Two-Stage Instrumental Variable Estimation of Linear Panel Data Models with Interactive Effects

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9 September 2020

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Abstract

This paper puts forward a new instrumental variables (IV) approach for linear panel data models with interactive effects in the error term and regressors. The instruments are transformed regressors and so it is not necessary to search for external instruments. The proposed method asymptotically eliminates the interactive effects in the error term and in the regressors *separately* in two stages. We propose a two-stage IV (2SIV) and a mean-group IV (MGIV) estimator for homogeneous and heterogeneous slope models, respectively. The asymptotic analysis for the models with homogeneous slopes reveals that: (i) the $\sqrt{NT}$-consistent 2SIV estimator is free from asymptotic bias that could arise due to the correlation between the regressors and the estimation error of the interactive effects; (ii) under the same set of assumptions, existing popular estimators, which eliminate interactive effects either *jointly* in the regressors and the error term, or *only* in the error term, can suffer from asymptotic bias; (iii) the proposed 2SIV estimator is asymptotically as efficient as the bias-corrected version of estimators that eliminate interactive effects jointly in the regressors and the error, whilst; (iv) the relative efficiency of the estimators that eliminate interactive effects only in the error term is indeterminate. A Monte Carlo study confirms good approximation quality of our asymptotic results and competent performance of 2SIV and MGIV in comparison with existing estimators. Furthermore, it demonstrates that the bias-corrections can be imprecise and noticeably inflate the dispersion of the estimators in finite samples.

JEL classification: C13, C15, C23, C26.

Key Words: Large panel data, interactive effects, common factors, principal components analysis, instrumental variables.

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1 Introduction

Panel data sets with large cross-section and time-series dimensions (N and T, respectively) have become increasingly available in the social sciences. As a result, regression analysis of large panels has gained an ever-growing popularity. A central issue in these models is how to properly control for rich sources of unobserved heterogeneity, including common shocks and interactive effects (see e.g. Sarafidis and Wansbeek (2020) for a recent overview). The present paper puts forward a novel estimation approach for this class of models and offers new insights into the literature.

Broadly speaking, there are two popular estimation approaches currently advanced in the field. The first one involves eliminating the interactive effects from the error term and the regressors *jointly* in a single stage. Representative methods include the Common Correlated Effects approach of Pesaran (2006), which involves least-squares on a regression model augmented by cross-sectional averages (CA) of observables; and the Principal Components (PC) estimator considered first by Kapetanios and Pesaran (2005) and analysed subsequently by Westerlund and Urbain (2015). The second approach asymptotically eliminates the interactive effects from the error term only. The representative method is the Iterative Principal Components (IPC) estimator of Bai (2009a), further developed by Moon and Weidner (2015, 2017), among many others. An attractive feature of CA (as well as PC) is that it permits estimation of models with heterogeneous slopes. On the other hand, an advantage of IPC is that it does not assume regressors are subject to a factor structure.

In models with homogeneous slopes, Westerlund and Urbain (2015) showed that both CA and PC estimators suffer from asymptotic bias due to the incidental parameter problem (recently, Juodis et al. (2020) provided additional results on the asymptotic properties of CA and some further insights). A similar outcome was shown by Bai (2009a) for the IPC estimator. Thus in all three cases, bias correction is necessary for asymptotically valid inferences. In addition, the CA estimator requires the so-called rank condition, which assumes that the number of factors does not exceed the rank of the (unknown) matrix of cross-sectional averages of the factor loadings. On the other hand, IPC involves non-linear optimisation, and so convergence to the global optimum might not be guaranteed (see e.g. Jiang et al. (2017)).

This paper puts forward an instrumental variables (IV) approach, which differs from the aforementioned ones because it asymptotically eliminates the interactive effects in the error term and the regressors *separately* in two stages. In particular, for models with homogeneous slopes, in the first stage we project out the interactive effects from the regressors and use the transformed regressors as instruments to obtain consistent estimates of the model parameters. Thus, it is not necessary to search for external instruments. In the second stage, we eliminate the interactive effects in the error term using the first-stage residuals, and run another IV regression. That is, IV regression is performed in both of two stages. The resulting two-stage IV (2SIV) estimator is shown to be $\sqrt{NT}$-consistent and asymptotically normal. For models with heterogeneous slopes, we propose a mean-group IV (MGIV) estimator and establish $\sqrt{N}$-consistency and asymptotic normality. Norkutė et al. (2020) adopted a similar approach for estimation of dynamic panels with interactive effects, assuming cross-sectional and serial independence of the idiosyncratic disturbances. The asymptotic results established in this paper is completely new as weak cross-section and time-series dependence in idiosyncratic errors are permitted. The weak dependence assumption is typically employed by the IPC literature such as Bai (2009a).

In this paper, we offer new insights into the literature by comparing the asymptotic properties of 2SIV with those of IPC, PC and CA. Such a task was not considered by Norkutė et al. (2020). To be more specific, we analytically clarify why our two-stage approach successfully makes the 2SIV estimator free from asymptotic bias, whilst IPC, PC and CA are subject to biases under the same conditions. In brief, this is because estimating factors separately in two stages for 2SIV makes the endogeneity caused by the estimation errors asymptotically negligible, whereas this is not the case for the remaining estimators. Moreover, our analysis reveals that 2SIV is asymptotically as efficient as the bias-corrected versions of PC and CA, whereas the relative efficiency of the bias-corrected IPC estimator is indeterminate, in general. This is because the IPC estimator does not necessarily eliminate the factors contained in the regressors and also it requires transformation of the regressors due to the estimation effects. A Monte Carlo study confirms good approximation quality of our asymptotic results and competent performance of 2SIV and MGIV in comparison with existing estimators. Furthermore, the results demonstrate that the bias-corrections of IPC and PC can be imprecise and noticeably inflate the dispersion of the estimators in finite samples.

The remainder of this paper is organised as follows. Section 2 introduces a panel data model with homogeneous slopes and interactive effects, and describes the set of assumptions employed. Section 3 studies the asymptotic properties of the proposed 2SIV estimator. Section 4 puts forward a mean-group IV estimator for models with heterogeneous slopes and establishes its properties in large samples. Section 5 provides an asymptotic comparison among 2SIV, IPC, CA and PC. Section 6 studies the finite sample performance of these estimators, and Section 7 concludes. Proofs of main theoretical results with necessary lemmas are provided in Appendices A-D. Proofs of auxiliary lemmas are relegated to Online Supplement. A Stata algorithm that implements our approach, has been recently developed by Kripfganz and Sarafidis (2020) and is available to all Stata users.[1]

Notation: Throughout, we denote the largest eigenvalues of the $N \times N$ matrix $\mathbf{A} = (a_{ij})$ by $\mu_{\max}(\mathbf{A})$, its trace by $\text{tr}(\mathbf{A}) = \sum_{i=1}^{N} a_{ii}$, its Frobenius norm by $\|\mathbf{A}\| = \sqrt{\text{tr}(\mathbf{A}'\mathbf{A})}$. The projection matrix on $\mathbf{A}'$ is $\mathbf{P}_{\mathbf{A}} = \mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'$ and $\mathbf{M}_{\mathbf{A}} = \mathbf{I} - \mathbf{P}_{\mathbf{A}}$. C is a generic positive constant large enough, C_{min} is a small positive constant sufficiently away from zero, $\delta_{NT}^2 = \min\{N, T\}$. We use $N, T \to \infty$ to denote that N and T pass to infinity jointly.

2 Model and assumptions

We consider the following panel data model:

$$\begin{aligned} y_{it} &= \mathbf{x}_{it}'\boldsymbol{\beta} + \mathbf{u}_i; \quad \mathbf{u}_i = \boldsymbol{\varphi}_i^{0\prime}\mathbf{h}_t^0 + \varepsilon_{it}, \\ \mathbf{x}_{it} &= \boldsymbol{\Gamma}_i^{0\prime}\mathbf{f}_t^0 + \mathbf{v}_{it}; \quad i = 1, ..., N; \quad t = 1, ..., T, \end{aligned} \tag{2.1}$$

where y_{it} denotes the value of the dependent variable for individual i at time t, $\mathbf{x}_{it}$ is a $k \times 1$ vector of regressors and $\boldsymbol{\beta}$ is the corresponding vector of slope coefficients. $\mathbf{u}_i$ follows a factor structure, where $\mathbf{h}_t^0$ denotes an $r_2 \times 1$ vector of latent factors, $\boldsymbol{\varphi}_i^0$ is the associated factor loading vector, and ε_{it} denotes an idiosyncratic error. The regressors are also subject to a factor model, where $\mathbf{f}_t^0$ denotes an $r_1 \times 1$ vector of latent factors, $\boldsymbol{\Gamma}_i^0$ is a $r_1 \times k$ matrix of factor loadings, and $\mathbf{v}_{it}$ is an idiosyncratic error of dimension $k \times 1$.

Remark 2.1 *Permitting different sets of interactive effects in $\mathbf{x}_{it}$ and u_{it} is essential in order to study in detail the properties of the estimators that eliminate the factors in the error term and*

[1] See http://www.kripfganz.de/stata/xtivdfreg.html.

the regressors separately (like our approach), or in the error term only (like the Iterative Principal Components (IPC) estimator of Bai (2009a)). This is because in practice there is no reason why $\mathbf{x}_{it}$ *and* u_{it} *should contain identical sets of factors or loadings. This remark does not apply to the estimators which extract factors in* $\mathbf{x}_{it}$ *and* u_{it} *jointly (like the estimators considered by Pesaran (2006) and Westerlund and Urbain (2015)).*

The model above has been employed in a wide variety of fields, including in economics and finance. Estimation of this model has been studied by Pesaran (2006), Bai and Li (2014), Westerlund and Urbain (2015), Juodis and Sarafidis (2020), Cui et al. (2020) to mention a few.

Stacking Eq. (2.1) over t, we have

$$\begin{aligned} \mathbf{y}_i &= \mathbf{X}_i \boldsymbol{\beta} + \mathbf{u}_i; \quad \mathbf{u}_i = \mathbf{H}^0 \boldsymbol{\varphi}_i^0 + \boldsymbol{\varepsilon}_i, \\ \mathbf{X}_i &= \mathbf{F}^0 \boldsymbol{\Gamma}_i^0 + \mathbf{V}_i, \end{aligned} \tag{2.2}$$

where $\mathbf{y}_i = (y_{i1}, \ldots, y_{iT})'$, $\mathbf{X}_i = (\mathbf{x}_{i1}, \cdots, \mathbf{x}_{iT})'$, $\mathbf{F}^0 = (\mathbf{f}_1^0, \cdots, \mathbf{f}_T^0)'$, $\mathbf{H}^0 = (\mathbf{h}_1^0, \cdots, \mathbf{h}_T^0)'$, $\boldsymbol{\varepsilon}_i = (\varepsilon_{i1}, \cdots, \varepsilon_{iT})'$ and $\mathbf{V}_i = (\mathbf{v}_{i1}, \ldots, \mathbf{v}_{iT})'$.

We propose an IV estimation approach that involves two stages. In the first stage, the common factors in $\mathbf{X}_i$ are asymptotically eliminated using principal components analysis. Next, the transformed regressors are used to construct instruments and estimate the model parameters. To illustrate the first-stage IV estimator, suppose for the moment that $\mathbf{F}^0$ is observed. Pre-multiplying $\mathbf{X}_i$ by $\mathbf{M}_{\mathbf{F}^0}$ yields

$$\mathbf{M}_{\mathbf{F}^0} \mathbf{X}_i = \mathbf{M}_{\mathbf{F}^0} \mathbf{V}_i. \tag{2.3}$$

Assuming $\mathbf{V}_i$ is independent of $\boldsymbol{\varepsilon}_i, \mathbf{H}^0$ and $\boldsymbol{\varphi}_i^0$, it is easily seen that $E[\mathbf{X}_i' \mathbf{M}_{\mathbf{F}^0} \mathbf{u}_i] = E[\mathbf{V}_i' \mathbf{M}_{\mathbf{F}^0} (\mathbf{H}^0 \boldsymbol{\varphi}_i^0 + \boldsymbol{\varepsilon}_i)] = \mathbf{0}$. Together with the fact that $\mathbf{M}_{\mathbf{F}^0} \mathbf{X}_i$ is correlated with $\mathbf{X}_i$ through $\mathbf{V}_i$, $\mathbf{M}_{\mathbf{F}^0} \mathbf{X}_i$ can be regarded as an instrument for $\mathbf{X}_i$.

The first-stage (infeasible) estimator is defined as

$$\widehat{\boldsymbol{\beta}}_{1SIV}^{inf} = \left(\sum_{i=1}^{N} \mathbf{X}_i' \mathbf{M}_{\mathbf{F}^0} \mathbf{X}_i \right)^{-1} \sum_{i=1}^{N} \mathbf{X}_i' \mathbf{M}_{\mathbf{F}^0} \mathbf{y}_i. \tag{2.4}$$

In the second stage, the space spanned by $\mathbf{H}^0$ is estimated from the residual $\widehat{\mathbf{u}}_i^{inf} = \mathbf{y}_i - \mathbf{X}_i \widehat{\boldsymbol{\beta}}^{inf}$ and then it is projected out. To illustrate, suppose for the moment that $\mathbf{H}^0$ is also observed; one can instrument $\mathbf{X}_i$ using $\mathbf{M}_{\mathbf{H}^0} \mathbf{M}_{\mathbf{F}^0} \mathbf{X}_i$. Note that $E[\mathbf{X}_i' \mathbf{M}_{\mathbf{F}^0} \mathbf{M}_{\mathbf{H}^0} \mathbf{u}_i] = E[\mathbf{V}_i' \mathbf{M}_{\mathbf{F}^0} \mathbf{M}_{\mathbf{H}^0} \boldsymbol{\varepsilon}_i] = \mathbf{0}$. The (infeasible) second-stage IV (2SIV) estimator of $\boldsymbol{\beta}$ is given by

$$\widehat{\boldsymbol{\beta}}_{2SIV}^{inf} = \left(\sum_{i=1}^{N} \mathbf{X}_i' \mathbf{M}_{\mathbf{F}^0} \mathbf{M}_{\mathbf{H}^0} \mathbf{X}_i \right)^{-1} \sum_{i=1}^{N} \mathbf{X}_i' \mathbf{M}_{\mathbf{F}^0} \mathbf{M}_{\mathbf{H}^0} \mathbf{y}_i. \tag{2.5}$$

In practice, $\mathbf{F}^0$ and $\mathbf{H}^0$ are typically unobserved. As it will be discussed in detail below, in practice we replace these quantities with estimates obtained using principal components analysis, as advanced in Bai (2003) and Bai (2009a).[2]

To obtain our theoretical results it is sufficient to make the following assumptions:

[2] r_1 and r_2 are treated as given. In practice, r_1 can be estimated from the raw data $\{\mathbf{X}_i\}_{i=1}^N$ using methods already available in the literature, such as the information criteria of Bai and Ng (2002) or the eigenvalue-based tests of Kapetanios (2010) and Ahn and Horenstein (2013). r_2 can be estimated in the same way from the residual covariance matrix. An asymptotic justification of such practice is provided in Bai (2009b, Section C.3). In the Monte Carlo section of the paper we show that these methods provide quite accurate determination of the number of factors.

Assumption A (idiosyncratic error in y) *We assume that*

1. $\mathbb{E}(\varepsilon_{it}) = 0$ *and* $\mathbb{E}|\varepsilon_{it}|^{8+\delta} \leq C$ *for some* $\delta > 0$*;*

2. *Let* $\sigma_{ij,st} \equiv \mathbb{E}(\varepsilon_{is}\varepsilon_{jt})$*. We assume that there exist* $\bar{\sigma}_{ij}$ *and* $\tilde{\sigma}_{st}$*,* $|\sigma_{ij,st}| \leq \bar{\sigma}_{ij}$ *for all* (s,t)*, and* $|\sigma_{ij,st}| \leq \tilde{\sigma}_{st}$ *for all* (i,j)*, such that*
$$N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\bar{\sigma}_{ij} \leq C; \quad T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\sigma}_{st} \leq C; \quad N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{s=1}^{T}\sum_{t=1}^{T}|\sigma_{ij,st}| \leq C\,.$$

3. *For every* (s,t)*,* $\mathbb{E}\left\|N^{-1/2}\sum_{i=1}^{N}[\varepsilon_{is}\varepsilon_{it} - \sigma_{ii,st}]\right\|^4 \leq C$*.*

4. *For each* j*,* $\mathbb{E}\left\|N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\left[\varepsilon_{it}\varepsilon_{jt} - \mathbb{E}(\varepsilon_{it}\varepsilon_{jt})\right]\boldsymbol{\varphi}_i^0\right\|^2 \leq C$*. Additionally, for each* s*,* $\mathbb{E}\left\|N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}[\varepsilon_{is}\varepsilon_{it} - \mathbb{E}(\varepsilon_{is}\varepsilon_{it})]\mathbf{g}_t^0\right\|^2 \leq C$*, where* $\mathbf{g}_t^0 = (\mathbf{f}_t^{0\prime}, \mathbf{h}_t^{0\prime})'$*.*

5. $N^{-1}T^{-2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{s_1=1}^{T}\sum_{s_2=1}^{T}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\left|\text{cov}(\varepsilon_{is_1}\varepsilon_{is_2}, \varepsilon_{jt_1}\varepsilon_{jt_2})\right| \leq C\,.$

Assumption B (idiosyncratic error in x) *Let* $\boldsymbol{\Sigma}_{ij,st} \equiv \mathbb{E}\left(\mathbf{v}_{is}\mathbf{v}_{jt}'\right)$*. We assume that*

1. $\mathbf{v}_{it}$ *is group-wise independent from* ε_{it}*,* $\mathbb{E}(\mathbf{v}_{it}) = 0$ *and* $\mathbb{E}\|\mathbf{v}_{it}\|^{8+\delta} \leq C$*;*

2. *There exist* $\bar{\tau}_{ij}$ *and* $\tilde{\tau}_{st}$*,* $\|\boldsymbol{\Sigma}_{ij,st}\| \leq \bar{\tau}_{ij}$ *for all* (s,t)*, and* $\|\boldsymbol{\Sigma}_{ij,st}\| \leq \tilde{\tau}_{st}$ *for all* (i,j)*, such that*
$$N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\bar{\tau}_{ij} \leq C; \quad T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\tau}_{st} \leq C; \quad N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{s=1}^{T}\sum_{t=1}^{T}\|\boldsymbol{\Sigma}_{ij,st}\| \leq C\,.$$
Additionally, the largest eigenvalue of $\mathbb{E}(\mathbf{V}_i\mathbf{V}_i')$ *is bounded uniformly in* i*.*

3. *For every* (s,t)*,* $\mathbb{E}\left\|N^{-1/2}\sum_{i=1}^{N}[\mathbf{v}_{is}\mathbf{v}_{it}' - \boldsymbol{\Sigma}_{ii,st}]\right\|^4 \leq C$*.*

4. *For each* j*,* $\mathbb{E}\left\|N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\boldsymbol{\varphi}_i^0 \otimes \left[\mathbf{v}_{it}\mathbf{v}_{jt}' - \mathbb{E}\left(\mathbf{v}_{it}\mathbf{v}_{jt}'\right)\right]\right\|^2 \leq C$*. Additionally, for each* s*,* $\mathbb{E}\left\|N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}[\mathbf{v}_{is}'\mathbf{v}_{it} - \mathbb{E}(\mathbf{v}_{is}'\mathbf{v}_{it})]\mathbf{g}_t^0\right\|^2 \leq C$*.*

5. $N^{-1}T^{-2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{s_1=1}^{T}\sum_{s_2=1}^{T}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\left|\text{cov}(\mathbf{v}_{is_1}'\mathbf{v}_{is_2}, \mathbf{v}_{jt_1}'\mathbf{v}_{jt_2})\right| \leq C\,.$

Assumption C (factors) $\mathbb{E}\|\mathbf{f}_t^0\|^4 \leq C$*,* $T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0 \xrightarrow{p} \boldsymbol{\Sigma}_F^0$ *as* $T \to \infty$ *for some non-random positive definite matrix* $\boldsymbol{\Sigma}_F^0$*.* $\mathbb{E}\|\mathbf{h}_t^0\|^4 \leq C$*,* $T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0 \xrightarrow{p} \boldsymbol{\Sigma}_H^0$ *as* $T \to \infty$ *for some non-random positive definite matrix* $\boldsymbol{\Sigma}_H^0$*.*

Assumption D (loadings) $\mathbb{E}\left\|\boldsymbol{\Gamma}_i^0\right\|^4 \leq C$*,* $\boldsymbol{\Upsilon}^0 = N^{-1}\sum_{i=1}^{N}\boldsymbol{\Gamma}_i^0\boldsymbol{\Gamma}_i^{0\prime} \xrightarrow{p} \bar{\boldsymbol{\Upsilon}}^0$ *as* $N \to \infty$*, and* $\mathbb{E}\left\|\boldsymbol{\varphi}_i^0\right\|^4 \leq C$*,* $\boldsymbol{\Upsilon}_\varphi^0 = N^{-1}\sum_{i=1}^{N}\boldsymbol{\varphi}_i^0\boldsymbol{\varphi}_i^{0\prime} > 0 \xrightarrow{p} \bar{\boldsymbol{\Upsilon}}_\varphi^0$ *as* $N \to \infty$ *for some non-random positive definite matrices* $\bar{\boldsymbol{\Upsilon}}^0$ *and* $\bar{\boldsymbol{\Upsilon}}_\varphi^0$*. In addition,* $\boldsymbol{\Gamma}_i^0$ *and* $\boldsymbol{\varphi}_i^0$ *are independent groups from* ε_{it}*,* $\mathbf{v}_{it}$*,* $\mathbf{f}_t^0$ *and* $\mathbf{h}_t^0$*.*

Assumption E (identification) *The matrix* $T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i$ *has full column rank and* $\mathbb{E}\left\|T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i\right\|^{2+2\delta} \leq C$ *for all* i*.*

Assumptions A and B permit weak cross-sectional and serial dependence in ε_{it} and $\mathbf{v}_{it}$, in a similar manner to Bai (2009a). Assumptions C and D on the moments and the limit variance of factors and factor loadings are standard and in line with Bai (2009a). Note that these assumptions permit that $T^{-1}\mathbf{G}^{0\prime}\mathbf{G}^0 \xrightarrow{p} \boldsymbol{\Sigma}_G^0$, a positive semi-definite matrix, where $\mathbf{G}^0 = (\mathbf{F}^0, \mathbf{H}^0)$. Assumption E is sufficient for identification of heterogeneous slope coefficients.

3 Estimation of models with homogeneous slopes

We propose the following two-stage IV procedure:

1. Estimate the span of $\mathbf{F}^0$ by $\widehat{\mathbf{F}}$, defined as $\sqrt{T}$ times the eigenvectors corresponding to the r_1 largest eigenvalues of the $T \times T$ matrix $N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{X}_i\mathbf{X}_i'$. Then estimate $\boldsymbol{\beta}$ as

$$\widehat{\boldsymbol{\beta}}_{1SIV} = \left(\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i\right)^{-1}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{y}_i. \tag{3.1}$$

2. Let $\widehat{\mathbf{u}}_i = \mathbf{y}_i - \mathbf{X}_i\widehat{\boldsymbol{\beta}}_{1SIV}$. Define $\widehat{\mathbf{H}}$ to be $\sqrt{T}$ times the eigenvectors corresponding to the r largest eigenvalues of the $T \times T$ matrix $(NT)^{-1}\sum_{i=1}^{N}\widehat{\mathbf{u}}_i\widehat{\mathbf{u}}_i'$. The second-stage estimator of $\boldsymbol{\beta}$ is defined as follows:[3]

$$\widehat{\boldsymbol{\beta}}_{2SIV} = \left(\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{X}_i\right)^{-1}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{y}_i. \tag{3.2}$$

In order to establish the asymptotic properties of these estimators, we first expand (3.1) as follows:

$$\sqrt{NT}(\widehat{\boldsymbol{\beta}}_{1SIV} - \boldsymbol{\beta}) = \left(\frac{1}{NT}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i\right)^{-1}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i. \tag{3.3}$$

The following Proposition demonstrates $\sqrt{NT}$-consistency of the first-stage estimator, $\widehat{\boldsymbol{\beta}}_{1SIV}$:

Proposition 3.1 *Under Assumptions A-E, we have*

$$N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i = N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i + \mathbf{b}_{0F} + \mathbf{b}_{1F} + \mathbf{b}_{2F} + O_p(\sqrt{NT}\delta_{NT}^{-3})$$

with

$$\begin{aligned}
\mathbf{b}_{0F} =& -N^{-1/2}T^{-1/2}\sum_{i=1}^{N}N^{-1}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i;\\
\mathbf{b}_{1F} =& -\sqrt{\frac{T}{N}}\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h)\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0\\
&+\sqrt{\frac{T}{N}}\frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{h=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbb{E}(\mathbf{V}_\ell'\mathbf{V}_h)\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0;\\
\mathbf{b}_{2F} =& -\sqrt{\frac{N}{T}}\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{F}^0}\mathbf{H}^0\boldsymbol{\varphi}_i^0,
\end{aligned}$$

where $\boldsymbol{\Upsilon}^0 = \sum_{i=1}^{N}\boldsymbol{\Gamma}_i^0\boldsymbol{\Gamma}_i^{0\prime}/N$, $\boldsymbol{\Sigma} = N^{-1}\sum_{i=1}^{N}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)$, *and* $N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i$, $\mathbf{b}_{0F}$, $\mathbf{b}_{1F}$ *and* $\mathbf{b}_{2F}$ *are* $O_p(1)$ *when* $N/T \longrightarrow C$. *Consequently,*

$$\sqrt{NT}(\widehat{\boldsymbol{\beta}}_{1SIV} - \boldsymbol{\beta}) = O_p(1).$$

[3] Letting $\widehat{\boldsymbol{\varphi}}_i = (\widehat{\mathbf{H}}'\widehat{\mathbf{H}})^{-1}\widehat{\mathbf{H}}'\widehat{\mathbf{u}}_i$, an alternative second-stage estimator can be defined by $\left(\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i\right)^{-1}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}(\mathbf{y}_i - \mathbf{P}_{\widehat{\mathbf{H}}}\widehat{\mathbf{u}}_i)$. We do not discuss this estimator since the finite sample performance was slightly worse than that of $\widehat{\boldsymbol{\beta}}_{2SIV}$.

Proposition 3.1 implies that $\widehat{\boldsymbol{\beta}}_{1SIV}$ is consistent but asymptotically biased. Rather than bias-correcting this estimator, we show that the second-stage IV estimator is free from asymptotic bias. To begin with, we make use of the following expansion:

$$\sqrt{NT}(\widehat{\boldsymbol{\beta}}_{2SIV}-\boldsymbol{\beta})=\left(\frac{1}{NT}\sum_{i=1}^{N}\mathbf{X}_i^{'}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{X}_i\right)^{-1}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_i^{'}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i. \tag{3.4}$$

The next proposition provides an asymptotic representation of $\widehat{\boldsymbol{\beta}}_{2SIV}$.

Proposition 3.2 *Under Assumptions A-E, as $N,T\to\infty$, $N/T\to C$, we have*

$$\begin{aligned}\sqrt{NT}(\widehat{\boldsymbol{\beta}}_{2SIV}-\boldsymbol{\beta})&=\left(\frac{1}{NT}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{H}^0}\mathbf{X}_i\right)^{-1}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i+O_p(\sqrt{NT}\delta_{NT}^{-3})\\&=\left(\frac{1}{NT}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{V}_i\right)^{-1}\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'\boldsymbol{\varepsilon}_i+O_p(\sqrt{NT}\delta_{NT}^{-3}).\end{aligned}$$

Proposition 3.2 shows that the effects of estimating $\mathbf{F}^0$ from $\mathbf{X}_i$ and $\mathbf{H}^0$ from $\widehat{\mathbf{u}}_i=\mathbf{y}_i-\mathbf{X}_i\widehat{\boldsymbol{\beta}}_{1SIV}$ are asymptotically negligible. Moreover, $\widehat{\boldsymbol{\beta}}_{2SIV}$ is asymptotically equivalent to a least-squares estimator obtained by regressing $(\mathbf{y}_i-\mathbf{H}^0\boldsymbol{\varphi}_i^0)$ on $(\mathbf{X}_i-\mathbf{F}^0\boldsymbol{\Gamma}_i^0)$.

To establish asymptotic normality under weak cross-sectional and serial error dependence, we place the following additional assumption, which is in line with Assumption E in Bai (2009a).

Assumption F plim$N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{V}_i'\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_j'\mathbf{V}_j/T=\mathbf{B}$, *and* $\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'\boldsymbol{\varepsilon}_i\xrightarrow{d}N(\mathbf{0},\mathbf{B})$, *for some non-random positive definite matrix* $\mathbf{B}$.

Using Proposition 3.2 and Assumption F, it is straightforward to establish the asymptotic distribution of $\widehat{\boldsymbol{\beta}}_{2SIV}$:

Theorem 3.1 *Under Assumptions A-F, as $N,T\to\infty$, $N/T\to C$, we have*

$$\sqrt{NT}(\widehat{\boldsymbol{\beta}}_{2SIV}-\boldsymbol{\beta})\xrightarrow{d}N(\mathbf{0},\boldsymbol{\Psi})$$

where $\boldsymbol{\Psi}=\mathbf{A}^{-1}\mathbf{B}\mathbf{A}^{-1}$.

Note that despite the fact that our assumptions permit serial correlation and heteroskedasticity in $\mathbf{v}_{it}$ and ε_{it}, $\widehat{\boldsymbol{\beta}}_{2SIV}$ is not subject to any asymptotic bias. We discuss this property in more detail in Section 5.

As discussed in Bai (2009a) and Norkutė et al. (2020), in general consistent estimation of $\boldsymbol{\Psi}$ is not feasible when the idiosyncratic errors are both cross-section and time-series dependent. Following Norkutė et al. (2020) and Cui et al. (2020), we propose using the following estimator:

$$\widehat{\boldsymbol{\Psi}}=\widehat{\mathbf{A}}^{-1}\widehat{\mathbf{B}}\widehat{\mathbf{A}}^{-1} \tag{3.5}$$

with

$$\widehat{\mathbf{A}}=\frac{1}{NT}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{X}_i;\quad \widehat{\mathbf{B}}=\frac{1}{NT}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\dot{\mathbf{u}}_i\dot{\mathbf{u}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i,$$

where $\dot{\mathbf{u}}_i=\mathbf{y}_i-\mathbf{X}_i\widehat{\boldsymbol{\beta}}_{2SIV}$. In line with the discussion in Hansen (2007), it can be shown that when $\{\mathbf{v}_{it}',\varepsilon_{it}\}$ follows a certain strong mixing process over t and is independent over i, $\widehat{\boldsymbol{\Psi}}-\boldsymbol{\Psi}\xrightarrow{p}\mathbf{0}$ as $N,T\to\infty$, $N/T\to C$.

4 Models with heterogeneous slopes

We now turn our focus on models with heterogeneous coefficients:

$$\begin{aligned} \mathbf{y}_i &= \mathbf{X}_i \boldsymbol{\beta}_i + \mathbf{H}^0 \boldsymbol{\varphi}_i^0 + \boldsymbol{\varepsilon}_i \,, \\ \mathbf{X}_i &= \mathbf{F}^0 \boldsymbol{\Gamma}_i^0 + \mathbf{V}_i . \end{aligned} \tag{4.1}$$

We first consider the following individual-specific estimator

$$\widehat{\boldsymbol{\beta}}_i = (\mathbf{X}_i' \mathbf{M}_{\widehat{\mathbf{F}}} \mathbf{X}_i)^{-1} \mathbf{X}_i' \mathbf{M}_{\widehat{\mathbf{F}}} \mathbf{y}_i .$$

Proposition 4.1 *Under Assumptions A-E, for each i we have*

$$\sqrt{T}(\widehat{\boldsymbol{\beta}}_i - \boldsymbol{\beta}_i) = \left(T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i\right)^{-1} \times T^{-1/2}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i + O_p\left(\delta_{NT}^{-1}\right) + O_p\left(T^{1/2}\delta_{NT}^{-2}\right)$$

and

$$T^{-1/2}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i \xrightarrow{d} \mathrm{N}(\mathbf{0}, \boldsymbol{\Omega}_i)$$

where $\boldsymbol{\Omega}_i = T^{-1}\mathrm{plim}_{T\to\infty} \sum_{s=1}^T \sum_{t=1}^T \tilde{u}_{is}\tilde{u}_{it}\mathbb{E}(\mathbf{v}_{is}\mathbf{v}_{it}')$ *and* $\tilde{\mathbf{u}}_i = \mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i \equiv (\tilde{u}_{i1}, \cdots, \tilde{u}_{iT})'$.

We also consider inference on the mean of $\boldsymbol{\beta}_i$. We make the following assumptions.

Assumption G (random coefficients) $\boldsymbol{\beta}_i = \boldsymbol{\beta} + \mathbf{e}_i$*, where* $\mathbf{e}_i$ *is independently and identically distributed over i with mean zero and variance* $\boldsymbol{\Sigma}_{\boldsymbol{\beta}}$*. Furthermore,* $\mathbf{e}_i$ *is independent with* $\boldsymbol{\Gamma}_j^0$, $\boldsymbol{\varphi}_j^0$, ε_{jt}, $\mathbf{v}_{jt}$, $\mathbf{f}_t^0$ *and* $\mathbf{h}_t^0$ *for all* i, j, t.

Assumption H (moments) *For each i,* $\mathbb{E}\left\|T^{-1/2}\mathbf{V}_i'\mathbf{F}^0\right\|^4 \leq C$, $\left\|T^{-1/2}\boldsymbol{\varepsilon}_i'\boldsymbol{\Sigma}\mathbf{F}^0\right\|^4 \leq C$, $\mathbb{E}\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^N \boldsymbol{\varepsilon}_i'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\|^4 \leq C$, $\mathbb{E}\left\|T^{-1/2}\sum_{t=1}^T [\mathbf{V}_i'\mathbf{V}_i - \boldsymbol{\Sigma}]\right\|^4 \leq C$, $\mathbb{E}\left\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^N \left(\mathbf{V}_i'\mathbf{V}_\ell - \mathbb{E}\big(\mathbf{V}_i'\mathbf{V}_\ell\big)\right)\boldsymbol{\Gamma}_\ell^{0\prime}\right\|^4 \leq C$*, and* $0 < C_{min} \leq \|\boldsymbol{\Sigma}\| \leq C$.

We propose the following mean-group IV (MGIV) estimator:

$$\widehat{\boldsymbol{\beta}}_{MGIV} = N^{-1}\sum_{i=1}^N \widehat{\boldsymbol{\beta}}_{\boldsymbol{i}}. \tag{4.2}$$

Theorem 4.1 *Under Assumptions A-E and G-H, we have*

$$\sqrt{N}(\widehat{\boldsymbol{\beta}}_{MGIV} - \boldsymbol{\beta}) = N^{-1/2}\sum_{i=1}^N \mathbf{e}_i + O_p(N^{3/4}T^{-1}) + O_p(NT^{-3/2}) + O_p(N^{1/2}\delta_{NT}^{-2}),$$

such that for $N^3/T^4 \to 0$ *as* $N, T \to \infty$*, we obtain*

$$\sqrt{N}(\widehat{\boldsymbol{\beta}}_{MGIV} - \boldsymbol{\beta}) \xrightarrow{d} N(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{\beta}}).$$

Furthermore, $\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\beta}} - \boldsymbol{\Sigma}_{\boldsymbol{\beta}} \xrightarrow{p} \mathbf{0}$*, where*

$$\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\beta}} = \frac{1}{N-1}\sum_{i=1}^N (\widehat{\boldsymbol{\beta}}_i - \widehat{\boldsymbol{\beta}}_{MGIV})(\widehat{\boldsymbol{\beta}}_i - \widehat{\boldsymbol{\beta}}_{MGIV})'. \tag{4.3}$$

5 Asymptotic comparison of $\widehat{\boldsymbol{\beta}}_{2SIV}$ with existing estimators

This section investigates asymptotic bias properties and relative efficiency of the 2SIV, IPC, PC and CA estimators for the models with homogeneous slopes. For this purpose, let $\mathbf{G}^0 = (\mathbf{F}^0, \mathbf{H}^0)$ denote a $T \times r$ matrix. We shall assume that $\mathbf{G}^{0\prime}\mathbf{G}^0/T \xrightarrow{p} \boldsymbol{\Sigma}_G^0 > 0$, a positive definite matrix. Note that, together with Assumption C, this implies that $\mathbf{F}^0$ and $\mathbf{H}^0$ are linearly independent of each other (and can be correlated), which is slightly stronger than Assumption C.

5.1 2SIV estimator

Recall that $\mathbf{X}_i = \mathbf{F}^0\boldsymbol{\Gamma}_i^0 + \mathbf{V}_i$ and $\mathbf{u}_i = \mathbf{H}^0\boldsymbol{\varphi}_i^0 + \boldsymbol{\varepsilon}_i$. Proposition 3.2 in Appendix B demonstrates that under Assumptions A-E $\left(N^{-1}T^{-1}\sum_{i=1}^N \mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{X}_i\right)\sqrt{NT}\left(\widehat{\boldsymbol{\beta}}_{2SIV} - \boldsymbol{\beta}\right)$ can be expanded as follows:

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^N \mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i = \frac{1}{\sqrt{NT}}\sum_{i=1}^N \mathbf{V}_i'\boldsymbol{\varepsilon}_i + \mathbf{b}_{0FH} + \mathbf{b}_{1FH} + \mathbf{b}_{2FH} + O_p\left(\sqrt{NT}\delta_{NT}^{-3}\right), \tag{5.1}$$

where

$$\begin{aligned}
\mathbf{b}_{0FH} &= -\frac{1}{N^{1/2}}\frac{1}{NT^{1/2}}\sum_{i=1}^N\sum_{j=1}^N(\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_j^0 + \boldsymbol{\varphi}_j^{0\prime}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0)\mathbf{V}_j'\boldsymbol{\varepsilon}_i;\\
\mathbf{b}_{1FH} &= -\frac{1}{N^{1/2}}\frac{1}{N^2T^{1/2}}\sum_{i=1}^N\sum_{\ell=1}^N\sum_{j=1}^N\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\left(\mathbf{V}_\ell'\boldsymbol{\varepsilon}_j\right)\boldsymbol{\varphi}_j^{0\prime}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0;\\
\mathbf{b}_{2FH} &= -\frac{1}{T^{1/2}}\frac{1}{N^{3/2}T}\sum_{i=1}^N\sum_{j=1}^N\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_j^0\mathbf{V}_j'\boldsymbol{\Sigma}_\varepsilon\mathbf{H}^0\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0,
\end{aligned}$$

with $\boldsymbol{\Sigma}_\varepsilon = \frac{1}{N}\sum_{j=1}^N \mathbb{E}\left(\boldsymbol{\varepsilon}_j\boldsymbol{\varepsilon}_j'\right)$. It is easily seen that (see proof of Proposition 3.2) $\mathbf{b}_{0FH} = O_p\left(N^{-1/2}\right)$, $\mathbf{b}_{1FH} = O_p\left(N^{-1/2}\right)$ and $\mathbf{b}_{2FH} = O_p\left(T^{-1/2}\right)$.
Hence, we have

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^N \mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i = \frac{1}{\sqrt{NT}}\sum_{i=1}^N \mathbf{V}_i'\boldsymbol{\varepsilon}_i + o_p(1).$$

5.2 Asymptotic bias of Bai's (2009a) IPC-type estimator

It is instructive to consider a PC estimator that is asymptotically equivalent to Bai (2009a) but avoids iterations:

$$\widetilde{\boldsymbol{\beta}}_{2SIV} = \left(\sum_{i=1}^N \mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{X}_i\right)^{-1}\sum_{i=1}^N \mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{y}_i.$$

Observe that this estimator projects out $\widehat{\mathbf{H}}$ from $(\mathbf{X}_i, \mathbf{y}_i)$, but it does not eliminate $\widehat{\mathbf{F}}$ from $\mathbf{X}_i$. $\widehat{\mathbf{H}}$ is estimated using the residuals of the first-stage IV estimator, $\widehat{\mathbf{u}}_i = \mathbf{y}_i - \mathbf{X}_i\widehat{\boldsymbol{\beta}}_{1SIV}$.

