Patterns of competitive interaction

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September 2020

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Abstract

We explore patterns of price competition in an oligopoly where consumers vary in the set of firms they consider for their purchase and buy from the lowest-priced firm they consider. We study a pattern of consideration, termed "symmetric interactions", that generalises models used in existing work (duopoly, symmetric firms, and firms with independent reach). Within this class, equilibrium profits are proportional to a firm's reach, firms with a larger reach set higher average prices, and a reduction in the number of firms (either by exit or by merger) harms consumers. We go on to study patterns of consideration with asymmetric interactions. In situations with disjoint reach and with nested reach we find equilibria in which price competition is "duopolistic": only two firms compete within each price range. We characterize equilibria in the three-firm case, and show how entry and merger can affect patterns of price competition in novel ways.

Keywords: Price competition, consideration sets, mixed strategies, entry and merger.

1 Introduction

We study oligopoly pricing in a setting where consumers differ in the set of firms they consider for their purchase, and who buy from the firm in their consideration set with the lowest price. Bertrand equilibrium then typically involves firms choosing their prices according to mixed strategies, and a firm chooses from a *range* of prices. The pattern of price competition could take many forms. Firms might all choose from a similar range of prices, or competition might be more segmented with only a small subset of firms competing at a given price. Who competes with whom at each price is determined in equilibrium.

*Both authors at Department of Economics and All Souls College, University of Oxford. We are grateful to Daisuke Hirata, Maarten Janssen, Jon Levin, Domenico Menicucci, Vlad Nora, Martin Obradovits, David Ronayne, Robert Somogyi, Giovanni Ursino, Jidong Zhou and Junjie Zhou for helpful comments. Armstrong thanks the European Research Council for financial support from Advanced Grant 833849.

How does the equilibrium pattern of price competition depend on the underlying pattern of consumer consideration?

The simplest situation in which this question arises is a duopoly in which each firm has some captive customers, while contested customers are able to pay the lower of the two firms' prices. A firm then has choice between "fighting", by competing against its rival for the contested consumer segment with a low price, or "retreating" towards its captive base by setting a high price, and in equilibrium these strategies yield the same profit. Even if the firms are asymmetric, they use the same interval range of prices. With more than two firms, though, richer patterns of consumer consideration become possible. Taking the interaction between two firms to be the overlap in the sets of consumers who consider buying from them relative to their reaches, different pairs of firms might have very different levels of interaction. With more than one rival a firm can compete on several fronts, and richer patterns of pricing also emerge. With a segmented pricing pattern, for instance, a firm might compete against one firm when it charges a low price and another firm when it charges a higher price.

The foundation of our model is the distribution of consideration sets among consumers. There are various reasons why different consumers have different sets of choices open to them. Perhaps following a prior stage of advertising by firms, some consumers become aware of a different set of suppliers than other consumers. For instance, Draganska and Klapper (2011) document limited and heterogeneous consumer awareness of various brands of ground coffee, while Honka, Hortacsu, and Vitorino (2017) do the same for retail banks. Likewise, consumers have different propensities to search (some consumers only obtain one price quote for a mortgage, some obtain two, and so on), and this leads to different degrees of consumer awareness. Personalised online search tools might mean that different consumers see different results even when using the same search terms. As in Spiegler (2006), there might be (extreme) horizontal product differentiation such that only a subset of products meet a consumer's needs. Some consumers might be constrained in their choices by transport costs. For instance, some models of spatial competition, such as Smith (2004), suppose that a consumer considers buying from those firms located within a specified radius of her. Consumers might also differ in their ability to make comparisons between offers, with confused consumers choosing randomly between suppliers or buying from a default seller (Piccione and Spiegler (2012), Chioveanu and Zhou (2013)). Our analysis does not take a view on the underlying reason why consumers have different consideration sets. Rather, it takes the distribution of consideration sets in the consumer population as given, and explores the consequences for competition.

A considerable literature has explored instances of this general framework, and some patterns of consideration are now well understood: (a) the case with symmetric firms; (b) the case with independent reach, and (c) the "one-or-all" case where consumers are either captive to one firm or can choose between all firms. Within case (a), which covers the great majority of existing models, Rosenthal (1980) and Varian (1980) considered the situation in which some consumers are randomly captive to particular firms, while others compare the prices of all firms and buy from the cheapest. (Thus these papers also fall under case (c).) Burdett and Judd (1983, section 3.3) analyze a more general symmetric model, in which arbitrary fractions of consumers consider one random firm, two random firms, and so on. In this framework there exists a symmetric equilibrium, and Johnen and Ronayne (2019) show that this symmetric equilibrium is the unique equilibrium if and only if there are some consumers who consider precisely two firms. Bergemann, Brooks, and Morris (2020) extend Burdett and Judd's model so that firms might have information about the number of firms a consumer considers. This information might be public, so all firms see the same information about a consumer, or different firms might have different information, and the authors derive the information structure which maximizes industry profit.

In case (b) with independent reach, the fact that a consumer considers a given firm does not affect the likelihood she considers any other firm. Then the firm that reaches the most consumers also has the largest proportion of captive consumers among the consumers within its reach—i.e., it has the highest captive-to-reach ratio. This model was studied by Ireland (1993) and McAfee (1994), who show that in equilibrium all firms use the same minimum price, but the maximum price charged is lower for firms with a smaller reach. Thus price supports are nested, so that smaller-reach firms only offer low prices while more ubiquitous firms offer the full range of prices. Since firms use the same minimum price, their profits are proportional to their reach.[1]

Case (c), where consumers either consider just one firm or consider the entire set of firms, was fully solved by Baye, Kovenock, and De Vries (1992). (This framework includes

[1]This equilibrium was subsequently shown by Szech (2011) to be unique. Spiegler (2006) studies the special case of this framework where all firms are equally likely to be considered (which therefore also fits into case (a) with symmetric firms). In an empirical study of the personal computer market, Sovinsky Goeree (2008) assumes that the reach of the various products is independent.

duopoly as a special case, which was studied by Narasimhan (1988).) In the symmetric version of the model (which coincides with the models of Varian and Rosenthal), when there are more than two firms many asymmetric equilibria exist alongside the symmetric equilibrium. With asymmetry, when firms have different numbers of captive customers, all but the two smallest firms choose the monopoly price for sure, while the two smallest firms compete using mixed strategies. Intuitively, the two firms with the fewest captive customers have the strongest incentive to fight, leaving firms with more captives with an incentive to retreat to their captive base. This is an extreme instance of the situation where some firms choose only high prices, which we discuss further in the following.

While these three special cases are natural benchmarks, in practice patterns of consumer consideration will fall outside these cases. For example, in their study of ground coffee Draganska and Klapper (2011) document in their sample that firms are not close to being symmetric, that consumer awareness is far from independent across brands, and that the consideration sets of many consumers consisted neither of a single firm nor of the whole set of firms. The aim of the present paper is to provide a unifying framework which encompasses patterns (a) to (c) as special cases, but which allows us to study richer situations outside these cases as well, and to discover new types of equilibrium interaction.

The analysis is organized as follows. In section 2 we present our general framework of consideration sets, and formalize the notion of "interaction" between sets of firms which captures the relevant measure of correlation between the reaches of firms. In section 3 we introduce a pattern of consumer consideration, which we term "symmetric interactions", which includes (a) to (c) as special cases. With symmetric interactions, the probability that a consumer considers firm i given that she considers firm j does not depend on the identity of j. In this case the unique equilibrium pattern of prices resembles that seen with independent reach, i.e., there is an increasing sequence of prices $\{p_i\}$, and the firm with the ith smallest reach uses prices in the interval $[p_0, p_i]$. In particular, all firms use the same minimum price p_0, profits are proportional to a firm's reach, and firms with a larger reach stochastically choose higher prices. When the set of firms is reduced, either by exit or by merger, the profits of remaining firms rise and consumers overall are harmed.

In the remainder of the paper we study patterns of consideration beyond symmetric interactions. In section 4 we analyse two situations—disjoint reach and nested reach—which are sure to have asymmetric interactions. To illustrate which might happen with disjoint reach we discuss competition between a chain store and a number of disjoint local rivals. In this case, the chain store uses prices from the whole market range, $[p_0, 1]$, and different local firms choose their prices from disjoint intervals within $[p_0, 1]$ (where local firms with greater interaction with the chain store use lower prices). Thus there is "duopolistic" price competition, and the chain store competes against a single rival at a given price. With nested reach, only the firm with the largest reach has any captive customers, and firms with larger reach chooses prices from a higher range. If the increments between successive firm reaches are non-decreasing we find equilibria with a different form of duopolistic pricing which we term "overlapping duopoly": there is an increasing sequence of prices $\{p_i\}$ such that the firm with the ith smallest reach uses prices in the interval $[p_{i-1}, p_{i+1}]$. Hence small firms charge only low prices while large firms charge only high prices, and firms compete only against their immediate neighbours in the nest.

In section 5 we provide a general analysis of the three-firm case. When interactions between pairs of firms are similar, as in section 3, all firms use a common lowest price. In some of these cases, however, we find that the price support of the least competitive firm need not be an interval—the firm might price high or low but not in an intermediate range. By contrast, when one pair of firms has significantly more interaction than other pairs, the equilibrium exhibits the duopolistic pricing seen in section 4—one firm prices low, one high, and one across the full price range. Intuitively, this pair mostly compete with each other, leaving the remaining firm with an incentive to set high prices. While entry into a duopoly market by a third firm often pushes down prices, there are natural patterns of interaction where, counter-intuitively, the opposite happens and consumers are harmed by entry. The reason is that more intense competition for contestable consumers induces incumbents to retreat towards their captive base. While a profitable merger in a three-firm market will always harm consumers (as with symmetric interactions), there are situations where a profitable merger between two firms with a strong interaction can reduce industry profit. The reason is that such a merger opens up a profitable front for the non-merging firms, and induces these firms to fight rather than retreat.

We conclude in section 6 by summarizing the main insights derived from our analysis, and suggesting avenues for further research on this topic.

2 A model with consideration sets

There are $n \geq 2$ firms that costlessly supply a homogeneous product. There is a population of consumers of total measure normalized to 1, each of whom has unit demand and is willing to pay up to 1 for a unit of the product.[2] Consumers differ according to which firms they consider for their purchase, and for each subset $S \subset N \equiv \{1, ..., n\}$ of firms the fraction of consumers who consider exactly the subset S is α_S. (We slightly abuse notation, and write α_1 for the fraction who consider only firm 1, $\alpha_{12} = \alpha_{21}$ for the fraction who consider only firms 1 and 2, and so on.) When there are only few firms the pattern of consideration sets can be illustrated using a Venn diagram, and Figure 1 depicts a market with three firms.[3] Here, a consumer considers a particular subset of firms if she lies inside the "circle" of each of those firms. For instance, a fraction α_{12} of consumers consider the two firms 1 and 2. A consumer is *captive* to firm i if she considers i but no other other firm, so there are α_i such consumers.

It is also useful also to write σ_S for the fraction of consumers who consider all firms in $S \subset N$, and possibly other firms too. (Formally $\sigma_S = \sum_{S' | S \subset S'} \alpha_{S'}$, so that with the three firms on Figure 1 we have $\sigma_{12} = \alpha_{12} + \alpha_{123}$, say.) Thus $\sigma_S \geq \alpha_S$, with equality for $S = N$, and σ_S weakly decreases with S in the set-theoretic sense. The *reach* of firm i is the set of consumers who consider the firm, i.e., the firm's potential customers, and the fraction of such consumers is σ_i. Finally, the *captive-to-reach* ratio of firm i is denoted $\rho_i \equiv \alpha_i / \sigma_i$.

Firms compete in a one-shot Bertrand manner, and a consumer buys from the firm she considers that has the lowest price (provided this price is no greater than 1). In particular, a firm offers a uniform price to consumers, and cannot make its price to a consumer contingent on her consideration set.[4] Thus if a firm sets a price strictly below all its rivals it sells to its reach and its demand is σ_i, while if it sets a price strictly above all rivals it sells only to its captive customers and its demand is α_i. If two or more firms choose the same lowest price, we suppose that the consumer is equally likely to buy from any

[2] The equilibrium analysis which follows is not affected if each consumer has a downward-sloping demand function $x(p)$, provided revenue $px(p)$ is an increasing function up to the monopoly price. However, welfare analysis (e.g., in our discussion of mergers and entry) requires adjustment with downward-sloping demand.

[3] In a spatial context this Venn diagram has a more literal interpretation: if consumers only consider buying from a firm within a specified distance, then the locations of firms determine the centre of the circles on the diagram. In a very different context, Prat (2018) uses a model of consideration sets similar to that presented in this paper.

[4] In Armstrong and Vickers (2019) we investigate the impact of firms being able to offer different prices to captive and contested customers.

such firm (although the details of the tie-breaking rule make no difference to the analysis). Since industry profit is a continuous function of the vector of prices chosen, Theorem 5 in Dasgupta and Maskin (1986) shows that an equilibrium exists. Since an individual firm's profit is usually discontinuous in the price vector, the equilibrium will usually involve mixed strategies for some firms.

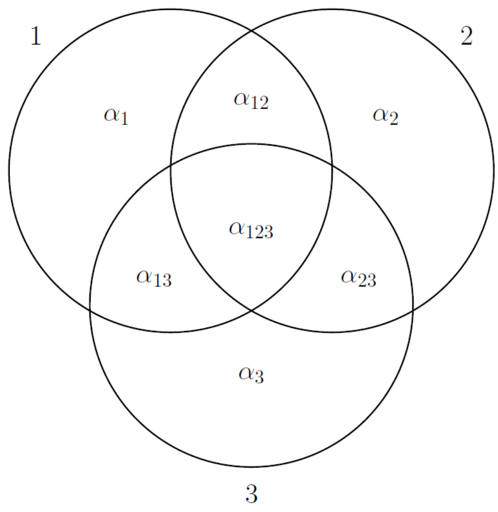

Figure 1: Consideration sets with three firms

We make two running assumptions to rule out some extreme and uninteresting configurations. The first requires that there be some competition between firms:

Assumption 1: Some consumers consider at least two firms.

(If all customers were captive, each firm chooses $p \equiv 1$ for sure.) The second assumption prohibits the possibility that a subset of firms choose the competitive price $p \equiv 0$ for sure, as such firms play no important role in the analysis:

Assumption 2: Every non-empty subset of firms S contains at least one firm with consumers within its reach who consider no other firm in S.

For instance, this assumption rules out the situation where two firms reach precisely the same set of consumers. Intuitively, Assumption 2 ensures that no subset S of firms will set $p \equiv 0$, since there is a firm in S which has some customers with no overlap with other firms in S, and this firm can profitably raise its price above zero. These two assumptions together imply that there is no equilibrium in pure strategies. As well as these two running assumptions, we sometimes make use of a third assumption:

Assumption 3: $\alpha_{ij} > 0$ for all i and j.

