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30 July 2020

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July 2020

http://ssrn.com/abstract=3678651

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ABSTRACT

This paper has proposed new option Greeks and new upper and lower bounds for European and American options. It also shows that because of the put-call parity, the Greeks of put and call options are interconnected and should be shown simultaneously. In terms of the theory of the firm, it is found that both the Black-Scholes-Merton and the binomial option pricing models implicitly assume that maximizing the market value of the firm is not equivalent to maximizing the equityholders' wealth. The binomial option pricing model implicitly assumes that further increasing (decreasing) the promised payment to debtholders affects neither the speed of decreasing (increasing) in the equity nor the speed of increasing (decreasing) in the insurance for the promised payment. The Black-Scholes-Merton option pricing model, on the other hand, implicitly assumes that further increasing (decreasing) in the promised payment to debtholders will: (1) decrease (increase) the speed of decreasing (increasing) in the equity though bounded by upper and lower bounds, and (2) increase (decrease) the speed of increasing (decreasing) in the insurance though bounded by upper and lower bounds. The paper also extends the put-call parity to include senior debt and convertible bond. It is found that when the promised payment to debtholders is approaching the market value of the firm and the risk-free interest rate is small, both the owner of the equity and the owner of the insurance will be more reluctant to liquidate the firm. The lower bound for the risky debt is: the promised payment to debtholders is greater or equal to the market value of the firm times one plus the risk-free interest rate.

Keywords: The put-call parity, option Greeks, the binomial option pricing model, risk level of debt.

JEL Classification: G13, G32.

1. Introduction

The seminal works of Black and Scholes (1973) and Merton (1973a) have presented the Black-Scholes-Merton option pricing model in continuous time. Cox, Ross and Rubinstein (1979) have proposed the binomial option pricing model in discrete time. Merton (1973a) also derives some option Greeks under the Black-Scholes-Merton model. Merton (1973b) modifies Stoll's (1969) put-call parity and gives the upper bound for European put option. These option pricing models and Greeks are very useful in hedging risks and pricing derivatives as well as other financial assets such as equity and debt of the firm.

This paper corrects and proposes new upper and lower bounds for both European and American options and new option Greeks. It also shows that because of the put-call parity, the Greeks of put and call options are interconnected and should be shown simultaneously. A put option can be regarded as an insurance to insure the promised payment to debtholders. In terms of the theory of the firm, it is found that both the Black-Scholes-Merton and the binomial option pricing models implicitly assume that maximizing the market value of the firm is not equivalent to maximizing the equityholders' wealth. The binomial option pricing model implicitly assumes that further increasing (decreasing) the promised payment to debtholders affects neither the speed of decreasing (increasing) in the equity nor the speed of increasing (decreasing) in the insurance. The Black-Scholes-Merton option pricing model, on the other hand, implicitly assumes that further increasing (decreasing) in the promised payment to debtholders will: (1) decrease (increase) the speed of decreasing (increasing) in the equity though bounded by upper and lower bounds, and (2) increase (decrease) the speed of increasing (decreasing) in the insurance though bounded by upper and lower bounds. The paper also extends the put-call parity to include senior debt and convertible bond. It is found that when the promised payment to debtholders is approaching the market value of the firm and the risk-free interest rate is small, both the owner of the equity and the owner of the insurance will be more reluctant to liquidate the firm. It also specifies the conditions that American put option will not be exercised before the expiration date. The lower bound for risky debt is: the promised payment to debtholders is greater or equal to the market value of the firm times one plus the risk-free interest rate.

The remainder of this paper is organized as follows. Section 2 presents the model-free option Greeks and upper and lower bounds for European and American options. The Greeks under the binomial option pricing model are discussed in Section 3. The extensions of the put-call parity are shown in Section 4. Concluding remarks appear in Section 5.

2. Model-Free Greeks

Options are rights and not obligations, and have a limited life time (e.g., from $t = 0$ to $t = T$). A call option gives the holder the right to buy the underlying for the strike (exercised) price K by the date T. A put option gives the holder the right to sell the underlying for the price K by the date T. At $t = 0$, the price of a European call option $c \geq 0$, and $c > 0$ if and only if people believe that at $t = T$, it is possible (i.e., there is a positive probability) to have $Max[S_T - K, 0] > 0$, where S_T is the price of the underlying asset. Similarly, at $t = 0$, the price of a European put option is $p \geq 0$, and $p > 0$ if and only if people believe that at $t = T$, it is possible to have $Max[K - S_T, 0] > 0$. If $K = 0$, all options do not exist. At $t = 0$, the price of an asset (again, a right and not an obligation) is $S_0 \geq 0$, and $S_0 = 0$ if and only if people believe $S_t = 0,\ \forall t > 0$. Since $S_t = 0,\ \forall t > 0$, is not a random variable, all options do not exist.

Suppose that one European call option and one European put option are related to the same one-unit underlying asset, and have the same expiration date T and the same strike price K. Then, at $t = 0$, the following put-call parity holds:[1]

$$c + \frac{K}{1+r} = S_0 + p \tag{1}$$

where r is the simple risk-free interest rate.

Rearrange eq. (1):

$$S_0 = c + \left(\frac{K}{1+r} - p\right) , \tag{2}$$

where S_0 can be interpreted as the market value of the levered firm, c as the equity of the firm, $\left(\frac{K}{1+r} - p\right)$ as the risky debt of the firm, and p as the insurance to insure the promised payment K to debtholders. In the case of riskless debt, the insurance $p = 0$ and

$$S_0 = c + \frac{K}{1+r} , \tag{3}$$

where $\frac{K}{1+r}$ is the riskless debt. Hence, for a given K, the magnitude of the put option (i.e., the insurance)

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[1] Consider two portfolios at $t = 0$:

Portfolio A: one European call option c with strike price K, and cash $\frac{K}{1+r}$ deposited in a bank;

Portfolio B: one European put option p with strike price K, and one unit of the underlying asset S_0.

On the expiration date $t = T$, both portfolios give exactly the same payoff: $Max[S_T, K]$. Thus, the costs of the two portfolios at $t = 0$ must be the same.

p is a measurement of the risk level of the risky debt $\left(\frac{K}{1+r} - p\right)$. At $t = T$, if the equityholders pay K to the debtholders, then the equityholders can have the firm, S_T.[2]

From eq. (1) we can have $c = p$ if and only if $S_0 = \frac{K}{1+r}$. However, it is meaningless to say that $c = p = 0$ implies $S_0 = \frac{K}{1+r}$. This is because $c = 0$ implies $S_T \leq K$ and $p = 0$ implies $K \leq S_T$, i.e., $S_T = K$ is not a random variable, all options do not exist. Also, in terms of firm's capital structure, $p = 0$ means the promised payment K is riskless (i.e., a riskless debt) and $c = 0$ implies $S_T \leq K$, and hence, the asset S_0 must be a default-less fixed-income asset: $S_0 = \frac{K}{1+r}$. Thus, we have: if options exist, $c = p > 0$ if and only if $S_0 = \frac{K}{1+r}$.

