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Abstract

This paper explores the possibility of cointegration existing between processes integrated at different frequencies. Using the demodulator operator, we show that such cointegration can exist and explore its form using both complex- and real-valued representations. A straightforward approach to test for the presence of cointegration between processes integrated at different frequencies is proposed, with a Monte Carol study and an application showing that the testing approach works well.

Keywords: Periodic Cointegration, Polynomial Cointegration, Demodulator Operator.

JEL codes: C32.

1 Introduction

To date, the vast literature on cointegration has focused primarily on the long-run characteristics of economic time series through the analysis of zero frequency unit roots. Nevertheless, economic and financial time series may exhibit unit roots at other frequencies; in particular, Engle, Granger, Hylleberg and Lee (1993), Johansen and Schaumburg (1999), Ahn and Reinsel (1994) and Bauer and Wagner (2012) analyze the seasonal case, while Bierens (2001) and Caporale, Cuñado and Gil-Alana (2013) consider unit roots associated with the business cycle. However, such analyses typically examine a specific frequency, without allowing the possibility that the responses of economic agents may vary in relation to the seasonal or business cycle.

The current paper studies long-run linkages between time series with unit roots at different frequencies. Thus, for example, we consider the nature of any cointegration between two series integrated at different harmonic frequencies, or where one series is integrated at the zero frequency and the other at a business cycle or seasonal frequency. To our knowledge, no previous study has examined the possibility or nature of such cointegration. Succinctly stating our main result, we show that cointegration can exist between time series that are integrated at different frequencies, with this being a specific type of time-varying polynomial cointegration. More specifically, the cointegrating relationship is dynamic with coefficients that exhibit cyclical variation, so that a long-run relationship can vary over the seasonal or business cycle.

Polynomial cointegration is discussed in the literature in the contexts of (so-called) seasonal cointegration and multicointegration (see Hylleberg, Engle, Granger and Yoo, 1990, and Granger and Lee, 1989, respectively), while Gregoir (1999a, 1999b) undertakes a general analysis of these cases. Cubadda (2001) provides an alternative representation of the polynomial cointegration arising in the seasonal case in terms of complex-valued cointegration, which is developed further by Cubadda and Omtzigt (2005), and Gregoir (2006, 2010). Although we take a similar approach to these latter authors, we relax the restrictions that cointegration applies only at a single frequency and that cointegrating vectors are time invariant.

As examined by Park and Hahn (1999) and Bierens and Martin (2010), time-varying cointegration allows the relevant coefficients to change smoothly over time in any direction. Such a general specification is, however, problematic in that it raises the question of what underlying mechanism drives these changes and hence it is not surprising that other authors place some economic structure on the nature of the temporal variation exhibited by the long-run relationship. For example, Hall, Psaradakis and Sola (1997) allow the

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cointegrating relationship to change with the economic environment through the use of a Markov-switching specification, while Birchenhall, Bladen-Hovell, Chui, Osborn and Smith (1989) apply periodic cointegration, in which the long-run coefficients vary with the time of the year.

The present paper generalizes periodic cointegration to show that temporal variation in the coefficients of a long-run relationship, with this variation being of a cyclic nature, can deliver cointegration between variables that are individually integrated at different frequencies. This approach encompasses not only variation associated with the seasons, but also over a cycle at a business cycle frequency, and the approach provides a specific form of regime-switching cointegration. Some of our results are implicit in analyses of periodic cointegration (see, in particular, Ghysels and Osborn, 2001, and Franses and Paap, 2004), but the cross-frequency cointegration implications have not previously been drawn out.

In our analysis, a central role is played by the complex demodulator operator, which transforms a real valued process integrated at a frequency different from zero to a complex valued process that is integrated at frequency zero. The idea of complex demodulation has a long history in time series analysis (see, e.g., chapter 6 of Bloomfield (1976) and the references therein) but, to the best of our knowledge, it has not been previously used to investigate the presence of cointegration among series that are integrated at different frequencies.

This paper is organized as follows. Section 2 reviews the notions of integration at a given frequency and the demodulator operator. Section 3 presents our theoretical results. First, we show that two complex valued processes integrated at different frequencies can cointegrated and the connection of this with the demodulator operator. Second, we examine in details the various forms of cointegration that may exist between real valued time series integrated at different frequencies. Third, we tackle inferential issues. In Section 4, a Monte Carlo simulation exercise documents the small sample properties of the tests that we suggest. Section 5 presents an empirical application to illustrate concepts and methods. Finally, Section 6 concludes.

It is useful to introduce some notation at this stage. Our analysis is concerned with a cyclical process which has N observations per cycle; for example, $N = 4$ for quarterly seasonal data or $N = 6$ for annual data following a six year business cycle. The analysis of the Appendix uses the vector of seasons (or, more generally, cycles) representation that indicates a specific observation within the cycle. This double subscript notation is also sometimes used below and it is important to appreciate that, in this vector notation, $x_{n\tau}$ indicates the n^{th} observation within the τ^{th} cycle; for example with quarterly data $x_{n\tau}$ is the n^{th} quarter of year τ within the available sample. Assuming that $t = 1$ represents the first period within a cycle, the identity $t = N(\tau - 1) + n$ provides the link between the usual time index and the vector notation.

2 Integration at a frequency

It is useful to have a notation for the operator that removes a single unit root at a spectral frequency $\omega \in [0, \pi]$. To this end, and following Gregoir (1999a) and Cubadda (1999), we adopt the notation

$$\Delta_\omega = \begin{cases} 1 - e^{-i\omega}L, & \omega = 0,\ \pi \\ 1 - 2\cos\omega L + L^2 = (1 - e^{-i\omega}L)(1 - e^{i\omega}L), & \omega \in (0, \pi) \end{cases} \tag{1}$$

where L is the conventional lag operator. Special cases include the conventional first difference operator $\Delta_0 = 1 - L$, while $\Delta_\pi = 1 + L$ and $\Delta_{\pi/2} = 1 + L^2$ remove unit roots at the semi-annual and annual frequencies, respectively, for a seasonally integrated quarterly process (Hylleberg *et al.*, 1990), and $\Delta_{\pi/3} = 1 - L + L^2$ remove as unit root corresponding to a cycle of six years duration in annual data.

