Technical efficiency and inefficiency: Reassurance of standard SFA models and a misspecification problem

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Technical efficiency and inefficiency: Reassurance of standard SFA models and a misspecification problem

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Abstract

This paper formally proves that if inefficiency (u) is modelled through the variance of u which is a function of z then marginal effects of z on technical inefficiency (TI) and technical efficiency (TE) have opposite signs. This is true in the typical setup with normally distributed random error v and exponentially or half-normally distributed u for both conditional and unconditional TI and TE.

We also provide an example to show that signs of the marginal effects of z on TI and TE may coincide for some ranges of z. If the real data comes from a bimodal distribution of u, and we estimate model with an exponential or half-normal distribution for u, the estimated efficiency and the marginal effect of z on TE would be wrong. Moreover, the rank correlations between the true and the estimated values of TE could be small and even negative for some subsamples of data. This result is a warning that the interpretation of

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the results of applying standard models to real data should take into account this possible problem. The results are demonstrated by simulations.

Keywords: Productivity and competitiveness, stochastic frontier analysis, model misspecification, efficiency, inefficiency

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1. Introduction

Stochastic frontier (SF) production model (Aigner et al., 1977; Meeusen and van den Broeck, 1977) is designed to estimate the observation-specific technical inefficiency TI. The SF models are increasingly used in both academic and non-academic studies. The main academic use is in economics and OR. They are also used in regulatory cases, viz., price control in electricity, water, transportation, post offices. etc., in all over Europe as well as in many other countries. The SF model has two separate error terms: a symmetrical statistical noise v and a non-negative error term u that represents the technical inefficiency. The complete specification of the SF model also includes the specification of distributions for v and u. If v has a normal distribution, and u has an exponential distribution, then the SF model is called normal-exponential, if v has a normal distribution, and u has a half-normal distribution, then the SF model is called normal-half-normal. To accommodate determinants of inefficiency z, the SF model is generalized to make u heteroscedastic (Kumbhakar and Lovell (2000); Wang (2003); (Daniel) Kao et al. (2019); Galán et al. (2014), among many others).

Our goal is to investigate marginal effects of z on TI as well as technical efficiency (TE) for the normal-exponential and normal-half-normal models. We assume u to be heteroscedastic, i.e., the variance of u is a function of z. Suppose that an increase in z leads to an increase in TI measured as $\mathrm{E}(u)$ or $\mathrm{E}(u|(v-u))$. Does it mean that TE measured as $TE = \mathrm{E}(e^{-u})$ or $TE = \mathrm{E}(e^{-u}|(v-u))$ (see Battese and Coelli (1988)) will decrease? Although it is intuitive, to the best of our knowledge there is no formal proof of this in the literature. We provide proof of this statement for the conditional means for the exponential and half-normal distributions of u.

A number of papers in the past have considered similar issues. For example, Wang (2002), Ray et al. (2015) derived an expression for marginal effects of the z variables on the expected value of inefficiency $\mathrm{E}(u)$. They showed that the sign of the marginal effects of z is determined by the sign of the marginal effects of z on the variance of u. Kumbhakar and Sun (2013) derived formulas for the marginal effect of exogenous factors on the observation-specific inefficiency $\mathrm{E}(u|(v-u))$ for the normal-truncated normal model with heteroscedasticity in both v and u. They demonstrated that, for this model, signs of the marginal effect may vary across observations.

In addition to the stochastic frontier model with exponential or half-normal distribution of the inefficiency term, we consider a model with a discrete distribution of the inefficiency term. Properties of these models can differ from the properties of the commonly used SF models (Kumbhakar and Lovell, 2000). First, for such models, an increase in z may increase both TI and TE, which is not possible in the usual normal-exponential and normal-half-normal models. It means that, if the true distribution of u is discrete, then applying the usual normal-exponential model may result in wrong conclusions on the directions of the marginal effects of the z variables on TE of the production units. Also, it may result in incorrect rankings of the production units by their estimated TE. More generally, the ranking of the production units by their estimated TE might be different from their rankings in terms of their "true" TE.

The impact of the model misspecification on the estimated TE was studied, using simulations, among other papers in Yu (1998); Ruggiero (1999); Ondrich and Ruggiero (2001); Andor and Parmeter (2017); Andor et al. (2019). Ruggiero (1999) concluded, that if data are generated by normal-half-normal model, then TE estimates by true (normal-half-normal) and misspecified (normal-exponential) models provide similar results. Thus this type of misspecification in incorrect choice of the error distribution is not problematic. Some papers (Yu, 1998; Ruggiero, 1999; Ondrich and Ruggiero, 2001) use rank correlation between true and estimated values of TE as a measure of the model misspecification. Other papers (Andor and Parmeter, 2017; Andor et al., 2019) use root mean square error (RMSE) measure as the distance between true and estimated TE for performance comparison of different models. Giannakas et al. (2003) demonstrated that predictions of TE are sensitive to the misspecification of the functional form of the production function in stochastic frontier regression.

The goal of this paper is twofold. First, we provide a formal proof that in the case of commonly used in applied papers normal-exponential and normal-half-normal models, signs of the marginal effects of z on TI and TE are opposite, which corresponds to intuition. Second, we provide an example of normal-discrete model, which demonstrate, that if the real data is generated according to this model, then results of commonly used normal-half-normal and normal-exponential models would be misleading. Signs of the marginal effects of z on TI and TE could coincide. This example is not an exotic one. The problem is related to the situation when the inefficiency term has a bimodal distribution. It can happen, for example, when there are two types of firms' managers. Discrete distribution is chosen just for the simplicity of calculations.[2]

The paper is organized as follows. In Section 2 we introduce the normal-exponential and normal-half-normal model and derive the formulas for computing the marginal effects of determinants of technical efficiency and technical inefficiency z. This is followed by Section 3 where we introduce the normal-discrete SF model and examine its properties. Section 4 concludes the paper. The proofs are provided in Appendix A.

2. Marginal effects of exogenous determinants on technical inefficiency and technical efficiency

For cross-sectional data, the basic SF model (Aigner et al. (1977); Meeusen and van den Broeck (1977)) is:

$$y_i = \beta_0 + f(x_i, \beta) + v_i - u_i, i = 1, \dots, N, \tag{1}$$

where y_i is log output, x_i is a $k \times 1$ vector of inputs (usually in logs), β is $k \times 1$ vector of coefficients; N is the number of observations. The production function $f(\cdot)$ usually takes the log-linear (Cobb-Douglas) or the transcendental logarithmic (translog) form. The noise and inefficiency terms, v_i and u_i, respectively, are assumed to be independent of each other and also independent of x. The sum $\varepsilon_i = v_i - u_i$ is often labeled as the composed error term. This assumption is relaxed in some recent papers, see Lai and Kumbhakar (2019) and the references therein.

To separate noise from inefficiency the SF models assume distributions for both v and u. The popular assumption on the noise term is that $v_i \sim i.i.d.\mathcal{N}(0, \sigma_v^2)$. Several alternative assumptions are made on the inefficiency term, u_i. The most popular ones are exponential and half-normal. We refer to these specifications as *the normal-exponential model* and *the normal-half-normal model*.

As an alternative we consider a model in which the inefficiency term follows a discrete distribution: u takes a value u_1 with probability p and a value u_2 with probability $1-p$. Here $u_1 > 0$, $u_2 > 0$, $0 < p < 1$. We refer to this specification as *the normal-discrete model*. We show that the behavior

[2] It could be approximated with bimodal continuous distribution — a mixture of two normal distributions, with same mean values as values of the discrete distribution takes, and small variances σ. By the continuity with respect to σ the "abnormal" behavior of TI and TE, which we found for the normal-discrete model, will also holds.

of this model can be richer than the behavior for the normal-exponential and normal-half-normal models.

Technical efficiency in model (1) can be defined in several ways. Aigner et al. (1977) suggested $\mathrm{E}(u)$ as the measure of the mean technical inefficiency. Later, Lee and Tyler (1978) proposed $\mathrm{E}(e^{-u})$ as the measure of the mean technical efficiency. Without determinants, these measures are not observation-specific. To make it observation-specific, Jondrow et al. (1982) suggested $\mathrm{E}(u_i|\varepsilon_i)$ as a predictor of TI. Following this procedure, Battese and Coelli (1988) suggested $\mathrm{E}(e^{-u_i}|\varepsilon_i)$ as a predictor of observation-specific measures of TE.

Since we model determinants of TI via the z variables in the variance of u, σ_u, we write $\sigma_u = \sigma_u(z)$. For convenience we consider only one z variable. A popular specification in the literature is $\sigma_u(z) = \exp(z'\gamma) = \exp(\gamma_0 + \gamma z) > 0$.

If $\gamma > 0$, then

$$\frac{\partial \sigma_u}{\partial z} = \sigma_u(z)\,\gamma > 0.$$

Thus, an increase in z causes σ_u to increase. Intuition tells us that, in this case, TI measured by either $\mathrm{E}(u(z))$ or $\mathrm{E}(u(z)|\varepsilon)$ will increase while TE measured by either $\mathrm{E}(e^{-u(z)})$ or $\mathrm{E}(e^{-u(z)}|\varepsilon)$ will decrease. Below, we show that it is true for the normal-exponential and the normal-half-normal models. However, the situation with the normal-discrete model can be different.

