.14;-1.11)***
Empirical	0	0.19	1.10	-0.18	0.18	0.20	-1.10
4-comp. Lognormal	11	0.19	1.10	-0.18	0.18	0.20	-1.10
		(0.19;0.19)	(1.10;1.10)	(-0.19;-0.18)	(0.17;0.18)	(0.18;0.22)	(-1.13;-1.08)
5-comp. Lognormal	14	0.19	1.10	-0.19	0.18	0.20	-1.10
		(0.19;0.19)	(1.10;1.10)	(-0.19;-0.18)	(0.17;0.19)	(0.17;0.22)	(-1.12;-1.07)
2-comp. Lognormal	5	0.19	1.10	-0.17	0.17	0.23	-1.13
		(0.19;0.19)	(1.10;1.10)	(-0.18;-0.17)***	(0.16;0.17)***	(0.22;0.25)***	(-1.15;-1.12)***
3-comp. Lognormal	8	0.19	1.10	-0.18	0.18	0.19	-1.09
		(0.19;0.19)	(1.10;1.10)	(-0.19;-0.18)	(0.17;0.18)	(0.16;0.22)	(-1.12;-1.06)
Lognormal-Pareto	3	0.22	1.10	-0.22	0.22	0.02	-0.90
		(0.20;0.21)***	(1.10;1.10)	(-0.22;-0.20)***	(0.20;0.22)***	(0.04;0.14)***	(-1.04;-0.93)***

Burr	3	-	1.10	-	-	-	-
		(0.20;0.21)***	(1.10;1.10)	(-0.21;-0.19)***	(0.19;0.21)***	(0.03;0.12)***	(-1.02;-0.92)***
2-comp. Burr	7	-	1.10	-	-	-	-
		(0.19;0.20)	(1.10;1.10)	(-0.20;-0.18)	(0.17;0.20)	(0.10;0.22)	(-1.12;-1.00)
3-comp. Burr	11	-	1.10	-	-	-	-
		(0.19;0.20)**	(1.10;1.10)	(-0.21;-0.18)	(0.18;0.21)	(0.08;0.20)	(-1.11;-0.97)
4-comp. Burr	15	-	1.10	-	-	-	-
		(0.19;0.21)**	(1.10;1.10)	(-0.21;-0.18)**	(0.18;0.21)**	(0.07;0.20)**	(-1.10;-0.96)**
5-comp. Burr	19	-	1.10	-	-	-	-
		(0.19;0.21)***	(1.10;1.10)	(-0.22;-0.19)***	(0.18;0.22)***	(0.05;0.19)***	(-1.09;-0.94)***
Burr-Pareto	4	-	1.10	-	-	-	-
		(0.20;0.21)***	(1.10;1.10)	(-0.21;-0.19)***	(0.19;0.21)***	(0.02;0.12)***	(-1.02;-0.91)***
Double-Pareto Lognormal	4	-	1.10	-	-	-	-
		(0.20;0.22)***	(1.10;1.10)	(-0.20;-0.19)***	(0.19;0.20)***	(0.02;0.09)***	(-0.98;-0.90)***
Fréchet	2	-	1.10	-	-	-	-
		(0.22;0.22)***	(1.10;1.10)	(-0.14;-0.08)***	(0.08;0.14)***	(0.00;0.01)***	(-0.89;-0.88)***
2-comp. Fréchet	5	-	1.10	-	-	-	-
		(0.22;0.22)***	(1.10;1.10)	(-0.15;-0.10)***	(0.10;0.15)***	(0.00;0.01)***	(-0.89;-0.88)***
3-comp. Fréchet	8	-	1.10	-	-	-	-
		(0.22;0.22)***	(1.10;1.10)	(-0.15;-0.11)**	(0.11;0.15)**	(0.00;0.01)***	(-0.89;-0.88)***
4-comp. Fréchet	11	-	1.10	-	-	-	-
		(0.22;0.22)***	(1.10;1.10)	(-0.15;-0.11)**	(0.11;0.15)**	(0.00;0.01)***	(-0.89;-0.88)***
5-comp. Fréchet	14	-	1.10	-	-	-	-
		(0.22;0.22)***	(1.10;1.10)	(-0.15;-0.10)***	(0.10;0.15)**	(0.00;0.01)***	(-0.89;-0.88)***
Fréchet-Pareto	3	-	1.10	-	-	-	-
		(0.22;0.22)***	(1.10;1.10)	(-0.14;-0.08)***	(0.08;0.14)***	(0.00;0.01)***	(-0.89;-0.88)***
Inv. Pareto-Burr	4	-	1.10	-	-	-	-
		(0.20;0.22)***	(1.10;1.10)	(-0.21;-0.19)***	(0.19;0.21)***	(0.02;0.11)***	(-1.00;-0.90)***
Inv. Pareto-Burr-Pareto	5	-	1.10	-	-	-	-
		(0.20;0.22)***	(1.10;1.10)	(-0.21;-0.19)***	(0.19;0.20)***	(0.02;0.11)***	(-1.00;-0.90)***
Inv. Pareto-Fréchet	3	-	1.10	-	-	-	-
		(0.22;0.22)***	(1.10;1.10)	(-0.14;-0.08)***	(0.08;0.14)**	(0.00;0.01)***	(-0.89;-0.88)***
Inv. Pareto-Fréchet-Pareto	4	-	1.10	-	-	-	-
		(0.21;0.22)***	(1.10;1.10)	(-0.19;-0.18)	(0.18;0.19)**	(0.01;0.07)***	(-0.96;-0.89)***
Inv. Pareto-Lognormal-Pareto	4	-	1.10	-	-	-	-
		(0.21;0.22)***	(1.10;1.10)	(-0.20;-0.18)	(0.18;0.20)***	(0.01;0.08)***	(-0.97;-0.89)***
Inv. Pareto-Weibull-Pareto	4	-	1.10	-	-	-	-

		(0.21;0.22)***	(1.10;1.10)	(-0.18;-0.16)**	(0.16;0.18)	(0.00;0.05)***	(-0.93;-0.88)***
Weibull-Pareto	3	-	1.10	-	-	-	-
		(0.21;0.22)***	(1.10;1.10)	(-0.18;-0.16)**	(0.16;0.18)	(0.00;0.05)***	(-0.93;-0.88)***

Notes: Values between parentheses report the 5th and 95th quantile of the parametric bootstrapped statistics with 999 replications. ***, **, * indicate the rejection of a signifcant overlap of the parametric bootstrapped statistic with the empirical statistic at 1%, 5% and 10% respectively.

Appendix B Motivation and identification of generative processes for mixture models

FMMs can be utilized in two ways. First, they can be used as a semi-parametric, flexible approximation of the overall distribution, which is the case in this paper. Second, they are model-based clustering methods when a certain distribution is imposed (Fop et al., 2018; Grün, 2018). While both applications rely on the idea that discrete subpopulations define the overall distribution, the semi-parametric approximation does not claim to identify these subpopulations. In this appendix, we conceptualize possible Data Generative Processes (DGPs) for FMMs based on theoretical and empirical work in the economics literature. We then elaborate on the identification difficulties/opportunities of the underlying mixture components in the context of productivity distributions.

B.1 Generative processes

Many economic models rely on the assumption that the firm size distribution originates from firm dynamics in productivity (see for instance Hopenhayn (1992); Luttmer (2007); Rossi-Hansberg and Wright (2007); Costantini and Melitz (2008); Arkolakis (2016)). In this section, we will use a simplified version of such productivity dynamics for explanatory purposes. Consider productivity dynamics specified as a first-order autoregressive process:

$$ln\omega_{bt} = c + \rho ln\omega_{bt-1} + \eta_{bt}, \tag{1}$$

where η_{bt} is a white noise process with zero mean and constant variance σ^2.