Upper and Lower Bounds for Options Prices

From eq. (1), we have: $p = \frac{K}{1+r} - S_0 + c \geq \frac{K}{1+r} - S_0$, and hence, the lower bound for the European put option is: $p \geq \frac{K}{1+r} - S_0$. Also, because the market value of risky debt cannot be negative, i.e., $\frac{K}{1+r} - p \geq 0$, we have: $p \leq \frac{K}{1+r}$. However, it is impossible to have $p = \frac{K}{1+r}$. This is because at $t = 0$ people will use $\frac{K}{1+r}$ to buy a default-less zero-coupon bond which gives K with certainty at $t = T$ rather than use $\frac{K}{1+r}$ to buy an insurance p which may give at most K at $t = T$. That is, an asset cannot sell for more than or equal to the present value of a sure payment of its maximum pay-off. Also, $K > 0$ if and only if the risky debt $\frac{K}{1+r} - p > 0$, and $K = 0$ implies that no options exist, and both the debt $\left(\frac{K}{1+r} - p\right)$ and the put option p do not exist. Thus, at $t = 0$, the upper and lower bounds for the European put option are:

$$\frac{K}{1+r} > p \geq Max\left[\frac{K}{1+r} - S_0, 0\right]. \tag{4}$$

For the American put option P, its price at $t = 0$ cannot be greater than K, i.e., $P \leq K$. But it is nonsense to have $P = K$ because at $t = 0$, people will not use K to buy an insurance P which later on may give at most K. Thus, at $t = 0$, the upper and the lower bounds for the American put option are:[3]

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[2] Chang (2015, p. 26) shows that because changes of K (i.e., higher or lower debt) do not affect S_0, the Modigliani-Miller First Proposition is a corollary of the put-call parity. Capital Structure Irrelevancy Proposition I should be written as: In a complete market with no transaction costs and no arbitrage, the market value of the firm is independent of its capital structure.

[3] The literature of derivatives (e.g., Hull (2018, p. 269) and Merton (1973b, p. 183)) incorrectly states the upper and lower

$$K > P \geq Max[K - S_0, 0]. \tag{5}$$

From eq. (1), we also have: $c = S_0 - \frac{K}{1+r} + p \geq S_0 - \frac{K}{1+r}$, and hence, the lower bound for European call option is: $c \geq Max\left[S_0 - \frac{K}{1+r}, 0\right]$. The European call c cannot be greater than the underlying asset S_0. At $t = T$, the best scenario for the holder of c is payoff S_T where $K = 0$. But $K = 0$ means that all options do not exist. Hence, we must have $c < S_0$. The upper and the lower bounds for the European call option at $t = 0$ are:

$$S_0 > c \geq Max\left[S_0 - \frac{K}{1+r}, 0\right]. \tag{6}$$

The upper and the lower bounds for the American call option C at $t = 0$ are:

$$S_0 > C \geq Max\left[S_0 - \frac{K}{1+r}, 0\right]. \tag{7}$$

When the insurances p and P decrease (i.e., debts become less risky), c and C approach their lower bounds as in eq. (3).

The Greeks

The put-call parity in eq. (1) shows the interconnection between call and put options. Thus, the Greeks of call and put options must be determined simultaneously. For example, we have: $\frac{\partial c}{\partial x} = \frac{\partial p}{\partial x}$ for any $x \notin \{S_0, r, K\}$ (see also Chang, 2015, p. 28).

Case 1. Either $c = 0$ or $p = 0$.

(i) With $p = 0$, $\frac{\partial p}{\partial S} = 0$ and $\frac{\partial c}{\partial S} = 1 > 0$ indicates that with riskless debt, maximizing the firm's market value is equivalent to maximizing the equityholders' wealth.

$\frac{\partial p}{\partial r} = 0$ and $\frac{\partial c}{\partial r} = \frac{K}{(1+r)^2} > 0$ (and $\frac{\partial}{\partial r}\left(\frac{K}{1+r}\right) = \frac{-K}{(1+r)^2} < 0$) indicates that with riskless debt, higher interest rate transfers the debtholders' wealth to the equityholders.

$\frac{\partial p}{\partial K} = 0$ and $\frac{\partial c}{\partial K} = \frac{-1}{1+r} < 0$ (and $1 > \frac{\partial}{\partial K}\left(\frac{K}{1+r}\right) = \frac{1}{1+r} > 0$) indicates that with riskless debt, higher promised payment K results in higher leveraged firm.

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bounds of the put options as:

$$\frac{K}{1+r} \geq p \geq Max\left[\frac{K}{1+r} - S_0, 0\right] \text{ and } K \geq P \geq Max[K - S_0, 0].$$

Hull (2018 , p. 270) and Merton (1973b, p. 183) erroneously argue that "the maximum pay-off to a European put is the exercise price, K, which occurs if the underlying asset price S_0 is zero". This argument is wrong because $S_0 = 0$ if and only if people believe $S_t = 0,\ \forall t > 0$. Since $S_t = 0,\ \forall t > 0$, is not a random variable, all options do not exist.

(ii) With $c = 0$, this implies that the owner of the firm is both the debtholder and the equitholder, i.e., at $t = T$, the debtholder obtains $Min[S_T, K]$ and the equityholder obtains $Max[S_0 - K, 0]$, and $Min[S_T, K] + Max[S_T - K, 0] = S_T$, where S_T is also the payoff of the totally equity-financed firm. $\frac{\partial c}{\partial s} = 0$ and $\frac{\partial p}{\partial s} = -1 < 0$; $\frac{\partial c}{\partial r} = 0$ and $\frac{\partial p}{\partial r} = \frac{-K}{(1+r)^2} < 0$; $\frac{\partial c}{\partial K} = 0$ and $1 > \frac{\partial p}{\partial K} = \frac{1}{1+r} > 0$.

Case 2. $c > 0$ and $p > 0$.

About $\frac{\partial c}{\partial s}$ and $\frac{\partial p}{\partial s}$:

(i) It is impossible to have: $\frac{\partial c}{\partial s} \leq 0$ and $\frac{\partial p}{\partial s} \geq 0$ because from eq. (1), we have: $\frac{\partial c}{\partial s} = 1 + \frac{\partial p}{\partial s}$. This indicates that increasing the market value of the firm cannot lead to: lower (or no change in) equity and lower (or no change in) debt.

(ii) $\frac{\partial c}{\partial s} > 0$ and $\frac{\partial p}{\partial s} < 0$ (where $0 < \frac{\partial c}{\partial s} < 1$ and $-1 < \frac{\partial p}{\partial s} < 0$). This indicates that increasing the firm's market value benefits both the equityholders and the debtholders (by reducing the risk level of the debt), and maximizing the market value of the firm is not equivalent to maximizing the equityholders' wealth. Both the Black-Scholes-Merton and the binomial option pricing models have this property (see the following section).

(iii) $\frac{\partial c}{\partial s} = 0$ and $\frac{\partial p}{\partial s} = -1 < 0$. This indicates that increasing the firm's market value only benefits the debtholders, and the equiltyholders get no benefit.

(iv) $\frac{\partial p}{\partial s} = 0$ and $\frac{\partial c}{\partial s} = 1 > 0$. This indicates that increasing the firm's market value only benefits the equiltyholders, and the debtholders get no benefit, i.e., maximizing the firm's market value is equivalent to maximizing the equityholders' wealth.