In the next subsections we examine these predictors of TI and TE for the two models: normal-exponential and normal-half-normal. In the next section we move to the normal-discrete model.

2.1. Exponential distribution of inefficiency

The two common models for $u \geq 0$ are an exponential distribution and a half-normal distribution. If u follows an exponential distribution it has the following probability density function:

$$f(u) = \frac{1}{\sigma_u(z)} \exp\left(-\frac{u(z)}{\sigma_u(z)}\right), \;\; u \geqslant 0, \tag{2}$$

Technical inefficiency TI and the technical efficiency TE can be predicted from:

$$\begin{aligned} \mathrm{E}(u) &= \sigma_u, \\ \mathrm{E}\left(e^{-u}\right) &= \frac{1}{\sigma_u + 1}. \end{aligned} \tag{3}$$

One can obtain marginal effects of z on the mean technical inefficiency TI and the mean technical efficiency TE from the equations which are:

$$\frac{\partial \mathrm{E}(u)}{\partial z} = \frac{\partial \sigma_u}{\partial z}, \tag{4}$$

$$\frac{\partial \mathrm{E}\left(e^{-u}\right)}{\partial z} = -\frac{1}{(\sigma_u+1)^2}\frac{\partial \sigma_u}{\partial z}. \tag{5}$$

Thus the signs of the marginal effects of z on $TI = \mathrm{E}(u)$ and $TE = \mathrm{E}(e^{-u})$ have opposite signs. If z increases inefficiency, it will decrease efficiency and vice versa.

Instead of using the unconditional means, one can use the conditional means Jondrow et al. (1982) to estimate TI, and the Battese and Coelli (1988) to estimate TE. These estimators can then be used to compute the marginal effects of z.

It is believed, that, for both the unconditional and conditional (observation specific) estimates of $TI = \mathrm{E}(u_i|\varepsilon_i)$ and $TE = \mathrm{E}\left(e^{-u_i}|\varepsilon_i\right)$, discussed below, the marginal effects of z on TI and TE have opposite signs. However, we failed to find proof of this result in the literature. We provide the proof of these results in four theorems below.

In the empirical literature, the conditional mean is widely used to estimate both TI and TE. The advantage of using the conditional means is that the resulting estimates of TI and TE are observation-specific without the z variables explaining inefficiency. However, since our focus is the marginal effects, we assume there are determinants.

The conditional mean (Jondrow et al., 1982) measure of TI and TE (Battese and Coelli, 1992) (after dropping the 'i' subscript to avoid clutter of notation) for the normal-exponential case are (Kumbhakar and Lovell, 2000):

$$TI = \mathrm{E}(u|\varepsilon) = \frac{\sigma_v \phi\left(\frac{\mu_*}{\sigma_v}\right)}{\Phi\left(\frac{\mu_*}{\sigma_v}\right)} + \mu_*, \tag{6}$$

$$TE = \mathrm{E}(e^{-u}|\varepsilon) = \frac{\exp\left(-\mu_* + \frac{\sigma_v^2}{2}\right)\Phi\left(\frac{\mu_*}{\sigma_v} - \sigma_v\right)}{\Phi\left(\frac{\mu_*}{\sigma_v}\right)}, \tag{7}$$

$$\mu_* = -\varepsilon - \frac{\sigma_v^2}{\sigma_u}, \tag{8}$$

where $\varepsilon = v - u$, $\phi(\cdot)$ is the probability density function and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal variable. In deriving this formula, v is assumed to be *i.i.d.* normal and u is *i.i.d.* exponential (see Kumbhakar and Lovell (2000)). Note that both TI and TE are observation-specific.

The marginal effects of z can be computed from $\frac{\partial \mathrm{E}(u|\varepsilon)}{\partial z}$ and $\frac{\partial \mathrm{E}(e^{-u}|\varepsilon)}{\partial z}$:

$$\frac{\partial \mathrm{E}(u|\varepsilon)}{\partial z} = \frac{\partial \mathrm{E}(u|\varepsilon)}{\partial \sigma_u(z)} \frac{\partial \sigma_u(z)}{\partial z}, \tag{9}$$

$$\frac{\partial \mathrm{E}(e^{-u}|\varepsilon)}{\partial z} = \frac{\partial \mathrm{E}(e^{-u}|\varepsilon)}{\partial \sigma_u(z)} \frac{\partial \sigma_u(z)}{\partial z}. \tag{10}$$

So, to prove that marginal effects of z on the technical inefficiency and the technical efficiency have opposite signs, it is enough to prove that the marginal effects of σ_u on TI and TE have opposite signs [3].

We derive these in Statements 1 and 2 and prove the result about signs in Theorems 1 and 2. To avoid notational clutter, from now on, we write σ_u instead of $\sigma_u(z)$.

Statement 1. *For the normal-exponential model* (1)–(2) *the marginal effect of the σ_u on the inefficiency* (6) *is:*

$$\frac{\partial \mathrm{E}(u|\varepsilon)}{\partial \sigma_u} = \frac{\sigma_v^2}{\sigma_u^2} \left(\frac{\Phi^2(t) - \phi^2(t) - t\phi(t)\Phi(t)}{\Phi^2(t)} \right), \tag{11}$$

where $t = \frac{\mu *}{\sigma_v} = -\frac{\varepsilon}{\sigma_v} - \frac{\sigma_v}{\sigma_u}$.

Proof.

$$\frac{\partial \mathrm{E}(u|\varepsilon)}{\partial \sigma_u} = \frac{\partial \mathrm{E}(u|\varepsilon)}{\partial t} \frac{\partial t}{\partial \sigma_u} = \frac{\sigma_v}{\sigma_u^2} \frac{\partial}{\partial t} \left(\sigma_v \frac{\phi(t)}{\Phi(t)} + z\sigma_v \right) =$$
$$= \frac{\sigma_v^2}{\sigma_u^2} \left(\frac{\Phi^2(t) - \phi^2(t) - t\phi(t)\Phi(t)}{\Phi^2(t)} \right).$$

□

Statement 2. *For the normal-exponential model* (1)–(2) *the marginal effect of the σ_u on technical efficiency* $TE = \mathrm{E}(\exp(-u)|\varepsilon)$ *equals:*

[3] In some papers (e.g. Ruggiero (1999); Ondrich and Ruggiero (2001)) efficiency is defined as $E(-u|\varepsilon)$, thus, these marginal effects are opposite by definition.

$$\frac{\partial TE}{\partial \sigma_u} = \frac{\sigma_v}{\sigma_u^2} \cdot \frac{\exp\left(-t\sigma_v + \frac{\sigma_v^2}{2}\right)}{\Phi^2(t)} \times$$
$$\times \left(-\sigma_v \Phi(t-\sigma_v)\Phi(t) + \phi(t-\sigma_v)\Phi(t) - \Phi(t-\sigma_v)\phi(t)\right), \qquad (12)$$

where as before $t = \frac{\mu*}{\sigma_v} = -\frac{\varepsilon}{\sigma_v} - \frac{\sigma_v}{\sigma_u}$.

Proof. From (7)–(8) we get:

$$TE = \mathrm{E}(e^{-u}|\varepsilon) = \frac{\exp\left(-t\sigma_v + \frac{\sigma_v^2}{2}\right)\Phi(t-\sigma_v)}{\Phi\left(t\right)},$$

thus

$$\frac{\partial TE}{\partial \sigma_u} = \frac{\partial TE}{\partial t}\frac{\partial t}{\partial \sigma_u} = \frac{\sigma_v}{\sigma_u^2}\frac{\partial}{\partial t}\frac{\exp\left(-t\sigma_v + \frac{\sigma_v^2}{2}\right)\Phi(t-\sigma_v)}{\Phi(t)}$$
$$= \frac{\sigma_v}{\sigma_u^2} \cdot \frac{\exp\left(-t\sigma_v + \frac{\sigma_v^2}{2}\right)}{\Phi^2(t)} \times \qquad (13)$$
$$\times \left(-\sigma_v \Phi(t-\sigma_v)\Phi(t) + \phi(z-\sigma_v)\Phi(t) - \Phi(t-\sigma_v)\phi(t)\right).$$

□

Theorem 1. *For the normal-exponential model defined by* (1) *and* (2) *the marginal effect of* σ_u *on* $E(u|\varepsilon)$ *is non-negative. That is, if* σ_u *increases, technical inefficiency estimated by* $E(u|\varepsilon)$ *also increases:*

$$\frac{\partial E(u|\varepsilon)}{\partial \sigma_u} \geqslant 0.$$

Theorem 2. *For the normal-exponential model defined by* (1) *and* (2) *the marginal effect of* σ_u *on* $TE = E(e^{-u}|\varepsilon)$ *is non-positive. That is, if* σ_u *increases,* TE *decreases:*

$$\frac{\partial \mathrm{E}(e^{-u}|\varepsilon)}{\partial \sigma_u} \leqslant 0.$$

Proofs of Theorems 1 and 2 are given in Appendix A.1.