There exists empirical evidence arguing that productivity dynamics, and therefore the resulting productivity distributions, are endogenous to exporting (De Loecker, 2013), importing (Kasahara and Rodrigue, 2008), innovation (Aw et al., 2011), management practices (Bloom and Reenen, 2011; Caliendo et al., 2020), ... Overall, there are "many sources of heterogeneity that support the idea of discrete subpopulations likely to differ in important characteristics ..." (Perline (2005),p.80). In the case of exporting, the endogenous evolution of productivity results in an exporting productivity premium. This can empirically be observed from the standard textbook comparison of cross-sectional productivity densities between exporting and non-exporting firms (see Figure 5). Building on equation 1, a simplified version of the empirical specification to identify such exporting productivity premium, and replicate Figure 5, is essentially a specifically parametrized FMM:

$$\begin{aligned} ln\omega_{bt} &= \alpha_0 + \beta_0 EXP_b + \alpha_1 ln\omega_{bt-1} + \beta_1 EXP_b \times ln\omega_{bt-1} + \eta_{bt} \\ &= EXP_b \left[\beta_0 + \beta_1 ln\omega_{bt-1}\right] + (1 - EXP_b)\left[\alpha_0 + \alpha_1 ln\omega_{bt-1}\right] + \eta_{bt}, \end{aligned} \tag{2}$$

with EXP_b a dummy variable that takes the value 1 when the firm b is an exporter and 0 otherwise.

Whereas the components are identified by means of an exporter dummy variable in this example,

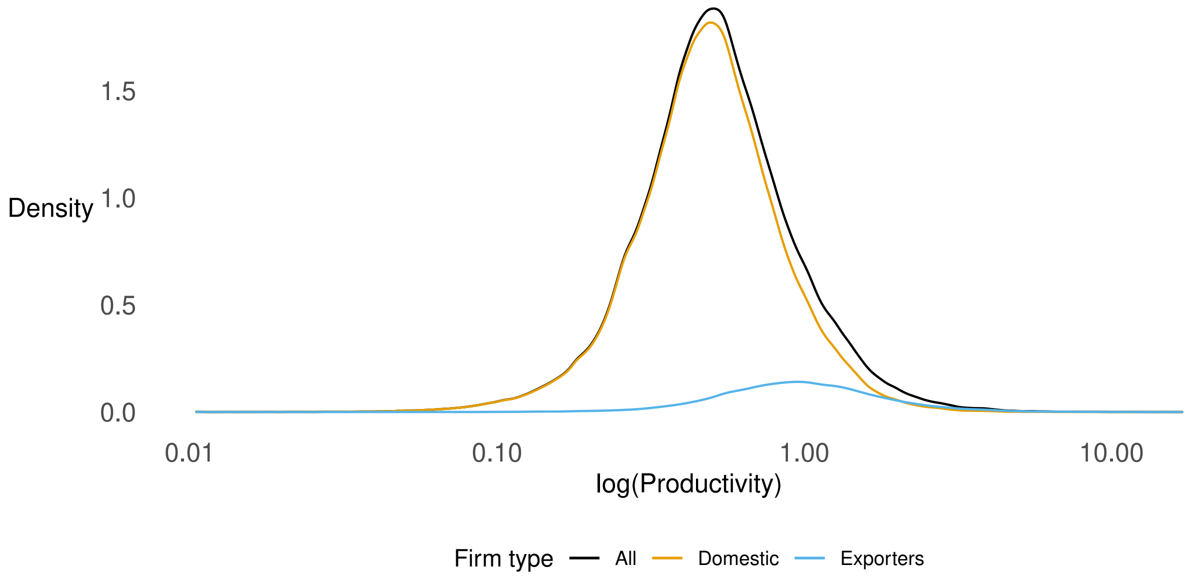

Figure 5: Productivity density of Portuguese firm productivity in 2006 for all, exporting- and non-exporting firms.
Notes: Productivity is measured as domestic sales (relative to the mean) to the power of $1/(\sigma - 1)$ with σ, the elasticity of substitution between varieties, set to four.

FMMs are a semi-parametric specification that remain agnostic about the (possibly multiple) determinants of the unobserved components and allow the data to determine these components:[1]

$$ln\omega_{bt} = \sum_{i=1}^{I} \mathbb{I}_b^i \left[\beta_0^i + \beta_1^i ln\omega_{bt-1}\right] + \eta_{bt}. \tag{3}$$

B.2 Identification

As stated before, the use of FMM's can focus on the semi-parametric, flexible approximation of the overall distribution or on model-based clustering. This paper purely focuses on the semi-parametric approximation, with good reason. First, we take no à-priori stance on distributional specification.[2] Second, even if one is willing to assume distributional specification such as the Lognormal, the underlying components remain unidentifiable in the current setting. As the overall distribution is

[1]Note that, for simplicity, we specify the variance to be constant between components. FMMs in the main analysis allow for the variance to differ between components.

[2]The empirical evidence in this paper seems to favor a Lognormal specification. This can be motivated from two perspectives. From the perspective of overall fit, a mixture of (log-) normal distributions with sufficient components is assumed to be able to approach all distributions (McLachlan and Peel, 2000). From a generative perspective for individual components, the Lognormal distribution is the realization of applying the Central Limit Theorem (CLT) in the log domain: firm heterogeneity will approximately be Lognormal if it is the multiplicative product of many independent random variables. Whereas firm heterogeneity reduces to firm-level productivity in the Melitz (2003)-model, it has been argued to be multi-dimensional when taking into consideration for instance the product dimension (Bernard et al., 2009) or uncertainty in demand and/or supply (see (De Loecker, 2011; Bas et al., 2017; Sager and Timoshenko, 2019; Gandhi et al., 0) . . .)

unimodal (see Figure 5), there is a large overlap between the underlying individual densities. These individual densities will therefore be poorly identified. Indeed, Figure 6 displays the posterior probability distribution for each component of the fitted 4-component Lognormal mixture from the main text. Whereas well-identified components have a large weight near zero and 1, average probabilities lie close to 0.25 in our case, and are therefore not well-identified. While the overall distribution can be closely approximated, the large overlap of individual densities results in a large uncertainty on which observation can be assigned to which density. Neither the parameter estimates used to characterize the clusters nor the partitions derived can therefore be uniquely determined, rendering the interpretation of results in terms of clustering futile (Follmann and Lambert, 1991; Hennig, 2000; Grün, 2018; Grün and Leisch, 2008).

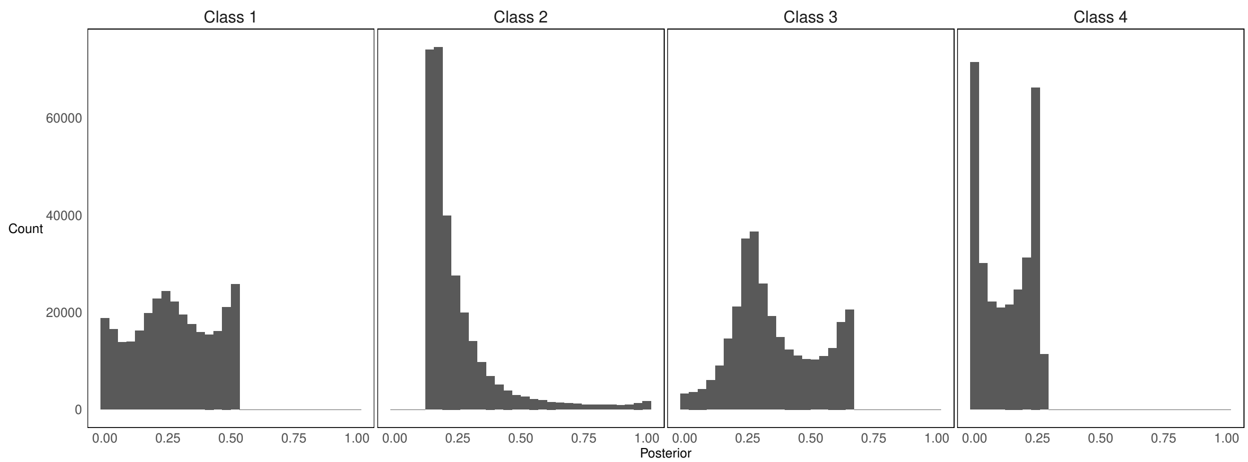

Figure 6: Posterior probability distribution for each component of the 4-components Lognormal mixture.

