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Abstract

This paper considers price volatility as the reason for description of the second-degree economic variables, trades and expectations aggregated during certain time interval *Δ*. We call it - the second-order economic theory. The *n-th* degree products of costs and volumes of trades, performed by economic agents during interval *Δ* determine price *n-th* statistical moments. First two price statistical moments define volatility. To model volatility one needs description of the squares of trades aggregated during interval *Δ*. To describe price probability one needs all *n-th* statistical moments of price but that is almost impossible. We define squares of agent's trades and macro expectations those approve the second-degree trades aggregated during interval *Δ*. We believe that agents perform trades under action of multiple expectations. We derive equations on the second-degree trades and expectations in economic space. As economic space we regard numerical continuous risk grades. Numerical risk grades are discussed at least for 80 years. We propose that econometrics permit accomplish risk assessment for almost all economic agents. Agents risk ratings distribute agents by economic space and define densities of macro second-degree trades and expectations. In the linear approximation we derive mean square price and volatility disturbances as functions of the first and second-degree trades disturbances. In simple approximation numerous expectations and their perturbations can cause small harmonic oscillations of the second-degree trades disturbances and induce harmonic oscillations of price and volatility perturbations.

Keywords: volatility, economic theory, market trades, expectations, price probability

JEL : C1, D4, E3, E4, G1, G2

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1. Introduction

"In everyday language, volatility refers to the fluctuations observed in some phenomenon over time" (Andersen et.al., 2005). Price and returns volatility and their impact on economy are studied for decades: Hall and Hitch (1939), Fama (1965), Stigler and Kindahl (1970), Pearce (1983), Tauchen and Pitts (1983), Schwert, (1988), Mankiw, Romer and Shapiro (1991), Campbell et.al. (1993), Heston (1993), Brock and LeBaron (1995), Bernanke and Gertler (1999), Andersen et.al. (2001), Engle and Patton (2001), Poon and Granger (2003), Andersen et.al. (2005), Daly (2008), Christiansen, Schmeling and Schrimpf (2012), Padungsaksawasdi and Daigler (2018), Bogousslavsky and Collin-Dufresne (2019). We refer a small part of infinite number of volatility studies to outline importance and complexity of the volatility problem.

As usual volatility is defined as standard deviations of certain probability distribution. This definition of volatility is the starting point for most studies. Each new volatility model considers properties of the proposed probability distribution and properties of the standard deviations. Description of volatility is concentrated on the properties of price probability that assumed to reflect properties of market in the best way. This approach separates studies of price uncertainties from studies of the financial markets itself. Nevertheless a lot of papers investigate correlations and mutual dependence between volatility, volume and number of market trades (Tauchen and Pitts 1983; Campbell et.al. 1993; Ito and Lin 1993; Avramov et.al. 2006; Ciner and Sackley 2007; Miloudi et.al. 2016; Takaishi and Chen 2017). Impact of market trades on price and volatility plays important role for volatility forecasting.

Volatility modeling uncovers important problem of economic theory. Each economic model describes economic and financial variables aggregated during certain time interval Δ. Time interval Δ can be equal minute, hour, day, month and etc. Any description of price volatility uses financial data aggregated during certain interval Δ. The choice of interval Δ and the method for aggregation the economic and financial data on price, assets, investment, credits, profits and etc., plays core role for adequacy of the macroeconomic modeling. The duration of time interval Δ determines main properties of the economic model – its accuracy, sustainability, scales of possible fluctuations and etc. Choice of time interval Δ and choice of data aggregation method are the most important initial steps for economic forecasting.

Modern macro theory describes evolution of first-degree macro variables, transactions and expectations aggregated during certain interval Δ. For example, macro investment, credits, assets are determined as sum of first-degree investment, credits and assets of economic agents (without doubling) during time interval *Δ*. Macro trades equal the sum of first-degree trades during interval *Δ*. These macro variables and trades are defined as sum of first-degree variables and trades. We call description of these first-degree variables and trades as the first-order economic theory. Almost all macroeconomic and financial variables are the first-degree variables. Price volatility is nearly only exception. Indeed, price volatility as the standard deviation describes difference between mean square price and square of mean price during certain interval *Δ*. This simple notion – mean square price - leads to numerous problems. During last decades mean price *p(1;t)* is determined as volume weighted average price (VWAP) (Berkowitz et.al 1988; Buryak and Guo, 2014; Guéant and Royer, 2014; Busseti and Boyd, 2015; Padungsaksawasdi and Daigler, 2018, CME Group, 2020). VWAP mean price *p(1;t)* equals the ratio of sum of costs C_i to sum of volumes U_i of all trades *i* during interval *Δ*. We call the pair *(U,C)* of volume *U* and cost *C* of trade as the first-degree trade. As the second-degree trade we call the pair (U^2,C^2) of squares of volume U^2 and squares of cost C^2 of trade performed by agent. We follow the economic meaning of VWAP and define the mean square price *p(2;t)* as the ratio of the sum of squares of costs C_i^2 to the sum of squares of volumes U_i^2 of trades *i* performed during interval *Δ*. We give formal definitions in Sec.2 but underline that description of mean square price *p(2;t)* requires modeling sums of squares of costs and volumes – sums of second-degree trades perfirmed during interval *Δ*.

It is obvious that price volatility is not the only variable that requires description of the second-degree variables and trades. Bank rates, inflation, currency exchange rates, return on investment, GDP growth rates and etc., are subject to fluctuations. Profits, supply and demand, production function and almost all other macro variables fluctuate. Description of volatility of macro variables requires model collective second-degree economic and financial variables, trades and expectations aggregated during interval *Δ*. We call this model as the second-order economic theory. Development of the second-order economic theory doubles the complexity of the current description of macroeconomics.

In this paper we only start studying the particular case of the second-order economic theory. We model price volatility and describe the second-degree trades collected during interval *Δ*. We introduce main notions and derive equations that describe evolution of squares of market trades. It is generally accepted that agents perform trades under impact of expectations. We describe expectations those approve the second-degree trades and derive equations that govern evolution of the second-degree expectations. We show how equations on the second-degree trades cause equations on mean square price *p(2;t)*. We believe that math complexity of our methods and equations corresponds the complexity of the economic problem.

In this paper we develop results on price volatility presented in (Olkhov, 2020). We introduce price statistical moments for certain averaging time interval *Δ*. First and second statistical moments determine price volatility. We describe price volatility as function of the number of trades, first and second-degree costs and volumes of trades performed during interval *Δ*.

In Sec.2 we present brief description of price volatility due to (Olkhov, 2020). In Sec.3 we describe second-degree macro trades and their flows. In Sec. 4 we introduce collective expectations, collective expected trades and their flows. In Sec 5 we describe disturbances of second-degree trades and model small price and volatility disturbances. Sec.6 - Conclusion. In Appendix A we derive relations on second-degree macro trades and their flows. In Appendix B we introduce math for collective expectations. In Appendix C we derive equations on second-degree trades. Appendix D describes perturbations of second-degree trades, price and volatility.

Equations (3.4) denote Section 3 and equation 4. We use roman letters *A, B, d* to denote scalars and bold ***B, P, v*** – to denote vectors.

2. Price and volatility

We take the price *p* definition mentioned at least by Fetter among 117 others (1912): "Ratio-of-exchange definitions of price in terms of value in the sense of a mere ratio of exchange" – price *p* is a coefficient between cost *C* and volume *U* of market trade:

$$C = pU \tag{2.1}$$

We call the pair *(U,C)* as the market trade. Relation (2.1) defines price *p* of a single trade. Meanwhile most studies treat price averaged during certain time interval *Δ*. Interval *Δ* can equal minute, hour, day, month and etc. Let's chose interval *Δ* and define number *N(t)* of market trades performed during interval *Δ* near moment *t*. For continuous time we define the number *N=N(t)* of trades during interval *Δ* as:

$$N(t) = \int_{-\Delta/2}^{+\Delta/2} N_{tr}(t+\tau)d\tau \tag{2.2}$$

Here $N_{tr}(t+\tau)$ - number of trades at *t+τ*. For discrete time model number *N=N(t)* of trades performed during interval *Δ* near moment *t* can be determined as:

$$N(t) = \sum_i \theta\left(t_i - \left(t - \frac{\Delta}{2}\right)\right)\theta\left(\left(\frac{\Delta}{2} + t\right) - t_i\right) \tag{2.3}$$

$$\theta(t) = 1\,,\ if\ t \geq 0\,;\ \theta(t) = 0\,,\ if\ t < 0 \tag{2.4}$$

Here t_i denotes the moment of trade. Relations (2.3, 2.4) collect all moment t_i of market trades during interval [t-*Δ/2*; t+*Δ/2*]. Now let's sum the *n-th* degree of cost $C^n(t_i)$ and *n-th* degree of volume $U^n(t_i)$ for all *N(t)* (2.2-2.4) of each trade during interval *Δ:*

$$C(n;t) = \sum_{i=1}^{N(t)} C^n(t_i) \quad ; \quad U(n;t) = \sum_{i=1}^{N(t)} U^n(t_i) \tag{2.5}$$

For interval *Δ* and *C(n;t)* and *U(n;t)* we define mean *n-th* degree price *p(n;t)* (2.1, 2.6) as*:*

$$C(n;t) = p(n;t)U(n;t) \tag{2.6}$$

For *n=1* relations (2.1, 2.5, 2.6) are identical to VWAP definition (Berkowitz et.al 1988; Buryak and Guo, 2014; Guéant and Royer, 2014; Busseti and Boyd, 2015; Padungsaksawasdi and Daigler, 2018; CME Group, 2020):

$$p(1;t) = \frac{1}{V(1;t)} \sum_{i=1}^{N(t)} p(t_i)\, U(t_i) = \frac{C(1;t)}{V(1;t)} \tag{2.7}$$

Price volatility as dispersion $\sigma_p^{\ 2}(t)$ of price probability takes form (Olkhov, 2020):

$$\sigma_p^2(t) = < \delta p^2(t) >= p(2;t) - p^2(1,t) \tag{2.8}$$

We use <…> to denote averaging procedure. To define price probability distribution one should describe all *n-th* statistical moments of random price process *p(t)* those match relations (2.1, 2.6). For time set $t_1,\ldots t_n$, we take the *n-th* price statistical moment $p(n;t_1,\ldots t_n)$:

$$p(n;t_1,..t_n) = < \prod_{j=1}^{n} p(t_j) > \tag{2.9}$$

$$C(n;t_1,..t_n) = p(n;t_1,..t_n)U(n;t_1,..t_n) \tag{2.10}$$

$$C(n;t_1,..t_n) = \sum \prod_{j=1}^{n} C(t_j;\tau_{ji}) \quad ; \quad U(n;t_1,..t_n) = \sum \prod_{j=1}^{n} U(t_j;\tau_{ji}) \tag{2.11}$$

Here $C(t_j;\ \tau_{ji})$ denote cost and $U(t_j;\ \tau_{ji})$ denote volume of trade performed at moment τ_{ji} during time intervals $[t_j-\Delta/2;\ t_j+\Delta/2]$ for *j=1,..n*. Sums in (2.11) collect costs *C* and volumes *U* of all trades performed at moments τ_{ji} for all *i* for each interval $[t_j-\Delta/2;\ t_j+\Delta/2]$, *j=1,..n*. Relations (2.9, 2.10) define the *n-th* price statistical moments $p(n;t_1,\ldots t_n)$ of price random process *p(t)* averaged during intervals $[t_j-\Delta/2;\ t_j+\Delta/2]$,], *j=1,..n*. The set of statistical moments $p(n;t_1,\ldots t_n)$ (2.9, 2.10) determine characteristic functional and gives complete description of all statistical properties of random process *p(t)*. We refer (Klyatskin, 2005; 2015) for technical details and usage of characteristic functional and functional calculus for description of random processes and stochastic dynamic systems. Characteristic functional *F(x(t))* of random process *p(t)* takes form:

$$F(x(t)) = \sum_{i=1}^{\infty} \frac{i^n}{n!} \int dt_1 \ldots dt_n\ p(n;t_1,\ldots t_n)\, x(t_1)..x(t_n) \tag{2.12}$$

We underline, that (2.12) describes all statistical properties of the price random process *p(t)* and (2.12) depends on choice of time averaging interval *Δ*. Thus to identify price probability by (2.12) one should describe evolution of collective costs $C(n;\ t_1,\ldots t_n)$ and volumes $U(n;\ t_1,\ldots t_n)$ (2.11) for all *n=1,2…* aggregated during interval *Δ*. In other words for each n=1,2,... one should develop *n-th* order macro theory that describe evolution of *n-th* degrees of trades, economic variables and expectations. This problem seems a little challenging to be solved ever and hence exact description of market price probability distribution will remain unattainable.

Forecasting of price volatility (2.8) requires description of $p(1;t)$ and $p(2;t)$ (2.6). Expression (2.8) describes price volatility $\sigma_p^2(t)$ as function of first and second moments. Relations (2.6) for $n=2$ define price second moment $p(2;t)$ (2.6) as function of sums of squares of cost and volumes of trades collected during interval Δ. To forecast price volatility one should describe the second-degree costs $C(2;t)$ and volumes $U(2;t)$ (2.5) aggregated during interval Δ.