This assumption states that each pair of firms has some mutually contested customers, regardless of the prices chosen by other firms. The assumption is satisfied with independent reach, but with more than two firms it rules out the "all-or-one" pattern of consideration.

Suppose that firm $j \in N$ chooses its price (independently) according to the cumulative distribution function (CDF) $F_j(p)$. These strategies then induce a demand function $q_i(p)$ for firm i, which is the expected fraction of consumers who buy its product when it chooses price p. (We present explicit formulas for this demand shortly.) Equilibrium occurs when for each firm i there exists a profit level π_i and a CDF $F_i(p)$ for its price with such that firm i's expected profit $pq_i(p)$ is equal to π_i for every price in the support of F_i and no higher than π_i for any price outside its support.[5]

It is also useful to introduce the notation $q_{ij}(p)$, which is the probability that a consumer considers both firms i and j and that any other firm she considers has price at least p. This represents the fraction of consumers contested by these two firms at price p, i.e., the increase in its demand if firm i slightly undercuts firm j at price p. (Just as the regular demand function q_i satisfies $\alpha_i \leq q_i(p) \leq \sigma_i$, here we have $\alpha_{ij} \leq q_{ij}(p) \leq \sigma_{ij}$.) Using this notation, say that in equilibrium "firms i and j compete at price p" if p lies in both firms' price supports and $q_{ij}(p) > 0$. If the two firms have disjoint reach then they do not compete even if they happen to use the same price. However, if Assumption 3 holds then $q_{ij}(p)$ is always positive, in which case firm i competes with j at p whenever the price lies in both supports.

The following result collects a number of preliminary observations about the structure of price competition in equilibrium, some of which are familiar from the existing literature.[6]

Lemma 1 *In any equilibrium:*
(i) firm i obtains profit $\pi_i \geq \alpha_i$, with equality for at least one firm, and the minimum price in its support is no smaller than ρ_i;
(ii) the minimum price ever chosen in the market, p_0, is strictly positive, and firm i obtains profit $\pi_i \geq \sigma_i p_0$;
(iii) each firm's price distribution is continuous (that is, has no "atoms") in the half-open interval $[p_0, 1)$;

[5] As usual, the support of firm i's price distribution is defined to be the smallest closed set $\mathcal{P} \subset [0, 1]$ such that the probability the firm chooses a price in $\mathcal{P}$ equals one.

[6] For instance, see McAfee (1994, page 28).

(iv) at least two firms compete at each price in the interval $[p_0, 1]$*;*
(v) if there are three or more firms, there is at least one price which lies in the support of three or more firms, and
(vi) p_0 *lies weakly between the second lowest* ρ_i *and the highest* ρ_i*. If the firm with the highest* ρ_i *has* p_0 *in its support then* p_0 *is equal to the highest* ρ_i*.*

Proof. All proofs are contained in the appendix. ■

Part (iii) of the lemma shows that ties do not occur with positive probability at any price $p < 1$. Moreover, there cannot be ties with positive probability even at $p = 1$ between firms which compete at this price.[7] When firm i chooses price $p \leq 1$ it will sell to a consumer when that consumer is within its reach and when none of the other firms the consumer considers offers a lower price. Therefore, when rival firms $k \neq i$ choose price according to the CDF $F_k(p)$, firm i's expected demand with price p is

$$q_i(p) \equiv \sum_{S|i \in S} \alpha_S \left(\prod_{k \in S/i} (1 - F_k(p)) \right) . \tag{1}$$

Here, the sum is over all consumer segments which consider firm i, and for each such segment the product is over all rivals for firm i in that segment. (We use the convention that a product taken over the empty set, i.e., when $S = \{i\}$ in (1), equals 1.) For a given rival firm j, firm i's demand (1) can be decomposed as

$$q_i(p) = q_{i/j}(p) - F_j(p) q_{ij}(p) , \tag{2}$$

where $q_{i/j}(p)$ is firm i's demand in the hypothetical situation where firm j is absent, which is

$$q_{i/j}(p) = \sum_{S|i,j \in S} \left[\alpha_S + \alpha_{S/j} \right] \left(\prod_{k \in S/\{i,k\}} (1 - F_k(p)) \right) , \tag{3}$$

while $q_{ij}(p)$ is the "overlap demand" previously discussed, which is

$$q_{ij}(p) \equiv \sum_{S|i,j \in S} \alpha_S \left(\prod_{k \in S/\{i,j\}} (1 - F_k(p)) \right) . \tag{4}$$

[7] If $q_{ij}(1) > 0$ then each of them obtains a discrete jump in demand if it reduces its price slightly from 1, which cannot occur in equilibrium. So the only way both can have an atom is if $q_{ij}(1) = 0$, in which case a firm's demand is not affected if its rival slightly undercuts it at $p = 1$.

It is often more convenient to express a firm's demand in terms of the σ_S rather than then α_S parameters, and demand (1) can be re-written as

$$q_i(p) = \sum_{S|i\in S} (-1)^{|S|-1} \sigma_S \left(\prod_{k\in S/i} F_k(p) \right) . \tag{5}$$

Here, $(-1)^{|S|-1}$ is plus or minus 1 according to whether i has an even or odd number of rivals. The equivalence of the expressions (1) and (5) can be seen by comparing terms: for each subset S containing firm i the term $\Pi_{k\in S/i} F_k$ appears in (1) for each set S' such that $S \subset S' \subset N$, and each time with the same sign $(-1)^{|S|-1}$, and so has total weight $(-1)^{|S|-1}\sigma_S$ in (1), just as in (5).

Finally, expression (5) can be re-written as

$$\frac{q_i(p)}{\sigma_i} = \sum_{S|i\in S} (-1)^{|S|-1} \gamma_S \left(\prod_{k\in S/i} G_k(p) \right) , \tag{6}$$

where we define

$$G_i(p) \equiv \sigma_i F_i(p)$$

and

$$\gamma_S \equiv \frac{\sigma_S}{\Pi_{i\in S}\sigma_i} . \tag{7}$$

Thus, $G_i(p)$ is the probability a consumer considers firm i and is offered a price below p from that firm. We say that γ_S is the *interaction* between firms in the subset $S \subset N$. These interaction parameters play a major role in the following analysis, and capture patterns of correlation (or "commonality") in the reaches of subsets of firms.[8] Clearly, $\gamma_i \equiv 1$. The reaches of two firms i and j are positively correlated if $\sigma_{ij} \geq \sigma_i\sigma_j$, i.e., if $\gamma_{ij} \geq 1$, while if two firms have disjoint reach then $\gamma_{ij} = 0$. From this perspective, patterns of consumer consideration are determined by firm reaches, σ_i, and the way these reaches overlap as captured by the interaction parameters, γ_S.

The patterns of consideration which have been examined in the existing literature, discussed in the introduction, can be interpreted using this notation. With symmetric firms, each of the parameters α_S, σ_S and γ_S depend only on the number of firms in S, not their identity. With independent reach, where a consumer considers firm i with independent

[8] This representation of the correlation structure of multiple binary random variables is in the spirit of Bahadur (1961), although the details of correlation measures differ.

probability σ_i, we have

$$\alpha_S = (\Pi_{i \in S} \sigma_i)(\Pi_{j \notin S}(1 - \sigma_j)) \ ; \ \sigma_S = \Pi_{i \in S} \sigma_i \ ; \gamma_S \equiv 1 \ .$$

In this case (6) factorizes to

$$\frac{q_i(p)}{\sigma_i} = \prod_{k \in N/i} (1 - G_k(p)) \ . \tag{8}$$

With "all-or-one" consideration, $\alpha_N > 0$ consumers consider all n firms and α_i consumers consider only firm i, so the reach of firm i is $\sigma_i = \alpha_N + \alpha_i$ and $\sigma_S = \alpha_N$ whenever S includes more than a single firm. The pair of firms consisting of the two smallest firms has the greatest pairwise interaction term γ_{ij}. With duopoly, demand in (6) satisfies

$$\frac{q_i(p)}{\sigma_i} = 1 - \gamma_{12} G_j(p) \ , \tag{9}$$

while with three firms demand (6) satisfies

$$\frac{q_i(p)}{\sigma_i} = 1 - \gamma_{ij} G_j(p) - \gamma_{ik} G_k(p) + \gamma_{123} G_j(p) G_k(p) \ . \tag{10}$$

At various points in the remainder of the paper we will discuss the impact of entry, exit and mergers on outcomes. We model entry by a new firm as the introduction of a new "circle" superimposed onto the existing Venn diagram. That is, entry does not affect which consumers consider which incumbent firms, and the reach of an incumbent firm is unaffected by entry, although its number of captive customers will weakly fall.[9] More generally, entry does not alter the interaction parameters γ_S in (7) between sets of incumbents. Since welfare—consumer surplus plus industry profit—is the total number of consumers reached, it follows that entry (if it is costless) will increase welfare by the number of captive customers of the entrant. For this reason, if entry reduces industry profit it will benefit consumers. Since the entrant's contribution to welfare (its captive base) is a lower bound on its profit, the external impact of entry on incumbent profit plus consumer surplus is weakly negative. Exit by a firm is just the removal of a "circle", and leaves the interactions γ_S between the remaining firms unchanged, and weakly reduces welfare. If exit increases industry profit, it will harm consumers.

Mergers also have a natural set-theoretic interpretation in this framework: when two or more firms merge we assume that the merged entity sets the same price to all its customers,

[9] In particular, there is no danger of "choice overload", whereby the number of consumers who compare prices falls when there are more firms, as discussed for instance in Spiegler (2011, page 150).

and that the reach of the merged entity is the union of the reaches of the separate firms.[10] Thus, a merger (with no accompanying cost synergies) has no impact on welfare, and harms consumers if and only if it increases industry profit. The fraction of consumers reached by the merged firm, which is $\sigma_i + \sigma_j - \sigma_{ij}$, is no greater than the sum of those reached by the separate firms, $\sigma_i + \sigma_j$, while the captive base of the merged firm, $\alpha_i + \alpha_j + \alpha_{ij}$, is no smaller than the sum of captives of the separate firms, $\alpha_i + \alpha_j$. A merger does not alter the interactions γ_S between sets of non-merging firms.

3 Symmetric interactions

In this section we suppose that interactions between sets of firms in (7) are *symmetric*, in the sense that γ_S depends only on the number, not the identity, of firms in S. The familiar cases of symmetric firms, independent reach and duopoly all fall within this more general class of consideration structures. In essence, this class combines the flexibility of the symmetric firm framework, which is to be able to choose the distribution of number of considered firms freely, with the flexibility of the independent reach framework, which allows for asymmetric firms where firm reaches can be chosen freely.[11] The class of symmetric interactions rules out richer patterns of consideration, however, such as where reach is nested, where some firms have disjoint reach, or where some pairs of firms have reach which is positively correlated and some where reach is negatively correlated.

One natural environment in which interactions are symmetric is when consumers differ systematically in *attentiveness*. Suppose that firm i sends a message to a consumer with probability $\tilde{\sigma}_i$ (independent across firms), and that a consumer with attentiveness θ receives such a message (given it has been sent to her) with probability θ (independent across all messages sent to her).[12] Then a type-θ consumer considers firm i with probability $\theta\tilde{\sigma}_i$, considers both firms i and j with probability $\theta^2\tilde{\sigma}_i\tilde{\sigma}_j$, and so on. In this situation there is positive correlation in reaches across firms, since if a consumer considers firm i she is

[10] An alternative approach would be for the merged entity to maintain separate brands and to be able to charge distinct prices for each brand.

[11] To illustrate, when there are three firms as on Figure 1, patterns of consumer consideration are described by seven parameters (the number of segments in the Venn diagram). If interactions are constrained to be symmetric the three pairwise interactions γ_{ij} must be equal, and so two degrees of freedom are lost leaving five degrees of freedom remaining (the three reaches plus the pairwise interaction and the three-firm interaction). The cases of symmetric firms and independent reach each have only three degrees of freedom.

[12] For instance, θ might capture how much "media", across which marketing messages are delivered, the consumer chooses to see.

likely to have a high θ and hence is more likely to consider firm j too. Integrating over the distribution of θ in the consumer population shows that we have

$$\gamma_S = \frac{\mathbb{E}_\theta[\Pi_{i\in S}(\theta\tilde{\sigma}_i)]}{\Pi_{i\in S}(\mathbb{E}_\theta[\theta\tilde{\sigma}_i])} = \frac{\mathbb{E}_\theta[\theta^{|S|}]}{(\mathbb{E}_\theta[\theta])^{|S|}} \ ,$$

which depends only on the cardinality of S. (In this case the interaction terms are all greater than 1.)

This assumption of symmetric interactions implies that the probability a consumer considers firm i given that she considers a second firm j (which is $\sigma_i\gamma_{ij}$) does not depend on the identity of j. Further useful implications of the assumption are reported in the following result.

Lemma 2 *Suppose interactions are symmetric. Then for any pair of firms i and j the demand functions in (6) satisfy*

$$\frac{q_i(p)}{\sigma_i} - \frac{q_j(p)}{\sigma_j} = \frac{q_{ij}(p)}{\sigma_i\sigma_j}[G_i(p) - G_j(p)] \ , \tag{11}$$

and captive-to-reach ratios satisfy

$$\rho_i - \rho_j = \frac{\alpha_{ij}}{\sigma_i\sigma_j}[\sigma_i - \sigma_j] \ . \tag{12}$$

Recall that $q_{ij}(p)$ is defined in (4), and if $q_{ij}(p)$ is positive, which is ensured by Assumption 3, then $q_i/\sigma_i - q_j/\sigma_j$ has the same sign as $G_i - G_j$. Likewise, with Assumption 3 expression (12) implies $\rho_i > \rho_j$ if and only if $\sigma_i > \sigma_j$, and so a firm with a larger reach has a higher captive-to-reach ratio.