Using similar derivations as in Section 5.1, Proposition 5.1 below shows that $\left(N^{-1}T^{-1}\sum_{i=1}^N \mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{X}_i\right)\times$ $\sqrt{NT}\left(\widetilde{\boldsymbol{\beta}}_{2SIV} - \boldsymbol{\beta}\right)$ has the following asymptotic expansion:

Proposition 5.1 *Under Assumptions A-E, we have*

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^N \mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i = \frac{1}{\sqrt{NT}}\sum_{i=1}^N \mathbf{X}_i'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i + \mathbf{b}_{0H} + \mathbf{b}_{1H} + \mathbf{b}_{2H} + O_p\left(\sqrt{NT}\delta_{NT}^{-3}\right) \tag{5.2}$$

with

$$\mathbf{b}_{0H} = -\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\mathbf{X}_j'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i;$$

$$\mathbf{b}_{1H} = -\sqrt{\frac{T}{N}}\frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N}\boldsymbol{\mathcal{X}}_i'\mathbf{H}^0\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\boldsymbol{\varphi}_j^0\mathbb{E}\left(\boldsymbol{\varepsilon}_j'\boldsymbol{\varepsilon}_i/T\right);$$

$$\mathbf{b}_{2H} = -\sqrt{\frac{N}{T}}\frac{1}{NT}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\Sigma}_{\varepsilon}\mathbf{H}^0\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\boldsymbol{\varphi}_i^0,$$

where $a_{ij} = \boldsymbol{\varphi}_j^{0\prime}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\boldsymbol{\varphi}_i^0$, $\boldsymbol{\mathcal{X}}_i = \mathbf{X}_i - N^{-1}\sum_{\ell=1}^{N}a_{i\ell}\mathbf{X}_\ell$ *and* $\boldsymbol{\Sigma}_{\varepsilon} = \frac{1}{N}\sum_{j=1}^{N}\mathbb{E}\left(\boldsymbol{\varepsilon}_j\boldsymbol{\varepsilon}_j'\right)$.

The above asymptotic bias terms are identical to those of the IPC estimator of Bai (2009a). As a result, it suffices to compare $\widehat{\boldsymbol{\beta}}_{2SIV}$ with $\widetilde{\boldsymbol{\beta}}_{2SIV}$. Incidentally, as shown in Bai (2009a), the term $\mathbf{b}_{0H}$ tends to a normal random vector, which necessitates the transformation of the regressor matrix to $\boldsymbol{\mathcal{X}}_i$; see equation (5.3) below.

The terms $\mathbf{b}_{0H}$, $\mathbf{b}_{1H}$ and $\mathbf{b}_{2H}$ in (5.2) are comparable to the terms $\mathbf{b}_{0FH}$, $\mathbf{b}_{1FH}$ and $\mathbf{b}_{2FH}$, respectively, in (5.1). One striking result is that $\mathbf{b}_{0H}$, $\mathbf{b}_{1H}$ and $\mathbf{b}_{2H}$ are not asymptotically ignorable, whereas $\mathbf{b}_{0FH}$, $\mathbf{b}_{1FH}$ and $\mathbf{b}_{2FH}$ are. This difference stems solely from the fact that $\widehat{\boldsymbol{\beta}}_{2SIV}$ asymptotically projects out $\mathbf{F}^0\boldsymbol{\Gamma}_i^0$ from $\mathbf{X}_i$ and $\mathbf{H}^0\boldsymbol{\varphi}_i^0$ from $\mathbf{u}_i$ separately, whereas $\widetilde{\boldsymbol{\beta}}_{2SIV}$ projects out $\mathbf{H}^0\boldsymbol{\varphi}_i^0$ from $\mathbf{u}_i$ *only*. Therefore, the asymptotic bias terms of $\widetilde{\boldsymbol{\beta}}_{2SIV}$, $\mathbf{b}_{0H}$, $\mathbf{b}_{1H}$ and $\mathbf{b}_{2H}$, contain correlations between the regressors $\mathbf{X}_i$ and the disturbance $\mathbf{u}_i(=\mathbf{H}^0\boldsymbol{\varphi}_i^0+\boldsymbol{\varepsilon}_i)$ since the estimation error of $\widehat{\mathbf{H}}$ contains $\mathbf{u}_i$. Recalling that $\mathbf{X}_i = \mathbf{F}^0\boldsymbol{\Gamma}_i^0 + \mathbf{V}_i$, such correlations are asymptotically non-negligible because $\mathbf{H}^{0\prime}\mathbf{F}^0/T = O_p(1)$ and $\sum_{i=1}^{N}\boldsymbol{\varphi}_i^{0\prime}\text{vec}\left(\boldsymbol{\Gamma}_i^0\right)/N = O_p(1)$.

On the other hand, $\widehat{\boldsymbol{\beta}}_{2SIV}$ asymptotically projects out $\mathbf{F}^0\boldsymbol{\Gamma}_i^0$ from $\mathbf{X}_i$ as well as $\mathbf{H}^0\boldsymbol{\varphi}_i^0$ from $\mathbf{u}_i$. Therefore, $\mathbf{b}_{0FH}$, $\mathbf{b}_{1FH}$ and $\mathbf{b}_{2FH}$ contain correlations between $\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i = \mathbf{M}_{\mathbf{F}^0}\mathbf{V}_i$ and $\mathbf{u}_i$. Since $\mathbf{V}_i$, $\mathbf{H}^0\boldsymbol{\varphi}_i^0$ and $\boldsymbol{\varepsilon}_i$ are independent of each other, such correlations are asymptotically negligible. As a result, our estimator $\widehat{\boldsymbol{\beta}}_{2SIV}$ does not suffer from asymptotic bias.

Using similar reasoning, it turns out that in some special cases, some of the bias terms of $\widetilde{\boldsymbol{\beta}}_{2SIV}$ may disappear as well. For instance, when $\mathbf{F}^0 \subseteq \mathbf{H}^0$, we have $\mathbf{M}_{\mathbf{H}^0}\mathbf{X}_j = \mathbf{M}_{\mathbf{H}^0}\mathbf{V}_j$ because $\mathbf{M}_{\mathbf{H}^0}\mathbf{F}^0 = \mathbf{0}$. Thus, $\mathbf{b}_{0H} = O_p\left(N^{-1/2}\right)$ and $\mathbf{b}_{2H} = O_p\left(T^{-1/2}\right)$ although $\mathbf{b}_{1H}$ remains $O_p(1)$. Note that under our assumptions all three bias terms, $\mathbf{b}_{0H}$, $\mathbf{b}_{1H}$ and $\mathbf{b}_{2H}$, are asymptotically negligible only if $\mathbf{H}^0 = \mathbf{F}^0$, which can be a highly restrictive condition in practice.[4]

5.3 Asymptotic bias of PC and CA estimators

Pesaran (2006) and Westerlund and Urbain (2015) put forward pooled estimators in which the whole set of factors in $\mathbf{X}_i$ and $\mathbf{u}_i$ are estimated *jointly*, rather than *separately*. This difference makes these estimators asymptotically biased. To show this, we rewrite the model as

$$\mathbf{Z}_i = (\mathbf{y}_i, \mathbf{X}_i) = \mathbf{G}^0\boldsymbol{\Lambda}_i^0 + \mathbf{U}_i,$$

where

$$\boldsymbol{\Lambda}_i^0 = \begin{pmatrix} \boldsymbol{\Gamma}_i^0\boldsymbol{\beta} & \boldsymbol{\Gamma}_i^0 \\ \boldsymbol{\varphi}_i^0 & \mathbf{0} \end{pmatrix}, \mathbf{U}_i = (\mathbf{V}_i\boldsymbol{\beta} + \boldsymbol{\varepsilon}_i, \mathbf{V}_i).$$

[4]When $\varepsilon_{it} \sim i.i.d.(0,\sigma^2)$, $\mathbf{b}_{0H}$ remains $O_p(1)$ whilst $\mathbf{b}_{1H}$ and $\mathbf{b}_{2H}$ become asymptotically negligible. See Corollary 1 in Bai (2009a).

Define

$$\boldsymbol{\Upsilon}_{\Lambda}^{0} = N^{-1}\sum_{i=1}^{N}\boldsymbol{\Lambda}_{i}^{0}\boldsymbol{\Lambda}_{i}^{0\prime} = \begin{pmatrix} N^{-1}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{i}^{0}\left(\boldsymbol{\beta}\boldsymbol{\beta}'+\mathbf{I}_{k}\right)\boldsymbol{\Gamma}_{i}^{0\prime}+\boldsymbol{\Upsilon}^{0} & N^{-1}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{i}^{0}\boldsymbol{\beta}\boldsymbol{\varphi}_{i}^{0\prime} \\ N^{-1}\sum_{i=1}^{N}\boldsymbol{\varphi}_{i}^{0}\boldsymbol{\beta}'\boldsymbol{\Gamma}_{i}^{0\prime} & \boldsymbol{\Upsilon}_{\varphi}^{0} \end{pmatrix}.$$

In the PC approach of Westerlund and Urbain (2015), a span of $\mathbf{G}^0$ is estimated as $\sqrt{T}$ times the eigenvectors corresponding to the first r largest eigenvalues of $\sum_{i=1}^{N}\mathbf{Z}_i\mathbf{Z}_i'/N$, which is denoted by $\widehat{\mathbf{G}}_z$. The resulting PC estimator is defined as

$$\widehat{\boldsymbol{\beta}}_{PC} = \left(\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{G}}_z}\mathbf{X}_i\right)^{-1}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{G}}_z}\mathbf{y}_i.$$

In line with Pesaran (2006), the CA estimator of Westerlund and Urbain (2015) approximates a span of $\mathbf{G}^0$ by a linear combination of $\bar{\mathbf{Z}} = N^{-1}\sum_{i=1}^{N}\mathbf{Z}_i$. The associated CA estimator is given by

$$\widehat{\boldsymbol{\beta}}_{CA} = \left(\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\bar{\mathbf{Z}}}\mathbf{X}_i\right)^{-1}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\bar{\mathbf{Z}}}\mathbf{y}_i.$$

As discussed in Westerlund and Urbain (2015), both PC and CA are asymptotically biased due to the correlation between the estimation error of $\widehat{\mathbf{G}}_z$ and $\{\mathbf{X}_i,\mathbf{u}_i\}$. The estimation error of $\widehat{\mathbf{G}}_z$ contains the error term of the system equation $\mathbf{U}_i$, which is a function of both $\mathbf{V}_i$ and $\boldsymbol{\varepsilon}_i$. Therefore, the estimation error of $\widehat{\mathbf{G}}_z$ is correlated with $\mathbf{M}_{\mathbf{G}}\mathbf{X}_i$ and $\mathbf{M}_{\mathbf{G}}\mathbf{u}_i$, which causes the asymptotic bias. In what follows, we shall focus on the PC estimator as the bias analysis for the CA estimator is very similar.

Following Westerlund and Urbain (2015), we expand $\left(N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{G}}_z}\mathbf{X}_i\right)\sqrt{NT}\left(\widehat{\boldsymbol{\beta}}_{PC}-\boldsymbol{\beta}\right)$ as follows:

Proposition 5.2 *Under Assumptions A-E*

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{G}}_z}\mathbf{u}_i = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'\boldsymbol{\varepsilon}_i + \mathbf{b}_{1G} + \mathbf{b}_{2G} + \mathbf{b}_{3G} + O_p\left(\sqrt{NT}\delta_{NT}^{-3}\right),$$

$$\begin{aligned}
\mathbf{b}_{1G} &= -\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\boldsymbol{\Gamma}_i^{0\prime},\mathbf{0}'\right)\left(\boldsymbol{\Upsilon}_{\Lambda}^{0}\right)^{-1}\boldsymbol{\Lambda}_j^0\mathbb{E}\left(\mathbf{U}_j'\boldsymbol{\varepsilon}_i/T\right); \\
\mathbf{b}_{2G} &= -\sqrt{\frac{T}{N}}\frac{1}{N^2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}\left(\boldsymbol{\Gamma}_i^{0\prime},\mathbf{0}'\right)\left(\boldsymbol{\Upsilon}_{\Lambda}^{0}\right)^{-1}\boldsymbol{\Lambda}_\ell^0\mathbb{E}\left(\mathbf{U}_\ell'\mathbf{U}_j/T\right)\boldsymbol{\Lambda}_j^{0\prime}(\boldsymbol{\Upsilon}_{\Lambda}^{0})^{-1}\left(\frac{\mathbf{G}^{0\prime}\mathbf{G}^0}{T}\right)^{-1}\frac{\mathbf{G}^{0\prime}\mathbf{H}^0}{T}\boldsymbol{\varphi}_i^0; \\
\mathbf{b}_{3G} &= -\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{V}_i'\mathbf{U}_j}{T}\boldsymbol{\Lambda}_j^{0\prime}(\boldsymbol{\Upsilon}_{\Lambda}^{0})^{-1}\left(\frac{\mathbf{G}^{0\prime}\mathbf{G}^0}{T}\right)^{-1}\frac{\mathbf{G}^{0\prime}\mathbf{H}^0}{T}\boldsymbol{\varphi}_i^0.
\end{aligned}$$

It is easily seen that $\mathbf{b}_{1G}$, $\mathbf{b}_{2G}$ and $\mathbf{b}_{3G}$ are all $O_p(1)$. Note that the asymptotic bias terms are functions of $\boldsymbol{\Lambda}_\ell^0$ and $\boldsymbol{\Upsilon}_{\Lambda}^0$, which depend on the slope coefficient vector $\boldsymbol{\beta}$.

5.4 Relative asymptotic efficiency of 2SIV, IPC, PC and CA estimators

Finally, we compare the asymptotic efficiency of the estimators. To make the problem tractable and as succinct as possible, we shall assume that ε_{it} is i.i.d. over i and t with $\mathbb{E}(\varepsilon_{it}) = 0$ and

$\mathbb{E}\left(\varepsilon_{it}^2\right) = \sigma_\varepsilon^2$. In this case, it is easily seen that the asymptotic variance of $\widehat{\boldsymbol{\beta}}_{2SIV}$ is

$$\boldsymbol{\Psi} = \sigma_\varepsilon^2 \left(\text{plim} N^{-1} T^{-1} \sum_{i=1}^N \mathbf{V}_i' \mathbf{V}_i \right)^{-1}.$$

Next, using Proposition 5.2, consider the bias-corrected PC estimator

$$\widehat{\boldsymbol{\beta}}_{PC}^* = \widehat{\boldsymbol{\beta}}_{PC} - N^{1/2} T^{1/2} \left(\sum_{i=1}^N \mathbf{V}_i' \mathbf{V}_i \right)^{-1} (\mathbf{b}_{1G} + \mathbf{b}_{2G} + \mathbf{b}_{3G}).$$

We can see that the asymptotic variance of the bias-corrected PC estimator is identical to $\boldsymbol{\Psi}$. Therefore, the 2SIV and the bias-corrected PC estimators are asymptotically equivalent.

Consider now $\widetilde{\boldsymbol{\beta}}_{2SIV}$. Noting that $\mathbf{b}_{0H}$ tends to a normal distribution, and following Bai (2009a), the bias-corrected estimator with transformed regressors can be expressed as:

$$\widetilde{\boldsymbol{\beta}}_{2SIV}^* = \widetilde{\boldsymbol{\beta}}_{2SIV}^+ - N^{1/2} T^{1/2} \left(\sum_{i=1}^N \boldsymbol{\mathcal{X}}_i' \mathbf{M}_{\mathbf{H}^0} \boldsymbol{\mathcal{X}}_i \right)^{-1} (\mathbf{b}_{1H} + \mathbf{b}_{2H}),$$

where

$$\widetilde{\boldsymbol{\beta}}_{2SIV}^+ = \left(\sum_{i=1}^N \boldsymbol{\mathcal{X}}_i' \mathbf{M}_{\mathbf{H}^0} \boldsymbol{\mathcal{X}}_i \right)^{-1} \sum_{i=1}^N \boldsymbol{\mathcal{X}}_i' \mathbf{M}_{\mathbf{H}^0} \mathbf{y}_i. \tag{5.3}$$

The asymptotic variance of this bias-corrected estimator is given by

$$\widetilde{\boldsymbol{\Psi}} = \sigma_\varepsilon^2 \left(\text{plim} N^{-1} T^{-1} \sum_{i=1}^N \boldsymbol{\mathcal{X}}_i' \mathbf{M}_{\mathbf{H}^0} \boldsymbol{\mathcal{X}}_i \right)^{-1}.$$

There exist two differences compared to $\boldsymbol{\Psi}$. First, in general $\mathbf{M}_{\mathbf{H}^0} \mathbf{X}_i \neq \mathbf{M}_{\mathbf{H}^0} \mathbf{V}_i$ as the factors in $\mathbf{X}_i$ may not be identical to the factors in $\mathbf{u}_i$. Second, regressors are to be transformed as $\boldsymbol{\mathcal{X}}_i = \mathbf{X}_i - N^{-1} \sum_{\ell=1}^N a_{i\ell} \mathbf{X}_\ell$. Therefore, $\boldsymbol{\Psi} - \widetilde{\boldsymbol{\Psi}}$ can be positive semi-definite or negative-semi-definite. Thus, the asymptotic efficiency of the bias-corrected IPC estimator of Bai (2009a) relative to 2SIV and the bias-corrected PC/CA estimators, is indeterminate. However, in the special case where $\mathbf{F}^0 \subseteq \mathbf{H}^0$, we have $\mathbf{M}_{\mathbf{H}^0} \boldsymbol{\mathcal{X}}_i = \mathbf{M}_{\mathbf{H}^0} \boldsymbol{\mathcal{V}}_i$, with $\boldsymbol{\mathcal{V}}_i = \mathbf{V}_i - N^{-1} \sum_{\ell=1}^N a_{i\ell} \mathbf{V}_\ell$. The second term of $\boldsymbol{\mathcal{V}}_i$ is $O_p(N^{-1/2})$ because $\mathbf{V}_\ell$ and $a_{i\ell}$ are independent. Hence, in this case $\widetilde{\boldsymbol{\Psi}} = \boldsymbol{\Psi}$, and the bias-corrected IPC estimator is asymptotically as efficient as the bias-corrected PC/CA estimator and 2SIV.

6 Monte Carlo Simulations

We conduct a small-scale Monte Carlo simulation exercise in order to assess the finite sample behaviour of the proposed approach in terms of bias, standard deviation (s.d.), root mean squared error (RMSE), empirical size and power of the t-test. More specifically, we investigate the performance of 2SIV, defined in (3.2), and MGIV defined in (4.2). For the purposes of comparison, we also consider the (bias-corrected) IPC of Bai (2009a) and the PC estimator, labeled as (BC-)IPC and (BC-)PC respectively, the CA estimator, as well as the mean-group versions of PC and CA (denoted as MGPC and MGCA), which were put forward by Pesaran (2006), Westerlund and Urbain (2015) and Reese and Westerlund (2018). The t-statistics for 2SIV and MGIV are computed using the variance estimators defined by (3.5) and (4.3), respectively. The t-statistics for IPC, PC and CA estimators and their MG versions (if any) employ analogous variance estimators.

6.1 Design

We consider the following panel data model:

$$y_{it} = \alpha_i + \sum_{\ell=1}^{k} \beta_{\ell i} x_{\ell it} + u_{it}; \quad u_{it} = \sum_{s=1}^{m_y} \gamma_{si}^0 f_{s,t}^0 + \varepsilon_{it}, \tag{6.1}$$

$i = 1, ..., N$, $t = -49, ..., T$, where the process for the covariates is given by

$$x_{\ell it} = \mu_{\ell i} + \sum_{s=1}^{m_x} \gamma_{\ell si}^0 f_{s,t}^0 + v_{\ell it}; \quad i = 1, 2, ..., N; \quad t = -49, -48, ..., T. \tag{6.2}$$

We set $k = 2$, $m_y = 2$ and $m_x = 3$. This implies that the first two factors in u_{it}, f_{1t}^0 and f_{2t}^0, are also in the DGP of $x_{\ell it}$ for $\ell = 1, 2$, while f_{3t}^0 is included in $x_{\ell it}$ only. Observe that, using notation of earlier sections, $\mathbf{h}_t^0 = (f_{1t}^0, f_{2t}^0)'$ and $\mathbf{f}_t^0 = (f_{1t}^0, f_{2t}^0, f_{3t}^0)'$. [5]

The factors $f_{s,t}^0$ are generated using the following AR(1) process:

$$f_{s,t}^0 = \rho_{fs} f_{s,t-1}^0 + (1 - \rho_{fs}^2)^{1/2} \zeta_{s,t}, \tag{6.3}$$

where $\rho_{fs} = 0.5$ and $\zeta_{s,t} \sim i.i.d.N(0, 1)$ for $s = 1, ..., 3$.

The idiosyncratic error of y_{it}, ε_{it}, is non-normal and heteroskedastic across both i and t, such that $\varepsilon_{it} = \varsigma_\varepsilon \sigma_{it}(\epsilon_{it} - 1)/\sqrt{2}$, $\epsilon_{it} \sim i.i.d.\chi_1^2$, with $\sigma_{it}^2 = \eta_i \varphi_t$, $\eta_i \sim i.i.d.\chi_2^2/2$, and $\varphi_t = t/T$ for $t = 0, 1, ..., T$ and unity otherwise. We define $\pi_u := \varsigma_\varepsilon^2 / \left(m_y + \varsigma_\varepsilon^2\right)$ which is the proportion of the average variance of u_{it} due to ε_{it}. This implies $\varsigma_\varepsilon^2 = \pi_u m_y (1 - \pi_u)^{-1}$. We set ς_ε^2 such that $\pi_u \in \{1/4, 3/4\}$.

The idiosyncratic errors of the covariates follow an AR(1) process

$$v_{\ell it} = \rho_{\upsilon,\ell} v_{\ell it-1} + (1 - \rho_{\upsilon,\ell}^2)^{1/2} \varpi_{\ell it}; \quad \varpi_{\ell it} \sim i.i.d.N(0, \varsigma_\upsilon^2), \tag{6.4}$$

for $\ell = 1, 2$. We set $\rho_{\upsilon,\ell} = 0.5$ for all ℓ.

We define the signal-to-noise ratio (SNR) as $SNR := \left(\beta_1^2 + \beta_2^2\right) \varsigma_\upsilon^2 \varsigma_\varepsilon^{-2}$ where $\rho_\upsilon = \rho_{\upsilon,\ell}$ for $\ell = 1, 2$. Solving for ς_υ^2 gives $\varsigma_\upsilon^2 = \varsigma_\varepsilon^2 SNR \left(\beta_1^2 + \beta_2^2\right)^{-1}$. We set $SNR = 4$, which lies within the values considered by Bun and Kiviet (2006) and Juodis and Sarafidis (2018).

The individual-specific effects are generated by drawing initially *mean-zero* random variables as

$$\mu_{\ell i}^* = \rho_{\mu,\ell} \alpha_i^* + (1 - \rho_{\mu,\ell}^2)^{1/2} \omega_{\ell i}, \tag{6.5}$$

where $\alpha_i^* \sim i.i.d.N(0, 1)$, $\omega_{\ell i} \sim i.i.d.N(0, 1)$, for $\ell = 1, 2$. We set $\rho_{\mu,\ell} = 0.5$ for $\ell = 1, 2$. Subsequently, we set

$$\alpha_i = \alpha + \alpha_i^*, \quad \mu_{\ell i} = \mu_\ell + \mu_{\ell i}^*, \tag{6.6}$$

where $\alpha = 1/2$, $\mu_1 = 1$, $\mu_2 = -1/2$, for $\ell = 1, 2$.

Similarly, the factor loadings in u_{it} are generated at first instance as *mean-zero* random variables such that $\gamma_{si}^{0*} \sim i.i.d.N(0, 1)$ for $s = 1, ..., m_y = 2$, $\ell = 1, 2$; the factor loadings in x_{1it} and x_{2it} are generated as

$$\gamma_{\ell si}^{0*} = \rho_{\gamma,\ell s} \gamma_{si}^{0*} + (1 - \rho_{\gamma,\ell s}^2)^{1/2} \xi_{\ell si}; \quad \xi_{\ell si} \sim i.i.d.N(0, 1); \tag{6.7}$$

[5] Tables E1-E3 in Appendix E present results for a different specification, where $m_y = 3$ and $m_x = 2$. To save space, we do not discuss these results here but it suffices to say that the conclusions are similar to those in Section 6.2.

$$\gamma_{13i}^{0*} = \rho_{\gamma,13}\gamma_{1i}^{0*} + (1-\rho_{\gamma,13}^2)^{1/2}\xi_{13i}; \quad \xi_{13i} \sim i.i.d.N(0,1); \tag{6.8}$$

$$\gamma_{23i}^{0*} = \rho_{\gamma,23}\gamma_{2i}^{0*} + (1-\rho_{\gamma,23}^2)^{1/2}\xi_{23i}; \quad \xi_{23i} \sim i.i.d.N(0,1). \tag{6.9}$$

The process (6.7) allows the factor loadings to $f_{1,t}^0$ and $f_{2,t}^0$ in x_{1it} and x_{2it} to be correlated with the factor loadings corresponding to the factor specific in u_{it}. On the other hand, (6.8) and (6.9) ensure that the factor loadings to $f_{3,t}^0$ in x_{1it} and x_{2it} can be correlated with the factor loadings corresponding to the factors $f_{1,t}^0$ and $f_{2,t}^0$ in u_{it}. We consider $\rho_{\gamma,11} = \rho_{\gamma,12} = \rho_{\gamma,21} = \rho_{\gamma,22} = \rho_{\gamma,13} = \rho_{\gamma,23} = 0.5$. The factor loadings that enter into the model are then generated as

$$\mathbf{\Gamma}_i^0 = \mathbf{\Gamma}^0 + \mathbf{\Gamma}_i^{0*} \tag{6.10}$$

where

$$\mathbf{\Gamma}_i^0 = \begin{pmatrix} \gamma_{1i}^0 & \gamma_{11i}^0 & \gamma_{21i}^0 \\ \gamma_{2i}^0 & \gamma_{12i}^0 & \gamma_{22i}^0 \\ 0 & \gamma_{13i}^0 & \gamma_{23i}^0 \end{pmatrix} \text{ and } \mathbf{\Gamma}_i^{0*} = \begin{pmatrix} \gamma_{1i}^{0*} & \gamma_{11i}^{0*} & \gamma_{21i}^{0*} \\ \gamma_{2i}^{0*} & \gamma_{12i}^{0*} & \gamma_{22i}^{0*} \\ 0 & \gamma_{13i}^{0*} & \gamma_{23i}^{0*} \end{pmatrix}.$$

Observe that, using notation of earlier sections, $\boldsymbol{\gamma}_{yi}^0 = (\gamma_{1i}^0, \gamma_{2i}^0)'$ and $\mathbf{\Gamma}_{x,i}^0 = (\boldsymbol{\gamma}_{1i}^0, \boldsymbol{\gamma}_{2i}^0, \boldsymbol{\gamma}_{3i}^0)'$ with $\boldsymbol{\gamma}_{\ell i}^0 = (\gamma_{\ell 1i}^0, \gamma_{\ell 2i}^0, \gamma_{\ell 3i}^0)'$ for $\ell = 1, 2$. It is easily seen that the average of the factor loadings is $E\left(\mathbf{\Gamma}_i^0\right) = \mathbf{\Gamma}^0$. We set

$$\mathbf{\Gamma}^0 = \begin{pmatrix} \gamma_1^0 & \gamma_{11}^0 & \gamma_{21}^0 \\ \gamma_2^0 & \gamma_{12}^0 & \gamma_{22}^0 \\ 0 & \gamma_{13}^0 & \gamma_{23}^0 \end{pmatrix} = \begin{pmatrix} 1/4 & 1/4 & -1 \\ 1/2 & -1 & 1/4 \\ 0 & 1/2 & 1/2 \end{pmatrix}. \tag{6.11}$$

The slope coefficients in (6.1) are generated as

$$\beta_{1i} = \beta_1 + \eta_{\beta_1 i}; \quad \beta_{2i} = \beta_2 + \eta_{\beta_2 i}, \tag{6.12}$$

such that $\beta_1 = 3$ and $\beta_2 = 1$. In the case of homogeneous slopes, we impose $\rho_i = \rho$, $\beta_{1i} = \beta_1$ and $\beta_{2i} = \beta_2$, whereas in the case of heterogeneous slopes, we specify $\eta_{\rho i} \sim i.i.d.U\left[-c, +c\right]$, and

$$\eta_{\beta_\ell i} = \left[(2c)^2/12\right]^{1/2} \rho_\beta \xi_{\beta_\ell i} + \left(1 - \rho_\beta^2\right)^{1/2} \eta_{\rho i},$$

where $\xi_{\beta_\ell i}$ is the standardised squared idiosyncratic errors in $x_{\ell it}$, computed as

$$\xi_{\beta_\ell i} = \frac{\overline{v_{\ell i}^2} - \overline{v_\ell^2}}{\left[N^{-1}\sum_{i=1}^N \left(\overline{v_{\ell i}^2} - \overline{v_\ell^2}\right)^2\right]^{1/2}},$$

with $\overline{v_{\ell i}^2} = T^{-1}\sum_{t=1}^T v_{\ell it}^2$, $\overline{v_\ell^2} = N^{-1}\sum_{i=1}^N \overline{v_{\ell i}^2}$, for $\ell = 1, 2$. We set $c = 1/5$, $\rho_\beta = 0.4$ for $\ell = 1, 2$.

We consider various combinations of (T, N), i.e. $T \in \{25, 50, 100, 200\}$ and $N \in \{25, 50, 100, 200\}$. The results are obtained based on 2,000 replications, and all tests are conducted at the 5% significance level. For the size of the "t-test", $H_0 : \beta_\ell = \beta_\ell^0$ for $\ell = 1, 2$, where β_1^0 and β_2^0 are the true parameter values. For the power of the test, $H_0 : \beta_\ell = \beta_\ell^0 + 0.1$ for $\ell = 1, 2$ against two sided alternatives are considered.

Prior to computing the estimators except for CA and MGCA , the data are demeaned using the within transformation in order to eliminate individual-specific effects. For the CA and MGCA estimators, the untransformed data are used, but a $T \times 1$ vector of ones is included along with the cross-sectional averages. The number of factors m_x and m_y are estimated in each replication using the eigenvalue ratio (ER) statistic proposed by Ahn and Horenstein (2013).

6.2 Results

Tables 1–3 report results for β_1 in terms of bias, standard deviation, RMSE, empirical size and power for the model in (6.1).[6]

Table 1 focuses on the case where $N = T = 200$ and π_u alternates between $\{1/4, 3/4\}$. Consider first the homogeneous model with $\pi_u = 3/4$. As we can see, the bias ($\times 100$) for 2SIV and MGIV is very close to zero and takes the smallest value compared to the remaining estimators. The bias of BC-IPC is larger in absolute value than that of IPC but of opposite sign. This may suggest that bias-correction over-corrects in this case. MGPC and PC perform similarly and exhibit larger bias than IPC. Last, both CA and MGCA are subject to substantial bias, which is not surprising as these estimators may require bias-correction in the present DGP.

In regards to the dispersion of the estimators, the standard deviation of 2SIV and PC is very similar, which is in line with our theoretical results. For this specific design, IPC takes the smallest s.d. value among the estimators under consideration. On the other hand, when it comes to the bias-corrected estimators, bias-correction appears to inflate dispersion and thus the standard deviation of BC-IPC and BC-PC is relatively large (equal to 0.805 and 0.885, respectively). As a result, 2SIV outperforms BC-IPC and BC-PC, with a s.d. value equal to 0.586.

In terms of RMSE, IPC appears to perform best, although this estimator is not recommended in practice due to its asymptotic bias. 2SIV takes the second smallest RMSE value, followed by MGIV. CA and MGCA exhibit the largest RMSE values, an outcome that reflects the large bias of these estimators.

Next, we turn our attention to the model with heterogeneous slopes and $\pi_u = 3/4$. In comparison to the homogeneous model, all estimators suffer a substantial increase in bias; the only exception is MGIV, which has the smallest bias. MGPC and MGCA are severely biased, both in absolute magnitude as well as relative to the remaining inconsistent estimators. The s.d. values of MGIV and MGPC are very similar and relatively small compared to the other estimators. The smallest RMSE value is that of MGIV.

We now discuss the results in the lower panel of Table 1, which correspond to $\pi_u = 1/4$. The relative performance of the estimators is similar to the case where $\pi_u = 3/4$, except for a noticeable improvement in the performance of BC-IPC. Thus, the results for BC-IPC and IPC are quite comparable, suggesting that the bias-correction term is close to zero and so over-correction is avoided. The results for 2SIV are very similar to those for $\pi_u = 3/4$, which indicates that the estimator is robust to different values of the variance ratio. The conclusions with heterogeneous slopes for $\pi_u = 1/4$ are similar to those for $\pi_u = 3/4$

In regards to inference, the size of the t-test associated with 2SIV and MGIV is close to the nominal value of 5% under the setting of homogeneous slopes. The same appears to hold true for BC-IPC when $\pi_u = 1/4$, although there are substantial distortions when $\pi_u = 3/4$. The t-test associated with BC-PC is oversized when $\pi_u = 3/4$ and the distortion becomes more severe with $\pi_u = 1/4$. CA and MGCA have the largest size distortions. In the case of heterogeneous slopes, MGIV performs well and size is close to 5%. MGPC and MGCA have substantial size distortions regardless of the value of π_u.

Table 2 presents results for the case where $(N, T) = (200, 25)$ (i.e. N is large relative to T)

[6]The results for β_2 are qualitatively similar and so we do not report them to save space. These results are available upon request.

and $(N, T) = (25, 200)$ (N is small relative to T) for $\pi_u = 3/4$. In the former case, 2SIV performs best in terms of bias. IPC has the smallest RMSE, followed by 2SIV. CA has the largest bias and RMSE. In the case of heterogeneous slopes, MGIV has smaller absolute bias than MGPC and MGCA. Therefore, MGIV is superior among mean-group type estimators, which are the only consistent estimators in this design. In the case where T is large relative to N, 2SIV and MGIV again outperform BC-IPC, BC-PC and CA in terms of bias, standard deviation and RMSE.

In regards to the properties of the t-test, 2SIV and MGIV have the smallest size distortions relative to the other estimators, and inference based on 2SIV and MGIV remains credible even for small values of N or T. Moreover, 2SIV and MGIV exhibit good power properties, whereas MGPC has the lowest power when N is small relative to T.

Table 3 shows the bias of the estimators, scaled by $\sqrt{NT}$ ($\sqrt{N}$) for different values of $N = T$ with $\pi_u = \{1/4, 3/4\}$ when the slopes are homogeneous (heterogeneous). The performance of 2SIV and MGIV is in agreement with our theoretical results. More specifically, the bias monotonically decreases as the sample size goes up. In contrast, for $\pi_u = 3/4$ it appears that a relatively large sample size is necessary so that bias-correction works for BC-IPC. BC-PC appears to require even larger sample sizes.

In a nutshell, the results presented in Tables 1-3 and the associated discussion above suggest that 2SIV and MGIV have good small sample properties and outperform existing popular estimators for the experimental designs considered here.

7 Conclusions

We put forward IV estimators for linear panel data models with interactive effects in the error term and regressors. The instruments are transformed regressors, and so it is not necessary to search for external instruments. Models with homogeneous and heterogeneous slope coefficients have been considered. In the former model, we propose a two-stage IV estimator. In the first stage, we asymptotically projects out the interactive effects from the regressors and use the defactored regressors as instruments. In the second stage, we asymptotically eliminate the interactive effects in the error term based on their estimates using the first-stage residuals. We established the $\sqrt{NT}$-consistency and the asymptotic normality of the 2SIV estimator. For the heterogeneous slopes, we put forward a mean-group IV estimator (MGIV) and established $\sqrt{N}$-consistency and asymptotic normality.

Having derived the theoretical properties of our IV estimators, we compared the asymptotic expressions of our 2SIV estimator, IPC of Bai (2009a), PC and CA of Westerlund and Urbain (2015) and Pesaran (2006), for the models with homogeneous slopes. Under the conditions similar to those in Bai (2009a), it has emerged that 2SIV is free from asymptotic bias, whereas the remaining estimators suffer from asymptotic bias. In addition, it is revealed that 2SIV is asymptotically as efficient as the bias-corrected versions of PC and CA, while the relative efficiency of the bias-corrected IPC estimator is generally indeterminate. The theoretical results are corroborated in a Monte Carlo simulation exercise, which shows that 2SIV and MGIV perform competently and can outperform existing estimators.

Table 1: Bias, root mean squared error (RMSE) of the estimators of β_1, and size and power of the associated t-tests when $\pi_u = \{1/4, 3/4\}$ and $N = T = 200$.

Estimator	Homogeneous Slopes					Heterogenous Slopes				
	Bias ($\times 100$)	S.D. ($\times 100$)	RMSE ($\times 100$)	Size	Power	Bias ($\times 100$)	S.D. ($\times 100$)	RMSE ($\times 100$)	Size	Power
	$\pi_u = 3/4$									
2SIV	0.003	0.586	0.586	5.5	100.0	0.583	0.960	1.122	7.9	100.0
BC-IPC	−0.149	0.805	0.818	21.9	100.0	0.238	1.246	1.268	10.0	100.0
IPC	0.020	0.528	0.528	6.1	100.0	0.408	1.061	1.137	6.4	100.0
BC-PC	0.306	0.885	0.937	19.7	100.0	0.891	1.181	1.479	17.9	100.0
PC	−0.638	0.589	0.868	21.2	100.0	−0.081	0.969	0.973	4.5	100.0
CA	1.859	0.806	2.026	80.1	100.0	2.469	1.131	2.716	64.3	100.0
MGIV	0.000	0.593	0.592	5.1	100.0	0.014	0.958	0.958	4.2	100.0
MGPC	−0.650	0.595	0.882	21.5	100.0	−0.636	0.963	1.154	8.7	100.0
MGCA	1.623	0.722	1.776	72.4	100.0	1.693	1.064	1.999	38.3	100.0
	$\pi_u = 1/4$									
2SIV	−0.002	0.573	0.572	6.0	100.0	0.559	0.992	1.138	9.0	100.0
BC-IPC	−0.073	0.438	0.444	6.1	100.0	0.100	1.645	1.648	8.7	100.0
IPC	−0.073	0.437	0.443	6.3	100.0	0.107	1.645	1.648	8.8	100.0
BC-PC	2.786	2.520	3.756	72.4	100.0	3.446	2.785	4.430	65.8	100.0
PC	−0.638	0.576	0.859	20.2	100.0	−0.097	0.993	0.998	4.7	100.0
CA	2.083	0.920	2.278	84.4	100.0	2.645	1.229	2.916	69.0	100.0
MGIV	−0.002	0.582	0.582	5.4	100.0	−0.008	0.980	0.979	4.5	100.0
MGPC	−0.646	0.586	0.872	20.3	100.0	−0.649	0.983	1.177	9.5	100.0
MGCA	1.789	0.788	1.955	76.5	100.0	1.827	1.111	2.138	42.4	100.0

Table 2: Bias, root mean squared error (RMSE) of the estimators of β_1, and size and power of the associated t-tests when $\pi_u = 3/4$, $N = 200$, $T = 25$ and $N = 25$, $T = 200$.