Let's clarify once more the main issue of our treatment of price statistical properties. As usual, researchers study price $p(t)$ aggregated during interval Δ that equals minute, hour, day and etc., and assume that this function $p(t)$ is random. Indeed, price $p(t)$ averaged during minute or hour behaves absolutely unpredictable during long time terms – days or weeks. Function $p(t)$ behaves like a random variable. Hence researchers propose that $p(t)$ obeys certain stochastic distribution and calculate price volatility and other price stochastic properties. However, we demonstrate that aggregation of price $p(t)$ during any interval Δ carries properties of the mean price $p(1;t)$ (2.6, 2.7) only. It is obvious, that during time terms $T >> \Delta$ function $p(1;t)$ can behaves irregular. But mean price $p(1;t)$ (2.6, 2.7) carries information about properties of the mean price only. Moreover, as we use VWAP procedure (2.5-2.7), thus one can derive mean price $p(1;t;T)$ aggregated during time interval $T > \Delta$ with help of corresponding aggregated first-degree volume $U(1;t)$ only. To do that in continuous approximation one should use first-degree costs $C(1;t;T)$ and first-degree volumes $U(1;t;T)$ (2.13, 2.14) aggregated during time interval $T > \Delta$ as:

$$C(1;t;T) = \int_{t-T/2}^{t+T/2} p(1;\tau)U(1;\tau)\,d\tau \ ; \quad U(1;t;T) = \int_{t-T/2}^{t+T/2} U(1;\tau)\,d\tau \qquad (2.13)$$

$$C(1;t;T) = p(1;t;T)U(1;t;T) \ ; \quad T > \Delta \qquad (2.14)$$

Mean price $p(1;t;T)$ can irregularly behave on time terms $T_2 > T$ and one can recursively apply same procedure (2.13, 2.14) for $T_2 > T$ to obtain mean price $p(1;t;T_2)$ and etc. Thus relations (2.6, 2.7, 2.13, 2.14) for $n=1$ define VWAP procedures and determine mean fist-degree price $p(1;t;T)$ only. To assess mean squares or n-th degree price statistical moments one must use (2.6, 2.7) complemented by relations (2.15, 2.16) similar to (2.13, 2.14):

$$C(n;t;T) = \int_{t-T/2}^{t+T/2} p(n;\tau)U(n;\tau)\,d\tau \ ; \quad U(n;t;T) = \int_{t-T/2}^{t+T/2} U(n;\tau)\,d\tau \qquad (2.15)$$

$$C(n;t;T) = p(n;t;T)U(n;t;T) \ ; \quad T > \Delta \qquad (2.16)$$

Transition of price $p(1;t;T_1)$, .. $p(n;t;T_1)$ from one averaging time interval T_1 to another interval $T_2 > T_1$ should follow (2.15, 2.16) for each $n=1, 2, \ldots$ independently. This procedure completely differs from common description of price random properties. We propose that procedures (2.5-2.16) should be treated as correct because they describe price statistical moments that correspond to economic meaning of price (2.1). Transition from averaging interval T_1 to interval $T_2 > T_1$ establishes certain relations between $p(1;t;T_1)$ and $p(1;t;T_2)$, $p(2;t;T_1)$ and $p(2;t;T_2)$ and etc., but we leave this for the future. For brevity below we omit interval T in $p(n;t;T)$, $C(n;t;T)$, $U(n;t;T)$ and denote these functions as $p(n;t)$, $C(n;t)$, $U(n;t)$.

Current macroeconomic and financial models describe interdependence of macroeconomic and financial variables of the first-degree. Macro variables are composed as sum of corresponding variables of all economic agents in economy during interval Δ. For example, macro investment is determined as investment made by all agents in economy during interval Δ. Macro credit, taxes, consumption, output, profits and etc., are determined as sum (without doubling) of credits, taxes, consumption, output, profits and etc., of all agents in economy during interval Δ. To describe price volatility (2.8) one should complement model of the first-degree macro variables and transactions with description of squares of trades aggregated during interval Δ. To begin with in the next section we consider methods that help describe the second-degree macro trades.

3. Second-degree macro trades

We describe the second-degree macro trades using the methods (Olkhov, 2016-2019) and refer these papers for details. For reader's convenience we present brief explanations. We base our model of the second-degree macro transactions on several assumptions.

First, we assume that it is possible develop unified methodology for agents risk assessments based on numerical continuous risk scoring. Up now risk rating noted by letters. Each agency - S&P, (2014, 2016), Moody's, (2010, 2018), Fitch, (2018) - introduces its own ratings methodology. However, numerical credit scoring was suggested eighty years ago by Durand (1941) and sixty years ago by Myers and Forgy (1963). Last years researchers use numerical risk rating to compare risk assessment provided by different rating agencies (Beaver et.al., 2006; Poon and Shen, 2020). We assume that no internal econometric problems prevent the transfer from letters to numerical risk scoring. This transition depends mostly on business of major ratings companies. Development of unified methodology that will be adopted by major business players should deliver benefits to rating agencies, business and economic authorities. Unified numerical continuous risk grades methodology will significantly improve methods for modeling and forecasting of macroeconomic and financial processes. Introduction of continuous risk grades is a natural development of numerical grades methodology. We don't specify any particular risks like credit or inflation risk but regard all set of economic, financial, political and etc., risks those impact economic and financial development.

Second, we assume that econometrics can provide sufficient data to assess risk ratings for almost all economic agents, for international banks, corporations and for local firms, companies and even households. That will distribute all agents of the economy by their risk ratings as coordinates of risk space.

Third, agents in economy always act under pressure of numerous risks. Assessment of agents ratings for one risk distribute agents over *1*-dimensional risk space. Simultaneous assessment of agents ratings for two or three risks distribute agents by two or three dimensional risk space. Absolute numerical values of risk ratings have no sense. For certainty, for any risk we take most safe rating equal zero and most risky rating equal 1. Thus agents ratings for a single risk fill unit interval [0,1] and agents ratings for m risks fill unit cube in space R^m.

Forth. Explicit description of numerous separate agents of the entire economy is almost impossible. It requires too much exact data of economic and financial variables for millions separate agents, data about all particular market trades between agents, risk assessment of all agents and etc. It is impossible to collect precise data for all agents and small perturbations could move the forecasts in the wrong directions. To solve the problem we reduce the accuracy of economic description. We aggregate description of agents in the risk economic space and derive continuous media approximation (Olkhov, 2016a, 2018; 2019a; 2019b) that describes collective market trades. Below we explain meaning of continuous media approximation for trades in more detail.

Let's explain the benefits of our approach to economic modeling. First, transition from letters to continuous numerical risk scoring uncovers the hidden motion of economic agents. Indeed, rating agencies for decades use transition matrix to describe probability of agent's transfer from particular risk grade to a different one (Belkin, 1998; Schuermann and Jafry, 2003; Ho et.al, 2017; S&P, 2018). Introduction of numerical continuous scoring defines the numerical distance between risk ratings and hence introduces the mean velocity of agent's motion during risk transition. Hence one describes the motion of agents in the risk economic space and describes flows of macro variables, trades and expectations induced by risk motion of agents. Below we use this approach to develop description of second-degree trades and refer to (Olkhov, 2016-2019) for all further details and definitions.

3.1. Second-degree macro trades

To describe second-degree trades between economic agents we use methods and models developed for description of the first-degree macro trades and expectations in Olkhov

(2019b). We regard economy as numerous economic agents those perform economic and financial trades. We assume that agents perform trades under action of *m* risks and it is possible to provide risk assessment for all agents in the economy. Each economic agent *i* is determined by it's numerical continuous risk ratings $\boldsymbol{x}$

$$\boldsymbol{x} = (x_1, \dots x_m) \ ; \quad 0 \le x_i \le 1 \ ; \quad i = 1, .. m \tag{3.1}$$

Due to (3.1) all agents fill economic domain of risk space R^m. Further we note risk space as economic space. We treat risk assessments of agents as procedures similar to measurements of coordinates of physical particles and call agents numerical continuous risk ratings $\boldsymbol{x}$ (3.1) as agents coordinates in economic domain

$$0 \le x_i \le 1 \ ; \quad i = 1, .. m \tag{3.2}$$

Let's take that agent *i* with risk coordinates $\boldsymbol{x}$ performs market transactions with agent *j* with risk coordinates $\boldsymbol{y}$. To simplify the model we assume that at moment *t* agent *i* at point $\boldsymbol{x}$ sells the volume U_{ij} of variable *E* to agent *j* at point $\boldsymbol{y}$. As *E* one can consider any commodities, goods, service, credits, investment, assets and etc. Let's assume that the volume U_{ij} of this particular trade between agents *i* and *j* cost C_{ij}. We define trade at moment t between two agents *i* and *j* as two component function $\boldsymbol{b}_{ij}$:

$$\boldsymbol{b}_{ij}(1; t, \boldsymbol{z}) = \left(U_{ij}(t, \boldsymbol{z}); C_{ij}(t, \boldsymbol{z})\right) \ ; \ \boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \tag{3.3}$$

Coordinates $\boldsymbol{x}$ and $\boldsymbol{y}$ of agents *i* and *j* involved into this trade establish economic domain of double dimension (3.1; 3.2; 3.4; 3.5) with coordinates $z=(\boldsymbol{x},\boldsymbol{y})$:

$$\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \ ; \quad \boldsymbol{x} = (x_1 \dots x_n) \ ; \quad \boldsymbol{y} = (y_1 \dots y_n) \tag{3.4}$$

$$0 \le x_i \le 1 \ ; \ 0 \le y_j \le 1 \ ; \quad i = 1, .. m \ ; \ j = 1, .. m \tag{3.5}$$

We define the price p_{ij} of the trade $\boldsymbol{b}_{ij}$ (3.3) between agents *i* and *j* similar to (2.1):

$$C_{ij}(t, \boldsymbol{z}) = p_{ij}(t, \boldsymbol{z}) U_{ij}(t, \boldsymbol{z}) \tag{3.6}$$

One need certain time *dt* to perform the trade $\boldsymbol{b}_{ij}$ (3.3). The volume U_{ij} of the trade $\boldsymbol{b}_{ij}$ (3.3) change the amount of economic or financial variable *E* of agents *i* and *j* involved into the trade during time *dt* and the cost C_{ij} change the amount of funds. We consider trades as a rate of change of economic and financial variables of economic agents. We define the second-degree trades $\boldsymbol{b}_{ij}(2;t,z)$ as:

$$\boldsymbol{b}_{ij}(2; t, \boldsymbol{z}) = \left(U_{ij}^2(t, \boldsymbol{z}); C_{ij}^2(t, \boldsymbol{z})\right) \tag{3.7}$$

and define trades $\boldsymbol{b}_{ij}(n;t,z)$ of degree *n* as

$$\boldsymbol{b}_{ij}(n; t, \boldsymbol{z}) = \left(U_{ij}^n(t, \boldsymbol{z}); C_{ij}^n(t, \boldsymbol{z})\right) \tag{3.8}$$

We determine the *n-th* degree price $p_{ij}(n;t,z)$ of trade $\boldsymbol{b}_{ij}(n;t,z)$ (3.8) as:

$$C_{ij}^{n}(t,\boldsymbol{z}) = p_{ij}^{n}(t,\boldsymbol{z})U_{ij}^{n}(t,\boldsymbol{z}) \quad ; \quad p_{ij}(n;t,\boldsymbol{z}) = p_{ij}^{n}(t,\boldsymbol{z}) \tag{3.9}$$

To avoid excess accuracy and to derive economic continuous media approximation for the second-degree trades we replace description of second-degree trades $\boldsymbol{b}_{ij}(2;t,z)$ between separate agents i at $\boldsymbol{x}$ and j at $\boldsymbol{y}$ by description of collective second-degree trades $\boldsymbol{B}(2;t,z)$, $z=(\boldsymbol{x},\boldsymbol{y})$ between points $\boldsymbol{x}$ and $\boldsymbol{y}$. To do that we introduce small scale d and a unit volume $dV(z)$ in economic domain (3.4, 3.5)

$$dV(\boldsymbol{z}) = dV(\boldsymbol{x})dV(\boldsymbol{y}) \; ; \; dV(\boldsymbol{x}) = d^{n} \; ; \; dV(\boldsymbol{y}) = d^{n} \; ; \; \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{3.10}$$

We assume that scale $d << 1$ but each unit volume $dV(\boldsymbol{x})$ and $dV(\boldsymbol{y})$ contains a lot of agents with risk coordinates inside $dV(\boldsymbol{x})$ and $dV(\boldsymbol{y})$. Let's assume that during time Δ agents inside $dV(\boldsymbol{x})$ and $dV(\boldsymbol{y})$ perform a lot of mutual trades. We define the collective second-degree trade $\boldsymbol{B}(2;t,z)$ between points $\boldsymbol{x}$ and $\boldsymbol{y}$ as a sum of all second-degree trades $\boldsymbol{b}_{ij}(2;t,z)$ of agents i with coordinates in a unit volume $dV(\boldsymbol{x})$ and agents j with coordinates in a unit volume $dV(\boldsymbol{y})$ (3.10) and then average this sum during the interval Δ as:

$$\boldsymbol{B}(2;t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta} \boldsymbol{b}_{i,j}(2;t,\boldsymbol{z}) \tag{3.11}$$

$$\sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta} \boldsymbol{b}_{i,j}(2;t,\boldsymbol{z}) = \frac{1}{\Delta}\int_{t-\Delta/2}^{t+\Delta/2} d\tau \, \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})} \boldsymbol{b}_{i,j}(2;\tau,\boldsymbol{z}) \tag{3.12}$$

$$\boldsymbol{B}(2;t,\boldsymbol{z}) = \big(U(2;t,\boldsymbol{z}); C(2;t,\boldsymbol{z})\big) \; ; \; \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{3.13}$$

$$U(2;t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta} U_{ij}^{2}(t,\boldsymbol{z}) \tag{3.14}$$

$$C(2;t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta} C_{ij}^{2}(t,\boldsymbol{z}) \tag{3.15}$$