Expression (11) almost immediately implies our main result concerning symmetric interactions, which is that in equilibrium all firms use the same minimum price, p_0. For if not, there is a strict subset A of firms with the market minimum price p_0 in their support, and let firm i be the firm outside A with the next lowest minimum price, $L > p_0$. Any firm $j \in A$ weakly prefers price p_0 (when it serves its entire reach) to the price L, so that $q_j(P)/\sigma_j \leq p_0/L$. Firm i, by contrast, weakly prefers L to p_0, and so $q_i(P)/\sigma_i \geq p_0/L$. In particular, $q_i(L)/\sigma_i \geq q_j(L)/\sigma_j$. But since $G_j(L) > 0$ and $G_i(L) = 0$, this contradicts (11) provided that $q_{ij}(L) > 0$. We deduce that in equilibrium all firms must have the same minimum price p_0, which from part (vi) of Lemma 1 is therefore the highest captive-to-reach ratio among the firms. As discussed above, this is the captive-to-reach ratio of the firm with the greatest reach. The next result provides further detail about the pricing pattern in equilibrium:

Proposition 1 *Suppose Assumption 3 holds and interactions are symmetric. Suppose firms are labelled in order of increasing reach, so that $\sigma_1 \leq ... \leq \sigma_n$. Then in the unique equilibrium firm i has interval price support $[p_0, p_i]$, where the minimum price p_0 is equal to the captive-to-reach ratio of the firm with the largest reach (firm n) and maximum prices satisfy $p_1 \leq ... \leq p_{n-1} = p_n = 1$. Firms with larger reach set stochastically higher prices, in the sense that $F_1(p) \geq ... \geq F_n(p)$ for $p \in [p_0, 1]$. Firm i obtains expected profit $\pi_i = \sigma_i p_0$.*

This result shows that with symmetric interactions the equilibrium pricing pattern takes the same form as with independent reach: all firms have the same minimum price and firms with smaller reach have a lower maximum price, so that price supports are nested. When a firm chooses a higher price from its support, it competes against fewer rivals. The role of Assumption 3 is as in Johnen and Ronayne (2019): if some pairs of firms have no unique consumers in common, the possibility arises that other equilibria exist where some firms price deterministically at $p = 1$.

Markets with symmetric interactions exhibit intuitive properties when the number of firms is reduced, either by exit or by merger. If a firm exits a market with symmetric interactions, then the firms that remain continue to have symmetric interactions and hence all use the same minimum price. This new minimum price will necessarily be higher than before exit. If two firms merge in a market with symmetric interactions, the new market does not necessarily continue to have symmetric interactions, although if the merger is profitable the minimum price will rise. In either case, the following result shows that consumers overall are harmed.

Proposition 2 *Suppose Assumption 3 holds and interactions are symmetric. Then:*
(i) If a firm exits the minimum price rises and consumer surplus falls;
(ii) If two firms merge profitably the minimum price rises and consumer surplus falls.

In the remainder of the paper we study in more detail situations outside the "regular" case of symmetric interactions, and show how the form of equilibrium can differ markedly. As well as finding situations where only a subset of firms use the minimum price, we will also see instances where a firm has a "gap" in its price support. In contrast to Proposition 2, we will find situations where exit by a firm or a profitable merger between firms can increase consumer surplus.

4 Asymmetric interactions

In this section we discuss equilibrium pricing patterns when consumer consideration involves asymmetric interactions. We will focus especially on markets where some firms have disjoint reach or where firms have nested reach.

Proposition 1 provided a sufficient condition for all firms to use the same minimum price. A sufficient condition for the converse, i.e., for only a subset of firms to use the minimum price, can also be derived. If all firms have the minimum price p_0 in their support, then $q_i(p)/\sigma_i = p_0/p$ for prices just above p_0, and so $q_i'(p_0)/\sigma_i = -1/p_0$ for each firm i. From (6), this entails $\sum_{j\neq i}\gamma_{ij}G_j'(p_0) = 1/p_0$ for each firm i. If this system of linear equations in $G_1'(p_0)$,...,$G_n'(p_0)$ has no solution with each $G_i'(p_0) \geq 0$, there can be no equilibrium where all firms use the same minimum price p_0. Farkas' Lemma implies that there is no such solution if and only if there exists a vector $(x_1, ..., x_n)$ that satisfies

$$\sum_{j=1}^{n} x_j < 0 \text{ and } \sum_{j\neq i}\gamma_{ij}x_j \geq 0 \text{ for all } i \ . \tag{13}$$

Thus, when the pairwise interactions are such that we can find a vector $(x_1, ..., x_n)$ satisfying (13) then any equilibrium has only a subset of firms using the minimum price p_0. (Note that condition (13) does not depend on the behaviour of γ_S for larger subsets S.) To illustrate the method for $n = 3$, suppose interactions are so asymmetric that $\gamma_{12} > \gamma_{13} + \gamma_{23}$. Then setting $(x_1, x_2, x_3) = (\gamma_{23}, \gamma_{13}, -\gamma_{12})$ satisfies (13), and so only two firms will use the minimum price. (In section 5 we will see that it is firms 1 and 2 which price low.)

A similar method can be employed to show that a given subset of firms cannot use the minimum price p_0 in equilibrium. We shortly do this in the case of nested reach to show when only a small number of firms will price low.

Chain store competition: An economically natural scenario with disjoint reach is when a "chain store", firm n, competes against $n-1 \geq 2$ disjoint local rivals (so there are $n \geq 3$ firms in all), as shown in Figure 2 when $n = 5$. With this pattern of consideration, $\gamma_S = 0$ for every S which contains two or more local firms. Lemma 1 shows that at least two firms compete at each price in the market range of prices $[p_0, 1]$. Since local firms do not compete between themselves, this implies that the chain store chooses its price from the whole range $[p_0, 1]$ and each price in $[p_0, 1]$ is also chosen by at least one local firm.

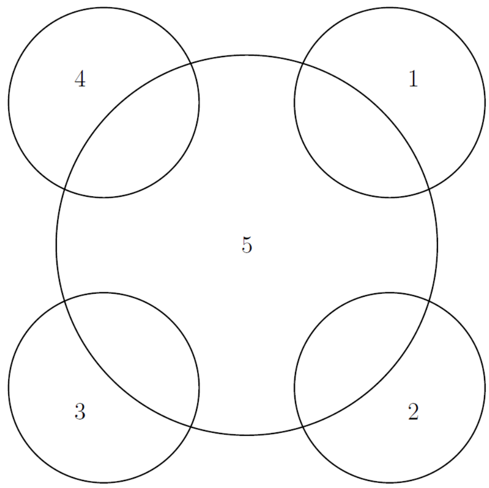

Figure 2: A chain store competing against disjoint local rivals

As in (9), the demand of local firm i is $q_i(p) = \sigma_i(1 - \gamma_{in}G_n(p))$, where γ_{in} is the pairwise interaction term for i and n, and $G_n(p) = \sigma_n F_n(p)$ is the chain store's adjusted CDF. To avoid knife-edge cases suppose local firms have distinct interaction terms, and label them as $\gamma_{1n} > ... > \gamma_{n-1,n}$. Suppose local firm i has price p_i in its support, and another local firm j has price p_j in its support. By revealed preference

$$p_i(1 - \gamma_{in}G_n(p_i)) \geq p_j(1 - \gamma_{in}G_n(p_j)) \ ; \ p_j(1 - \gamma_{jn}G_n(p_j)) \geq p_i(1 - \gamma_{jn}G_n(p_i)) \ ,$$

and subtracting yields

$$(\gamma_{jn} - \gamma_{in})[p_i G_n(p_i) - p_j G_n(p_j)] \geq 0 \ .$$

Since $pG_n(p)$ is a strictly increasing function, it follows that firms with larger i (i.e., with a weaker interaction with n) choose weakly higher prices. Moreover, if both firms have both the prices p_i and p_j in their supports the argument shows that

$$(\gamma_{jn} - \gamma_{in})[p_i G_n(p_i) - p_j G_n(p_j)] = 0 \ ,$$

which implies $p_i = p_j$. Therefore, there can be no overlap beyond a single price in the price supports of two local firms.

In economic terms, the local firms all compete against the same price distribution from the chain store, and local firms with greater interaction with the chain store have more elastic demand and so offer lower prices. In technical terms, the only way that a single price distribution from the chain store can maintain indifference for each local rival is for different rivals to have disjoint price supports.

Putting this together yields the following result. (Note that we have weak inequalities in $p_0 < p_1 \leq ... \leq p_{n-1} = 1$ since it is possible that a number of local firms, but not all, choose the monopoly price $p = 1$ for sure.)

Proposition 3 *Suppose a chain store, firm n, competes with a number of local rivals, $i = 1, ..., n-1$, where $\gamma_{1n} > ... > \gamma_{n-1,n}$. Then in equilibrium there are threshold prices $p_0 < p_1 \leq ... \leq p_{n-1} = 1$ such that the chain store has price support $[p_0, 1]$ and local firm $i < n$ has price support $[p_{i-1}, p_i]$.*

Thus with this pattern of consideration, the chain store which interacts with all local firms uses all prices while firms with disjoint reach use disjoint sets of prices. This is therefore a situation where the pattern of consumer consideration leads very directly to the equilibrium pattern of price competition. In contrast to the pricing pattern with symmetric interactions in Proposition 1, here price competition is "duopolistic", in the sense that only two firms use a given price range.[13] As will be seen later in section 5, this "chain store" pattern of prices can arise in equilibrium even when no firms have disjoint reach.

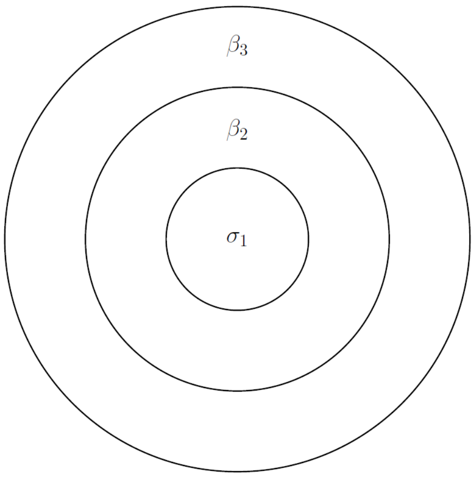

Figure 3: Three firms with nested reach

Nested reach: A second situation which involve asymmetric interactions is when consideration sets are nested, in the sense that if firm i reaches a greater fraction of consumers than firm j, then all consumers reached by j also consider firm i. This is a natural configuration if consumers consider options in an *ordered* fashion, as may be the case with internet

[13]With the exception of the threshold prices $p_1, ..., p_{n-2}$, which are in the support of three firms.

search results (where some consumers just consider the first result, others consider the first two, and so on). With nested reach, only the largest firm has any captive customers, $\sigma_S = \min_{i\in S}\{\sigma_i\}$, and pairwise interactions are $\gamma_{ij} = 1/\max\{\sigma_i, \sigma_j\}$.

Suppose there are $n \geq 3$ firms with nested reach. Let firm i have reach σ_i, where firms are labelled as $\sigma_1 < \sigma_2 < ... < \sigma_n$, and for $i \geq 2$ write $\beta_i = \sigma_i - \sigma_{i-1}$ for the incremental reach of firm i. (Figure 3 depicts the case with three nested firms.) While it is hard to calculate the precise equilibrium in all nested situations, the following result describes general features of equilibrium pricing and also derives an equilibrium in the particular case where where incremental reach is larger for larger firms.

Proposition 4 *Suppose $n \geq 3$ firms, labelled as $\sigma_1 < \sigma_2 < ... < \sigma_n$, have nested reach.*
(i) If L_i and H_i are respectively the minimum and maximum price in firm i's support then $p_0 = L_1 = L_2 \leq L_3 \leq ... \leq L_n$, $H_1 < H_2 < ... < H_{n-1} = H_n = 1$, and $H_i > L_{i+1}$ for $i < n$.
(ii) If $\sigma_k > 2\sigma_2$ then no firm $i \geq k$ uses the minimum price p_0 (i.e., $L_k > p_0$).
(iii) If

$$\beta_2 \leq \;...\; \leq \beta_n \tag{14}$$

there is an equilibrium with price thresholds $p_0 = p_1 < p_2 < ... < p_{n-1} < p_n = p_{n+1} = 1$ such that the price support of firm i is $[p_{i-1}, p_{i+1}]$. The thresholds are determined recursively by $p_2 = \frac{\sigma_1+\beta_2}{\beta_2}p_1$ and for $1 < i < n$

$$p_{i+1} = p_i + \frac{\beta_i}{\beta_{i+1}} p_{i-1} \ , \tag{15}$$

where $p_0 = p_1$ is chosen to make $p_n = 1$. The profit of firm 1 is $\pi_1 = \sigma_1 p_1$ and the profit of firm $i > 1$ is $\pi_i = \beta_i p_i$.

In part (i), the reason that a smaller firm will not choose a price strictly above the maximum price used by a larger firm is that will then have no demand or profit. The observation that minimum prices weakly increase with size is less immediate, but relates to the intuition that smaller firms face more elastic demand. The implication is that if only a subset of firms use the minimum price p_0 that subset will consist of the smaller firms. Since $H_i > L_{i+1}$ the price supports of successive firms overlap. Part (ii) shows that if reaches are spread out then only few firms will price low. In particular, if $\sigma_3 > 2\sigma_2$ then only the two smallest firms will price low. Together, parts (i) and (ii) shows that if reaches are spread out then the pricing pattern is such that smaller firms only choose low prices while larger firms only choose high prices. Finally, part (iii) shows that if the reaches are spread out enough that (14) holds, then as with the chain store scenario price competition is duopolistic, with only two firms competing at a given price. More precisely, the pricing pattern takes the form of "overlapping duopolies", where a firm only competes against its two immediate neighbours in the nest.[14]

To illustrate part (iii), consider the case where reach decays with a constant rate of attrition, so that the reach of firm $i = 1, ..., n$ is $\sigma_i = \delta^{n-i}$. In this case $\beta_i = \delta^{n-i}(1-\delta)$ which increases with i as required, and equation (15) becomes $p_{i+1} = p_i + \delta p_{i-1}$. When $n = 2$ the two firms have reaches $\sigma_1 = \delta$ and $\sigma_2 = 1$, and Proposition 1 shows that the minimum price is $1-\delta$ and industry profit is $1-\delta^2$. Proposition 4 can be used to obtain the equilibrium for any n. However, the analysis simplifies in the limit with many firms, when one can show that the threshold prices are given by the geometric progression $\kappa, \kappa^2, ...$, where $\kappa = \frac{2}{1+\sqrt{1+4\delta}} \leq 1$, and the minimum price p_0 converges to zero. Industry profit is $(1-\delta)(1+\delta\kappa+(\delta\kappa)^2+...) = \frac{1-\delta}{1-\delta\kappa}$. Perhaps surprisingly, this profit with many firms is higher than that with two firms, $1-\delta^2$, even though the set of consumers served is the same. Put another way, exit by all but the two largest firms causes industry profit to fall. Since welfare is unchanged after exit, it follows that exit benefits consumers in aggregate, in contrast to Proposition 2. We discuss how entry can harm consumers further in the next section, using a more transparent framework with symmetric incumbents.