2.2. Half-normal distribution of inefficiency

If u follows a half-normal distribution it has the following probability density function:

$$f(u) = \frac{\sqrt{2}}{\sqrt{\pi}\sigma_u(z)} \exp\left(-\frac{u(z)^2}{2\sigma_u^2(z)}\right), \quad u \geqslant 0, \tag{14}$$

The technical inefficiency TI and the technical efficiency TE can be measured as (see, e.g. Kumbhakar and Lovell (2000)):

$$\begin{aligned} \mathrm{E}(u) &= \sigma_u \sqrt{\frac{2}{\pi}}, \\ \mathrm{E}\left(e^{-u}\right) &= 2\left(1 - \Phi(\sigma_u\right) \exp\left(\frac{\sigma_u^2}{2}\right). \end{aligned} \tag{15}$$

One can obtain marginal effects of z on the mean technical inefficiency TI and the mean technical efficiency TE from the equations:

$$\frac{\partial \mathrm{E}(u)}{\partial z} = \sqrt{\frac{2}{\pi}} \frac{\partial \sigma_u}{\partial z}, \tag{16}$$

$$\frac{\partial \mathrm{E}\left(e^{-u}\right)}{\partial z} = 2 \frac{\partial \sigma_u}{\partial z} \exp\left(\frac{\sigma_u^2}{2}\right) \left(\sigma_u - \phi(\sigma_u) - \Phi(\sigma_u)\sigma_u\right). \tag{17}$$

Since $\phi(x)/(1 - \Phi(x)) > x$ (see inequality (2) in (Sampford, 1953)) we have $\sigma_u - \phi(\sigma_u) - \Phi(\sigma_u)\sigma_u < 0$, thus marginal effects (16) and (17) have different signs, as expected.

The conditional mean measure of TI (Jondrow et al., 1982) and TE (Battese and Coelli, 1992) for the normal-half-normal case are (Kumbhakar and Lovell, 2000):

$$\mathrm{E}(u|\varepsilon) = \frac{\sigma_* \phi\left(\frac{\mu_*}{\sigma_*}\right)}{\Phi\left(\frac{\mu_*}{\sigma_*}\right)} + \mu_*, \tag{18}$$

$$TE = \mathrm{E}(e^{-u}|\varepsilon) = \frac{\exp\left(-\mu_* + \frac{\sigma_*^2}{2}\right) \Phi\left(\frac{\mu_*}{\sigma_*} - \sigma_*\right)}{\Phi\left(\frac{\mu_*}{\sigma_*}\right)}, \tag{19}$$

$$\mu_* = \frac{-\sigma_u^2 \varepsilon}{\sigma_v^2 + \sigma_u^2}, \tag{20}$$

$$\sigma_*^2 = \frac{\sigma_v^2 \sigma_u^2}{\sigma_v^2 + \sigma_u^2}. \tag{21}$$

Theorem 3. *For the normal-half-normal model, defined by* (1) *and* (14), *the marginal effect of* σ_u *on* $E(u|\varepsilon)$ *is non-negative. That is, if* σ_u *increases, technical inefficiency estimated by* $E(u|\varepsilon)$ *also increases:*

$$\frac{\partial E(u|\varepsilon)}{\partial \sigma_u} \geqslant 0.$$

Theorem 4. *For the normal-half-normal model, defined by* (1) *and* (14), *the marginal effect of* σ_u *on* $TE = E(e^{-u}|\varepsilon)$ *is non-positive. That is, if* σ_u *increases,* TE *decreases:*

$$\frac{\partial \mathrm{E}(e^{-u}|\varepsilon)}{\partial \sigma_u} \leqslant 0.$$

Thus, taking into account (9), (10) and Theorems 1–4, we conclude that for the normal-exponential model (1), (2), as well as for the normal-half-normal model (1), (14), signs of marginal effects of z on $E(u|\varepsilon)$ and $TE = E(e^{-u}|\varepsilon)$ are opposite, i.e.,

$$\mathrm{sign}\frac{\partial \mathrm{E}(u|\varepsilon)}{\partial z} = -\mathrm{sign}\frac{\partial \mathrm{E}(e^{-u}|\varepsilon)}{\partial z}. \tag{22}$$

Proofs of Theorems 3 and 4 are given in Appendix A.2.

3. Discrete distribution of inefficiency error

3.1. Discrete model

To come up with a counter-example of the above result, we now consider an example of a discrete distribution for $u > 0$ with the support that consists of two values u_1 and u_2:

$$u = \begin{cases} u_1, \text{ with } P(u = u_1) = p, \\ u_2, \text{ with } P(u = u_2) = 1 - p, \end{cases} \tag{23}$$

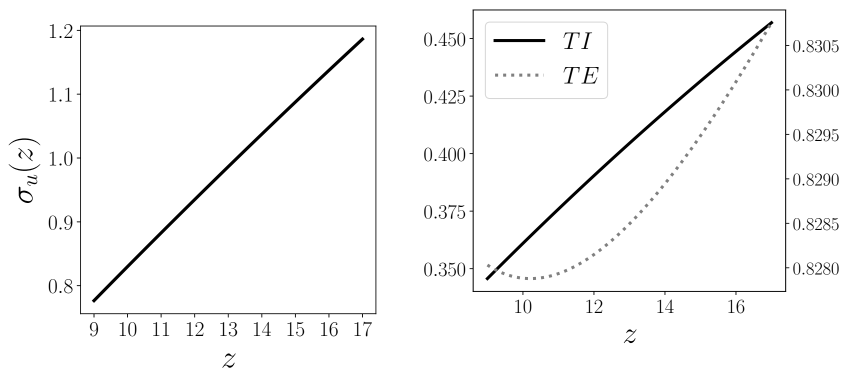
with $u_1 > 0$, $u_2 > 0$, $0 < p < 1$.

For the distribution of u in (23) we have:

$$\mathrm{E}(u) = u_1 p + u_2(1 - p),$$
$$\mathrm{Var}(u) = \sigma_u^2 = p(1 - p)(u_1 - u_2)^2, \tag{24}$$
$$TE(u) = \mathrm{E}(e^{-u}) = pe^{-u_1} + (1 - p)e^{-u_2}. \tag{25}$$

The proposed normal-discrete model is an identifiable model, as our study in Appendix B shows.

In contrast to the exponential distribution (2), standard deviation σ_u of this distribution, depends on three parameters u_1, u_2, and p.



(a) Variance σ_u is an increasing function of z for the considered normal-discrete model

(b) $TI = \mathrm{E}(u)$ (left scale) and $TE = \mathrm{E}(e^{-u})$ (right scale) in the range $10.3 \leq z \leq 17$ are both monotonically increasing function of z and thus of σ_u

Figure 1: Unusual behavior of the discrete normal model

3.2. Numerical experiments

Use of this discrete distribution can result in unexpected behavior of TI and TE with an increase in σ_u induced by an increase in z.

To show this we consider an example with the factor variable z, such that $9 \leq z \leq 17$ and

$$\begin{cases} p &= 0.9 + 0.001z, \\ u_1 &= 0.1, \\ u_2 &= 1 + 0.2z. \end{cases} \tag{26}$$

so that $\sigma_u(z)$ is an increasing function of z (left pane of Fig. 1). But, in the range $10.3 \leq z \leq 17$, the behavior of TI and TE is "abnormal", see the right pane of Fig. 1. In this range both TI and TE are increasing functions of σ_u. The variance σ_u is an increasing function of z. That is, an increase in z causes an increase of σ_u which causes a simultaneous increase of TI and TE. For values $z \leq 10.3$ this abnormal effect doesn't exist, because difference between u_1 and u_2 is "too small".

Suppose that the real data are generated with model (1) with v generated from a normal distribution, and u from the discrete distribution in (23) with parameters in (26). Then, if one one applies the normal-exponential model (1) and (2), the estimates are likely to suffer from model misspecification. Use of the normal-exponential model, according to (9) and (10), an increase in z causes a decrease of TI, while the real situation is the opposite.

3.3. Discrete distribution. Mean TE

To illustrate the aforementioned problem, we run simulations with the following specifications. We choose the sample size $N = 1000$. The single input x_i is generated from a uniform distribution on the interval $[2, 7]$. The noise term $v_i \sim N(0, 0.25)$. A variable z_i comes from an uniform distribution on the interval $[9, 17]$. The parameters of the discrete distribution of u in (23) are: $u_{i,1} = 0.1$; $u_{i,2} = 1 + 0.2z_i$; $p_i = 0.9 + 0.001z_i$. To simulate u_i, we also use a uniformly distributed random variable $r_i \sim U[0, 1]$ for each i. We then assign $u_i = u_{i,1}$ if $r_i < p_i$ and $u_i = u_{i,2}$ otherwise. Finally we generate output y_i according to $y_i = 1 + x_i + v_i - u_i$.

Using the generated data we estimated the parameters of normal-exponential model, (1) and (2), with the following specification for $\sigma_u(z)$, viz., $\ln \sigma_u(z_i) = \gamma_0 + \gamma z_i$, and obtained

$$\hat{\sigma}_{u_i} = \exp(-0.618 + 0.025z_i).$$

We used this estimate of $\sigma_u(z_i)$ to get estimate of TE using (3), i.e., $\widehat{TE}_i = 1/(1 + \hat{\sigma}_{ui})$.