(v) $\frac{\partial c}{\partial s} > 0$ and $\frac{\partial p}{\partial s} > 0$ (where $\frac{\partial c}{\partial s} > 1$, and $\frac{\partial p}{\partial s} > 0$ leads to $\frac{\partial c}{\partial s} > 0$). This indicates that increasing the firm's market value not only benefits the equiltyholders but transfers parts of the debtholders' wealth to the equiltyholders by increasing p, the risk level of the debt.

(vi) $\frac{\partial c}{\partial s} < 0$ and $\frac{\partial p}{\partial s} < 0$ (where $\frac{\partial p}{\partial s} < -1$, and $\frac{\partial c}{\partial s} < 0$ leads to $\frac{\partial p}{\partial s} < 0$). This indicates that increasing the firm's market value not only benefits the debtholders but transfers parts of the equiltyholders' wealth to the debtholders by reducing p, the risk level of the debt.

About $\frac{\partial c}{\partial r}$ and $\frac{\partial p}{\partial r}$:

(i) It is impossible to have: $\frac{\partial c}{\partial r} \leq 0$ and $\frac{\partial p}{\partial r} \geq 0$ because from eq. (1), we have: $\frac{\partial c}{\partial r} - \frac{K}{(1+r)^2} = \frac{\partial p}{\partial r}$. This indicates that with higher risk-free interest rate, keeping the same market value of the firm cannot lead to: lower (or no change in) equity and higher (or no change in) insurance p.

(ii) $\frac{\partial c}{\partial r} > 0$ and $\frac{\partial p}{\partial r} < 0$ (where $0 < \frac{\partial c}{\partial r} < \frac{K}{(1+r)^2}$ and $\frac{-K}{(1+r)^2} < \frac{\partial p}{\partial r} < 0$). This indicates that with higher risk-free interest rate, keeping the same market value of the firm leads to higher equity (and hence, lower debt) and lower insurance. Both the Black-Scholes-Merton and the binomial option pricing models have this property (see the following section).

(iii) $\frac{\partial c}{\partial r} = 0$ and $\frac{\partial p}{\partial r} = -\frac{K}{(1+r)^2} < 0$ (or $\frac{\partial}{\partial r}\left(\frac{K}{1+r} - p\right) = 0$). This indicates that with higher risk-free interest rate, keeping the same market value of the firm leads to the same equity (hence, the same debt), and reduce the risk level of the debt.

(iv) $\frac{\partial p}{\partial r} = 0$ and $\frac{\partial c}{\partial r} = \frac{K}{(1+r)^2} > 0$. This indicates that with higher risk-free interest rate, keeping the same market value of the firm leads to higher equity (and hence, lower debt) and the same insurance.

(v) $\frac{\partial c}{\partial r} > 0$ and $\frac{\partial p}{\partial r} > 0$ (where $\frac{\partial c}{\partial r} > \frac{K}{(1+r)^2}$, and $\frac{\partial p}{\partial r} > 0$ leads to $\frac{\partial c}{\partial r} > 0$). This indicates that with higher risk-free interest rate, keeping the same market value of the firm leads to higher equity (and hence, lower debt) and higher insurance.

(vi) $\frac{\partial c}{\partial r} < 0$ and $\frac{\partial p}{\partial r} < 0$ (where $\frac{\partial p}{\partial r} < -\frac{K}{(1+r)^2}$, and $\frac{\partial c}{\partial r} < 0$ leads to $\frac{\partial p}{\partial r} < 0$). This indicates that with higher risk-free interest rate, keeping the same market value of the firm leads to lower equity (and hence, higher debt) and lower insurance.

About $\frac{\partial c}{\partial K}$ and $\frac{\partial p}{\partial K}$:

Because at $t = T$, the payoff of c is $Max[S_T - K, 0]$, and the payoff of p is $Max[K - S_T, 0]$, we must have: $\frac{\partial c}{\partial K} < 0$ and $\frac{\partial p}{\partial K} > 0$. Also, from eq. (1), we have: $\frac{\partial c}{\partial K} + \frac{1}{1+r} = \frac{\partial p}{\partial K}$, and hence, $\frac{-1}{1+r} < \frac{\partial c}{\partial K} < 0$ and $\frac{1}{1+r} > \frac{\partial p}{\partial K} > 0$. This indicates that higher promised payment K leads to lower equity (and hence, higher debt, i.e., $\frac{\partial}{\partial K}\left(\frac{K}{1+r} - p\right) > 0$) and higher insurance.

An Example

Assume a four states of nature world with probabilities: π_1, π_2, π_3, $\pi_4 > 0$, and $\sum_{i=1}^{4} \pi_i = 1$. At $t = 0$, the firm (the underlying asset) with current price S_0 has four possible market prices ($10,000, $8,000, $4,000, $3,000, respectively) at $t = T$. With the promised payment (the strike price) $K = \$6{,}000$, at $t = T$, the possible prices of the equity (the call) c, the insurance (the put) p, and the risky debt

$\left(\frac{K}{1+r} - p\right)$ are shown in the following table:

$t = 0$		S_0	c	p	$\frac{K}{1+r}$	$\frac{K}{1+r} - p$
$t = T$	Probabilities					
State 1	π_1	10,000	4,000	0	6,000	6,000
State 2	π_2	8,000	2,000	0	6,000	6,000
State 3	π_3	4,000	0	2,000	6,000	4,000
State 4	π_4	3,000	0	3,000	6,000	3,000

Note that in the option Greeks, changes in the current price S_0 (or r changes but S_0 remains constant) cannot happen without a cause. These may be caused by changes in probabilities π_i, in future possible payoffs, or in both. Suppose the probabilities change. The Greeks of call and put options are as the follows.

About $\frac{\partial c}{\partial s}$ and $\frac{\partial p}{\partial s}$:

(i) $\frac{\partial c}{\partial s} > 0$ and $\frac{\partial p}{\partial s} < 0$. That S_0 increases (decreases), c increases (decreases) and p decreases (increases) may be caused by: π_2 becomes $\pi_2' = \pi_2 + (-)\varepsilon$, π_3 becomes $\pi_3' = \pi_3 - (+)\varepsilon$, and π_1 and π_4 remain the same, where ε is a very small positive number. This is the case when the market value of the firm S_0 increases (decreases), the equity c increases (decreases), and debt $\left(\frac{K}{1+r} - p\right)$ increases (decreases) because the insurance p decreases (increases).

(ii) $\frac{\partial c}{\partial s} = 0$ and $\frac{\partial p}{\partial s} = -1 < 0$. That S_0 increases (decreases), c remains constant and p decreases (increases) may be caused by: π_3 becomes $\pi_3' = \pi_3 + (-)\varepsilon$, π_4 becomes $\pi_4' = \pi_4 - (+)\varepsilon$, and π_1 and π_2 remain the same, where ε is a very small positive number. This is the case when the market value of the firm S_0 increases (decreases), the equity c remains constant, and debt $\left(\frac{K}{1+r} - p\right)$ increases (decreases) because the insurance p decreases (increases).