Future research might resolve the identifiability problem relying on panel rather than cross-sectional data. The problem as specified now is a problem in levels (the cross-section), where it appears there is insufficient distance between different components for them to be identified. From empirical evidence, however, it can be deduced that the different components likely originate from differences in growth rates (Kasahara and Rodrigue, 2008; Aw et al., 2011; De Loecker, 2013; Caliendo et al., 2020). Tracking the growth rates of individual firms over time might allow for the variation needed to identify the components of the overall distribution.

This observation can be easily illustrated using simulated data. Building on the example of the previous paragraph, imagine $ln\omega_{bt}$ follows an AR(1)-process with an exporting productivity premium of 20%:

$$ln\omega_{bt} = 1 + 1.2 \times EXP_b + 0.7 \times ln\omega_{bt-1} + \eta_{bt},$$

with $\eta_{bt} \sim \mathcal{N}(0, 0.3)$. We simulate this evolution for 200 exporters ($EXP_b = 1$) and 800 purely domestic businesses over 10 years.[3] The firm densities of the simulated data will look similar to

[3]When simulating, we allow for a run-in period of 90 years.

Figure 5, with two densities largely overlapping but the exporter productivity density located on the right of domestic firms density.

If we fit, as in our main analysis, a FMM on the cross-sectional data of a selected (the first) year, we obtain a familiar posterior probability distribution (see Figure 7). Individual clusters are not well-identified. Exploiting the panel dimension of the data,[4] however, results in well-identified components. As can be observed in Figure 8, the posterior probabilities predominantly take the values zero or one. Once components are well-identified, one can try to determine which mechanisms motivate the existence of FMMs from a generative perspective.

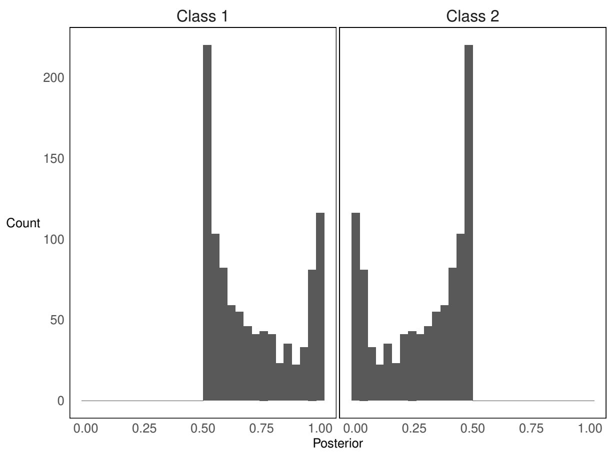

Figure 7: *Cross-sectional* posterior probability distribution for each component of the simulated 2-components normal mixture.

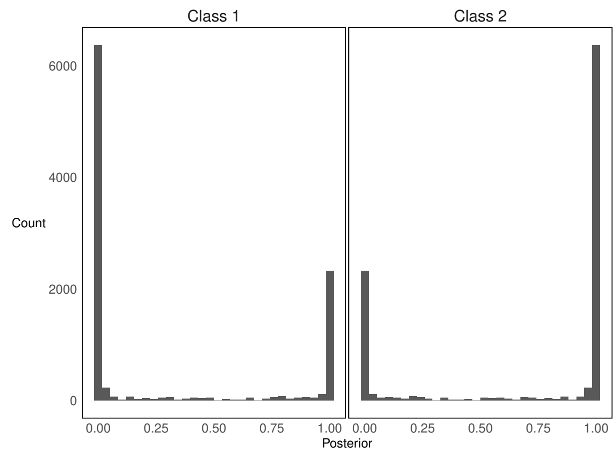

Figure 8: *Panel* posterior probability distribution for each component of the simulated 2-components normal mixture.

[4]Specifically, the EM estimation procedure is adapted to take into account panel data. The component probabilities in our main analysis are specified over the complete data (eq. 9):

$$\pi_{bi}^{(s)} = E\left[z_{bi}|\omega_b, \mathbf{\Psi}^{(s-1)}\right] = \frac{\pi_i^{(s-1)} m_i(\omega_b|\boldsymbol{\theta}_i^{(s-1)})}{\sum_{i=1}^{I} \pi_i^{(s-1)} m_i(\omega_b|\boldsymbol{\theta}_i^{(s-1)})}.$$

When working with panel data, we adapt this specification to take into account the time dimension:

$$\pi_{bi}^{(s)} = E\left[z_{bi}|\omega_{bt}, \mathbf{\Psi}^{(s-1)}\right] = \frac{\pi_i^{(s-1)} \prod_{t=1}^{T} m_{it}(\omega_{bt}|\boldsymbol{\theta}_i^{(s-1)})}{\sum_{i=1}^{I} \pi_i^{(s-1)} \prod_{t=1}^{T} m_{it}(\omega_{bt}|\boldsymbol{\theta}_i^{(s-1)})}.$$

Note that the probabilities are specified to be constant over time, meaning that we do not allow for regime switching in this exercise.

Appendix C Heterogeneous firms model

This appendix provides a detailed description of the heterogeneous firms models relied upon in the paper. We follow (Dewitte, 2020) in presenting a firm heterogeneous open economy model of Melitz (2003) with a finite number of firms. The model features Constant Elasticity of Substitution (CES)-demand and monopolistic competition between a finite number of firms who ignore their aggregate impact (Dixit and Stiglitz, 1977; Krugman, 1980; di Giovanni and Levchenko, 2012), while remaining agnostic on the parametric specification of firm-level heterogeneity. For the number of firms going to infinity, the model is equivalent to the Melitz (2003)- model.

C.1 Setup

Demand Consumer preferences in country $j \in J$ are defined over a finite number of horizontally differentiated varieties ($\varpi \in \Omega^i$) originating from country $i \in I$ and are assumed to take the Constant Elasticity of Substitution (CES) utility (U) form

$$U^j = \left(\sum_{i=1}^{I} \sum_{\varpi \in \Omega^i} q^{ij} \left(\varpi\right)^{\frac{\sigma-1}{\sigma}} d\varpi \right)^{\frac{\sigma}{\sigma-1}}, \tag{4}$$

with σ the elasticity of substitution between varieties. Utility maximization defines the optimal consumption and expenditure decisions over the individual varieties

$$\frac{q^{ij}(\varpi)}{Q^j} = \left[\frac{p^{ij}(\varpi)}{P^j} \right]^{-\sigma}, \tag{5}$$

where the set of varieties consumed is considered as an aggregate good $Q \equiv U$ and P is the CES aggregate price index.

Supply There is a finite number of businesses ($b \in B$) which choose to supply a distinct horizontally-differentiated variety. They are heterogeneous in terms of their productivity $\omega_b \in [0, \infty]$ drawn from the unconditional Cumulative Distribution Function (CDF) $G(\omega_b)$ after paying a fixed cost f^{ie} in terms of production factor L^i to enter the market.[5] There is zero probability of firm death.[6] Supply of the production factor to the individual firm is perfectly elastic, so that firms are effectively price (W^i) takers on the input markets. Once active, firms from country i have to pay a fixed cost f^{ij} to produce goods destined for country j. The cost function of the firm involves a fixed production cost, iceberg trade costs $\tau^{ij} > 1$ and a constant marginal costs that depends on its productivity: $f^{ij} + \left(\frac{\tau^{ij} q^{ij}}{\omega}\right) W^i$. Profit maximization of the firm, then:

[5]As ω_b is the sole heterogeneity component identifying individual firms, we drop the subscript b in further derivations.