Plot of true σ_{u_i} using (24) and estimated $\hat{\sigma}_{u_i}$ against z is presented in Figure 2. Similarly, plot of true TE_i calculated using (25) and estimated $\widehat{TE}_i$ against z is presented in Figure 3. It can be seen from the figures that, while $\hat{\sigma_u}$ increases with z, like σ_u, true TE and the estimate of TE move in opposite directions. In this case, the model misspecification leads to the wrong conclusion of the negative effect of z on TE.

3.4. Discrete distribution. Observation-specific TE

We continue with the discrete case to provide another counter-example when TE is estimated from the conditional mean. For this, we consider a discrete random variable $u > 0$, which takes values $u_i = z\, u_{i0}$, $i = 1, 2$ with probabilities p_1, p_2, such that $p_1 + p_2 = 1$, and $u_{i0} > 0$, $i = 1, 2$, $z > 0$.

$$P(u_i = zu_{i0}) = p_i, \;\; i = 1, 2. \tag{27}$$

Variance of u_i depends on z, i.e.,

$$\sigma^2_{ui} = z^2 p_1 p_2 (u_{10} - u_{20})^2 = z^2 c^2, \;\; c > 0, \tag{28}$$

where $c = p_1 p_2 (u_{10} - u_{20})$. Thus

$$\sigma_u = z\, c, \text{ and } \frac{\partial \sigma_u}{\partial z} = c > 0, \tag{29}$$

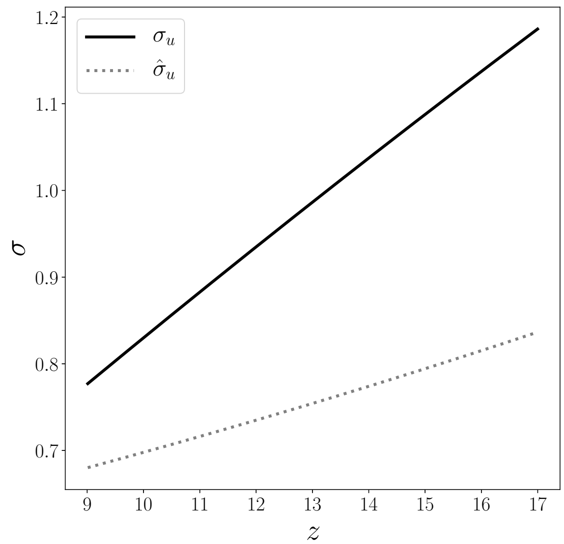


Figure 2: σ_u and $\hat{\sigma_u}$ behave in a similar way for the normal-discrete model

Statement 3. *Consider the SF model* (1) *with* $v_i \sim \mathcal{N}(0, \sigma_v^2)$ *and a one-parameter distribution for* u *in* (27)*. Then the sign of the marginal effect of* z *on* TE *defined as* $TE = E(e^{-u}|\varepsilon)$ *is:*

$$\begin{aligned}
\frac{\partial TE}{\partial z} &= \frac{\partial \mathrm{E}(e^{-u}|\varepsilon)}{\partial z} \\
&= -\frac{1}{\sum_{i=1}^{2} p_i e^{-w_i}} \sum_{i=1}^{2} p_i e^{-z\, u_{i0}} e^{-w_i} (u_{i0} + w_i') \\
&+ \frac{1}{\left(\sum_{i=1}^{2} p_i e^{-w_i}\right)^2} \left(\sum_{i=1}^{2} p_i e^{-z\, u_{i0}} e^{-w_i}\right) \left(\sum_{i=1}^{2} p_i e^{-w_i} w_i'\right),
\end{aligned}$$

where $w_i = \frac{(z\, u_{i0} + \varepsilon)^2}{2\sigma_v^2}$ *and* $w_i' = \frac{\partial}{\partial z} w_i = \frac{z\, u_{i0}^2 + \varepsilon\, u_{i0}}{\sigma_v^2}$.

The proof is presented in the Appendix A.3.

Note that the marginal effect of z on TE, in the normal-exponential model, is negative if $\frac{\partial \sigma_u}{\partial z} > 0$ (see Theorem 2). However, in the normal-discrete model, the sign of the marginal effect of z depends on the value of ε. That is, the value of the marginal effect, as well as its sign, depends on the value of ε.

We illustrate this with the plot of $\frac{\partial TE}{\partial z}$ against ε for these values of the model parameters: $z = 8.5$; $\sigma_v = 1$; $u_1 = 0.1$; $u_2 = 0.89$; $p_1 = 0.99$; $p_2 = 0.01$.

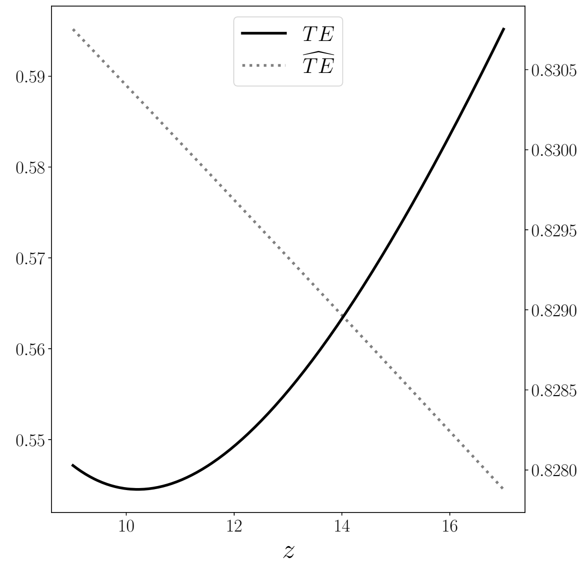


Figure 3: TE (left scale) and $\widehat{TE}$ (right scale) as a function of z, behave in a different way for the normal-discrete model

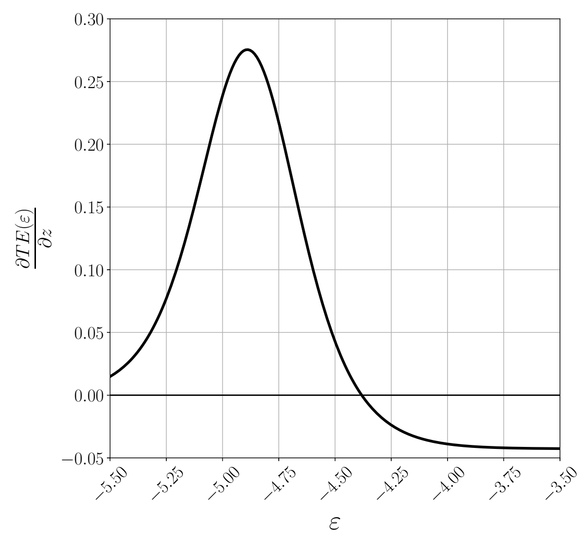


Figure 4: Marginal effect $\frac{\partial TE}{\partial z}$ as function of ε for $z = 8.5$; $\sigma_v = 1$; $u_1 = 0.1$; $u_2 = 0.89$; $p_1 = 0.99$; $p_2 = 0.01$

From Figure 4 one can see that if the normal-discrete model is the true model, then the sign of the marginal effect may vary across observations.

But for the normal-exponential model the marginal effect is always negative if $\frac{\partial \sigma_u}{\partial z} > 0$. Thus if the normal-exponential model is used, where the true model is normal-discrete, one can come to the wrong conclusion regarding the sign of the marginal effect.

Sometimes the focus is not on the individual values of TE but their rankings. To examine how the true values of TE are related to their estimated counterparts, we consider the following simulations. We used $N = 1000$, generated input x_i from a uniformly distributed random variable in the interval $[2, 7]$. The noise term is generated from standard normal distribution $v_i \sim N(0,\, 1)$. The z_i variable is generated from a normal distribution with mean 4.5 and variance 0.25, $z_i \sim N(4.5,\, 0.25)$ uniformly distributed random variable in the interval $[8, 9.4]$. The parameters of the discrete distribution of u are chosen as: $p_1 = 0.5, p_2 = 0.5; u_{(1)} = 0.1, u_{(2)} = 0.89$. We also generated a variable r_i, which is uniformly distributed in the interval $[0, 1]$. Then we generated $u_{i0} = u_{(2)}$ if $r_i < p_2$ and $u_{i0} = u_{(1)}$ otherwise, and assume $u_i = z_i\, u_{i0}$. Finally we generated output y_i as: $y_i = 1 + x_i + v_i - u_i$.

Note that for each unit i distribution of the inefficiency term $u_i|z_i$ is discrete. But since parameters of this distribution vary with z, the distribution of u in our sample is continuous, bimodal distribution, mixture of two normal distributions. Kernel density of u with Gaussian kernel is presented in Figure 5. Suppose, for example, that our units are firms and there are two types of firm managers. Thus this kind of inefficiency distribution could be observed.