Estimator	Homogeneous Slopes					Heterogeneous Slopes				
	Bias ($\times 100$)	S.D. ($\times 100$)	RMSE ($\times 100$)	Size	Power	Bias ($\times 100$)	S.D. ($\times 100$)	RMSE ($\times 100$)	Size	Power
	$N = 200$, $T = 25$									
2SIV	0.126	1.941	1.944	6.7	99.8	1.519	2.156	2.637	12.4	100.0
BC-IPC	−1.180	2.610	2.864	23.6	97.6	−0.070	2.911	2.911	17.1	98.1
IPC	0.374	1.870	1.906	8.7	99.9	1.301	2.234	2.585	12.9	100.0
BC-PC	0.825	2.746	2.867	12.7	99.8	2.185	2.842	3.584	20.9	100.0
PC	−0.211	2.756	2.763	11.6	99.7	1.145	2.842	3.063	12.6	99.8
CA	2.084	2.000	2.888	21.4	100.0	3.404	2.218	4.062	37.8	100.0
MGIV	0.482	2.534	2.578	9.9	99.4	0.606	2.687	2.754	10.8	99.6
MGPC	−0.414	2.554	2.587	9.0	99.0	−0.279	2.737	2.751	9.9	98.0
MGCA	1.850	2.127	2.819	15.9	100.0	1.914	2.334	3.018	14.8	100.0
	$N = 25$, $T = 200$									
2SIV	0.016	1.715	1.715	9.2	99.9	0.480	2.736	2.777	8.7	97.7
BC-IPC	−2.552	9.303	9.644	65.0	79.1	−2.679	10.032	10.381	51.4	69.5
IPC	0.639	2.883	2.953	14.8	98.2	0.939	3.885	3.996	13.2	91.1
BC-PC	2.547	5.525	6.083	29.5	95.7	2.910	6.102	6.759	24.5	87.7
PC	−5.703	2.103	6.078	82.5	57.8	−5.413	3.011	6.194	42.6	33.2
CA	5.971	3.267	6.805	64.3	100.0	6.277	4.086	7.489	39.9	99.7
MGIV	0.038	1.742	1.742	6.6	99.9	0.036	2.725	2.725	5.6	94.7
MGPC	−6.047	2.179	6.427	83.6	48.3	−5.997	3.018	6.713	48.3	26.5
MGCA	4.705	2.610	5.380	54.6	100.0	4.689	3.416	5.801	32.0	99.5

Table 3: Scaled bias of the estimators of β_1.

	Homogeneous Slopes ($\sqrt{NT}\times$Bias)				Heterogeneous Slopes ($\sqrt{N}\times$Bias)			
Estimator\\$N=T$	25	50	100	200	25	50	100	200
				$\pi_u=3/4$				
2SIV	0.162	0.044	0.015	0.005	0.094	0.068	0.075	0.082
BC-IPC	−0.142	−1.228	−0.771	−0.298	0.003	−0.148	−0.027	0.034
IPC	0.551	0.384	0.116	0.040	0.145	0.092	0.053	0.058
BC-PC	0.753	0.771	0.604	0.612	0.195	0.166	0.143	0.126
PC	−1.061	−1.390	−1.317	−1.277	−0.174	−0.137	−0.063	−0.011
CA	1.509	2.353	3.157	3.718	0.356	0.387	0.383	0.349
MGIV	0.258	0.072	0.025	−0.001	0.058	0.006	−0.001	0.002
MGPC	−1.229	−1.463	−1.351	−1.301	−0.240	−0.205	−0.138	−0.090
MGCA	1.228	1.891	2.604	3.245	0.256	0.266	0.262	0.239
				$\pi_u=1/4$				
2SIV	−0.037	−0.011	−0.015	−0.003	0.067	0.069	0.080	0.079
BC-IPC	−0.068	−0.031	−0.061	−0.146	0.003	0.004	0.013	0.014
IPC	−0.047	−0.018	−0.055	−0.146	0.008	0.007	0.015	0.015
BC-PC	7.650	6.461	5.771	5.571	1.698	0.962	0.668	0.487
PC	−1.643	−1.408	−1.332	−1.276	−0.267	−0.138	−0.054	−0.014
CA	2.157	3.261	3.875	4.167	0.484	0.507	0.459	0.374
MGIV	−0.009	0.025	−0.031	−0.003	0.009	−0.004	0.004	−0.001
MGPC	−1.740	−1.462	−1.376	−1.292	−0.339	−0.210	−0.129	−0.092
MGCA	1.610	2.425	3.036	3.578	0.335	0.339	0.314	0.258

Appendices: Proofs of the main theoretical results

In Appendices A-D, proof of main theoretical results with necessary Lemmas are provided. Proofs of used lemmas are available in Online Supplement.

Appendix A Lemmas and proof of Proposition 3.1

Throughout the appendix, we use C to denote a generic finite constant large enough, which need not to be the same at each appearance. Denote the projection matrix $\mathbf{P_A} = \mathbf{A}(\mathbf{A}'\mathbf{A})^{-1}\mathbf{A}'$ and the residual maker $\mathbf{M_A} = \mathbf{I} - \mathbf{P_A}$ for a matrix $\mathbf{A}$. Let $\mathbf{\Xi}$ be $r_1 \times r_1$ diagonal matrix that consist of the first r_1 largest eigenvalues of the $T \times T$ matrix $(NT)^{-1}\sum_{i=1}^{N}\mathbf{X}_i\mathbf{X}_i'$. Then by the definition of eigenvalues and $\widehat{\mathbf{F}}$, $\widehat{\mathbf{F}}\mathbf{\Xi} = (NT)^{-1}\sum_{i=1}^{N}\mathbf{X}_i\mathbf{X}_i'\widehat{\mathbf{F}}$. It's easy to show that $\mathbf{\Xi}$ is invertible following the proof of Lemma A.3 in Bai (2003). Then

$$\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{F}^0\mathbf{\Gamma}_i^0\mathbf{V}_i'\widehat{\mathbf{F}}\mathbf{\Xi}^{-1} + \frac{1}{NT}\sum_{i=1}^{N}\mathbf{V}_i\mathbf{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\widehat{\mathbf{F}}\mathbf{\Xi}^{-1} + \frac{1}{NT}\sum_{i=1}^{N}\mathbf{V}_i\mathbf{V}_i'\widehat{\mathbf{F}}\mathbf{\Xi}^{-1} \tag{A.1}$$

where $\mathbf{R} = (NT)^{-1}\sum_{i=1}^{N}\mathbf{\Gamma}_i^0\mathbf{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\widehat{\mathbf{F}}\mathbf{\Xi}^{-1}$. Following the proof of Lemma A.3 in Bai (2003) again, we can show that $\mathbf{R}$ is invertible.

Lemma A.1 *Under Assumptions B to D, we have*

(*a*) $T^{-1}\|\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}\|^2 = O_p(\delta_{NT}^{-2})$,

(*b*) $T^{-1}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\mathbf{F}^0 = O_p(\delta_{NT}^{-2})$, $T^{-1}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\mathbf{H}^0 = O_p(\delta_{NT}^{-2})$, $T^{-1}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\widehat{\mathbf{F}} = O_p(\delta_{NT}^{-2})$,

(*c*) $\mathbf{\Xi} = O_p(1)$, $\mathbf{R} = O_p(1)$, $\mathbf{\Xi}^{-1} = O_p(1)$, $\mathbf{R}^{-1} = O_p(1)$,

(*d*) $\mathbf{R}\mathbf{R}' - (T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1} = O_p(\delta_{NT}^{-2})$,

(*e*) $\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0} = O_p(\delta_{NT}^{-1})$,

(*f*) $N^{-1}T^{-1}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}) = O_p(N^{-1}) + O_p(N^{-1/2}\delta_{NT}^{-2})$,

Lemma A.2 *Under Assumptions A to D, we have*

$$(a)\quad N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|T^{-1}\boldsymbol{\varepsilon}_i'(\mathbf{F}^0 - \widehat{\mathbf{F}}\mathbf{R}^{-1})\| = O_p(\delta_{NT}^{-2})$$

$$(b)\quad N^{-1}\sum_{i=1}^{N}\|\boldsymbol{\varphi}_i^0\|\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\| = O_p(\delta_{NT}^{-2})$$

$$(c)\quad N^{-1}\sum_{i=1}^{N}\|T^{-1}\boldsymbol{\varepsilon}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\|\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\| = O_p(\delta_{NT}^{-4})$$

Lemma A.3 *Under Assumptions A to D, we have*

$$\begin{aligned}
&N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
=& -N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
&-N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0)^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i + O_p(N^{-1/2}T^{1/2}\delta_{NT}^{-2}) + O_p(\delta_{NT}^{-2})
\end{aligned}$$

Lemma A.4 *Under Assumptions A to D, we have*

$$
\begin{aligned}
& - N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
=& - N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i \\
& + N^{-5/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{V}_h\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i + O_p(T^{1/2}\delta_{NT}^{-2})
\end{aligned}
$$

Lemma A.5 *Under Assumptions A to D, we have*

$$
\begin{aligned}
& N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0)^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
=& N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i + O_p(T^{1/2}\delta_{NT}^{-2}) + O_p(N^{1/2}T^{-1/2}\delta_{NT}^{-1})
\end{aligned}
$$

Lemma A.6 *Under Assumptions A to D, we have*

$$
\begin{aligned}
& N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
=& N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i - N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\mathbf{V}_h\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i + O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3})
\end{aligned}
$$

Lemma A.7 *Under Assumptions A to D, we have*

$$
\begin{aligned}
& \frac{1}{NT}\sum_{i=1}^{N}\sum_{h=1}^{N}\boldsymbol{\mathcal{V}}_i'\mathbf{V}_h\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i) = \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h)\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0 \\
& - \frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{h=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbb{E}(\mathbf{V}_\ell'\mathbf{V}_h)\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0 + O_p(T^{-1/2}) \\
& \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i = \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{F}^0}\mathbf{H}^0\boldsymbol{\varphi}_i^0 + O_p(\delta_{NT}^{-1})
\end{aligned}
$$

which are $O_p(1)$, *where* $\boldsymbol{\mathcal{X}}_i = \mathbf{X}_i - N^{-1}\sum_{\ell=1}^{N}\mathbf{X}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_i^0$, $\boldsymbol{\mathcal{V}}_i = \mathbf{V}_i - N^{-1}\sum_{\ell=1}^{N}\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_i^0$, $\boldsymbol{\Upsilon}^0 = N^{-1}\sum_{i=1}^{N}\boldsymbol{\Gamma}_i^0\boldsymbol{\Gamma}_i^{0\prime}$ *and* $\boldsymbol{\Sigma} = N^{-1}\sum_{\ell=1}^{N}\mathbb{E}(\mathbf{V}_\ell\mathbf{V}_\ell')$.

Proof of Proposition 4.1. With Lemmas A.3, A.4, A.5, we have

$$
\begin{aligned}
& N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
=& - N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i \\
& + N^{-5/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{V}_h\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i \\
& - N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i + O_p(T^{1/2}\delta_{NT}^{-2}) + O_p(N^{1/2}T^{-1/2}\delta_{NT}^{-1}),
\end{aligned}
$$

and with Lemma A.6,

$$
\begin{aligned}
&N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
=&N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i - N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\mathbf{V}_h\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i + O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3})\,.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
&N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
=&N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\boldsymbol{\mathcal{X}}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i - N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\boldsymbol{\mathcal{V}}_i'\mathbf{V}_h\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i \\
&- N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i + O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3}) \\
=&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\boldsymbol{\mathcal{V}}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i + \sqrt{\frac{T}{N}}\mathbf{a}_1 + \sqrt{\frac{N}{T}}\mathbf{a}_2 + O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3})
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{a}_1 =& -\frac{1}{NT}\sum_{i=1}^{N}\sum_{h=1}^{N}\boldsymbol{\mathcal{V}}_i'\mathbf{V}_h\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i) \\
\mathbf{a}_2 =& -\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i
\end{aligned}
$$

with $\boldsymbol{\mathcal{V}}_i = \mathbf{V}_i - N^{-1}\sum_{\ell=1}^{N}\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_i^0$, $\boldsymbol{\Upsilon}^0 = N^{-1}\sum_{i=1}^{N}\boldsymbol{\Gamma}_i^0\boldsymbol{\Gamma}_i^{0\prime}$ and $\boldsymbol{\Sigma} = N^{-1}\sum_{\ell=1}^{N}\mathbb{E}(\mathbf{V}_\ell\mathbf{V}_\ell')$. Applying Lemma A.7 to $\mathbf{a}_1$ and $\mathbf{a}_2$, we can derive that $N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i = O_p(N^{-1/2}T^{1/2}) + O_p(N^{1/2}T^{-1/2}) + O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3})$ and

$$
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\boldsymbol{\mathcal{V}}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i + \mathbf{b}_1 + \mathbf{b}_2 + O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3})
$$

where

$$
\begin{aligned}
\mathbf{b}_1 =& -\sqrt{\frac{T}{N}}\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h)\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0 \\
&+\sqrt{\frac{T}{N}}\frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{h=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbb{E}(\mathbf{V}_\ell'\mathbf{V}_h)\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0 \\
\mathbf{b}_2 =& -\sqrt{\frac{N}{T}}\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{F}^0}\mathbf{H}^0\boldsymbol{\varphi}_i^0
\end{aligned}
$$

as required. With the facts that $N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{X}_i\|^2 = O_p(1)$, $\|\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0}\| = O_p(\delta_{NT}^{-1})$ and $N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{V}_i - N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{V}_i = O_p(T^{-1})$ we have

$$
N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i - N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{V}_i = O_p(\delta_{NT}^{-1}) + O_p(T^{-1})
$$

so that, with continuous mapping theorem, $\sqrt{NT}(\widehat{\boldsymbol{\beta}}_{1SIV} - \boldsymbol{\beta}) = O_p(1)$. This completes the proof. □

Appendix B Lemmas and proofs of Proposition 3.2 and Theorem 3.1

Let $\boldsymbol{\Xi}$ be $r_2 \times r_2$ diagonal matrix that consist of the first r_2 largest eigenvalues of the $T \times T$ matrix $N^{-1}T^{-1}\sum_{i=1}^{N}\widehat{\mathbf{u}}_i\widehat{\mathbf{u}}_i'$. Then by the definition of eigenvalues and $\widehat{\mathbf{H}}$, $\widehat{\mathbf{H}}\boldsymbol{\Xi} = N^{-1}T^{-1}\sum_{i=1}^{N}\widehat{\mathbf{u}}_i\widehat{\mathbf{u}}_i'\widehat{\mathbf{H}}$. It's easy to show that $\boldsymbol{\Xi}$ is invertible following the proof of Proposition A.1 (i) in Bai (2009a) given $\widehat{\boldsymbol{\beta}}_{1SIV} - \boldsymbol{\beta} = O_p(N^{-1/2}T^{-1/2})$. Then

$$
\begin{aligned}
&\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}} \\
=&N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{X}_i(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}_{1SIV})(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}_{1SIV})'\mathbf{X}_i'\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1} \\
&+ N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{X}_i(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}_{1SIV})\mathbf{u}_i'\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1} + N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{u}_i(\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}_{1SIV})'\mathbf{X}_i'\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1} \\
&+ N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{H}^0\boldsymbol{\varphi}_i^0\boldsymbol{\varepsilon}_i'\widehat{\mathbf{H}} + N^{-1}T^{-1}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i\boldsymbol{\varphi}_i^{0\prime}\mathbf{H}^{0\prime}\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1} + N^{-1}T^{-1}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i'\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1}
\end{aligned}
\tag{B.1}
$$

where $\boldsymbol{\mathcal{R}} = T^{-1}\boldsymbol{\Upsilon}_\varphi\mathbf{H}'\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1}$ with $\boldsymbol{\Upsilon}_\varphi^0 = N^{-1}\sum_{i=1}^{N}\boldsymbol{\varphi}_i^0\boldsymbol{\varphi}_i^{0\prime}$. Following the proof of Proposition A.1 (ii) in Bai (2009a), we can show that $\boldsymbol{\mathcal{R}}$ is invertible.

Lemma B.1 *Under Assumptions A to D, we have*

(a) $T^{-1}\|\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}\|^2 = O_p(\delta_{NT}^{-2})$,

(b) $T^{-1}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0 = O_p(\delta_{NT}^{-2})$, $T^{-1}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\widehat{\mathbf{H}} = O_p(\delta_{NT}^{-2})$,

(c) $\boldsymbol{\Xi} = O_p(1)$, $\boldsymbol{\mathcal{R}} = O_p(1)$, $\boldsymbol{\Xi}^{-1} = O_p(1)$, $\boldsymbol{\mathcal{R}}^{-1} = O_p(1)$.

(d) $\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - (T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1} = O_p(\delta_{NT}^{-2})$,

(e) $\mathbf{M}_{\widehat{\mathbf{H}}} - \mathbf{M}_{\mathbf{H}^0} = O_p(\delta_{NT}^{-1})$,

(f) $N^{-1}T^{-1}\sum_{\ell=1}^{N}\boldsymbol{\varphi}_\ell\boldsymbol{\varepsilon}_\ell'(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}) = O_p(N^{-1}) + O_p(N^{-1/2}\delta_{NT}^{-2})$.

Lemma B.2 *Under Assumptions A to D, we have*

(a) $N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i\|\|T^{-1}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i\| = O_p(\delta_{NT}^{-2})$,

(b) $N^{-1}\sum_{i=1}^{N}\|\boldsymbol{\varphi}_i^0\|\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})\| = O_p(\delta_{NT}^{-2})$,

(c) $N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})\|\|T^{-1}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i\| = O_p(\delta_{NT}^{-4})$.

Lemma B.3 *Under Assumptions A to D, we have*

$$
\begin{aligned}
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i =& \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\
&- N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0 + O_p(\delta_{NT}^{-1})
\end{aligned}
$$

Lemma B.4 *Under Assumptions A to D, we have*

$$N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i$$
$$= - N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i$$
$$- N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0)^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i + O_p(\delta_{NT}^{-2}) + O_p(N^{-1/2}T^{1/2}\delta_{NT}^{-2})$$

Lemma B.5 *Under Assumptions A to D, we have*

$$N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0)^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i = O_p(T^{1/2}\delta_{NT}^{-2}) + O_p(N^{1/2}T^{-1/2}\delta_{NT}^{-1})$$

Lemma B.6 *Under Assumptions A to D, we have*

$$-\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i = -\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i^0$$
$$-N^{-5/2}T^{-1/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}(\mathbf{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0$$
$$- N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{\Sigma}_\varepsilon\mathbf{H}(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}(\mathbf{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0$$
$$+O_p(T^{1/2}\delta_{NT}^{-2})$$

Proof of Proposition 3.2. By Lemmas B.3, B.4, B.5 and B.6 and the fact that $\mathbf{M}_{\mathbf{H}^0}\mathbf{X}_i = \mathbf{M}_{\mathbf{H}^0}\mathbf{V}_i$, we can derive that

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'\boldsymbol{\varepsilon}_i + \mathbf{b}_{0FH} + \mathbf{b}_{1FH} + \mathbf{b}_{2FH} + o_p(1)$$

with

$$\begin{aligned}
\mathbf{b}_{0FH} &= -\frac{1}{N^{1/2}}\frac{1}{NT^{1/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}(\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_j^0 + \boldsymbol{\varphi}_j^{0\prime}(\mathbf{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0)\mathbf{V}_j'\boldsymbol{\varepsilon}_i \\
\mathbf{b}_{1FH} &= -\frac{1}{N^{1/2}}\frac{1}{N^2T^{1/2}}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{j=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\left(\mathbf{V}_\ell'\boldsymbol{\varepsilon}_j\right)\boldsymbol{\varphi}_j^{0\prime}(\mathbf{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0 \\
\mathbf{b}_{2FH} &= -\frac{1}{T^{1/2}}\frac{1}{N^{3/2}T}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{\Gamma}_i^{0\prime}\mathbf{\Upsilon}^{-1}\mathbf{\Gamma}_j^0\mathbf{V}_j'\mathbf{\Sigma}_\varepsilon\mathbf{H}^0\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}(\mathbf{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0,
\end{aligned}$$

$\mathbf{\Sigma}_\varepsilon = \frac{1}{N}\sum_{j=1}^{N}\mathbb{E}\left(\boldsymbol{\varepsilon}_j\boldsymbol{\varepsilon}_j'\right)$, where $\mathbf{b}_{0FH} = O_p\left(N^{-1/2}\right)$, $\mathbf{b}_{1FH} = O_p\left(N^{-1/2}\right)$ and $\mathbf{b}_{2FH} = O_p\left(T^{-1/2}\right)$. Hence, we have

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'\boldsymbol{\varepsilon}_i + o_p(1).$$

In addition, it is easily shown that

$$\frac{1}{NT}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{X}_i - \frac{1}{NT}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{V}_i = O_p(\delta_{NT}^{-1})$$

This completes the proof.□

Proof of Theorem 3.1. By Proposition 3.2 we have

$$\sqrt{NT}(\widehat{\boldsymbol{\beta}}_{2SIV}-\boldsymbol{\beta})=(N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{V}_i)^{-1}\cdot N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{V}_i'\boldsymbol{\varepsilon}_i+O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3}).$$

$N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{V}_i \xrightarrow{p} \mathbf{A}$ and $N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{V}_i'\boldsymbol{\varepsilon}_i \xrightarrow{d} N(\mathbf{0},\mathbf{B})$ by Assumption F, together with continuous mapping theorem yield the required result. □

Appendix C Lemmas and proofs of Proposition 4.1 and Theorem 4.1

Lemma C.1 *Under Assumptions A to D, we have*

$$(a)\quad \|T^{-1}\boldsymbol{\varepsilon}_i'(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1})\|=O_p(\delta_{NT}^{-2})$$
$$(b)\quad \|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|=O_p(\delta_{NT}^{-2})$$

Lemma C.2 *Under Assumptions A-E and G-H, we have*

$$T^{-1/2}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i=T^{-1/2}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i+O_p(T^{1/2}\delta_{NT}^{-2}).$$

Proof of Proposition 4.1. It's easy to show that

$$\|T^{-1}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i-T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i\|\le\|T^{-1/2}\mathbf{X}_i\|^2\|\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\|=O_p(\delta_{NT}^{-1})$$

with Lemma A.1(f), we have $\hat{\boldsymbol{\beta}}_i-\boldsymbol{\beta}_i=O_p(T^{-1/2})+O_p(\delta_{NT}^{-2})$, we can derive that

$$\begin{aligned}\sqrt{T}(\widehat{\boldsymbol{\beta}}_i-\boldsymbol{\beta}_i)&=\left(T^{-1}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i\right)^{-1}\times T^{-1/2}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\\&=\left(T^{-1}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i\right)^{-1}\times T^{-1/2}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i+O_p\left(T^{1/2}\delta_{NT}^{-2}\right)\\&=\left(T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i\right)^{-1}\times T^{-1/2}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i+O_p\left(\delta_{NT}^{-1}\right)+O_p\left(T^{1/2}\delta_{NT}^{-2}\right)\end{aligned}$$

which implies that Theorem. As $\tilde{\mathbf{u}}_i=\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i$, $T^{-1/2}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i=T^{-1/2}\mathbf{V}_i'\tilde{\mathbf{u}}_i=T^{-1/2}\sum_{t=1}^{T}\mathbf{v}_{it}\tilde{u}_{it}$, and the term

$$T^{-1/2}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i\xrightarrow{d}\mathrm{N}(\mathbf{0},\boldsymbol{\Omega}_i)$$

where $\boldsymbol{\Omega}_i=T^{-1}\mathrm{plim}_{T\to\infty}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{u}_{is}\tilde{u}_{it}\mathbb{E}(\mathbf{v}_{is}\mathbf{v}_{it}')$. This completes the proof. □

Lemma C.3 *Under Assumptions A-E and G-H, we have*

$$(a)\quad \sup_{1\le i\le N}\|\boldsymbol{\Gamma}_i^0\|=O_p(N^{1/4})$$

$$(b)\quad \sup_{1\le i\le N}\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\|=\sup_{1\le i\le N}\|N^{-1/2}T^{-1/2}\sum_{t=1}^{T}\sum_{\ell=1}^{N}\varepsilon_{it}\mathbf{v}_{\ell t}'\boldsymbol{\Gamma}_\ell^{0\prime}\|=O_p(N^{1/4})$$

$$(c)\quad \sup_{1\le i\le N}\|N^{-1}T^{-1}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{F}^0\|=O_p(N^{1/4})$$

$$(d)\quad \sup_{1\le i\le N}N^{-1/2}T^{-1/2}\|\sum_{\ell=1}^{N}\left(\mathbf{V}_i'\mathbf{V}_\ell-\mathbb{E}\big(\mathbf{V}_i'\mathbf{V}_\ell\big)\right)\boldsymbol{\Gamma}_\ell^{0\prime}\|=O_p(N^{1/4})$$

Lemma C.4 *Under Assumptions A-E and G-H, we have*

$$(a)\quad \sup_{1\le i\le N}\|T^{-1}\boldsymbol{\varepsilon}_i'(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1})\|=O_p(\delta_{NT}^{-2})+O_p(N^{1/4}T^{-1})+O_p(N^{-1/4}T^{-1/2})$$

$$(b)\quad \sup_{1\le i\le N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|=O_p(N^{1/4}\delta_{NT}^{-2})$$

Lemma C.5 *Under Assumptions A-E and G-H, we have*

$$(a)\quad N^{-1/2}T^{-1}\sum_{i=1}^{N}\|\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i-\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i\|=O_p(N^{1/2}\delta_{NT}^{-2}).$$

$$(b)\quad \sup_{1\le i\le N}\|T^{-1}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i-T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i\|=O_p(N^{1/2}\delta_{NT}^{-2}).$$

Proof of Theorem 4.1. Under Assumptions A-E and G-H, we have

$$\sqrt{N}(\widehat{\boldsymbol{\beta}}_{MGIV}-\boldsymbol{\beta})=N^{-1/2}\sum_{i=1}^{N}(\widehat{\boldsymbol{\beta}}_i-\boldsymbol{\beta})=N^{-1/2}\sum_{i=1}^{N}(\widehat{\boldsymbol{\beta}}_i-\boldsymbol{\beta}_i)+N^{-1/2}\sum_{i=1}^{N}\mathbf{e}_i$$

where

$$\begin{aligned}
&N^{-1/2}\sum_{i=1}^{N}(\widehat{\boldsymbol{\beta}}_i-\boldsymbol{\beta}_i)=N^{-1/2}\sum_{i=1}^{N}(\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\\
=&N^{-1/2}\sum_{i=1}^{N}(\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i+N^{-1/2}\sum_{i=1}^{N}\left[(\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i-(\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i\right]\\
=&N^{-1/2}\sum_{i=1}^{N}(\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i+N^{-1/2}\sum_{i=1}^{N}\left[(\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i)^{-1}-(\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}\right]\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i\\
&+N^{-1/2}\sum_{i=1}^{N}\left[(\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i)^{-1}-(\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}\right](\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i-\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i)\\
&+N^{-1/2}\sum_{i=1}^{N}(\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}(\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i-\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i)\\
=&\mathbb{D}_1+\mathbb{D}_2+\mathbb{D}_3+\mathbb{D}_4
\end{aligned}$$

We first consider the terms $\mathbb{D}_2$, $\mathbb{D}_3$, and $\mathbb{D}_4$. Since

$$\begin{aligned}
&T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i=T^{-1}\mathbf{V}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{V}_i=T^{-1}\mathbf{V}_i'\mathbf{V}_i-T^{-1}\mathbf{V}_i'\mathbf{F}^0(\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{V}_i\\
=&T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)+(T^{-1}\mathbf{V}_i'\mathbf{V}_i-T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i))-T^{-1}\mathbf{V}_i'\mathbf{F}^0(\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{V}_i
\end{aligned}$$

we have

$$\begin{aligned}
&\sup_{1\le i\le N}\|T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i-T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)\|\\
=&T^{-1/2}\sup_{1\le i\le N}\|T^{-1/2}(\mathbf{V}_i'\mathbf{V}_i-\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i))\|+\sup_{1\le i\le N}\|T^{-1/2}\mathbf{V}_i'\mathbf{F}^0\|^2\|(\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|\\
=&O_p(N^{1/4}T^{-1/2})+O_p(N^{1/2}T^{-1}).
\end{aligned}$$

Furthermore, since

$$\begin{aligned}
&(T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}-[T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}\\
=&(T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}[T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)-T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i][T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}\\
=&[T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}[T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)-T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i][T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}\\
&+[(T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}-[T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}][T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)-T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i][T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}
\end{aligned}$$

we have

$$\begin{aligned}
&\sup_{1\le i\le N}\|(T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}-[T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}\|\\
\le&\sup_{1\le i\le N}\|[T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}\|^2\sup_{1\le i\le N}\|T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)-T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i]\|\\
&+\sup_{1\le i\le N}\|[(T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}-[T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}]\|\sup_{1\le i\le N}\|T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)-T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i\|\sup_{1\le i\le N}\|[T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}\|\\
=&\sup_{1\le i\le N}\|T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)-T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i]\|\cdot C^2\\
&+\sup_{1\le i\le N}\|[(T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}-[T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}]\|\cdot[O_p(N^{1/4}T^{-1/2})+O_p(N^{1/2}T^{-1})]\cdot C
\end{aligned}$$

Since $\sqrt{N}/T\to 0$, we can see that the second term is $\sup_{1\le i\le N}\|[(T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}-[T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}]\|\cdot o_p(1)$, which means that the first term dominates the second term, thus

$$\sup_{1\le i\le N}\|(T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}-[T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}\|=O_p(N^{1/4}T^{-1/2})+O_p(N^{1/2}T^{-1})\tag{C.1}$$

and

$$\sup_{1\le i\le N} \|(T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}\| = O_p(1) \tag{C.2}$$

as $\sup_{1\le i\le N} \|[T^{-1}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}\| \le C_{min}^{-1} \le C$. Similarly, by Lemma C.5 (b), we can show that

$$\sup_{1\le i\le N} \|(T^{-1}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i)^{-1} - (T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}\| = O_p(N^{1/2}\delta_{NT}^{-2}) \tag{C.3}$$

With the above facts, we have

$$\begin{aligned}
\|\mathbb{D}_2\| \le& N^{-1/2}\sum_{i=1}^N \|T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i\| \cdot \sup_{1\le i\le N} \|(T^{-1}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i)^{-1} - (T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}\| \\
=& O_p(NT^{-1/2}\delta_{NT}^{-2}) \\
\|\mathbb{D}_3\| \le& N^{-1/2}T^{-1}\sum_{i=1}^N \|\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i - \mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i\| \cdot \sup_{1\le i\le N} \|(T^{-1}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i)^{-1} - (T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}\| \\
=& O_p(N\delta_{NT}^{-4})
\end{aligned}$$

and

$$\|\mathbb{D}_4\| \le N^{-1/2}T^{-1}\sum_{i=1}^N \|\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i - \mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i\| \cdot \sup_{1\le i\le N} \|(T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}\| = O_p(N^{1/2}\delta_{NT}^{-2}).$$

Consider $\mathbb{D}_1$. Since

$$\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i = \mathbf{V}_i'\mathbf{H}^0\boldsymbol{\varphi}_i^0 - \mathbf{V}_i'\mathbf{F}^0(\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0 + \mathbf{V}_i'\boldsymbol{\varepsilon}_i - \mathbf{V}_i'\mathbf{F}^0(\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i$$

we have

$$\begin{aligned}
\mathbb{D}_1 =& N^{-1/2}\sum_{i=1}^N (\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i \\
=& N^{-1/2}\sum_{i=1}^N [(\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i)^{-1} - (\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i))^{-1}]\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i + N^{-1/2}\sum_{i=1}^N [\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}\mathbf{V}_i'\mathbf{H}^0\boldsymbol{\varphi}_i^0 \\
&- N^{-1/2}\sum_{i=1}^N [\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}]\mathbf{V}_i'\mathbf{F}^0(\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0 + N^{-1/2}\sum_{i=1}^N [\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}\mathbf{V}_i'\boldsymbol{\varepsilon}_i \\
&- N^{-1/2}\sum_{i=1}^N [\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}\mathbf{V}_i'\mathbf{F}^0(\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i
\end{aligned}$$

With (C.1), we can show that the first term is $O_p(N^{3/4}T^{-1}) + O_p(NT^{-3/2})$. It's easy to show that the last term is $O_p(N^{1/2}T^{-1})$. For the second term, we have

$$\begin{aligned}
&\mathbb{E}\|N^{-1/2}\sum_{i=1}^N [\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}\mathbf{V}_i'\mathbf{H}^0\boldsymbol{\varphi}_i^0\|^2 = \mathbb{E}\|N^{-1/2}\sum_{i=1}^N\sum_{s=1}^T [\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}\mathbf{v}_{is}\mathbf{h}_s^{0\prime}\boldsymbol{\varphi}_i^0\|^2 \\
=& \mathrm{tr}\Big(N^{-1/2}\sum_{i=1}^N\sum_{j=1}^N\sum_{s=1}^T\sum_{t=1}^T [\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}\mathbb{E}(\mathbf{v}_{is}\mathbf{v}_{jt}')\mathbb{E}(\mathbf{h}_s^{0\prime}\boldsymbol{\varphi}_i^0\boldsymbol{\varphi}_j^{0\prime}\mathbf{h}_t^0)[\mathbb{E}(\mathbf{V}_j'\mathbf{V}_j)]^{-1}\Big) \\
\le& CN^{-1}\sum_{i=1}^N\sum_{j=1}^N\sum_{s=1}^T\sum_{t=1}^T \Big\|[\mathbb{E}(\mathbf{V}_i'\mathbf{V}_i)]^{-1}\Big\|\Big\|[\mathbb{E}(\mathbf{V}_j'\mathbf{V}_j)]^{-1}\Big\|\|\boldsymbol{\Sigma}_{ij,st}\|\mathbb{E}(\|\mathbf{h}_s^0\|\|\mathbf{h}_t^0\|)\mathbb{E}(\|\boldsymbol{\varphi}_i^0\|\|\boldsymbol{\varphi}_j^0\|) \\
\le& CN^{-1}T^{-2}\sum_{i=1}^N\sum_{j=1}^N\sum_{s=1}^T\sum_{t=1}^T \|\boldsymbol{\Sigma}_{ij,st}\| \le CT^{-1}
\end{aligned}$$

by Lemma B.2. then the second term is $O_p(T^{-1/2})$. Analogously, the third term can be proved to be $O_p(T^{-1/2})$. Thus, $\mathbb{D}_1 = O_p(N^{3/4}T^{-1}) + O_p(NT^{-3/2}) + O_p(T^{-1/2})$.