Proposition 4 describes the precise equilibrium only for cases where incremental reach weakly increases. In the next section we analyse the case of triopoly, and obtain results implying for the case of nested reach that (a) when $\beta_3 > \beta_2$ the equilibrium in part (iii) of Proposition 4 is unique and (b) when $\beta_3 < \beta_2$ the equilibrium instead has all three firms using the same minimum price.

5 The three-firm problem

So far we have encountered two contrasting patterns of pricing: (i) all firms use the same minimum price (the case of symmetric interactions in section 3), and (ii) duopolistic pricing

[14]A similar pattern of overlapping duopoly pricing is seen in Bulow and Levin (2006). They study a matching model where n heterogeneous firms each wish to hire a single worker from a pool with n heterogeneous workers, where the payoff from a match is (in the simplest version of their model) the product of qualities of the firm and worker. In equilibrium, firms offer wages according to mixed strategies, where higher quality firms offer wages in a higher range than lower quality firms.

where only two firms use prices in a given price range (cases of disjoint and nested reach in section 4). In this section we study in detail the situation with three firms, and show that the pricing patterns in (i) and (ii) are (generically) the *only* possibilities. That is, if pricing is duopolistic for low prices, it continues to be duopolistic throughout the whole price range.[15]

We first describe a sufficient condition for regime (i) to apply, using a similar argument to that for symmetric interactions in section 3. This argument will also reveal which pair of firms price low when regime (ii) applies. Suppose that an equilibrium has firms 1 and 2 using minimum price p_0, while 3's minimum price is $L > p_0$. Then both firms 1 and 2 have prices $[p_0, L]$ in their support, and by revealed preference

$$\frac{q_1(L)}{\sigma_1} = \frac{q_2(L)}{\sigma_2} = \frac{p_0}{L} \leq \frac{q_3(L)}{\sigma_3} \ .$$

From (10), demand functions for distinct firms i, j and k satisfy

$$\frac{q_i(p)}{\sigma_i} - \frac{q_j(p)}{\sigma_j} = (\gamma_{jk} - \gamma_{ik})G_k(p) + (\gamma_{ij} - \gamma G_k(p))(G_i(p) - G_j(p)) \ , \tag{16}$$

where for notational simplicity we have written $\gamma \equiv \gamma_{123}$. Note that the term $(\gamma_{ij} - \gamma G_k(p))$ is equal to $q_{ij}(p)/(\sigma_i\sigma_j) \geq 0$, and so expression (16) reduces to (11) when interactions are symmetric.

Since 1 and 2 compete alone in the range $[p_0, L)$, we must have $\gamma_{12} > 0$, and since $G_3(L) = 0$ expression (16) with $k = 3$ implies $G_1(L) = G_2(L)$. Applying (16) with $k = 1$ then implies

$$0 \geq \gamma_{13} + \gamma_{23} - \gamma_{12} - \gamma G_1(L) \geq \gamma_{13} + \gamma_{23} - \gamma_{12} - \gamma\sigma_1 \ . \tag{17}$$

The same argument with $k = 2$ in (16) shows that $0 \geq \gamma_{13} + \gamma_{23} - \gamma_{12} - \gamma\sigma_2$, and so if only firms 1 and 2 use the minimum price p_0 it is necessary that

$$\gamma \min\{\sigma_1, \sigma_2\} \geq \gamma_{13} + \gamma_{23} - \gamma_{12} \ . \tag{18}$$

Since $\gamma_{13} \geq \gamma\sigma_2 \geq \gamma \min\{\sigma_1, \sigma_2\}$ and $\gamma_{23} \geq \gamma\sigma_1 \geq \gamma \min\{\sigma_1, \sigma_2\}$, expression (18) implies that $\gamma_{12} \geq \max\{\gamma_{13}, \gamma_{23}\}$. Thus, if the equilibrium involves only two firms pricing low those firms must have the greatest interaction among the three pairs. We deduce that if firms are labelled so that firms 1 and 2 have the greatest interaction, then when (18) is violated the equilibrium must have all three firms using the same minimum price.

[15] As discussed after Proposition 5, other possibilities arise on the knife edge between these two regimes.

This discussion is summarized in part (i) of the following result. This result also shows that when only two firms price low, the outcome is duopolistic pricing of the sort seen in section 4. In particular, it cannot be an equilibrium that two firms choose prices over the range $[p_0, 1]$ while the third firm chooses from an intermediate or upper range of prices.

Proposition 5 *Suppose that firms are labelled so that firms 1 and 2 have the greatest interaction, i.e.,* $\gamma_{12} \geq \max\{\gamma_{13}, \gamma_{23}\}$*, and that* $\sigma_1 \leq \sigma_2$*.*
(i) If

$$\gamma\sigma_1 < \gamma_{13} + \gamma_{23} - \gamma_{12} \tag{19}$$

then in equilibrium all firms use the same minimum price p_0*, which is the highest captive-to-reach ratio among the firms;*
(ii) If

$$\gamma\sigma_1 > \gamma_{13} + \gamma_{23} - \gamma_{12} \tag{20}$$

then in equilibrium price competition is duopolistic: there are prices $p_0 < p_1 \leq 1$ *such that firm 1 has price support* $[p_0, p_1]$*, firm 2 has support* $[p_0, 1]$ *and firm 3 has support* $[p_1, 1]$*. Expressions for the thresholds* p_0 *and* p_1 *and for firm profits are given in the proof.*

Part (i) of this result applies when interactions are similar across pairs of firms, as is the case with symmetric interactions, so long as Assumption 3 holds so that $\gamma\sigma_1 < \gamma_{23}$ in (19). With nested reach the two smallest firms have the strongest interaction and condition (19) reduces to $\beta_2 > \beta_3$. Thus with three nested firms, those cases not covered by Proposition 4 have all firms using the same minimum price in equilibrium. Condition (19) implies that $\gamma_{12} \leq \gamma_{13} + \gamma_{23}$, as is consistent with the discussion in section 4 showing that when $\gamma_{12} > \gamma_{13} + \gamma_{23}$ only two firms will price low.

Part (ii) applies when one pair of firms has significantly stronger interaction than other pairs. The two scenarios covered in section 4 (the chain store case and nested reach with increasing incremental reach), which exhibit duopolistic pricing, both satisfy (20).[16] If firms 1 and 2 are considered by almost the same set of consumers (so their circles on the Venn diagram almost coincide), and if $\alpha_3 > 0$, then firms 1 and 2 have the greatest interaction and condition (20) is satisfied, and firm 3 chooses price $p \approx 1$. Intuitively, when two firms reach nearly the same set of consumers, they compete fiercely between themselves, leaving

[16] Although in general the "chain store" and the "overlapping duopoly" pricing patterns in section 4 differ, they coincide when there are three firms.

the remaining firm to price at or near the monopoly level. Another situation where (20) holds is the asymmetric "one-or-all" specification in Baye *et al.* (1992, Section V), where no consumer considers exactly two firms and $\alpha_1 \le \alpha_2 < \alpha_3$, when the two smallest firms 1 and 2 have the greatest interaction and firm 3 chooses $p \equiv 1$.

When part (ii) applies, firm 3 is sure to set a higher price than firm 1. As such, these two firms do not compete in equilibrium, even though in general their reaches overlap. The outcome for consumers and firms in this market is exactly the same as with an alternative pattern of consideration where consumers who previously considered $\{1, 3\}$ now just consider $\{1\}$ and consumers who previous considered $\{1, 2, 3\}$ now consider $\{1, 2\}$. This alternative pattern is "coarser", involving only four segments in the Venn diagram, and is of the chain store form whereby firms 1 and 3 have disjoint reach.

In the knife-edge case where

$$\gamma\sigma_1 = \gamma_{13} + \gamma_{23} - \gamma_{12} \; , \tag{21}$$

which is not covered by Proposition 5, there is the possibility that both kinds of equilibrium coexist. For instance, this is so in the symmetric Varian-type market where $\alpha_{12} = \alpha_{13} = \alpha_{23} = 0$ and $\alpha_1 = \alpha_2 = \alpha_3$, where there is a symmetric equilibrium where all firms price low and also asymmetric equilibria where one of the firms chooses $p \equiv 1$. (See Baye *et al.* (1992) for the full range of equilibria in this market.) The chain store case where two local firms are symmetric also satisfies (21) and has multiple equilibria.[17]

Equilibrium strategies when all firms use the same minimum price: Proposition 5 provides much information about equilibria in this model—it characterizes equilibrium profit and consumer surplus in the two regimes, and it describes equilibrium strategies when part (ii) applies. However, it does not describe equilibrium pricing strategies for part (i), and the equilibrium patterns of prices turn out to have interesting economic features.

Proposition 1 shows that when interactions are precisely symmetric the pricing pattern is such that all three firms are active for low prices, and for higher prices only the two larger firms compete. In the earlier version of this paper (Armstrong and Vickers, 2018, Proposition 2) we calculated an equilibrium whenever part (i) of Proposition 5 applied

[17] Essentially, only the *sum* of the CDFs of the two local firms are determined in equilibrium in this case. It is then possible that one local firm uses prices in the whole range, while the other uses prices in an interior or upper interval, say. These knife-edge cases have a different pricing pattern from those found generically in Proposition 5.

(without showing if it was unique), and this took one of two forms: either (a) the three firms were active in a lower price range and then two were active in a range of higher prices (as with symmetric interactions), or (b) the three firms were active in a lower price range, then only the most interactive pair continued to be active in an intermediate price range, and then another pair of firms were active in a higher range. In particular, in situation (b) one firm (firm 3 using the labelling in Proposition 5) chose low and high prices, but not intermediate prices.

The general analysis was complicated, and here we merely report an example to show the possibility. Suppose three firms have nested reach, where $\sigma_1 = \frac{1}{2}$, $\sigma_2 = \frac{4}{5}$ and $\sigma_3 = 1$. We show in the appendix that equilibrium with this pattern of consideration has all firms choosing prices in the range $[\frac{1}{5}, \frac{9}{25}]$, firms 1 and 2 choosing prices in the range $[\frac{9}{25}, \frac{16}{25}]$ and firms 2 and 3 choosing prices in the range $[\frac{16}{25}, 1]$.

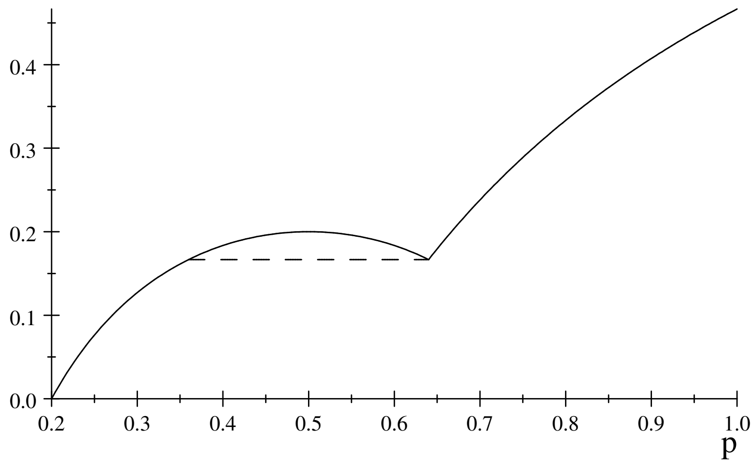

Figure 4: "Ironing" in a nested market with $\sigma_1 = 1/2$, $\sigma_2 = 4/5$, $\sigma_3 = 1$

The reason why the largest firm has non-convex price support can be explained as follows. When all firms price low in equilibrium, so that part (i) of Proposition 5 applies, one can calculate that the three CDFs increase in p for prices just above p_0, the minimum price. (The condition for the three CDFs to increase in p above p_0 is no vector (x_1, x_2, x_3) can be found to satisfy (13), which corresponds to $\gamma_{12} < \gamma_{13} + \gamma_{23}$ and which is implied by condition (19).) One can also calculate the smallest price, p_1 say, at which some CDF reaches 1 and above which the two remaining firms compete as duopolists for prices up to 1. (In the nested case, it is the smallest firm's CDF which first reaches 1, although in the general model more detailed analysis is required to determine which firm first drops out.)

However, in some cases—as in this example—firm 3's candidate CDF (i.e., when we ignore the monotonicity constraint on the CDF) starts to *decrease* in p before the largest CDF reaches 1, which cannot therefore be a valid CDF. Figure 4 illustrates this. The correct CDF for this firm is then obtained by "ironing" this curve as shown on the figure, so that the largest firm does not choose prices in the interval denoted by the dashed line, which in this example is the interval $(\frac{9}{25}, \frac{16}{25})$.[18]

These equilibria with ironing provide insight into the relationship between the two seemingly contrasting parts of Proposition 5. A configuration which is "well inside" the parameter space defined by (19) will have a pattern of prices as with symmetric interactions: all three firms choose low prices, then firm 1 drops out leaving firms 2 and 3 to compete in the range with high prices. As parameters change to approach the boundary (21), the candidate CDF for firm 3 will start to decrease before firm 1's CDF reaches 1. In this case, the "ironing" procedure is used so that firm 3's price support has a gap in the middle. As the boundary (21) is reached, the lower price range where all three firms are active shrinks and ultimately vanishes, leaving an equilibrium with duopolistic pricing.

The impact of entry: As an application of this analysis, consider entry by a third firm into a duopoly market. If the three firms post-entry have symmetric interactions, then Proposition 2 shows entry will increase consumer surplus. Beyond this case, however, the analysis is less clear cut. Entry might induce an incumbent to retreat towards its captive base by raising its price, thereby harming its captive customers. This is the case, for example, when the set of consumers reached by the entrant approximately coincides with the set reached by one of the incumbents. Then these firms will set prices $p \approx 0$, while the other incumbent chooses $p \approx 1$ and almost fully exploits its captive customers. Nevertheless, since entry of this form reduces industry profit, consumers overall will benefit.

In section 4 we have already seen how "nested" entry, which does not affect the number of captive customers in the market, might harm consumers overall. More generally, when entry only occurs within contested segments there is a tendency for entry to harm consumers overall. To illustrate, suppose the incumbents are symmetric and the entrant

[18] The CDF on Figure 4 does not reach 1 since this firm has an atom at $p = 1$ in equilibrium. The ironing procedure shown in Figure 4 is also used in the (otherwise distantly related) analysis in de Clippel, Eliaz, and Rozen (2014). Some of the asymmetric equilibria in Baye *et al.* (1992) also have a gap in one firm's support: two firms use the whole range $[p_0, 1]$, while the third chooses prices in a lower range $[p_0, p_1]$, with $p_1 < 1$, but also has an atom at $p = 1$.

is considered only by those consumers who already consider both incumbents, as shown on Figure 5. This is a natural pattern of consideration if only "savvy" consumers consider buying from the entrant, and these are the consumers who anyway consider both incumbents. In this case part (i) of Proposition 5 applies to the post-entry market (provided the entrant's reach lies strictly inside the incumbents' overlap). The minimum price is equal to an incumbent's captive-to-reach ratio, which is unchanged with entry. Thus, entry of this form leaves welfare and incumbent profit unaffected, increases industry profit due to the profit obtained by the entrant, and so harms consumers. In fact, it is possible that even the consumers who consider all three firms are harmed by this form of entry, despite being able to choose among more firms, as the higher prices offered by incumbents leave the entrant relatively free to set high prices too.