[6]The static specification in which there is zero probability of firm death follows most of the international trade literature.

$$
\begin{aligned}
\max_{q^{ij}} \pi^{ij} &= \max_{q^{ij}} \left[p^{ij} q^{ij} - \left(f^{ij} - \frac{\tau^{ij} q^{ij}}{\omega} \right) W^i \right] \\
&= \max_{q^{ij}} \left[\left(q^{ij}\right)^{\frac{\sigma-1}{\sigma}} \left(Q^j\right)^{\frac{1}{\sigma}} P^j - \left(f^{ij} - \frac{\tau^{ij} q^{ij}}{\omega} \right) W^i \right], \qquad (6)
\end{aligned}
$$

results in an optimal quantity produced:

$$
\begin{gathered}
\frac{\partial \pi^{ij}}{\partial q^{ij}} = 0 \\
\Leftrightarrow \\
\frac{\sigma - 1}{\sigma} \left(q^{ij}\right)^{-\frac{1}{\sigma}} \left(Q^j\right)^{\frac{1}{\sigma}} P^j = \frac{\tau^{ij} W^i}{\omega} \\
\Leftrightarrow \\
q^{ij} = \left(\frac{\sigma}{\sigma - 1} \frac{\tau^{ij} W^i}{\omega} \right)^{-\sigma} Q^j \left(P^j\right)^{\sigma}. \qquad (7)
\end{gathered}
$$

and an equilibrium price as a constant markup over marginal costs $p^{ij} = \frac{\sigma}{\sigma-1} \frac{\tau^{ij} W^i}{\omega}$:

$$
\begin{gathered}
\left(\frac{q^{ij}}{(Q^j)} \right)^{\frac{-1}{\sigma}} P^j = p^{ij} \\
p^{ij} = \frac{\sigma}{\sigma - 1} \frac{\tau^{ij} W^i}{\omega}. \qquad (8)
\end{gathered}
$$

The realized revenue expression for firms from country i selling in destination j at time t can then be expressed as:

$$
\begin{aligned}
x^{ij} = p^{ij} q^{ij} &= \left(q^{ij}\right)^{\frac{\sigma-1}{\sigma}} \left(Q^j\right)^{\frac{1}{\sigma}} P^j \\
&= \left(\frac{\sigma}{\sigma - 1} \frac{\tau^{ij} W^i}{\omega} \right)^{1-\sigma} Q^j \left(P^j\right)^{\sigma} \qquad (9)
\end{aligned}
$$

C.2 Operating decisions

In line with (Dixit and Stiglitz, 1977; Krugman, 1980; di Giovanni and Levchenko, 2012), we assume that the marginal firm ignores the impact its own production level on the aggregate economy. The necessary productivity levels for serving each market are then determined by the zero cutoff profit conditions.

$$
\begin{aligned}
\pi^{ij} = 0 &= p^{ij}q^{ij} - \left(f^{ij} - \frac{\tau^{ij}q^{ij}}{\omega^{ij*}}\right)W^{i}, \\
&= \left(\frac{\sigma}{\sigma-1}\frac{\tau^{ij}W^{i}}{\omega^{ij*}}\right)^{1-\sigma} Q^{j}\left(P^{j}\right)^{\sigma} - f^{ij}W^{i} - \left(\frac{\sigma}{\sigma-1}\frac{\tau^{ij}W^{i}}{\omega^{ij*}}\right)^{-\sigma} Q^{j}\left(P^{j}\right)^{\sigma}\frac{\tau^{ij}}{\omega^{ij*}}W^{i}, \\
&= \left(\frac{\sigma}{\sigma-1}\frac{\tau^{ij}W^{i}}{\omega^{ij*}}\right)^{1-\sigma} Q^{j}\left(P^{j}\right)^{\sigma} - f^{ij}W^{i} - \left(\frac{\sigma}{\sigma-1}\right)^{-\sigma}\left(\frac{\tau^{ij}W^{i}}{\omega^{ij*}}\right)^{1-\sigma} Q^{j}\left(P^{j}\right)^{\sigma}, \\
&= \left(1 - \frac{\sigma-1}{\sigma}\right)\left(\frac{\sigma}{\sigma-1}\frac{\tau^{ij}W^{i}}{\omega^{ij*}}\right)^{1-\sigma} Q^{j}\left(P^{j}\right)^{\sigma} - f^{ij}W^{i}, \\
&\Leftrightarrow \\
\sigma f^{ij}W^{i} &= \left(\frac{\sigma}{\sigma-1}\frac{\tau^{ij}W^{i}}{\omega^{ij*}}\right)^{1-\sigma} Q^{j}\left(P^{j}\right)^{\sigma}. \qquad (10)
\end{aligned}
$$

Combining the zero cutoff profit conditions allows us to write the export cutoff as a function of a foreign domestic productivity cutoff, variable and fixed costs and the wages:

$$
\omega^{ij*} = \left(\frac{W^{i}}{W^{j}}\right)^{\frac{\sigma}{\sigma-1}}\left(\frac{f^{ij}}{f^{jj}}\right)^{\frac{1}{\sigma-1}}\left(\frac{\tau^{ij}}{\tau^{jj}}\right)\omega^{jj*}. \qquad (11)
$$

Similarly, we can combine the zero cutoff profit conditions from a single origin country, linking the domestic and export productivity cutoffs:

$$
\omega^{ij*} = \frac{\tau^{ij}}{\tau^{ii}}\left(\frac{P^{j}}{P^{i}}\right)^{\frac{\sigma}{1-\sigma}}\left(\frac{Q^{i}}{Q^{j}}\frac{f^{ij}}{f^{ii}}\right)^{\frac{1}{\sigma-1}}\omega^{ii*}. \qquad (12)
$$

In this paper, we focus on parameter values such that there is, in line with empirical evidence, selection into exporting ($\omega^{ij*} > \omega^{ii*}$). This implies

- A large fixed cost of exporting relative to the fixed cost of production. The revenue required to cover the fixed export cost is then large relative to the revenue required to cover the fixed production cost, implying that only firms of high productivity find it profitable to serve both markets.
- A high home price index relative to the foreign price index, and a large home market relative to the foreign market. Only high productivity firms receive enough revenue in the relatively small and competitive foreign market to cover the fixed cost of exporting.
- Variable trade costs increase the exporting productivity cutoff relative to the zero-profit productivity cutoff by increasing prices and reducing revenue in the export market.

The equilibrium value of these cutoffs are uniquely determined by the free entry condition, requiring the probability of successful entry times the expected future value of entry conditional upon successful entry to equal the sunk entry cost:

$$\sum_{j=1}^{J} \mathbb{E}\left[\pi^{ij} | \omega > \omega^{ij*}\right] = f^{ie} W^i$$

$$\sum_{j=1}^{J} \frac{1}{B} \sum_{b=1}^{B} \mathbb{I}\left(\omega > \omega^{ij*}\right) \pi^{ij} = f^{ie} W^i$$

$$\sum_{j=1}^{J} \frac{1}{B} \sum_{b=1}^{B} \mathbb{I}\left(\omega > \omega^{ij*}\right) \left[\frac{1}{\sigma}\left(\frac{\sigma}{\sigma-1}\frac{\tau^{ij} W^i}{\omega}\right)^{1-\sigma} Q^j \left(P^j\right)^\sigma - f^{ij} W^i\right] = f^{ie} W^i$$

$$\sum_{j=1}^{J} f^{ij} W^i \frac{1}{B} \sum_{b=1}^{B} \mathbb{I}\left(\omega > \omega^{ij*}\right) \left[\left(\frac{\omega}{\omega^{ij*}}\right)^{\sigma-1} - 1\right] = f^{ie} W^i$$

$$\sum_{j=1}^{J} f^{ij} \left[\left(\omega^{ij*}\right)^{1-\sigma} \frac{1}{B} \sum_{b=1}^{B} \mathbb{I}\left(\omega > \omega^{ij*}\right) \omega^{\sigma-1} - \frac{1}{B} \sum_{b=1}^{B} \mathbb{I}\left(\omega > \omega^{ij*}\right) \omega^0\right] = f^{ie}$$

$$\sum_{j=1}^{J} f^{ij} \left[\left(\omega^{ij*}\right)^{1-\sigma} m^{\sigma-1}_{\omega^{ij*}} - m^0_{\omega^{ij*}}\right] = f^{ie}, \tag{13}$$

where we denote by m^r_y the y-bounded, r-th sample moment of the productivity distribution. For the number of firms going to infinity, the law of large numbers kicks in such that we replace these sample moments with their continuous equivalent $\left(\mu^r(y) = \int_y^\infty \omega^r g(\omega) d\omega\right)$, providing us with the well-known continuous free-entry equation as specified by (Melitz, 2003).