Combining the above terms, we have

$$N^{-1/2}\sum_{i=1}^{N}(\widehat{\boldsymbol{\beta}}_i-\boldsymbol{\beta}_i)=O_p(N^{3/4}T^{-1})+O_p(NT^{-3/2})+O_p(N^{1/2}\delta_{NT}^{-2}).$$

Consequently, we obtain

$$\sqrt{N}(\widehat{\boldsymbol{\beta}}_{MGIV}-\boldsymbol{\beta})=N^{-1/2}\sum_{i=1}^{N}\mathbf{e}_i+o_p(1)$$

and by a standard central limit theorem

$$\sqrt{N}(\widehat{\boldsymbol{\beta}}_{MGIV}-\boldsymbol{\beta})\xrightarrow{d}N(\mathbf{0},\boldsymbol{\Sigma}_{\boldsymbol{\beta}}).$$

Finally consider $\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\beta}}$. Since $\widehat{\boldsymbol{\beta}}_i-\widehat{\boldsymbol{\beta}}_{MGIV}=\widehat{\boldsymbol{\beta}}_i-\boldsymbol{\beta}_i-\left(\widehat{\boldsymbol{\beta}}_{MGIV}-\boldsymbol{\beta}\right)+\mathbf{e}_i$, we have

$$\begin{aligned}
\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\beta}} &= \frac{1}{N-1}\sum_{i=1}^{N}\left(\widehat{\boldsymbol{\beta}}_i-\widehat{\boldsymbol{\beta}}_{MGIV}\right)\left(\widehat{\boldsymbol{\beta}}_i-\widehat{\boldsymbol{\beta}}_{MGIV}\right)' \\
&= \frac{1}{N-1}\sum_{i=1}^{N}\mathbf{e}_i\mathbf{e}_i'+\frac{1}{N-1}\sum_{i=1}^{N}\left(\widehat{\boldsymbol{\beta}}_i-\boldsymbol{\beta}_i\right)\left(\widehat{\boldsymbol{\beta}}_i-\boldsymbol{\beta}_i\right)'+\frac{1}{N-1}\sum_{i=1}^{N}\left(\widehat{\boldsymbol{\beta}}_{MGIV}-\boldsymbol{\beta}\right)\left(\widehat{\boldsymbol{\beta}}_{MGIV}-\boldsymbol{\beta}\right)' \\
&\quad -\frac{1}{N-1}\sum_{i=1}^{N}\left(\widehat{\boldsymbol{\beta}}_i-\boldsymbol{\beta}_i\right)\mathbf{e}_i'-\frac{1}{N-1}\sum_{i=1}^{N}\mathbf{e}_i\left(\widehat{\boldsymbol{\beta}}_i-\boldsymbol{\beta}_i\right)' \\
&\quad -\frac{1}{N-1}\sum_{i=1}^{N}\left(\widehat{\boldsymbol{\beta}}_{MGIV}-\boldsymbol{\beta}\right)\mathbf{e}_i'-\frac{1}{N-1}\sum_{i=1}^{N}\mathbf{e}_i'\left(\widehat{\boldsymbol{\beta}}_{MGIV}-\boldsymbol{\beta}\right)' \\
&\quad -\frac{1}{N-1}\sum_{i=1}^{N}\left(\widehat{\boldsymbol{\beta}}_i-\boldsymbol{\beta}_i\right)\left(\widehat{\boldsymbol{\beta}}_{MGIV}-\boldsymbol{\beta}\right)'-\frac{1}{N-1}\sum_{i=1}^{N}\left(\widehat{\boldsymbol{\beta}}_{MGIV}-\boldsymbol{\beta}\right)\left(\widehat{\boldsymbol{\beta}}_i-\boldsymbol{\beta}_i\right)' \\
&= \mathbb{J}_1+...+\mathbb{J}_9.
\end{aligned}$$

Since $\widehat{\boldsymbol{\beta}}_i-\boldsymbol{\beta}_i=O_p\left(T^{-1/2}\right)$ and $\widehat{\boldsymbol{\beta}}_{MGIV}-\boldsymbol{\beta}=O_p\left(N^{-1/2}\right)$, $\mathbb{J}_2=O_p\left(T^{-1}\right)$, $\mathbb{J}_3=O_p\left(N^{-1}\right)$, $\mathbb{J}_8=O_p\left(N^{-1/2}T^{-1/2}\right)$ and $\mathbb{J}_9=O_p\left(N^{-1/2}T^{-1/2}\right)$. Next, as $\frac{1}{\sqrt{N-1}}\sum_{i=1}^{N}\mathbf{e}_i\xrightarrow{d}N\left(\mathbf{0},\boldsymbol{\Sigma}_{\boldsymbol{\beta}}\right)$, $\mathbb{J}_6=O_p\left(N^{-1/2}\right)$ and $\mathbb{J}_7=O_p\left(N^{-1/2}\right)$. Noting that $\mathbf{e}_i$ is independent of $\mathbf{X}_i$ and $\mathbf{u}_i$, a similar argument for Theorem 4.1 yields that

$$\begin{aligned}
\frac{1}{\sqrt{N-1}}\sum_{i=1}^{N}\left(\widehat{\boldsymbol{\beta}}_i-\boldsymbol{\beta}_i\right)\mathbf{e}_i' &= \frac{1}{\sqrt{N-1}}\sum_{i=1}^{N}\left(T^{-1}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i\right)^{-1}T^{-1}\mathbf{X}_i\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\mathbf{e}_i' \\
&= \frac{1}{\sqrt{N-1}}\sum_{i=1}^{N}\left[T^{-1}\mathbb{E}\left(\mathbf{V}_i'\mathbf{V}_i\right)\right]^{-1}T^{-1}\mathbf{V}_i'\mathbf{u}_i\mathbf{e}_i' \\
&= O_p\left(N^{3/4}T^{-1}\right)+O_p\left(NT^{-3/2}\right)+O_p\left(N^{1/2}\delta_{NT}^{-2}\right),
\end{aligned}$$

thus, $\mathbb{J}_4$ and $\mathbb{J}_5$ are $O_p\left(N^{1/4}T^{-1}\right)+O_p\left(N^{1/2}T^{-3/2}\right)+O_p\left(\delta_{NT}^{-2}\right)$. Combining all, we have $\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\beta}}=\frac{1}{N-1}\sum_{i=1}^{N}\mathbf{e}_i\mathbf{e}_i'+o_p\left(1\right)$ so long as $NT^{-3}\to 0$ as $N,T\to\infty$. As $\frac{1}{N-1}\sum_{i=1}^{N}\mathbf{e}_i\mathbf{e}_i'-\boldsymbol{\Sigma}_{\boldsymbol{\beta}}=O_p\left(N^{-1/2}\right)$, $\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{\beta}}-\boldsymbol{\Sigma}_{\boldsymbol{\beta}}\xrightarrow{p}\mathbf{0}$ as required.□

Appendix D Lemmas and Proofs of Propositions 5.1 and 5.2

Lemma D.1 *Under Assumptions A-E*

$$\begin{aligned}
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{H}^0\boldsymbol{\varphi}_i &= -\frac{1}{N^{3/2}T^{1/2}}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ij}\mathbf{X}_j'\mathbf{M}_{\widehat{\mathbf{H}}}\boldsymbol{\varepsilon}_i \\
&\quad -\frac{1}{N^{1/2}T^{3/2}}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\boldsymbol{\Sigma}_{\varepsilon}\widehat{\mathbf{H}}\left(\frac{\mathbf{H}'\widehat{\mathbf{H}}}{\boldsymbol{T}}\right)^{-1}(\boldsymbol{\Upsilon}_{\varphi}^{0})^{-1}\boldsymbol{\varphi}_i^0
\end{aligned}$$

$$+ O_p\left(\sqrt{NT}\delta_{NT}^{-3}\right)$$

where $a_{ij} = \boldsymbol{\varphi}_j^{0\prime}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0$ *and* $\boldsymbol{\Sigma}_\varepsilon = \frac{1}{N}\sum_{j=1}^N \mathbb{E}\left(\boldsymbol{\varepsilon}_j\boldsymbol{\varepsilon}_j'\right)$.

Lemma D.2 *Under Assumptions A-E*

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^N \boldsymbol{\mathcal{X}}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\boldsymbol{\varepsilon}_i = \frac{1}{\sqrt{NT}}\sum_{i=1}^N \boldsymbol{\mathcal{X}}_i'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i - \sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N \frac{\boldsymbol{\mathcal{X}}_i'\mathbf{H}^0}{T}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_j^0\mathbb{E}\left(\boldsymbol{\varepsilon}_j'\boldsymbol{\varepsilon}_i/T\right) + O_p\left(\sqrt{NT}\delta_{NT}^{-3}\right)$$

where $\boldsymbol{\mathcal{X}}_i = \mathbf{X}_i - N^{-1}\sum_{j=1}^N a_{ij}\mathbf{X}_j'$.

Lemma D.3 *Under Assumptions A-E*

$$\frac{1}{N^{1/2}T^{3/2}}\sum_{i=1}^N \mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\boldsymbol{\Sigma}_\varepsilon\widehat{\mathbf{H}}\left(\frac{\mathbf{H}^{0\prime}\widehat{\mathbf{H}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0 = \sqrt{\frac{N}{T}}\frac{1}{NT}\sum_{i=1}^N \mathbf{X}_i'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\Sigma}_\varepsilon\mathbf{H}^0\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0 + O_p\left(\sqrt{NT}\delta_{NT}^{-3}\right).$$

Proof of Proposition 5.1 Under Assumptions A-E, by Lemmas D.1, D.2 and D.3, we have

$$\begin{aligned}\frac{1}{\sqrt{NT}}\sum_{i=1}^N \mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i &= \frac{1}{\sqrt{NT}}\sum_{i=1}^N \mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{H}^0\boldsymbol{\varphi}_i^0 + \frac{1}{\sqrt{NT}}\sum_{i=1}^N \mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{H}}}\boldsymbol{\varepsilon}_i \\ &= \frac{1}{\sqrt{NT}}\sum_{i=1}^N \boldsymbol{\mathcal{X}}_i'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\ &\quad - \sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^N\sum_{j=1}^N \frac{\boldsymbol{\mathcal{X}}_i'\mathbf{H}^0}{T}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_j^0\mathbb{E}\left(\boldsymbol{\varepsilon}_j'\boldsymbol{\varepsilon}_i/T\right) \\ &\quad - \sqrt{\frac{N}{T}}\frac{1}{NT}\sum_{i=1}^N \mathbf{X}_i'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\Sigma}_\varepsilon\mathbf{H}^0\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0 + O_p\left(\sqrt{NT}\delta_{NT}^{-3}\right)\end{aligned}$$

as required.□

Proof of Proposition 5.2 Under Assumptions A-E, following Westerlund and Urbain (2015) we have

$$N^{-1/2}T^{-1/2}\sum_{i=1}^N \mathbf{V}_i'\mathbf{M}_{\widehat{\mathbf{G}}_z}\boldsymbol{\varepsilon}_i = N^{-1/2}T^{-1/2}\sum_{i=1}^N \mathbf{V}_i'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i + O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3})$$

$$N^{-1/2}T^{-1/2}\sum_{i=1}^N \mathbf{V}_i'\mathbf{M}_{\widehat{\mathbf{G}}_z}\mathbf{H}^0\boldsymbol{\varphi}_i^0 = -N^{-3/2}T^{-1/2}\sum_{i=1}^N\sum_{h=1}^N \mathbf{V}_i'\mathbf{U}_h\boldsymbol{\Lambda}_h^{0\prime}(\boldsymbol{\Upsilon}_\Lambda^0)^{-1}\boldsymbol{\varphi}_i^0 + O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3})$$

$$N^{-1/2}T^{-1/2}\sum_{i=1}^N \boldsymbol{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{G}}_z}\boldsymbol{\varepsilon}_i = -N^{-3/2}T^{-1/2}\sum_{i=1}^N\sum_{j=1}^N [\boldsymbol{\Gamma}_i^{0\prime}, \mathbf{0}'](\boldsymbol{\Upsilon}_\Lambda^0)^{-1}\boldsymbol{\Lambda}_j^0\mathbb{E}(\mathbf{U}_j'\boldsymbol{\varepsilon}_i) + O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3})$$

$$N^{-1/2}T^{-1/2}\sum_{i=1}^N \boldsymbol{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{G}}_z}\mathbf{H}^0\boldsymbol{\varphi}_i^0 = N^{-5/2}T^{-1/2}\sum_{i=1}^N\sum_{j=1}^N\sum_{h=1}^N [\boldsymbol{\Gamma}_i^{0\prime}, \mathbf{0}'](\boldsymbol{\Upsilon}_\Lambda^0)^{-1}\boldsymbol{\Lambda}_j^0\mathbb{E}(\mathbf{U}_j'\mathbf{U}_h)\boldsymbol{\Lambda}_h'(\boldsymbol{\Upsilon}_\Lambda^0)^{-1}\boldsymbol{\varphi}_i^0$$
$$+ O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3})$$

where $\boldsymbol{\Upsilon}_\Lambda^0 = N^{-1}\sum_{i=1}^N \boldsymbol{\Lambda}_i^0\boldsymbol{\Lambda}_i^{0\prime}$, and

$$N^{-1/2}T^{-1/2}\sum_{i=1}^N \mathbf{V}_i'\mathbf{M}_{\bar{\mathbf{Z}}}\boldsymbol{\varepsilon}_i = N^{-1/2}T^{-1/2}\sum_{i=1}^N \mathbf{V}_i'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i + O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3})$$

$$N^{-1/2}T^{-1/2}\sum_{i=1}^N \mathbf{V}_i'\mathbf{M}_{\bar{\mathbf{Z}}}\mathbf{H}^0\boldsymbol{\varphi}_i^0 = -N^{-1/2}T^{-1/2}\sum_{i=1}^N \mathbf{V}_i'\bar{\mathbf{U}}\bar{\boldsymbol{\Lambda}}^-\boldsymbol{\varphi}_i^0 + O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3})$$

$$N^{-1/2}T^{-1/2}\sum_{i=1}^N \boldsymbol{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\bar{\mathbf{Z}}}\boldsymbol{\varepsilon}_i = -N^{-1/2}T^{-1/2}\sum_{i=1}^N [\boldsymbol{\Gamma}_i^{0\prime},\mathbf{0}']\bar{\boldsymbol{\Lambda}}^{-\prime}\mathbb{E}(\bar{\mathbf{U}}'\boldsymbol{\varepsilon}_i) + O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3})$$

$$N^{-1/2}T^{-1/2}\sum_{i=1}^N \boldsymbol{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\bar{\mathbf{Z}}}\mathbf{H}^0\boldsymbol{\varphi}_i^0 = N^{-1/2}T^{-1/2}\sum_{i=1}^N [\boldsymbol{\Gamma}_i^{0\prime},\mathbf{0}']\bar{\boldsymbol{\Lambda}}^{-\prime}\mathbb{E}(\bar{\mathbf{U}}'\bar{\mathbf{U}})\bar{\boldsymbol{\Lambda}}^-\boldsymbol{\varphi}_i^0 + O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3})$$

where $\bar{\mathbf{U}} = N^{-1}\sum_{i=1}^N \mathbf{U}_i$ and $\bar{\boldsymbol{\Lambda}}^- = \bar{\boldsymbol{\Lambda}}'(\bar{\boldsymbol{\Lambda}}\bar{\boldsymbol{\Lambda}}')^{-1}$ with $\bar{\boldsymbol{\Lambda}} = N^{-1}\sum_{i=1}^N \boldsymbol{\Lambda}_i^0$.

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1 Proofs of Lemmas

A Proofs of Lemmas in Appendix A

Proof of Lemma A.1. For the proofs of (a) to (d), and (f), see Proof of Lemma 4 in Supplemental Material, Norkutė et al. (2020). For (e), we decompose the left hand side term as

$$\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}=-T^{-1}\widehat{\mathbf{F}}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'-T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}-T^{-1}\mathbf{F}^0\Big(\mathbf{R}\mathbf{R}'-(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\Big)\mathbf{F}^{0\prime}$$

then it will be bounded in norm by

$$\begin{aligned}&\|T^{-1/2}\widehat{\mathbf{F}}\|\|T^{-1/2}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|+\|\mathbf{R}\|\|T^{-1/2}\mathbf{F}^0\|\|T^{-1/2}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|+\|T^{-1/2}\mathbf{F}^0\|^2\|\mathbf{R}\mathbf{R}'-(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|\\ =&O_p(\delta_{NT}^{-1})\end{aligned}$$

with (a), (c), (d) and the facts that $\|T^{-1/2}\widehat{\mathbf{F}}\|^2=r_1$ and $\mathbb{E}\|T^{-1/2}\mathbf{F}^0\|^2\le C$ by Assumption C. This completes the proof. □

Proof of Lemma A.2. Consider (a). With the equation (A.1), we have

$$\begin{aligned}&N^{-1}\sum_{i=1}^N\left\|\boldsymbol{\Gamma}_i^0\right\|\|T^{-1}\boldsymbol{\varepsilon}_i'(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1})\|\\ \le& N^{-2}T^{-2}\sum_{i=1}^N\left\|\boldsymbol{\Gamma}_i^0\right\|\|\sum_{\ell=1}^N\boldsymbol{\varepsilon}_i'\mathbf{F}^0\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\widehat{\mathbf{F}}\|\|\boldsymbol{\Xi}^{-1}\mathbf{R}^{-1}\|+N^{-2}T^{-2}\sum_{i=1}^N\left\|\boldsymbol{\Gamma}_i^0\right\|\|\sum_{\ell=1}^N\boldsymbol{\varepsilon}_i'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\mathbf{F}^{0\prime}\widehat{\mathbf{F}}\|\|\boldsymbol{\Xi}^{-1}\mathbf{R}^{-1}\|\\ &+N^{-2}T^{-2}\sum_{i=1}^N\|\boldsymbol{\Gamma}_i^0\|\|\sum_{\ell=1}^N\boldsymbol{\varepsilon}_i'\mathbf{V}_\ell\mathbf{V}_\ell'\widehat{\mathbf{F}}\|\|\boldsymbol{\Xi}^{-1}\mathbf{R}^{-1}\|\end{aligned}$$

Since $\boldsymbol{\Xi}^{-1}=O_p(1)$ and $\mathbf{R}^{-1}=O_p(1)$ by Lemma A.1 (c), we omit $\|\boldsymbol{\Xi}^{-1}\mathbf{R}^{-1}\|$, which is $O_p(1)$, in the following analysis. The first term is bounded in norm by

$$T^{-1/2}\Big(N^{-1}\sum_{i=1}^N\left\|\boldsymbol{\Gamma}_i^0\right\|\|T^{-1/2}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\|\Big)\Big\|N^{-1}T^{-1}\sum_{\ell=1}^N\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\widehat{\mathbf{F}}\Big\|$$

With Assumptions A and C, we have

$$\mathbb{E}\|T^{-1/2}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\|^2=T^{-1}\sum_{s=1}^T\sum_{t=1}^T\mathbb{E}(\varepsilon_{is}\varepsilon_{it})\mathbb{E}(\mathbf{f}_s^{0\prime}\mathbf{f}_t^0)\le T^{-1}\sum_{s=1}^T\sum_{t=1}^T\tilde{\sigma}_{st}\sqrt{\mathbb{E}\|\mathbf{f}_s^0\|^2\mathbb{E}\|\mathbf{f}_t^0\|^2}\le C.$$

then $\mathbb{E}(N^{-1}\sum_{i=1}^N\|\boldsymbol{\Gamma}_i^0\|\|T^{-1/2}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\|)\le N^{-1}\sum_{i=1}^N\sqrt{\mathbb{E}\|\boldsymbol{\Gamma}_i^0\|^2\mathbb{E}\|T^{-1/2}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\|^2}\le C$ by Assumption D, which then implies that

$$N^{-1}\sum_{i=1}^N\|\boldsymbol{\Gamma}_i^0\|\|T^{-1/2}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\|=O_p(1)\qquad\text{(A.2)}$$

By Lemma A.1 (h), we have

$$N^{-1}T^{-1}\sum_{\ell=1}^{N}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime} = N^{-1}T^{-1}\sum_{\ell=1}^{N}\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime} + N^{-1}T^{-1}\sum_{\ell=1}^{N}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime}$$
$$= O_p(N^{-1/2}T^{-1/2}) + O_p(N^{-1})\,. \tag{A.3}$$

With the above two equations, the first term is $O_p(N^{-1/2}T^{-1}) + O_p(N^{-1}T^{-1/2})$. The second term is bounded in norm by

$$N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime}\|\|T^{-1/2}\mathbf{F}^0\|\|T^{-1/2}\widehat{\mathbf{F}}\| = O_p(N^{-1/2}T^{-1/2})$$

where $N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime}\| = O_p(1)$ can be proved by following the way in the proof of (C.4). Consider the third term. Easily, we can prove $\mathbb{E}\|T^{-1/2}\boldsymbol{\varepsilon}_i\|^2 \le C$. By Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\mathbb{E}\big\|N^{-1}T^{-1}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbb{E}(\mathbf{V}_\ell\mathbf{V}_\ell')\mathbf{F}^0\big\|^2 = \mathbb{E}\Big\|T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\big(N^{-1}\sum_{l=1}^{N}\mathbb{E}(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t})\big)\varepsilon_{is}\mathbf{f}_t^0\Big\|^2\\
\le& T^{-2}\sum_{s_1=1}^{T}\sum_{t_1=1}^{T}\sum_{s_2=1}^{T}\sum_{t_2=1}^{T}|N^{-1}\sum_{\ell=1}^{N}\mathbb{E}(\mathbf{v}_{\ell s_1}'\mathbf{v}_{\ell t_1})||N^{-1}\sum_{\ell=1}^{N}\mathbb{E}(\mathbf{v}_{\ell s_2}'\mathbf{v}_{\ell t_2})|\mathbb{E}\big(\|\varepsilon_{is_1}\mathbf{f}_{t_1}^0\|\|\varepsilon_{is_2}\mathbf{f}_{t_2}^0\|\big)\\
\le& T^{-2}\sum_{s_1=1}^{T}\sum_{t_1=1}^{T}\sum_{s_2=1}^{T}\sum_{t_2=1}^{T}\tilde{\tau}_{s_1t_1}\tilde{\tau}_{s_2t_2}\sqrt[4]{\mathbb{E}\varepsilon_{is_1}^4\,\mathbb{E}\varepsilon_{is_2}^4\,\mathbb{E}\left\|\mathbf{f}_{t_1}^0\right\|^4\mathbb{E}\left\|\mathbf{f}_{t_2}^0\right\|^4}\\
\le& C\cdot(T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\tau}_{st})^2 \le C^3\,,
\end{aligned} \tag{A.4}$$

by Assumptions A, B2, and C. With Assumption B5, we can follow the way of the proof of Lemma A.2(i) in Bai (2009a) to show that $\mathbb{E}\|N^{-1/2}T^{-1}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\big[\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\big]\mathbf{F}^0\|^2 \le C$. Using the similar argument of (C.4), with the above three moment conditions, we obtain

$$\begin{aligned}
&N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|T^{-1/2}\boldsymbol{\varepsilon}_i\| = O_p(1)\\
&N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|N^{-1}T^{-1}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{F}^0\| = O_p(1)\\
&N^{-1}\sum_{i=1}^{N}\left\|\mathbf{\Gamma}_i^0\right\|\|N^{-1/2}T^{-1}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\big[\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\big]\mathbf{F}^0\| = O_p(1)
\end{aligned}$$

Thus, with Lemma A.1 (a), the third term is bounded in norm by

$$\begin{aligned}
&N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|T^{-1/2}\boldsymbol{\varepsilon}_i\|\cdot\|N^{-1}T^{-1}\sum_{\ell=1}^{N}\mathbf{V}_\ell\mathbf{V}_\ell'\|\|T^{-1/2}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\\
&+T^{-1}\cdot N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|N^{-1}T^{-1}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{F}^0\|\|\mathbf{R}\|\\
&+N^{-1/2}T^{-1}\cdot N^{-1}\sum_{i=1}^{N}\left\|\mathbf{\Gamma}_i^0\right\|\|N^{-1/2}T^{-1}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\big[\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\big]\mathbf{F}^0\|\|\mathbf{R}\| = O_p(\delta_{NT}^{-2})
\end{aligned}$$

because

$$\|N^{-1}T^{-1}\sum_{\ell=1}^{N}\mathbf{V}_\ell\mathbf{V}_\ell'\| = O_p(\delta_{NT}^{-1})\,. \tag{A.5}$$

which suggest from

$$
\begin{aligned}
&\|N^{-1}T^{-1/2}\sum_{\ell=1}^{N}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\|^2 = T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\Big|N^{-1}\sum_{i=1}^{N}\mathbb{E}\left(\mathbf{v}_{is}'\mathbf{v}_{it}\right)\Big|^2 \\
\leq &CN^{-1}T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\sum_{i=1}^{N}\big|\mathbb{E}(\mathbf{v}_{is}'\mathbf{v}_{it})\big| \leq CT^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\sigma}_{st} \leq C^2\,,
\end{aligned}
\tag{A.6}
$$

and

$$
\mathbb{E}\Big\|N^{-1/2}T^{-1}\sum_{\ell=1}^{N}\left[\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\Big\|^2 = T^{-2}\sum_{s=1}^{T}\sum_{t=1}^{T}\mathbb{E}\big[N^{-1/2}\sum_{\ell=1}^{N}(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t} - \mathbb{E}(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}))\big]^2 \leq C\,,
\tag{A.7}
$$

given $|N^{-1}\sum_{i=1}^{N}\mathbb{E}(\mathbf{v}_{is}'\mathbf{v}_{it})| \leq N^{-1}\sum_{i=1}^{N}|\mathbb{E}(\mathbf{v}_{is}'\mathbf{v}_{it})| \leq N^{-1}\sum_{i=1}^{N}\sqrt{\mathbb{E}\|\mathbf{v}_{is}\|^2\mathbb{E}\|\mathbf{v}_{it}\|^2} \leq C$ and Assumption B. Collecting the above three terms, the claim holds.

Consider (b). Replacing $\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R}$ by its expression (A.1), we have

$$
\begin{aligned}
&N^{-1}\sum_{i=1}^{N}\|\boldsymbol{\varphi}_i^0\|\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\| \\
\leq &N^{-2}T^{-2}\sum_{i=1}^{N}\|\boldsymbol{\varphi}_i^0\|\|\sum_{\ell=1}^{N}\mathbf{V}_i'\mathbf{F}^0\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\widehat{\mathbf{F}}\|\|\boldsymbol{\Xi}^{-1}\| + N^{-2}T^{-2}\sum_{i=1}^{N}\|\boldsymbol{\varphi}_i^0\|\|\sum_{\ell=1}^{N}\mathbf{V}_i'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\mathbf{F}^{0\prime}\widehat{\mathbf{F}}\|\|\boldsymbol{\Xi}^{-1}\| \\
&+ N^{-2}T^{-2}\sum_{i=1}^{N}\|\boldsymbol{\varphi}_i^0\|\|\sum_{\ell=1}^{N}\mathbf{V}_i'\mathbf{V}_\ell\mathbf{V}_\ell'\widehat{\mathbf{F}}\|\|\boldsymbol{\Xi}^{-1}\|
\end{aligned}
$$

Ignoring $\|\boldsymbol{\Xi}^{-1}\|$ and following the arguments of the first term in the proof of (a), the first term is $O_p(N^{-1/2}T^{-1}) + O_p(N^{-1}T^{-1/2}) + O_p(N^{-1/2}T^{-1/2}\delta_{NT}^{-2})$. The second term is bounded in norm by

$$
\begin{aligned}
&N^{-2}T^{-1}\sum_{i=1}^{N}\|\boldsymbol{\varphi}_i^0\|\|\sum_{\ell=1}^{N}\mathbf{V}_i'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\|\cdot\|T^{-1/2}\mathbf{F}^0\|\|T^{-1/2}\widehat{\mathbf{F}}\| \\
= &N^{-2}T^{-1}\sum_{i=1}^{N}\|\boldsymbol{\varphi}_i^0\|\|\sum_{\ell=1}^{N}\mathbf{V}_i'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\| \times O_p(1) \\
\leq &N^{-2}T^{-1}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\|\boldsymbol{\varphi}_i^0\|\|\mathbb{E}\big(\mathbf{V}_i'\mathbf{V}_\ell\big)\|\|\boldsymbol{\Gamma}_\ell^0\| + N^{-2}T^{-1}\sum_{i=1}^{N}\|\boldsymbol{\varphi}_i^0\|\|\sum_{\ell=1}^{N}\left(\mathbf{V}_i'\mathbf{V}_\ell - \mathbb{E}\big(\mathbf{V}_i'\mathbf{V}_\ell\big)\right)\boldsymbol{\Gamma}_\ell^{0\prime}\| \\
\leq &N^{-2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\bar{\sigma}_{i\ell}\|\boldsymbol{\varphi}_i^0\|\|\boldsymbol{\Gamma}_\ell^0\| + N^{-2}T^{-1}\sum_{i=1}^{N}\|\boldsymbol{\varphi}_i^0\|\|\sum_{\ell=1}^{N}\left(\mathbf{V}_i'\mathbf{V}_\ell - \mathbb{E}\big(\mathbf{V}_i'\mathbf{V}_\ell\big)\right)\boldsymbol{\Gamma}_\ell^{0\prime}\| \\
= &O_p(N^{-1}) + O_p(N^{-1/2}T^{-1/2})\,,
\end{aligned}
$$

since

$$
\mathbb{E}\Big(N^{-2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\bar{\sigma}_{i\ell}\|\boldsymbol{\varphi}_i^0\|\|\boldsymbol{\Gamma}_\ell^0\|\Big) \leq N^{-2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\bar{\sigma}_{i\ell}\sqrt{\mathbb{E}\|\boldsymbol{\varphi}_i^0\|^2\mathbb{E}\|\boldsymbol{\Gamma}_\ell^0\|^2} \leq CN^{-2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\bar{\sigma}_{i\ell} \leq C^2N^{-1}
$$

by Assumption B2 and

$$
N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\|\boldsymbol{\varphi}_i^0\|\|\sum_{\ell=1}^{N}\left(\mathbf{V}_i'\mathbf{V}_\ell - \mathbb{E}\big(\mathbf{V}_i'\mathbf{V}_\ell\big)\right)\boldsymbol{\Gamma}_\ell^{0\prime}\| = O_p(1)
$$

given $\mathbb{E}\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\left(\mathbf{V}_i'\mathbf{V}_\ell-\mathbb{E}(\mathbf{V}_i'\mathbf{V}_\ell)\right)\mathbf{\Gamma}_\ell^{0\prime}\|^2\leq C$ by Assumption B4. With (A.5) and Lemma A.1 (a) and (d), the third term is bounded in norm by

$$
\begin{aligned}
&N^{-1}\sum_{i=1}^{N}\|\boldsymbol{\varphi}_i^0\|\|T^{-1/2}\mathbf{V}_i\|\|N^{-1}T^{-1}\sum_{\ell=1}^{N}\mathbf{V}_\ell\mathbf{V}_\ell'\|\|T^{-1/2}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\\
&+N^{-1}T^{-1}\sum_{i=1}^{N}\|\boldsymbol{\varphi}_i^0\|\|N^{-1}T^{-1}\sum_{\ell=1}^{N}\mathbf{V}_i'\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{F}^0\|\|\mathbf{R}\|\\
&+N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\|\boldsymbol{\varphi}_i^0\|\|T^{-1/2}\mathbf{V}_i\|\left\|N^{-1/2}T^{-1}\sum_{\ell=1}^{N}\left[\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}(\mathbf{V}_\ell\mathbf{V}_\ell')\right]\mathbf{F}^0\right\|\|\mathbf{R}\|=O_p(\delta_{NT}^{-2})
\end{aligned}
$$

because $\mathbb{E}\|N^{-1}T^{-1}\sum_{\ell=1}^{N}\mathbf{V}_i'\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{F}^0\|^2\leq C$, which can be proved by following the way of the proof of (C.6), and

$$
\mathbb{E}\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}(\mathbf{V}_\ell\mathbf{V}_\ell'))\mathbf{F}^0\|^2=T^{-1}\sum_{s=1}^{T}\mathbb{E}\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{N}\sum_{t=1}^{T}\left[\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}-\mathbb{E}\left(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}\right)\right]\mathbf{f}_t^0\right\|^2\leq C \tag{A.8}
$$

by Assumption B2. Combining the above three terms, (b) holds.

Consider (c). In the proof of (b), we only require $\mathbb{E}\|\boldsymbol{\varphi}_i^0\|^2\leq C$ with respect to $\boldsymbol{\varphi}_i$. Then, we can follow the argument in the proof of (b) to show that

$$
\begin{aligned}
&N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\|T^{-1/2}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\|=O_p(\delta_{NT}^{-2})\\
&N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime}\|=O_p(\delta_{NT}^{-2})\\
&N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\|T^{-1/2}\boldsymbol{\varepsilon}_i\|=O_p(\delta_{NT}^{-2})\\
&N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\|N^{-1}T^{-1}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{F}^0\|=O_p(\delta_{NT}^{-2})\\
&N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\|N^{-1/2}T^{-1}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\left[\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\mathbf{F}^0\|=O_p(\delta_{NT}^{-2})
\end{aligned}
$$

With the above equations, the proof of (c) is analogous to that of (a), in which we replace $\|\mathbf{\Gamma}_i^0\|$ by $\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|$. This completes the proof. □

Proof of Lemma A.3. As $\mathbf{M}_{\widehat{\mathbf{F}}}\widehat{\mathbf{F}}=\mathbf{0}$, the term on the left hand is equal to $N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1})'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i$. With the equation (A.1), it can be decomposed as

$$
\begin{aligned}
&-N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}\mathbf{R}^{-1\prime}\mathbf{\Xi}^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i-N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}\mathbf{R}^{-1\prime}\mathbf{\Xi}^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\\
&-N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}\mathbf{R}^{-1\prime}\mathbf{\Xi}^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i=\mathbb{A}_1+\mathbb{A}_2+\mathbb{A}_3
\end{aligned}
$$

We consider $\mathbb{A}_1$. It's easy to show that $N^{-1}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|\boldsymbol{\varphi}_i^0\| = O_p(1)$, then,

$$
\begin{aligned}
&N^{-1/2}T^{-1/2}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1})'\mathbf{u}_i\| \\
\le& N^{1/2}T^{1/2}\Big(N^{-1}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|\boldsymbol{\varphi}_i^0\|\|T^{-1}(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1})'\mathbf{H}^0\| + N^{-1}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|T^{-1}(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1})'\boldsymbol{\varepsilon}_i\|\Big) \\
=& O_p(N^{1/2}T^{1/2}\delta_{NT}^{-2})
\end{aligned}
\tag{A.9}
$$

by Lemma A.1(b), A.2(a) and $\mathbf{u}_i = \mathbf{H}^0\boldsymbol{\varphi}_i^0 + \boldsymbol{\varepsilon}_i$. In addition, we have

$$
\begin{aligned}
&N^{-1/2}T^{-1/2}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|\widehat{\mathbf{F}}'\mathbf{u}_i\| \\
\le& \sqrt{NT}\Big(N^{-1}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|\boldsymbol{\varphi}_i^0\|\cdot\|T^{-1/2}\widehat{\mathbf{F}}\|\|T^{-1/2}\mathbf{H}^0\| + N^{-1}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|T^{-1/2}\boldsymbol{\varepsilon}_i\|\cdot\|T^{-1/2}\widehat{\mathbf{F}}\|\Big) \\
=& O_p(N^{1/2}T^{1/2})
\end{aligned}
\tag{A.10}
$$

by Assumption C and $\|T^{-1/2}\widehat{\mathbf{F}}\| = \sqrt{r_1}$. Since $\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F} = \mathbf{M}_{\widehat{\mathbf{F}}}(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1})$, $N^{-1/2}T^{-1/2}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1})'\mathbf{u}_i\| = O_p(N^{1/2}T^{1/2}\delta_{NT}^{-2})$ and $N^{-1/2}T^{-1/2}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|\widehat{\mathbf{F}}'\mathbf{u}_i\| = O_p(N^{1/2}T^{1/2})$, $\mathbb{A}_1$ is bounded in norm by

$$
\begin{aligned}
&N^{-1/2}T^{-1/2}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1})'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\|\cdot\|N^{-1}T^{-1}\sum_{\ell=1}^N \widehat{\mathbf{F}}'\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}\|\|\mathbf{R}^{-1}\|\|\boldsymbol{\Xi}^{-1}\| \\
=& N^{-1/2}T^{-1/2}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1})'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\|\cdot\big[O_p(N^{-1})+O_p(N^{-1/2}T^{-1/2})\big] \\
\le& N^{-1/2}T^{-1/2}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1})'\mathbf{u}_i\|\cdot\big[O_p(N^{-1})+O_p(N^{-1/2}T^{-1/2})\big] \\
&+ N^{-1/2}T^{-1/2}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|\widehat{\mathbf{F}}'\mathbf{u}_i\|\cdot\|T^{-1}(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1})'\widehat{\mathbf{F}}\|\cdot\big[O_p(N^{-1})+O_p(N^{-1/2}T^{-1/2})\big] \\
=& O_p(N^{-1/2}T^{1/2}\delta_{NT}^{-2}) + O_p(\delta_{NT}^{-2})
\end{aligned}
$$

with (C.5) and Lemma A.1(b). With the definition of $\mathbf{R}$, $\mathbb{A}_2$ and $\mathbb{A}_3$ can be reformulated as

$$
\begin{aligned}
\mathbb{A}_2 &= -N^{-3/2}T^{-1/2}\sum_{i=1}^N\sum_{\ell=1}^N \boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i \\
\mathbb{A}_3 &= -N^{-3/2}T^{-1/2}\sum_{i=1}^N\sum_{\ell=1}^N \boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0)^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i
\end{aligned}
$$

Combining the above three terms, we can complete the proof. □

Proof of Lemma A.4. Note that $\mathbf{M}_{\mathbf{F}^0} - \mathbf{M}_{\widehat{\mathbf{F}}} = \mathbf{P}_{\widehat{\mathbf{F}}} - \mathbf{P}_{\mathbf{F}^0}$ and $\mathbf{P}_{\widehat{\mathbf{F}}} = T^{-1}\widehat{\mathbf{F}}\widehat{\mathbf{F}}'$. We can derive that

$$
\begin{aligned}
&-N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i-\big(-N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i\big)\\
=&N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{u}_i+N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\\
&+N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\\
&+N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0(\mathbf{R}\mathbf{R}'-(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1})\mathbf{F}^{0\prime}\mathbf{u}_i\\
=&\mathbb{B}_1+\mathbb{B}_2+\mathbb{B}_3+\mathbb{B}_4
\end{aligned}
$$

We first consider the last three terms. Consider the term $\mathbb{B}_2$. By Lemmas A.1 (c) and (A.9), $\mathbb{B}_2$ is bounded in norm by

$$
N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\|\cdot\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0\|\cdot\|(\mathbf{\Upsilon}^0)^{-1}\|\|\mathbf{R}\|=O_p(\delta_{NT}^{-2})
$$

given the fact that $N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0=O_p(1)$. $\mathbb{B}_3$ is bounded in norm by

$$
\begin{aligned}
&N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\|\cdot\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\cdot\|(\mathbf{\Upsilon}^0)^{-1}\|\\
=&O_p(\delta_{NT}^{-2})\cdot\Big[O_p\big(N^{-1/2}T^{1/2}\big)+O_p\big(T^{1/2}\delta_{NT}^{-2}\big)\Big]=O_p\big(N^{-1/2}T^{1/2}\delta_{NT}^{-2}\big)+O_p\big(T^{1/2}\delta_{NT}^{-4}\big)
\end{aligned}
$$

by Lemmas A.1 (f) and (A.9). $\mathbb{B}_4$ is bounded in norm by

$$
N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|T^{-1/2}\mathbf{u}_i\|\cdot\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0\|\|\mathbf{R}\mathbf{R}'-(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|\|T^{-1/2}\mathbf{F}^0\|\|(\mathbf{\Upsilon}^0)^{-1}\|=O_p(\delta_{NT}^{-2})
$$

by Lemma A.1 (d), and $N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|T^{-1/2}\mathbf{u}_i\|=O_p(1)$ which can be proved similar to (A.10). $\mathbb{B}_1$ is decomposed as

$$
\begin{aligned}
\mathbb{B}_1=&N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'(\widehat{\mathbf{F}}\mathbf{R}^{-1}-\mathbf{F}^0)(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\\
&+N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'(\widehat{\mathbf{F}}\mathbf{R}^{-1}-\mathbf{F}^0)(\mathbf{R}\mathbf{R}'-(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1})\mathbf{F}^{0\prime}\mathbf{u}_i\\
=&\mathbb{B}_{1.1}+\mathbb{B}_{1.2}
\end{aligned}
$$

Since $N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|T^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\|=O_p(1)$, which can be proved by following the argument in (A.10), the term $\mathbb{B}_{1.2}$ is bounded in norm by

$$
\begin{aligned}
&N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|T^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\|\cdot\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'(\widehat{\mathbf{F}}\mathbf{R}^{-1}-\mathbf{F}^0)\|\|\mathbf{R}\mathbf{R}'-(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|\|(\mathbf{\Upsilon}^0)^{-1}\|\\
=&O_p\big(N^{-1/2}T^{1/2}\delta_{NT}^{-2}\big)+O_p\big(T^{1/2}\delta_{NT}^{-4}\big)
\end{aligned}
$$

by Lemmas A.1 (d), (f).