Second-degree collective trades $\boldsymbol{B}(2;t,z)$ (3.11-3.15) define second-degree price $p(2;t,z)$ of trades between points $\boldsymbol{x}$ and $\boldsymbol{y}$ in domain (3.4, 3.5) averaged during interval Δ.

$$C(2;t,\boldsymbol{z}) = p(2;t,\boldsymbol{z})U(2;t,\boldsymbol{z}) \tag{3.16}$$

Meanings of second-degree price $p(2,t)$ (2.6) for $n=2$ and $p(2;t,z)$ (3.16) are simple. Relation (3.16) describes mean second-degree price $p(2;t,z)$ between points $\boldsymbol{x}$ and $\boldsymbol{y}$ averaged during interval Δ. Relation (2.6) describes mean second-degree price $p(2;t)$ of all trades in the economy averaged during interval Δ. To derive (2.6) one should take integral by collective trade $\boldsymbol{B}(2;t,z)$ of the second-degree over economic domain (3.4, 3.5) – derive the sum over all agents involved into trades during time interval Δ:

$$\boldsymbol{B}(2;t) = \big(U(2;t); C(2;t)\big) \tag{3.17}$$

$$U(2;t) = \int d\boldsymbol{z}\, U(2;t,\boldsymbol{z}) \quad ; \quad C(2;t) = \int d\boldsymbol{z}\, C(2;t,\boldsymbol{z}) \tag{3.18}$$

$$C(2;t) = p(2;t)U(2;t) \tag{3.19}$$

Relations (3.19) define price $p(2;t)$ in the same way as (2.6) for $n=2$. Description of price (2.6) requires models of the second-degree trades $\boldsymbol{B}(2;t)$ and $\boldsymbol{B}(2;t,z)$.

3.2. Flows of the second-degree trades

Let's remind briefly the meaning of trades flows described in (Olkhov, 2018-2019b). We start with the definition of agent's velocities in the economic domain (3.1, 3.2). As we mentioned above any agent's economic activity is the source of risks and agents always act under the pressure of risks of different nature – credit, inflation, market, political and etc. Rating agencies for decades provide assessments of risk transition matrices (Belkin, 1998; Schuermann and Jafry, 2003; Ho et.al, 2017; S&P, 2018). Up now risk grades are determined by letters and elements a_{ij} of transition matrices have sense of probability of transition from grade x_i to x_j during time interval T. As usual interval T equals one, two three years. If and when ratings methodology use numerical continuous grades then transition matrices will get completely different and much more important economic meaning. For numerical continuous grades the transition from rating x_i to x_j defines numerical interval l_{ij} :

$$l_{ij} = x_j - x_i \tag{3.20}$$

Transition from x_i to x_j takes time T. Hence element a_{ij} of transition matrices define probability of agent's motion in the economic domain (3.2) during time T with velocity v_{ij} :

$$v_{ij} = \frac{l_{ij}}{T} \ \ with\, probability\ a_{ij}; \quad \sum_j a_{ij} = 1 \tag{3.21}$$

As we mentioned above, introduction of agent's velocities in the economic domain is important contribution of the transfer from letters to numerical risk grades. Hence transition matrices define mean velocity of agent in point x_i during time interval T as:

$$v(t, x_i) = < v_{ij} >_j = \sum_{j=1}^{K} v_{ij} a_{ij} = \frac{1}{T} \sum_{j=1}^{K} l_{ij} a_{ij} \tag{3.22}$$

Here K means the number of different numerical risk grades that defines the degree KxK of the transition matrix. Hence, the transition matrices can define mean velocities of agents in the economic domain. Risk motion of agents induces flows of economic and financial variables and flows of market trades as well (Olkhov, 2018-2019b).

To describe flows of second-degree trades $\boldsymbol{B}(2;t,z)$ let's assume that in economic domain (3.2) agent i at a moment t have risk coordinates $\boldsymbol{x}=(x_1,...x_m)$ and velocities $\boldsymbol{v_x}=(v_{x1},\ldots v_{xm})$. Let's take that agent i at point $\boldsymbol{x}$ performs trade $\boldsymbol{b}_{ij}(2;t,z)$ with agent j at point $\boldsymbol{y}=(y_1,\ldots y_m)$ and velocities $\boldsymbol{v_y}=(v_{y1},\ldots v_{ym})$. Similar to (Olkhov, 2018-2019b) let's define the flows $\boldsymbol{p}_{ij}(2;t,z)$ of the trades $\boldsymbol{b}_{ij}(2;t,z)$ (3.7) between agents i and j as:

$$\boldsymbol{p}_{ij}(2; t, \boldsymbol{z}) = \left(\boldsymbol{p}_{Uij}(2; t, \boldsymbol{z}), \boldsymbol{p}_{Cij}(2; t, \boldsymbol{z})\right) \tag{3.23}$$

$$\boldsymbol{p}_{Uij}(2; t, \boldsymbol{z}) = \left(\boldsymbol{p}_{Uxij}(2; t, \boldsymbol{z});\ \boldsymbol{p}_{Uyij}(2; t, \boldsymbol{z})\right)\ ;\ \boldsymbol{p}_{Cij}(2; t, \boldsymbol{z}) = \left(\boldsymbol{p}_{Cxij}(2; t, \boldsymbol{z});\ \boldsymbol{p}_{Cyij}(2; t, \boldsymbol{z})\right) \tag{3.24}$$

$$\boldsymbol{p}_{Uxij}(2; t, \boldsymbol{z}) = U_{ij}^2(t, \boldsymbol{z})\boldsymbol{v}_{xi}(t, \boldsymbol{x}) \ \ ; \ \ \boldsymbol{p}_{Uyij}(t, \boldsymbol{z}) = U_{ij}^2(t, \boldsymbol{z})\boldsymbol{v}_{yj}(t, \boldsymbol{y}) \tag{3.25}$$

$$\boldsymbol{p}_{Cxij}(t,\boldsymbol{z}) = C_{ij}^2(t,\boldsymbol{z})\boldsymbol{v}_{xi}(t,\boldsymbol{x}) \quad ; \quad \boldsymbol{p}_{Cijy}(t,\boldsymbol{z}) = C_{ij}^2(t,\boldsymbol{z})\boldsymbol{v}_{yj}(t,\boldsymbol{y}) \tag{3.26}$$

The flows $\boldsymbol{p}_{ij}(2;t,z)$ (3.23) define the flows $\boldsymbol{p}_{Uij}(2;t,z)$ (3.25) that carry the square of volume U_{ij}^2 of the trade $\boldsymbol{b}_{ij}(2;t,z)$ (3.7). The flows $\boldsymbol{p}_{Cij}(t,z)$ (3.26) carry the square of the cost C_{ij}^2 of the trade $\boldsymbol{b}_{ij}(2;t,z)$ (3.7). We determine the cumulative flows $\boldsymbol{P}(2;t,z)$ of the macro second-degree trade $\boldsymbol{B}(2;t,z)$, $z=(\boldsymbol{x},\boldsymbol{y})$ between points $\boldsymbol{x}$ and $\boldsymbol{y}$ similar to (3.11-3.15) as aggregation of flows $\boldsymbol{p}_{ij}(2;t,z)$ (3.23) of all second-degree trades $\boldsymbol{b}_{ij}(2;t,z)$ (3.7) between agents in small volumes $dV(\boldsymbol{x})$ and $dV(\boldsymbol{y})$ (3.10) in economic domain (3.4, 3.5) and averaging during interval Δ. Due to (3.11-3.15) we introduce flows $\boldsymbol{P}(2;t,z)$ and velocities $\boldsymbol{v}(2;t,z)$ as:

$$\boldsymbol{P}(2;t,\boldsymbol{z}) = \big(\boldsymbol{P}_U(2;t,\boldsymbol{z}), \boldsymbol{P}_C(2;t,\boldsymbol{z})\big) \; ; \; \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{3.27}$$

$$\boldsymbol{P}_U(2;t,\boldsymbol{z}) = \textstyle\sum_{i\in dV(\boldsymbol{x}); j\in dV(\boldsymbol{y})\,\Delta} \boldsymbol{p}_{Uij}(2;t,\boldsymbol{z}) \tag{3.28}$$

$$\boldsymbol{P}_C(t,\boldsymbol{z}) = \textstyle\sum_{i\in dV(\boldsymbol{x}); j\in dV(\boldsymbol{y})\,\Delta} \boldsymbol{p}_{Cij}(2;t,\boldsymbol{z}) \tag{3.29}$$

$$\boldsymbol{P}_U(2;t,\boldsymbol{z}) = \Big(\boldsymbol{P}_{Ux}(t,\boldsymbol{z}); \boldsymbol{P}_{Uy}(t,\boldsymbol{z})\Big) \; ; \; \boldsymbol{P}_C(t,\boldsymbol{z}) = \Big(\boldsymbol{P}_{Cx}(t,\boldsymbol{z}); \boldsymbol{P}_{Cy}(t,\boldsymbol{z})\Big) \tag{3.30}$$

$$\boldsymbol{P}_{Ux}(2;t,\boldsymbol{z}) = \textstyle\sum_{i\in dV(\boldsymbol{x}); j\in dV(\boldsymbol{y})\,\Delta} U_{ij}^2(t,\boldsymbol{z})\boldsymbol{v}_i(t,\boldsymbol{x}) = U(2;t,\boldsymbol{z})\boldsymbol{v}_{Ux}(t,\boldsymbol{z}) \tag{3.31}$$

$$\boldsymbol{P}_{Uy}(2;t,\boldsymbol{z}) = \textstyle\sum_{i\in dV(\boldsymbol{x}); j\in dV(\boldsymbol{y})\,\Delta} U_{ij}^2(t,\boldsymbol{z})\boldsymbol{v}_j(t,\boldsymbol{y}) = U(2;t,\boldsymbol{z})\boldsymbol{v}_{Uy}(t,\boldsymbol{z}) \tag{3.32}$$

$$\boldsymbol{P}_{Cx}(2;t,\boldsymbol{z}) = \textstyle\sum_{i\in dV(\boldsymbol{x}); j\in dV(\boldsymbol{y})\,\Delta} C_{ij}^2(t,\boldsymbol{z})\boldsymbol{v}_i(t,\boldsymbol{x}) = C(2;t,\boldsymbol{z})\boldsymbol{v}_{Cx}(t,\boldsymbol{z}) \tag{3.33}$$

$$\boldsymbol{P}_{Cy}(2;t,\boldsymbol{z}) = \textstyle\sum_{i\in dV(\boldsymbol{x}); j\in dV(\boldsymbol{y})\,\Delta} C_{ij}^2(t,\boldsymbol{z})\boldsymbol{v}_j(t,\boldsymbol{y}) = C(2;t,\boldsymbol{z})\boldsymbol{v}_{Cy}(t,\boldsymbol{z}) \tag{3.34}$$

$$\boldsymbol{v}(2;t,\boldsymbol{z}) = \big(\boldsymbol{v}_U(2;t,\boldsymbol{z});\; \boldsymbol{v}_C(2;t,\boldsymbol{z})\big) \tag{3.35}$$

$$\boldsymbol{v}_U(2;t,\boldsymbol{z}) = \Big(\boldsymbol{v}_{Ux}(2;t,\boldsymbol{z}) \,; \boldsymbol{v}_{Uy}(2;t,\boldsymbol{z})\Big) \; ; \; \boldsymbol{v}_C(2;t,\boldsymbol{z}) = \Big(\boldsymbol{v}_{Cx}(2;t,\boldsymbol{z}) \,; \boldsymbol{v}_{Cy}(2;t,\boldsymbol{z})\Big) \tag{3.36}$$

The flows $\boldsymbol{P}(2;t,z)$ (3.27-3.34) of the second-degree trade $\boldsymbol{B}(2;t,z)$ between points $\boldsymbol{x}$ and $\boldsymbol{y}$ describe the amounts of second-degree volume $U(2;t,z)$ (3.14) and cost $C(2;t,z)$ (3.15) of trades $\boldsymbol{B}(2;t,z)$ (3.11) carried by trades velocities $\boldsymbol{v}(2;t,z)$ (3.35, 3.36) through $2m$-dimensional economic domain (3.4, 3.5). Velocity $\boldsymbol{v}_U(2;t,z)$ (3.36) defines the motion of second-degree volume $U(2;t,z)$ (3.14) and may be different from velocity $\boldsymbol{v}_C(2;t,z)$ (3.36) that describes the motion of second-degree cost $C(2;t,z)$ (3.15).

Integral of trades $\boldsymbol{B}(2;t,z)$ and their flows $\boldsymbol{P}(2;t,z)$ by $d\boldsymbol{y}$ over entire economic domain (3.4, 3.5) defines the second-degree sell-trades $\boldsymbol{B}_s(2;t,\boldsymbol{x})$ from point $\boldsymbol{x}$. Integral by $d\boldsymbol{x}$ defines buy-trades $\boldsymbol{B}_s(2;t,\boldsymbol{y})$ at point $\boldsymbol{y}$. Integral by $dz=d\boldsymbol{x}d\boldsymbol{y}$ defines total trades $\boldsymbol{B}(2;t)$ in the entire economy at moment t and their flows (see Appendix A). Long definitions (Appendix A) demonstrate the hidden complexities of the second-degree economic processes in economy and are useful for description of macroeconomic evolution.