Using the relation between productivity cutoffs (eq. 11), the free entry condition (eq. 13) determines a unique equilibrium values of these cutoffs.[7] Thus, a parametrization of the Melitz (2003)-model in relation to firm heterogeneity relies solely on the bounded (by the respective productivity cutoffs) 0th and $(\sigma-1)$th moments of the productivity distribution (Nigai, 2017; Dewitte, 2020).

C.3 Aggregation

Summing equation 9 across all active firms, we obtain an expression for aggregate trade between country i and j:

$$X^{ij} = \left(\frac{\sigma}{\sigma-1}\tau^{ij} W^i\right)^{1-\sigma} Q^j \left(P^j\right)^\sigma M^{ie} m^{\sigma-1}_{\omega^{ij*}} \tag{14}$$

The number of successful entrants $\left[1 - G(\omega^{ii*})\right] M^{ie}$ is specified as the ratio of aggregate over average revenue:

[7] Sufficient conditions for this equilibrium to exist are that the term in brackets of equation (13) is (i) finite and (ii) a decreasing function of the cutoffs (Melitz, 2003, p.1704). The second condition corresponds to $\frac{g(x)x}{1-G(x)}$ increasing to infinity on $(0, \infty)$.

$$M^i = \left[1 - G(\omega^{ii*})\right] M^{ie} = \frac{X^i}{\mathbb{E}\left[x^i\right]}. \tag{15}$$

We can rewrite this number of firms, using the free entry condition, goods and labor market clearing $\left(X^i = W^i L^i\right)$, as a function of exogenous variables:

$$\begin{aligned} M^i &= \frac{W^i L^i}{\sigma\left(\frac{f^{ie}}{1-G(\omega^{ii*})} + \sum_{j=1}^{J} \frac{1-G(\omega^{ij*})}{1-G(\omega^{ii*})} f^{ij}\right) W^i} \\ &= \frac{L^i}{\sigma\left(\frac{f^{ie}}{1-G(\omega^{ii*})} + \sum_{j=1}^{J} \frac{1-G(\omega^{ij*})}{1-G(\omega^{ii*})} f^{ij}\right)}. \end{aligned} \tag{16}$$

Assuming a two-country symmetric economy and setting the wage of the composite factor as the numeraire, welfare can be calculated as the inverse of the price index

$$\mathbb{W}^i = (P^i)^{-1}. \tag{17}$$

The price index can be deduced from equation 14:

$$P^j = \left[\left(\frac{\sigma}{\sigma - 1}\tau^{ij} W^i\right)^{1-\sigma} \frac{1}{\lambda_{ij}} \frac{M^i}{1 - G(\omega^{ii*})} m_{\omega^{ij*}}^{\sigma-1}\right]^{\frac{1}{1-\sigma}}, \tag{18}$$

where we denote the bilateral trade share by $\lambda^{ij} = \frac{X^{ij}}{X^j}$.

The percentage changes in welfare from a change in variable trade costs ($\boldsymbol{\tau} \rightarrow \boldsymbol{\tau'}$) can then written as:

$$\begin{aligned} 100 \times ln\frac{(\mathbb{W}^i)'}{\mathbb{W}^i} &= 100 \times -ln\frac{(P^i)'}{P^i} \\ &= 100 \times -ln\frac{(P^j)'}{P^j} \\ &= 100 \times -\left[ln\frac{(\tau^{ij})'}{(\tau^{ij})} - \frac{1}{\sigma - 1}\left(ln\frac{(M^i)'}{M^i} - ln\frac{1 - G(\omega^{ii*})'}{1 - G(\omega^{ii*})} + ln\frac{(m_{\omega^{ij*}}^{\sigma-1})'}{m_{\omega^{ij*}}^{\sigma-1}} - ln\frac{(\lambda^{ij})'}{\lambda^{ij}}\right)\right] \\ &= 100 \times -\left[ln\frac{(\tau^{ij})'}{(\tau^{ij})} - \frac{1}{\sigma - 1}\left(ln\frac{(M^i)'}{M^i} - ln\frac{(m_{\omega^{ij*}}^{0})'}{m_{\omega^{ij*}}^{0}} + ln\frac{(m_{\omega^{ij*}}^{\sigma-1})'}{m_{\omega^{ij*}}^{\sigma-1}} - ln\frac{(\lambda^{ij})'}{\lambda^{ij}}\right)\right]. \end{aligned} \tag{19}$$

C.4 Parametrization

In order to parametrize the previously described model, we need to parametrize two statistics related to the productivity distribution: the 0th and $(\sigma - 1)$the y-bounded moments of the *productivity* distribution (Nigai, 2017). As described in (Dewitte, 2020), this corresponds to the 0th and 1st y-bounded moments of the *sales* distribution if the parametric distribution is stable under power-law transformations.

Assuming a parametric distribution and under the assumption of an *infinite* number of firms, we can calculate the necessary analytical expressions using the distributional parameters from our empirical analysis to capture heterogeneity. This is the standard approach in the literature. Following (Nigai, 2017; Dewitte, 2020), we can also capture heterogeneity directly from the empirical, *finite*, data. To allow comparison between GFT obtained assuming a parametric distribution and GFT obtained from the finite data, we perform a parametric bootstrap. This parametric bootstrap generates a range of finite sample estimates under the hypothesis that the observed data is generated by a certain parametric distribution (Dewitte, 2020).

C.4.1 Continuum of firms

When there are an infinite number of firms, the parametrization of the heterogeneity distribution consists of calculating the y-bounded 0th and 1st population moments of the sales distribution:

$$\mu_y^r = \int_y^\infty x^r g(x) dx. \tag{20}$$

The analytical expressions of these parametric implied population moments are gathered in Table 4 and 5 for all distributions considered. As bounded moments are not generally available, the mathematical elaboration on how to obtain these expressions can be found in the section D.