We consider the term $\mathbb{B}_{1.1}$. By (A.1), $\mathbb{B}_{1.1}$ is decomposed as

$$
\begin{aligned}
&N^{-5/2}T^{-5/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{F}^0\mathbf{\Gamma}_h^0\mathbf{V}_h^\prime\mathbf{F}^0\mathbf{R}(T^{-1}\mathbf{F}^{0\prime}\hat{\mathbf{F}})^{-1}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\\
&+N^{-5/2}T^{-5/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{F}^0\mathbf{\Gamma}_h^0\mathbf{V}_h^\prime(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(T^{-1}\mathbf{F}^{0\prime}\hat{\mathbf{F}})^{-1}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\\
&+N^{-5/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{V}_h\mathbf{\Gamma}_h^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\\
&+N^{-5/2}T^{-5/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{V}_h\mathbf{V}_h^\prime\widehat{\mathbf{F}}(T^{-1}\mathbf{F}^{0\prime}\hat{\mathbf{F}})^{-1}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\\
=&\mathbb{B}_{1.1.1}+\mathbb{B}_{1.1.2}+\mathbb{B}_{1.1.3}+\mathbb{B}_{1.1.4}
\end{aligned}
$$

The term $\mathbb{B}_{1.1.1}$ is bounded in norm by

$$
\begin{aligned}
&N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|T^{-1/2}\mathbf{u}_i\|\cdot\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{F}^0\|^2\\
&\times\|(T^{-1}\mathbf{F}^{0\prime}\widehat{\mathbf{F}})^{-1}\|\|(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|\|T^{-1/2}\mathbf{F}^0\|\|\mathbf{R}\|\|(\mathbf{\Upsilon}^0)^{-1}\|^2=O_p(N^{-1/2}T^{-1/2})
\end{aligned}
$$

by Lemma A.1 (e). Similarly, we can show the term $\mathbb{B}_{1.1.2}$ is bounded in norm by

$$
\begin{aligned}
&N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|T^{-1/2}\mathbf{u}_i\|\cdot\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{F}^0\|\cdot\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\\
&\times\|(T^{-1}\mathbf{F}^{0\prime}\widehat{\mathbf{F}})^{-1}\|\|(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|\|T^{-1/2}\mathbf{F}^0\|\|(\mathbf{\Upsilon}^0)^{-1}\|^2=O_p(N^{-1})+O_p(N^{-1/2}\delta_{NT}^{-2})
\end{aligned}
$$

by Lemma A.1 (f). $\mathbb{B}_{1.1.3}$ is the leading term, which is reformulated as

$$
N^{-1/2}T^{1/2}\cdot N^{-1}\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\big(N^{-1}T^{-1}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{V}_h\mathbf{\Gamma}_h^{0\prime}\big)(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i)
$$

For the term $\mathbb{B}_{1.1.4}$, it is bounded in norm by

$$
\begin{aligned}
&N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|T^{-1/2}\mathbf{u}_i\|\cdot\|(T^{-1}\mathbf{F}^{0\prime}\widehat{\mathbf{F}})^{-1}\|\|(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|\|T^{-1/2}\mathbf{F}^0\|\|\mathbf{R}\|\|(\mathbf{\Upsilon}^0)^{-1}\|^2\\
&\times\big\|N^{-3/2}T^{-3/2}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{V}_h\mathbf{V}_h^\prime\widehat{\mathbf{F}}\big\|=\big\|N^{-3/2}T^{-3/2}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbf{V}_h\mathbf{V}_h^\prime\widehat{\mathbf{F}}\big\|\cdot O_p(1)\\
\leq&\big\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\big\|\big\|N^{-1}T^{-1}\sum_{h=1}^{N}\big(\mathbf{V}_h\mathbf{V}_h^\prime\widehat{\mathbf{F}}-\mathbb{E}(\mathbf{V}_h\mathbf{V}_h^\prime)\mathbf{F}^0\mathbf{R}\big)\big\|\cdot O_p(1)\\
&+\big\|N^{-3/2}T^{-3/2}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbb{E}(\mathbf{V}_h\mathbf{V}_h^\prime)\mathbf{F}^0\big\|\|\mathbf{R}\|\cdot O_p(1)\\
\leq&\big\|N^{-1}T^{-1}\sum_{h=1}^{N}\big(\mathbf{V}_h\mathbf{V}_h^\prime\widehat{\mathbf{F}}-\mathbb{E}(\mathbf{V}_h\mathbf{V}_h^\prime)\mathbf{F}^0\mathbf{R}\big)\big\|\cdot O_p(1)+\big\|N^{-3/2}T^{-3/2}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell^\prime\mathbb{E}(\mathbf{V}_h\mathbf{V}_h^\prime)\mathbf{F}^0\big\|\cdot O_p(1)
\end{aligned}
$$

the first term is bounded in norm by

$$
N^{-1/2}\big\|N^{-1/2}T^{-1}\sum_{h=1}^{N}\big(\mathbf{V}_h\mathbf{V}_h^\prime-\mathbb{E}(\mathbf{V}_h\mathbf{V}_h^\prime)\big)\mathbf{F}^0\big\|\|\mathbf{R}\|+\big\|N^{-1}T^{-1/2}\sum_{h=1}^{N}\mathbf{V}_h\mathbf{V}_h^\prime\big\|\|T^{-1/2}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|
$$

$$=O_p(N^{-1/2}) + O_p(T^{1/2}\delta_{NT}^{-2})$$

by Lemmas A.1 (a), (A.5) and (A.8). The second term is bounded in norm by

$$\begin{aligned}
&T^{-1/2}\Big\|T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}(N^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{v}_{\ell s})\cdot\mathbf{f}_t^{0\prime}\mathrm{tr}(N^{-1}\sum_{h=1}^{N}\mathbf{\Sigma}_{hh,ts})\Big\|\\
\leq& kT^{-3/2}\sum_{s=1}^{T}\sum_{t=1}^{T}\|N^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{v}_{\ell s}\|\|\mathbf{f}_t^0\|\tilde{\tau}_{ts} = O_p(T^{-1/2})
\end{aligned}$$

since $\mathrm{tr}(A)\leq k\|A\|$ for any $k\times k$ matrix A, Assumption B2 and

$$\begin{aligned}
&\mathbb{E}\big(T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\|N^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{v}_{\ell s}\|\|\mathbf{f}_t^0\|\tilde{\tau}_{ts}\big)\\
\leq& T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\tau}_{ts}\sqrt{\mathbb{E}\left\|\mathbf{f}_t^0\right\|^2\mathbb{E}\Big\|N^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{v}_{\ell s}\Big\|^2} \leq CT^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\tau}_{ts}\sqrt{N^{-1}\sum_{\ell=1}^{N}\sum_{j=1}^{N}\mathrm{tr}\left(\mathbf{\Sigma}_{\ell j,ss}\mathbb{E}(\mathbf{\Gamma}_j^{0\prime}\mathbf{\Gamma}_\ell^0)\right)}\\
\leq& kCT^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\tau}_{ts}\sqrt{N^{-1}\sum_{\ell=1}^{N}\sum_{j=1}^{N}\|\mathbf{\Sigma}_{\ell j,ss}\|\sqrt{\mathbb{E}\left\|\mathbf{\Gamma}_j^0\right\|^2\mathbb{E}\left\|\mathbf{\Gamma}_\ell^0\right\|^2}} \leq kCT^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\tau}_{ts}\sqrt{N^{-1}\sum_{\ell=1}^{N}\sum_{j=1}^{N}\bar{\tau}_{\ell j}} \leq C
\end{aligned}$$

by Assumptions B2, C and D. Then $\mathbb{B}_{1.1.4} = O_p(T^{1/2}\delta_{NT}^{-2}) + O_p(\delta_{NT}^{-1})$. Combining the above terms, we complete the proof. □

Proof of Lemma A.5. By substracting and adding terms, we have

$$\begin{aligned}
&N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0)^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\\
&-N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i\\
=&N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0)^{-1}\widehat{\mathbf{F}}'\left(\mathbf{V}_\ell\mathbf{V}_\ell'-\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\\
&+N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\left[(T^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0)^{-1}\widehat{\mathbf{F}}'-(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\right]\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\\
&+N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0})\mathbf{u}_i\\
=&\mathbb{C}_1+\mathbb{C}_2+\mathbb{C}_3
\end{aligned}$$

Consider $\mathbb{C}_1$. As $\mathbf{M}_{\widehat{\mathbf{F}}} = \mathbf{I}_T - T^{-1}\widehat{\mathbf{F}}\widehat{\mathbf{F}}'$, it is bounded in norm by

$$
\begin{aligned}
&N^{-3/2}T^{-3/2}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|\widehat{\mathbf{F}}'\sum_{\ell=1}^N \left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\| \cdot \|(\boldsymbol{\Upsilon}^0)^{-1}\|\|(T^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0)^{-1}\| \\
\leq & N^{-3/2}T^{-3/2}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|\widehat{\mathbf{F}}'\sum_{\ell=1}^N \left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{u}_i\| \cdot O_p(1) \\
&+ N^{-3/2}T^{-5/2}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|\widehat{\mathbf{F}}'\sum_{\ell=1}^N \left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\widehat{\mathbf{F}}\widehat{\mathbf{F}}'\mathbf{u}_i\| \cdot O_p(1) \\
\leq & N^{-3/2}T^{-3/2}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|\boldsymbol{\varphi}_i^0\| \cdot \|\widehat{\mathbf{F}}'\sum_{\ell=1}^N \left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{H}^0\| \cdot O_p(1) \\
&+ N^{-3/2}T^{-3/2}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|\widehat{\mathbf{F}}'\sum_{\ell=1}^N \left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\boldsymbol{\varepsilon}_i\| \cdot O_p(1) \\
&+ N^{-3/2}T^{-5/2}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|\mathbf{u}_i\| \cdot \|\widehat{\mathbf{F}}'\sum_{\ell=1}^N \left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\widehat{\mathbf{F}}\|\|\widehat{\mathbf{F}}\| \cdot O_p(1)
\end{aligned}
$$

Consider the first term. Following the argument in the proof of Lemma A.2(i) in Bai (2009a), we can show that $\|N^{-1/2}T^{-1}\sum_{\ell=1}^N \mathbf{F}^{0\prime}\big(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}(\mathbf{V}_\ell\mathbf{V}_\ell')\big)\mathbf{H}^0\| = O_p(1)$. In addition, similar to (A.8), we can show that $\|N^{-1/2}T^{-1}\sum_{\ell=1}^N \big(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}(\mathbf{V}_\ell\mathbf{V}_\ell')\big)\mathbf{H}^0\| = O_p(1)$. The first term is bounded in norm by

$$
\begin{aligned}
&T^{-1/2}\cdot N^{-1}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|\boldsymbol{\varphi}_i^0\| \cdot \|N^{-1/2}T^{-1}\mathbf{F}^{0\prime}\sum_{\ell=1}^N \left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{H}^0\| \cdot \|\mathbf{R}\| \\
&+ N^{-1}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|\boldsymbol{\varphi}_i^0\| \cdot \|T^{-1/2}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\|\|N^{-1/2}T^{-1}\sum_{\ell=1}^N \left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{H}^0\| = O_p(\delta_{NT}^{-1})
\end{aligned}
$$

by Lemma A.1 (a) and (c). Similar to the proof of the first term, the second term is bounded in norm by

$$
\begin{aligned}
&T^{-1/2}\cdot N^{-1}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|N^{-1/2}T^{-1}\mathbf{F}^{0\prime}\sum_{\ell=1}^N \left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\boldsymbol{\varepsilon}_i\| \cdot \|\mathbf{R}\| \\
&+ N^{-1}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|N^{-1/2}T^{-1}\sum_{\ell=1}^N \left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\boldsymbol{\varepsilon}_i\| \cdot \|T^{-1/2}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\| = O_p(\delta_{NT}^{-1})
\end{aligned}
$$

by Lemma A.1 (a). The third term is bounded in norm by

$$
\begin{aligned}
&T^{-1/2}\cdot N^{-1}\sum_{i=1}^N \|\boldsymbol{\Gamma}_i^0\|\|T^{-1/2}\mathbf{u}_i\| \cdot \|T^{-1/2}\widehat{\mathbf{F}}\|\|N^{-1/2}T^{-1}\widehat{\mathbf{F}}'\sum_{\ell=1}^N \left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\widehat{\mathbf{F}}\| \\
=& O_p(T^{-1/2}) \cdot \|N^{-1/2}T^{-1}\widehat{\mathbf{F}}'\sum_{\ell=1}^N \left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\widehat{\mathbf{F}}\| = O_p(\delta_{NT}^{-1}) + O_p(T^{1/2}\delta_{NT}^{-2})
\end{aligned}
$$

because

$$
\begin{aligned}
&\|N^{-1/2}T^{-1}\widehat{\mathbf{F}}'\sum_{\ell=1}^{N}\left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\widehat{\mathbf{F}}\| \\
\leq&\|N^{-1/2}T^{-1}\mathbf{F}^{0\prime}\sum_{\ell=1}^{N}\left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\mathbf{F}^0\|\|\mathbf{R}\|^2 \\
&+2T^{1/2}\cdot\|N^{-1/2}T^{-1}\mathbf{F}^{0\prime}\sum_{\ell=1}^{N}\left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\|\|T^{-1/2}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\|\mathbf{R}\| \\
&+T\cdot\|N^{-1/2}T^{-1}\sum_{\ell=1}^{N}\left(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\|\|T^{-1/2}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|^2 \\
=&O_p(T^{1/2}\delta_{NT}^{-1})+O_p(T\delta_{NT}^{-2})
\end{aligned}
$$

by (A.7), (A.8) and Lemmas A.1 (a), (c), and the fact that $\|N^{-1/2}T^{-1}\sum_{\ell=1}^{N}\mathbf{F}^{0\prime}\big(\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}(\mathbf{V}_\ell\mathbf{V}_\ell')\big)\mathbf{F}^0\| = O_p(1)$ by following the argument in the proof of Lemma A.2(i) in Bai (2009a). Collecting the above terms, $\mathbb{C}_1 = O_p(\delta_{NT}^{-1}) + O_p(T^{1/2}\delta_{NT}^{-2})$.

Consider $\mathbb{C}_2$ and $\mathbb{C}_3$. Note that by Assumptions B2, we have

$$
\begin{aligned}
&\|\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\| \leq \mu_{\max}\left(\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\|\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\| \leq C\|\mathbf{u}_i\| \\
&\|\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\left(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\right)\mathbf{u}_i\| \leq \mu_{\max}\left(\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right)\|\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\|\|\mathbf{u}_i\| = O_p(\delta_{NT}^{-1})\|\mathbf{u}_i\|
\end{aligned}
$$

In addition,

$$
\begin{aligned}
&\|(T^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0)^{-1}(T^{-1/2}\widehat{\mathbf{F}})' - (T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}(T^{-1/2}\mathbf{F}^0)'\| = \|(T^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0)^{-1}(T^{-1/2}\widehat{\mathbf{F}})'\mathbf{M}_{\mathbf{F}^0}\| \\
=&\|(T^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0)^{-1}\big(T^{-1/2}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\big)'\mathbf{M}_{\mathbf{F}^0}\| \leq \|(T^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0)^{-1}\|\|T^{-1/2}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\| = O_p\left(\delta_{NT}^{-1}\right)
\end{aligned}
$$

then $\mathbb{C}_2$ is bounded in norm by

$$
\begin{aligned}
&\sqrt{\frac{N}{T}}\frac{1}{N^2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\|\boldsymbol{\Gamma}_i^0\|\|T^{-1/2}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\|\big\|(T^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0)^{-1}(T^{-1/2}\widehat{\mathbf{F}})' - (T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}(T^{-1/2}\mathbf{F}^0)'\big\|\|(\boldsymbol{\Upsilon}^0)^{-1}\| \\
\leq&\sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\|\boldsymbol{\Gamma}_i^0\|\|T^{-1/2}\mathbf{u}_i\|\big\|(T^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0)^{-1}(T^{-1/2}\widehat{\mathbf{F}})' - (T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}(T^{-1/2}\mathbf{F}^0)'\big\|\|(\boldsymbol{\Upsilon}^0)^{-1}\| \\
=&O_p(N^{1/2}T^{-1/2}\delta_{NT}^{-1})
\end{aligned}
$$

and $\mathbb{C}_3$ is bounded in norm by

$$
\begin{aligned}
&\sqrt{\frac{N}{T}}\frac{1}{N^2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\|\boldsymbol{\Gamma}_i^0\|\Big\|T^{-1/2}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\left(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\right)\mathbf{u}_i\Big\|\Big\|(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\Big\|\Big\|T^{-1/2}\mathbf{F}^0\Big\|\,\|(\boldsymbol{\Upsilon}^0)^{-1}\| \\
=&O_p(N^{1/2}T^{-1/2}\delta_{NT}^{-1})
\end{aligned}
$$

by the above three facts. This completes the proof. □

Proof of Lemma A.6. Since $\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0} = -T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime} - T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})' -$

$T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'-T^{-1}\mathbf{F}^0(\mathbf{R}\mathbf{R}'-\big(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\big)^{-1})\mathbf{F}^{0\prime}$, we have

$$
\begin{aligned}
&N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{V}_i'(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0})\mathbf{u}_i\\
=&-N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'(\widehat{\mathbf{F}}\mathbf{R}^{-1}-\mathbf{F}^0)(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\\
&-N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'(\widehat{\mathbf{F}}\mathbf{R}^{-1}-\mathbf{F}^0)(\mathbf{R}\mathbf{R}'-(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1})\mathbf{F}^{0\prime}\mathbf{u}_i\\
&-N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i-N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\\
&-N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{F}^0(\mathbf{R}\mathbf{R}'-(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1})\mathbf{F}^{0\prime}\mathbf{u}_i\\
=&\mathbb{D}_1+\mathbb{D}_2+\mathbb{D}_3+\mathbb{D}_4+\mathbb{D}_5
\end{aligned}
$$

We first consider the last four terms. Following the argument in the proof of Lemma A.2(b), we can prove that $N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\|\boldsymbol{\varphi}_i^0\|=O_p(\delta_{NT}^{-2})$ and $N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\|T^{-1/2}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\|=O_p(\delta_{NT}^{-2})$. Then with Lemma A.1 (a) and (d), $\mathbb{D}_2$ is bounded in norm by

$$
\begin{aligned}
&N^{1/2}T^{1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\|\boldsymbol{\varphi}_i^0\|\cdot\|T^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\|\|\mathbf{R}\mathbf{R}'-(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|\|\mathbf{R}^{-1}\|\\
&+N^{1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\|T^{-1/2}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\|\cdot\|\mathbf{R}\mathbf{R}'-(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|\|\mathbf{R}^{-1}\|\\
=&O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3})
\end{aligned}
$$

Following the argument in the proof of Lemma A.2(a), we derive $N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i'\mathbf{F}^0\|\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\boldsymbol{\varepsilon}_i\|=O_p(\delta_{NT}^{-2})$. Then, $\mathbb{D}_3$ is bounded in norm by

$$
\begin{aligned}
&N^{1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i'\mathbf{F}^0\|\|\boldsymbol{\varphi}_i^0\|\cdot\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{H}^0\|\|\mathbf{R}\|\\
&+N^{1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i'\mathbf{F}^0\|\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\boldsymbol{\varepsilon}_i\|\cdot\|\mathbf{R}\|=O_p(N^{1/2}\delta_{NT}^{-2})
\end{aligned}
$$

by Lemmas A.1 (b), (c). Since $N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\|\boldsymbol{\varphi}_i^0\|=O_p(\delta_{NT}^{-2})$, $\mathbb{D}_4$ is bounded in norm by

$$
\begin{aligned}
&N^{1/2}T^{1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\|\boldsymbol{\varphi}_i^0\|\cdot\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{H}^0\|\\
&+N^{1/2}T^{1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\boldsymbol{\varepsilon}_i\|=O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3})
\end{aligned}
$$

by Lemmas A.1(b) and A.2(c). $\mathbb{D}_5$ is bounded in norm by

$$
\begin{aligned}
&N^{1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i'\mathbf{F}^0\|\|\boldsymbol{\varphi}_i^0\|\cdot\|T^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\|\|\mathbf{R}\mathbf{R}'-(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|\\
&+N^{1/2}T^{-1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i'\mathbf{F}^0\|\|T^{-1/2}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\|\cdot\|\mathbf{R}\mathbf{R}'-(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|=O_p(N^{1/2}\delta_{NT}^{-2})
\end{aligned}
$$

by Lemma A.1(d).

Now consider the term $\mathbb{D}_1$. With (A.1), we decompose the term $\mathbb{D}_1$ as follows

$$
\begin{aligned}
& -N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\mathbf{V}_h\mathbf{\Gamma}_h^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i \\
& -N^{-3/2}T^{-5/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\mathbf{F}^0\mathbf{\Gamma}_h^0\mathbf{V}_h'\widehat{\mathbf{F}}(T^{-1}\mathbf{F}^{0\prime}\hat{\mathbf{F}})^{-1}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i \\
& -N^{-3/2}T^{-5/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbb{E}\left(\mathbf{V}_i'\mathbf{V}_h\right)\mathbf{V}_h'\widehat{\mathbf{F}}(T^{-1}\mathbf{F}^{0\prime}\hat{\mathbf{F}})^{-1}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i \\
& -N^{-3/2}T^{-5/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\big(\mathbf{V}_i'\mathbf{V}_h-\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h)\big)\mathbf{V}_h'\widehat{\mathbf{F}}(T^{-1}\mathbf{F}^{0\prime}\hat{\mathbf{F}})^{-1}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i \\
=& \mathbb{D}_{1.1}+\mathbb{D}_{1.2}+\mathbb{D}_{1.3}+\mathbb{D}_{1.4}
\end{aligned}
$$

We consider the last three terms $\mathbb{D}_{1.2}$, $\mathbb{D}_{1.3}$, and $\mathbb{D}_{1.4}$. $\mathbb{D}_{1.2}$ is bounded in norm by

$$
\begin{aligned}
& T^{-1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i'\mathbf{F}^0\|\|T^{-1/2}\mathbf{u}_i\|\cdot\|N^{-1/2}T^{-1/2}\sum_{h=1}^{N}\mathbf{\Gamma}_h^0\mathbf{V}_h'\widehat{\mathbf{F}}\| \\
& \times\|(T^{-1}\mathbf{F}^{0\prime}\hat{\mathbf{F}})^{-1}\|\|(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|\|T^{-1/2}\mathbf{F}^0\|\|(\mathbf{\Upsilon}^0)^{-1}\| \\
=& O_p(T^{-1/2})\cdot\|N^{-1/2}T^{-1/2}\sum_{h=1}^{N}\mathbf{\Gamma}_h^0\mathbf{V}_h'\widehat{\mathbf{F}}\|=O_p(\delta_{NT}^{-1})
\end{aligned}
$$

by (C.5). As $\left\|\mathbb{E}(T^{-1}\mathbf{V}_i'\mathbf{V}_h)\right\|\le\bar{\tau}_{ih}$ by Assumption B2, $\mathbb{D}_{1.3}$ is bounded in norm by

$$
\begin{aligned}
& N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\|T^{-1/2}\mathbf{u}_i\|\|\mathbb{E}\left(\mathbf{V}_i'\mathbf{V}_h\right)\|\|\mathbf{V}_h'\widehat{\mathbf{F}}\|\cdot\|(T^{-1}\mathbf{F}^{0\prime}\hat{\mathbf{F}})^{-1}\|\|(\mathbf{\Upsilon}^0)^{-1}\|\|(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|\|T^{-1/2}\mathbf{F}^0\| \\
=& N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\|T^{-1/2}\mathbf{u}_i\|\|\mathbb{E}\left(\mathbf{V}_i'\mathbf{V}_h\right)\|\|\mathbf{V}_h'\widehat{\mathbf{F}}\|\cdot O_p(1) \\
\le& N^{-3/2}\sum_{i=1}^{N}\sum_{h=1}^{N}(\|T^{-1/2}\mathbf{H}^0\|\|\boldsymbol{\varphi}_i^0\|+\|T^{-1/2}\boldsymbol{\varepsilon}_i\|)\|T^{-1}\mathbb{E}\left(\mathbf{V}_i'\mathbf{V}_h\right)\|\|T^{-1/2}\mathbf{V}_h'\mathbf{F}^0\|\|\mathbf{R}\|\cdot O_p(1) \\
& +N^{-3/2}T^{1/2}\sum_{i=1}^{N}\sum_{h=1}^{N}(\|T^{-1/2}\mathbf{H}^0\|\|\boldsymbol{\varphi}_i^0\|+\|T^{-1/2}\boldsymbol{\varepsilon}_i\|)\|T^{-1}\mathbb{E}\left(\mathbf{V}_i'\mathbf{V}_h\right)\|\|T^{-1/2}\mathbf{V}_h\|\|T^{-1/2}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\cdot O_p(1) \\
\le& N^{-3/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\bar{\tau}_{ih}\|\boldsymbol{\varphi}_i^0\|\|T^{-1/2}\mathbf{V}_h'\mathbf{F}^0\|\cdot O_p(1)+N^{-3/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\bar{\tau}_{ih}\|T^{-1/2}\boldsymbol{\varepsilon}_i\|\|T^{-1/2}\mathbf{V}_h'\mathbf{F}^0\|\cdot O_p(1) \\
& +N^{-3/2}T^{1/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\bar{\tau}_{ih}\|\boldsymbol{\varphi}_i^0\|\|T^{-1/2}\mathbf{V}_h\|\cdot O_p(\delta_{NT}^{-1})+N^{-3/2}T^{1/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\bar{\tau}_{ih}\|T^{-1/2}\boldsymbol{\varepsilon}_i\|\|T^{-1/2}\mathbf{V}_h\|\cdot O_p(\delta_{NT}^{-1})
\end{aligned}
$$

The first term is $O_p(N^{-1/2})$ since $\mathbb{E}\big(N^{-1}\sum_{i=1}^{N}\sum_{h=1}^{N}\bar{\tau}_{ih}\|\boldsymbol{\varphi}_i^0\|\|T^{-1/2}\mathbf{V}_h'\mathbf{F}^0\|\big)$ is bounded by

$$
N^{-1}\sum_{i=1}^{N}\sum_{h=1}^{N}\bar{\tau}_{ih}\sqrt{\mathbb{E}\|\boldsymbol{\varphi}_i^0\|^2\mathbb{E}\|T^{-1/2}\mathbf{V}_h'\mathbf{F}^0\|^2}\le CN^{-1}\sum_{i=1}^{N}\sum_{h=1}^{N}\bar{\tau}_{ih}\le C
$$

by Assumption B2, C and D. Similarly, we can show that the second term is $O_p(N^{-1/2})$, while the third and the fourth terms both are $O_p(N^{-1/2}T^{1/2}\delta_{NT}^{-1})$. Thus, $\mathbb{D}_{1.3}$ is $O_p(N^{-1/2})+O_p(N^{-1/2}T^{1/2}\delta_{NT}^{-1})$.

$\mathbb{D}_{1.4}$ is decomposed as following

$$
\begin{aligned}
&-N^{-3/2}T^{-5/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\big(\mathbf{V}_i'\mathbf{V}_h-\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h)\big)\mathbf{V}_h'\mathbf{F}^0\mathbf{R}(T^{-1}\mathbf{F}^{0\prime}\hat{\mathbf{F}})^{-1}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0\\
&-N^{-3/2}T^{-5/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\big(\mathbf{V}_i'\mathbf{V}_h-\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h)\big)\mathbf{V}_h'\mathbf{F}^0\mathbf{R}(T^{-1}\mathbf{F}^{0\prime}\hat{\mathbf{F}})^{-1}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\\
&-N^{-3/2}T^{-5/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\big(\mathbf{V}_i'\mathbf{V}_h-\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h)\big)\mathbf{V}_h'\big(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\big)(T^{-1}\mathbf{F}^{0\prime}\hat{\mathbf{F}})^{-1}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0\\
&-N^{-3/2}T^{-5/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\big(\mathbf{V}_i'\mathbf{V}_h-\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h)\big)\mathbf{V}_h'\big(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}\big)(T^{-1}\mathbf{F}^{0\prime}\hat{\mathbf{F}})^{-1}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\\
=&\mathbb{D}_{1.4.1}+\mathbb{D}_{1.4.2}+\mathbb{D}_{1.4.3}+\mathbb{D}_{1.4.4}
\end{aligned}
$$

Consider $\mathbb{D}_{1.4.1}$. As $\operatorname{vec}(\mathbf{ABC})=(\mathbf{C}'\otimes\mathbf{A})\operatorname{vec}(\mathbf{B})$ for any comfortable matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, $\mathbb{D}_{1.4.1}$ is bounded in norm by

$$
\begin{aligned}
&\Big\|\operatorname{vec}\big(N^{-3/2}T^{-5/2}\sum_{h=1}^{N}\sum_{i=1}^{N}(\mathbf{V}_i'\mathbf{V}_h-\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h))\cdot\mathbf{V}_h'\mathbf{F}^0\mathbf{R}(T^{-1}\mathbf{F}^{0\prime}\hat{\mathbf{F}})^{-1}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\cdot\boldsymbol{\varphi}_i^0\big)\Big\|\\
=&\Big\|N^{-3/2}T^{-5/2}\sum_{h=1}^{N}\Big(\sum_{i=1}^{N}\boldsymbol{\varphi}_i^{0\prime}\otimes(\mathbf{V}_i'\mathbf{V}_h-\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h))\Big)\\
&\quad\times\operatorname{vec}\Big(\mathbf{I}_k\cdot\mathbf{V}_h'\mathbf{F}^0\cdot\mathbf{R}(T^{-1}\mathbf{F}^{0\prime}\hat{\mathbf{F}})^{-1}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\Big)\Big\|\\
=&\Big\|N^{-3/2}T^{-5/2}\sum_{h=1}^{N}\Big(\sum_{i=1}^{N}\boldsymbol{\varphi}_i^{0\prime}\otimes(\mathbf{V}_i'\mathbf{V}_h-\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h))\Big)\\
&\quad\times\Big(\big(\mathbf{R}(T^{-1}\mathbf{F}^{0\prime}\hat{\mathbf{F}})^{-1}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\big)'\Big)\otimes\mathbf{I}_k\cdot\operatorname{vec}(\mathbf{V}_h'\mathbf{F}^0)\Big\|\\
\le&N^{-3/2}T^{-3/2}\sum_{h=1}^{N}\Big\|\sum_{i=1}^{N}\boldsymbol{\varphi}_i^{0\prime}\otimes(\mathbf{V}_i'\mathbf{V}_h-\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h))\Big\|\cdot\|\operatorname{vec}(\mathbf{V}_h'\mathbf{F}^0)\|\\
&\quad\times\|\mathbf{R}\|\|(T^{-1}\mathbf{F}^{0\prime}\hat{\mathbf{F}})^{-1}\|\|(\boldsymbol{\Upsilon}^0)^{-1}\|\|(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|\|T^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\|\|\mathbf{I}_k\|\\
=&O_p(T^{-1/2})\cdot N^{-1}\sum_{h=1}^{N}\big\|N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\boldsymbol{\varphi}_i^{0\prime}\otimes(\mathbf{v}_{it}\mathbf{v}_{ht}'-\mathbb{E}(\mathbf{v}_{it}\mathbf{v}_{ht}'))\big\|\|T^{-1/2}\sum_{s=1}^{T}\mathbf{v}_{hs}\mathbf{f}_s^{0\prime}\|\\
=&O_p(T^{-1/2})
\end{aligned}
$$

by Lemma A.1 (c) and because

$$
\begin{aligned}
&\mathbb{E}\Big(N^{-1}\sum_{h=1}^{N}\big\|N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\boldsymbol{\varphi}_i^{0\prime}\otimes\big(\mathbf{v}_{it}\mathbf{v}_{ht}'-\mathbb{E}(\mathbf{v}_{it}\mathbf{v}_{ht}')\big)\big\|\|T^{-1/2}\sum_{s=1}^{T}\mathbf{v}_{hs}\mathbf{f}_s^{0\prime}\|\Big)\\
\le&N^{-1}\sum_{h=1}^{N}\sqrt{\mathbb{E}\big\|N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\boldsymbol{\varphi}_i^{0\prime}\otimes\big(\mathbf{v}_{it}\mathbf{v}_{ht}'-\mathbb{E}(\mathbf{v}_{it}\mathbf{v}_{ht}')\big)\big\|^2\mathbb{E}\|T^{-1/2}\sum_{s=1}^{T}\mathbf{v}_{hs}\mathbf{f}_s^{0\prime}\|^2}\le C
\end{aligned}
$$

by Assumption B4 and $\mathbb{E}\|T^{-1/2}\sum_{s=1}^{T}\mathbf{v}_{hs}\mathbf{f}_s^{0\prime}\|^2\le C$, which can be proved easily by Assumption B2. Similarly, we can show that $\mathbb{D}_{1.4.2}$ is $O_p(T^{-1/2})$, while $\mathbb{D}_{1.4.3}$ and $\mathbb{D}_{1.4.4}$ both are $O_p(\delta_{NT}^{-1})$.

Collecting $\mathbb{D}_{1.4.1}$ to $\mathbb{D}_{1.4.4}$, $\mathbb{D}_{1.4} = O_p\left(\delta_{NT}^{-1}\right)$. Thus

$$
\begin{aligned}
&N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{V}_i'(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0})\mathbf{u}_i \\
=&-N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\mathbf{V}_h\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i+O_p(N^{1/2}T^{1/2}\delta_{NT}^{-3})
\end{aligned}
$$

This completes the proof. □

Proof of Lemma A.7. Denote

$$
\begin{aligned}
\mathbf{a}_1 &= -\frac{1}{NT}\sum_{i=1}^{N}\sum_{h=1}^{N}\boldsymbol{\mathcal{V}}_i'\mathbf{V}_h\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i) \\
\mathbf{a}_2 &= -\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i
\end{aligned}
$$

with $\boldsymbol{\mathcal{X}}_i = \mathbf{X}_i - N^{-1}\sum_{\ell=1}^{N}\mathbf{X}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_i^0$, $\boldsymbol{\mathcal{V}}_i = \mathbf{V}_i - N^{-1}\sum_{\ell=1}^{N}\mathbf{V}_\ell\boldsymbol{\Gamma}_\ell^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_i^0$, $\boldsymbol{\Upsilon}^0 = N^{-1}\sum_{i=1}^{N}\boldsymbol{\Gamma}_i^0\boldsymbol{\Gamma}_i^{0\prime}$. In addition, $\boldsymbol{\Sigma} = N^{-1}\sum_{\ell=1}^{N}\mathbb{E}(\mathbf{V}_\ell\mathbf{V}_\ell')$.

Here we investigate the stochastic order of $\mathbf{a}_1$ and $\mathbf{a}_2$. $\mathbf{a}_1$ is decomposed as

$$
\begin{aligned}
&-\frac{1}{NT}\sum_{i=1}^{N}\sum_{h=1}^{N}\boldsymbol{\mathcal{V}}_i'\mathbf{V}_h\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i) \\
=&-\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h)\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0 \\
&+\frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{h=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbb{E}(\mathbf{V}_\ell'\mathbf{V}_h)\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0 \\
&-\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{h=1}^{N}\big(\mathbf{V}_i'\mathbf{V}_h-\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h)\big)\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0 \\
&+\frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{h=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\big(\mathbf{V}_\ell'\mathbf{V}_h-\mathbb{E}(\mathbf{V}_\ell'\mathbf{V}_h)\big)\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0 \\
&-\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\mathbf{V}_h\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i \\
&+\frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{h=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{V}_h\boldsymbol{\Gamma}_h^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i
\end{aligned}
$$

As $\|\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h)\| \le T\bar{\tau}_{ih}$ by Assumption B2, the first term is bounded in norm by

$$
\begin{aligned}
&N^{-1}T^{-1}\sum_{i=1}^{N}\sum_{h=1}^{N}\|\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h)\|\|\boldsymbol{\varphi}_i^0\|\|\boldsymbol{\Gamma}_h^0\|\cdot\|(\boldsymbol{\Upsilon}^0)^{-1}\|\|(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|\|T^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\| \\
\le& N^{-1}\sum_{i=1}^{N}\sum_{h=1}^{N}\bar{\tau}_{ih}\|\boldsymbol{\varphi}_i^0\|\|\boldsymbol{\Gamma}_h^0\|\cdot O_p(1) = O_p(1)
\end{aligned}
$$

with $N^{-1}\sum_{i=1}^{N}\sum_{h=1}^{N}\bar{\tau}_{ih}\mathbb{E}\|\boldsymbol{\varphi}_i^0\|\|\boldsymbol{\Gamma}_h^0\| \le N^{-1}\sum_{i=1}^{N}\sum_{h=1}^{N}\bar{\tau}_{ih}\sqrt{\mathbb{E}\|\boldsymbol{\varphi}_i^0\|^2\mathbb{E}\|\boldsymbol{\Gamma}_h^0\|^2} \le CN^{-1}\sum_{i=1}^{N}\sum_{h=1}^{N}\bar{\tau}_{ih} \le C^2$ by Assumption B2. Similarly, we can show that the second term is $O_p(1)$. The third term is bounded in norm by

$$
\begin{aligned}
&\left\|\text{vec}\Big(N^{-1}T^{-2}\sum_{i=1}^{N}\sum_{h=1}^{N}\big(\mathbf{V}_i'\mathbf{V}_h-\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h)\big)\mathbf{\Gamma}_h^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0\Big)\right\|\\
=&T^{-1/2}\cdot\left\|N^{-1}T^{-1/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\boldsymbol{\varphi}_i^{0\prime}\otimes\Big(\big(\mathbf{V}_i'\mathbf{V}_h-\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h)\big)\mathbf{\Gamma}_h^{0\prime}\Big)\text{vec}\big((\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}T^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\big)\right\|\\
\leq&T^{-1/2}\cdot\left\|N^{-1}T^{-1/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\sum_{t=1}^{T}\boldsymbol{\varphi}_i^{0\prime}\otimes\Big(\big(\mathbf{v}_{it}\mathbf{v}_{ht}'-\mathbb{E}(\mathbf{v}_{it}\mathbf{v}_{ht}')\big)\mathbf{\Gamma}_h^{0\prime}\Big)\right\|\cdot\|(\mathbf{\Upsilon}^0)^{-1}\|\|(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|\|T^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\|\\
=&O_p(T^{-1/2})
\end{aligned}
$$

Similarly, we can show that the forth term is $O_p(T^{-1/2})$. The fifth term is bounded in norm by

$$
\left\|\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\mathbf{V}_h\mathbf{\Gamma}_h^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\right\|+\left\|\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{h=1}^{N}(\mathbf{V}_i'\mathbf{V}_h-\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h))\mathbf{\Gamma}_h^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\right\|
$$

Similar to the argument in the proof of the first term, the former term is $O_p(T^{-1/2})$. Similar to the argument in the proof of the third term, the latter is $O_p(T^{-1})$. Then the fifth term is $O_p(T^{-1/2})$. The sixth term is bounded in norm by

$$
T^{-1/2}\cdot N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|T^{-1/2}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\|\cdot\|(\mathbf{\Upsilon}^0)^{-1}\|^2\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\|\|(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|=O_p(T^{-1/2})
$$

thus $\mathbf{a}_1=O_p(1)$ and

$$
\begin{aligned}
\mathbf{a}_1=&-\frac{1}{NT^2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbb{E}(\mathbf{V}_i'\mathbf{V}_h)\mathbf{\Gamma}_h^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0\\
&+\frac{1}{N^2T^2}\sum_{i=1}^{N}\sum_{h=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbb{E}(\mathbf{V}_\ell'\mathbf{V}_h)\mathbf{\Gamma}_h^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0+O_p(T^{-1/2}).
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
\mathbf{a}_2=&-\frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{\Sigma}\mathbf{M}_{\mathbf{F}^0}\mathbf{H}^0\boldsymbol{\varphi}_i^0\\
&-\frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{\Sigma}\boldsymbol{\varepsilon}_i+\frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{\Sigma}\mathbf{F}^0(\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i.
\end{aligned}
$$

The first term is bounded in norm by

$$
\begin{aligned}
&N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|\boldsymbol{\varphi}_i^0\|\cdot\|(\mathbf{\Upsilon}^0)^{-1}\|\|(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|\cdot T^{-1}\|\mathbf{F}^{0\prime}\mathbf{\Sigma}\mathbf{M}_{\mathbf{F}^0}\mathbf{H}^0\|=T^{-1}\|\mathbf{F}^{0\prime}\mathbf{\Sigma}\mathbf{M}_{\mathbf{F}^0}\mathbf{H}^0\|\cdot O_p(1)\\
\leq&T^{-1}\|\mathbf{F}^{0\prime}\mathbf{\Sigma}\mathbf{H}^0\|\cdot O_p(1)+T^{-1}\|\mathbf{F}^{0\prime}\mathbf{\Sigma}\mathbf{F}^0\|\|(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|T^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\|\cdot O_p(1)\\
=&T^{-1}\|\mathbf{F}^{0\prime}\mathbf{\Sigma}\mathbf{H}^0\|\cdot O_p(1)+T^{-1}\|\mathbf{F}^{0\prime}\mathbf{\Sigma}\mathbf{F}^0\|\cdot O_p(1)=O_p(1)
\end{aligned}
$$

because $T^{-1}\|\mathbf{F}^{0\prime}\mathbf{\Sigma}\mathbf{H}^0\|=O_p(1)$ and $T^{-1}\|\mathbf{F}^{0\prime}\mathbf{\Sigma}\mathbf{F}^0\|=O_p(1)$, where the former holds because

$$
\begin{aligned}
&T^{-1}\mathbb{E}\left\|\mathbf{F}^{0\prime}\mathbf{\Sigma}\mathbf{H}^0\right\|=N^{-1}T^{-1}\mathbb{E}\left\|\sum_{s=1}^{T}\sum_{t=1}^{T}\sum_{i=1}^{N}\mathbf{f}_s^0\mathbf{f}_t^{0\prime}\mathbb{E}(\mathbf{v}_{is}'\mathbf{v}_{it})\right\|\leq N^{-1}T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\sum_{i=1}^{N}\mathbb{E}(\|\mathbf{f}_s^0\|\|\mathbf{f}_t^0\|)\tilde{\tau}_{st}\\
\leq&N^{-1}T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\sum_{i=1}^{N}\sqrt{\mathbb{E}\|\mathbf{f}_s^0\|^2\mathbb{E}\|\mathbf{f}_t^0\|^2}\tilde{\tau}_{st}\leq CT^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\tau}_{st}\leq C
\end{aligned}
$$

and the latter also holds whose proof is similar to that of the former. The second is equal to

$$
\begin{aligned}
&- N^{-2}T^{-1}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\mathbf{f}_t^0\mathbb{E}(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t})\boldsymbol{\varepsilon}_{it}\\
=&- N^{-2}T^{-1}\sum_{\ell=1}^{N}\sum_{s=1}^{T}\sum_{t=1}^{T}\mathbb{E}(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t})\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}\varepsilon_{it}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{f}_t^0
\end{aligned}
$$

which is bounded in norm by

$$
\begin{aligned}
&N^{-2}T^{-1}\sum_{\ell=1}^{N}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\tau}_{st}\|\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}\varepsilon_{it}\|\|\mathbf{f}_t^0\|\cdot\|(\mathbf{\Upsilon}^0)^{-1}\|\|(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|\\
\le&N^{-1/2}\cdot\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\tau}_{st}\|N^{-1/2}\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}\varepsilon_{it}\|\|\mathbf{f}_t^0\|\cdot O_p(1)=O_p(N^{-1/2})
\end{aligned}
$$

because $T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\tau}_{st}\mathbb{E}\|N^{-1/2}\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}\varepsilon_{it}\|\|\mathbf{f}_t^0\|\le T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\tau}_{st}\sqrt{\mathbb{E}\|N^{-1/2}\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}\varepsilon_{it}\|^2\mathbb{E}\|\mathbf{f}_t^0\|^2}\le CT^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\tau}_{st}\le C$ by Assumption B2, A1, C and D. The third term is bounded in norm by

$$
T^{-1/2}\cdot N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|T^{-1/2}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\|\cdot\|(\mathbf{\Upsilon}^0)^{-1}\|\|(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|^2\|T^{-1}\mathbf{F}^{0\prime}\mathbf{\Sigma}\mathbf{F}^0\|=O_p(T^{-1/2})
$$

Then $\mathbf{a}_2=O_p(1)$ and

$$
\mathbf{a}_2=-\frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{\Sigma}\mathbf{M}_{\mathbf{F}^0}\mathbf{H}^0\boldsymbol{\varphi}_i^0+O_p(\delta_{NT}^{-1}).
$$

This completes the proof.