Introduction of the second-degree trades and their flows as functions of risk in economic domain (3.4,3.5) outlines the main contribution of our approach to macroeconomics, finance, and price volatility modeling. Relations on the second-degree trades define different forms of the price. Relations (3.6, 3.9) define price $p_{ij}(t,z)$ and $p_{ij}^{n}(t,z)$ of particular transaction between agents i and j at points $\boldsymbol{x}$ and $\boldsymbol{y}$, $z=(\boldsymbol{x},\boldsymbol{y})$. Relations (3.16) define average square price $p(2;t,z)$ at moment t of all transactions between agents in small volumes near points $\boldsymbol{x}$ and $\boldsymbol{y}$, averaged during interval Δ. Relations (3.19) introduce average squared price $p(2;t)$ of all trades in the economy averaged during time interval Δ. Relations (A.1-A.4) in Appendix A allow introduce average square price $p_s(2;t,\boldsymbol{x})$ for all sell-trades from point $\boldsymbol{x}$ and average square price $p_b(2;t,\boldsymbol{y})$ for all buy-trades at point $\boldsymbol{y}$ at moment t averaged during time Δ as:

$$C_s(2;t,\boldsymbol{x}) = p_s(2;t,\boldsymbol{x})U_s(2;t,\boldsymbol{x}) \qquad (3.37)$$

$$C_b(2;t,\boldsymbol{y}) = p_b(2;t,\boldsymbol{y})U_b(2;t,\boldsymbol{y}) \qquad (3.38)$$

Relations (3.6, 3.9, 3.16, 3.19, 3.37, 3.38) describe different forms of price and can be derived without long expressions for market trades. Key contribution of our approach to macroeconomic modeling: evolution of market trades depends on trades flows. Description of price must take into account trades flows – these new economic factors impact price dynamics. Below we derive equations that govern the second-degree trades and their flows.

4. Collective expectations

It is generally considered that agents take trade decisions under press of their expectations. Expectations play the crucial role for market trades performance. Decades of research describe impact of expectations on financial markets, price dynamics and economic evolution (Muth, 1961; Lucas, 1972; Blume and Easley, 1984; Brunnermeier and Parker, 2005; Dominitz and Greenwood and Shleifer, 2014). Observations and measurements of expectations is a difficult problem (Manski, 2004; Dominitz and Manski, 2005; Janžek and Ziherl, 2013; Manski, 2017). A lot of efforts are spent to observe, identify, estimate and measure the expectations. These problems are very complex. We suggest simplify the description of expectations and their impact on performance of market trades. To do that we regard expectations as agents assumptions and forecasts of numerous economic and financial variables, price and inflation trends, market trade activity, expectations of other agents, forecasts of any factors that can impact economy and etc. We assume that agents take decisions on market trades under action of multiple expectations. Each agent may have multiple expectations and agent can perform each market trade under different expectations. Below we model action of multiple expectations of agents on market trades.

It seems that our assumption on multiplicity of expectations increases the complexity of the expectation modeling: the difficulties of identification and measurement of expectations are increased by huge amount of different expectations of each agent. To overcome that we aggregate expectations of different agents alike to aggregation of market trades (3.11-3.15) performed by agents with risk coordinates inside the unit volume $dV(\boldsymbol{z})$ (3.10) in economic domain (3.4, 3.5). We rough the description of expectations of separate agents and introduce smooth collective expectations. This approach to expectations modeling is similar to description of macro variables and trades developed in (Olkhov, 2016-2019) via continuous economic approximation.

Multiplicity of expectations arises the problem of their comparative effect on agent's trades. Agent's expectations differ from agent's economic and financial variables. Agents variables like investment, assets, consumption – are additive. Sum of assets of group of agents (without overlapping) define assets of the group of agents. But expectations are not additive. Sum of expectations have no economic sense. To collect agents expectations those approve the second-degree trades (3.7) one must take into account the economic value of the trades approved by these expectations. Economic impact of the expectation that approve 1 bln.$ deal must be a little more valuable than impact of the expectation that approve 1$ deal. One can't simply sum agent's expectations to derive collective expectations of numerous agents. We state that market value of the collective expectation with regard to definite second-degree trade should be proportional to the second-degree trades (3.7) made under selected expectations. We assume that agents take decisions on squares of volume U^2 and cost C^2 of the trade under different expectations. Thus expectation that approve U^2 of the second-degree trade (3.7) should be weighted by square volume U^2. Expectations those approve the C^2 of the second-degree trade (3.7) must be weighted by square cost C^2. We refer to (Olkhov, 2019b) for details and apply this method for description of collective expectations of the second-degree trades $\boldsymbol{b}_{ij}(2;t,z)$ (3.7). Let's regard description of collective expectations in a more formal manner.

Let's assume that agents in economy have $j=1,\ldots K$ different expectations. Each agent involved into market trade $\boldsymbol{b}_{ij}(2;t,z)$ (3.7) between agents i and j should take decisions on volume U_{ij} and cost C_{ij}. Thus any trade is performed under action of four expectation – two expectations of agent i on volume U_{ij} and cost C_{ij} and two expectations of agent j. Let's propose that agent i take decision on volume U_{ij} under expectation $ex_{Ui}(k;t,\boldsymbol{x})$ of type k, and cost C_{ij} under expectation $ex_{Ci}(l;t,\boldsymbol{x})$ of type l , k, $l=1,..K$. We propose that expectations $ex_{Ui}(k;t,\boldsymbol{x})$ that approve the volume $U_{ij}(k;t,\boldsymbol{z})$ of trade may depend on expectations $ex_{Ci}(l;t,\boldsymbol{x})$ that approve the cost $C_{ij}(l;t,\boldsymbol{z})$ and vice versa. Hence we propose that the volume U_{ij} and the cost C_{ij} of the trade performed by the seller can depend on both expectations k and l:

$ex_{Ui}(k,l;t,\boldsymbol{x})$ and $ex_{Ci}(k,l;t,\boldsymbol{x})$. Let's call as second-degree trade $\boldsymbol{b}_{ij}(2;\boldsymbol{k};t,z)$ the pair (U_{ij}^{2}, C_{ij}^{2}) of squares of volume and cost made under sellers expectations of type $\boldsymbol{k}=(k,l)$ from seller i at point $\boldsymbol{x}$ with buyer j at point $\boldsymbol{y}$ as:

$$\boldsymbol{bs}_{ij}(2;\boldsymbol{k};t,\boldsymbol{z}) = \left(U_{ij}^{2}(\boldsymbol{k};t,\boldsymbol{z});C_{ij}^{2}(\boldsymbol{k};t,\boldsymbol{z})\right)\;;\boldsymbol{k}=(k,l)\;;\;k,l=1,\ldots K;\;\boldsymbol{z}=(\boldsymbol{x},\boldsymbol{y}) \quad (4.1)$$

and expectations $\boldsymbol{ex}(\boldsymbol{k};t,\boldsymbol{x})$ those approve trades (4.1) :

$$\boldsymbol{ex}(\boldsymbol{k};t,\boldsymbol{x}) = \left(ex_{Ui}(\boldsymbol{k};t,\boldsymbol{x});ex_{Ci}(\boldsymbol{k};t,\boldsymbol{x})\right) \quad (4.2)$$

We propose that all expectations $\boldsymbol{ex}(\boldsymbol{k};t,\boldsymbol{x})$ have same measure and sum of two expectations has meaning of expectation. Thus we propose that one can sum different expectations and derive sum of expectations weighted by volume or cost. Relations (4.1, 4.2) define second-degree trades and expectations of separate agent i as a seller at point x. Similar relations define second-degree trades $\boldsymbol{b}_{ij}(2;t,z;\boldsymbol{l})$ and expectations $\boldsymbol{ex}(t,\boldsymbol{y};\boldsymbol{l})$ of buyer at point $\boldsymbol{y}$.

$$\boldsymbol{bs}_{ij}(2;t,\boldsymbol{z};\boldsymbol{l}) = \left(U_{ij}^{2}(t,\boldsymbol{z};\boldsymbol{l});C_{ij}^{2}(t,\boldsymbol{z};\boldsymbol{l})\right)\;;\boldsymbol{l}=(k,l)\;;\;k,l=1,\ldots K;\;\boldsymbol{z}=(\boldsymbol{x},\boldsymbol{y}) \quad (4.3)$$

and expectations $\boldsymbol{ex}(t,\boldsymbol{y};\boldsymbol{l})$ those approve transactions (4.2) :

$$\boldsymbol{ex}(t,\boldsymbol{y};\boldsymbol{l}) = \left(ex_{Ui}(t,\boldsymbol{y};\boldsymbol{l});ex_{Ci}(t,\boldsymbol{y};\boldsymbol{l})\right)\;;\boldsymbol{l}=(k,l)\;;\;k,l=1,\ldots K \quad (4.4)$$

To derive collected expectations of agents with coordinates inside (3.10) we define new additive factor - expected trades. We weight sellers $\boldsymbol{ex}(\boldsymbol{k};t,\boldsymbol{x})$ (4.2) and buyers $\boldsymbol{ex}(t,\boldsymbol{y},\boldsymbol{l})$ (4.4) expectations by squares of volumes and squares of cost of trades and define the second-degree sellers expected trades $\boldsymbol{et}_{ij}(\boldsymbol{k};t,z)$ as:

$$\boldsymbol{et}_{ij}(\boldsymbol{k};t,\boldsymbol{z}) = \left(et_{Uij}(\boldsymbol{k};t,\boldsymbol{z})\;;\;et_{Cij}(\boldsymbol{k};t,\boldsymbol{z})\right) \quad (4.5)$$

$$et_{Uij}(\boldsymbol{k};t,\boldsymbol{z}) = ex_{Ui}(\boldsymbol{k};t,\boldsymbol{x})U_{ij}^{2}(\boldsymbol{k};t,\boldsymbol{z}) \quad (4.6)$$

$$et_{Cij}(\boldsymbol{k};t,\boldsymbol{z}) = ex_{Cj}(\boldsymbol{k};t,\boldsymbol{y})C_{ij}^{2}(\boldsymbol{k};t,\boldsymbol{z}) \quad (4.7)$$

The similar relations define the second-degree buyers expected trades $\boldsymbol{et}_{ij}(t,z;\boldsymbol{l})$. The second-degree sellers expected trades of seller $\boldsymbol{et}_{ij}(\boldsymbol{k};t,z;\boldsymbol{l})$ and buyer $\boldsymbol{et}_{ij}(t,z;\boldsymbol{l})$ are additive functions and we use the procedure similar to (3.11-3.15) and (3.27-3.36) to define aggregate second-degree expected trades and their flows (Appendix B). We define collective sellers $Ex(\boldsymbol{k};t,z)$ (B.12-B14) and buyers $Ex(t,z;\boldsymbol{l})$ (B.15-B.17) expectations. Integral by economic domain (3.4, 3.5) introduce collective sellers $Ex(\boldsymbol{k};t)$ (B.20, B.210) and buyers $Ex(t;\boldsymbol{l})$ (B.22, B.23) expectations as function of time t and type of expectation (4.3, 4.4). Below we use these expectations to model oscillations of price $p(2;t)$ (2.6).

5. Price and volatility oscillations

In Appendix C we derive the system of equations on the second-degree trades $\boldsymbol{B}(2;t,z)$ (3.11-3.15) and their flows $\boldsymbol{P}(2;t,z)$ (3.28-3.37) and on expected trades according to (Olkhov, 2019b). The system of equation (C.1-C.5) in economic domain (3.4, 3.5) and aggregated form of the equations (C.6-C.9) that describe collective trades of the entire economy as functions of type of expectations $\boldsymbol{k}$ and time t only are very complex. We present the simplest consequence of the equations (C.5-C.9) to demonstrate possible origin of price fluctuations and present equations that describe price and volatility evolution.

In Olkhov (2019b) we show how simple equations on first-degree trades imply equations on price $p(1,t)$ (2.7). Here we derive similar equations on the price $p(2,t)$ (2.6) for $n=2$. To do that we neglect the action of flows in (C.6, C.7) and take equations on trades as:

$$\frac{d}{dt}U(2,\boldsymbol{k};t) = F_U(\boldsymbol{k};t) \ \ ; \ \frac{d}{dt}C(2,\boldsymbol{k};t) = F_C(\boldsymbol{k};t) \tag{5.1}$$

Let's remind the form of second-degree price $p(2,\boldsymbol{k};t)$ as

$$C(2,\boldsymbol{k};t) = p(2,\boldsymbol{k};t)U(2,\boldsymbol{k};t) \tag{5.2}$$

Equations (5.1, 5.2) on $U(2;\boldsymbol{k},t)$ and $C(2;\boldsymbol{k},t)$ allow present equations on $p(2;\boldsymbol{k},t)$ as:

$$U(2,\boldsymbol{k};t)\frac{d}{dt}p(2,\boldsymbol{k};t) + p(2,\boldsymbol{k};t)F_U(\boldsymbol{k};t) = F_C(\boldsymbol{k};t) \tag{5.3}$$

Equations (5.1, 5.3) show that dynamics of price $p(2,\boldsymbol{k};t)$ or similar equations (5.4) for price $p(2;t)$ and volume $U(2;t)$ (3.18, 3.19):

$$\frac{d}{dt}U(2;t) = F_U(t) \ \ ; \ \ U(2;t)\frac{d}{dt}p(2;t) + p(2;t)F_U(t) = F_C(t) \tag{5.4}$$

are determined by functions F_U, F_C. As we discussed above, F_U, F_C can depend on expectations, expected transactions and their flows. Even in the simplest case if one neglects impact of transactions flows P_U, P_C (A.7-A.10) and flows of expected transactions (B.28, B.29) and their equations (C.3, C.5) the equations (5.3, 5.4) on price $p(2,\boldsymbol{k};t)$ or price $p(2;t)$ depend on functions F_U, F_C that model dynamics of collective the second-degree trades. Equations on $p(2,\boldsymbol{k};t)$ or $p(2;t)$ and similar equations on $p(1;t)$ (Olkhov, 2019) describe evolution of price volatility $\sigma_p^{\ 2}$ (2.8) determined by first and second-degree trades.