(iii) $\frac{\partial p}{\partial s} = 0$ and $\frac{\partial c}{\partial s} = 1 > 0$. That S_0 increases (decreases), p remains constant and c increases (decreases) may be caused by: π_1 becomes $\pi_1' = \pi_1 + (-)\varepsilon$, $\pi_2' = \pi_2 - (+)\varepsilon$, and π_3 and π_4 remain the same, where ε is a very small positive number. This is the case when the market value of the firm S_0 increases (decreases), the equity c increases (decreases), and debt $\left(\frac{K}{1+r}-p\right)$ remains the same because the insurance p does not change.

(iv) $\frac{\partial c}{\partial s}>0$ and $\frac{\partial p}{\partial s}>0$. That S_0 increases (decreases), c increases (decreases) and p increases (decreases) may be caused by: π_1 becomes $\pi_1'=\pi_1+(-)\varepsilon$, π_2 becomes $\pi_2'=\pi_2-(+)\varepsilon$, π_3 becomes $\pi_3'=\pi_3-(+)\varepsilon$, and $\pi_4'=\pi_4+(-)\varepsilon$ where ε is a very small positive number. This is the case when the market value of the firm S_0 increases (decreases), the equity c increases (decreases), and debt $\left(\frac{K}{1+r}-p\right)$ decreases (increases) because the insurance p increases (decreases).

(v) $\frac{\partial c}{\partial s}<0$ and $\frac{\partial p}{\partial s}<0$. That S_0 increases (decreases), c decreases (increases) and p decreases (increases) may be caused by: π_1 becomes $\pi_1'=\pi_1-(+)\varepsilon$, π_2 becomes $\pi_2'=\pi_2+(-)\varepsilon$, π_3 becomes $\pi_3'=\pi_3+(-)\frac{\pi_4}{2}$, and π_4 becomes $\pi_4'=\pi_4-(+)\frac{\pi_4}{2}$, where ε is a very small positive number. This is the case when the market value of the firm S_0 increases (decreases), the equity c decreases (increases), and debt $\left(\frac{K}{1+r}-p\right)$ increases (decreases) because the insurance p decreases (increases).

About $\frac{\partial c}{\partial r}$ and $\frac{\partial p}{\partial r}$:

(i) $\frac{\partial c}{\partial r}>0$ and $\frac{\partial p}{\partial r}<0$ (where $0<\frac{\partial c}{\partial r}<\frac{K}{(1+r)^2}$ and $\frac{-K}{(1+r)^2}<\frac{\partial p}{\partial r}<0$). That r increases (decreases) but S_0 remains constant, c increases (decreases), and both $\left(\frac{K}{1+r}-p\right)$ and p decrease (increase) may be caused by: π_1 becomes $\pi_1'=\pi_1+(-)\delta$, $\pi_2'=\pi_2-(+)\delta$, and π_3 and π_4 remain the same, where δ is a very small positive number. In this case, because π_3 and π_4 do not change, for the risky debt: $\frac{\partial}{\partial r}\left(\frac{K}{1+r}-p\right)<0$ and for the insurance: $\frac{\partial p}{\partial r}<0$.

(ii) $\frac{\partial c}{\partial r}=0$ and $\frac{\partial p}{\partial r}=-\frac{K}{(1+r)^2}<0$. That r increases (decreases) but S_0 remains constant, both c and $\left(\frac{K}{1+r}-p\right)$ remain constant, and p decreases (increases) may be caused by: π_1 becomes $\pi_1'=\pi_1+(-)\frac{\delta}{2}$, π_2 becomes $\pi_2'=\pi_2-(+)\frac{\delta}{2}$, π_3 becomes $\pi_3'=\pi_3+(-)\omega$, and π_4 becomes $\pi_4'=\pi_4-(+)\omega$, where δ and ω are very small positive numbers. In this case, for the risky debt: $\frac{\partial}{\partial r}\left(\frac{K}{1+r}-p\right)=0$ and for the insurance: $\frac{\partial p}{\partial r}<0$.

(iii) $\frac{\partial p}{\partial r}=0$ and $\frac{\partial c}{\partial r}=\frac{K}{(1+r)^2}>0$. That r increases (decreases) but S_0 remains constant, c increases (decreases), and p remains constant may be caused by: π_1 becomes $\pi_1'=\pi_1+(-)2\delta$, π_2

becomes $\pi_2' = \pi_2 - (+)2\delta$, π_3 becomes $\pi_3' = \pi_3 + (-)\frac{\omega}{2}$, and π_4 becomes $\pi_4' = \pi_4 - (+)\frac{\omega}{2}$, where δ and ω are very small positive numbers.

(iv) $\frac{\partial c}{\partial r} > 0$ and $\frac{\partial p}{\partial r} > 0$ (where $\frac{\partial c}{\partial r} > \frac{K}{(1+r)^2}$, and $\frac{\partial p}{\partial r} > 0$ leads to $\frac{\partial c}{\partial r} > 0$). That r increases (decreases) but S_0 remains constant, c increases (decreases), and p increases (decreases) may be caused by: π_1 becomes $\pi_1' = \pi_1 + (-)3\delta$, π_2 becomes $\pi_2' = \pi_2 - (+)3\delta$, π_3 becomes $\pi_3' = \pi_3 - (+)\omega$, and π_4 becomes $\pi_4' = \pi_4 + (-)\omega$, where δ and ω are very small positive numbers. In this case, because S_0 remains constant, $\frac{\partial p}{\partial r} > 0$ implies that for the risky debt: $\frac{\partial}{\partial r}\left(\frac{K}{1+r} - p\right) < 0$, and hence, for the equity: $\frac{\partial c}{\partial r} > 0$.

(v) $\frac{\partial c}{\partial r} < 0$ and $\frac{\partial p}{\partial r} < 0$ (where $\frac{\partial p}{\partial r} < -\frac{K}{(1+r)^2}$, and $\frac{\partial c}{\partial r} < 0$ leads to $\frac{\partial p}{\partial r} < 0$). That r increases (decreases) but S_0 remains constant, c decreases (increases), and p decreases (increases) may be caused by: π_3 becomes $\pi_3' = \pi_3 + (-)\delta$, π_4 becomes $\pi_4' = \pi_4 - (+)\delta$, and π_1 and π_2 remain the same, where δ is a very small positive number. In this case, because π_1 and π_2 do not change, for the equity: $\frac{\partial c}{\partial r} < 0$, and because S_0 remains constant, for the risky debt: $\frac{\partial}{\partial r}\left(\frac{K}{1+r} - p\right) > 0$, and for the insurance: $\frac{\partial p}{\partial r} < 0$.