C.4.2 Finite number of firms

Under the assumption of a finite number of firms in the economy, the parametrization of the model consists of calculating the y-bounded 0th and 1st moment of the sales distribution:

$$m_y^r = \frac{1}{B} \sum_{b=1}^{B} \mathbb{I}\left(x > y\right) x^r. \tag{21}$$

These moments can easily be retrieved if the data is available. To allow comparison between GFT obtained assuming a parametric distribution and GFT obtained from the finite data, we perform a parametric bootstrap. This parametric bootstrap generates a range of finite sample estimates under the hypothesis that the observed data is generated by a certain parametric distribution:

1. Assume B i.i.d. random variables with distribution $G(\cdot|\boldsymbol{\theta})$, with empirical finite sample moments m_y^r for $r = 0, 1$, as specified in equation 21 and corresponding GFT_B;

2. Estimate the parameters $\boldsymbol{\theta}$ of the distribution using MLE, calculate the parametric plug-in population moments as specified in equation 20, $\hat{\mu}^r(y|\hat{\boldsymbol{\theta}})$ for $r = 0, 1$, and corresponding $G\hat{F}T(\hat{\boldsymbol{\theta}})$;

3. $H_0 : GFT = G\hat{F}T(\hat{\boldsymbol{\theta}})$;

4. Draw N bootstrap samples of size B from $G(\cdot|\hat{\boldsymbol{\theta}})$;

5. For each sample of the parametric distribution, calculate the bootstrapped sample moments $(m_y^r)^*$ and calculate the corresponding GFT_B^*.[8]

6. The p-value for the left-, and right-tailed test is then respectively specified as:

$$\hat{p}_l = \frac{1}{N+1}\left[\sum_{n=1}^{N} \mathbb{I}\left(GFT_B^* \geq GFT_B\right) + 1\right]; \qquad \hat{p}_r = \frac{1}{N+1}\left[\sum_{n=1}^{N} \mathbb{I}\left(GFT_B^* \leq GFT_B\right) + 1\right]. \tag{22}$$

The bootstrap exercise should therefore be interpreted as ‘the likelihood of observing GFT as small or as large as GFT_B under the null hypothesis that the observed data originates from the parametric distribution $G(\cdot|\boldsymbol{\theta})$’, allowing us to evaluate whether the distributional assumption provides a good fit to calculate GFT within the proposed model.

When calculating the bounded sample moments, complications can arise related to the lower bound y. This lower bound is ex ante unknown, can take values not observed in the data and/or resides in an unrepresentative part of the finite dataset.[9] We address each issue below and argue that these complications have little influence on our results.

1. y can take values within the boundaries of the data but are not observed. We use the ‘approx-fun‘ interpolation function of the R base distribution to approximate the statistics for such lower bounds.[10] As the calculation of Gains From Trade (GFT) relies on domestic cutoffs residing in the dense part of the productivity distribution, the influence of interpolation is negligible.

2. y can take values below the lowest observed value in the data ($y < x_{min}$):

$$\mu_y^r = \underbrace{\sum \mathbb{I}\left(y < x < x_{min}\right) x^r}_{\text{unobserved}} + \underbrace{\frac{1}{B}\sum_{b=1}^{B} \mathbb{I}\left(x \geq x_{min}\right) x^r}_{observed}. \tag{23}$$

[8]Note that we do not re-fit the parametric distribution to the bootstrap sample. The vastness of the dataset at our availability in the empirical section results both in a large computational burden but also a very precise estimation of the distribution parameters. The influence of not refitting the parametric distribution to the bootstrap sample is therefore negligent.

[9]We thank Gonzague Vannoorenberghe for pointing this out.

[10]All code available on request.

The error arising from neglecting the unobserved part of the distribution is likely small as (i) the smallest observation x_{min} in our dataset is rather small, (ii) the density in the unobserved part is most likely very low and (iii) the relative weight of the observations in the unobserved part is small (see also Figure 1).

3. As the presented model is a stylized model, it is conceivable firms produce below the model's implied zero-profit productivity cutoff, for instance when there is a positive expectation of future profits (Impullitti et al., 2013). This can explain very low observed productivity values, but will result in an unrepresentative left tail of the distribution (the lower the actual zero-profit productivity cutoff, the more firms will have a positive expectation of future profits and the denser the left tail of the distribution will be). This issue affects both the nonparametric and parametric estimates, as the parametric distribution is fitted to the observed distribution. Also in this case, however, provided the low density in the left tail of the distribution and the low relative weight of the observations in the left tail, the influence of this issue is likely small.

Appendix D Analytical expressions of μ_y^r

D.1 Pareto

$$\begin{aligned}\mu_y^r &= \int_y^\infty x^r \frac{kx_{min}^k}{x^{k+1}} dx \\ &= kx_{min}^k \frac{-y^{r-k}}{r-k} \qquad \text{if } k > r\end{aligned} \tag{24}$$

D.2 Inverse Pareto

$$\begin{aligned}\mu_y^r &= \int_y^{x_{max}} x^r \frac{kx_{max}^{-k}}{x^{-k+1}} dx \\ &= kx_{max}^{-k} \frac{x_{max}^{r+k} - y^{r+k}}{r+k}\end{aligned} \tag{25}$$

D.3 Lognormal

$$\begin{aligned}\mu_y^r &= \int_y^\infty x^r \frac{1}{xVar\sqrt{2\pi}} e^{-(lnx-\mu)^2/2Var^2} dx \\ &= \int_y^\infty e^{rlnx} \frac{1}{xVar\sqrt{2\pi}} e^{-(lnx-\mu)^2/2Var^2} dx\end{aligned} \tag{26}$$

Note that

$$\begin{aligned}rlnx - (lnx-\mu)^2/2Var^2 &= \frac{2Var^2rlnx - (lnx)^2 - \mu^2 + 2\mu lnx}{2Var^2} \\ &= -\frac{(lnx)^2 - 2(Var^2r+\mu)lnx + \left((Var^2r+\mu)\right)^2 - (Var^2r+\mu)^2 + \mu^2}{2Var^2} \\ &= -\frac{\left[lnx - \left(Var^2r+\mu\right)\right]^2}{2Var^2} + \frac{(Var^2r+\mu)^2 - \mu^2}{2Var^2} \\ &= -\frac{\left[lnx - \left(Var^2r+\mu\right)\right]^2}{2Var^2} + \frac{r\left(rVar^2+2\mu\right)}{2}\end{aligned}$$

so that

$$\mu_y^r = e^{\frac{r\left(rVar^2+2\mu\right)}{2}} \int_y^{\infty} \frac{1}{xVar\sqrt{2\pi}} e^{-\frac{\left[lnx-\left(Var^2 r+\mu\right)\right]^2}{2Var^2}} dx$$

$$\text{let } z = \frac{lnx - \left(rVar^2+\mu\right)}{Var},\ dz = \frac{dx}{xVar}$$

$$= e^{\frac{r\left(rVar^2+2\mu\right)}{2}} \int_{\frac{lny-\left(rVar^2+\mu\right)}{Var}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dx$$

$$= e^{\frac{r\left(rVar^2+2\mu\right)}{2}} \left[1 - \Phi\left(\frac{lny - \left(rVar^2+\mu\right)}{Var}\right)\right] \tag{27}$$

D.4 Weibull[11]

$$\mu_y^r = \int_y^{\infty} x^r \frac{k}{s}\left(\frac{x}{s}\right)^{k-1} e^{-\left(\frac{x}{s}\right)^k} dx$$

$$\text{let } z = \left(\frac{x}{s}\right)^k, dz = \frac{k}{s}\left(\frac{x}{s}\right)^{k-1} dx$$

$$\text{s.t. } x = sz^{\frac{1}{k}}$$

$$= \int_{\left(\frac{y}{s}\right)^k}^{\infty} s^r z^{\frac{r}{k}} e^{-z} dz$$

$$= s^r \int_{\left(\frac{y}{s}\right)^k}^{\infty} z^{\left(\frac{r}{k}+1\right)-1} e^{-z} dz$$

$$= s^r \Gamma\left(\frac{r}{k}+1, \left(\frac{y}{s}\right)^k\right) \tag{28}$$

where $\Gamma(,)$ denotes the upper incomplete gamma function.

[11]The bounded moments of the exponential distribution are obtained setting k=1.