Equations (5.1) allow describe fluctuations of the price $p(2,\boldsymbol{k};t)$. To do that assume that in linear approximation by perturbations functions F_U and F_C in (5.1) depend on perturbations of collective expected trades $Et_U(\boldsymbol{k};t)$ and $Et_C(\boldsymbol{k};t)$ (B.19) as (D.9). Let's take equations on expected trades $Et_U(\boldsymbol{k};t)$ and $Et_C(\boldsymbol{k};t)$ as (C.8) :

$$\frac{d}{dt}Et_U(\boldsymbol{k};t) = W_U(\boldsymbol{k};t) \ \ ; \quad \frac{d}{dt}Et_C(\boldsymbol{k};t) = W_C(\boldsymbol{k};t) \tag{5.5}$$

and assume that in linear approximation by perturbations functions W_U and W_C in (5.5) depend on perturbations of volume $U(2,\boldsymbol{k};t)$ and cost $C(2,\boldsymbol{k};t)$ as (D.10). Assumptions (5.1, 5.5) describe small oscillations of collective transactions $U(2,k;t)$ and $C(2,k;t)$ and hence describe fluctuations of price $p(2,\boldsymbol{k};t)$ (5.2). We give model of fluctuations in Appendix D.

Relations (D.4) express mean square price *p(2;t)* (3.19) disturbances though disturbances of squares volumes *u(2,**k**;t)* (D.1) and squares costs *c(2,**k**;t)* (D.1). Relations (D.7, D.8) describe *p(2;t)* (3.19) dependence on partial price disturbances *π(2,**k**;t)* (D.6) and volume disturbances *u(2,**k**;t)* (D.1). Disturbances of *p(2;t)* (D.4, D.8) and disturbances of *p(1;t)* (D.26) describe price volatility $\sigma_p^{\,2}$ (2.8) in linear approximation by disturbances (D.27):

$$\sigma_p^2(t) = \sigma_{p0}^2\{1 + \sum_{k,l=1}^{K} \sigma_2[\mu_{2\boldsymbol{k}}c(2,\boldsymbol{k};t) - \lambda_{2k}u(2,\boldsymbol{k};t)] - \sigma_1[\mu_{1\boldsymbol{k}}c(1,\boldsymbol{k};t) - \lambda_{1k}u(1,\boldsymbol{k};t)]\}$$

We leave further development of these problems for future.

6. Conclusion

Macroeconomics describes mutual dependence of macro variables and market trades aggregated during time interval *Δ*. All macro variables and trades fluctuate. To describe volatility of these fluctuations one should describe the second-degree variables and trades aggregated during interval *Δ*. Methods for aggregating variables during interval *Δ* must have economic sense. We call model of the second-degree variables and trades as the second-order economic theory. Development of the second-order economic theory is a though problem and in this paper we only start study price volatility and the second-degree trades.

We take VWAP as the only example of the price aggregation procedure with economic sense. VWAP defines mean price *p(1;t)* as ratio of sum of costs and sum of volumes of trades performed during interval *Δ*. We generalize the VWAP procedure and use the similar approach to define mean *n-th* degree price *p(n;t)* (2.5, 2.6) and price statistical moments $p(n;t_1,..t_n)$ (2.10, 2.11). To define price probability one should model all price statistical moments. Thus price probability depends on description of collective *n-th* degree products of trades (2.5). It seems almost impossible develop description of *n-th* degree trades, economic variables and expectations for all *n=1,2,…* . Hence the bright dreams to find out the price probability distribution fortunately or unfortunately unattainable.

The *n-th* degree price *p(n;t)* (2.5, 2.6) and price statistical moments $p(n;t_1,..t_n)$ (2.10, 2.11) depend on the time interval *Δ*. For time terms *T>>Δ* the *n-th* degree price *p(n;t)* (2.5, 2.6) and statistical moments can behave irregular. One can recursively perform transitions from one averaging time interval *Δ* to another averaging interval $T_1 > \Delta$ and further from T_1 *to* $T_2 > T_1$. These transitions imply change of the *n-th* degree price *p(n;t)* (2.5, 2.6) and price statistical moments $p(n;t_1,..t_n)$ (2.10, 2.11) determined by averaging time interval T_1, T_2 and etc. Each change of averaging interval cause change of *n-th* degree price *p(n;t;T)* according to (2.15, 2.16) and corresponding change of characteristic functional (2.12). Relations between *n-th* degree prices *p(n;t;T)* and characteristic functionals (2.12) for averaging intervals T_1, T_2... should be studied further.

We underline that for each *n* price *p(n;t)* (2.6) caries information about the *n-th* price degree only. In simple words: mean price *p(1;t)* (2.5, 2.6) for *n=1* carries information about the mean price only. Function *p(1;t)* can behave irregular, "like random" function at time *T>Δ* but nevertheless averaging procedure over time *T> Δ* give the mean price for time interval *T* only. To assess mean square price *p(2;t)* or mean *n-th* degree price *p(n;t)* for time interval *T>Δ* one must use *p(2;t)* or *p(n;t).* Usage of irregular "random" behavior of *p(1;t)* during time *T>Δ* to assess mean square price, price volatility and etc., has no economic sense.

Price volatility is determined by first two statistical moments *n=1,2* (2.6) and depends on sums of squares of volumes and costs of trades. All sums are taken during interval *Δ* and the choice of interval *Δ* determines properties of second-degree trades and price volatility.

To describe skewness of price probability one should model *3-rd* statistical moments. Price skewness depends on description on *3-rd* degree trades and development of the third-order macro theory – theory that takes into account the *3-rd* degree economic and financial variables and *3-rd* degree trades aggregated during interval *Δ.*

We begin our study of price volatility with model of the second-degree trades. To do that we introduce numerical continuous risk grades and suggest that modern econometric data provide sufficient info for risk assessments of almost all economic agents. 80 years ago Durand, (1941) and 60 years ago Myers and Forgy, (1963) already introduced numerical risk grades. We regard usage letters risk grades as a way to protect the business of major rating agencies – S&P, Moody's, Fitch. However we are sure that development and usage of unified numerical continuous rating methodology should boost market value of major rating companies by several times. Introduction of numerical continuous risk grades enlarge the meaning of risk transition matrices. Transition matrices could describe not only the probability to change risk rating from current state to another one during certain time term, but also determine mean velocities of economic agents in the risk economic space. Exact description of numerous separate agents and their economic and financial variables is too difficult problem. We rough accuracy of the description and aggregate variables and trades of agents with coordinates inside small volume *dV* in economic space and average during time interval *Δ*. This replace description of economic variables and trades those belong to separate agents by description of economic variables and trades as functions of coordinates in economic space. Motion of agents in the risk economic space induces collective motion of aggregated economic variables and trades. We show that agent's motion cause flows of economic variables and flows of trades. Flows of trades and variables are new issues in economic theory. We develop continuous economic media approximation that describes densities of economic variables, trades and their flows in economic space as functions of time and coordinates. Aggregation over economic space describes economic variables and trades as functions of time only.

To describe the second-degree trades we define trades performed by separate agents and collective trades and their flows performed by agents in small volume *dV* as functions of coordinates in economic space. We follow the general economic consensus that agents make trades under action of their expectations. We propose that for the model of second-degree trades economic "weight" of agents expectations should be proportional to squares of volumes and costs of trades performed under expectations. We define aggregated expectations and derive equations that govern macro expectations and their flows.

Introduction of macro second-degree trades defines mean square price as function of risk coordinates. Aggregation of trades over economic space defines mean square price as function of time only.

For the simple model case in the linear approximation by perturbations we derive equations on perturbations of the second-degree trades and expectations. We derive mean square price as function of perturbations of the second-degree costs and volumes of trades. Simples assumptions allow describe the second-degree trade perturbations as set of harmonic oscillations induced by perturbations of multiple expectations. We describe similar perturbations of the VWAP mean price *p(1;t)* in (Olkhov, 2019b). Both results give price volatility perturbations through perturbations of first and second-degree trades (D.27). These expressions demonstrate dependence of price volatility on trade volume – the problem studied for a long time on base of market econometric data.

We hope that further development of the second-order economic theory may bring interesting economic problems and useful financial solutions.

Appendix A. Macro trades and their flows

Integral of the second-degree trade $\boldsymbol{B}(2;t,z)$ (3.11-3.15) by $d\boldsymbol{y}$ over economic domain (3.2) determines all sell-trades $\boldsymbol{B}_s(2;t,\boldsymbol{x})$ from point $\boldsymbol{x}$:

$$\boldsymbol{B}_s(2;t,\boldsymbol{x}) = \int d\boldsymbol{y}\ \boldsymbol{B}(2;t,\boldsymbol{x},\boldsymbol{y}) = \big(U_s(2;t,\boldsymbol{x}); C_s(2;t,\boldsymbol{x})\big) \tag{A.1}$$

$$U_s(2;t,\boldsymbol{x}) = \int d\boldsymbol{y}\ U(2;t,\boldsymbol{x},\boldsymbol{y})\ ;\ C_s(2;t,\boldsymbol{x}) = \int d\boldsymbol{y}\ C(2;t,\boldsymbol{x},\boldsymbol{y}) \tag{A.2}$$

Integral $\boldsymbol{B}(2;t,z)$ by $d\boldsymbol{x}$ over economic domain (3.2) determines all buy-trades $\boldsymbol{B}_b(2;t,\boldsymbol{y})$ at $\boldsymbol{y}$

$$\boldsymbol{B}_b(2;t,\boldsymbol{y}) = \int d\boldsymbol{x}\ \boldsymbol{B}(2;t,\boldsymbol{x},\boldsymbol{y}) = \big(U_b(2;t,\boldsymbol{y}); C_b(2;t,\boldsymbol{y})\big) \tag{A.3}$$

$$U_b(2;t,\boldsymbol{y}) = \int d\boldsymbol{x}\ U(2;t,\boldsymbol{x},\boldsymbol{y})\ ;\ C_b(2;t,\boldsymbol{y}) = \int d\boldsymbol{x}\ C(2;t,\boldsymbol{x},\boldsymbol{y}) \tag{A.4}$$

Integral of the second-degree trade *B(2;t,z)* by $dz=d\boldsymbol{x}d\boldsymbol{y}$ over economic domain (3.2) define all macro second-degree trades $\boldsymbol{B}(2;t)$ as function of time *t* only:

$$\boldsymbol{B}(2;t) = \int d\boldsymbol{z}\ \boldsymbol{B}(2;t,\boldsymbol{z}) = \big(U(2;t); C(2;t)\big) \tag{A.5}$$

$$U(2;t) = \int d\boldsymbol{z}\ U(2;t,\boldsymbol{z})\ ;\ C(2;t) = \int d\boldsymbol{z}\ C(2;t,\boldsymbol{z}) \tag{A.6}$$

Relations (A.1-A.6) introduce flows and velocities for corresponding second-degree trades. Due to (3.31-3.32) obtain:

$$\int d\boldsymbol{y}\ U(2;t,\boldsymbol{z})\,\boldsymbol{v}_{Ux}(2;t,\boldsymbol{z})$$

$$\boldsymbol{P}_{Ux}(2;t,\boldsymbol{x}) = \int d\boldsymbol{y}\,\boldsymbol{P}_{Ux}(2;t,\boldsymbol{x},\boldsymbol{y}) = \textstyle\sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta}\int d\boldsymbol{y}\,U_{ij}^2(t,\boldsymbol{x},\boldsymbol{y})\boldsymbol{v}_i(t,\boldsymbol{x}) \tag{A.7}$$

$$\boldsymbol{P}_{Ux}(2;t,\boldsymbol{x}) = \int d\boldsymbol{y}\ U(2;t,\boldsymbol{z})\,\boldsymbol{v}_{Ux}(2;t,\boldsymbol{z}) = U_s(2;t,\boldsymbol{x})\boldsymbol{v}_{Usx}(2;t,\boldsymbol{x}) \tag{A.8}$$

$$\boldsymbol{P}_{Uy}(2;t,\boldsymbol{x}) = \int d\boldsymbol{y}\,\boldsymbol{P}_{Uy}(2;t,\boldsymbol{x},\boldsymbol{y}) = \textstyle\sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta}\int d\boldsymbol{y}\,U_{ij}^2(t,\boldsymbol{x},\boldsymbol{y})\boldsymbol{v}_j(t,\boldsymbol{y}) \tag{A.9}$$

$$\boldsymbol{P}_{Uy}(2;t,\boldsymbol{x}) = \int d\boldsymbol{y}\ U(2;t,\boldsymbol{z})\,\boldsymbol{v}_{Uy}(2;t,\boldsymbol{z}) = U_s(2;t,\boldsymbol{x})\boldsymbol{v}_{Uby}(2;t,\boldsymbol{x}) \tag{A.10}$$