3. The Greeks under the Binomial Option Pricing Model

The binomial option pricing model may be presented as the follows (where $K = 8{,}900$).

$$c_u = Max[S_0u - K, 0] = Max[10{,}000 - 8{,}900, 0]$$

$$p_u = Max[K - S_0u, 0] = Max[8{,}900 - 10{,}000, 0]$$

π → $S_0 \cdot u = 8{,}000 \times \frac{10{,}000}{8{,}000} = 10{,}000$

$S_0 = 8{,}000$

$r = 10\%$ $1 - \pi$ → $S_0 \cdot d = 8{,}000 \times \frac{8{,}400}{8{,}000} = 8{,}400$

$c = ?$ $c_d = Max[S_0d - K, 0] = Max[8{,}400 - 8{,}900, 0]$

$p = ?$ $p_d = Max[K - S_0d, 0] = Max[8{,}900 - 8{,}400, 0]$

$t = 0$		S_0 $= 8{,}000$	c	p	$\frac{K}{1+r}$ $= \frac{8{,}900}{1+0.1}$	$\frac{K}{1+r} - p$ $= \frac{8{,}900}{1+0.1} - p$
$t = T$	Probabilities					
State 1	π	10,000	1,100	0	8,900	8,900
State 2	$1 - $ π	8,400	0	500	8,900	8,400

The Greeks can be easily derived:

(i) $\frac{\partial c}{\partial s} > 0$ and $\frac{\partial p}{\partial s} < 0$. When r, $S_0 u$ and $S_0 d$ remain constant, S_0 cannot increase (decrease) unless π increases (decreases), i.e., $\frac{\partial \pi}{\partial s} > 0$. Therefore, higher (lower) π leads to higher (lower) c and lower (higher) p.

(ii) $\frac{\partial c}{\partial r} > 0$ and $\frac{\partial p}{\partial r} < 0$. For any asset, an increase (decrease) in r can decrease (increase) its present value. But when r increases (decreases) and $S_0 u$ and $S_0 d$ remain constant, constant S_0 cannot happen unless π increases (decreases), i.e., $\frac{\partial \pi}{\partial r} > 0$. Therefore, an increase (decrease) in π leads to lower (higher) p, i.e., $\frac{\partial p}{\partial r} < 0$. Also,

Case (1). Suppose r increases (decreases), but π and $(1 - \pi)$ remain the same (and hence, S_0 decreases (increases)). Then, because r increases (decreases), the riskless debt $\frac{K}{1+r}$ will decrease (increase) more than the insurance p, i.e., $\frac{\partial}{\partial r}\left(\frac{K}{1+r} - p\right) = \frac{\partial}{\partial r}\left(\frac{K}{1+r}\right) - \frac{\partial p}{\partial r} < 0$.

Case (2). In the binomial option pricing model, when r increases (decreases), and S_0, $S_0 u$ and $S_0 d$ remain constant, an increase (decrease) in π implies that the riskless debt $\frac{K}{1+r}$ will decrease (increase) further more than the insurance p, i.e., $\frac{\partial}{\partial r}\left(\frac{K}{1+r} - p\right) = \frac{\partial}{\partial r}\left(\frac{K}{1+r}\right) - \frac{\partial p}{\partial r} < 0$, and hence, the equity c will increase (decrease), i.e., $\frac{\partial c}{\partial r} > 0$.

(iii) $\frac{\partial c}{\partial u} = \frac{\partial p}{\partial u} > 0$. When u increases (decreases) to u', i.e., $S_0 u' > (<)\ S_0 u$, and r and $S_0 d$ remain constant, constant S_0 cannot happen unless π decreases (increases), i.e., $\frac{\partial \pi}{\partial u} < 0$. Hence, higher (lower) $1 - \pi$ leads to higher (lower) p. Because $\frac{\partial c}{\partial x} = \frac{\partial p}{\partial x}$ for any $x \notin \{S_0, r, K\}$, we must have: $\frac{\partial c}{\partial u} = \frac{\partial p}{\partial u} > 0$.

(iv) $\frac{\partial c}{\partial d} = \frac{\partial p}{\partial d} < 0$. When d decreases (increases) to d', i.e., $S_0 d' < (>)\ S_0 d$, and r and $S_0 u$ remain constant, constant S_0 cannot happen unless π increases (decreases), i.e., $\frac{\partial \pi}{\partial d} < 0$. Hence, higher (lower) π leads to higher (lower) c. Because $\frac{\partial c}{\partial x} = \frac{\partial p}{\partial x}$ for any $x \notin \{S_0, r, K\}$, we must have: $\frac{\partial c}{\partial d} = \frac{\partial p}{\partial d} < 0$.

A more rigorous proof for the Greeks is as the follows.

Chang (2015, p. 41) has used the Gordan theory to price assets:[4]

$$\begin{cases} \text{Money market:} \quad 1 = \frac{1}{1+r}[\pi(1+r) + (1-\pi)(1+r)] = \frac{1}{1+0.1}\left[\frac{1}{4}(1.1) + \frac{3}{4}(1.1)\right] \\ \text{The firm: } S_0 = 8{,}000 = \frac{1}{1+r}[\pi \cdot S_0 u + (1-\pi) \cdot S_0 d] = \frac{1}{1+0.1}\left[\frac{1}{4}(10{,}000) + \frac{3}{4}(0)\right] \\ \text{Call option: } c = 250 = \frac{1}{1+r}[\pi \cdot (S_0 u - K) + (1-\pi) \cdot 0] = \frac{1}{1+0.1}\left[\frac{1}{4}(1{,}100) + \frac{3}{4}(1.1)\right] \\ \text{Put option: } p = \frac{3{,}750}{11} = \frac{1}{1+r}[\pi \cdot 0 + (1-\pi) \cdot (K - S_0 d)] = \frac{1}{1+0.1}\left[\frac{1}{4}(0) + \frac{3}{4}(500)\right] \end{cases} \quad (8)$$

where $\pi = \frac{(1+r)-d}{u-d}$ and $1 - \pi = \frac{u-(1+r)}{u-d}$. Thus, $\frac{\partial \pi}{\partial r} = \frac{1}{u-d} > 0$, $\frac{\partial \pi}{\partial u} = \frac{d-(1+r)}{(u-d)^2} < 0$ and $\frac{\partial \pi}{\partial d} = \frac{(1+r)-u}{(u-d)^2} < 0$.[5]

(i) $1 > \frac{\partial c}{\partial s} > 0$ and $-1 < \frac{\partial p}{\partial s} < 0$; and $\frac{\partial^2 c}{\partial s^2} = \frac{\partial^2 p}{\partial s^2} = 0$. Let $S_0' = \frac{1}{1+r}[\pi' \cdot S_0 u + (1-\pi') \cdot S_0 d]$

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[4] The Gordan theory is:

Let A be an $m \times n$ matrix. Then, exactly one of the following systems has a solution:

System 1: $Ax > 0$ for some $x \in R^n$

System 2: $A^T \pi = 0$ for some $\pi \in R^m$, $\pi \geq 0$, $e^T \pi = 1$ where $e = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}$.