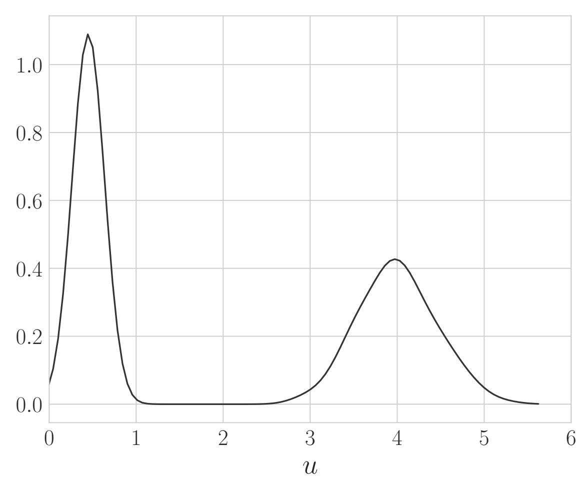
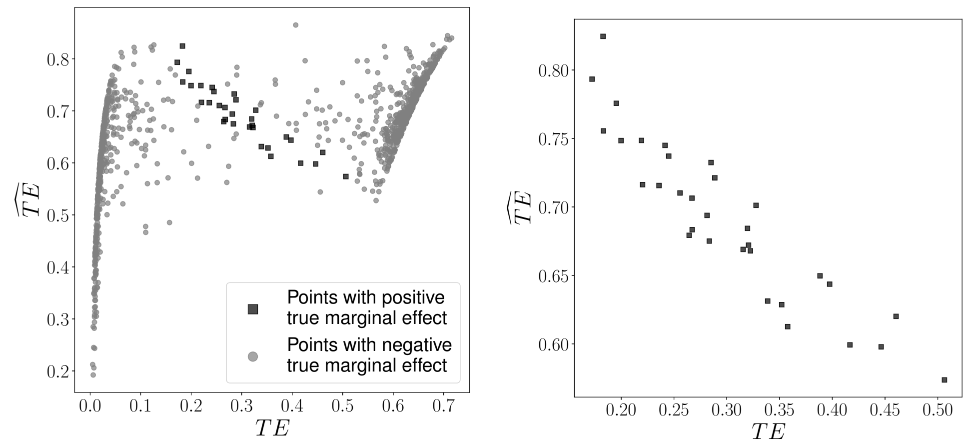


Figure 5: Kernel density of inefficiency term u distribution



(a) Comparison of all estimates of $\widehat{TE}$ and true TE

(b) Selected points with ε between -2.3 and -2.1

Figure 6: Scatter plot of $\widehat{TE}$ and true TE

We used these data to estimate the parameters of the normal-exponential model (1)–(2) with the following specification of σ_u: $\ln \sigma_{ui} = \gamma_0 + \gamma z_i$, and obtained the estimates of the observation specific technical efficiencies $\widehat{TE}_i$.

For each i true TE_i was calculated as

$$TE_i = \mathrm{E}(e^{-u}|\varepsilon_i) = \frac{\left(\sum_{i=1}^{k} p_i e^{-z_i\, u_i} e^{-w_i}\right)}{\left(\sum_{i=1}^{k} p_i e^{-w_i}\right)}, \tag{30}$$

where $w_i = \frac{(z_i\, u_i + \varepsilon_i)^2}{2\sigma_v^2}$.

A scatter plot of the estimated $\widehat{TE}_i$ against true TE_i is provided in Figure 6. It can be seen that for some subsets of data increase in true TE corresponds to the decrease of the estimated $\widehat{TE}$. Thus applying the traditional normal-exponential model for the situation when true distribution of the inefficiency term is discrete of bimodal one can come to wrong conclusions on the ranking of firms by their techincal efficiency.

In Appendix B we compare in more details the estimates of technical efficiency obtained by normal-exponential, normal-half-normal and (true) normal-discrete models using these data, generated by normal-discrete model. Plot of the distribution of residuals in normal-exponential model B.8 could be used as diagnostic plot for this type of misspecification.

4. Conclusions and discussions

In this paper we derived the formula for computing the marginal effects of determinants of inefficiency (z) on both the unconditional and conditional means of technical inefficiency and technical efficiency for the normal-exponential and for the normal-half-normal stochastic frontier models. We proved that, for the normal-exponential and normal-half-normal models, the signs of the marginal effects of z on the technical inefficiency and technical efficiency are of opposite signs.

We considered an example of discrete distribution for technical inefficiency and showed that the relationship between the true and estimated technical efficiency for the normal-discrete model can be substantially different from the normal-exponential model, at least for some values of z. These results illustrate that, if the real world data on noise comes from a normal and inefficiency comes from a discrete distribution, or bimodal continuous distribution, and a researcher estimates the model assuming that the errors are normal and exponential instead, results on estimated efficiency, its marginal effect and rankings, might all be wrong. Such situation may occurs if we estimate technical efficiency of firms and there are two types of managers. That is, the consequence of misspecification of inefficiency distribution can be quite serious.

Acknowledgments

We are grateful for many invaluable comments provided by participants at the Sixteenth European Workshop on Efficiency and Productivity Analysis in London, 2019.

Appendix A. Proofs

Appendix A.1. Proof of Theorems 1 and 2

First we reproduce a proof of the Lemma 1 from (Sampford (1953)):

Lemma 1. *Let $\phi(z)$ and $\Phi(z)$ be the probability density function and the cumulative density function of the standard normal distribution $\mathcal{N}(0,1)$, and $\lambda(z) = \frac{\phi(z)}{\Phi(z)}$. Then it holds:*

1. $1 - z\lambda(z) - \lambda(z)^2 \geq 0$.
2. $\lambda(z)$ *is a decreasing function and its derivative* $\lambda'(z) \in (-1, 0)$.

Proof. Obviously $f(t) = \frac{\phi(t)}{\Phi(z)} = \frac{\phi(t)}{P(Z \le z)}$ is a probability density function of a random variable X defined at the interval $(-\infty, z)$.

$$\mathrm{E}(X) = \int_{-\infty}^{z} t\frac{\phi(t)}{\Phi(z)}dt = \frac{1}{\Phi(z)}\int_{-\infty}^{z} t\phi(t)dt = -\frac{1}{\Phi(z)}\int_{-\infty}^{z} \phi'(t)dt = -\frac{\phi(z)}{\Phi(z)} = -\lambda(z),$$

$$\mathrm{E}(X^2) = \int_{-\infty}^{z} t^2\frac{\phi(t)}{\Phi(z)}dt = \frac{1}{\Phi(z)}\int_{-\infty}^{z} t^2\phi(t)dt = -\frac{1}{\Phi(z)}\int_{-\infty}^{z} t\phi'(t)dt =$$

$$= -\frac{1}{\Phi(z)}\left(t\phi(t)\Big|_{-\infty}^{z} - \int_{-\infty}^{z}\phi(t)dt\right) = -\frac{1}{\Phi(z)}\big(z\phi(z) - \Phi(z)\big) = 1 - z\lambda(z).$$

Hence, the variance is

$$\mathrm{Var}(X) = 1 - z\lambda(z) - (-\lambda(z))^2 = 1 - z\lambda(z) - \lambda(z)^2 \ge 0.$$

Since

$$\lambda'(z) = \left(\frac{\phi(z)}{\Phi(z)}\right)' = \frac{1}{\Phi(z)^2}\big(\phi(z)'\Phi(z) - \phi(z)\Phi(z)'\big) = -z\lambda(z) - \lambda(z)^2$$
$$= \mathrm{Var}(X) - 1,$$

we have $-1 \le \lambda'(z) \le 0$. □

Appendix A.1.1. Proof of Theorem 1

Proof. From Statement 1 we have

$$\frac{\partial \mathrm{E}(u|\varepsilon)}{\partial \sigma_u} = \frac{\sigma_v^2}{\sigma_u^2}\frac{\Phi^2(z) - \phi^2(z) - z\phi(z)\Phi(z)}{\Phi^2(z)} = \frac{\sigma_v^2}{\sigma_u^2}\big(1 - z\lambda(z) - z\lambda(z)^2\big),$$

which is non-negative by Lemma 1. □

Appendix A.1.2. Proof of Theorem 2

Proof. From Statement 2 we have

$$\frac{\partial TE}{\partial \sigma_u} = \frac{\sigma_v}{\sigma_u^2}\cdot\frac{\exp\left(-z\sigma_v + \frac{\sigma_v^2}{2}\right)}{\Phi^2(z)}\Phi(z)\Phi(z - \sigma_v)\big(-\sigma_v + \lambda(z - \sigma_v) - \lambda(z)\big). \tag{A.1}$$

Since the first factors in (A.1) and σ_v are greater or equal to 0, it is enough to prove that

$$f(t) = -t + \lambda(z - t) - \lambda(z) \le 0 \text{ for all } t \ge 0.$$

We have $f(0) = 0$, and $f'(t) = -1 - \lambda'(z - t) \le 0$ since $-1 \le \lambda(t) \le 0$ for all t (Lemma 1). Thus $f(t) \le 0$, and Theorem 2 is proven. □

Appendix A.2. Proof of Theorems 3 and 4

Statement 4. *For $\lambda(z) = \frac{\phi(z)}{\Phi(z)}$ it holds that:*

$$2\lambda^2(z) > 1 - z^2 - 3z\lambda(z) \text{ for } z < 0. \tag{A.2}$$

Proof. From the proof of Theorem 9 in (Gasull and Utzet (2014)) we find

$$2 + x^2a^2(x) - a^2(x) - 3xa(x) > 0 \text{ for } x > 0,$$

where

$$a(x) = \frac{1 - \Phi(x)}{\phi(x)} = \frac{1}{\lambda(-x)}.$$

So,

$$2 + x^2\frac{1}{\lambda^2(-x)} - \frac{1}{\lambda^2(-x)} - 3x\frac{1}{\lambda(-x)} > 0 \text{ for } x > 0.$$

By the change of variable $z = -x$ we get:

$$2 + z^2\frac{1}{\lambda^2(z)} - \frac{1}{\lambda^2(z)} + 3z\frac{1}{\lambda(z)} > 0 \text{ for } z < 0.$$

Moving $\frac{1}{\lambda^2(z)}$ we obtain the following inequality:

$$\frac{1}{\lambda^2(z)}\left[2\lambda^2(z) + z^2 - 1 + 3z\lambda(z)\right] > 0 \text{ for } z < 0.$$

As $\lambda^2(z) > 0$, this inequality is equivalent to:

$$2\lambda^2(z) + z^2 - 1 + 3z\lambda(z) > 0 \text{ for } z < 0.$$

Moving two terms to the right side of the inequality we get:

$$2\lambda^2(z) > 1 - z^2 - 3z\lambda(z) \text{ for } z < 0.$$

□

Proof of Theorem 3.