Relations (A.7, A.8) introduce aggregate flow of sales $\boldsymbol{P}_{Ux}(2;t,\boldsymbol{x})$ of second-degree volumes $U_s(2;t,\boldsymbol{x})$ and collective velocity $\boldsymbol{v}_{Usx}(2;t,\boldsymbol{x})$ of sellers at point $\boldsymbol{x}$ along axis X to all buyers in economy. Relations (A.9, A.10) introduce aggregate flow of sales $\boldsymbol{P}_{Uy}(2;t,\boldsymbol{x})$ of second-degree volumes $U_s(2;t,\boldsymbol{x})$ and collective velocity $\boldsymbol{v}_{Usy}(2;t,\boldsymbol{x})$ of all buyers in economy along axis Y from collective sellers at point $\boldsymbol{x}$. Corresponding relations for the second-degree cost flows $\boldsymbol{P}_{Csx}(2;t,\boldsymbol{x})$ and $\boldsymbol{P}_{Csy}(2;t,\boldsymbol{x})$ take form:

$$\boldsymbol{P}_{Csx}(2;t,\boldsymbol{x}) = \int d\boldsymbol{y}\,\boldsymbol{P}_{Cx}(2;t,\boldsymbol{x},\boldsymbol{y}) = \textstyle\sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta}\int d\boldsymbol{y}\,C_{ij}^2(t,\boldsymbol{x},\boldsymbol{y})\boldsymbol{v}_i(t,\boldsymbol{x}) \tag{A.11}$$

$$\boldsymbol{P}_{Csx}(2;t,\boldsymbol{x}) = C_s(2;t,\boldsymbol{x})\boldsymbol{v}_{Csx}(2;t,\boldsymbol{x}) \tag{A.12}$$

$$\boldsymbol{P}_{Csy}(2;t,\boldsymbol{x}) = \int d\boldsymbol{y}\,\boldsymbol{P}_{Cy}(2;t,\boldsymbol{x},\boldsymbol{y}) = \textstyle\sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta}\int d\boldsymbol{y}\,C_{ij}^2(t,\boldsymbol{x},\boldsymbol{y})\boldsymbol{v}_j(t,\boldsymbol{y}) \tag{A.13}$$

$$\boldsymbol{P}_{Csy}(2;t,\boldsymbol{x}) = C_s(2;t,\boldsymbol{x})\boldsymbol{v}_{Csy}(2;t,\boldsymbol{x}) \tag{A.14}$$

The similar relations define the buyer's flows. To derive buyer's the second-degree volume and cost flows $\boldsymbol{P}_{Ubx}(2;t,\boldsymbol{y})$ and $\boldsymbol{P}_{Uby}(2;t,\boldsymbol{y})$, $\boldsymbol{P}_{Cbx}(2;t,\boldsymbol{y})$ and $\boldsymbol{P}_{Cby}(2;t,\boldsymbol{y})$ one should take integral for (3.32-3.35) by $d\boldsymbol{x}$ over economic domain (3.4, 3.5). For brevity we omit them here.

Integrals by ***dxdy*** for (3.31-3.35) over entire economy – over economic domain (3.4,3.5) introduce macro flows of second-degree trades as functions of time *t* only:

$$\boldsymbol{P}_{Ux}(2;t) = \int d\boldsymbol{z}\,\boldsymbol{P}_{Ux}(2;t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} \int d\boldsymbol{x}d\boldsymbol{y}\, U_{ij}^2(t,\boldsymbol{x},\boldsymbol{y})\boldsymbol{v}_i(t,\boldsymbol{x}) \quad (A.15)$$

$$\boldsymbol{P}_{Ux}(2;t) = U(2;t)\boldsymbol{v}_{Ux}(2;t) \quad (A.16)$$

$$\boldsymbol{P}_{Uy}(2;t) = \int d\boldsymbol{z}\,\boldsymbol{P}_{Uy}(2;t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} \int d\boldsymbol{x}d\boldsymbol{y}\, U_{ij}^2(t,\boldsymbol{x},\boldsymbol{y})\boldsymbol{v}_j(t,\boldsymbol{y}) \quad (A.17)$$

$$\boldsymbol{P}_{Uy}(2;t) = U(2;t)\boldsymbol{v}_{Uy}(2;t) \quad (A.18)$$

$$\boldsymbol{P}_{Cx}(2;t) = \int d\boldsymbol{z}\,\boldsymbol{P}_{Cx}(2;t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} \int d\boldsymbol{x}d\boldsymbol{y}\, C_{ij}^2(t,\boldsymbol{x},\boldsymbol{y})\boldsymbol{v}_i(t,\boldsymbol{x}) \quad (A.19)$$

$$\boldsymbol{P}_{Cx}(2;t) = C(2;t)\boldsymbol{v}_{Cx}(2;t) \quad (A.20)$$

$$\boldsymbol{P}_{Cy}(2;t) = \int d\boldsymbol{z}\,\boldsymbol{P}_{Cy}(2;t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} \int d\boldsymbol{x}d\boldsymbol{y}\, C_{ij}^2(t,\boldsymbol{x},\boldsymbol{y})\boldsymbol{v}_j(t,\boldsymbol{y}) \quad (A.21)$$

$$\boldsymbol{P}_{Cy}(2;t) = C(2;t)\boldsymbol{v}_{Cy}(2;t) \quad (A.22)$$

Relations (A.15-A.22) introduce new economic concepts – collective flows of all agents of the economy involved into the second-degree trades in economic domain (3.4, 3.5). Collective flows of second-degree volumes ($\boldsymbol{P}_{Ux}(2;t)$; $\boldsymbol{P}_{Uy}(2;t)$), and flows of second-degree costs ($\boldsymbol{P}_{Cx}(2;t)$; $\boldsymbol{P}_{Cy}(2;t)$) define corresponding collective velocities of volume ($\boldsymbol{v}_{Ux}$, $\boldsymbol{v}_{Uy}$) and cost ($\boldsymbol{v}_{Cx}$, $\boldsymbol{v}_{Cy}$). Velocities $\boldsymbol{v}_{Ux}$ define collective motion of all sellers in economy weighted by volumes of trades during interval *Δ*. Velocities $\boldsymbol{v}_{Uy}$ define collective motion of all buyers in economy weighted by volumes of trades during interval *Δ*. Velocities ($\boldsymbol{v}_{Cx}$, $\boldsymbol{v}_{Cy}$) define collective motion of all sellers and all buyers weighted by cost of all trades during interval *Δ*. These velocities describe collective motion of the entire economy in the economic domain (3.4, 3.5) reduced by its borders at 0 and 1 by all axes. Hence these collective velocities can't have the constant direction and must oscillate. Collective velocities of the volume ($\boldsymbol{v}_{Ux}$, $\boldsymbol{v}_{Uy}$) and the cost ($\boldsymbol{v}_{Cx}$, $\boldsymbol{v}_{Cy}$) occasionally change their direction and describe collective motion of sellers and buyers from the safe area of economic domain to the risky one and back. We suggest that such fluctuations of the second-degree trades accompanied with rise and fall of market activity, production functions, demand and supply, price variations and etc., establish ground for economic fluctuations alike to business cycles. We describe motion of collective trades in the economic domain (3.4, 3.5) as hidden properties of the business cycles (Olkhov, 2017b, 2019a)

Appendix B. Collective expectations

To define collective expectations we introduce new issue: collective expected trades. To do that we use (4.1) to define collective second-degree trades $U(2,\boldsymbol{k};t,z)$, $C(2,\boldsymbol{k};t,z)$ approved by sellers expectations of type $\boldsymbol{k}=(k,l)$ as:

$$U(2,\boldsymbol{k};t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} U_{ij}^{2}(\boldsymbol{k};t,\boldsymbol{z}) \tag{B.1}$$

$$C(2,\boldsymbol{k};t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} C_{ij}^{2}(\boldsymbol{k};t,\boldsymbol{z}) \tag{B.2}$$

Definitions of collective flows $\boldsymbol{P}_U(2,\boldsymbol{k};t,z)$ and $P_C(2,\boldsymbol{k};t,z)$ of trades with volume $U(2,\boldsymbol{k};t,z)$ and cost $C(2,\boldsymbol{k};t,z)$ (B.1,B.2) are similar to (A.7-A.10) and we omit it here for brevity. Trades $U(2,\boldsymbol{k};t,z)$, $C(2,\boldsymbol{k};t,z)$ (B.1, B.2) obey obvious relations (3.13-3.15):

$$U(2;t,\boldsymbol{z}) = \sum_{k,l=1}^{K} U(2,\boldsymbol{k};t,\boldsymbol{z}) \;\; ; \;\; C(2;t,\boldsymbol{z}) = \sum_{k,l=1}^{K} C(2,\boldsymbol{k};t,\boldsymbol{z}) \tag{B.3}$$

Collective buyers second-degree trades $U(2;t,z;\boldsymbol{l})$, $C(2;t,z;\boldsymbol{l})$ approved by buyers expectations take the similar form:

$$U(2;t,\boldsymbol{z};\boldsymbol{l}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} U_{ij}^{2}(t,\boldsymbol{z};\boldsymbol{l}) \tag{B.4}$$

$$C(2;t,\boldsymbol{z};\boldsymbol{l}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} C_{ij}^{2}(t,\boldsymbol{z};\boldsymbol{l}) \tag{B.5}$$

Buyer's trades obey same relations (B.3). Similar to (3.11-3.15) we use (4.5-4.7) and define sellers collective expected trades $\boldsymbol{Et}(\boldsymbol{k};t,z)$ as:

$$\boldsymbol{Et}(k;t,\boldsymbol{z}) = \left(Et_U(\boldsymbol{k};t,\boldsymbol{z}); Et_C(\boldsymbol{k};t,\boldsymbol{z})\right) \tag{B.6}$$

$$Et_U(\boldsymbol{k};t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta} ex_{Ui}(\boldsymbol{k};t,\boldsymbol{x}) U_{ij}^{2}(\boldsymbol{k};t,\boldsymbol{z}) \tag{B.7}$$

$$Et_C(\boldsymbol{k};t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta} ex_{Ci}(\boldsymbol{k};t,\boldsymbol{x}) C_{ij}^{2}(\boldsymbol{k};t,\boldsymbol{z}) \tag{B.8}$$

Buyer's collective second-degree expected trades $\boldsymbol{Et}(t,z;\boldsymbol{l})$,

$$\boldsymbol{Et}(t,\boldsymbol{z};\boldsymbol{l}) = \left(Et_U(t,\boldsymbol{z};\boldsymbol{l}); Et_C(t,\boldsymbol{z};\boldsymbol{l})\right) \tag{B.9}$$

$$Et_U(t,\boldsymbol{z};\boldsymbol{l}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta} ex_{Uj}(t,\boldsymbol{y};\boldsymbol{l}) U_{ij}^{2}(t,\boldsymbol{z};\boldsymbol{l}) \tag{B.10}$$

$$Et_C(t,\boldsymbol{z};\boldsymbol{l}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y});\,\Delta} ex_{Cj}(t,\boldsymbol{y};\boldsymbol{l}) C_{ij}^{2}(t,\boldsymbol{z};\boldsymbol{l}) \tag{B.11}$$

Relations (B.6-B.8) and (B.1,B.2) introduce collective sellers expectations of the second-degree $\boldsymbol{Ex}(\boldsymbol{k};t,z)$, as:

$$Ex(\boldsymbol{k};t,\boldsymbol{z}) = \left(Ex_U(\boldsymbol{k};t,\boldsymbol{z}), Ex_C(\boldsymbol{k};t,\boldsymbol{z})\right) \tag{B.12}$$

$$Ex_U(\boldsymbol{k};t,\boldsymbol{z})U(2,\boldsymbol{k};t,\boldsymbol{z}) = Et_U(\boldsymbol{k};t,\boldsymbol{z}) \tag{B.13}$$

$$Ex_C(\boldsymbol{k};t,\boldsymbol{z})C(2,\boldsymbol{k};t,\boldsymbol{z}) = Et_C(\boldsymbol{k};t,\boldsymbol{z}) \tag{B.14}$$

Collective buyers expectations take similar form:

$$Ex(t,\boldsymbol{z};\boldsymbol{l}) = \left(Ex_U(t,\boldsymbol{z};\boldsymbol{l}), Ex_C(t,\boldsymbol{z};\boldsymbol{l})\right) \tag{B.15}$$

$$Ex_U(t,\boldsymbol{z};\boldsymbol{l})U(2;t,\boldsymbol{z};\boldsymbol{l}) = Et_U(t,\boldsymbol{z};\boldsymbol{l}) \tag{B.16}$$

$$Ex_C(t,\boldsymbol{z};\boldsymbol{l})C(2;t,\boldsymbol{z};\boldsymbol{l}) = Et_C(t,\boldsymbol{z};\boldsymbol{l}) \tag{B.17}$$