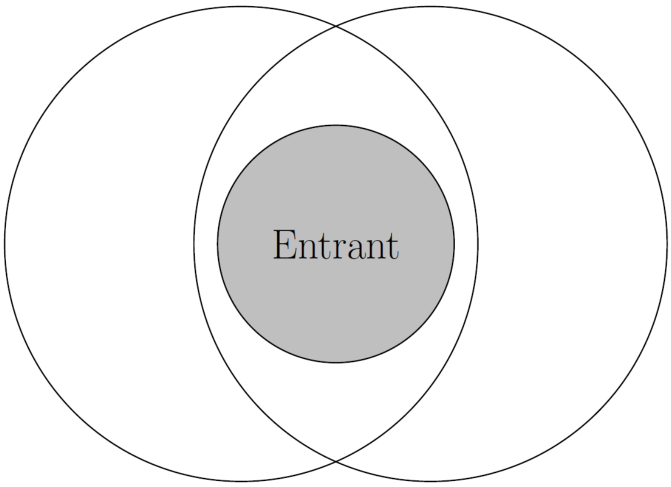

Figure 5: Entry into the contested market

This result is related to Rosenthal (1980), where entry by a new firm causes the average price paid by both captive and informed consumers to rise. However, in his model the entrant arrives with its own new pool of captive customers, thus raising welfare, whereas the effect arises in our scenario despite the entrant having none.[19]

[19] Relatedly, in a setting with differentiated products, Chen and Riordan (2008) show how entry to a monopoly market can induce the incumbent to raise its price. For instance, entry by generic pharmaceuticals might cause a branded incumbent to raise its price, as it prefers to focus on those relatively "captive" customers who care particularly about its brand. Pazgal, Soberman, and Thomadsen (2016) study a situation where two incumbents are located on a Hotelling line, and a third firm enters at a location to the left of the left-hand incumbent. Because the left-hand incumbent has less incentive to serve consumers to its left, it may raise its price, with the effect that industry profit could rise. Closer to the consideration set framework is Chen and Riordan (2007), who study a model with symmetric firms, where consumers either consider a single random firm or consider a random pair of firms. Among other results, they show that the equilibrium price can increase when an additional firm enters.

The impact of a merger: When a market has symmetric interactions, Proposition 2 shows that a profitable merger harms consumers overall. One can show that the same is always true when the initial market has three firms. Specifically, one can show that a profitable merger between two firms necessarily increases the third firm's profit. For instance, if the non-merging firm, say firm F, used the minimum price p_0 before merger, then the merger between the other two will increase F's profit because the minimum price must rise for the merger to be profitable.[20] The remaining case to consider is a merger between firms 1 and 2 under the conditions of part (ii) of Proposition 5. More detailed calculations reveal that here too firm 3's profit rises when firms 1 and 2 merge. We deduce that any profitable 3-to-2 merger is detrimental to consumers.

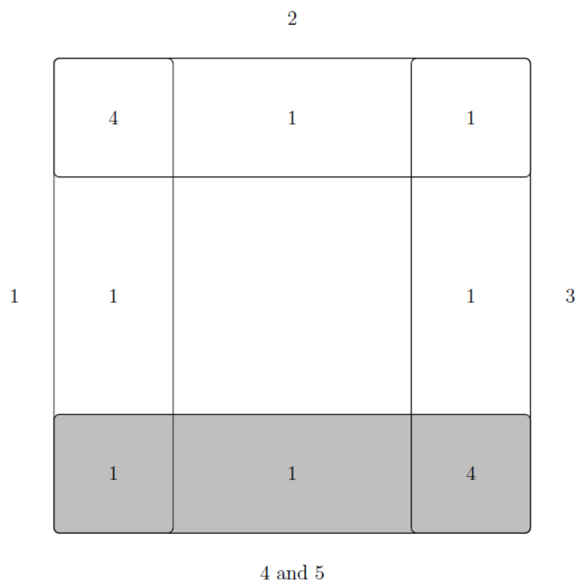

Figure 6: A profitable merger which benefits consumers

However, it is not always true, with more than three firms, that profitable mergers harm consumers. It may be, for example, that a merger between two firms with a strong interaction—which is therefore likely to be profitable—might induce non-merging firms to fight for the newly-profitable consumer segment, with the result that overall industry profits might fall and consumers are made better off. To illustrate this possibility consider the following example, which draws from our analysis of triopoly. Figure 6 depicts the

[20] Before the merger, the combined profit of the merging firms, say firms A and B, was at least $(\sigma_A + \sigma_B)p_0$, and since their combined reach falls after the merger, for the merger to be profitable the minimum price must rise.

pattern of consumer consideration. There are initially five firms, where firms 4 and 5 reach precisely the same set of consumers (depicted as the shaded set) and hence set price $p = 0$ in equilibrium.[21] Firms 1, 2 and 3 each have a single captive customer, a single consumer considers each set of firms $\{2, 3\}$, $\{4, 5\}$ and $\{1, 4, 5\}$, while four consumers consider each set of firms $\{1, 2\}$ and $\{3, 4, 5\}$. (No consumers consider more than three firms.)

Since firms 4 and 5 set price zero, firms 1, 2 and 3 compete as triopolists as if the shaded row on Figure 6 was eliminated. Here, firms 1 and 2 have the greatest interaction in this triopoly, and Proposition 5 implies that the equilibrium involves duopolistic pricing with firms 1 and 2 setting low prices and firms 2 and 3 setting high prices. The proof of part (ii) of the Proposition shows that firm 3 obtains its captive profit ($\pi_1 = 1$), while firms 1 and 2 obtain respective profits $\pi_1 = 5p_0$ and $\pi_2 = 6p_0$, where $p_0 = \frac{2}{7}$ is the minimum price. Since firms 4 and 5 make zero profit industry profit is $\frac{29}{7}$. Now suppose firms 4 and 5 merge. (Clearly this is a profitable merger, as before the firms obtained no profit.) Since the four firms are symmetrically placed, each firm now obtains its captive profit ($\pi_i = 1$), in which case industry profit falls to 4 after the merger, and consumers overall are better off.[22] Intuitively, before the merger the market was highly asymmetric, which allowed firms to enjoy high profits, and the merger brings more intense symmetric competition to the market. This example shows that not all profitable mergers in our setting are detrimental to consumers, but such competition-enhancing mergers appear to be relatively rare.

6 Conclusions

The aim of this paper has been to explore, in a parsimonious framework with price-setting firms and homogeneous products, how the pattern of consumer consideration determines the pattern of price competition.

In general, patterns of consideration were determined by firm reaches, σ_i, and how those reaches overlap as captured by the interaction parameters, γ_S. We introduced a relatively flexible class of consideration patterns, which we termed symmetric interactions, where the interaction parameters did not depend on the identity of firms in S. This class

[21] This violates Assumption 2. However, the same argument works if firms 4 and 5 have reaches which nearly coincide.

[22] Similar analysis shows that imposing a price ceiling on one firm in a market (which is akin to de-merging firms 4 and 5 on the figure) can cause industry profits to increase. Likewise, this example shows that entry might increase the profit of an incumbent firm: if the market initially consisted of firms 1 to 4, then entry by firm 5 strictly increases the profits of both firms 1 and 2.

includes cases studied in the existing literature—symmetric firms, independent reach, and duopoly—as particular cases. Within this class, in equilibrium all firms use the same minimum price, which implies that profits are proportional to a firm's reach. Firms choose their prices from an interval, and firms with smaller reach choose lower average prices. Markets within this class have intuitive properties with respect to exit and profitable merger, both of which raise the minimum price and harm consumers.

Outside the class of symmetric interactions, other pricing patterns and more novel comparative statics can emerge. We found equilibria with duopolistic pricing patterns, i.e., where only two firms compete in a given price range. In a market in which a chain store competed against local rivals, the former used the full range of prices while local firms used disjoint price ranges. With nested reach and increasing differences there was an overlapping duopoly price pattern where small firms only use low prices and large firms only use high prices. In the three-firm case we established that if all firms do not use the same minimum price then pricing was necessarily duopolistic. When one pair of firms had significantly greater interaction than other pairs, firms with a strong interaction focus their competitive efforts against each other, leaving a third firm able to set high prices. For some parameter configurations we found equilibria with a gap in one firm's price support, so that that firm sometimes prices high, and sometimes low, but never in between. We found plausible patterns of consumer consideration in which entry is detrimental to consumers because it softens competition between incumbents, leading them to retreat towards their captive base. Profitable mergers were shown always to be detrimental to consumers in the three-firm case, as with symmetric interactions, but not more generally.

The analysis could be extended in a number of directions. One way would be to allow firms to offer multiple "brands", where they are able to charge different prices for different brands and where consumers might consider some of its brands but not others. In this situation it would be interesting to understand when a multi-brand firm chooses distinct prices for its brands, and, if so, the resulting pattern of pricing within a firm. A second extension would be to endogenise the pattern of consideration by introducing search by consumers, word-of-mouth communication between consumers, or advertising by firms. For instance, Butters (1977), Ireland (1993) and McAfee (1994) studied a market where firms choose both their reach (via costly advertising) and price, under the assumption that reach was independent. Using the analysis in this paper, one could study the same issue but with alternative patterns of interaction (i.e., where $\gamma_S \neq 1$), such as when consumers differed in their attentiveness to advertising or when reach was nested rather than independent. Alternatively, if advertising is mediated by platforms and firms pay platforms for reach, those platforms may have an incentive to control the interactions γ_S (e.g., to choose which adverts are shown together) in order to stimulate or stifle competition between firms.

References

Armstrong, M., and J. Vickers (2018): "Patterns of Competition with Captive Customers," Department of Economics Working Paper No. 864, University of Oxford.

——— (2019): "Discriminating Against Captive Customers," *American Economic Review: Insights*, 1(3), 257–272.

Bahadur, R. (1961): "A Representation of the Joint Distribution of Responses to n Dichotomous Items," in *Studies in Item Analysis and Prediction*, ed. by H. Solomon, pp. 158–168. Stanford University Press.

Baye, M., D. Kovenock, and C. De Vries (1992): "It Takes Two to Tango: Equilibria in a Model of Sales," *Games and Economic Behavior*, 4, 493–510.

Bergemann, D., B. Brooks, and S. Morris (2020): "Search, Information and Prices," Cowles Foundation Working Paper 2224R.

Bulow, J., and J. Levin (2006): "Matching and Price Competition," *American Economic Review*, 96(3), 652–668.

Burdett, K., and K. Judd (1983): "Equilibrium Price Dispersion," *Econometrica*, 51(4), 955–969.

Butters, G. (1977): "Equilibrium Distributions of Sales and Advertising Prices," *Review of Economic Studies*, 44(3), 465–491.

Chen, Y., and M. Riordan (2007): "Price and Variety in the Spokes Model," *Economic Journal*, 117, 897–921.

——— (2008): "Price-Increasing Competition," *Rand Journal of Economics*, 39(4), 1042–1058.

Chioveanu, I., and J. Zhou (2013): "Price Competition with Consumer Confusion," *Management Science*, 59(11), 2450–2469.

Dasgupta, P., and E. Maskin (1986): "The Existence of Equilibrium in Discontinuous Economic Games, I: Theory," *Review of Economic Studies*, 53(1), 1–26.

de Clippel, G., K. Eliaz, and K. Rozen (2014): "Competiting for Consumer Inattention," *Journal of Political Economy*, 122(6), 1203–1234.

Draganska, M., and D. Klapper (2011): "Choice Set Heterogeneity and the Role of Advertising: An Analysis with Micro and Macro Data," *Journal of Marketing Research*, 48(4), 653–669.

Honka, E., A. Hortacsu, and M. A. Vitorino (2017): "Advertising, Consumer Awareness, and Choice: Evidence From the U.S. Banking Industry," *Rand Journal of Economics*, 48(3), 611–646.

Ireland, N. (1993): "The Provision of Information in a Bertrand Oligopoly," *Journal of Industrial Economics*, 41(1), 61–76.

Johnen, J., and D. Ronayne (2019): "The Only Dance in Town: Unique Equilibrium in a Generalised Model of Price Competition," mimeo.

McAfee, R. P. (1994): "Endogneous Availability, Cartels, and Merger in an Equilibirium Price Dispersion," *Journal of Economic Theory*, 62, 24–47.

Narasimhan, C. (1988): "Competitive Promotional Strategies," *Journal of Business*, 61, 427–449.

Pazgal, A., D. Soberman, and R. Thomadsen (2016): "Profit-Increasing Asymmetric Entry," *International Journal of Research in Marketing*, 33(1), 107–122.

Piccione, M., and R. Spiegler (2012): "Price Competition Under Limited Comparability," *Quarterly Journal of Economics*, 127(1), 97–135.

Prat, A. (2018): "Media Power," *Journal of Political Economy*, 126(4), 1747–1782.

Rosenthal, R. (1980): "A Model in which an Increase in the Number of Sellers Leads to a Higher Price," *Econometrica*, 48(6), 1575–1579.

Smith, H. (2004): "Supermarket Choice and Supermarket Competition in Market Equilibrium," *Review of Economic Studies*, 71(1), 235–263.

Sovinsky Goeree, M. (2008): "Limited Information and Advertising in the U.S. Personal Computer Industry," *Econometrica*, 76(5), 1017–1074.

Spiegler, R. (2006): "The Market for Quacks," *Review of Economic Studies*, 73(4), 1113–1131.

——— (2011): *Bounded Rationality and Industrial Organization*. Oxford University Press, Oxford, UK.

Szech, N. (2011): "Welfare in Markets where Consumers Rely on Word of Mouth," mimeo.

Varian, H. (1980): "A Model of Sales," *American Economic Review*, 70(4), 651–659.

Technical Appendix

Sketch proof of Lemma 1: We first discuss arguments to do with deletion of dominated prices. In any equilibrium we have $\pi_i \geq \alpha_i$, since firm i can ensure at least this profit by choosing price equal to 1 and serving its captive customers. For this reason, no firm would ever offer a price below ρ_i, its captive-to-reach ratio, since if it did so it would obtain profit below α_i even if it supplied its entire reach.