[5] Chang (2017) has shown that because an asset's current price (e.g., $S_0 = 8{,}000$) is determined by people's expectation of the asset's future possible payoffs and their probabilities, the probabilities of the Gordan theory derived from S_0 (e.g., π and $1 - \pi$ in eq. (8)) are the actual world (not the risk-neutral world) probabilities.

and $S_0 = \frac{1}{1+r}[\pi \cdot S_0 u + (1-\pi) \cdot S_0 d]$. We have: $\frac{\pi' - \pi}{S_0' - S_0} = \frac{1+r}{S_0 u - S_0 d} > 0$, i.e., $\frac{\partial \pi}{\partial S} = \frac{1+r}{S_0 u - S_0 d} > 0$ and $\frac{\partial^2 \pi}{\partial S^2} = 0$. For the call option, $c' = \frac{1}{1+r}[\pi' \cdot (S_0 u - K)]$ and $c = \frac{1}{1+r}[\pi \cdot (S_0 u - K)]$, and hence, $1 > \frac{c' - c}{S_0' - S_0} = \frac{S_0 u - K}{S_0 u - S_0 d} > 0$, i.e., $1 > \frac{\partial c}{\partial S} = \frac{S_0 u - K}{S_0 u - S_0 d} > 0$ and $\frac{\partial^2 c}{\partial S^2} = 0$. For the put option, $p' = \frac{1}{1+r}[(1 - \pi') \cdot (K - S_0 d)]$ and $p = \frac{1}{1+r}[(1-\pi) \cdot (K - S_0 d)]$, and hence, $-1 < \frac{p' - p}{S_0' - S_0} = \frac{-(K - S_0 d)}{S_0 u - S_0 d} < 0$, i.e., $-1 < \frac{\partial p}{\partial S} = \frac{-(K - S_0 d)}{S_0 u - S_0 d} < 0$ and $\frac{\partial^2 p}{\partial S^2} = 0$. Both the binomial and the Black-Scholes-Merton option pricing models have: $1 > \frac{\partial c}{\partial S} > 0$ and $-1 < \frac{\partial p}{\partial S} < 0$, which indicate both models implicitly assume that maximizing the market value of the firm is not equivalent to maximizing the equityholders' wealth and that increasing (decreasing) the market value of the firm can decrease (increase) the risk level of the debt. Also, $\frac{\partial^2 c}{\partial S^2} = \frac{\partial^2 p}{\partial S^2} = 0$ of the binomial option pricing model indicates that the model implicitly assumes that further increasing (decreasing) in the market value of the firm affects neither the speed of increasing (decreasing) in the equity nor the speed of decreasing (increasing) in the risk level of the debt. The Black-Scholes-Merton option pricing model, on the other hand, has $\frac{\partial^2 c}{\partial S^2} = \frac{\partial^2 p}{\partial S^2} > 0$ which indicates the model implicitly assumes that further increasing (decreasing) the market value of the firm will: (1) increase (decrease) the speed of increasing (decreasing) in the equity though bounded by $1 > \frac{\partial c}{\partial S} > 0$; and (2) decrease (increase) the speed of decreasing (increasing) in the risk level of the debt though bounded by $-1 < \frac{\partial p}{\partial S} < 0$.

(ii) $\frac{K}{(1+r)^2} > \frac{\partial c}{\partial r} > 0$ and $\frac{-K}{(1+r)^2} < \frac{\partial p}{\partial r} < 0$; and $\frac{\partial^2 c}{\partial r^2} < 0$ and $\frac{\partial^2 p}{\partial r^2} > 0$. $\frac{\partial c}{\partial r} = \frac{\partial}{\partial r}\left[\frac{\pi \cdot (S_0 u - K)}{1+r}\right] = \frac{d \cdot (S_0 u - K)}{(1+r)^2 (u-d)} > 0$, and $\frac{\partial^2 c}{\partial r^2} = \frac{-2d(S_0 u - K)}{(1+r)^3 (u-d)} < 0$. $\frac{\partial p}{\partial r} = \frac{\partial}{\partial r}\left[\frac{(1-\pi)(K - S_0 d)}{1+r}\right] = \frac{-u \cdot (K - S_0 d)}{(1+r)^2 (u-d)} < 0$, and $\frac{\partial^2 p}{\partial r^2} = \frac{2u(K - S_0 d)}{(1+r)^3 (u-d)} > 0$. Both the binomial and the Black-Scholes-Merton option pricing models have: $\frac{\partial c}{\partial r} > 0$ and $\frac{\partial p}{\partial r} < 0$, which indicates both models implicitly assume that increasing (decreasing) in r can increase (decrease) the equity c and decrease (increase) both the risky debt $\left(\frac{K}{1+r} - p\right)$ and the insurance p. Also, both the binomial and the Black-Scholes-Merton option pricing models have: $\frac{\partial^2 c}{\partial r^2} < 0$ and $\frac{\partial^2 p}{\partial r^2} > 0$, which indicate both models implicitly assume that further increasing (decreasing) in r will: (1) decrease (increase) the speed of increasing (decreasing) in the equity though bounded by $0 < \frac{\partial c}{\partial r} < \frac{K}{(1+r)^2}$; and (2) decrease (increase) the speed of decreasing (increasing) in the insurance though bounded by $\frac{-K}{(1+r)^2} < \frac{\partial p}{\partial r} < 0$.

(iii) $\frac{-1}{1+r} < \frac{\partial c}{\partial K} < 0$ and $\frac{1}{1+r} > \frac{\partial p}{\partial K} > 0$; and $\frac{\partial^2 c}{\partial K^2} = \frac{\partial^2 p}{\partial K^2} = 0$. $\frac{\partial c}{\partial K} = \frac{\partial}{\partial K}\left[\frac{\pi\cdot(S_0 u-K)}{1+r}\right] = \frac{-\pi}{1+r} < 0$ and hence, $\frac{\partial^2 c}{\partial K^2} = 0$. $\frac{\partial p}{\partial K} = \frac{\partial}{\partial K}\left[\frac{(1-\pi)(K-S_0 d)}{1+r}\right] = \frac{1-\pi}{1+r} > 0$ and hence, $\frac{\partial^2 p}{\partial K^2} = 0$. This indicates that the binomial model implicitly assumes that further increasing (decreasing) K affects neither the speed of decreasing (increasing) in the equity c nor the speed of increasing (decreasing) in the insurance p.[6] The Black-Scholes-Merton option pricing model, on the other hand, has $\frac{\partial^2 c}{\partial K^2} = \frac{\partial^2 p}{\partial K^2} > 0$,[7] which indicates the model implicitly assumes that further increasing (decreasing) in K will: (1) decrease (increase) the speed of decreasing (increasing) in the equity though bounded by $\frac{-1}{1+r} < \frac{\partial c}{\partial K} < 0$; and (2) increase (decrease) the speed of increasing (decreasing) in the insurance though bounded by $\frac{1}{1+r} > \frac{\partial p}{\partial K} > 0$.

(iv) $\frac{\partial c}{\partial u} = \frac{\partial p}{\partial u} > 0$; and $\frac{\partial^2 c}{\partial u^2} = \frac{\partial^2 p}{\partial u^2} < 0$. $\frac{\partial c}{\partial u} = \frac{\partial}{\partial u}\left[\frac{\pi\cdot(S_0 u-K)}{1+r}\right] = \frac{\partial p}{\partial u} = \frac{\partial}{\partial u}\left[\frac{(1-\pi)\cdot(K-S_0 d)}{1+r}\right] = \frac{(K-S_0 d)(1+r-d)}{(1+r)(u-d)^2} > 0$; and $\frac{\partial^2 c}{\partial u^2} = \frac{\partial^2 p}{\partial u^2} = \frac{-2(K-S_0 d)(1+r-d)}{(1+r)(u-d)^3} < 0$.