B Proofs of Lemmas in Appendix B

Proof of Lemma B.1. With $\widehat{\beta}_{1SIV}-\beta=O_p(N^{-1/2}T^{-1/2})$, we can follow the argument in the proof of Proposition A.1(ii), Lemma A.3 and Lemma A.4(iii) to prove this lemma. Thus, we omitted the details. □

Proof of Lemma B.2. Consider (a). With (B.1), we have

$$
\begin{aligned}
&N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i\|\|T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i\|\\
\le&N^{-2}T^{-2}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i\|\|\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{X}_\ell(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})'\mathbf{X}_\ell'\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1}\|\\
&+N^{-2}T^{-2}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i\|\|\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{X}_\ell(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}})\mathbf{u}_\ell'\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1}\|\\
&+N^{-2}T^{-2}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i\|\|\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{u}_\ell(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})'\mathbf{X}_\ell'\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1}\|+N^{-2}T^{-2}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i\|\|\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{H}^0\boldsymbol{\varphi}_l^0\boldsymbol{\varepsilon}_\ell'\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1}\|\\
&+N^{-2}T^{-2}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i\|\|\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\boldsymbol{\varepsilon}_\ell\boldsymbol{\varphi}_l^{0\prime}\mathbf{H}^{0\prime}\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1}\|+N^{-2}T^{-2}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i\|\|\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\boldsymbol{\varepsilon}_\ell\boldsymbol{\varepsilon}_\ell'\widehat{\mathbf{H}}\boldsymbol{\Xi}^{-1}\|
\end{aligned}
$$

The first term is bounded in norm by

$$
N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i\|\|T^{-1/2}\boldsymbol{\varepsilon}_i\|\cdot N^{-1}\sum_{\ell=1}^{N}\|T^{-1/2}\mathbf{X}_\ell\|^2\cdot\|T^{-1/2}\widehat{\mathbf{H}}\|\|\boldsymbol{\Xi}^{-1}\|\cdot\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|^2=O_p(\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|^2)
$$

thus, the first term is $O_p(\delta_{NT}^{-4})$. Similarly, we can prove that the second and the third terms both are $O_p(\|\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}_{1SIV}\|) = O_p(N^{-1/2}T^{-1/2})$. Following the argument in the proof of Lemma A.2(b), the last three terms are $O_p(\delta_{NT}^{-2})$. Consequently, $N^{-1}\sum_{i=1}^N \|T^{-1/2}\mathbf{V}_i\|\|T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i\| = O_p(\delta_{NT}^{-2})$.

Consider (b). With (B.1), $N^{-1}\sum_{i=1}^N \|\boldsymbol{\varphi}_i^0\|\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})\|$ can be decomposed into six terms. The three terms involved of $\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}_{1SIV}$ are $O_p(\|\boldsymbol{\beta} - \widehat{\boldsymbol{\beta}}_{1SIV}\|) = O_p(N^{-1/2}T^{-1/2})$. The remaining three terms can be proved to be $O_p(\delta_{NT}^{-2})$ by following the argument in the proof of Lemma A.2(a), then we have (b).

Following the way in the proof of Lemma A.2(c), we can prove (c). This completes the proof. □

Proof of Lemma B.3. Since

$$\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}} - \mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{H}^0} = \mathbf{M}_{\mathbf{F}^0}(\mathbf{M}_{\widehat{\mathbf{H}}} - \mathbf{M}_{\mathbf{H}^0}) + (\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0})\mathbf{M}_{\mathbf{H}^0} + (\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0})(\mathbf{M}_{\widehat{\mathbf{H}}} - \mathbf{M}_{\mathbf{H}^0})$$

we have

$$\begin{aligned}
&N^{-1/2}T^{-1/2}\sum_{i=1}^N \mathbf{V}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i - N^{-1/2}T^{-1/2}\sum_{i=1}^N \mathbf{V}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\
=&N^{-1/2}T^{-1/2}\sum_{i=1}^N \mathbf{V}_i'\mathbf{M}_{\mathbf{F}^0}(\mathbf{M}_{\widehat{\mathbf{H}}} - \mathbf{M}_{\mathbf{H}^0})\mathbf{u}_i + N^{-1/2}T^{-1/2}\sum_{i=1}^N \mathbf{V}_i'(\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0})\mathbf{M}_{\mathbf{H}^0}\mathbf{u}_i \\
&+ N^{-1/2}T^{-1/2}\sum_{i=1}^N \mathbf{V}_i'(\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0})(\mathbf{M}_{\widehat{\mathbf{H}}} - \mathbf{M}_{\mathbf{H}^0})\mathbf{u}_i \\
=&\mathbb{F}_1 + \mathbb{F}_2 + \mathbb{F}_3
\end{aligned}$$

Now we consider the term $\mathbb{F}_1$. Since $\mathbf{M}_{\widehat{\mathbf{H}}} - \mathbf{M}_{\mathbf{H}^0} = -T^{-1}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime} - T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})' - T^{-1}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})' - T^{-1}\mathbf{H}^0(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1})\mathbf{H}^{0\prime}$, we have

$$\begin{aligned}
&N^{-1/2}T^{-1/2}\sum_{i=1}^N \mathbf{V}_i'\mathbf{M}_{\mathbf{F}^0}(\mathbf{M}_{\widehat{\mathbf{H}}} - \mathbf{M}_{\mathbf{H}^0})\mathbf{u}_i \\
=& - N^{-1/2}T^{-3/2}\sum_{i=1}^N \mathbf{V}_i'(\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1} - \mathbf{H}^0)(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i \\
&- N^{-1/2}T^{-3/2}\sum_{i=1}^N \mathbf{V}_i'(\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1} - \mathbf{H}^0)(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - (T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1})\mathbf{H}^{0\prime}\mathbf{u}_i \\
&- N^{-1/2}T^{-3/2}\sum_{i=1}^N \mathbf{V}_i'\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i - N^{-1/2}T^{-3/2}\sum_{i=1}^N \mathbf{V}_i'(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i \\
&- N^{-1/2}T^{-3/2}\sum_{i=1}^N \mathbf{V}_i'\mathbf{H}^0(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - (T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1})\mathbf{H}^{0\prime}\mathbf{u}_i \\
&+ N^{-1/2}T^{-3/2}\sum_{i=1}^N \mathbf{V}_i'\mathbf{F}^0\left(\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\mathbf{F}^{0\prime}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})\widehat{\mathbf{H}}'\mathbf{u}_i \\
&+ N^{-1/2}T^{-3/2}\sum_{i=1}^N \mathbf{V}_i'\mathbf{F}^0\left(\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i \\
&+ N^{-1/2}T^{-3/2}\sum_{i=1}^N \mathbf{V}_i'\mathbf{F}^0\left(\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - (T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1})\mathbf{H}^{0\prime}\mathbf{u}_i \\
=&\mathbb{F}_{1.1} + \mathbb{F}_{1.2} + \mathbb{F}_{1.3} + \mathbb{F}_{1.4} + \mathbb{F}_{1.5} + \mathbb{F}_{1.6} + \mathbb{F}_{1.7} + \mathbb{F}_{1.8}
\end{aligned}$$

We first consider the terms $\mathbb{F}_{1.2}$ to $\mathbb{F}_{1.8}$. Note that $\mathbf{u}_i = \mathbf{H}^0\boldsymbol{\varphi}_i^0 + \boldsymbol{\varepsilon}_i$, $\mathbb{F}_{1.2}$ is bounded in norm by

$$\begin{aligned}
&N^{1/2}T^{1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\|\|\boldsymbol{\varphi}_i^0\|\cdot\|T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\|\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\|\|\boldsymbol{\mathcal{R}}^{-1}\|\\
&+N^{1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\|\|T^{-1/2}\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i\|\cdot\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\|\|\boldsymbol{\mathcal{R}}^{-1}\|\\
=&O_p(N^{1/2}T^{1/2}\delta_{NT}^{-4})
\end{aligned}$$

by Lemma B.1(e), (f) and Lemma B.2 (b) and $N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\|\|T^{-1/2}\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i\| = O_p(\delta_{NT}^{-2})$, which can be proved similar to the argument in the proof of Lemma B.2 (b). Similar to the proof of Lemma B.2(a), we have $N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i'\mathbf{H}^0\|\|T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i\| = O_p(\delta_{NT}^{-2})$, then $\mathbb{F}_{1.3}$ is bounded in norm by

$$\begin{aligned}
&N^{1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i'\mathbf{H}^0\|\|\boldsymbol{\varphi}_i^0\|\cdot\|T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0\|\|\boldsymbol{\mathcal{R}}\|\\
&+N^{1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i'\mathbf{H}^0\|\|T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i\|\cdot\|\boldsymbol{\mathcal{R}}\| = O_p(N^{1/2}\delta_{NT}^{-2})
\end{aligned}$$

by Lemmas B.1(b), (c). $\mathbb{F}_{1.4}$ is bounded in norm by

$$\begin{aligned}
&N^{1/2}T^{1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\|\|\boldsymbol{\varphi}_i^0\|\cdot\|T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0\|\\
&+N^{1/2}T^{1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\|\|T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i\| = O_p(N^{1/2}T^{1/2}\delta_{NT}^{-4})
\end{aligned}$$

by Lemmas B.1(b) and Lemmas B.2 (b), (c). $\mathbb{F}_{1.5}$ is bounded in norm by

$$\begin{aligned}
&N^{1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i'\mathbf{H}^0\|\|\boldsymbol{\varphi}_i^0\|\cdot\|T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\|\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\|\\
&+N^{1/2}T^{-1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i'\mathbf{H}^0\|\|T^{-1/2}\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i\|\cdot\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\| = O_p(N^{1/2}\delta_{NT}^{-2})
\end{aligned}$$

by Lemma B.1(f). Similarly, we can prove that $\mathbb{F}_{1.6}$, $\mathbb{F}_{1.7}$ and $\mathbb{F}_{1.8}$ both are $O_p(N^{1/2}\delta_{NT}^{-2})$. Consider the term $\mathbb{F}_{1.1}$. With (B.1) and the definition of $\boldsymbol{\mathcal{R}}$, we have

$$\begin{aligned}
&\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1}-\mathbf{H}^0\\
=&N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{X}_i(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})'\mathbf{X}_i'\widehat{\mathbf{H}}(T^{-1}\mathbf{H}^{0\prime}\widehat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\\
&+N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{X}_i(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})\mathbf{u}_i'\widehat{\mathbf{H}}(T^{-1}\mathbf{H}^{0\prime}\widehat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\\
&+N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{u}_i(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})'\mathbf{X}_i'\widehat{\mathbf{H}}(T^{-1}\mathbf{H}^{0\prime}\widehat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}+N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{H}^0\boldsymbol{\varphi}_i^0\boldsymbol{\varepsilon}_i'\widehat{\mathbf{H}}(T^{-1}\mathbf{H}^{0\prime}\widehat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\\
&+N^{-1}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i\boldsymbol{\varphi}_i^{0\prime}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}+N^{-1}T^{-1}\sum_{i=1}^{N}\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i'\widehat{\mathbf{H}}(T^{-1}\mathbf{H}^{0\prime}\widehat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}
\end{aligned}$$

we can decompose the term $\mathbb{F}_{1.1}$ as follows

$$
\begin{aligned}
&-N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'(\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1}-\mathbf{H}^0)(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i\\
=&-N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i\\
&-N^{-3/2}T^{-5/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\mathbf{H}\boldsymbol{\varphi}_h^{0}\boldsymbol{\varepsilon}_h'\widehat{\mathbf{H}}(T^{-1}\mathbf{H}^{0\prime}\hat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i\\
&-N^{-3/2}T^{-5/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\boldsymbol{\varepsilon}_h\boldsymbol{\varepsilon}_h'\widehat{\mathbf{H}}(T^{-1}\mathbf{H}^{0\prime}\hat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i\\
&-N^{-3/2}T^{-5/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\mathbf{X}_h(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})'\mathbf{X}_h'\widehat{\mathbf{H}}(T^{-1}\mathbf{H}^{0\prime}\hat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i\\
&-N^{-3/2}T^{-5/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\mathbf{X}_h(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})\mathbf{u}_h'\widehat{\mathbf{H}}(T^{-1}\mathbf{H}^{0\prime}\hat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i\\
&-N^{-3/2}T^{-5/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\mathbf{u}_h(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})'\mathbf{X}_h'\widehat{\mathbf{H}}(T^{-1}\mathbf{H}^{0\prime}\hat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i\\
=&\mathbb{F}_{1.1.1}+\mathbb{F}_{1.1.2}+\mathbb{F}_{1.1.3}+\mathbb{F}_{1.1.4}+\mathbb{F}_{1.1.5}+\mathbb{F}_{1.1.6}
\end{aligned}
$$

We consider the last five terms $\mathbb{F}_{1.1.2}$ to $\mathbb{F}_{1.1.6}$. $\mathbb{F}_{1.1.2}$ is bounded in norm by

$$
\begin{aligned}
&N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i'\mathbf{H}^0\|\|T^{-1/2}\mathbf{u}_i\|\cdot\|N^{-1/2}T^{-1}\sum_{h=1}^{N}\boldsymbol{\varphi}_h^0\boldsymbol{\varepsilon}_h'\widehat{\mathbf{H}}\|\|(T^{-1}\mathbf{H}^{0\prime}\hat{\mathbf{H}})^{-1}\|\|(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\|\|T^{-1/2}\mathbf{H}^0\|\|(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\|\\
=&O_p(1)\cdot\|N^{-1/2}T^{-1}\sum_{h=1}^{N}\boldsymbol{\varphi}_h^0\boldsymbol{\varepsilon}_h'\widehat{\mathbf{H}}\|\\
\leq&O_p(T^{-1/2})\cdot\|N^{-1/2}T^{-1}\sum_{h=1}^{N}\boldsymbol{\varphi}_h^0\boldsymbol{\varepsilon}_h'\mathbf{H}^0\|\|\boldsymbol{\mathcal{R}}\|+O_p(1)\cdot\|N^{-1/2}T^{-1/2}\sum_{h=1}^{N}\boldsymbol{\varphi}_h^0\boldsymbol{\varepsilon}_h'\|\|\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}}\|=O_p(\delta_{NT}^{-1})
\end{aligned}
$$

by Lemmas B.1(a). As $\mathbb{E}\left(\mathbf{V}_i'\boldsymbol{\varepsilon}_h\right)=0$, we can follow the argument in the proof of $\mathbb{D}_{1.4}$, we can prove that $\mathbb{F}_{1.1.3}=O_p\left(\delta_{NT}^{-1}\right)$. $\mathbb{F}_{1.1.4}$ is bounded in norm by

$$
\begin{aligned}
&N^{1/2}T^{1/2}\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|^2\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1/2}\mathbf{V}_i\|\|T^{-1/2}\mathbf{u}_i\|\\
&\quad\times N^{-1}\sum_{h=1}^{N}\|T^{-1/2}\mathbf{X}_h\|^2\|(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\|\|(T^{-1}\mathbf{H}^{0\prime}\hat{\mathbf{H}})^{-1}\|\|(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\|\|T^{-1/2}\widehat{\mathbf{H}}\|\|T^{-1/2}\mathbf{H}^0\|\\
=&O_p(N^{-1/2}T^{-1/2})
\end{aligned}
$$

With definitions of $\mathbf{u}_h$ and $\mathbf{u}_i$, $\mathbb{F}_{1.1.5}$ is bounded in norm by

$$
\begin{aligned}
&\Big\|\mathrm{vec}\big(N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\mathbf{X}_h(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\boldsymbol{\varphi}_i^0\big)\Big\|\\
&+\Big\|N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\mathbf{X}_h(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})\boldsymbol{\varepsilon}_h'\widehat{\mathbf{H}}(T^{-1}\mathbf{H}^{0\prime}\hat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\boldsymbol{\varphi}_i^0\Big\|\\
&+\Big\|N^{-3/2}T^{-5/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\mathbf{X}_h(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})\mathbf{u}_h'\widehat{\mathbf{H}}(T^{-1}\mathbf{H}^{0\prime}\hat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i\Big\|
\end{aligned}
$$

The first term is equal to $\big\|(N^{-1/2}T^{-1/2}\sum_{i=1}^N \boldsymbol{\varphi}_i^{0\prime}\otimes\mathbf{V}_i')\mathrm{vec}\big(N^{-1}\sum_{h=1}^N \mathbf{X}_h(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\big)\big\|$, then is bounded in norm by

$$
\begin{aligned}
&\|N^{-1/2}T^{-1/2}\sum_{i=1}^N \boldsymbol{\varphi}_i^{0\prime}\otimes\mathbf{V}_i'\|\cdot N^{-1}\sum_{h=1}^N\|T^{-1/2}\mathbf{X}_h\|\|\boldsymbol{\varphi}_h^0\|\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\|\cdot T^{1/2}\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|\\
=&O_p(T^{1/2}\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|)=O_p(N^{-1/2})
\end{aligned}
$$

The second term is bounded in norm by

$$
\begin{aligned}
&N^{1/2}\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|\cdot N^{-1}\sum_{i=1}^N\|T^{-1/2}\mathbf{V}_i\|\|\boldsymbol{\varphi}_i^0\|\cdot\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\|\|(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\|\cdot N^{-1}\sum_{h=1}^N\|T^{-1/2}\mathbf{X}_h\|\|T^{-1/2}\boldsymbol{\varepsilon}_h'\widehat{\mathbf{H}}\|\\
=&O_p(N^{1/2}\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|)\cdot N^{-1}\sum_{h=1}^N\|T^{-1/2}\mathbf{X}_h\|\|T^{-1/2}\boldsymbol{\varepsilon}_h'\widehat{\mathbf{H}}\|\\
\le&O_p(N^{1/2}\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|)\cdot N^{-1}\sum_{h=1}^N\|T^{-1/2}\mathbf{X}_h\|\|T^{-1/2}\boldsymbol{\varepsilon}_h'\mathbf{H}^0\|\|\boldsymbol{\mathcal{R}}\|\\
&+O_p(N^{1/2}\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|)\cdot T^{1/2}N^{-1}\sum_{h=1}^N\|T^{-1/2}\mathbf{X}_h\|\|T^{-1/2}\boldsymbol{\varepsilon}_h'\|\|T^{-1/2}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\|\\
=&O_p(T^{-1/2})+O_p(\delta_{NT}^{-1})
\end{aligned}
$$

by Lemma B.1(a), (c). The third term is bounded in norm by

$$
\begin{aligned}
&N^{1/2}\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|\cdot N^{-1}\sum_{i=1}^N\|T^{-1/2}\mathbf{V}_i\|\|T^{-1/2}\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i\|\cdot N^{-1}\sum_{h=1}^N\|T^{-1/2}\mathbf{X}_h\|\|T^{-1/2}\mathbf{u}_h\|\\
&\cdot\|T^{-1/2}\widehat{\mathbf{H}}\|\|(T^{-1}\mathbf{H}^{0\prime}\widehat{\mathbf{H}})^{-1}\|\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\|\|(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\|=O_p(N^{1/2}\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|)=O_p(T^{-1/2})
\end{aligned}
$$

Then $\mathbb{F}_{1.1.5}=O_p(\delta_{NT}^{-1})$.

With the definitions of $\mathbf{u}_h$ and $\mathbf{u}_i$, $\mathbb{F}_{1.1.6}$ is bounded in norm by

$$
\begin{aligned}
&\Big\|N^{-3/2}T^{-3/2}\sum_{i=1}^N\sum_{h=1}^N\mathbf{V}_i'\mathbf{H}^0\boldsymbol{\varphi}_h^0(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})'\mathbf{X}_h'\widehat{\mathbf{H}}(T^{-1}\mathbf{H}^{0\prime}\widehat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0\Big\|\\
&+\Big\|N^{-3/2}T^{-3/2}\sum_{i=1}^N\sum_{h=1}^N\mathbf{V}_i'\boldsymbol{\varepsilon}_h(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})'\mathbf{X}_h'\widehat{\mathbf{H}}(T^{-1}\mathbf{H}^{0\prime}\widehat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0\Big\|\\
&+\Big\|N^{-3/2}T^{-5/2}\sum_{i=1}^N\sum_{h=1}^N\mathbf{V}_i'\mathbf{u}_h(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})'\mathbf{X}_h'\widehat{\mathbf{H}}(T^{-1}\mathbf{H}^{0\prime}\widehat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i\Big\|\\
\le&N^{1/2}\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|\cdot N^{-1}\sum_{i=1}^N\|T^{-1/2}\mathbf{V}_i'\mathbf{H}^0\|\|\boldsymbol{\varphi}_i^0\|\cdot N^{-1}\sum_{h=1}^N\|\boldsymbol{\varphi}_h\|\|T^{-1/2}\mathbf{X}_h\|\|T^{-1}\widehat{\mathbf{H}}\|\|(T^{-1}\mathbf{H}^{0\prime}\widehat{\mathbf{H}})^{-1}\|\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\|\\
&+N^{-1/2}\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|\cdot N^{-2}\sum_{h=1}^N\sum_{i=1}^N\|\boldsymbol{\varphi}_i^0\|\|T^{-1/2}\mathbf{V}_i'\boldsymbol{\varepsilon}_h\|\|T^{-1/2}\mathbf{X}_h\|\cdot\|T^{-1/2}\widehat{\mathbf{H}}\|\|(T^{-1}\mathbf{H}^{0\prime}\widehat{\mathbf{H}})^{-1}\|\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\|\\
&+N^{1/2}\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|\cdot N^{-1}\sum_{i=1}^N\|T^{-1/2}\mathbf{V}_i'\mathbf{H}^0\|\|T^{-1/2}\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i\|\cdot N^{-1}\sum_{h=1}^N\|T^{-1/2}\mathbf{u}_h\|\|T^{-1/2}\mathbf{X}_h\|\\
&\times\|T^{-1/2}\widehat{\mathbf{H}}\|\|(T^{-1}\mathbf{H}^{0\prime}\widehat{\mathbf{H}})^{-1}\|\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\|\|(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\|\\
=&O_p(N^{1/2}\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|)=O_p(T^{-1/2})\,.
\end{aligned}
$$

Combining the above terms and noting that $\mathbb{F}_1 = -N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i = O_p(T^{-1/2})$, we can show that

$$\mathbb{F}_1 = -N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{h=1}^{N}\mathbf{V}_i'\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_{\varphi}^0)^{-1}\boldsymbol{\varphi}_i^0 + O_p(\delta_{NT}^{-1}).$$

Consider the term $\mathbb{F}_2$. Since $\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0} = -T^{-1}\mathbf{F}^0\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}(\widehat{\mathbf{F}}\mathbf{R}^{-1}-\mathbf{F}^0)' - \mathbf{F}^0\left[\mathbf{R}\mathbf{R}' - \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right](\widehat{\mathbf{F}}\mathbf{R}^{-1} - \mathbf{F}^0)' - T^{-1}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime} - T^{-1}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' - T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}' - \left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}$ we have

$$\begin{aligned}
&N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{V}_i'(\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0})\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\
&= -N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{F}^0\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}(\widehat{\mathbf{F}}\mathbf{R}^{-1} - \mathbf{F}^0)'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\
&- N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{F}^0\left[\mathbf{R}\mathbf{R}' - \left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right](\widehat{\mathbf{F}}\mathbf{R}^{-1} - \mathbf{F}^0)'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\
&- N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\
&- N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\
&- N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{F}^0\left(\mathbf{R}\mathbf{R}' - \left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\
&= \mathbb{F}_{2.1} + \mathbb{F}_{2.2} + \mathbb{F}_{2.3} + \mathbb{F}_{2.4} + \mathbb{F}_{2.5}.
\end{aligned}$$

For the terms $\mathbb{F}_{2.2}$ to $\mathbb{F}_{2.5}$, we can easily show that

$$\begin{aligned}
\|\mathbb{F}_{2.2}\| &\le \sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\mathbf{R}\mathbf{R}' - \left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\frac{(\widehat{\mathbf{F}}\mathbf{R}^{-1} - \mathbf{F}^0)'\boldsymbol{\varepsilon}_i}{T}\right\| \\
&+ \sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\mathbf{R}\mathbf{R}' - \left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\frac{(\widehat{\mathbf{F}}\mathbf{R}^{-1} - \mathbf{F}^0)'\mathbf{H}^0}{T}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\| \\
&= O_p\left(\sqrt{N}\delta_{NT}^{-4}\right),
\end{aligned}$$

$$\begin{aligned}
\|\mathbb{F}_{2.3}\| &\le N^{1/2}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})}{T}\right\|\|\mathbf{R}'\|\left\|\frac{\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\| \\
&+ N^{1/2}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})}{T}\right\|\|\mathbf{R}'\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\| \\
&= O_p\left(N^{1/2}\delta_{NT}^{-2}\right),
\end{aligned}$$

$$\begin{aligned}
\|\mathbb{F}_{2.4}\| &\le N^{1/2}T^{1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\|\|T^{-1}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\boldsymbol{\varepsilon}_i\| \\
&+ N^{1/2}\cdot N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\|\|T^{-1/2}\mathbf{H}'\boldsymbol{\varepsilon}_i\|\cdot\|T^{-1}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\mathbf{H}^0\|\|(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\|
\end{aligned}$$

$$= O_p(N^{1/2}T^{1/2}\delta_{NT}^{-4}),$$

$$\begin{aligned}\|\mathbb{F}_{2.5}\| &\leq \sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\mathbf{R}\mathbf{R}' - \left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\| \\ &\quad + \sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\mathbf{R}\mathbf{R}' - \left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\| \\ &\quad = O_p\left(\sqrt{\frac{N}{T}}\delta_{NT}^{-2}\right),\end{aligned}$$

by Lemmas B.1 (b), Lemma B.2 (i). Then we have $\mathbb{F}_2 = O_p(N^{1/2}\delta_{NT}^{-2}) + O_p(N^{1/2}T^{1/2}\delta_{NT}^{-4})$. Using

$$\begin{aligned}\widehat{\mathbf{F}}\mathbf{R}^{-1} - \mathbf{F}^0 = N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{F}^0\boldsymbol{\Gamma}_i^0\mathbf{V}_i'\widehat{\mathbf{F}}\left(\frac{\mathbf{F^{0\prime}}\widehat{\mathbf{F}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1} + N^{-1}\sum_{i=1}^{N}\mathbf{V}_i\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1} \\ + N^{-1}T^{-1}\sum_{i=1}^{N}\mathbf{V}_i\mathbf{V}_i'\widehat{\mathbf{F}}\left(\frac{\mathbf{F^{0\prime}}\widehat{\mathbf{F}}}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\end{aligned}$$

$$\begin{aligned}\mathbb{F}_{2.1} &= -N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{F}^0\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}N^{-1}T^{-1}\sum_{j=1}^{N}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F^{0\prime}}\widehat{\mathbf{F}}}{T}\right)^{-1}\widehat{\mathbf{F}}'\mathbf{V}_j\boldsymbol{\Gamma}_j^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\mathbf{H}^0}\varepsilon_i \\ &\quad - N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{F}^0\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}N^{-1}\sum_{j=1}^{N}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_j^0\mathbf{V}_j'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\ &\quad - N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{F}^0\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}N^{-1}T^{-1}\sum_{j=1}^{N}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F^{0\prime}}\widehat{\mathbf{F}}}{T}\right)^{-1}\widehat{\mathbf{F}}'\mathbf{V}_j\mathbf{V}_j'\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\ &\quad = \mathbb{F}_{2.1.1} + \mathbb{F}_{2.1.2} + \mathbb{F}_{2.1.3}.\end{aligned}$$

$$\begin{aligned}\|\mathbb{F}_{2.1.1}\| &\leq \left\|N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{F}^0\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}N^{-1}T^{-1}\sum_{j=1}^{N}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F^{0\prime}}\widehat{\mathbf{F}}}{T}\right)^{-1}\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{V}_j\boldsymbol{\Gamma}_j^{0\prime}\mathbf{F}^{0\prime}\varepsilon_i\right\| \\ &\quad + \left\|N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{F}^0\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}N^{-1}T^{-1}\sum_{j=1}^{N}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F^{0\prime}}\widehat{\mathbf{F}}}{T}\right)^{-1}\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{V}_j\boldsymbol{\Gamma}_j^{0}\mathbf{F}^{0\prime}\mathbf{P}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i\right\| \\ &\quad + \left\|N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{F}^0\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}N^{-1}T^{-1}\sum_{j=1}^{N}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F^{0\prime}}\widehat{\mathbf{F}}}{T}\right)^{-1}\frac{(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\mathbf{V}_j}{T}\boldsymbol{\Gamma}_j^{0\prime}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\right\| \\ &\quad + \left\|N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{F}^0\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}N^{-1}T^{-1}\sum_{j=1}^{N}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F^{0\prime}}\widehat{\mathbf{F}}}{T}\right)^{-1}\frac{(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\mathbf{V}_j}{T}\boldsymbol{\Gamma}_j^{0\prime}\mathbf{F}^{0\prime}\mathbf{P}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i\right\|\end{aligned}$$

$$\begin{aligned}&\leq \sqrt{\frac{N}{T^2}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\frac{\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\|(\boldsymbol{\Upsilon}^0)^{-1}\|\left\|\left(\frac{\mathbf{F^{0\prime}}\widehat{\mathbf{F}}}{T}\right)^{-1}\right\|\|\mathbf{R}'\|\,N^{-1}\sum_{j=1}^{N}\left\|\frac{\mathbf{F}^{0\prime}\mathbf{V}_j}{\sqrt{T}}\right\|\|\boldsymbol{\Gamma}_j'\| \\ &+ \sqrt{\frac{N}{T^2}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\|(\boldsymbol{\Upsilon}^0)^{-1}\|\left\|\left(\frac{\mathbf{F^{0\prime}}\widehat{\mathbf{F}}}{\boldsymbol{T}}\right)^{-1}\right\|\|\mathbf{R}'\|\end{aligned}$$

$$
\begin{aligned}
&\times N^{-1}\sum_{j=1}^{N}\left\|\frac{\mathbf{F}^{0\prime}\mathbf{V}_j}{\sqrt{T}}\right\|\|\boldsymbol{\Gamma}_j'\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\| \\
&+\sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\frac{\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\left(\frac{\mathbf{F^{0\prime}\widehat{F}}}{\boldsymbol{T}}\right)^{-1}\right\|\|(\boldsymbol{\Upsilon}^0)^{-1}\|\,N^{-1}\sum_{j=1}^{N}\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{V}_j}{T}\right\|\|\boldsymbol{\Gamma}_j^{0\prime}\| \\
&+\sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\left\|\left(\frac{\mathbf{F^{0\prime}\widehat{F}}}{\boldsymbol{T}}\right)^{-1}\right\|\|(\boldsymbol{\Upsilon}^0)^{-1}\|\,N^{-1}\sum_{j=1}^{N}\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{V}_j}{T}\right\|\|\boldsymbol{\Gamma}_j^{0\prime}\| \\
&\times\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\| \\
&=O_p\left(\sqrt{\frac{N}{T^2}}\right)+O_p\left(\sqrt{\frac{N}{T}}\delta_{NT}^{-2}\right).
\end{aligned}
$$

$$
\begin{aligned}
\|\mathbb{F}_{2.1.2}\| &\le \frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\|\boldsymbol{\Upsilon}^{-1}\|\left\|\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\boldsymbol{\Gamma}_j^0\frac{\mathbf{V}_j'\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\| \\
&+\sqrt{\frac{N}{T^2}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right\|\|(\boldsymbol{\Upsilon}^0)^{-1}\|\,N^{-1}\sum_{j=1}^{N}\|\boldsymbol{\Gamma}_j^0\|\left\|\frac{\mathbf{V}_j'\mathbf{H}^0}{\sqrt{T}}\right\|\left\|\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\| \\
&=O_p\left(T^{-1/2}\right)+O_p\left(\sqrt{\frac{N}{T^2}}\right).
\end{aligned}
$$

By a similar derivation for Lemma A.5, we can show that

$$
\begin{aligned}
\mathbb{F}_{2.1.3} &= -N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{F}^0\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}N^{-1}T^{-1}\sum_{j=1}^{N}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F^{0\prime}F^0}}{\boldsymbol{T}}\right)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_j\mathbf{V}_j'\right)\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i \\
&+O_p\left(\delta_{NT}^{-1}\right)+O_p\left(\sqrt{\frac{N}{T}}\delta_{NT}^{-1}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&-\sqrt{\frac{N}{T^2}}\frac{1}{N}\sum_{i=1}^{N}\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F^{0\prime}F^0}}{\boldsymbol{T}}\right)^{-1}N^{-1}\sum_{j=1}^{N}\frac{\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_j\mathbf{V}_j'\right)\boldsymbol{\varepsilon}_i}{T} \\
&-\sqrt{\frac{N}{T^3}}\frac{1}{N}\sum_{i=1}^{N}\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}(\boldsymbol{\Upsilon}^0)^{-1}\left(\frac{\mathbf{F^{0\prime}F^0}}{\boldsymbol{T}}\right)^{-1}N^{-1}\sum_{j=1}^{N}\frac{\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_j\mathbf{V}_j'\right)\mathbf{H}^0}{T}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}} \\
&=O_p\left(\sqrt{\frac{N}{T^2}}\right),
\end{aligned}
$$

hence, $\mathbb{F}_{2.1.3}=O_p\left(\delta_{NT}^{-1}\right)+O_p\left(\sqrt{\frac{N}{T^2}}\right)$. Therefore, $\mathbb{F}_2=O_p\left(\delta_{NT}^{-1}\right)$.