Proof. Denote $A = \frac{\mu_*}{\sigma_*}$. Thus,

$$A = -\varepsilon\frac{\sigma_u^2}{\sigma_u^2 + \sigma_v^2} \cdot \frac{\sqrt{\sigma_u^2 + \sigma_v^2}}{\sigma_u\sigma_v} = -\varepsilon\frac{\sigma_u}{\sigma_v}\frac{1}{\sqrt{\sigma_u^2 + \sigma_v^2}} =$$
$$= -\varepsilon\frac{1}{\sigma_v^2}\frac{\sigma_u\sigma_v}{\sqrt{\sigma_u^2 + \sigma_v^2}} = -\varepsilon\frac{\sigma_*}{\sigma_v^2}.$$

Using this notation we get:

$$\mathrm{E}(u|\varepsilon) = \sigma_* \frac{\phi(A)}{\Phi(A)} + \sigma_* A = \sigma_* \left[\frac{\phi(A)}{\Phi(A)} + A \right].$$

The desired partial derivative has the form:

$$\frac{\partial}{\partial \sigma_*}\mathrm{E}(u|\varepsilon) = \frac{\partial}{\partial \sigma_*}\left[\sigma_* \left(\frac{\phi(A)}{\Phi(A)} + A\right)\right] = \frac{\partial}{\partial \sigma_*}\left[\sigma_*(\lambda(A) + A)\right] =$$
$$= \lambda(A) + A + \sigma_*(\lambda'(A) + 1)\frac{\partial A}{\partial \sigma_*} = \lambda(A) + A + (1 + \lambda'(A))\sigma_* \left(\frac{-\varepsilon}{\sigma_v^2}\right) =$$
$$= \lambda(A) + A + (1 + \lambda'(A))A = \lambda(A) + 2A + A\lambda'(A) =$$
$$= \frac{\phi(A)\Phi(A) + 2A\Phi^2(A) + A(-A\phi(A)\Phi(A) - \phi^2(A)}{\Phi^2(A)} =$$
$$= \frac{1}{\Phi^2(A)}\left(\phi(A)\Phi(A) + 2A\Phi^2(A) - A^2\phi(A)\Phi(A) - A\phi^2(A)\right),$$

since

$$\lambda'(z) = \frac{\partial}{\partial z}\frac{\phi(z)}{\Phi(z)} = \frac{\phi'(z)\Phi(z) - \phi(z)\Phi'(z)}{\Phi^2(z)} = \frac{-z\phi(z)\Phi(z) - \phi^2(z)}{\Phi^2(z)}$$
$$= -z\lambda(z) - \lambda^2(z),$$

and

$$\frac{\partial A}{\partial \sigma_*} = \frac{\partial}{\partial \sigma_*}\left(-\varepsilon\frac{\sigma_*}{\sigma_v^2}\right) = -\frac{\varepsilon}{\sigma_v^2}.$$

So, to prove the theorem it is sufficient to prove that

$$\psi(z) = \phi(z)\Phi(z) + 2z\Phi^2(z) - z^2\phi(z)\Phi(z) - z\phi^2(z) > 0, \ \forall z.$$

It is equivalent to proving

$$\lambda(z) + 2z - z^2\lambda(z) - z\lambda^2(z) > 0. \tag{A.3}$$

We start with the case $z < 0$.

Multiplying the inequality by 2 we get an equivalent inequality:

$$2\lambda(z) + 4z - 2z^2\lambda(z) - 2z\lambda^2(z) > 0.$$

From (A.2) in Statement 4 above:

$$\begin{aligned} &2\lambda(z) + 4z - 2z^2\lambda(z) - 2z\lambda^2(z) \\ &\quad > 2\lambda(z) + 4z - 2z^2\lambda(z) - z\left(1 - z^2 - 3z\lambda(z)\right) \\ &\quad = 2\lambda(z) + 4z - 2z^2\lambda(z) - z + z^3 + 3z^2\lambda(z) \\ &\quad = 2\lambda(z) + 3z + z^2\lambda(z) + z^3 = (2 + z^2)\lambda(z) + 3z + z^3. \end{aligned}$$

So, it is sufficient to prove, that for $z < 0$:

$$(2+z^2)\lambda(z) + 3z + z^3 > 0. \tag{A.4}$$

From (Baricz (2008)) the following inequality holds:

$$\frac{1}{\lambda(-x)} < \frac{4}{\sqrt{x^2+8}+3x}, x > 0.$$

Using the change of variables $z = -x$ we get:

$$\frac{1}{\lambda(z)} < \frac{4}{\sqrt{z^2+8}-3z}, z \leq 0,$$

which implies

$$\lambda(z) > \frac{1}{4}\left(\sqrt{z^2+8} - 3z\right), z \leq 0. \tag{A.5}$$

The inequality (A.4) is equivalent to:

$$\lambda(z) > \frac{-3z - z^3}{2+z^2}.$$

So, using the bound (A.5) it is sufficient to prove, that for $z < 0$:

$$\frac{1}{4}\left(\sqrt{z^2+8} - 3z\right) > \frac{-3z - z^3}{2+z^2}.$$

For $x = -z \geq 0$ we get an equivalent inequality:

$$\frac{1}{4}\left(\sqrt{x^2+8} + 3x\right) > \frac{3x + x^3}{2+x^2}.$$

Rearranging the terms we get the inequality:

$$\sqrt{x^2+8} > 4\frac{3x+x^3}{2+x^2} - 3x. \tag{A.6}$$

For the right side of (A.6) we have:

$$4\frac{3x+x^3}{2+x^2} - 3x = \frac{12x + 4x^3 - 3x^3 - 6x}{2+x^2} = \frac{x^3+6x}{2+x^2} = x + \frac{4x}{x^2+2}.$$

Both parts of (A.6) are positive, so (A.6) is equivalent to:

$$x^2 + 8 > \left(x + \frac{4x}{x^2+2}\right)^2.$$

Moving x^2 to the right side we get:

$$8 > \frac{8x^2}{x^2+2} + \frac{16x^2}{(x^2+2)^2}.$$

Moving the first term at the right side to the left we get:

$$8\left(1 - \frac{x^2}{x^2+2}\right) > \frac{16x^2}{(x^2+2)^2}.$$

Subtracting $\frac{x^2}{x^2+2}$ from 1 we obtain:

$$8\frac{2}{x^2+2} > \frac{16x^2}{(x^2+2)^2}.$$

Since

$$1 > \frac{x^2}{x^2+2}.$$

we proved (A.3) for $z < 0$. The remaining part is the proof of (A.3) for $z > 0$.

Since

$$-1 \leq \lambda'(z) \leq 0,$$

we have:

$$\lambda(z)+2z-z^2\lambda(z)-z\lambda^2(z) = \lambda(z)+2z+z\lambda'(z) \geq \lambda(z)+2z+(-z) = \lambda(z)+z > 0.$$

QED. □

Appendix A.2.1. Proof of Theorem 4

Proof.

$$\mathrm{E}(e^{-u}|\varepsilon) = \frac{\Phi\left(\frac{\mu_*}{\sigma_*} - \sigma_*\right)}{\Phi\left(\frac{\mu_*}{\sigma_*}\right)} e^{-\mu_* + \frac{1}{2}\sigma_*^2}.$$

For A we have

$$A = \frac{\mu_*}{\sigma_*} = -\varepsilon\frac{\sigma_*}{\sigma_v^2}.$$

Then the partial derivative with respect to σ_* has the form:

$$\frac{\partial A}{\partial \sigma_*} = -\frac{\varepsilon}{\sigma_v^2}.$$

For $E(e^{-u}|\varepsilon)$ we obtain:

$$\mathrm{E}(e^{-u}|\varepsilon) = \frac{\Phi(A-\sigma_*)}{\Phi(A)} e^{-A\sigma_* + \frac{1}{2}\sigma_*^2}.$$