To see that the minimum price p_0 is positive we invoke Assumption 2. There is at least one firm i which has captive customers, and which will not set price below $\rho_i > 0$. From the remaining firms, at least one firm j has captive customers in the subset of firms excluding i, and so this firm can set price ρ_i and be sure to obtain positive profit. Firm j therefore also has a positive lower bound on its prices. Following the same argument, a firm in the subset of firms excluding both i and j can obtain positive profit, and so on until the set of firms is exhausted. In particular, each firm's minimum price is strictly above zero and hence so is p_0. If firm i chooses price p_0 (or just below), it will undercut all rivals and sell to its reach, so achieving profit $\sigma_i p_0$. This proves part (ii).

If price $p < 1$ is in firm i's support then its expected demand $q_i(\cdot)$ cannot be flat in a neighbourhood of p, for otherwise the firm could obtain strictly greater profit by raising its price above p. This implies that this price must be in the support of at least one other firm. More precisely, if price $p < 1$ is in firm i's support then i must compete against some other firm j at this price.

We next turn to arguments concerning the possibility of "atoms" in the price distributions. First observe that two firms cannot both have an atom at price p if they compete at this price (for otherwise each would have an incentive to undercut the price p and gain a discrete jump in demand).

To see that each firm's price distribution is continuous in the interval $[p_0, 1)$, suppose by contrast that firm i has an atom at some price $0 < p < 1$ in its support. We claim that firm's i demand $q_i(\cdot)$ must then be locally flat above p. As noted above, there cannot be another firm which competes with i at price p which also has an atom at p, and so q_i does not jump down discretely at p. In addition, any firm which competes with i at p obtains a discrete increase in demand if it slightly undercuts p, and so such a firm would never choose a price immediately above p. Since there is no firm which competes with i at prices immediately above p, firm i loses no demand if it raises its price slightly above p, which is not compatible with p maximizing the firm's profit. Therefore, firm i cannot have an atom at $p < 1$, and this completes the proof of part (iii). This implies that each firm's demand $q_i(p)$ is continuous in the interval $[p_0, 1)$.

Similarly, if p_0 is the minimum price ever chosen in the market, then all prices in the interval $[p_0, 1]$ are sometimes chosen. If p is in firm i's support but no firm is active in an interval (p, p') above p, then firm i has flat demand over the range (p, p'), and this cannot occur in equilibrium. This completes the proof of part (iv).

Suppose now that there are at least three firms. Let P_{ij} denote the set of prices in $[p_0, 1]$ which are in the supports of both firm i and firm j, which is a closed set. Part (iv) implies that the collection $\{P_{ij}\}$ covers the interval $[p_0, 1]$, and since each firm participates, at least two of the sets in $\{P_{ij}\}$ are non-empty. If there were no price in the support of three or more firms then the collection $\{P_{ij}\}$ would consist of disjoint sets. However, since $[p_0, 1]$ is connected it cannot be covered by two or more disjoint closed sets, and we deduce that at least two sets in $\{P_{ij}\}$ must overlap, which proves part (v).

Firms can have an atom at the reservation price $p = 1$. However, as noted above, if firm i has an atom at $p = 1$ then no potential competitor can also have an atom at 1, which implies that when firm i chooses $p = 1$ it sells *only* to its captive customers and so its profit is precisely $\pi_i = \alpha_i$. If no firm has an atom at $p = 1$ then any firm with $p = 1$

in its support (and there must be at two such firms from part (iv)) has profit equal to α_i. This completes the proof for part (i).

Let firm j be a firm which obtains profit equal to α_j. Then the minimum price ever chosen, p_0, must be no higher than ρ_j (for otherwise firm j could obtain more profit by choosing $p = p_0$), and so p_0 cannot exceed the highest ρ_i. Since no firm sets a price below its ρ_i, the minimum price p_0 (which from part (iv) is sometimes chosen by at least two firms) must be weakly above the second lowest ρ_i. Finally, if the firm with the highest ρ_i has p_0 in its support, then p_0 cannot be strictly lower than this highest ρ_i, and so must equal this highest ρ_i. This completes the proof for part (vi).

Proof of Lemma 2: Consider firm i's demand in (2), which can be expressed as

$$\frac{q_i(p)}{\sigma_i} = \frac{q_{i/j}(p)}{\sigma_i} - G_j(p)\frac{q_{ij}(p)}{\sigma_i\sigma_j} \ . \tag{22}$$

Note that the demand expression $q_{i/j}$ in (3) can be written in terms of the interaction parameters as

$$\frac{q_{i/j}(p)}{\sigma_i} = \sum_{S|i,j\notin S} (-1)^{|S|}\gamma_{S\cup i}\left(\Pi_{k\in S}G_k(p)\right) \ . \tag{23}$$

Because the interactions are symmetric, $\gamma_{S\cup i} = \gamma_{S\cup j}$ for each set S which excludes i and j, and so $q_{i/j}(p)/\sigma_i = q_{j/i}(p)/\sigma_j$. Expression (22) therefore implies (11).

Firm i's captive-to-reach ratio is $\alpha_i/\sigma_i = (\alpha_i + \alpha_{ij})/\sigma_i - \alpha_{ij}/\sigma_i$. Similarly to (23) we have

$$\frac{\alpha_i + \alpha_{ij}}{\sigma_i} = \sum_{S|i,j\notin S} (-1)^{|S|}\gamma_{S\cup i}\left(\Pi_{k\in S}\sigma_k\right) = \sum_{S|i,j\notin S} (-1)^{|S|}\gamma_{S\cup j}\left(\Pi_{k\in S}\sigma_k\right) = \frac{\alpha_j + \alpha_{ij}}{\sigma_j} \ ,$$

and (12) follows.

Proof of Proposition 1: The argument in the main text proves that all firms use the same minimum price p_0. Firm i's profit is therefore $\sigma_i p_0$, and if price p is also in its support then $q_i(p)/\sigma_i = p_0/p$. Therefore, expression (11) implies that if both firm i and firm j have p in their support then $G_i(p) = G_j(p)$ in equilibrium.

We claim that this implies that no firm can have a "gap" in its price support, so that firm i's support takes the form $[p_0, p_i]$ for some maximum price p_i. For if not, there is at least one firm with a gap, and let firm i be the firm with a gap which starts at the *lowest* price, say a, and where $b > a$ is the next price in firm i's support. Therefore, firm i does not use prices in the interval (a, b) and so $G_i(a) = G_i(b)$. If a firm is not active at price a, it will never use a higher price, and so any firm which is active at b must also be active at a. Part (iv) of Lemma 1 implies that at least two firms are active at any price in (a, b), and in particular, some firm, say j, which is active at b must also be active at prices immediately below b as well (as well as being active at a). Therefore, for this firm $G_j(b) > G_j(a)$, which contradicts $G_i(a) = G_i(b)$ given that both i and j are active at both a and b.

Since no firm has a gap, the price support of firm i takes the form $[p_0, p_i]$. Since $G_i(p_i) \equiv \sigma_i$, if $p_i > p_j$ then both i and j are active at p_j and so $\sigma_i > G_i(p_j) = G_j(p_j) = \sigma_j$ and firms with a higher maximum price must have a larger reach, and so with the given labelling of firms we have $p_1 \leq ... \leq p_n$. Since at least two firms must be active at $p = 1$ we have $p_{n-1} = p_n = 1$ as claimed. Finally, we show stochastic dominance in price strategies. For $p \in [p_0, p_1]$ all firms have price p in their support. Therefore, $G_1(p) = ... = G_n(p)$ and so $F_1(p) \geq ... \geq F_n(p)$. For $p \in [p_{i-1}, p_i]$, firms $j < i$ have $F_j(p) = 1$ while firms $j \geq i$ have p in their support. Therefore, $G_i(p) = ... = G_n(p)$, and so again $F_1(p) \geq ... \geq F_n(p)$ as claimed.

Proof of Proposition 2: First observe that when $n = 2$, exit or merger leaves a monopoly and so consumer obtain zero surplus and the result holds trivially. Therefore, in the following suppose $n \geq 3$.

(i) Welfare is the total fraction of consumers reached by any firm, which falls by α_i when firm i exits. Consumer surplus is welfare minus industry profit. Suppose that before exit the firm with the largest reach is labelled n, in which case industry profit is $\rho_n(\Sigma_i \sigma_i)$.

Suppose first that the firm which exits is not the largest, so $i \neq n$. Then the minimum price increases from α_n/σ_n to $(\alpha_n + \alpha_{in})/\sigma_n$, and the change in consumer surplus as a result of exit is

$$-\alpha_i + \frac{\alpha_n}{\sigma_n}\left(\sum_j \sigma_j\right) - \frac{\alpha_n + \alpha_{in}}{\sigma_n}\left(\sum_{j \neq i} \sigma_j\right) = -\frac{\alpha_{in}}{\sigma_n}\left(\sum_{j \neq n} \sigma_j\right) < 0 \ ,$$

where the equality follows from (12). If instead it is the largest firm, firm n, which exits, expression (12) implies that all remaining firms have captive-to-reach ratio which is greater than α_n/σ_n, and so again the minimum price rises after exit. Since firm n's profit was α_n, which was its contribution to welfare, the change in consumer surplus is the opposite to the change in the remaining firms' profit, and the latter is positive since the minimum price rises. This completes the proof for part (i).

(ii) Before the merger all firms used the same minimum price, p_0. Suppose two firms, i and j, merge to form firm M, and the new minimum price in the market is $\hat{p}_0$. The reach of firm M is $\sigma_M = \sigma_i + \sigma_j - \sigma_{ij}$, which is less than the sum of the reaches of i and j. We will show that $\hat{p}_0 > p_0$, which in turn implies that the impact of the merger on the profits of the non-merging firms must be positive (since the profit of a non-merging firm k is at least $\sigma_k \hat{p}_0$, which exceeds its pre-merger profit). Therefore, if the merger is profitable, industry profits rise and consumers are harmed.

Since the post-merger market need not exhibit symmetric interactions, it may be that only a subset of firms have $\hat{p}_0$ in their price support. First suppose that firm M uses the minimum price $\hat{p}_0$ and hence obtains profit $\sigma_M \hat{p}_0$. Since $\sigma_M < \sigma_i + \sigma_j$, in order for the merger to be profitable it must be that $\hat{p}_0 > p_0$ as claimed.

Next suppose that firm M does not use the minimum price $\hat{p}_0$, and its minimum price is $P > \hat{p}_0$. The demand function of the merged firm is $q_M(p) = q_{i/j}(p) + q_{j/i}(p) - q_{ij}(p)$. Let firm k be a non-merging firm which does price at p_0. A similar argument to that used to derive (11) shows that

$$\frac{q_{i/j}(P)}{\sigma_i} - \frac{q_k(P)}{\sigma_k} = \frac{q_{i/j}(P)}{\sigma_i} - \frac{q_{k/j}(P)}{\sigma_k} = q_{ik/j}(P)[\frac{F_M(P)}{\sigma_k} - \frac{F_k(P)}{\sigma_i}] < 0 \ .$$

Here, the first equality follows since P is the minimum price charged by the merged firm, and so firm k's demand at P is the same as if j was absent, while the second equality follows since the market without j continues to have symmetric interactions, and so the appropriate adjustment of (11) applies. (The function $q_{ik/j}$ is the demand from consumers who consider both i and k in the market absent j.) The final inequality follows since P is the merged firm's minimum price, so $F_M(P) = 0 < F_k(P)$. The same argument for $q_{j/i}$ shows that $q_{j/i}(P)/\sigma_j < q_k(P)/\sigma_k$.

Therefore,

$$p_0(\sigma_i + \sigma_j) \leq Pq_M(P) \leq P[q_{i/j}(P) + q_{j/i}(P)] < \frac{P}{\sigma_k} q_k(P)(\sigma_i + \sigma_j) \leq \hat{p}_0(\sigma_i + \sigma_j) \ ,$$

and hence $p_0 < \hat{p}_0$ as claimed. Here, the first inequality follows since post-profit, $Pq_M(P)$, is not below the pre-merger combined profit, and the final inequality follows since firm k weakly prefers the price $\hat{p}_0$ to P in the post-merger market.

Proof of Proposition 4: (i) No firm will ever choose a price strictly above the maximum price used by a larger firm, for it will then be undercut by the larger firm and so have no demand or profit. Therefore, $H_1 \leq ... \leq H_n$. Since at least two firms use the price $p = 1$, we have $H_{n-1} = H_n = 1$. (Firm n must have an atom at $p = 1$ for otherwise firm $n-1$ would have zero profit when its chooses price 1. Firm $n-1$ cannot also have an atom at $p = 1$ for otherwise these firms have an incentive to undercut each other.) Since price distributions are continuous, if firm $i < n-1$ choose the price H_{i+1} it will also have zero demand and profit, and so the inequality is strict for all firms except the largest two.

Turning to the minimum prices L_i, assume for contradiction that $L_{i+1} < L_i$ for some $i < n$. Define $A(p) \equiv \Pi_{k>i+1}(1 - F_k(p))$ as the probability that all firms larger than $i+1$ price above p. (As usual, if $i + 1 = n$ then set $A(p) \equiv 1$.) As a preliminary point, note that for all p, the ratio

$$\frac{q_i(p)}{A(p)(1 - F_{i+1}(p))} = [\beta_i + \beta_{i-1}(1 - F_{i-1}(p)) + \beta_{i-2}(1 - F_{i-1}(p))(1 - F_{i-2}(p)) + ...]$$

is weakly decreasing in p. In particular, since $F_{i+1}(L_{i+1}) = 0 < F_{i+1}(L_i)$

$$\frac{q_i(L_{i+1})}{A(L_{i+1})} \geq \frac{q_i(L_i)}{A(L_i)(1 - F_{i+1}(L_i))} > \frac{q_i(L_i)}{A(L_i)} \; . \tag{24}$$

By revealed preference

$$L_i q_i(L_i) \geq L_{i+1} q_i(L_{i+1}) \tag{25}$$

and

$$\begin{aligned} L_{i+1} q_{i+1}(L_{i+1}) &= L_{i+1}[\beta_{i+1} A(L_{i+1}) + q_i(L_{i+1})] \\ &\geq L_i q_{i+1}(L_i) = L_i \left[\beta_{i+1} A(L_i) + \frac{q_i(L_i)}{1 - F_{i+1}(L_i)}\right] \\ &> L_i[\beta_{i+1} A(L_i) + q_i(L_i)] \; , \end{aligned} \tag{26}$$

since $F_{i+1}(L_i) > 0$. From (25) and (26) we have

$$L_{i+1} A(L_{i+1}) > L_i A(L_i) \; . \tag{27}$$

But multiplying each side of (24) by the corresponding side of (27) implies that

$$L_{i+1} q_i(L_{i+1}) > L_i q_i(L_i)$$

contrary to (25). We deduce that minimum prices L_i weakly increase with i.