Now we consider the term $\mathbb{F}_3$. We have

$$
\begin{aligned}
\mathbb{F}_3 &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'\left(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0}\right)\left(\mathbf{M}_{\widehat{\mathbf{H}}}-\mathbf{M}_{\mathbf{H}^0}\right)\mathbf{u}_i \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\mathbf{u}_i
\end{aligned}
$$

$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i$$
$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i$$
$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{u}_i$$

$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\mathbf{u}_i$$
$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i$$
$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i$$
$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{u}_i$$

$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\mathbf{u}_i$$
$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i$$
$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i$$
$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'T^{-1}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{u}_i$$

$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\mathbf{u}_i$$
$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i$$
$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i$$
$$+\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{u}_i$$

$$\begin{aligned}&=\mathbb{F}_{3.1.1}+\mathbb{F}_{3.1.2}+\mathbb{F}_{3.1.3}+\mathbb{F}_{3.1.4}\\&+\mathbb{F}_{3.2.1}+\mathbb{F}_{3.2.2}+\mathbb{F}_{3.2.3}+\mathbb{F}_{3.2.4}+\\&+\mathbb{F}_{3.3.1}+\mathbb{F}_{3.3.2}+\mathbb{F}_{3.3.3}+\mathbb{F}_{3.3.4}+\\&+\mathbb{F}_{3.4.1}+\mathbb{F}_{3.4.2}+\mathbb{F}_{3.4.3}+\mathbb{F}_{3.4.4}.\end{aligned}$$

$$\|\mathbb{F}_{3.1.1}\| \leq \sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\left\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\right\|\|\mathbf{R}'\|\left\|T^{-1}\mathbf{F}^{0\prime}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\right\|\|\boldsymbol{\mathcal{R}}'\|\left\|\frac{\mathbf{H}^{0\prime}\mathbf{u}_i}{T}\right\|$$
$$= O_p\left(\sqrt{NT}\delta_{NT}^{-3}\right).$$

by Lemma A.2 (b), Lemma B.1 (b) and Lemma B.2 (d).

$$\begin{aligned}\|\mathbb{F}_{3.1.2}\| &= \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i\right\| \\ &\leq \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}\varphi_i\right\| \\ &+ \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i\right\|\end{aligned}$$

$$\begin{aligned}&\leq \sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{T}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\mathcal{R}}\|^2\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i^0\| \\ &+ \sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{T}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\mathcal{R}}\|^2\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i}{T}\right\| \\ &= O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)\end{aligned}$$

by Lemma A.2 (b), Lemma B.1 (b) and Lemma B.2 (c).

$$\begin{aligned}\|\mathbb{F}_{3.1.3}\| &= \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i\right\| \\ &\leq \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0\boldsymbol{\varphi}_i^0\right\| \\ &+ \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i\right\|\end{aligned}$$

$$\begin{aligned}&\leq \sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{T}\right\|\|\mathbf{R}'\|\left\|\frac{\mathbf{F}^{0\prime}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})}{T}\right\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i^0\| \\ &+ \sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{T}\right\|\|\mathbf{R}'\|\left\|\frac{\mathbf{F}^{0\prime}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})}{T}\right\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i}{T}\right\| \\ &= O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^5}\right)\end{aligned}$$

by Lemma A.2 (b), Lemma B.1 (b) and Lemma B.2 (c).

$$\|\mathbb{F}_{3.1.4}\| = \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{u}_i\right\|$$

$$\leq \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_i' T^{-1}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime} T^{-1}\mathbf{H}^0 \left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1} \right) \mathbf{H}^{0\prime}\mathbf{H}^0 \boldsymbol{\varphi}_i^0 \right\|$$
$$+ \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_i' T^{-1}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime} T^{-1}\mathbf{H}^0 \left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1} \right) \mathbf{H}^{0\prime}\varepsilon_i \right\|$$

$$\leq \sqrt{NT}\frac{1}{N}\sum_{i=1}^{N} \left\| \frac{\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})}{T} \right\| \|\mathbf{R}'\| \left\| \frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T} \right\| \left\| \boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1} \right\| \left\| \frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right\| \|\boldsymbol{\varphi}_i^0\|$$
$$+ \sqrt{N}\frac{1}{N}\sum_{i=1}^{N} \left\| \frac{\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})}{T} \right\| \|\mathbf{R}'\| \left\| \frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T} \right\| \left\| \boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1} \right\| \left\| \frac{\mathbf{H}^{0\prime}\varepsilon_i}{\sqrt{T}} \right\|$$
$$= O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3} \right)$$

by Lemma A.2 (b), Lemma B.1 (d) and Lemma B.2 (d).

$$\|\mathbb{F}_{3.2.1}\| = \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_i' T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' T^{-1}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\mathbf{u}_i \right\|$$
$$\leq \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_i' T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' T^{-1}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0 \right\|$$
$$+ \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_i' T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' T^{-1}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\varepsilon_i \right\|$$

$$\leq \sqrt{N}\frac{1}{N}\sum_{i=1}^{N} \left\| \frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}} \right\| \|\mathbf{R}\| \left\| (\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})/\sqrt{T} \right\| \left\| (\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})/\sqrt{T} \right\| \|\boldsymbol{\mathcal{R}}'\| \left\| \frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right\| \|\boldsymbol{\varphi}_i^0\|$$
$$+ \sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N} \left\| \frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}} \right\| \|\mathbf{R}\| \left\| (\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})/\sqrt{T} \right\| \left\| (\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})/\sqrt{T} \right\| \|\boldsymbol{\mathcal{R}}'\| \left\| \frac{\mathbf{H}^{0\prime}\varepsilon_i}{\sqrt{T}} \right\|$$
$$= O_p\left(\frac{\sqrt{N}}{\delta_{NT}^2} \right)$$

by Lemma A.1 (b), Lemma A.2 (d) and Lemma B.1 (b).

$$\|\mathbb{F}_{3.2.2}\| = \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_i' T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' T^{-1}\mathbf{H}\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i \right\|$$
$$\leq \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_i' T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0\boldsymbol{\varphi}_i^0 \right\|$$
$$+ \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_i' T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\varepsilon_i \right\|$$

$$\leq \sqrt{N}\frac{1}{N}\sum_{i=1}^{N} \left\| \frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}} \right\| \|\mathbf{R}\| \left\| \frac{(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\mathbf{H}^0}{T} \right\| \|\boldsymbol{\mathcal{R}}\| \left\| \frac{(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0}{T} \right\| \|\boldsymbol{\varphi}_i^0\|$$
$$+ \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} \left\| \frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}} \right\| \|\mathbf{R}\| \left\| \frac{(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\mathbf{H}^0}{T} \right\| \|\boldsymbol{\mathcal{R}}\| \left\| \frac{(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\varepsilon_i}{T} \right\|$$

$$= O_p\left(\frac{\sqrt{N}}{\delta_{NT}^4}\right)$$

by Lemma B.1 (b) and Lemma B.2 (e).

$$\begin{aligned}
\|\mathbb{F}_{3.2.3}\| &= \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i' T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})' T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0\boldsymbol{\varphi}_i^0\right\| \\
&+ \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i' T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})' T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i\right\| \\
&\le \sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\|\mathbf{R}\|\left\|(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})/\sqrt{T}\right\|\left\|(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})/\sqrt{T}\right\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i^0\| \\
&+ \sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\|\mathbf{R}\|\left\|(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})/\sqrt{T}\right\|\left\|(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})/\sqrt{T}\right\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i}{T}\right\| \\
&= O_p\left(\frac{\sqrt{N}}{\delta_{NT}^4}\right)
\end{aligned}$$

by Lemma A.1 (a), Lemma B.1 (a) (b) and Lemma B.2 (e).

$$\begin{aligned}
\|\mathbb{F}_{3.2.4}\| &\le \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i' T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})' T^{-1}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0\right\| \\
&+ \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i' T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})' T^{-1}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}\right)^{-1}\right)\mathbf{H}^{0\prime}\varepsilon_i\right\| \\
&\le \sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\|\mathbf{R}\|\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right\|\left\|\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i^0\| \\
&+ \sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\|\mathbf{R}\|\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{H}^0}{T}\right\|\left\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right\|\left\|\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\| \\
&= O_p\left(\frac{\sqrt{N}}{\delta_{NT}^4}\right)
\end{aligned}$$

by Lemma A.2 (b), Lemma B.1 (b) (d) and Lemma B.2 (d).

$$\begin{aligned}
\|\mathbb{F}_{3.3.1}\| &\le \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i' T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})' T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0\right\| \\
&+ \left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i' T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})' T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}'\varepsilon_i\right\| \\
&\le \sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{T}\right\|\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{\sqrt{T}}\right\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})}{\sqrt{T}}\right\|\|\boldsymbol{\mathcal{R}}'\|\left\|\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i\| \\
&+ \sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{T}\right\|\left\|\frac{(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})}{\sqrt{T}}\right\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})}{\sqrt{T}}\right\|\|\boldsymbol{\mathcal{R}}'\|\left\|\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\| \\
&= O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^3}\right)
\end{aligned}$$

by Lemma A.1 (a), Lemma A.2 (b), Lemma B.1 (a) (b) and Lemma B.2 (d).

$$\begin{aligned}
\|\mathbb{F}_{3.3.2}\| &\leq \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_i' T^{-1} (\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' T^{-1} \mathbf{H}^0 \boldsymbol{\mathcal{R}} (\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})' \mathbf{H}^0 \boldsymbol{\varphi}_i^0 \right\| \\
&+ \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_i' T^{-1} (\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' T^{-1} \mathbf{H}^0 \boldsymbol{\mathcal{R}} (\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})' \boldsymbol{\varepsilon}_i \right\| \\
&\leq \sqrt{NT} \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})}{T} \right\| \left\| \frac{(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\mathbf{H}^0}{T} \right\| \|\boldsymbol{\mathcal{R}}\| \left\| \frac{(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0}{T} \right\| \|\boldsymbol{\varphi}_i^0\| \\
&+ \sqrt{NT} \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})}{T} \right\| \left\| \frac{(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\mathbf{H}^0}{T} \right\| \|\boldsymbol{\mathcal{R}}\| \left\| \frac{(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i}{T} \right\| \\
&= O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^5} \right)
\end{aligned}$$

by Lemma A.2 (b), Lemma B.1 (b) and Lemma B.2 (c).

$$\begin{aligned}
\|\mathbb{F}_{3.3.3}\| &\leq \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_i' T^{-1} (\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' T^{-1} (\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})' \mathbf{H}^0 \boldsymbol{\varphi}_i^0 \right\| \\
&+ \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_i' T^{-1} (\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' T^{-1} (\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})' \boldsymbol{\varepsilon}_i \right\| \\
&\leq \sqrt{NT} \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})}{T} \right\| \left\| \frac{(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})}{\sqrt{T}} \right\| \left\| \frac{(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})}{\sqrt{T}} \right\| \left\| \frac{(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0}{T} \right\| \|\boldsymbol{\varphi}_i^0\| \\
&+ \sqrt{NT} \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})}{T} \right\| \left\| \frac{(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})}{\sqrt{T}} \right\| \left\| \frac{(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})}{\sqrt{T}} \right\| \left\| \frac{(\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i}{T} \right\| \\
&= O_p\left(\frac{\sqrt{NT}}{\delta_{NT}^5} \right)
\end{aligned}$$

by Lemma A.1 (a), Lemma A.2 (b), Lemma B.1 (a) (b) and Lemma B.2 (c).

$$\begin{aligned}
\|\mathbb{F}_{3.3.4}\| &\leq \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_i' T^{-1} (\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' T^{-1} \mathbf{H}^0 \left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0 \right)^{-1} \right) \mathbf{H}^{0\prime}\mathbf{H}^0 \boldsymbol{\varphi}_i^0 \right\| \\
&+ \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_i' T^{-1} (\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})' T^{-1} \mathbf{H}^0 \left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0 \right)^{-1} \right) \mathbf{H}^{0\prime} \boldsymbol{\varepsilon}_i \right\| \\
&\leq \sqrt{NT} \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})}{T} \right\| \left\| \frac{(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\mathbf{H}^0}{T} \right\| \left\| \boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0 \right)^{-1} \right\| \left\| \frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T} \right\| \|\boldsymbol{\varphi}_i^0\| \\
&+ \sqrt{N} \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})}{T} \right\| \left\| \frac{(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\mathbf{H}^0}{T} \right\| \left\| \boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}' - \left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0 \right)^{-1} \right\| \left\| \frac{\mathbf{H}^{0\prime}\varepsilon_i}{\sqrt{T}} \right\| \\
&= O_p\left(\frac{\sqrt{NT}}{\delta^5} \right)
\end{aligned}$$

by Lemma A.2 (b), Lemma B.1 (b) (d) and Lemma B.2 (d).

$$\|\mathbb{F}_{3.4.1}\| \leq \left\| \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_i' T^{-1} \mathbf{F}^0 \left(\mathbf{R}\mathbf{R}' - \left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0 \right)^{-1} \right) \mathbf{F}^{0\prime} T^{-1} (\widehat{\mathbf{H}} - \mathbf{H}^0\boldsymbol{\mathcal{R}}) \boldsymbol{\mathcal{R}}' \mathbf{H}^{0\prime}\mathbf{H}^0 \boldsymbol{\varphi}_i^0 \right\|$$

$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\boldsymbol{\mathcal{R}}'\mathbf{H}^{0\prime}\varepsilon_i\right\|$$
$$\leq\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})}{T}\right\|\|\boldsymbol{\mathcal{R}}'\|\left\|\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i^0\|$$
$$+\sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})}{T}\right\|\|\boldsymbol{\mathcal{R}}'\|\left\|\frac{\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|$$
$$=O_p\left(\frac{\sqrt{N}}{\delta_{NT}^4}\right)$$

by Lemma A.1 (d) and Lemma B.1 (b).

$$\|\mathbb{F}_{3.4.2}\|\leq\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0\boldsymbol{\varphi}_i^0\right\|$$
$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\varepsilon_i\right\|$$
$$\leq\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\mathcal{R}}\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i^0\|$$
$$+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\mathcal{R}}\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i}{T}\right\|$$
$$=O_p\left(\frac{\sqrt{N}}{\delta_{NT}^4}\right)$$

by Lemma A.1 (d), Lemma B.1 (b) and Lemma B.2 (e).

$$\|\mathbb{F}_{3.4.3}\|\leq\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0\boldsymbol{\varphi}_i\right\|$$
$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i\right\|$$
$$\leq\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})}{T}\right\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i^0\|$$
$$+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\frac{\mathbf{F}^{0\prime}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})}{T}\right\|\left\|\frac{(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\boldsymbol{\varepsilon}_i}{T}\right\|$$
$$=O_p\left(\frac{\sqrt{N}}{\delta_{NT}^6}\right)$$

by Lemma A.1 (d), Lemma B.1 (b) and Lemma B.3 (e).

$$\|\mathbb{F}_{3.4.4}\|\leq\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right)\mathbf{H}^{0\prime}\mathbf{H}^0\boldsymbol{\varphi}_i^0\right\|$$
$$+\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}T^{-1}\mathbf{H}^0\left(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right)\mathbf{H}^{0\prime}\varepsilon_i\right\|$$
$$\leq\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\left\|\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right\|\left\|\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right\|\|\boldsymbol{\varphi}_i^0\|$$

$$+\sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_i'\mathbf{F}^0}{\sqrt{T}}\right\|\left\|\frac{\mathbf{F}^{0\prime}\mathbf{H}^0}{T}\right\|\left\|\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right\|\left\|\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-\left(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0\right)^{-1}\right\|\left\|\frac{\mathbf{H}^{0\prime}\varepsilon_i}{\sqrt{T}}\right\|$$
$$=O_p\left(\frac{\sqrt{N}}{\delta_{NT}^4}\right)$$

by Lemma A.1 (d), Lemma B.1 (d) and Lemma B.2 (f).

Combining the above terms from $\mathbb{F}_{3.1.1}$ to $\mathbb{F}_{3.4.4}$, we derive that $\mathbb{F}_3 = O_p\left(\sqrt{NT}\delta_{NT}^{-3}\right)$. This completes the proof. □

Proof of Lemma B.4. The proof can be completed following the argument in the proof of Lemma A.3. □

Proof of Lemma B.5. Follow the way of the proof of Lemma A.5, we can show that

$$N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0)^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i$$
$$=N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i+O_p(T^{1/2}\delta_{NT}^{-2})+O_p(N^{1/2}T^{-1/2}\delta_{NT}^{-1})$$

The first term is equal to

$$\begin{aligned}
&\mathrm{vec}\Big(N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i\Big)\\
=&\mathrm{vec}\Big(N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}\cdot(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\cdot\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\boldsymbol{\varepsilon}_i\Big)\\
&-\mathrm{vec}\Big(N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}\cdot(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{F}^0(\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\cdot\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\Big)\\
&-\mathrm{vec}\Big(N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}\cdot(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{H}(\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\cdot\mathbf{H}^{0\prime}\boldsymbol{\varepsilon}_i\Big)\\
&+\mathrm{vec}\Big(N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}\cdot(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{H}^0(\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\mathbf{F}^0(\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\cdot\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\Big)
\end{aligned}$$

$$\begin{aligned}
=&N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\left(\boldsymbol{\varepsilon}_i'\mathbb{E}(\mathbf{V}_\ell\mathbf{V}_\ell')\mathbf{F}^0\right)\otimes(\boldsymbol{\Gamma}_i^{0\prime})\cdot\mathrm{vec}((\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1})\\
&-N^{-1/2}T^{-3/2}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_i'\mathbf{F}^0)\otimes(\boldsymbol{\Gamma}_i^{0\prime})\cdot\mathrm{vec}\Big((\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\cdot N^{-1}T^{-1}\sum_{\ell=1}^{N}\mathbf{F}^{0\prime}\mathbb{E}(\mathbf{V}_\ell\mathbf{V}_\ell')\mathbf{F}^0\cdot(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\Big)\\
&-N^{-1/2}T^{-3/2}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_i'\mathbf{H}^0)\otimes(\boldsymbol{\Gamma}_i^{0\prime})\cdot\mathrm{vec}\Big((\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\cdot N^{-1}T^{-1}\sum_{\ell=1}^{N}\mathbf{F}^{0\prime}\mathbb{E}(\mathbf{V}_\ell\mathbf{V}_\ell')\mathbf{H}^0\cdot(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\Big)\\
&+N^{-1/2}T^{-3/2}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_i'\mathbf{H}^0)\otimes(\boldsymbol{\Gamma}_i^{0\prime})\cdot\mathrm{vec}\Big((\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\cdot N^{-1}T^{-1}\sum_{\ell=1}^{N}\mathbf{F}^{0\prime}\mathbb{E}(\mathbf{V}_\ell\mathbf{V}_\ell')\mathbf{F}^0\\
&\times(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}T^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\Big)
\end{aligned}$$

$$=N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\left(\boldsymbol{\varepsilon}_i'\mathbb{E}(\mathbf{V}_\ell\mathbf{V}_\ell')\mathbf{F}^0\right)\otimes(\boldsymbol{\Gamma}_i^{0\prime})\cdot O_p(1)+N^{-1/2}T^{-3/2}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_i'\mathbf{F}^0)\otimes(\boldsymbol{\Gamma}_i^{0\prime})\cdot O_p(1)$$
$$+N^{-1/2}T^{-3/2}\sum_{i=1}^{N}(\boldsymbol{\varepsilon}_i'\mathbf{H}^0)\otimes(\boldsymbol{\Gamma}_i^{0\prime})\cdot O_p(1)$$
$$=N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{s=1}^{T}\sum_{t=1}^{T}\mathbb{E}\left(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}\right)\varepsilon_{it}\mathbf{f}_s^{0\prime}\otimes\boldsymbol{\Gamma}_i^{0\prime}\cdot O_p(1)+N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\varepsilon_{it}\mathbf{f}_t^{0\prime}\otimes\boldsymbol{\Gamma}_i^{0\prime}\cdot O_p(1)$$
$$+N^{-1/2}T^{-3/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\varepsilon_{it}\mathbf{h}_t^{0\prime}\otimes\boldsymbol{\Gamma}_i^{0\prime}\cdot O_p(1)$$
$$=O_p(T^{-1/2})$$

because

$$\mathbb{E}\Big\|N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{s=1}^{T}\sum_{t=1}^{T}\mathbb{E}\left(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}\right)\varepsilon_{it}\mathbf{f}_s^{0\prime}\otimes\boldsymbol{\Gamma}_i^{0\prime}\Big\|^2$$
$$=N^{-3}T^{-3}\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\sum_{s_1=1}^{T}\sum_{s_2=1}^{T}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\mathbb{E}\left(\mathbf{v}_{\ell_1 s_1}'\mathbf{v}_{\ell_1 t_1}\right)\mathbb{E}\left(\mathbf{v}_{\ell_2 s_2}'\mathbf{v}_{\ell_2 t_2}\right)\mathbb{E}(\varepsilon_{i_1t_1}\varepsilon_{i_2t_2})\mathbb{E}\left(\mathbf{f}_{s_1}^{0\prime}\mathbf{f}_{s_2}^0\right)\operatorname{tr}(\mathbb{E}(\boldsymbol{\Gamma}_{i_1}^{0\prime}\boldsymbol{\Gamma}_{i_2}^0))$$
$$\le kN^{-3}T^{-3}\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\sum_{s_1=1}^{T}\sum_{s_2=1}^{T}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\tilde{\tau}_{s_1t_1}\tilde{\tau}_{s_2t_2}\bar{\sigma}_{i_1i_2}\sqrt{\mathbb{E}\|\mathbf{f}_{s_1}^0\|^2\mathbb{E}\|\mathbf{f}_{s_2}^0\|^2\mathbb{E}\|\boldsymbol{\Gamma}_{i_1}^0\|^2\mathbb{E}\|\boldsymbol{\Gamma}_{i_2}^0\|^2}$$
$$\le kCT^{-1}\cdot N^{-2}\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}1\cdot T^{-1}\sum_{s_1=1}^{T}\sum_{t_1=1}^{T}\tilde{\tau}_{s_1t_1}\cdot T^{-1}\sum_{s_2=1}^{T}\sum_{t_2=1}^{T}\tilde{\tau}_{s_2t_2}\cdot N^{-1}\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\bar{\sigma}_{i_1i_2}\le CT^{-1}$$

by Assumption A2 and B2, C and D, and

$$T^{-1}\cdot N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\varepsilon_{it}\mathbf{f}_t^{0\prime}\otimes\boldsymbol{\Gamma}_i^{0\prime}=O_p(T^{-1}),\, T^{-1}\cdot N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\sum_{t=1}^{T}\varepsilon_{it}\mathbf{h}_t^{0\prime}\otimes\boldsymbol{\Gamma}_i^{0\prime}=O_p(T^{-1}),$$

Thus, we derive that

$$N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}(T^{-1}\widehat{\mathbf{F}}'\mathbf{F}^0)^{-1}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i=O_p(T^{1/2}\delta_{NT}^{-2})+O_p(N^{1/2}T^{-1/2}\delta_{NT}^{-1})$$

This completes the proof. □

Proof of Lemma B.6. We can follow the way of the proof of Lemma A.4 to show that

$$\begin{aligned}&-N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i\\
=&-N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i\\
&+N^{-1/2}T^{1/2}\cdot N^{-1}\sum_{i=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\big(N^{-1}T^{-1}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{V}_h\boldsymbol{\Gamma}_h^{0\prime}\big)(\boldsymbol{\Upsilon}^0)^{-1}(\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i\\
&+O_p(T^{1/2}\delta_{NT}^{-2})\end{aligned}\tag{B.2}$$

Consider the second term on the right hand side in (B.2). Denoting

$$\mathbf{Q}=(\boldsymbol{\Upsilon}^0)^{-1}\big(N^{-1}T^{-1}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{V}_h\boldsymbol{\Gamma}_h^{0\prime}\big)(\boldsymbol{\Upsilon}^0)^{-1}(\mathbf{F}^{0\prime}\mathbf{F}^0/T)^{-1},$$

by a similar derivation of Lemma B.5, we can show that

$$N^{-1}\cdot N^{-1/2}T^{-1/2}\sum_{i=1}^{N}\mathbf{\Gamma}_i^{0\prime}\mathbf{Q}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i=O_p(N^{-1})$$

Then, we consider the first term on the right hand side in (B.2). Noting that $\mathbf{M}_{\mathbf{H}^0}\mathbf{u}_i=\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i$ and $\mathbf{M}_{\mathbf{H}^0}-\mathbf{M}_{\widehat{\mathbf{H}}}=T^{-1}\widehat{\mathbf{H}}\widehat{\mathbf{H}}'-\mathbf{P}_{\mathbf{H}^0}$, We can derive that

$$\begin{aligned}
&-N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i-\big(-N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i\big)\\
=&N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\widehat{\mathbf{H}}\widehat{\mathbf{H}}'\mathbf{u}_i-N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{P}_{\mathbf{H}^0}\mathbf{u}_i\\
=&N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}(\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1}-\mathbf{H}^0)(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i\\
&+N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}(\widehat{\mathbf{H}}\boldsymbol{\mathcal{R}}^{-1}-\mathbf{H}^0)\big(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\big)\mathbf{H}^{0\prime}\mathbf{u}_i\\
&+N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{H}^0\boldsymbol{\mathcal{R}}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i\\
&+N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})'\mathbf{u}_i\\
&+N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{H}(\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}'-(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1})\mathbf{H}^{0\prime}\mathbf{u}_i\\
=&\mathbb{G}_1+\mathbb{G}_2+\mathbb{G}_3+\mathbb{G}_4+\mathbb{G}_5
\end{aligned}$$

With the facts that

$$N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{H}^0=N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{H}^0-N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}(\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0=O_p(1)$$

and

$$\begin{aligned}
&\big\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\big\|\\
\le&\big\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\big\|+\big\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{F}^0\big\|\big\|(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\big\|\big\|T^{-1}\mathbf{F}^{0\prime}(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})\big\|\\
=&O_p(N^{-1/2}T^{1/2})+O_p(T^{1/2}\delta_{NT}^{-2})
\end{aligned}$$

by Lemma B.1(b). We can closely follow the arguments in the proofs of $\mathbb{B}_{1.2}$, $\mathbb{B}_2$, $\mathbb{B}_3$ and $\mathbb{B}_4$, to show that $\mathbb{G}_2=O_p(N^{-1/2}T^{1/2}\delta_{NT}^{-2})+O_p(T^{1/2}\delta_{NT}^{-4})$, $\mathbb{G}_3=O_p(\delta_{NT}^{-2})$, $\mathbb{G}_4=O_p(N^{-1/2}T^{1/2}\delta_{NT}^{-2})+O_p(T^{1/2}\delta_{NT}^{-4})$ and $\mathbb{G}_5=O_p(\delta_{NT}^{-2})$.

With (B.1), $\mathbb{G}_1$ is decomposed as follows

$$N^{-5/2}T^{-5/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{H}^0\boldsymbol{\varphi}_h^0\boldsymbol{\varepsilon}_h'\mathbf{H}^0\boldsymbol{\mathcal{R}}(T^{-1}\mathbf{H}^{0\prime}\hat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i$$
$$+N^{-5/2}T^{-5/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{H}^0\boldsymbol{\varphi}_h^0\boldsymbol{\varepsilon}_h'(\widehat{\mathbf{H}}-\mathbf{H}^0\boldsymbol{\mathcal{R}})(T^{-1}\mathbf{H}^{0\prime}\hat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i$$
$$+N^{-5/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i$$
$$+N^{-5/2}T^{-5/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\boldsymbol{\varepsilon}_h\boldsymbol{\varepsilon}_h'\widehat{\mathbf{H}}(T^{-1}\mathbf{H}^{0\prime}\hat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i$$
$$+N^{-5/2}T^{-5/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_h(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})'\mathbf{X}_h'\widehat{\mathbf{H}}$$
$$\times\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\left(\frac{\mathbf{H}^{0\prime}\mathbf{H}^0}{T}\right)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i$$
$$+N^{-5/2}T^{-5/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_h(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})\mathbf{u}_h'\widehat{\mathbf{H}}(T^{-1}\mathbf{H}^{0\prime}\hat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i$$
$$+N^{-5/2}T^{-5/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_h(\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV})'\mathbf{X}_h'\widehat{\mathbf{H}}(T^{-1}\mathbf{H}^{0\prime}\hat{\mathbf{H}})^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\mathbf{H}^{0\prime}\mathbf{u}_i$$
$$=\mathbb{G}_{1.1}+\mathbb{G}_{1.2}+\mathbb{G}_{1.3}+\mathbb{G}_{1.4}+\mathbb{G}_{1.5}+\mathbb{G}_{1.6}+\mathbb{G}_{1.7}$$

Following the argument in the proof of $\mathbb{B}_{1.1}$, we can show that $\mathbb{G}_{1.1}+\mathbb{G}_{1.2}=O_p(T^{1/2}\delta_{NT}^{-2})+O_p(N^{-1/2})$. In a similar manner, it can be shown that

$$\mathbb{G}_{1.4}=N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\boldsymbol{\Sigma}_\varepsilon\mathbf{H}^0(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0$$
$$+O_p(T^{1/2}\delta_{NT}^{-2})+O_p(T^{-1/2})+O_p(N^{-1/2},$$

the first term of which is $O_p(T^-1/2)$. As $\|\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_h\|\leq\|\mathbf{X}_h\|$, $\mathbb{G}_{1.6}$ is bounded in norm by

$$T^{1/2}\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|\cdot N^{-1}\sum_{i=1}^{N}\|\boldsymbol{\Gamma}_i^0\|\|T^{-1/2}\mathbf{u}_i\|\cdot N^{-1}\sum_{h=1}^{N}\|T^{-1/2}\mathbf{X}_h\|\|T^{-1/2}\mathbf{u}_h\|$$
$$\cdot\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell\|\|T^{-1/2}\widehat{\mathbf{H}}\|\|T^{-1/2}\mathbf{H}^0\|\|(\boldsymbol{\Upsilon}^0)^{-1}\|\|(T^{-1}\mathbf{H}^{0\prime}\hat{\mathbf{H}})^{-1}\|\|(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\|\|(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}\|$$
$$=O_p(T^{1/2}\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|)=O_p(N^{-1/2}).$$

A similar derivation yields that $\mathbb{G}_{1.7}=O_p(T^{1/2}\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|)=O_p(N^{-1/2})$. Also $\mathbb{G}_{1.5}=O_p(N^{1/2}T^{1/2}\|\boldsymbol{\beta}-\widehat{\boldsymbol{\beta}}_{1SIV}\|^2)=O_p(N^{-1/2}T^{-1/2})$. Finally

$$\mathbb{G}_{1.3}=N^{-5/2}T^{-1/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\boldsymbol{\Gamma}_i^{0\prime}(\boldsymbol{\Upsilon}^0)^{-1}\boldsymbol{\Gamma}_\ell^0\mathbf{V}_\ell'\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}(\boldsymbol{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0+O_p(T^{-1/2}),$$

which is bounded in norm by

$$N^{-1/2}N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^0\|\|\boldsymbol{\varphi}_i^0\|\cdot\|\|(\mathbf{\Upsilon}^0)^{-1}\|\|(\mathbf{\Upsilon}_\varphi^0)^{-1}\|\cdot\big\|N^{-1}T^{-1/2}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}\|$$
$$=N^{-1/2}O_p(1)\cdot\big\|N^{-1}T^{-1/2}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}\|$$
$$=O_p(N^{-1/2})$$

because

$$\mathbb{E}\Big(\mathbb{E}\big(\|N^{-1}T^{-1/2}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\sum_{t=1}^{T}\mathrm{vec}(\mathbf{\Gamma}_\ell^0\mathbf{v}_{\ell t}\varepsilon_{ht}\boldsymbol{\varphi}_h^{0\prime})\|^2\big|\{\mathbf{\Gamma}_i^0,\boldsymbol{\varphi}_i^0\}_{i=1}^N\big)\Big)$$
$$=\mathbb{E}\Big(\mathbb{E}\big(\|N^{-1}T^{-1/2}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\sum_{t=1}^{T}(\boldsymbol{\varphi}_h^0\otimes\mathbf{\Gamma}_\ell)^0\mathbf{v}_{\ell t}\varepsilon_{ht}\|^2\big|\{\mathbf{\Gamma}_i^0,\boldsymbol{\varphi}_i^0\}_{i=1}^N\big)\Big)$$
$$=\mathbb{E}\Big(\mathrm{tr}\big(N^{-2}T^{-1}\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\sum_{h_1=1}^{N}\sum_{h_2=1}^{N}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}(\boldsymbol{\varphi}_{h_1}^0\otimes\mathbf{\Gamma}_{\ell_1}^0)\mathbb{E}(\mathbf{v}_{\ell_1t_1}\mathbf{v}_{\ell_2t_2}')\mathbb{E}(\boldsymbol{\varepsilon}_{h_1t_1}\boldsymbol{\varepsilon}_{h_2t_2}^0)(\boldsymbol{\varphi}_{h_2}^{0\prime}\otimes\mathbf{\Gamma}_{\ell_2}^{0\prime})\big)\Big)$$
$$=CN^{-2}T^{-1}\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\sum_{h_1=1}^{N}\sum_{h_2=1}^{N}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\|\mathbb{E}(\mathbf{v}_{\ell_1t_1}\mathbf{v}_{\ell_2t_2}')\||\mathbb{E}(\boldsymbol{\varepsilon}_{h_1t_1}\boldsymbol{\varepsilon}_{h_2t_2}^0)|\mathbb{E}\big(\|\boldsymbol{\varphi}_{h_1}^0\|\|\boldsymbol{\varphi}_{h_2}^0\|\|\mathbf{\Gamma}_{\ell_1}^0\|\|\mathbf{\Gamma}_{\ell_2}^0\|\big)$$
$$\leq CN^{-2}T^{-1}\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\sum_{h_1=1}^{N}\sum_{h_2=1}^{N}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}\bar{\tau}_{\ell_1\ell_2}|\sigma_{h_1h_2,t_1t_2}|\sqrt[4]{\mathbb{E}\|\boldsymbol{\varphi}_{h_1}^0\|^4\mathbb{E}\|\boldsymbol{\varphi}_{h_2}^0\|^4\mathbb{E}\|\mathbf{\Gamma}_{\ell_1}^0\|^4\mathbb{E}\|\mathbf{\Gamma}_{\ell_2}^0\|^4}$$
$$\leq C^2\cdot N^{-1}\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\bar{\tau}_{\ell_1\ell_2}\cdot N^{-1}T^{-1}\sum_{h_1=1}^{N}\sum_{h_2=1}^{N}\sum_{t_1=1}^{T}\sum_{t_2=1}^{T}|\sigma_{h_1h_2,t_1t_2}|$$
$$\leq C$$

by Assumption A2, B2 and D. Collecting the above terms, we can derive that

$$-N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\widehat{\mathbf{H}}}\mathbf{u}_i=-N^{-3/2}T^{-1/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{M}_{\mathbf{F}^0}\mathbf{M}_{\mathbf{H}^0}\boldsymbol{\varepsilon}_i$$
$$-N^{-5/2}T^{-1/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\boldsymbol{\varepsilon}_h\boldsymbol{\varphi}_h^{0\prime}(\mathbf{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0$$
$$-N^{-3/2}T^{-3/2}\sum_{i=1}^{N}\sum_{\ell=1}^{N}\sum_{h=1}^{N}\mathbf{\Gamma}_i^{0\prime}(\mathbf{\Upsilon}^0)^{-1}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\mathbf{\Sigma}_\varepsilon\mathbf{H}(T^{-1}\mathbf{H}^{0\prime}\mathbf{H}^0)^{-1}(\mathbf{\Upsilon}_\varphi^0)^{-1}\boldsymbol{\varphi}_i^0$$
$$+O_p(T^{1/2}\delta_{NT}^{-2}) \tag{B.3}$$

Consequently, with (B.3), we complete the proof. □

C Proofs of Lemmas in Appendix C

Proof of Lemma C.1. Consider (a). With the equation (A.1), we have

$$\|T^{-1}\boldsymbol{\varepsilon}_i'(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1})\|$$
$$\leq N^{-1}T^{-2}\|\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\widehat{\mathbf{F}}\|\|\mathbf{\Xi}^{-1}\mathbf{R}^{-1}\|+N^{-1}T^{-2}\|\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime}\mathbf{F}^{0\prime}\widehat{\mathbf{F}}\|\|\mathbf{\Xi}^{-1}\mathbf{R}^{-1}\|$$
$$+N^{-1}T^{-2}\|\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{V}_\ell\mathbf{V}_\ell'\widehat{\mathbf{F}}\|\|\mathbf{\Xi}^{-1}\mathbf{R}^{-1}\|.$$

Since $\mathbf{\Xi}^{-1} = O_p(1)$ and $\mathbf{R}^{-1} = O_p(1)$, we omit $\|\mathbf{\Xi}^{-1}\mathbf{R}^{-1}\|$ in the following analysis. The first term is bounded in norm by

$$T^{-1/2} \cdot \|T^{-1/2}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\| \cdot \Big\|N^{-1}T^{-1}\sum_{\ell=1}^{N}\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\widehat{\mathbf{F}}\Big\|$$

With Assumptions A and C, we have $\mathbb{E}\|T^{-1/2}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\|^2 = T^{-1}\sum_{t=1}^{T}\mathbb{E}\|\boldsymbol{\varepsilon}_i\|^2\mathbb{E}\|\mathbf{f}_t^0\|^2 \le C$, which then implies that

$$\|T^{-1/2}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\| = O_p(1). \tag{C.4}$$

By Lemma A.1 (h), we have

$$\begin{aligned} N^{-1}T^{-1}\sum_{\ell=1}^{N}\widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime} =& N^{-1}T^{-1}\sum_{\ell=1}^{N}\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime} + N^{-1}T^{-1}\sum_{\ell=1}^{N}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime} \\ =& O_p(N^{-1/2}T^{-1/2}) + O_p(N^{-1}) + O_p(N^{-1/2}\delta_{NT}^{-2}). \end{aligned} \tag{C.5}$$

With (C.4) and (C.5), the first term is $O_p(N^{-1/2}T^{-1}) + O_p(N^{-1}T^{-1/2}) + O_p(N^{-1/2}T^{-1/2}\delta_{NT}^{-2})$. The second term is bounded in norm by

$$N^{-1/2}T^{-1/2}\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime}\|\|T^{-1/2}\mathbf{F}^0\|\|T^{-1/2}\widehat{\mathbf{F}}\| = O_p(N^{-1/2}T^{-1/2})$$

where $\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime}\| = O_p(1)$ can be proved by following the way in the proof of (C.4). Consider the third term. Easily, we can prove $\mathbb{E}\|T^{-1/2}\boldsymbol{\varepsilon}_i\|^2 \le C$. By Cauchy-Schwartz inequality, we have

$$\begin{aligned} &\mathbb{E}\big\|N^{-1}T^{-1}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbb{E}(\mathbf{V}_\ell\mathbf{V}_\ell')\mathbf{F}^0\big\|^2 = \mathbb{E}\Big\|T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\big(N^{-1}\sum_{l=1}^{N}\mathbb{E}(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t})\big)\varepsilon_{is}\mathbf{f}_t^0\Big\|^2 \\ \le& T^{-2}\sum_{s_1=1}^{T}\sum_{t_1=1}^{T}\sum_{s_2=1}^{T}\sum_{t_2=1}^{T}|N^{-1}\sum_{\ell=1}^{N}\mathbb{E}(\mathbf{v}_{\ell s_1}'\mathbf{v}_{\ell t_1})||N^{-1}\sum_{\ell=1}^{N}\mathbb{E}(\mathbf{v}_{\ell s_2}'\mathbf{v}_{\ell t_2})|\mathbb{E}\big(\|\varepsilon_{is_1}\mathbf{f}_{t_1}^0\|\|\varepsilon_{is_2}\mathbf{f}_{t_2}^0\|\big) \\ \le& T^{-2}\sum_{s_1=1}^{T}\sum_{t_1=1}^{T}\sum_{s_2=1}^{T}\sum_{t_2=1}^{T}\tilde{\sigma}_{s_1t_1}\tilde{\sigma}_{s_2t_2}\sqrt[4]{\mathbb{E}\varepsilon_{is_1}^4\mathbb{E}\varepsilon_{is_2}^4\mathbb{E}\left\|\mathbf{f}_{t_1}^0\right\|^4\mathbb{E}\left\|\mathbf{f}_{t_2}^0\right\|^4} \le C(T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\sigma}_{st})^2 \le C^3 \end{aligned} \tag{C.6}$$

by Assumptions A, B2, and C. With Assumption B5, we can follow the way of the proof of Lemma A.2(i) in Bai (2009a) to show that $\mathbb{E}\|N^{-1/2}T^{-1}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\big[\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\big]\mathbf{F}^0\|^2 \le C$. With the above three moment conditions, we obtain

$$\begin{aligned} &\|T^{-1/2}\boldsymbol{\varepsilon}_i\| = O_p(1) \\ &\|N^{-1}T^{-1}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{F}^0\| = O_p(1) \\ &\|N^{-1/2}T^{-1}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\big[\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\big]\mathbf{F}^0\| = O_p(1) \end{aligned}$$