D.5 Fréchet

$$
\begin{aligned}
\mu_y^r &= \int_y^\infty x^r \frac{k}{s}\left(\frac{x}{s}\right)^{-1-k} e^{-\left(\frac{x}{s}\right)^{-k}} dx \\
&\qquad \text{let } z = \left(\frac{x}{s}\right)^{-k}, dz = \frac{-k}{s}\left(\frac{x}{s}\right)^{-k-1} dx \\
&\qquad \text{s.t. } x = s z^{-\frac{1}{k}} \\
&= -\int_{\left(\frac{y}{s}\right)^{-k}}^{0} s^r z^{-\frac{r}{k}} e^{-z} dz, \qquad \text{if } k > 0 \\
&= \int_0^{\left(\frac{y}{s}\right)^{-k}} s^r z^{-\frac{r}{k}} e^{-z} dz \\
&= s^r \int_0^{\left(\frac{y}{s}\right)^{-k}} z^{1-\left(\frac{r}{k}\right)-1} e^{-z} dz \\
&= s^r \left[1 - \Gamma\left(1 - \frac{r}{k}, \left(\frac{y}{s}\right)^{-k}\right)\right] \qquad \text{if } k > r \qquad (29)
\end{aligned}
$$

D.6 Burr

$$
\begin{aligned}
\mu_y^r &= \int_y^\infty x^r \frac{\frac{kc}{s}\left(\frac{x}{s}\right)^{c-1}}{\left(1+\left(\frac{x}{s}\right)^c\right)^{k+1}} dx \\
&\qquad \text{let } z = \left(\frac{x}{s}\right)^c, dz = \frac{c}{s}\left(\frac{x}{s}\right)^{c-1} dx \\
&\qquad \text{s.t. } x = s z^{\frac{1}{c}} \\
&= \int_{\left(\frac{y}{s}\right)^c}^\infty s^r z^{\frac{r}{c}} \frac{k}{(1+z)^{k+1}} dz, \qquad \text{if } c > 0 \\
&= s^r k \int_{\left(\frac{y}{s}\right)^c}^\infty z^{\frac{r}{c}} \frac{1}{(1+z)^{k+1}} dz \\
&= s^r k \int_{\left(\frac{y}{s}\right)^c}^\infty z^{\left(\frac{r}{c}+1\right)-1} \frac{1}{(1+z)^{k+1}} dz \\
&= s^r k \int_{\left(\frac{y}{s}\right)^c}^\infty z^{\left(\frac{r}{c}+1\right)-1} \frac{1}{(1+z)^{k+1}} dz \\
&= s^r k \left[\int_0^\infty z^{\left(\frac{r}{c}+1\right)-1} \frac{1}{(1+z)^{k+1}} dz - \int_0^{\left(\frac{y}{s}\right)^c} z^{\left(\frac{r}{c}+1\right)-1} \frac{1}{(1+z)^{k+1}} dz\right] \\
&\quad u = \frac{z}{1+z}, du = \frac{1}{(1+z)^2} \\
&\quad z = \frac{u}{1-u} \\
&= s^r k \left[\int_0^1 \left(\frac{u}{1-u}\right)^{\left(\frac{r}{c}+1\right)-1} \frac{1}{\left(1+\frac{u}{1-u}\right)^{k-1}} du - \int_0^{\frac{\left(\frac{y}{s}\right)^c}{1+\left(\frac{y}{s}\right)^c}} \left(\frac{u}{1-u}\right)^{\left(\frac{r}{c}+1\right)-1} \frac{1}{\left(1+\frac{u}{1-u}\right)^{k-1}} du\right] \\
&= s^r k \Bigg[\int_0^1 u^{\left(\frac{r}{c}+1\right)-1} (1-u)^{k-1-\left(\frac{r}{c}+1\right)+1} du \\
&\qquad\qquad - \int_0^{\frac{\left(\frac{y}{s}\right)^c}{1+\left(\frac{y}{s}\right)^c}} \int_0^{\left(\frac{y}{s}\right)^c} u^{\left(\frac{r}{c}+1\right)-1} (1-u)^{k-1-\left(\frac{r}{c}+1\right)+1} du\Bigg] \\
&= s^r k \left[\int_0^1 u^{\left(\frac{r}{c}+1\right)-1} (1-u)^{k-\left(\frac{r}{c}+1\right)} du - \int_0^{\frac{\left(\frac{y}{s}\right)^c}{1+\left(\frac{y}{s}\right)^c}} u^{\left(\frac{r}{c}+1\right)-1} (1-u)^{k-\left(\frac{r}{c}+1\right)} duz\right] \\
&= s^r k \left[\boldsymbol{B}\left(\frac{r}{c}+1, k-\frac{r}{c}\right) - \boldsymbol{B}\left(\frac{\left(\frac{y}{s}\right)^c}{1+\left(\frac{y}{s}\right)^c}; \frac{r}{c}+1, k-\frac{r}{c}\right)\right] \qquad \text{if } c > r, kc > r \qquad (30)
\end{aligned}
$$

where $\boldsymbol{B}(a,b)$ stands for the beta function, while $\boldsymbol{B}(x,a,b)$ stands for the lower incomplete beta function with upper bound x.

D.7 Generalized Gamma[12]

$$
\begin{aligned}
\mu_y^r &= \int_y^\infty x^r \frac{c}{s^k\Gamma(\frac{k}{c})} x^{k-1} e^{-(\frac{x}{s})^c} dx \\
&\qquad \text{let } z = \left(\frac{x}{s}\right)^c, dz = \frac{c}{s}\left(\frac{x}{s}\right)^{c-1} dx \\
&\qquad \text{s.t. } x = s z^{\frac{1}{c}} \\
&= \int_{(\frac{y}{s})^c}^\infty s^r \frac{z^{\frac{r}{c}}}{\Gamma(\frac{k}{c})} \left(\frac{s z^{\frac{1}{c}}}{s}\right)^{(k-1)-(c-1)} e^{-z} dz, \qquad \text{if } c > 0 \\
&= \frac{s^r}{\Gamma(\frac{k}{c})} \int_{(\frac{y}{s})^c}^\infty z^{\frac{r+k}{c}-1} e^{-z} dz \\
&= \frac{s^r}{\Gamma(\frac{k}{c})} \Gamma\left(\frac{r+k}{c}, \left(\frac{y}{s}\right)^c\right)
\end{aligned} \tag{31}
$$

D.8 Finite Mixture Model

The statistics for a Finite Mixture Model can easily be obtained from the calculated statistics for the underlying individual distributions on which the mixture consists. For a mixture of the form:

$$
g(x|\boldsymbol{\Psi}) = \sum_{i=1}^{I} \pi_i m_i(x|\boldsymbol{\theta}_i), \qquad \pi_i \geq 0, \quad \sum_{i=1}^{I} \pi_i = 1, \tag{32}
$$

we obtain, due to its additivity and applying the sum rule in integration:

$$
\mu_y^r = \int_y^\infty x^r g(x|\boldsymbol{\Psi}) dx = \int_y^\infty x^r \sum_{i=1}^{I} \pi_i m_i(x|\boldsymbol{\theta_i}) dx = \sum_{i=1}^{I} \pi_i \int_y^\infty x^r m_i(x) dx = \sum_{i=1}^{I} \pi_i (\mu_i)_y^r. \tag{33}
$$

D.9 Piecewise composite

$$
\begin{aligned}
\mu_y^r &= \int_y^\infty x^r g(x|\boldsymbol{\theta}) dx \\
&= \begin{cases}
\frac{\alpha_1}{1+\alpha_1+\alpha_2} \frac{(\mu_1)_y^r - (\mu_1)_{c_1}^r}{M_1(c_1)} + \frac{1}{1+\alpha_1+\alpha_2} \frac{(\mu_2)_{c_1}^r - (\mu_2)_{c_2}^r}{M_2(c_2)-M_2(c_1)} + \frac{\alpha_2}{1+\alpha_1+\alpha_2} \frac{(\mu_3)_y^r}{1-M_3(c_2)} & \text{if } \ 0 < y \leq c_2 \\
\frac{1}{1+\alpha_1+\alpha_2} \frac{(\mu_2)_y^r - (\mu_2)_{c_2}^r}{M_2(c_2)-M_2(c_1)} + \frac{\alpha_2}{1+\alpha_1+\alpha_2} \frac{(\mu_3)_{c_2}^r}{1-M_3(c_2)} & \text{if } \ c_1 < y \leq c_2 \\
\frac{\alpha_2}{1+\alpha_1+\alpha_2} \frac{(\mu_3)_y^r}{1-M_3(c_2)} & \text{if } \ c_2 < y < \infty
\end{cases}
\end{aligned} \tag{34}
$$

[12] The bounded moments of the Gamma distribution are obtained setting c=1.