Then the partial derivative has the form:

$$\frac{\partial \mathrm{E}(e^{-u}|\varepsilon)}{\partial \sigma_*} = \frac{\partial}{\partial \sigma_*}\left[\frac{\Phi(A-\sigma_*)}{\Phi(A)}\right] e^{-A\sigma_* + \frac{1}{2}\sigma_*^2} + \frac{\Phi(A-\sigma_*)}{\Phi(A)} e^{-A\sigma_* + \frac{1}{2}\sigma_*^2} \frac{\partial}{\partial \sigma_*}\left(-A\sigma_* + \frac{1}{2}\sigma_*^2\right).$$

We continue to expand the terms above using the following:

$$-A\sigma_* + \frac{1}{2}\sigma_*^2 = \frac{\varepsilon\sigma_*^2}{\sigma_v^2} + \frac{1}{2}\sigma_*^2 = \sigma_*^2\left(\frac{1}{2} + \frac{\varepsilon}{\sigma_v^2}\right).$$

So,

$$\begin{aligned}\frac{\partial \mathrm{E}(e^{-u}|\varepsilon)}{\partial \sigma_*} &= \frac{1}{\Phi^2(A)}\Bigg(\phi(A-\sigma_*)\left(-\frac{\varepsilon}{\sigma_v^2} - 1\right)\Phi(A) \\ &- \Phi(A-\sigma_*)\phi(A)\left(-\frac{\varepsilon}{\sigma_v^2}\right)\Bigg) e^{-A\sigma_* + \frac{1}{2}\sigma_*^2} + \\ &+ \frac{\Phi(A-\sigma_*)}{\Phi(A)} e^{\sigma_*^2\left(\frac{1}{2} + \frac{\varepsilon}{\sigma_v^2}\right)} 2\sigma_*\left(\frac{1}{2} + \frac{\varepsilon}{\sigma_v^2}\right) = \\ &= \frac{\Phi(A-\sigma_*)}{\Phi(A)} e^{\sigma_*^2\left(\frac{1}{2} + \frac{\varepsilon}{\sigma_v^2}\right)} \frac{1}{\sigma_*}\sigma_*\Bigg(\lambda(A-\sigma_*)\left(-\frac{\varepsilon}{\sigma_*^2} - 1\right) - \\ &- \lambda(A)\left(-\frac{\varepsilon}{\sigma_v^2}\right) + 2\sigma_*\left(\frac{1}{2} + \frac{\varepsilon}{\sigma_v^2}\right)\Bigg).\end{aligned}$$

Therefore, we need to prove that:

$$\sigma_*\left(\lambda(A-\sigma_*)\left(-\frac{\varepsilon}{\sigma_*^2} - 1\right) - \lambda(A)\left(-\frac{\varepsilon}{\sigma_v^2}\right) + 2\sigma_*\left(\frac{1}{2} + \frac{\varepsilon}{\sigma_v^2}\right)\right) < 0.$$

Or equivalently:

$$\lambda(A-\sigma_*)(A-\sigma_*) - \lambda(A)A + \sigma_*^2 - 2A\sigma_* < 0.$$

If $x = A - \sigma_*$, then $A = x + \sigma_* = x + t, t > 0$ and we have:

$$\lambda(x)x - \lambda(x+t)(x+t) + t^2 - 2(x+t)t < 0,$$

$$\lambda(x)x - \lambda(x+t)(x+t) - t^2 - 2xt < 0.$$

So, we need to prove that for $t > 0$ and any arbitrary x:

$$\psi(x,t) = (x+t)\lambda(x+t) - x\lambda(x) + t^2 + 2xt > 0.$$

Since $\psi(x,0) = 0$, it is sufficient to prove that the function is increasing, i.e., the corresponding partial derivative is positive:

$$\frac{\partial\psi(x,t)}{\partial t} = \lambda(x+t) + (x+t)\lambda'(x+t) + 2t + 2x > 0.$$

Using the change of variables $z = x + t$ we get the inequality for $z \in (-\infty, +\infty)$:

$$\lambda(z) + z\lambda'(z) + 2z > 0.$$

For $z > 0$ it is obvious that:

$$z(1 + \lambda'(z)) + (z + \lambda(z)) > 0,$$

since $0 < 1 + \lambda'(z) < 1$ and $z + \lambda(z) > 0$.

For $z < 0$ it is more complicated. We need to prove, that for $z < 0$

$$\lambda(z) + 2z - z^2\lambda(z) - z\lambda^2(z) > 0.$$

Substituting $\lambda(z)$ by $\frac{\phi(z)}{\Phi(z)}$ we get:

$$\phi(z)\Phi(z) + 2z\Phi^2(z) - z^2\phi(z)\phi(z) - z\phi^2(z) > 0.$$

We apply the change of variables $x = -z$, so for $x > 0$ we want to prove:

$$\phi(x)\Phi(-x) - 2x\Phi^2(-x) - x^2\phi(x)\Phi(-x) + x\phi^2(x) > 0.$$

Let $F(x) = \Phi(-x)$. Then we need to prove for $x > 0$:

$$\phi(x)F(x) + 2xF^2(x) - x^2\phi(x)F(x) + x\phi^2(x) > 0.$$

Rearranging terms we get the inequality:

$$(1 - x^2)\phi(x)F(x) + x\phi(x)^2 - 2xF(x)^2 > 0 \text{ for } x > 0, \tag{A.7}$$

where $F(x) = 1 - \Phi(x)$. To prove it we split the whole interval $(0, \infty)$ into two: $(0, 1]$ and $(1, \infty)$.

$x \in (1, \infty)$. In this case $1 - x^2 < 0$, and to prove (A.7) it is sufficient to prove:

$$(1-x^2)\frac{4}{\sqrt{x^2+8}+3x} + x - 2x\frac{16}{(\sqrt{x^2+8}+3x)^2} > 0,$$

because $F(x) \leq \frac{4}{\sqrt{x^2+8}+3x}\phi(x)$ according to (Baricz, 2007).

By multiplying both sides by $(\sqrt{x^2+8}+3x)^2$ we get:

$$\begin{aligned}
&4(1-x^2)(\sqrt{x^2+8}+3x) + x(\sqrt{x^2+8}+3x)^2 - 32x \\
&= 4\sqrt{x^2+8} + 12x - 4x^2\sqrt{x^2+8} - 12x^3 - 32x \\
&\quad + x(x^2+8+9x^2+6x\sqrt{x^2+8}) \\
&= 4\sqrt{x^2+8} - 4x^2\sqrt{x^2+8} - 20x - 12x^3 + 10x^3 + 8x + 6x^2\sqrt{x^2+8} \\
&= 4\sqrt{x^2+8} + 2x^2\sqrt{x^2+8} - 12x - 2x^3.
\end{aligned}$$

So, we need to prove that:

$$\begin{aligned}
&4\sqrt{x^2+8} + 2x^2\sqrt{x^2+8} > 12x + 2x^3 \Leftrightarrow \\
&\sqrt{x^2+8}(2+x^2) > 6x + x^3.
\end{aligned}$$

As the left side and the right side of inequality are positive for $x > 0$ it is equivalent to the inequalities for the squares of both sides:

$$\begin{aligned}
&(x^2+8)(2+x^2)^2 > (6x+x^3)^2 \Leftrightarrow \\
&(x^2+8)(4+4x^2+x^4) > 36x^2 + 12x^4 + x^6 \Leftrightarrow \\
&4x^2 + 4x^4 + x^6 + 32 + 32x^2 + 8x^4 > 36x^2 + 12x^4 + x^6 \Leftrightarrow \\
&32 + 36x^2 + 12x^4 + x^6 > 36x^2 + 12x^4 + x^6 \Leftrightarrow \\
&32 > 0.
\end{aligned}$$

We proved the inequality for the case $x > 1$.

$x \in (0, 1]$. We use the following strategy: we split to smaller intervals, for each interval we provide a bound $\phi(x) > cF(x)$ defined by the left edge of the interval as $(\phi(x)/F(x))' > 0$ according to Lemma 1, and then get a quadratic inequality or a linear inequality, which is easy to check.

Let's start with $x \in (0.9, 1]$. $\phi(x) > 1.44F(x)$, then

$$\begin{aligned}
&(1-x^2)\phi(x)F(x) + x\phi(x)^2 - 2xF^2(x) > \\
&(1-x^2)1.44F^2(x) + 2.07xF^2(x) - 2xF^2(x) \geq \\
&2.07xF^2(x) - 2xF^2(x) > 0.07xF^2(x) > 0.
\end{aligned}$$

We proceed in a similar way for other intervals. If $x \in (0.83, 0.9]$, then $\phi(x) > 1.39F(x)$. Then

$$(1-x^2)\phi(x)F(x) + x\phi(x)^2 - 2xF(x)^2$$
$$> 1.39(1-x^2)F(x)^2 + 1.93xF(x)^2 - 2xF(x)^2 \geq 0.$$

If $x \in (0.65, 0.83]$, then $\phi(x) > 1.25F(x)$. Then

$$(1-x^2)\phi(x)F(x) + x\phi(x)^2 - 2xF(x)^2$$
$$> 1.25(1-x^2)F(x)^2 + 1.5625xF(x)^2 - 2xF(x)^2 \geq 0.$$

If $x \in (0.4, 0.65]$, then $\phi(x) > 1.05F(x)$. Then

$$(1-x^2)\phi(x)F(x) + x\phi(x)^2 - 2xF(x)^2$$
$$> 1.05(1-x^2)F(x)^2 + 1.1025xF(x)^2 - 2xF(x)^2 \geq 0.$$