$\frac{\partial c}{\partial d} = \frac{\partial p}{\partial d} < 0$ and $\frac{\partial^2 c}{\partial d^2} = \frac{\partial^2 p}{\partial d^2} < 0$. $\frac{\partial c}{\partial d} = \frac{\partial}{\partial d}\left[\frac{\pi\cdot(S_0 u-K)}{1+r}\right] = \frac{\partial p}{\partial d} = \frac{\partial}{\partial d}\left[\frac{(1-\pi)\cdot(K-S_0 d)}{1+r}\right] = \frac{-(S_0 u-K)[u-(1+r)]}{(1+r)(u-d)^2} < 0$; and $\frac{\partial^2 c}{\partial d^2} = \frac{\partial^2 p}{\partial d^2} = \frac{-2(S_0 u-K)[u-(1+r)]}{(1+r)(u-d)^3} < 0$.[8]

4. Some Extensions of the Put-Call Parity

Define the time value of European call option as: $TV^c = c - Max[S_0 - K, 0]$. Note that if $r = 0$,

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[6] Chang (2016, 2017) have shown that in the binomial case, under risky debt, increasing the debt-equity ratio does not affect the probability density function of the rate of return on equity. This result refutes the Modigliani-Miller second proposition that the expected rate of return on the equity of the levered firm increases in proportion to the debt-equity ratio.

[7] The Dupire formula.

[8] Chang (2015, pp. 49-51) has shown that when both u and d change, and let $(S_0 u - S_0 d)$ be the range, the sign of $\frac{\partial c}{\partial (S_0 u - S_0 d)} = \frac{\partial p}{\partial (S_0 u - S_0 d)}$ could be positive or negative. The Black-Scholes-Merton option pricing model, on the other hand, has: $\frac{\partial c}{\partial \sigma} = \frac{\partial p}{\partial \sigma} > 0$, where σ is the volatility. Ross (1993, p. 470) and Chang (2014) have shown that with complete market, no transaction costs and no arbitrage, the Black-Scholes-Merton option pricing model has the restriction: $r = \mu + \frac{1}{2}\sigma^2$.

we still have: $c + K = S_0 + p$. Since from the put-call parity, $c = S_0 + p - \frac{K}{1+r} \geq Max[S_0 - \frac{K}{1+r}, 0]$, we have: $TV^c \geq 0$ if $r \geq 0$, and $TV^c > 0$ if $c > 0$, $p > 0$ and $r \geq 0$. This indicates that American call option on a non-dividend-paying stock will never be exercised before the expiration date. The time value of European put option is defined as: $TV^p = p - Max[K - S_0, 0]$. When $r = 0$, from $c + K = S_0 + p$, we have: $TV^p \geq 0$. When $K \leq S_0$, $TV^p = p \geq 0$. When $K > S_0$, if $r > 0$ and $c \geq \frac{rK}{1+r}$, then $TV^p \geq 0$. This is because $\frac{K}{1+r} + c \geq \frac{rK}{1+r} + \frac{K}{1+r} = K$, and hence, $S_0 + p = \frac{K}{1+r} + c \geq K$ or $TV^p = p - (K - S_0) \geq 0$. This indicates that if $r = 0$; or $K \leq S_0$; or $r > 0$, $K > S_0$ and $c \geq \frac{rK}{1+r}$, American put option on a non-dividend-paying stock will never be exercised before the expiration date.

Property 1. When the promised payment K is approaching the market value of the firm S_0 and the risk-free interest rate r is small, both the owner of the equity c and the owner of the insurance p will be more reluctant to liquidate the firm.[9]

From eq. (1), we have: $\frac{\partial c}{\partial K} + \frac{1}{1+r} = \frac{\partial p}{\partial K}$, where $\frac{-1}{1+r} < \frac{\partial c}{\partial K} < 0$ and $\frac{1}{1+r} > \frac{\partial p}{\partial K} > 0$. For $K < S_0$, $\frac{\partial TV^c}{\partial K} = \frac{\partial}{\partial K}(c - S_0 + K) = \frac{\partial c}{\partial K} + 1 > 0$; and $\frac{\partial TV^p}{\partial K} = \frac{\partial p}{\partial K} > 0$. For $K > S_0$, $\frac{\partial T\ ^c}{\partial K} = \frac{\partial c}{\partial K} < 0$; and $\frac{\partial TV^p}{\partial K} = \frac{\partial}{\partial K}(p - K + S_0) = \frac{\partial p}{\partial K} - 1 < 0$. That is, when $K = S_0$, both TV^c and TV^p will be the highest, and hence, both the equityholder and the owner of the insurance will be very reluctant to liquidate the firm.

Property 2. For a leveraged firm, the lower bound for the risky debt is: $K \geq S_0(1 + r)$, i.e., the upper bound for the riskless debt is: $K < S_0(1 + r)$.

From the put-call parity: $= S_0 + p - \frac{K}{1+r}$, if $p = 0$, i.e., the debt is riskless, we have: $c = S_0 - \frac{K}{1+r} \geq 0$. However, the promised payment $K = S_0(1 + r)$ cannot be riskless.[10] For example, let $K = 1{,}100$, $r = 10\%$ and $S_0 = 1{,}000$. If at $t = 0$, people believe that $K = 1{,}100$ is riskless, then the same people must believe $S_T \geq 1{,}100$ at $t = T$. This contradicts $S_0 = 1{,}000$ because if people believe $S_T \geq 1{,}100$ at $t = T$, its current price S_0 must be greater than 1,000, i.e., $S_0 > 1{,}000$.

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[9] Chang (2015, p. 28) finds that even without changing the expiration dates, issuers of European options can adjust the exercise price K to change the time values of European options.

[10] If options exist, we must have: $K = S_0(1 + r)$ if and only if $c = p > 0$.

Property 3. Every non-fixed income asset can be transformed into a portfolio of fixed income assets as long as there are insurance markets.

For a totally equity-financed firm, from the put-call parity $S_0 = c + \left(\frac{K}{1+r} - p\right)$, let $K = S_0(1+r)$ so that $c = p > 0$, the equityholder can sell part of the firm as equity c and keep $\left(\frac{K}{1+r} - p\right)$ as debt and buy the insurance p. The buyer of this firm's equity c who now becomes the equityholder of a totally equity-financed firm c can do the same thing: from $c = c' + \left(\frac{K'}{1+r} - p'\right)$, let $K' = c(1+r)$ so that $c' = p' > 0$, the equityholder can sell part of the firm as equity c' and keep $\left(\frac{K'}{1+r} - p'\right)$ as debt and buy the insurance p'. Continue this way, all non-fixed income assets can be transformed into fixed income assets.