To show that $H_i > L_{i+1}$, note first that if $H_i < L_{i+1}$ then because L_j and H_j increase with j there is an interval of prices (H_i, L_{i+1}) used by no firm, which is contrary to Lemma 1. If $H_i = L_{i+1}$, Lemma 1 shows that at least one firm k other than i uses prices just below H_i. Since L_j increases with j we must have $k < i$. But since $i < n$ has no atom at $p = H_i$, firm k will almost surely be undercut by i when it uses $p \approx H_i$ in which case it makes zero profit (contrary to Lemma 1). We deduce that $H_i > L_{i+1}$.

(ii) If exactly $k \geq 3$ firms use the minimum price p_0, part (i) above shows that these will be the firms $i = 1, ..., k$. Following the same argument as at the start of section 4, $q_i'(p_0)/\sigma_i = -1/p_0$ for each firm $i \leq k$. From (6), this entails $\sum_{j\in\{1,...,k\}/i} \gamma_{ij} G_j'(p_0) = 1/p_0$ for each $i \leq k$. By Farkas's Lemma, this system of linear equations has no solution with each $G_i'(p_0) \geq 0$ if and only if there exists a vector $(x_1, ...x_k)$ satisfying

$$\sum_{j=1}^{k} x_j < 0 \text{ and } \sum_{j\in\{1,...,k\}/i} \gamma_{ij} x_j \geq 0 \text{ for each } i \leq k \ . \tag{28}$$

With nested reach we have $\gamma_{ij} = 1/\max\{\sigma_i, \sigma_j\}$. Let $x_1 = x_2 = 1$, $x_3 = ... = x_{n-1} = 0$ and $x_n = -(2 + \varepsilon)$, which has a negative sum. Then we have $\sum_{j\in\{1,...,k\}/1} \gamma_{1j} x_j = \sum_{j\in\{1,...,k\}/2} \gamma_{2j} x_j = \frac{1}{\sigma_2} - \frac{2+\varepsilon}{\sigma_k}$, for $2 < i < n$ we have $\sum_{j\in\{1,...,k\}/i} \gamma_{ij} x_j = \frac{2}{\sigma_i} - \frac{2+\varepsilon}{\sigma_k}$, and finally $\sum_{j\in\{1,...,k\}/k} \gamma_{jk} x_j = \frac{2}{\sigma_k}$. If $\sigma_k > 2\sigma_2$ then each of these terms is positive for sufficiently small $\varepsilon > 0$, and so (28) holds. Therefore, if $\sigma_k > 2\sigma_2$ it is not possible that k firms use the minimum price. Moreover, if $\tilde{k} > k$ then $\sigma_{\tilde{k}} > 2\sigma_2$ and so it is also impossible to have larger number of firms using the minimum price.

(iii) We construct an equilibrium of the stated form. The profit of the largest firm n is $\pi_n = \beta_n$, its number of captive customers, and denote the profit of smaller firms by π_i. In the highest interval $[p_{n-1}, 1]$ used by the two largest firms, these firms are sure to be undercut by all smaller rivals, and so in this range their CDFs must satisfy

$$\beta_n + \beta_{n-1}(1 - F_{n-1}(p)) = \frac{\beta_n}{p} \ ; \ \beta_{n-1}(1 - F_n(p)) = \frac{\pi_{n-1}}{p} \ .$$

Since $F_n(p_{n-1}) = 0$ it follows that p_{n-1} and π_{n-1} are related as $\pi_{n-1} = \beta_{n-1} p_{n-1}$. We have $F_{n-1}(1) = 1$, while the largest firm has an atom at $p = 1$ with probability $1 - F_n(1) = \pi_{n-1}/\beta_{n-1} = p_{n-1}$.

In the lowest interval $[p_1, p_2]$ used by the two smallest firms, these firms are sure to undercut all larger rivals, and so in this range their CDFs must satisfy

$$\beta_2 + \sigma_1(1 - F_1(p)) = \frac{\pi_2}{p} \ ; \ \sigma_1(1 - F_2(p)) = \frac{\pi_1}{p}$$

and since $F_1(p_1) = F_2(p_1) = 0$ it follows that

$$\pi_1 = \sigma_1 p_1 \ ; \ \pi_2 = (\sigma_1 + \beta_2) p_1 \ .$$

Since $F_1(p_2) = 1$ we have $\pi_2 = \beta_2 p_2$, which combined with the previous expression for π_2 implies that

$$p_2 = \frac{\sigma_1 + \beta_2}{\beta_2} p_1 \ . \tag{29}$$

If there are just three firms, these are the two price intervals in the equilibrium. With more than three firms there are intermediate intervals, and in the interval $[p_i, p_{i+1}]$, where $1 < i < n - 1$, firms i and $i + 1$ are active and will be undercut by smaller rivals and undercut their larger rivals. Therefore, in this range their CDFs must satisfy

$$\beta_{i+1} + \beta_i (1 - F_i(p)) = \frac{\pi_{i+1}}{p} \ ; \ \beta_i (1 - F_{i+1}(p)) = \frac{\pi_i}{p} \ . \tag{30}$$

Since $F_{i+1}(p_i) = 0$ it follows that $\pi_i = \beta_i p_i$.

An intermediate firm i, where $2 \leq i \leq n - 1$, is active in both the intervals $[p_{i-1}, p_i]$ and $[p_i, p_{i+1}]$, and its CDF F_i needs to be continuous across the threshold price p_i. At the price p_i we therefore require that

$$\frac{\pi_{i-1}}{\beta_{i-1} p_i} = 1 - F_i(p_i) = \frac{1}{\beta_i} \left(\frac{\pi_{i+1}}{p_i} - \beta_{i+1} \right) \ , \tag{31}$$

where in the case of $i = 2$ we have written $\beta_1 = \sigma_1$. If we write $p_n = 1$ then we have $\pi_i = \beta_i p_i$ for all firms $1 \leq i \leq n$, and so for $2 \leq i \leq n-1$ expression (31) entails expression (15). This is a second-order difference equation in p_i where p_1 is free, p_2 is given in (29), and the terminal condition $p_n = 1$ serves to pin down p_1. It is clear from (29) and (15) that the sequence $p_1, p_2, p_3, \ldots$ is an increasing sequence of price thresholds. This completes the description of the candidate equilibrium.

We next show that firms have no incentive to deviate from these strategies. By construction, firm i is indifferent between choosing any price in the interval $[p_{i-1}, p_{i+1}]$, assuming its rivals follow the stated strategies. We need to check that a firm's profit is no higher if it chooses a price outside this interval. Consider first an upward price deviation, which is only relevant if $i < n - 1$. If $i < n - 2$ and firm i chooses a price above p_{i+2} is has no demand since firm $i + 1$ is sure to set a lower price and all firm i's potential customers also consider firm $(i + 1)$'s price. Suppose then that $i < n - 1$ and firm i chooses a price

$p \in [p_{i+1}, p_{i+2}]$, in which case it has demand β_i if its price is below the prices of both rivals $i+1$ and $i+2$. Therefore, from (30) its profit with such a price is

$$p\beta_i[1 - F_{i+1}(p)][1 - F_{i+2}(p)] = \frac{\beta_i \pi_{i+1}}{\beta_{i+1}^2}\left(\frac{\pi_{i+2}}{p} - \beta_{i+2}\right) = p_{i+1}\frac{\beta_i\beta_{i+2}}{\beta_{i+1}}\left(\frac{p_{i+2}}{p} - 1\right) .$$

This profit decreases from $\pi_i = \beta_i p_i$ at $p = p_{i+1}$ to zero at $p = p_{i+2}$. We deduce that firm i cannot increase its profit by choosing a price above p_{i+1}.

Next consider a downward price deviation, so that firm i chooses a price below p_{i-1} (which is only relevant when $i > 2$). Suppose that this firm chooses a price in the interval $[p_j, p_{j+1}]$, where $j \leq i-2$. The firm will undercut all firms larger than firm $j+1$, and so obtain demand at least $\beta_{j+2} + ... + \beta_i$. It will also serve the segment β_{j+1} if it undercuts firm $j+1$ and it will additionally serve the segment β_j if it undercuts both firms j and $j+1$. Putting this together implies that the firm's profit with price $p \in [p_j, p_{j+1}]$ is

$$p\left\{\beta_{j+2} + ... + \beta_i + (1 - F_{j+1}(p))(\beta_{j+1} + \beta_j(1 - F_j(p))\right\} . \tag{32}$$

Given the CDFs in (30), this profit is a convex function of p and so must be maximized in this range either at p_j or at p_{j+1}. Therefore, we can restrict our attention to deviations by firm $i > 2$ to the threshold prices $\{p_1, p_2, ..., p_{i-2}\}$. If it chooses price p_j where $2 \leq j \leq i-2$, expression (32) implies its profit is $p_j\left\{\beta_{j+1} + ... + \beta_i + \beta_j(1 - F_j(p_j))\right\}$. Expression (31) implies that $\beta_j(1 - F_j(p_j))$ is equal to $\beta_{j+1}(\frac{p_{j+1}}{p_j} - 1)$, in which case the above deviation profit with price p_j is

$$p_j\left(\beta_{j+1} + ... + \beta_i + \beta_{j+1}(\frac{p_{j+1}}{p_j} - 1)\right) = \beta_{j+1}p_{j+1} + (\beta_{j+2} + ... + \beta_i)p_j . \tag{33}$$

One can check that expression (33) holds also for $j = 1$. We need to show that (33) is no higher than firm i's equilibrium profit, which is $\pi_i = \beta_i p_i$. We do this in two steps: (i) we show that (33) is increasing in j given i, so that $j = i-2$ is the most tempting of these deviations for firm i, and (ii) we show (33) is below $\beta_i p_i$ when $j = i-2$.

To show (i), suppose that $i \geq 4$, which is the only relevant case, and suppose that $1 \leq j \leq i-3$. Then firm i's deviation profit with price p_{j+1} from (33) is

$$\beta_{j+2}p_{j+2} + (\beta_{j+3} + ... + \beta_i)p_{j+1} = \beta_{j+1}p_j + \beta_{j+2}p_{j+1} + (\beta_{j+3} + ... + \beta_i)p_{j+1}$$

$$\geq \beta_{j+1}p_j + \beta_{j+2}p_{j+1} + (\beta_{j+3} + ... + \beta_i)p_{j+1} - (\beta_{j+2} - \beta_{j+1})(p_{j+1} - p_j)$$

$$= \beta_{j+1}p_{j+1} + \beta_{j+2}p_j + (\beta_{j+3} + ... + \beta_i)p_{j+1} \geq \beta_{j+1}p_{j+1} + (\beta_{j+2} + ... + \beta_i)p_j$$

where the final expression is the firm's deviation profit with price p_j, which proves claim (i). (Here, the first equality follows from (15), the first inequality follows from (14) and the fact that $\{p_j\}$ is an increasing sequence, while the final inequality follows from $\{p_j\}$ being an increasing sequence.)

To show claim (ii), suppose that $i \geq 3$ which is the only relevant case, and observe that

$$\begin{aligned} \beta_i p_i &= \beta_{i-1} p_{i-2} + \beta_i p_{i-1} \\ &\geq \beta_{i-1} p_{i-2} + \beta_i p_{i-1} - (\beta_i - \beta_{i-1})(p_{i-1} - p_{i-2}) \\ &= \beta_{i-1} p_{i-1} + \beta_i p_{i-2} \end{aligned}$$

where the final expression is (33) when $j = i - 2$. (Here, the first equality follows from (15) and the inequality follows from $\{\beta_i\}$ being an increasing sequence.) This completes the proof that the stated strategies constitute an equilibrium.

Proof of Proposition 5: Part (i) was demonstrated in the main text, so suppose now that (20) is satisfied. Lemma 1 shows that there is at least one price in all three price supports, and let L and H denote respectively the lowest and highest price among the prices in all three supports. (The set of prices in all three supports is closed.) If condition (20) holds we will show that $L = H$ so there is only one price in all three supports.

Suppose by contradiction that we have $H > L$. Either all three firms have the same minimum price p_0 (i.e., $L = p_0$) or only two firms do, and in the latter case the proof for part (i) shows that it must be firms 1 and 2 which price low. In either case firms 1 and 2 use p_0, and in either case we have $G_1(L) = G_2(L) = \delta \geq 0$. Let $g = \max\{G_1(H), G_2(H)\}$. Since we cannot have only one firm active in the open interval (L, H), one or both of 1 and 2 must choose prices in (L, H), and so $\delta < g$.

Firms 1 and 2 obtain respective profits $p_0\sigma_1$ and $p_0\sigma_2$, and let $\pi_3 \geq p\sigma_3$ denote firm 3's profit. Expression (10) shows that a price p in firm 3's support satisfies

$$\frac{\pi_3}{\sigma_3 p} = 1 - \gamma_{13} G_1(p) - \gamma_{23} G_2(p) + \gamma G_1(p) G_2(p) \ ,$$

and setting $p = L, H$ in the above and subtracting implies that

$$\begin{aligned} \frac{\pi_3}{\sigma_3}\left(\frac{1}{L} - \frac{1}{H}\right) &= \gamma_{13} G_1(H) + \gamma_{23} G_2(H) - \gamma G_1(H) G_2(H) \\ &\quad - \gamma_{13} G_1(L) - \gamma_{23} G_2(L) + \gamma G_1(L) G_2(L) \end{aligned}$$

$$\leq \gamma_{13}g + \gamma_{23}g - \gamma g^2 - \gamma_{13}\delta - \gamma_{23}\delta + \gamma\delta^2$$
$$= (g-\delta)(\gamma_{13} + \gamma_{23} - \gamma(g+\delta)) \ . \tag{34}$$

Here, the inequality follows since $\gamma_{13} \geq \gamma G_2(H)$ and $\gamma_{23} \geq \gamma G_1(H)$, and so the initial expression is weakly increased if we replace $G_1(H)$ and $G_2(H)$ by $g = \max\{G_1(H), G_2(H)\}$. Likewise, and using that fact that $G_3(L) = 0$, for firm $j = 1, 2$ we have

$$\begin{aligned} p_0\left(\frac{1}{L} - \frac{1}{H}\right) &= \gamma_{12}G_i(H) + \gamma_{j3}G_3(H) - \gamma G_3(H)G_i(H) - \gamma_{12}G_i(L) \\ &\geq \gamma_{12}g + \gamma_{j3}G_3(H) - \gamma g G_3(H) - \gamma_{12}\delta \\ &\geq (g-\delta)\gamma_{12} \ . \end{aligned}$$

Since $\pi_3 \geq \sigma_3 p_0$ and $g - \delta > 0$, it follows that

$$\gamma(g+\delta) \leq \gamma_{13} + \gamma_{23} - \gamma_{12} \ . \tag{35}$$

If $\gamma = 0$ this inequality contradicts (20), so we deduce that it is not possible to have $H > L$ when (20) holds and $\gamma = 0$. Therefore, suppose henceforth that $\gamma > 0$. Then since $g > 0$ the inequality (35) contradicts the first inequality in (17) which holds whenever $L > p_0$. We deduce that if $H > L$ then all three firms use the same minimum price p_0.