Thus, the third term is bounded in norm by

$$\begin{aligned} &\|T^{-1/2}\boldsymbol{\varepsilon}_i\|\|N^{-1}T^{-1}\sum_{\ell=1}^{N}\mathbf{V}_\ell\mathbf{V}_\ell'\|\|T^{-1/2}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\| + T^{-1}\|N^{-1}T^{-1}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{F}^0\|\|\mathbf{R}\| \\ &+ N^{-1/2}T^{-1}\|N^{-1/2}T^{-1}\sum_{\ell=1}^{N}\boldsymbol{\varepsilon}_i'\big[\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\big]\mathbf{F}^0\|\|\mathbf{R}\| = O_p(\delta_{NT}^{-2}) \end{aligned}$$

where

$$\|N^{-1}T^{-1}\sum_{\ell=1}^{N}\mathbf{V}_\ell\mathbf{V}_\ell'\| = O_p(\delta_{NT}^{-1})\,. \tag{C.7}$$

which suggest from

$$\begin{aligned}&\|N^{-1}T^{-1/2}\sum_{\ell=1}^{N}\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\|^2 = T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\Big|N^{-1}\sum_{i=1}^{N}\mathbb{E}\left(\mathbf{v}_{is}'\mathbf{v}_{it}\right)\Big|^2\\
\leq &CN^{-1}T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\sum_{i=1}^{N}\left|\mathbb{E}(\mathbf{v}_{is}'\mathbf{v}_{it})\right| \leq CT^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}\tilde{\sigma}_{st} \leq C^2\,,\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\Big\|N^{-1/2}T^{-1}\sum_{\ell=1}^{N}\left[\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\Big\|^2 = \mathbb{E}\Big(T^{-2}\sum_{s=1}^{T}\sum_{t=1}^{T}\left[N^{-1/2}\sum_{\ell=1}^{N}\left(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t} - \mathbb{E}\left(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}\right)\right)\right]^2\Big)\\
=&T^{-2}\sum_{s=1}^{T}\sum_{t=1}^{T}\mathbb{E}\big[N^{-1/2}\sum_{\ell=1}^{N}(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t} - \mathbb{E}(\mathbf{v}_{\ell s}'\mathbf{v}_{\ell t}))\big]^2 \leq C\,,\end{aligned}$$

given $|N^{-1}\sum_{i=1}^{N}\mathbb{E}(\mathbf{v}_{is}'\mathbf{v}_{it})| \leq N^{-1}\sum_{i=1}^{N}|\mathbb{E}(\mathbf{v}_{is}'\mathbf{v}_{it})| \leq N^{-1}\sum_{i=1}^{N}\sqrt{\mathbb{E}\|\mathbf{v}_{is}\|^2\mathbb{E}\|\mathbf{v}_{it}\|^2} \leq C$ and Assumption B. Collecting the above three terms, the claim holds. Similarly, we can prove (b), details are omitted. This completes the proof. □

Proof of Lemma C.2. We first consider $T^{-1/2}\mathbf{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i$, which is

$$T^{-1/2}\mathbf{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i = O_p(1)\times T^{-1/2}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i$$

Note that $\mathbf{M}_{\mathbf{F}^0}\mathbf{F}^0 = \mathbf{0}$, we have $\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F}^0 = (\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0})\mathbf{F}^0$. We expand $\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0}$ as

$$-\frac{1}{T}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}-\frac{1}{T}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'-\frac{1}{T}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'-\frac{1}{T}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right)\mathbf{F}^{0\prime}\,,$$

then

$$\begin{aligned}\frac{1}{T^{1/2}}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i =& -\frac{1}{T^{3/2}}\mathbf{F}^{0\prime}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{u}_i - \frac{1}{T^{3/2}}\mathbf{F}^{0\prime}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\\
&-\frac{1}{T^{3/2}}\mathbf{F}^{0\prime}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i - \frac{1}{T^{3/2}}\mathbf{F}^{0\prime}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right)\mathbf{F}^{0\prime}\mathbf{u}_i\\
=&\mathbb{A}_1+\mathbb{A}_2+\mathbb{A}_3+\mathbb{A}_4\,.\end{aligned}$$

Consider $\mathbb{A}_1$. Given $\mathbf{u}_i = \mathbf{H}^0\boldsymbol{\varphi}_i^0 + \boldsymbol{\varepsilon}_i$, we have

$$\|T^{-1/2}\mathbf{u}_i\| \leq \|T^{-1/2}\mathbf{F}^0\| + \|\boldsymbol{\varphi}_i^0\|\|T^{-1/2}\mathbf{H}^0\| + \|T^{-1/2}\boldsymbol{\varepsilon}_i\|\,.$$

Since $\mathbb{E}\|\boldsymbol{\varphi}_i^0\| \leq C$ and the condition $\mathbb{E}\|T^{-1/2}\mathbf{H}^0\|^2 \leq C$ by Assumptions C and D, the first term is $O_p(1)$. Similarly, we can prove that the second and the third term both are $O_p(1)$. The above facts suggest $\|T^{-1/2}\mathbf{u}_i\| = O_p(1)$. Thus, $\mathbb{A}_1$ is bounded in norm by

$$\|\mathbf{u}_i\|\times\|T^{-1}\mathbf{F}^{0\prime}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\|\mathbf{R}\|\|T^{-1/2}\mathbf{F}^0\| = O_p(T^{1/2}\delta_{NT}^{-2})$$

Similarly, we can show that $\mathbb{A}_3 = O_p(T^{1/2}\delta_{NT}^{-3})$ and $\mathbb{A}_4 = O_p(T^{1/2}\delta_{NT}^{-2})$. Consider $\mathbb{A}_2$. The term is bounded in norm by $T^{1/2}\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\|\times\|T^{-1/2}\mathbf{F}^0\|^2\|\mathbf{R}\|$, which is $O_p(T^{1/2})\times\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\|$. Furthermore, $\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\|$ is bounded in norm by

$$\|\boldsymbol{\varphi}_i^0\|\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{H}^0\| + \|T^{-1}(\widehat{\mathbf{F}}\mathbf{R}^{-1}-\mathbf{F}^0)'\boldsymbol{\varepsilon}_i\|\|\mathbf{R}\| = O_p(\delta_{NT}^{-2})$$

by Lemmas A.1(b) and A.2(a). Thus, $\mathbb{A}_2 = O_p(T^{1/2}\delta_{NT}^{-2})$. Collecting the above four terms, we have $T^{-1/2}\mathbf{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i = O_p(T^{1/2}\delta_{NT}^{-2})$.

Next, we tend to prove that $T^{-1/2}\mathbf{V}_i'(\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0})\mathbf{u}_i = O_p(T^{1/2}\delta_{NT}^{-2})$. Since $\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0} = -T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime} - T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})' - T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})' - T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}' - \left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}$, we have

$$\begin{aligned}
&T^{-1/2}\mathbf{V}_i'(\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0})\mathbf{u}_i \\
=& - T^{-3/2}\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{u}_i - T^{-3/2}\mathbf{V}_i'\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\mathbf{u}_i \\
&- T^{-3/2}\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\mathbf{u}_i - T^{-3/2}\mathbf{V}_i'\mathbf{F}^0(\mathbf{R}\mathbf{R}' - \left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1})\mathbf{F}^{0\prime}\mathbf{u}_i \\
=&\mathbb{A}_5 + \mathbb{A}_6 + \mathbb{A}_7 + \mathbb{A}_8.
\end{aligned}$$

$\mathbb{A}_5$ in norm by

$$\begin{aligned}
&T^{1/2}\|\boldsymbol{\varphi}_i^0\| \cdot \|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\|\|T^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\|\|\mathbf{R}\| \\
&+ \|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\| \cdot \|T^{-1/2}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\|\|\mathbf{R}\| = O_p(T^{1/2}\delta_{NT}^{-2}).
\end{aligned}$$

With Lemma A.2(a), $\mathbb{A}_6$ is bounded in norm by

$$\begin{aligned}
&\|T^{-1/2}\mathbf{V}_i'\mathbf{F}^0\|\|T^{-1}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\boldsymbol{\varepsilon}_i\| \, \|\mathbf{R}\| \\
&+ \|T^{-1/2}\mathbf{V}_i'\mathbf{F}^0\|\|\boldsymbol{\varphi}_i^0\|\|T^{-1}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\mathbf{H}^0\| \, \|\mathbf{R}\| = O_p(\delta_{NT}^{-2})
\end{aligned}$$

by Lemmas A.1(b). Given Lemmas A.1(b), A.2(a) and A.2(b), $\mathbb{A}_7$ is bounded in norm by

$$\begin{aligned}
&T^{1/2}\|\boldsymbol{\varphi}_i^0\|\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\|\|T^{-1}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\mathbf{H}^0\| \\
&+ T^{1/2}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})\|\|T^{-1}(\widehat{\mathbf{F}} - \mathbf{F}^0\mathbf{R})'\boldsymbol{\varepsilon}_i\| = O_p(T^{1/2}\delta_{NT}^{-4})
\end{aligned}$$

$\mathbb{A}_8$ is bounded in norm by

$$\begin{aligned}
&\|T^{-1/2}\mathbf{V}_i'\mathbf{F}^0\|\|\boldsymbol{\varphi}_i^0\|\|T^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\|\|\mathbf{R}\mathbf{R}' - (T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\| \\
&+ T^{-1/2} \times \|T^{-1/2}\mathbf{V}_i'\mathbf{F}^0\|\|T^{-1/2}\mathbf{F}'\boldsymbol{\varepsilon}_i\|\|\mathbf{R}\mathbf{R}' - (T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\| = O_p(\delta_{NT}^{-2})
\end{aligned}$$

by Lemma A.1(e). With the stochastic orders of the above eight terms, we derive that

$$T^{-1/2}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i = T^{-1/2}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i + O_p(T^{1/2}\delta_{NT}^{-2})$$

we complete the proof. □

Proof of Lemma C.3. The results are immediately obtained from Assumptions D and H. □

Proof of Lemma C.4. Consider (a). With the equation (A.1), we have

$$\begin{aligned}
&\sup_{1\le i\le N}\|T^{-1}\boldsymbol{\varepsilon}_i'(\mathbf{F}^0 - \widehat{\mathbf{F}}\mathbf{R}^{-1})\| \\
\le& \sup_{1\le i\le N} N^{-1}T^{-2}\|\sum_{\ell=1}^N \boldsymbol{\varepsilon}_i'\mathbf{F}^0\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\widehat{\mathbf{F}}\|\|\mathbf{\Xi}^{-1}\mathbf{R}^{-1}\| + \sup_{1\le i\le N} N^{-1}T^{-2}\|\sum_{\ell=1}^N \boldsymbol{\varepsilon}_i'\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime}\mathbf{F}^{0\prime}\widehat{\mathbf{F}}\|\|\mathbf{\Xi}^{-1}\mathbf{R}^{-1}\| \\
&+ \sup_{1\le i\le N} N^{-1}T^{-2}\|\sum_{\ell=1}^N \boldsymbol{\varepsilon}_i'\mathbf{V}_\ell\mathbf{V}_\ell'\widehat{\mathbf{F}}\|\|\mathbf{\Xi}^{-1}\mathbf{R}^{-1}\|
\end{aligned}$$

Since $\mathbf{\Xi}^{-1} = O_p(1)$ and $\mathbf{R}^{-1} = O_p(1)$, we omit $\|\mathbf{\Xi}^{-1}\mathbf{R}^{-1}\|$ in the following analysis. The first term is bounded in norm by

$$T^{-1/2} \cdot \sup_{1\le i\le N}\|T^{-1/2}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\| \cdot \left\|N^{-1}T^{-1}\sum_{\ell=1}^N \mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\widehat{\mathbf{F}}\right\|$$

Since $\mathbb{E}\|T^{-1/2}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\|^4 \leq C$, we have

$$\sup_{1\leq i\leq N}\|T^{-1/2}\boldsymbol{\varepsilon}_i'\mathbf{F}^0\| = O_p(N^{1/4}) \tag{C.8}$$

Note that $N^{-1}T^{-1}\sum_{\ell=1}^N \widehat{\mathbf{F}}'\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime} = O_p(N^{-1/2}T^{-1/2}) + O_p(N^{-1}) + O_p(N^{-1/2}\delta_{NT}^{-2})$ by (C.5), and with (C.8), the first term is $O_p(N^{-1/4}T^{-1}) + O_p(N^{-3/4}T^{-1/2}) + O_p(N^{-1/4}T^{-1/2}\delta_{NT}^{-2})$. The second term is bounded in norm by

$$N^{-1/2}T^{-1/2}\sup_{1\leq i\leq N}\|N^{-1/2}T^{-1/2}\sum_{\ell=1}^N \boldsymbol{\varepsilon}_i'\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime}\|\cdot\|T^{-1/2}\mathbf{F}^0\|\|T^{-1/2}\widehat{\mathbf{F}}\| = O_p(N^{-1/4}T^{-1/2})$$

by Lemma C.3(b). Consider the third term. We have

$$\sup_{1\leq i\leq N}\|T^{-1/2}\boldsymbol{\varepsilon}_i\|^2 = \sup_{1\leq i\leq N}T^{-1}\sum_{t=1}^T\varepsilon_{it}^2 \leq \mathbb{E}\varepsilon_{it}^2 + T^{-1/2}\cdot\sup_{1\leq i\leq N}T^{-1/2}\sum_{t=1}^T(\varepsilon_{it}^2-\mathbb{E}\varepsilon_{it}^2) = O_p(1) + O_p(N^{1/3}T^{-1/2})$$

since $\mathbb{E}\|T^{-1/2}\sum_{t=1}^T(\varepsilon_{it}^2-\mathbb{E}\varepsilon_{it}^2)\|^3 \leq C$. With (A.8), we can show that $\mathbb{E}\|N^{-1/2}T^{-1}\sum_{\ell=1}^N\left[\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\mathbf{F}^0\|^2 \leq C$. Thus, the third term is bounded in norm by

$$\begin{aligned}
&\sup_{1\leq i\leq N}T^{-1/2}\|\boldsymbol{\varepsilon}_i\|\cdot\|N^{-1}T^{-1}\sum_{\ell=1}^N\mathbf{V}_\ell\mathbf{V}_\ell'\|\|T^{-1/2}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\| + T^{-1}\cdot\sup_{1\leq i\leq N}\|N^{-1}T^{-1}\sum_{\ell=1}^N\boldsymbol{\varepsilon}_i'\mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\mathbf{F}^0\|\cdot\|\mathbf{R}\|\\
&+N^{-1/2}T^{-1/2}\cdot\sup_{1\leq i\leq N}T^{-1/2}\|\boldsymbol{\varepsilon}_i\|\cdot\|N^{-1/2}T^{-1}\sum_{\ell=1}^N\left[\mathbf{V}_\ell\mathbf{V}_\ell' - \mathbb{E}\left(\mathbf{V}_\ell\mathbf{V}_\ell'\right)\right]\mathbf{F}^0\|\|\mathbf{R}\|\\
=&O_p(\delta_{NT}^{-2}) + O_p(N^{1/4}T^{-1}).
\end{aligned}$$

Collecting the above three terms, the claim holds.

Consider (b). Replacing $\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R}$ by its expression (A.1), we have

$$\begin{aligned}
&\sup_{1\leq i\leq N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\\
\leq&\sup_{1\leq i\leq N}N^{-1}T^{-2}\|\sum_{\ell=1}^N\mathbf{V}_i'\mathbf{F}^0\mathbf{\Gamma}_\ell^0\mathbf{V}_\ell'\widehat{\mathbf{F}}\|\|\mathbf{\Xi}^{-1}\| + \sup_{1\leq i\leq N}N^{-1}T^{-2}\|\sum_{\ell=1}^N\mathbf{V}_i'\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime}\mathbf{F}^{0\prime}\widehat{\mathbf{F}}\|\|\mathbf{\Xi}^{-1}\|\\
&+\sup_{1\leq i\leq N}N^{-1}T^{-2}\|\sum_{\ell=1}^N\mathbf{V}_i'\mathbf{V}_\ell\mathbf{V}_\ell'\widehat{\mathbf{F}}\|\|\mathbf{\Xi}^{-1}\|
\end{aligned}$$

Ignoring $\|\mathbf{\Xi}^{-1}\|$ and following the arguments of the first term and the third term in the proof of (a), the first term is $O_p(N^{-1/4}T^{-1}) + O_p(N^{-3/4}T^{-1/2}) + O_p(N^{-1/4}T^{-1/2}\delta_{NT}^{-2})$ and the third term is $O_p(\delta_{NT}^{-2}) + O_p(N^{1/4}T^{-1})$. The second term is bounded in norm by

$$\begin{aligned}
&\sup_{1\leq i\leq N}N^{-1}T^{-1}\|\sum_{\ell=1}^N\mathbf{V}_i'\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime}\|\|T^{-1/2}\mathbf{F}^0\|\|T^{-1/2}\widehat{\mathbf{F}}\| = \sup_{1\leq i\leq N}N^{-1}T^{-1}\|\sum_{\ell=1}^N\mathbf{V}_i'\mathbf{V}_\ell\mathbf{\Gamma}_\ell^{0\prime}\|\times O_p(1)\\
\leq&\sup_{1\leq i\leq N}N^{-1}T^{-1}\sum_{\ell=1}^N\|\mathbb{E}\big(\mathbf{V}_i'\mathbf{V}_\ell\big)\|\|\mathbf{\Gamma}_\ell^0\| + \sup_{1\leq i\leq N}N^{-1}T^{-1}\|\sum_{\ell=1}^N\big(\mathbf{V}_i'\mathbf{V}_\ell - \mathbb{E}\big(\mathbf{V}_i'\mathbf{V}_\ell\big)\big)\mathbf{\Gamma}_\ell^{0\prime}\|\\
\leq&N^{-1}\cdot\sup_{1\leq i\leq N}\sum_{\ell=1}^N\bar{\sigma}_{i\ell}\cdot\sup_{1\leq\ell\leq N}\|\mathbf{\Gamma}_\ell^0\| + \sup_{1\leq i\leq N}N^{-1}T^{-1}\|\sum_{\ell=1}^N\big(\mathbf{V}_i'\mathbf{V}_\ell - \mathbb{E}\big(\mathbf{V}_i'\mathbf{V}_\ell\big)\big)\mathbf{\Gamma}_\ell^{0\prime}\|\\
=&O_p(N^{-3/4}) + O_p(N^{-1/4}T^{-1/2}),
\end{aligned}$$

Combining the above three terms, (b) holds. This completes the proof. □

Proof of Lemma C.5. Consider (a). The left hand is bounded in norm by

$$N^{-1}T^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\| + N^{-1}T^{-1}\sum_{i=1}^{N}\|\mathbf{V}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i - \mathbf{V}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{u}_i\|$$

We first consider $N^{-1}T^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\|$, which is bounded by $N^{-1}T^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^{0}\|\|\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\|$. Note that $\mathbf{M}_{\mathbf{F}^0}\mathbf{F}^0 = \mathbf{0}$, we have $\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F}^0 = (\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0})\mathbf{F}^0$. We expand $\mathbf{M}_{\widehat{\mathbf{F}}} - \mathbf{M}_{\mathbf{F}^0}$ as following

$$-\frac{1}{T}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}-\frac{1}{T}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'-\frac{1}{T}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'-\frac{1}{T}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}^0}{T}\right)^{-1}\right)\mathbf{F}^{0\prime},$$

then

$$\begin{aligned}
&N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^{0}\|\|T^{-1}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\| \\
\leq&N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^{0}\|\|T^{-2}\mathbf{F}'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{u}_i\| + N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^{0}\|\|T^{-2}\mathbf{F}^{0\prime}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\| \\
&+N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^{0}\|\|T^{-2}\mathbf{F}'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\| + N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^{0}\|\|T^{-2}\mathbf{F}^{0\prime}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(\frac{\mathbf{F}^{0\prime}\mathbf{F}}{T}\right)^{-1}\right)\mathbf{F}^{0\prime}\mathbf{u}_i\| \\
=&\mathbb{B}_1+\mathbb{B}_2+\mathbb{B}_3+\mathbb{B}_4\,.
\end{aligned}$$

Consider $\mathbb{B}_1$. Given $\mathbf{u}_i = \mathbf{H}^0\boldsymbol{\varphi}_i^0 + \boldsymbol{\varepsilon}_i$, we have

$$\begin{aligned}
&N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^{0}\|\|T^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\| \\
\leq&N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^{0}\|\|\boldsymbol{\varphi}_i^0\|\|T^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\| + N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^{0}\|\|T^{-1}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\| \\
=&O_p(1)
\end{aligned}$$

Thus, $\mathbb{B}_1$ is bounded in norm by

$$N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^{0}\|\|T^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\| \times \|T^{-1}\mathbf{F}^{0\prime}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\|\mathbf{R}\| = O_p(\delta_{NT}^{-2})$$

Similarly, we can show that $\mathbb{B}_4 = O_p(\delta_{NT}^{-2})$.

Consider $\mathbb{B}_2$. The term is bounded in norm by $N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^{0}\|\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\|\cdot\|T^{-1/2}\mathbf{F}^0\|^2\|\mathbf{R}\|$, which is $O_p(1)\cdot N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^{0}\|\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\|$. Furthermore, $N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^{0}\|\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\|$ is bounded in norm by

$$\begin{aligned}
&N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^{0}\|\|\boldsymbol{\varphi}_i^0\|\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{H}^0\| \\
&+N^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^{0}\|\|T^{-1}(\widehat{\mathbf{F}}\mathbf{R}^{-1}-\mathbf{F}^0)'\boldsymbol{\varepsilon}_i\|\|\mathbf{R}\| \\
=&O_p(\delta_{NT}^{-2})
\end{aligned}$$

by Lemmas A.1(b) and A.2(b). Thus, $\mathbb{B}_2 = O_p(\delta_{NT}^{-2})$. Analogously, we have $\mathbb{B}_3 = O_p(\delta_{NT}^{-4})$. Collecting the above four terms, we have $N^{-1}T^{-1}\sum_{i=1}^{N}\|\mathbf{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{u}_i\| = O_p(\delta_{NT}^{-2})$.

Next, we tend to prove that $N^{-1}T^{-1}\sum_{i=1}^{N}\|\mathbf{V}_i'(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0})\mathbf{u}_i\| = O_p(\delta_{NT}^{-2})$. Since $\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0} = -T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}-T^{-1}\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'-T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'-T^{-1}\mathbf{F}^0\left(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1}\right)\mathbf{F}^{0\prime}$, we have

$$
\begin{aligned}
&N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0})\mathbf{u}_i\| \\
=&N^{-1}\sum_{i=1}^{N}\|T^{-2}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{u}_i\| + N^{-1}\sum_{i=1}^{N}\|T^{-2}\mathbf{V}_i'\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\| \\
&+N^{-1}\sum_{i=1}^{N}\|T^{-2}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\| + N^{-1}\sum_{i=1}^{N}\|T^{-2}\mathbf{V}_i'\mathbf{F}^0(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1})\mathbf{F}^{0\prime}\mathbf{u}_i\| \\
=&\mathbb{B}_5+\mathbb{B}_6+\mathbb{B}_7+\mathbb{B}_8
\end{aligned}
$$

We bound $\mathbb{B}_5$ in norm by

$$
\begin{aligned}
&N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\|T^{-1}\mathbf{F}^{0\prime}\mathbf{u}_i\|\|\mathbf{R}\| \\
\leq&N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\|\boldsymbol{\varphi}_i^0\|\|\mathbf{R}\|\|T^{-1}\mathbf{F}^{0\prime}\mathbf{H}^0\| \\
&+N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\|T^{-1}\mathbf{F}^{0\prime}\boldsymbol{\varepsilon}_i\|\|\mathbf{R}\| \\
=&O_p(\delta_{NT}^{-2})
\end{aligned}
$$

by Lemma A.2. With Lemma A.2(a), $\mathbb{B}_6$ is bounded in norm by

$$
N^{-1}\sum_{i=1}^{N}\|T^{-1}\mathbf{V}_i'\mathbf{F}^0\|\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{u}_i\|\|\mathbf{R}\| = O_p(T^{-1/2}\delta_{NT}^{-2})
$$

by Lemmas A.1(b). Similarly, we can show that $\mathbb{B}_7 = O_p(\delta_{NT}^{-4})$ and $\mathbb{B}_8 = O_p(T^{-1/2}\delta_{NT}^{-2})$. With the stochastic orders of the above eight terms, we obtain (a).

Consider (b). The term is bounded in norm by

$$
\begin{aligned}
&\sup_{1\leq i\leq N}\|T^{-1}\mathbf{X}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{X}_i - T^{-1}\mathbf{X}_i'\mathbf{M}_{\mathbf{F}^0}\mathbf{X}_i\| \\
\leq&\sup_{1\leq i\leq N}\|T^{-1}\boldsymbol{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F}^0\boldsymbol{\Gamma}_i^0\| + 2\sup_{1\leq i\leq N}\|T^{-1}\mathbf{V}_i'\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F}^0\boldsymbol{\Gamma}_i^0\| + \sup_{1\leq i\leq N}\|T^{-1}\mathbf{V}_i'(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0})\mathbf{V}_i\| \\
=&\mathbb{C}_1+\mathbb{C}_2+\mathbb{C}_3
\end{aligned}
$$

$\mathbb{C}_1$ is bounded in norm by

$$
\begin{aligned}
&\sup_{1\leq i\leq N}\|T^{-1}\boldsymbol{\Gamma}_i^{0\prime}\mathbf{F}^{0\prime}\mathbf{M}_{\widehat{\mathbf{F}}}\mathbf{F}^0\boldsymbol{\Gamma}_i^0\| = \sup_{1\leq i\leq N}\|T^{-1}\boldsymbol{\Gamma}_i^{0\prime}(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1})'\mathbf{M}_{\widehat{\mathbf{F}}}(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1})\boldsymbol{\Gamma}_i^0\| \\
\leq&(\sup_{1\leq i\leq N}\|\boldsymbol{\Gamma}_i^0\|)^2\cdot\|T^{-1/2}(\mathbf{F}^0-\widehat{\mathbf{F}}\mathbf{R}^{-1})\|^2 = O_p(N^{1/2}\delta_{NT}^{-2})
\end{aligned}
$$

Ignoring the scale 2, $\mathbb{C}_2$ is bounded in norm by

$$
\begin{aligned}
&\sup_{1\leq i\leq N}\|T^{-1}\mathbf{V}_i'(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0})\mathbf{F}^0\boldsymbol{\Gamma}_i^0\| \\
=&\sup_{1\leq i\leq N}\|T^{-2}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{F}^0\boldsymbol{\Gamma}_i^0\| + \sup_{1\leq i\leq N}\|T^{-2}\mathbf{V}_i'\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{F}^0\boldsymbol{\Gamma}_i^0\| \\
&+\sup_{1\leq i\leq N}\|T^{-2}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{F}^0\boldsymbol{\Gamma}_i^0\| + \sup_{1\leq i\leq N}\|T^{-2}\mathbf{V}_i'\mathbf{F}^0(\mathbf{R}\mathbf{R}'-\left(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\right)^{-1})\mathbf{F}^{0\prime}\mathbf{F}^0\boldsymbol{\Gamma}_i^0\|
\end{aligned}
$$

We bound the first term in norm by

$$\sup_{1\le i\le N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\cdot\sup_{1\le i\le N}\|\mathbf{\Gamma}_i^0\|\|\mathbf{R}\|\|T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\|=O_p(N^{1/2}\delta_{NT}^{-2})$$

With Lemma A.2(a), the second term is bounded in norm by

$$\sup_{1\le i\le N}\|T^{-1}\mathbf{V}_i'\mathbf{F}^0\|\cdot\sup_{1\le i\le N}\|\mathbf{\Gamma}_i^0\|\|\mathbf{R}\|\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{F}^0\|=O_p(N^{1/2}T^{-1/2}\delta_{NT}^{-2})$$

by Lemmas A.1(b). Given Lemmas A.1(b), A.2(a) and A.2(b), the third is bounded in norm by

$$\sup_{1\le i\le N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\cdot\sup_{1\le i\le N}\|T^{-1}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{F}^0\mathbf{\Gamma}_i^0\|=O_p(N^{1/2}\delta_{NT}^{-4})$$

the forth term is bounded in norm by

$$\sup_{1\le i\le N}\|T^{-1}\mathbf{V}_i'\mathbf{F}^0\|\cdot\sup_{1\le i\le N}\|\mathbf{\Gamma}_i^0\|\|\mathbf{R}\mathbf{R}'-(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0)^{-1}\|\|T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\|=O_p(N^{1/2}T^{-1/2}\delta_{NT}^{-2})$$

by Lemma A.1(e). Thus $\mathbb{C}_2$ is $O_p(N^{1/2}\delta_{NT}^{-2})$. $\mathbb{C}_3$ is bounded in norm by

$$\begin{aligned}&\sup_{1\le i\le N}\|T^{-1}\mathbf{V}_i'(\mathbf{M}_{\widehat{\mathbf{F}}}-\mathbf{M}_{\mathbf{F}^0})\mathbf{V}_i\|\\ =&\sup_{1\le i\le N}\|T^{-2}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\mathbf{R}'\mathbf{F}^{0\prime}\mathbf{V}_i\|+\sup_{1\le i\le N}\|T^{-2}\mathbf{V}_i'\mathbf{F}^0\mathbf{R}(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{V}_i\|\\ &+\sup_{1\le i\le N}\|T^{-2}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})'\mathbf{V}_i\|+\sup_{1\le i\le N}\|T^{-2}\mathbf{V}_i'\mathbf{F}^0(\mathbf{R}\mathbf{R}'-\big(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\big)^{-1})\mathbf{F}^{0\prime}\mathbf{V}_i\|\end{aligned}$$

The first term is bounded in norm by

$$\sup_{1\le i\le N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|\cdot\sup_{1\le i\le N}\|T^{-1}\mathbf{F}^{0\prime}\mathbf{V}_i\|\cdot\|\mathbf{R}\|=O_p(N^{1/2}T^{-1/2}\delta_{NT}^{-2})$$

Similarly, the second term is $O_p(N^{1/2}T^{-1/2}\delta_{NT}^{-2})$. The third term is bounded in norm by

$$(\sup_{1\le i\le N}\|T^{-1}\mathbf{V}_i'(\widehat{\mathbf{F}}-\mathbf{F}^0\mathbf{R})\|)^2=O_p(N^{1/2}\delta_{NT}^{-4})$$

The fourth term is bounded in norm by

$$T^{-1}\cdot\sup_{1\le i\le N}\|T^{-1/2}\mathbf{V}_i'\mathbf{F}^0\|^2\cdot\|\mathbf{R}\mathbf{R}'-\big(T^{-1}\mathbf{F}^{0\prime}\mathbf{F}^0\big)^{-1}\|=O_p(N^{1/2}T^{-1}\delta_{NT}^{-2})$$

With the above terms, we have $\mathbb{C}_3=O_p(N^{1/2}T^{-1/2}\delta_{NT}^{-2})+O_p(N^{1/2}\delta_{NT}^{-4})$. Then, we have (b). Thus, we complete the proof. □

D Proofs of Lemmas in Appendix D

Proof of Lemmas D.1, D.2 and D.3. The results are straightforwardly derived following the proofs in Bai (2009) and Lemmas B.1-B.6, thus, details are omitted.

E Additional Experimental Results

Table E.1: Bias, root mean squared error (RMSE) of the estimators of β_1, and size and power of the associated t-tests when $\pi_u = \{1/2, 3/4\}$ and $N = T = 200$.

Estimator	Homogeneous Slopes					Heterogeneous Slopes				
	Bias (×100)	S.D. (×100)	RMSE (×100)	Size	Power	Bias (×100)	S.D. (×100)	RMSE (×100)	Size	Power
					$\pi_u = 3/4$					
2SIV	0.003	0.589	0.589	4.6	100.0	0.584	0.957	1.121	7.7	100.0
BC-IPC	0.176	1.112	1.126	30.9	100.0	0.799	1.366	1.582	20.9	100.0
IPC	−0.021	0.575	0.576	5.3	100.0	0.556	0.950	1.100	8.4	100.0
BC-PC	−1.624	1.096	1.959	58.9	100.0	−1.040	1.333	1.691	19.4	100.0
PC	−2.438	0.907	2.601	87.6	100.0	−1.874	1.180	2.214	38.4	100.0
CA	−0.588	0.606	0.844	18.0	100.0	−0.005	0.963	0.963	3.5	100.0
MGIV	0.009	0.683	0.683	4.7	100.0	0.014	0.996	0.996	3.3	100.0
MGPC	−2.426	0.898	2.587	86.6	100.0	−2.416	1.170	2.684	58.5	100.0
MGCA	−0.572	0.608	0.834	17.2	100.0	−0.558	0.954	1.105	7.3	100.0
					$\pi_u = 1/4$					
2SIV	0.003	0.767	0.767	5.1	100.0	0.573	1.059	1.204	6.9	100.0
BC-IPC	−0.062	0.585	0.588	5.3	100.0	0.530	0.949	1.087	7.5	100.0
IPC	−0.062	0.586	0.589	5.3	100.0	0.530	0.950	1.088	7.5	100.0
BC-PC	−12.528	7.205	14.451	95.4	50.1	−12.048	7.262	14.066	92.8	46.2
PC	−15.139	6.422	16.444	99.7	66.8	−14.651	6.459	16.011	94.3	57.6
CA	−0.694	0.626	0.935	22.7	100.0	−0.106	0.983	0.989	4.4	100.0
MGIV	0.018	1.279	1.278	4.8	100.0	0.004	1.426	1.426	3.5	100.0
MGPC	−13.780	5.503	14.837	99.8	57.7	−13.821	5.530	14.885	96.2	52.8
MGCA	−0.673	0.628	0.920	20.9	100.0	−0.646	0.970	1.166	8.6	100.0

Notes: The DGP is the same as the one for Table 1, except $m_x = 2$, $m_y = 3$, $\gamma_{si}^{0*} \sim i.i.d.N(0,1)$ for $s = 1, ..., m_y$, $\gamma_{1si}^{0*} = \rho_{\gamma,1s}\gamma_{3i}^{0*} + (1-\rho_{\gamma,1s}^2)^{1/2}\xi_{1si}$; $\xi_{1si} \sim i.i.d.N(0,1)$, $\gamma_{2si}^{0*} = \rho_{\gamma,2s}\gamma_{si}^{0*} + (1-\rho_{\gamma,2s}^2)^{1/2}\xi_{2si}$; $\xi_{2si} \sim i.i.d.N(0,1)$, and $\gamma_1^0 = 1/4$, $\gamma_2^0 = \gamma_3^0 = 1/2$, $\gamma_{11}^0 = \gamma_{22}^0 = 1/4$ and $\gamma_{12}^0 = \gamma_{21}^0 = -1$.

Table E.2: Bias, root mean squared error (RMSE) of the estimators of β_1, and size and power of the associated t-tests when $\pi_u = 3/4$, $N = 200$, $T = 25$ and $N = 25$, $T = 200$.

Estimator	Homogeneous Slopes					Heterogenous Slopes				
	Bias (×100)	S.D. (×100)	RMSE (×100)	Size	Power	Bias (×100)	S.D. (×100)	RMSE (×100)	Size	Power
					$N = 200$, $T = 25$					
2SIV	−0.600	2.470	2.541	11.1	96.9	0.743	2.620	2.723	11.9	98.0
BC-IPC	0.321	2.679	2.697	18.2	98.9	1.794	2.823	3.344	23.9	99.3
IPC	−0.741	2.172	2.294	11.4	99.0	0.672	2.378	2.470	10.2	99.2
BC-PC	−1.591	2.905	3.311	19.8	91.7	−0.280	3.067	3.079	13.3	95.0
PC	−2.330	2.844	3.676	25.2	88.7	−1.022	3.024	3.191	13.6	92.4
CA	−0.603	1.951	2.042	6.7	99.8	0.823	2.089	2.245	6.1	100.0
MGIV	−0.539	2.724	2.776	9.7	95.1	−0.516	2.832	2.878	9.3	93.3
MGPC	−2.235	2.832	3.608	21.0	86.7	−2.235	2.959	3.707	19.7	83.0
MGCA	−0.581	2.152	2.229	5.6	99.3	−0.523	2.266	2.325	5.4	98.6
					$N = 25$, $T = 200$					
2SIV	0.060	1.823	1.823	9.5	99.9	0.431	2.774	2.806	8.1	97.0
BC-IPC	1.076	10.587	10.639	63.3	82.8	1.177	11.497	11.554	55.6	76.3
IPC	−1.147	3.373	3.562	17.3	86.7	−0.801	4.069	4.146	14.9	79.1
BC-PC	−1.200	5.547	5.674	31.6	73.5	−0.798	5.836	5.889	19.1	62.4
PC	−7.264	3.015	7.865	82.6	29.4	−7.017	3.625	7.898	54.0	18.2
CA	−1.936	2.655	3.285	17.3	88.2	−1.623	3.429	3.793	9.6	73.2
MGIV	0.017	2.068	2.068	7.0	99.3	−0.101	2.888	2.889	4.5	89.8
MGPC	−7.410	3.047	8.012	81.8	26.7	−7.467	3.631	8.303	58.2	14.5
MGCA	−1.648	2.449	2.952	15.0	92.1	−1.802	3.195	3.668	10.3	74.7

Notes: The DGP is the same as the one for Table 2, except the differences explained bellow the Table E.1.

Table E.3: Scaled bias of the estimators of β_1.

	Homogeneous Slopes ($\sqrt{NT}\times$Bias)				Heterogeneous Slopes ($\sqrt{N}\times$Bias)			
Estimator\$N=T$	25	50	100	200	25	50	100	200
				$\pi_u = 3/4$				
2SIV	−0.215	−0.074	−0.039	0.005	0.021	0.067	0.079	0.083
BC-IPC	−0.254	0.587	0.923	0.352	0.010	0.147	0.159	0.113
IPC	−0.611	−0.599	−0.222	−0.041	−0.052	−0.011	0.062	0.079
BC-PC	−0.511	−0.732	−1.632	−3.249	−0.049	−0.037	−0.075	−0.147
PC	−1.841	−2.249	−3.235	−4.877	−0.314	−0.255	−0.242	−0.265
CA	−0.470	−0.760	−1.056	−1.177	−0.035	−0.040	−0.029	−0.001
MGIV	−0.183	−0.059	−0.054	0.018	−0.040	−0.010	−0.004	0.002
MGPC	−1.836	−2.245	−3.235	−4.852	−0.360	−0.323	−0.317	−0.342
MGCA	−0.398	−0.692	−1.000	−1.144	−0.083	−0.102	−0.099	−0.079
				$\pi_u = 1/4$				
2SIV	−0.029	−0.045	−0.016	0.006	0.077	0.072	0.079	0.081
BC-IPC	−0.207	−0.104	−0.065	−0.123	0.032	0.062	0.073	0.075
IPC	−0.209	−0.105	−0.065	−0.124	0.031	0.062	0.073	0.075
BC-PC	−0.666	−4.955	−12.699	−25.056	−0.028	−0.593	−1.164	−1.704
PC	−5.921	−9.876	−17.430	−30.277	−1.126	−1.333	−1.669	−2.072
CA	−1.258	−1.405	−1.365	−1.388	−0.193	−0.133	−0.062	−0.015
MGIV	−0.101	−0.055	−0.005	0.035	0.007	−0.012	−0.009	0.001
MGPC	−5.537	−8.995	−15.826	−27.559	−1.093	−1.279	−1.584	−1.955
MGCA	−1.020	−1.250	−1.284	−1.346	−0.195	−0.177	−0.128	−0.091

Notes: The DGP is the same as the one for Table 3, except the differences explained bellow the Table E.1.