Let's take integrals over economic domain (3.4,3.5) of *U(2,k;t,z)*, *C(2,k;t,z)* (B.1, B.2):

$$U(2,\boldsymbol{k};t) = \int d\boldsymbol{z}\ U(2,\boldsymbol{k};t,\boldsymbol{z})\ ;\ \ C(2,\boldsymbol{k};t) = \int d\boldsymbol{z}\ C(2,\boldsymbol{k};t,\boldsymbol{z}) \quad \text{(B.18)}$$

Relations (B.18) define collective sellers second-degree trades *U(2,k;t)*, *C(2,k;t)* performed under expectations of type $\boldsymbol{k}=(k,l)$. Then integral by economic domain (3.4, 3.5) of expected trades (B.13, B.14) define collective sellers expected trades $Et_U(\boldsymbol{k};t)$, $Et_C(\boldsymbol{k};t)$ of the entire economy

$$Et_U(\boldsymbol{k};t) = \int d\boldsymbol{z}\ Et_U(\boldsymbol{k};t,\boldsymbol{z})\ ;\ \ Et_C(\boldsymbol{k};t) = \int d\boldsymbol{z}\ Et_C(\boldsymbol{k};t,\boldsymbol{z}) \quad \text{(B.19)}$$

(B.18, B.19) define collective sellers expectations $Ex_U(\boldsymbol{k};t)$, $Ex_C(\boldsymbol{k};t)$ of the entire economy

$$Ex_U(\boldsymbol{k};t)U(2,\boldsymbol{k};t) = Et_U(\boldsymbol{k};t) \quad \text{(B.20)}$$

$$Ex_C(\boldsymbol{k};t)C(2,\boldsymbol{k};t) = Et_C(\boldsymbol{k};t) \quad \text{(B.21)}$$

The similar relations define collective buyers expected trades $Et_U(t;\boldsymbol{l})$, $Et_C(t;\boldsymbol{l})$ and expectations $Ex_U(t;\boldsymbol{l})$, $Ex_C(t;\boldsymbol{l})$ of the entire economy:

$$Ex_U(t;\boldsymbol{l})U(2;t;\boldsymbol{l}) = Et_U(t;\boldsymbol{l}) \quad \text{(B.22)}$$

$$Ex_C(t;\boldsymbol{l})C(2;t;\boldsymbol{l}) = Et_C(t;\boldsymbol{l}) \quad \text{(B.23)}$$

Relations (B.20) introduce collective seller expected trades $Et_{Us}(t)$, $Et_{Cs}(t)$:

$$Et_{Us}(t) = \sum_{k,l=1}^{K} Et_U(\boldsymbol{k};t)\ \ ;\ \ Et_{Cs}(t) = \sum_{k,l=1}^{K} Et_C(\boldsymbol{k};t) \quad \text{(B.24)}$$

Relations (B.24) define collective sellers expectations $Ex_{Us}(t)$, $Ex_{Cs}(t)$:

$$Ex_{Us}(t)U(2;t)\ = Et_{Us}(t)\ \ ;\ \ Ex_{Cs}(t)C(2;t)\ = Et_{Cs}(t) \quad \text{(B.25)}$$

The similar relations define collective buyers expectations of the entire economy. We omit them for brevity. It is clear that collective sellers expectations those approve squares of volume U^2 and squares of cost C^2 of transactions may be different. Collective seller's expectations may be different from buyer's expectations. Such diversity uncovers hidden complexity of the mutual interactions between trades and expectations. We present above set of long definitions to outline multiplicity of different states of collective trades and expectations. One easy defines collective flows of sellers and buyers expected trades similar to (Olkhov, 2019b). For brevity we present only definition of sellers volume flow of expected trades $\boldsymbol{Pe}_{Ux}$, $\boldsymbol{Pe}_{Uy}$ in (B.26-B2.9):

$$\boldsymbol{pe}_{Uxij}(\boldsymbol{k};t,\boldsymbol{z}) = et_{Uij}(\boldsymbol{k};t,\boldsymbol{z})\boldsymbol{v}_i(t,\boldsymbol{x}) = ex_i(\boldsymbol{k};t,\boldsymbol{x})U_{ij}^2(\boldsymbol{k};t,\boldsymbol{z})\boldsymbol{v}_i(t,\boldsymbol{x}) \quad \text{(B.26)}$$

$$\boldsymbol{pe}_{Uyij}(\boldsymbol{k};t,\boldsymbol{z}) = et_{Uij}(\boldsymbol{k};t,\boldsymbol{z})\boldsymbol{v}_j(t,\boldsymbol{y}) = ex_i(\boldsymbol{k};t,\boldsymbol{x})U_{ij}^2(\boldsymbol{k};t,\boldsymbol{z})\boldsymbol{v}_j(t,\boldsymbol{y}) \quad \text{(B.27)}$$

$$\boldsymbol{Pe}_{Ux}(\boldsymbol{k};t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} et_{Uij}(\boldsymbol{k};t,\boldsymbol{z})\boldsymbol{v}_i(t,\boldsymbol{x}) = Et_U(\boldsymbol{k};t,\boldsymbol{z})\boldsymbol{ve}_{Ux}(\boldsymbol{k};t,\boldsymbol{z}) \quad \text{(B.28)}$$

$$\boldsymbol{Pe}_{Uy}(\boldsymbol{k};t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x});j\in dV(\boldsymbol{y})\,\Delta} et_{Uij}(\boldsymbol{k};t,\boldsymbol{z})\boldsymbol{v}_j(t,\boldsymbol{y}) = Et_U(\boldsymbol{k};t,\boldsymbol{z})\boldsymbol{ve}_{Uy}(\boldsymbol{k};t,\boldsymbol{z}) \quad \text{(B.29)}$$

For brevity we don't' define full set of flows and refer (Olkhov, 2019b) for further detail.

Appendix C. Equations on the second-degree trades

For brevity we present here only equations on second-degree trades ***B****(2;t,z)* (3.11-3.15) and refer for details (Olkhov, 2018-2019b). Equations on second-degree volume *U(2,**k**;t,z)* and cost *C(2,**k**;t,z)* (B.1,B.2) take form:

$$\frac{\partial}{\partial t}U(2,\boldsymbol{k};t,\boldsymbol{z})+\nabla\cdot\big(U(2,\boldsymbol{k};t,\boldsymbol{z})\,\boldsymbol{v}_U(2,\boldsymbol{k};t,\boldsymbol{z})\big)=F_U(\boldsymbol{k};t,\boldsymbol{z}) \quad (C.1)$$

$$\frac{\partial}{\partial t}C(2,\boldsymbol{k};t,\boldsymbol{z})+\nabla\cdot\big(C(2,\boldsymbol{k};t,\boldsymbol{z})\,\boldsymbol{v}_C(2,\boldsymbol{k};t,\boldsymbol{z})\big)=F_C(\boldsymbol{k};t,\boldsymbol{z}) \quad (C.2)$$

Equations on flows $\boldsymbol{P}_{Ux}(2;t,z)$ (3.32) of second-degree volume *U(2;t,z)* take form:

$$\frac{\partial}{\partial t}\boldsymbol{P}_{Ux}(\boldsymbol{k};t,\boldsymbol{z})+\nabla\cdot\big(\boldsymbol{P}_{Ux}(\boldsymbol{k};t,\boldsymbol{z})\,\boldsymbol{v}_{Ux}(\boldsymbol{k};t,\boldsymbol{z})\big)=\boldsymbol{G}_{Ux}(\boldsymbol{k};t,\boldsymbol{z}) \quad (C.3)$$

Equations on flows (3.33-3.35) have similar form. Similar equations valid for expected trades $Ex_U(\boldsymbol{k};t,z)$ (B.6-B.8) and their flows $\boldsymbol{Pe}_{Ux}(\boldsymbol{k};t,\boldsymbol{z})$:

$$\frac{\partial}{\partial t}Et_U(\boldsymbol{k};t,\boldsymbol{z})\ +\nabla\cdot\big(Et_U(\boldsymbol{k};t,\boldsymbol{z})\,\boldsymbol{ve}_{Ux}(\boldsymbol{k};t,\boldsymbol{z})\big)=W_U(\boldsymbol{k};t,\boldsymbol{z}) \quad (C.4)$$

$$\frac{\partial}{\partial t}\boldsymbol{Pe}_{Ux}(\boldsymbol{k};t,\boldsymbol{z})+\nabla\cdot\big(\boldsymbol{Pe}_{Ux}(\boldsymbol{k};t,\boldsymbol{z})\,\boldsymbol{ve}_{Ux}(\boldsymbol{k};t,\boldsymbol{z})\big)=\boldsymbol{R}_{Ux}(\boldsymbol{k};t,\boldsymbol{z}) \quad (C.5)$$

Integrals of divergence by *dz* over domain (3.4,3.5) equal zero as no economic agents and no fluxes exist outside (3.4,3.5). Hence integrals of (C.1-C.5) by *dz* over domain (3.4,3.5) give:

$$\int d\boldsymbol{z}\left[\frac{\partial}{\partial t}U(2,\boldsymbol{k};t,\boldsymbol{z})+\nabla\cdot\big(U(2,\boldsymbol{k};t,\boldsymbol{z})\,\boldsymbol{v}_U(2,\boldsymbol{k};t,\boldsymbol{z})\big)\right]=\frac{d}{dt}U(2,\boldsymbol{k};t)=F_U(\boldsymbol{k};t)=\int d\boldsymbol{z}\,F_U(t,\boldsymbol{z}) \quad (C.6)$$

$$\frac{d}{dt}C(2,\boldsymbol{k};t)=F_C(\boldsymbol{k};t) \quad ; \quad \frac{d}{dt}\boldsymbol{P}_{Ux}(\boldsymbol{k};t)=\boldsymbol{G}_{Ux}(\boldsymbol{k};t) \quad (C.7)$$

$$\frac{d}{dt}Et_U(\boldsymbol{k};t)=W_U(\boldsymbol{k};t) \quad ; \quad \frac{d}{dt}\boldsymbol{Pe}_{Ux}(\boldsymbol{k};t)=\boldsymbol{R}_{Ux}(\boldsymbol{k};t) \quad (C.8)$$

(C.6-C.8) describe equations on collective second-degree trades, expectations and their flows as functions of time *t* and type of expectations ***k***. Formal simplicity of (C.6-C.8) hides complexity of the functions $F_U(t)$, $F_U(t)$, $\boldsymbol{G}_{Ux}$, $\boldsymbol{W}_{Ux}$, $\boldsymbol{R}_{Ux}$, and etc. These functions describe impact of expectations of the second-degree trades and their flows and impact of trades on the expected trades and their flows. These functions model the economic and financial processes, social expectations, technology forecast, and flows of these parameters those impact the second-degree trades ***B****(2,t)* (3.17) and flows ***P****(2,t)* and back action of trades on expected trades and their flows. Integrals in right side of (C.6-C.8) can describe integrals by flows and their products. Such factors are completely new for macroeconomic theory and were never taken into account before.

Appendix D. Price fluctuations

Let's follow (Olkhov, 2019b) and neglect the impact of trades and expected trades flows. Let's study small perturbations of $U(2,\boldsymbol{k};t)$, $C(2,\boldsymbol{k};t)$ and $Ex_U(\boldsymbol{k};t)$ and $Ex_C(\boldsymbol{k};t)$ as:

$$U(2,\boldsymbol{k};t) = U_{\boldsymbol{k}}^2\big(1+u(2,\boldsymbol{k};t)\big)\ ;\ C(2,\boldsymbol{k};t) = C_{\boldsymbol{k}}^2\big(1+c(2,\boldsymbol{k};t)\big) \tag{D.1}$$

$$Et_U(\boldsymbol{k};t) = Et_{U\boldsymbol{k}}\big(1+et_u(\boldsymbol{k};t)\big);\ Et_C(\boldsymbol{k};t) = Et_{C\boldsymbol{k}}\big(1+et_c(\boldsymbol{k};t)\big) \tag{D.2}$$

Relations (D.1, D.2) describe small dimensionless disturbances $u,c,et_u\ et_c$. Relations (3.19, 5.2, D.1) present the second-degree volume $U(2;t)$, cost $C(2;t)$ and price $p(2,t)$ (3.19) as:

$$U(2;t) = \sum_{k,l=1}^{K} U(2,\boldsymbol{k};t)\ ;\ \ C(2;t) = \sum_{k,l=1}^{K} C(2,\boldsymbol{k};t)\ ;\ \ C(2;t) = p(2;t)U(2;t) \tag{D.3}$$

$$p(2;t) = \frac{C(2;t)}{U(2;t)} = \frac{\sum_{k,l} C_{\boldsymbol{k}}^2\big(1+c(2,\boldsymbol{k};t)\big)}{\sum_{k,l} U_{\boldsymbol{k}}^2\big(1+u(2,\boldsymbol{k};t)\big)}$$

In linear approximation by the disturbances $u(2,\boldsymbol{k};t)$ and $c(2,\boldsymbol{k};t)$, the price $p(2;t)$ takes form:

$$p(2;t) = p_{20}[1+\pi(2;t)] = p_{20}\big[1+\sum_{k,l=1}^{K}(\mu_{2\boldsymbol{k}}c(2,\boldsymbol{k};t) - \lambda_{2k}u(2,\boldsymbol{k};t))\big] \tag{D.4}$$

$$U_{20} = \sum\nolimits_{k,l} U_{\boldsymbol{k}}^2\ ;\ C_{20} = \sum\nolimits_{k,l} C_{\boldsymbol{k}}^2\ ;\quad C_{20} = p_{20}U_{20}$$

$$\lambda_{2\boldsymbol{k}} = \frac{U_{\boldsymbol{k}}^2}{U_{20}}\ ;\quad \mu_{2\boldsymbol{k}} = \frac{C_{\boldsymbol{k}}^2}{C_{20}}\ ;\ \sum \lambda_{2\boldsymbol{k}} = \sum \mu_{2\boldsymbol{k}} = 1$$