We show next that if all three firms use the same minimum price, then (20) cannot hold. Suppose that all three firms use p_0 and that H is the highest price in all three price supports. Then $\frac{q_1(H)}{\sigma_1} = \frac{q_2(H)}{\sigma_2} = \frac{q_3(H)}{\sigma_3}$ and (16) implies

$$(\gamma_{12} - \gamma_{23})G_2(H) + (\gamma_{13} - \gamma G_2(H))(G_3(H) - G_1(H)) = 0 \ , \tag{36}$$

$$(\gamma_{12} - \gamma_{13})G_1(H) + (\gamma_{23} - \gamma G_1(H))(G_3(H) - G_2(H)) = 0 \ . \tag{37}$$

Condition (20) implies that $\gamma_{12} > \max\{\gamma_{13}, \gamma_{23}\}$, and so these expressions imply that $G_3(H) < \min\{G_1(H), G_2(H)\}$ and also that the terms $(\gamma_{13} - \gamma G_2(H))$ and $(\gamma_{23} - \gamma G_1(H))$ are strictly positive. At least one of the $G_i(H)$ must equal σ_i.

Suppose $G_1(H) = \sigma_1$. If $\sigma_1 \geq G_2(H)$ then (37) implies that $G_3(H)$ has the sign of

$$(\gamma_{23} - \gamma\sigma_1)G_2(H) - (\gamma_{12} - \gamma_{13})\sigma_1 \leq (\gamma_{23} - \gamma\sigma_1)\sigma_1 - (\gamma_{12} - \gamma_{13})\sigma_1 < 0 \ ,$$

where the final strict inequality follows from (20). Since G_3 cannot be negative, this is not possible. Likewise, if $\sigma_1 \leq G_2(H)$ then (36) implies that $G_3(H)$ has the sign

$$(\gamma_{13} - \gamma G_2(H))\sigma_1 - (\gamma_{12} - \gamma_{23})G_2(H) \leq (\gamma_{13} - \gamma\sigma_1)\sigma_1 - (\gamma_{12} - \gamma_{23})\sigma_1 < 0 \ .$$

Therefore, $G_1(H) < \sigma_1$. If instead $G_2(H) = \sigma_2$ then we must have $G_1(H) \leq \sigma_2$. Then (36) again implies that $G_3(H) < 0$. We conclude that $G_3(H) = \sigma_3$.

If $H = 1$ then both firms 1 and 2 have an atom at $p = 1$, which requires $\alpha_{12} = 0$. However, if $\alpha_{12} = 0$ then $\gamma_{12} = \gamma\sigma_3$ and so (20) implies that $\gamma(\sigma_1 + \sigma_3) > \gamma_{13} + \gamma_{23} \geq \gamma(\sigma_1 + \sigma_2)$, and so $\sigma_3 \geq \sigma_2$, which contradicts the previous condition $\sigma_3 = G_3(H) < \min\{G_1(H), G_2(H)\}$. Therefore, we deduce that $H < 1$.

Since only firms 1 and 2 are active above price H, we must have $\alpha_{12} > 0$ and so $\gamma_{12} > \gamma\sigma_3$. Expression (16) with $k = 3$ then implies that $(G_1(p) - G_2(p))$ is constant for $p \in [H, 1]$, so that $G_1(1) - G_1(H) = G_2(1) - G_2(H) \equiv \eta > 0$. For $k \neq 3$ we have $\sigma_k p_0 = Hq_k(H) = q_k(1)$ and hence

$$p_0\left(\frac{1}{H} - 1\right) = \frac{q_k(H) - q_k(1)}{\sigma_k} = (\gamma_{12} - \gamma\sigma_3)\eta \ , \tag{38}$$

whereas $\sigma_3 p_0 = Hq_3(H) \leq q_3(1)$ and so

$$\begin{aligned} p_0\left(\frac{1}{H} - 1\right) &\leq \frac{q_3(H) - q_3(1)}{\sigma_3} \\ &= (\gamma_{13} + \gamma_{23})\eta - \gamma[G_1(1)G_2(1) - G_1(H)G_2(H)] \\ &= [(\gamma_{13} + \gamma_{23}) - \gamma(G_1(H) + G_2(H) + \eta)]\eta \ . \end{aligned}$$

With (38) this implies

$$\begin{aligned} \gamma_{12} - \gamma_{13} - \gamma_{23} &\leq \gamma[\sigma_3 - (G_1(H) + G_2(H) + \eta)] \\ &< -\gamma(G_k(H) + \eta) = -\gamma G_k(1) \end{aligned}$$

for $k = 1, 2$. (Here, the strict inequality follows since $\sigma_3 = G_3(H) < \min\{G_1(H), G_2(H)\}$.) As 1 and 2 cannot both have atoms at $p = 1$, $G_k(1) = \sigma_k \geq \sigma_1$ for some k, and (20) is contradicted.

In sum, we have shown that when (20) holds, there is only one price in the support of all three firms, say p_1, which strictly exceeds p_0, and only firms 1 and 2 are active in the range $[p_0, p_1)$. If $p_1 = 1$ then the proof is complete. If $p_1 < 1$ then only two firms are active in this range, one of which is firm 3. The remaining issue is which of firms 1 and 2 is the other firm active above p_1. Expression (16) implies that $\sigma_1 F_1(p) = \sigma_2 F_2(p)$ in the range $[p_0, p_1]$. If $\sigma_1 = \sigma_2$ then $F_1 = F_2$, and so one of these firms cannot drop out before the other and we must have $p_1 = 1$. If $\sigma_2 > \sigma_1$ then in the range $[p_0, p_1]$ we have $F_2 > F_1$ and so it is firm 1 which drops out first.

The final step in the proof is to determine the profits of the three firms, as well as the price thresholds p_0 and p_1. Since firms 1 and 2 have p_0 as their minimum price in this equilibrium, their profits are $\pi_1 = \sigma_1 p_0$ and $\pi_2 = \sigma_2 p_0$. In the range $[p_0, p_1]$ their adjusted CDFs are given by

$$\gamma_{12} G_j(p) = 1 - \frac{p_0}{p} \ ,$$

and firm 1 drops out at price p_1, so that the ratio p_0/p_1 satisfies

$$\gamma_{12}\sigma_1 = 1 - \frac{p_0}{p_1} \tag{39}$$

and so

$$G_2(p_1) = \sigma_1 \ . \tag{40}$$

Either firm 2 or 3 (or both) obtains exactly its captive profit.[23] Suppose first that firm 3 obtains its captive profit, so that $\pi_3 = \alpha_3$. For prices in the upper range $[p_1, 1]$ only firms 2 and 3 compete and are sure to be undercut by firm 1, so from (10) firm 2's CDF satisfies

$$1 - \gamma_{13}\sigma_1 - \gamma_{23} G_2 + \gamma \sigma_1 G_2 = \frac{\rho_3}{p} \ ,$$

where recall that ρ_3 is firm 3's captive-to-reach ratio. In order for $G_2(\cdot)$ to be continuous at the threshold price p_1, (40) implies that

$$1 - \gamma_{13}\sigma_1 - \gamma_{23}\sigma_1 + \gamma\sigma_1^2 = \frac{\rho_3}{p_1} \ ,$$

which determines p_1. Expression (39) in turn implies that

$$p_0 = p_1(1 - \gamma_{12}\sigma_1) = \frac{\rho_3(1 - \gamma_{12}\sigma_1)}{1 - \gamma_{13}\sigma_1 - \gamma_{23}\sigma_1 + \gamma\sigma_1^2} \ . \tag{41}$$

It is convenient to write P for the right-hand side above, so that

$$P = \frac{\rho_3(1 - \gamma_{12}\sigma_1)}{1 - \gamma_{13}\sigma_1 - \gamma_{23}\sigma_1 + \gamma\sigma_1^2} = \frac{\alpha_3(\alpha_2 + \alpha_{23})}{\alpha_3\sigma_2 + \alpha_{23}(\sigma_2 - \sigma_1)} \ , \tag{42}$$

where the second equality follows by routine manipulation. Note from the first expression for P above that the condition $P < \rho_3$ corresponds to (20). Expression (41) implies

$$p_1 = \frac{\alpha_3\sigma_2}{\alpha_3\sigma_2 + \alpha_{23}(\sigma_2 - \sigma_1)} \ . \tag{43}$$

[23] If one of these firms has no atom at $p = 1$ then the other obtains its captive profit when it chooses $p = 1$. If both have an atom at $p = 1$ then for neither to have an incentive to undercut the other we must have $\alpha_{23} = 0$, in which case both firms obtain their captive profit at $p = 1$.

Alternatively, suppose firm 2 obtains its captive profit, so that $\pi_2 = \alpha_2$. Since the firm has p_0 as its lowest price it follows that

$$p_0 = \rho_2 \ . \tag{44}$$

Expression (39) then implies that

$$p_1 = \frac{\alpha_2}{\alpha_2 + \alpha_{23}} \ . \tag{45}$$

For prices in the upper range $[p_1, 1]$ firm 2's CDF now satisfies

$$1 - \gamma_{13}\sigma_1 - \gamma_{23}G_2 + \gamma\sigma_1 G_2 = \frac{\pi_3}{\sigma_3 p} \ ,$$

where π_3 is firm 3's profit. For G_2 to be continuous at $p_1 = \alpha_2/(\alpha_2 + \alpha_{23})$, (40) implies

$$1 - \gamma_{13}\sigma_1 - \gamma_{23}\sigma_1 + \gamma\sigma_1^2 = \frac{\alpha_2 + \alpha_{23}}{\alpha_2} \cdot \frac{\pi_3}{\sigma_3} \ ,$$

which determines π_3. This can be expressed as

$$\pi_3 = \frac{\alpha_3 \rho_2}{P} \tag{46}$$

where P is given in (42).

Finally, we determine when it is that firm 2 or firm 3 obtains its captive profit. When firm 3 obtains its captive profit, firm 2's minimum price is P in (42), which must be no lower than ρ_2 if firm 2 is willing to offer this price. Therefore, if $P < \rho_2$ the equilibrium must instead have firm 2 obtaining its captive profit, in which case the threshold prices and firm 3's profit are given respectively by (44), (45) and (46). Conversely, when firm 2 obtains its captive profit, firm 3's profit is given in (46). This profit cannot be below its captive profit α_3, which therefore requires $P \leq \rho_2$. Therefore, if $P > \rho_2$ the equilibrium must involve firm 3 obtaining its captive profit, so $\pi_3 = \alpha_3$, and the threshold prices p_0 and p_1 are given respectively by (42) and (43). Finally, in the knife-edge case where $P = \rho_2$ the two equilibria coincide, and firms 2 and 3 each obtain their captive profit. This completes the proof.

Details for the nested example in section 5: Recall that the example for Figure 4 has nested reach with $\sigma_1 = \frac{1}{2}$, $\sigma_2 = \frac{4}{5}$ and $\sigma_3 = 1$. Then part (i) of Proposition 5 applies, and all firms use the same minimum price $p_0 = \frac{1}{5}$ and have profits $\sigma_i p_0$. In this example we have

$\gamma_{13} = \gamma_{23} = 1$ and $\gamma_{12} = \gamma = \frac{5}{4}$, and so expression (10) implies that for any price in the support of all three firms we have

$$1 - \frac{5}{4}G_2 - G_3 + \frac{5}{4}G_2G_3 = 1 - \frac{5}{4}G_1 - G_3 + \frac{5}{4}G_1G_3 = 1 - G_1 - G_2 + \frac{5}{4}G_1G_2 = \frac{1}{5p} \ . \quad (47)$$

These simultaneous equations can be solved to give

$$G_1(p) = G_2(p) = \frac{4}{5} - \frac{2}{5}\sqrt{\frac{1-p}{p}} \ ; \ G_3(p) = 1 - \frac{2}{5}\sqrt{\frac{1}{p(1-p)}}. \quad (48)$$

Here, each G_i is zero at $p = p_0$ and G_1 and G_2 increase with p for prices above p_0.

A candidate solution is that all three firms choose prices in the range $[p_0, p_1]$, then firm 1 drops out leaving firms 2 and 3 active in the range $[p_1, 1]$. Here F_1 reaches 1, i.e., G_1 reaches $\sigma_1 = \frac{1}{2}$, at $p_1 = \frac{16}{25}$. For prices above p_1 firms 2 and 3 compete alone, with firm 1 sure to undercut them, in which case the required adjusted CDFs in (47) are given by

$$G_2(p) = \frac{4}{3} - \frac{8}{15p} \ ; \ G_3(p) = 1 - \frac{8}{15p} \ .$$

The problem with this candidate solution, however, is that G_3 $(= F_3)$ in (48) increases with p only for prices below $\frac{1}{2}$, and thereafter it decreases with p as depicted as the solid curve on Figure 4. The correct solution is then obtained by "ironing" this CDF as shown as the dashed line on the figure so that F_3 is flattened to be no greater than the level $F_3(p_1) = \frac{1}{6}$ for prices below p_1. The smaller root of $G_3 = \frac{1}{6}$ in (48) is $\hat{p} = \frac{9}{25}$.

In this example, all three firms are active in the price range $[\frac{1}{5}, \frac{9}{25}]$, only firms 1 and 2 are active in the interior range $[\frac{9}{25}, \frac{16}{25}]$, and then only firms 2 and 3 are active in the range $[\frac{16}{25}, 1]$. In the interior range $[\frac{9}{25}, \frac{16}{25}]$, the adjusted CDFs G_1 and G_2 need modifying from (48) to reflect that they will be undercut by firm 3 with the constant probability $F_3(p_1) = \frac{1}{6}$ in this range (in which case they have no demand), so that

$$G_1(p) = G_2(p) = \frac{4}{5} - \frac{24}{125p} \ .$$

(Again, G_1 reaches $\sigma_1 = \frac{1}{2}$ at $p_1 = \frac{15}{25}$.) With these CDFs, one can check that firm 3 does not gain by choosing a price in the interior range $[\frac{9}{25}, \frac{16}{25}]$, and that firm 1 has no incentive to choose a price above $p_1 = \frac{16}{25}$, so that this is indeed an equilibrium.