Property 4. Suppose that from the put-call parity: $c + \frac{K}{1+r} = S_0 + p$, the promised payment K is divided equally into three portions, i.e., $K = 3 \times \frac{K}{3}$, for three debtholders: $D_i = \frac{K}{3(1+r)} - p_i, i = 1, 2,$ and 3, where $D_1 = D_2$ are senior debts, and D_3 is a junior debt. The put-call parity can be rewritten as:

$$E + \frac{K}{3(1+r)} + \frac{K}{3(1+r)} + \frac{K}{3(1+r)} = S_0 + p_1 + p_2 + p_3$$

$$\text{or} \quad S_0 = E + \left[\frac{K}{3(1+r)} - p_1\right] + \left[\frac{K}{3(1+r)} - p_2\right] + \left[\frac{K}{3(1+r)} - p_3\right] , \tag{9}$$

where $E = c$ is the equity. At $t = T$, the payoffs to the equity, the debts and the insurances are:

for E: $Max[S_T - K, 0]$; for $D_1 = D_2$: $\frac{1}{2}Min\left[\frac{2}{3}K, S_T\right]$, for D_3: $Min\left\{Max\left[S_T - \frac{2}{3}K, 0\right], \frac{1}{3}K\right\}$, for $p_1 = p_2$: $\frac{1}{2}Max\left[\frac{2}{3}K - S_T, 0\right]$; and for p_3: $Max\left\{\frac{1}{3}K - Max[S_T - \frac{2}{3}K, 0], 0\right\}$.

$t = 0$	E	D_1	D_2	D_3	p_1	p_2	p_3
$t = T$							
$S_T < \frac{2}{3}K$	0	$\frac{1}{2}S_T$	$\frac{1}{2}S_T$	0	$\frac{1}{3}K - \frac{1}{2}S_T$	$\frac{1}{3}K - \frac{1}{2}S_T$	$\frac{1}{3}K$
$\frac{2}{3}K \leq S_T < K$	0	$\frac{1}{3}K$	$\frac{1}{3}K$	$S_T - \frac{2}{3}K$	0	0	$K - S_T$
$S_T \geq K$	$S_T - K$	$\frac{1}{3}K$	$\frac{1}{3}K$	$\frac{1}{3}K$	0	0	0

It shows that the first claim between senior and junior debts does not affect the non-fixed income asset (i.e., the equity). Also, because of the seniority (first claim) of D_1 and D_2 where in every scenario the future payoff of D_1 or D_2 is no less than that of D_3 and $p_1 = p_2 < p_3$, the present value of the senior debt is greater than that of the junior debt, i.e., $D_1 = D_2 > D_3$. Thus, at the good time, i.e., $S_T \geq K$, the maximum rate of return of the junior debt is greater than that of the senior debt. However, this does not mean that higher risk (i.e., $p_3 > p_1 = p_2$) means higher expected rate of return, or that the expected rate of return of the junior debt is greater than that of the senior debt even though all the debts have the same promised payment $\frac{1}{3}K$.

Property 5. Suppose that in eq. (9), the senior debt D_1 is changed into a convertible bond CB which at $t = T$ may be converted to another equity. Then, the put-call parity can be rewritten as:

$$E + CB + \frac{K}{3(1+r)} + \frac{K}{3(1+r)} = S_0 + p_2 + p_3$$

$$\text{or} \quad S_0 = E + CB + \left[\frac{K}{3(1+r)} - p_2\right] + \left[\frac{K}{3(1+r)} - p_3\right] \ , \qquad (10)$$

where $CB = D_1 + p'$, and p' is a put option. At $t = T$, the payoffs to the equity, the convertible bond, the debts, and the insurances are:

for E: $Max\left\{Min\left[\frac{1}{2}Max(S_T - \frac{2}{3}K, 0), S_T - K\right], 0\right\}$; for D_2: $\frac{1}{2}Min\left[\frac{2}{3}K, S_T\right]$;

for p_2: $\frac{1}{2}Max\left[\frac{2}{3}K - S_T, 0\right]$; for D_3: $Min\left\{Max\left[S_T - \frac{2}{3}K, 0\right], \frac{1}{3}K\right\}$;

for p_3: $Max\left\{\frac{1}{3}K - Max[S_T - \frac{2}{3}K, 0], 0\right\}$;

for CB: $Max\left\{\frac{1}{2}Max\left[S_T - \frac{2}{3}K, 0\right], \frac{1}{2}Min\left[\frac{2}{3}K, S_T\right]\right\}$; and for p': $\frac{1}{2}Max\left[S_T - \frac{4}{3}K, 0\right]$.

$t = 0$	E	CB	D_2	D_3	p'	p_2	p_3
$t = T$							
$S_T < \frac{2}{3}K$	0	$\frac{1}{2}S_T$	$\frac{1}{2}S_T$	0	0	$\frac{1}{3}K - \frac{1}{2}S_T$	$\frac{1}{3}K$
$\frac{2}{3}K \leq S_T < K$	0	$\frac{1}{3}K$	$\frac{1}{3}K$	$S_T - \frac{2}{3}K$	0	0	$K - S_T$
$\frac{4}{3}K \geq S_T \geq K$	$S_T - K$	$\frac{1}{3}K$	$\frac{1}{3}K$	$\frac{1}{3}K$	0	0	0
$S_T > \frac{4}{3}K$	$\frac{1}{2}(S_T - \frac{2}{3}K)$	$\frac{1}{2}(S_T - \frac{2}{3}K)$	$\frac{1}{3}K$	$\frac{1}{3}K$	$\frac{1}{2}S_T - \frac{2}{3}K$	0	0

It shows that when the senior debt D_1 is transformed into a convertible bond CB, other debts D_2 and D_3 as well as p_2 and p_3 will not be affected though the original equity E is worse. Thus, at $t = 0$, $E < CB$ and $CB > D_2 > D_3$. At the good time, i.e., $S_T > \frac{4}{3}K$, the maximum rate of return of the equity is greater than that of the convertible bond, and the maximum rate of return of the convertible bond is greater than that of the senior bond. But it does not mean that the expected rate of return of the equity is greater than that of the convertible bond, or the expected rate of return of the convertible bond is greater than that of the senior bond.

5. Concluding Remarks

This paper has proposed new upper and lower bounds for European and American options and new option Greeks. It also shows that because of the put-call parity, the Greeks of put and call options are interconnected and should be shown simultaneously. In terms of the theory of the firm, it is found that both the Black-Scholes-Merton and the binomial option pricing models implicitly assume that maximizing the market value of the firm is not equivalent to maximizing the equityholders' wealth. The binomial option pricing model implicitly assumes that further increasing (decreasing) the promised payment to debtholders affects neither the speed of decreasing (increasing) in the equity nor the speed of increasing (decreasing) in the insurance. The Black-Scholes-Merton option pricing model, on the other hand, implicitly assumes that further increasing (decreasing) in the promised payment to debtholders will: (1) decrease (increase) the speed of decreasing (increasing) in the equity though bounded by upper and lower bounds, and (2) increase (decrease) the speed of increasing (decreasing) in the insurance though bounded by upper and lower bounds. The paper also extends the put-call parity to include senior debt and convertible bond. It is found that when the promised payment to debtholders is approaching the market value of the firm and the risk-free interest rate is small, both the owner of the equity and the owner of the insurance will be more reluctant to liquidate the firm. It also specifies the conditions that American put option will not be exercised before the expiration date. The lower bound for risky debt is: the promised payment to debtholders is greater or equal to the market value of the firm times one plus the risk-free interest rate.

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