We use index 2 to underline that p_{20} and other const describe properties of the price $p(2;t)$. Let's take into account (D.1) and present price $p(2,\boldsymbol{k},t)$ (5.2) disturbances as

$$C_{\boldsymbol{k}}^2[1+c(2,\boldsymbol{k};t)] = p_{2k}[1+\pi(2,\boldsymbol{k};t)]U_{\boldsymbol{k}}^2[1+u(2,\boldsymbol{k};t)] \tag{D.5}$$

Then in linear approximation by disturbances obtain:

$$C_{\boldsymbol{k}}^2 = p_{2k}U_{\boldsymbol{k}}^2\ ;\ \pi(2;\boldsymbol{k};t) = c(2,\boldsymbol{k};t) - u(2,\boldsymbol{k};t) \tag{D.6}$$

Now substitute (D.6) into (D.4) and obtain dependence of price disturbances $\pi(2;t)$ on volume disturbances $u(2,\boldsymbol{k};t)$:

$$\pi(2;t) = \sum_{k,l}\mu_{2k}\pi(2,\boldsymbol{k};t) + \sum_{k,l}(\mu_{2k}-\lambda_{2k})\,u(2,\boldsymbol{k};t) \tag{D.7}$$

$$p(2;t) = p_{20}[1+\pi(2;t)] = p_{20}\big[1+\sum_{k,l}\mu_{2k}\pi(2,\boldsymbol{k};t) + (\mu_{2k}-\lambda_{2k})u(2,\boldsymbol{k};t)\big] \tag{D.8}$$

(D.7) describes fluctuations $\pi(2;t)$ (D.4) of mean square price $p(2;t)$ as function of partial square price disturbances $\pi(2,\boldsymbol{k};t)$ (D.6) and disturbances of squares volume $u(2,\boldsymbol{k};t)$ (D.1).

Now let's study simple model that can describe volume, cost and price disturbances (D.1, D.4, D.8). Let's assume that U_k^2, C_k^2, Et_{Uk} and Et_{Ck} in (D.1, D.2) are constant or their changes are slow to compare with changes of the small dimensionless disturbances $u(2,\boldsymbol{k};t)$, $c(2,\boldsymbol{k};t)$, $et_q(\boldsymbol{k};t)$ and $et_c(\boldsymbol{k};t)$. We assume that in the linear approximation by perturbations F_U, F_C in (5.1) and W_U, W_C in (5.5) take form:

$$F_U(\boldsymbol{k};t) = a_{uk}Et_{U\boldsymbol{k}}\big(1+et_u(\boldsymbol{k};t)\big)\ ;\ \ F_C(\boldsymbol{k};t) = a_{ck}Et_{C\boldsymbol{k}}\big(1+et_c(\boldsymbol{k};t)\big) \tag{D.9}$$

$$W_U(\boldsymbol{k};t) = b_{uk}U_{\boldsymbol{k}}^2\big(1 + u(2,\boldsymbol{k};t)\big) \quad ; \quad W_C(\boldsymbol{k};t) = b_{ck}C_{\boldsymbol{k}}^2\big(1 + c(2,\boldsymbol{k};t)\big) \tag{D.10}$$

Taking into account (B.20, B.21) and (D.1, D.2) equations (5.1, 5.6) in linear approximation by disturbances take form:

$$U_{\boldsymbol{k}}^2 \frac{d}{dt}u(2,\boldsymbol{k};t) = a_{uk}Et_{U\boldsymbol{k}}et_u(\boldsymbol{k};t) \quad ; \quad C_{\boldsymbol{k}}^2 \frac{d}{dt}c(2,\boldsymbol{k};t) = a_{ck}Et_{C\boldsymbol{k}}et_c(\boldsymbol{k};t) \tag{D.11}$$

$$Et_{U\boldsymbol{k}} \frac{d}{dt}et_U(\boldsymbol{k};t) = b_{uk}U_{\boldsymbol{k}}^2u(2,\boldsymbol{k};t) \; ; \quad Et_{C\boldsymbol{k}} \frac{d}{dt}et_C(\boldsymbol{k};t) = b_{ck}C_{\boldsymbol{k}}^2c(2,\boldsymbol{k};t) \tag{D.12}$$

For

$$\omega_{uk}^2 = -a_{uk}\, b_{uk} > 0 \quad ; \quad \omega_{ck}^2 = -a_{ck}\, b_{ck} > 0 \tag{D.13}$$

simple solutions for disturbances $u(2,\boldsymbol{k};t)$, $c(2,\boldsymbol{k};t)$, $et_u(\boldsymbol{k};t)$, $et_c(\boldsymbol{k};t)$ take harmonic oscillations:

$$u(2,\boldsymbol{k};t) = u_{1k}sin\omega_{uk}t + \; u_{2k}cos\omega_{uk}t \quad ; \quad u_{1k}, u_{2k} \; \ll 1 \tag{D.14}$$

$$c(2,\boldsymbol{k};t) = c_{1k}sin\omega_{ck}t + \; c_{2k}cos\omega_{ck}t \quad ; \quad c_{1k}, c_{2k} \; \ll 1 \tag{D.15}$$

Solutions (D.8, D.9) are similar to relations derived for the first-degree trades disturbances (Olkhov, 2019b). For the case

$$\gamma_{uk}^2 = -a_{uk}\, b_{uk} < 0 \quad ; \quad \gamma_{ck}^2 = -a_{ck}\, b_{ck} < 0 \tag{D.16}$$

equations (D.11, D.12) describe exponential growth or dissipations of perturbations

$$u(2,\boldsymbol{k};t) \sim \exp \gamma_{uk}t \quad ; c(2,\boldsymbol{k};t) \sim \exp \gamma_{ck}t \tag{D.17}$$

and perturbations $\pi(2,\boldsymbol{k};t)$ (D.6) also may follow exponential growth or decline. For different expectations $\boldsymbol{k}=(k,l)$ (4.1, B.7, B.8) solutions for disturbances $u(2,\boldsymbol{k};t)$, $c(2,\boldsymbol{k};t)$ can take (D.14, D.15) or (D.17). Combinations of perturbations determined by (D.14, D.15) or (D.17) define perturbations of price $p(2;t)$ (D.4). Thus simple equations (5.1, 5.5) on the second-degree volume and cost may describe unexpected exponential growth of price $p(2;t)$ perturbations $\pi(2;t)$ and hence exponential growth of price volatility $\sigma_p{}^2$ (2.8).

Now let's assume that $u(2,\boldsymbol{k};t)$ and $c(2,\boldsymbol{k};t)$ disturbances depend on disturbances of expectations $ex_u(\boldsymbol{k};t)$, $ex_c(\boldsymbol{k};t)$ (see B.20, B.21):

$$Ex_U(\boldsymbol{k};t) = Ex_{U\boldsymbol{k}}\big(1 + ex_u(\boldsymbol{k};t)\big); \; Ex_C(\boldsymbol{k};t) = Ex_{C\boldsymbol{k}}\big(1 + ex_c(\boldsymbol{k};t)\big) \tag{D.18}$$

Let's take into account (D.1) and in the linear approximation by disturbances obtain:

$$Et_U(\boldsymbol{k};t) = Ex_{U\boldsymbol{k}}U_{\boldsymbol{k}}^2\big(1 + ex_u(\boldsymbol{k};t) + u(2,\boldsymbol{k};t)\big) \tag{D.19}$$

$$Et_C(\boldsymbol{k};t) = Ex_{C\boldsymbol{k}}U_{\boldsymbol{k}}^2\big(1 + ex_c(\boldsymbol{k};t) + c(2,\boldsymbol{k};t)\big) \tag{D.20}$$

Now use (D.19, D.20) and replace (D.9) by (D.21, D.22):

$$F_U(\boldsymbol{k};t) = a_{uk}Et_{U\boldsymbol{k}}U_{\boldsymbol{k}}^2\big(1 + ex_u(\boldsymbol{k};t) + u(2,\boldsymbol{k};t)\big) \tag{D.21}$$

$$F_C(\boldsymbol{k};t) = a_{ck}Et_{C\boldsymbol{k}}C_{\boldsymbol{k}}^2\big(1 + ex_c(\boldsymbol{k};t) + c(2,\boldsymbol{k};t)\big) \tag{D.22}$$

It is easy to show that for

$$\omega_{uk}^2 = -\frac{a_{uk}\, b_{uk}}{U_{\boldsymbol{k}}} > 0 \quad ; \quad \omega_{ck}^2 = -\frac{a_{ck}\, b_{ck}}{C_{\boldsymbol{k}}} > 0 \tag{D.23}$$

$$\gamma_{uk} = \ a_{uk} \frac{Ex_{Uk}}{U_{\boldsymbol{k}}} \ ; \ \ \gamma_{ck} = \ a_{ck} \frac{Ex_{Ck}}{C_{\boldsymbol{k}}} \qquad \text{(D.24)}$$

equations (5.1, 5.5, D.10, D.21, D.22) describe damped harmonic oscillators:

$$\left[\frac{d^2}{dt^2} + \gamma_{uk}\frac{d}{dt} + \omega_{uk}^2\right]u(\boldsymbol{k};t) = 0 \ ; \ \ \left[\frac{d^2}{dt^2} + \gamma_{ck}\frac{d}{dt} + \omega_{ck}^2\right]c(\boldsymbol{k};t) = 0$$

$$u(\boldsymbol{k};t) \sim \sin\varphi_{uk} t \exp -\frac{\gamma_{uk}}{2} t \ \ ; \ \ c(\boldsymbol{k};t) \sim \sin\varphi_{ck} t \exp -\frac{\gamma_{ck}}{2} t \qquad \text{(D.25)}$$

$$\varphi_{uk}^2 = \omega_{uk}^2 - \frac{\gamma_{uk}^2}{4} \ ; \ \ \varphi_{ck}^2 = \omega_{ck}^2 - \frac{\gamma_{ck}^2}{4}$$

If γ_{uk} >0 and γ_{ck} >0 than (D.25) and (D.4) describe trade and price *p(2;t)* disturbances as harmonic oscillations with dissipating amplitudes. But if γ_{uk} <0 or γ_{ck} <0 than (D.25) and *p(2;t)* in (D.4) and volatility (5.6) have harmonic modes with exponentially growing amplitudes. It is interesting that perturbations of expectations $Ex_u(\boldsymbol{k};t)$, $Ex_c(\boldsymbol{k};t)$ (D.18) can generate disturbances of trades (D.25), price *p(2;t)* (D.4) and volatility σ_p^2 (5.6) like harmonic oscillations with amplitude growth by exponent in time. It is obvious that such exponential rise of perturbations *π(2;t)* and price volatility σ_p^2 (2.8) is reduced by applicability of model equations (5.1, 5.5) and relations (D.1, D.2, D.4, D.14, D.15, D.17). This approximation has sense till perturbations remain small to compare with 1.

Relations (D.4, D.8) describe mean square price *p(2;t)* (3.19) disturbances. We derive similar relations on mean price *p(1;t)* disturbances (Olkhov, 2019b):

$$p(1;t) = p_{10}\left[1 + \sum_{k,l=1}^{K}(\mu_{1\boldsymbol{k}} c(1,\boldsymbol{k};t) - \lambda_{1k} u(1,\boldsymbol{k};t))\right] \qquad \text{(D.26)}$$

Coefficients p_{10}, μ_{1k}, λ_{1k} and functions $c(1,\boldsymbol{k};t)$, $u(1,\boldsymbol{k};t)$ (D.26) describe price *p(1;t)* and first-degree cost and volume in a way similar to p_{10}, μ_{1k}, λ_{1k} (D.4) and functions $c(1,\boldsymbol{k};t)$, $u(1,\boldsymbol{k};t)$ (D.5) for *p(2;t)* (D.4) and second-degree cost and volume (D.1). Disturbances of *p(2;t)* (D.4) and *p(1;t)* (D.26) describe price volatility σ_p^2 (2.8) in linear approximation by disturbances as:

$$\sigma_p^2(t) = \sigma_{p0}^2\{1 + \sum_{k,l=1}^{K} \sigma_2[\mu_{2\boldsymbol{k}} c(2,\boldsymbol{k};t) - \lambda_{2k} u(2,\boldsymbol{k};t)] - \sigma_1[\mu_{1\boldsymbol{k}} c(1,\boldsymbol{k};t) - \lambda_{1k} u(1,\boldsymbol{k};t)]\}$$

$$\sigma_{p0}^2 = p_{20} - p_{10}^2 \ ; \ \ \sigma_2 = \frac{p_{20}}{\sigma_{p0}^2} \ ; \ \ \sigma_1 = 2\frac{p_{10}^2}{\sigma_{p0}^2} \qquad \text{(D.27)}$$

In (D.27) constants and functions with index 2 describe mean square price *p(2;t)* and its disturbances and index 1 describes mean VWAP price *p(1;t)* (Olkhov, 2019b) and its disturbances.

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