D.10 Right-Pareto Lognormal

$$\begin{aligned}
\mu_y^r &= \int_y^\infty x^r k_2 x^{-k_2-1} e^{k_2\mu+\frac{k_2^2Var^2}{2}} \Phi\left(\frac{lnx-\mu-k_2Var^2}{Var}\right)dx \\
&= k_2 e^{k_2\mu+\frac{k_2^2Var^2}{2}} \int_y^\infty x^{\sigma-k_2-2} \Phi\left(\frac{lnx-\mu-k_2Var^2}{Var}\right)dx \\
&\qquad dv = x^{\sigma-k_2-2}dx, v = \frac{x^{\sigma-k_2-1}}{\sigma-k_2-1} \\
&\qquad u = \Phi\left(\frac{lnx-\mu-k_2Var^2}{Var}\right), du = d\Phi\left(\frac{lnx-\mu-k_2Var^2}{Var}\right) \\
&= k_2 e^{k_2\mu+\frac{k_2^2Var^2}{2}} \left[\frac{x^{\sigma-k_2-1}}{\sigma-k_2-1}\Phi\left(\frac{lnx-\mu-k_2Var^2}{Var}\right)\right]_y^\infty \\
&\quad - k_2 e^{k_2\mu+\frac{k_2^2Var^2}{2}} \int_y^\infty \frac{x^{\sigma-k_2-1}}{\sigma-k_2-1} d\Phi\left(\frac{lnx-\mu-k_2Var^2}{Var}\right) \\
&= k_2 e^{k_2\mu+\frac{k_2^2Var^2}{2}} \left[0-\frac{x_{ij^*}^{\sigma-k_2-1}}{\sigma-k_2-1}\Phi\left(\frac{lny-\mu-k_2Var^2}{Var}\right)\right] \\
&\quad - k_2 e^{k_2\mu+\frac{k_2^2Var^2}{2}} \int_y^\infty \frac{x^{\sigma-k_2-1}}{\sigma-k_2-1} \frac{1}{xVar\sqrt{2\pi}} e^{-\frac{\left[lnx-\mu-k_2Var^2\right]^2}{2Var^2}} dx
\end{aligned} \tag{35}$$

The last integral resembles the bounded moment condition of the Lognormal distribution solved earlier with moment $(r-k_2)$ and mean $(\mu+k_2Var^2)$ so that

$$\begin{aligned}
\mu_y^r = &-k_2 e^{k_2\mu+\frac{k_2^2Var^2}{2}} \frac{x_{ij^*}^{\sigma-k_2-1}}{\sigma-k_2-1}\Phi\left(\frac{lny-\mu-k_2Var^2}{Var}\right) \\
&- \frac{k_2 e^{k_2\mu+\frac{k_2^2Var^2}{2}}}{r-k_2} e^{\frac{(r-k_2)\left((r-k_2)Var^2+2(\mu+k_2Var^2)\right)}{2}} \Bigg[1 \\
&\qquad -\Phi\left(\frac{lny-\left((r-k_2)Var^2-(\mu+k_2Var^2)\right)}{Var}\right)\Bigg]
\end{aligned} \tag{36}$$

Note that

$$\begin{aligned}
&e^{k_2\mu+\frac{k_2^2Var^2}{2}+\frac{(r-k_2)\left((r-k_2)Var^2+2(\mu+k_2Var^2)\right)}{2}} \\
&e^{\frac{2k_2\mu+k_2^2Var^2+(r-k_2)\left[rVar^2+2\mu+k_2Var^2\right]}{2}} \\
&e^{\frac{2k_2\mu+k_2^2Var^2+r^2Var^2+2\mu r+k_2rVar^2-k_2rVar^2-2\mu k_2+k_2^2Var^2}{2}} \\
&e^{\frac{r^2Var^2+2\mu r}{2}}
\end{aligned}$$

so that we get

$$\mu_y^r = -k_2 e^{k_2\mu + \frac{k_2^2 Var^2}{2}} \frac{x_{ij^*}^{\sigma-k_2-1}}{\sigma - k_2 - 1}\Phi\left(\frac{lny - \mu - k_2 Var^2}{Var}\right)$$
$$- \frac{k_2}{r - k_2} e^{\frac{r^2 Var^2 + 2\mu r}{2}} \Phi^c\left(\frac{lny - rVar^2 - \mu}{Var}\right) \quad (37)$$

D.11 Left-Pareto Lognormal

$$\mu_y^r = \int_y^\infty x^r x^{k_1-1} e^{-k_1\mu + \frac{k_1^2 Var^2}{2}} \Phi^c\left(\frac{lnx - \mu + k_1 Var^2}{Var}\right) dx$$
$$= k_1 e^{k_2\mu + \frac{k_2^2 Var^2}{2}} \left[\left(\frac{-(y)^{\sigma-k_2-1}}{\sigma - k_2 - 1}\right) - e^{-k_1\mu + \frac{k_1^2 Var^2}{2}} \int_y^\infty x^{\sigma-2+k_1} \Phi\left(\frac{lnx - \mu + k_1 Var^2}{Var}\right) dx\right]$$
$$= -k_1 e^{-k_1\mu + \frac{k_1^2 Var^2}{2}} \frac{x_{ij^*}^{\sigma+k_1-1}}{\sigma + k_1 - 1}\Phi^c\left(\frac{lny - \mu + k_1 Var^2}{Var}\right)$$
$$+ \frac{k_1}{r + k_1} e^{\frac{r^2 Var^2 + 2\mu r}{2}} \Phi^c\left(\frac{lny - rVar^2 + \mu}{Var}\right) \quad (38)$$

D.12 Double-Pareto Lognormal

$$\mu_y^r = \frac{k_2 k_1}{k_2 + k_1} \int_y^\infty x^r x^{-k_2-1} e^{k_2\mu + \frac{k_2^2 Var^2}{2}} \Phi\left(\frac{lnx - \mu - k_2 Var^2}{Var}\right) dx$$
$$+ \frac{k_2 k_1}{k_2 + k_1} \int_y^\infty x^r x^{k_1-1} e^{-k_1\mu + \frac{k_1^2 Var^2}{2}} \Phi^c\left(\frac{lnx - \mu + k_1 Var^2}{Var}\right) dx$$
$$= -\frac{k_2 k_1}{k_2 + k_1} e^{k_2\mu + \frac{k_2^2 Var^2}{2}} \frac{x_{ij^*}^{\sigma-k_2-1}}{\sigma - k_2 - 1}\Phi\left(\frac{lny - \mu - k_2 Var^2}{Var}\right)$$
$$- \frac{k_2 k_1}{k_2 + k_1} \frac{1}{r - k_2} e^{\frac{r^2 Var^2 + 2\mu r}{2}} \Phi^c\left(\frac{lny - rVar^2 - \mu}{Var}\right)$$
$$- \frac{k_2 k_1}{k_2 + k_1} e^{-k_1\mu + \frac{k_1^2 Var^2}{2}} \frac{x_{ij^*}^{\sigma+k_1-1}}{\sigma + k_1 - 1}\Phi\left(\frac{lny - \mu + k_1 Var^2}{Var}\right)$$
$$- \frac{k_2 k_1}{k_2 + k_1} \frac{1}{r + k_1} e^{\frac{r^2 Var^2 + 2\mu r}{2}} \Phi^c\left(\frac{lny - rVar^2 - \mu}{Var}\right) \quad (39)$$

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