If $x \in [0, 0.4]$, then $\phi(x) > 0.75F(x)$. Then

$$(1-x^2)\phi(x)F(x) + x\phi(x)^2 - 2xF(x)^2$$
$$> 0.75(1-x^2)F(x)^2 + 0.5625xF(x)^2 - 2xF(x)^2 \geq 0.$$

QED. □

Appendix A.3. Proof of the Statement 3

Proof. We consider a discrete random variable u. It takes values $u_i = z\,u_{i0}$, $i = 1,2$ with probabilities p_1, p_2 correspondingly, where $u_{i0} > 0$, $i = 1,2$. Since $v \sim \mathcal{N}(0, \sigma_v^2)$ and u are independent and $\varepsilon = v - u$, the joint distribution of u, ε has the form

$$f(u = u_i, \varepsilon) = p_i \frac{1}{\sqrt{2\pi}\sigma_v} \exp\left(-\frac{(u_i+\varepsilon)^2}{2\sigma_v^2}\right).$$

Thus, the marginal pdf of ε has the form:

$$f(\varepsilon) = \sum_{i=1}^{2} p_i \frac{1}{\sqrt{2\pi}\sigma_v} \exp\left(-\frac{(u_i+\varepsilon)^2}{2\sigma_v^2}\right). \tag{A.8}$$

The conditional distribution has the form:

$$P(u = u_i|\varepsilon) = \frac{p_i \frac{1}{\sqrt{2\pi}\sigma_v} \exp\left(-\frac{(u_i+\varepsilon)^2}{2\sigma_v^2}\right)}{\sum_{i=1}^{2} p_i \frac{1}{\sqrt{2\pi}\sigma_v} \exp\left(-\frac{(u_i+\varepsilon)^2}{2\sigma_v^2}\right)} = \frac{p_i e^{-w_i}}{p_1 e^{-w_1} + p_2 e^{-w_2}}, \quad i = 1, 2,$$

where $w_i = \frac{(u_i+\varepsilon)^2}{2\sigma_v^2} = \frac{(z\,u_{i0}+\varepsilon)^2}{2\sigma_v^2}$.

Then observation-specific technical efficiency is

$$TE = \mathrm{E}\left(e^{-u}|\varepsilon\right) = \sum_{i=1}^{2} e^{-u_i}\frac{p_i e^{-w_i}}{\sum_{j=1}^2 p_j e^{-w_j}} = \frac{\sum_{i=1}^2 p_i e^{-u_i} e^{-w_i}}{\sum_{j=1}^2 p_j e^{-w_j}}$$

$$= \frac{\sum_{i=1}^2 p_i e^{-z\,u_{i0}} e^{-w_i}}{\sum_{j=1}^2 p_j e^{-w_j}}. \tag{A.9}$$

Then the marginal effect $\frac{\partial TE}{\partial z}$ equals:

$$\frac{\partial TE}{\partial z} = \frac{1}{\sum_{j=1}^2 p_j e^{-w_j}}\frac{\partial}{\partial z}\sum\nolimits_{i=1}^2 p_i e^{-z\,u_{i0}} e^{-w_i}$$

$$- \frac{1}{\left(\sum_{j=1}^2 p_j e^{-w_j}\right)^2}\left(\sum\nolimits_{i=1}^2 p_i e^{-z\,u_{i0}} e^{-w_i}\right)\frac{\partial}{\partial z}\sum\nolimits_{j=1}^2 p_j e^{-w_j}$$

$$= -\frac{1}{\sum_{j=1}^2 p_j e^{-w_j}}\sum\nolimits_{i=1}^2 p_i e^{-z\,u_{i0}} e^{-w_i}(u_{i0} + w_i')$$

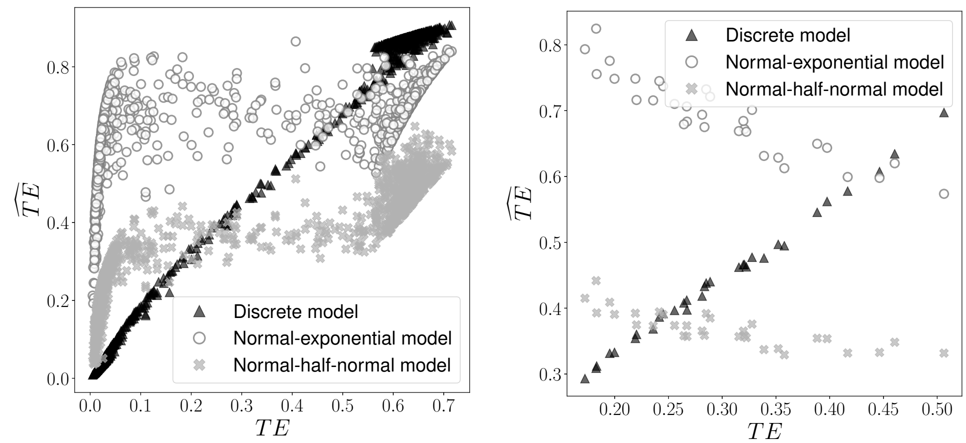
$$+ \frac{1}{\left(\sum_{j=1}^2 p_j e^{-w_j}\right)^2}\left(\sum\nolimits_{i=1}^2 p_i e^{-z\,u_{i0}} e^{-w_i}\right)\left(\sum\nolimits_{j=1}^2 p_j e^{-w_j} w_j'\right),$$

where $w_i' = \frac{\partial}{\partial z} w_i = \frac{z\,u_{i0}^2 + \varepsilon\,u_{i0}}{\sigma_v^2}$. □

Appendix B. Identifiability of the normal-discrete model

We examined the discrete model in a number of ways. The most important issue is to check identifiability of the model.

We use the dataset of size 1000, generated with the normal-discrete model, which we used for Fig.6 in Section 3. We use the maximum likelihood approach with p.d.f. from (A.8) to estimate the normal-discrete model. $\widehat{TE}_i$ for this model were calculated from (A.9). Also, for this data, we estimated two misspecified models: normal-half-normal and normal-exponential and derived predicted technical efficiencies $\widehat{TE}_i$ for these models. Figures B.7 contain comparisons of the true values of TE and their three estimates $\widehat{TE}_i$ using three different models. We see that, if the model is correctly specified, the obtained estimates are close to the real ones. While, if we start to use common, but misspecified normal-half-normal and normal-exponential models, the estimates are worse.



(a) Comparison of all estimates of $\widehat{TE}$ and true TE

(b) Selected points with ε between -2.3 and -2.1

Figure B.7: Comparison of estimates $\widehat{TE}$ using a normal-discrete, a normal-half-normal and a normal-exponential models and true TE obtained using a normal-discrete model

Model	Correlation	Correlation $-2.3 < \varepsilon < -2.1$	Correlation $-2.5 < \varepsilon < -1.9$
% of observations	100%	3.2%	7.5%
Normal-discrete	0.9816	0.9971	0.9982
Normal-half-normal	0.9451	-0.8768	-0.5739
Normal-exponential	0.7616	-0.9285	-0.8104

Table B.1: Spearman rank correlations for true values and the three estimates of TE if the true model is normal-discrete

Spearman rank correlation between the true TE and the three predicted $\widehat{TE}$ are provided in Table B.1. The highest rank correlation is obtained when the true model is estimated. The correlation is smaller for the normal-half-normal model and is even worse for the normal-exponential model. But for the subset of observations selected by the condition $-2.3 < \varepsilon < -2.1$ both misspecified models provide strongly negative rank correlations between predicted $\widehat{TE}$ and true values of the technical efficiency TE. Is this subset very unusual? For comparison in Table B.1 rank correlations in the last column are presented for a wider range of ε, which includes 7.5% of units in the sample. Correlations are negative and statistically significant. Thus for this part of the sample using the traditional normal-half-normal or normal-exponential model one can come to the wrong conclusion on ranking the units by technical efficiency. The reason for this is bimodal distribution of the inefficiency term u around the sample. One can meet such situation, for example, if there are two types of firm managers.

How is it possible to foresee this kind of the problem? One possible solution is to examine the plot of residuals i.e. for the normal-exponential model. For this data the plot is presented at the Figure B.8. One can see a bimodal distribution of residuals, which is a signal to the problem.

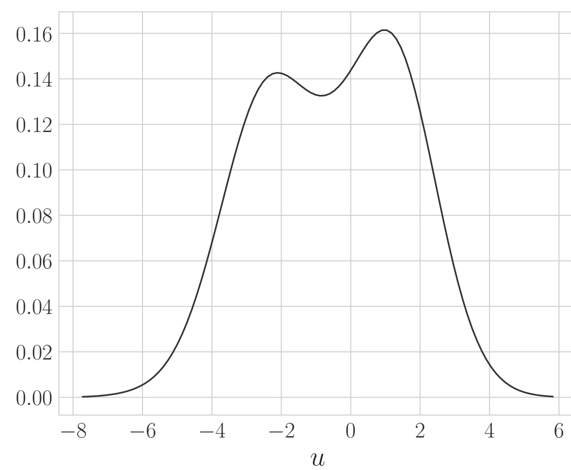


Figure B.8: kernel density estimate of the residuals of the normal-exponential model

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