To pin down the concept of integration at some frequency ω, we adopt the following definition, used by Gregoir (1999a):

Definition 1. *A purely nondeterministic real-valued random process x_t is integrated of order d, for non-negative integer d, at frequency $\omega \in [0, \pi]$ if $\Delta^d_\omega x_t$ is a covariance stationary process such that, for zero mean white noise ε_t, its Wold representation*

$$\Delta^d_\omega x_t = c(L)\varepsilon_t = \sum_{i=0}^{\infty} c_i \varepsilon_{t-i}$$

satisfies $\sum_{i=0}^{\infty} c_i^2 < \infty$ and $c(e^{i\omega}) \neq 0$.

Following Hylleberg *et al.* (1990) and Gregoir (1999a), a process x_t satisfying Definition 1 is denoted

$$x_t \sim I_\omega(d).$$

Although Gil-Alana (2001) and Gray, Zhang and Woodward (1989) allow fractional d, which is particularly relevant for financial time series, we are interested in cointegration for unit root economic time series which are typically $I_\omega(1)$ after taking account of deterministic effects. Obviously, $x_t \sim I_0(1)$ corresponds to a conventional (single) unit root process integrated at the zero frequency.

Although the differencing operator Δ_ω of (1) is defined for a real valued series, it is useful for the analysis that follows to consider complex valued processes. Specifically, when $x_t \sim I_\omega(1)$ we consider individually each of the two factors $\left(1 - e^{\pm i\omega}L\right)$ of Δ_ω for $\omega \in (0, \pi)$. Then, for

$$x_t = 2\cos(\omega)\, x_{t-1} - x_{t-2} + \nu_t, \tag{2}$$

with real-valued $\nu_t \sim I_\omega(0)$, define the complex valued process

$$x_t^- = x_t - e^{i\omega} x_{t-1}. \tag{3}$$

It is then straightforward to see that

$$x_t^- = e^{-i\omega} x_{t-1}^- + \nu_t. \tag{4}$$

Successive substitution from (4) yields

$$\begin{aligned} x_t^- &= e^{-i\omega t} x_0^- + \sum_{s=0}^{t-1} e^{-i\omega s} \nu_{t-s} \\ &= e^{-i\omega t} \left[x_0^- + \sum_{\ell=1}^{t} e^{i\omega \ell} \nu_\ell \right] \end{aligned} \tag{5}$$

where $e^{-i\omega t}$ is the demodulator operator and x_0^- is assumed to be $O_p(1)$ (with x_0^- been part of the starting value of x_t^- i.e. $e^{-i\omega t} x_0^-$).

As noted by Gregoir (1999a, 2006) and del Barrio Castro, Rodrigues and Taylor (2018, 2019), (5) is equivalent to

$$x_t^- = e^{-i\omega t} x_t^{(0)-} \tag{6}$$

where[1]

$$x_t^{(0)-} = x_0^- + \sum_{\ell=1}^{t} e^{i\omega \ell} \nu_\ell \sim I_0(1).$$

Note that, here it is clearly shown that x_0^- is the starting value of $x_t^{(0)-}$ and that $e^{-i\omega t} x_0^-$ is the stating value of x_t^-. The demodulator operator of (5) therefore shifts the zero frequency peak of the complex-valued process $x_t^{(0)-}$ to frequency ω, leading to a complex-valued $x_t^- \sim I_\omega(1)$. The demodulator operator provides the key to cointegration between processes integrated at different frequencies, examined in subsequent sections[2]. Further note that, using the identity $e^{\pm i\omega \ell} = \cos(\omega\, \ell) \pm i \sin(\omega\, \ell)$, (5) can also be written as

$$x_t^- = e^{-i\omega t} \left[x_0^- + \sum_{\ell=1}^{t} \cos(\omega\, \ell)\, \nu_\ell + i \sum_{\ell=1}^{t} \sin(\omega\, \ell)\, \nu_\ell \right]. \tag{7}$$

Following an analogous line of argument, the complex-valued process $x_t^+ \sim I_\omega(1)$ can also be constructed, where

$$x_t^+ = e^{i\omega} x_{t-1}^+ + \nu_t = e^{i\omega t} x_t^{(0)+} \tag{8}$$

and $x_t^{(0)+} = x_0^+ + \sum_{\ell=1}^{t} e^{-i\omega \ell} \nu_\ell \sim I_0(1)$. It is also clear that

$$x_t^+ = e^{i\omega t} \left[x_0^+ + \sum_{\ell=1}^{t} \cos(\omega\, \ell)\, \nu_\ell - i \sum_{\ell=1}^{t} \sin(\omega\, \ell)\, \nu_\ell \right], \tag{9}$$

where the starting value x_0^+ is the complex conjugate of x_0^- in (5). Hence x_t^- and x_t^+ form a complex conjugate pair of processes.

Lemma 1 in the Appendix summarizes the stochastic characteristics of (4) at the frequency $\omega_j = 2\pi j/N$, $j = 1, \ldots, (N-1)/2$, corresponding to a cycle of N/j periods. In particular, when appropriately scaled,

[1] Note that as $\nu_t \sim I_\omega(0)$ is the ω-frequency first difference of the real valued $I_\omega(1)$ process (2), $e^{i\omega k}\nu_t$ acts as the complex valued increment of the $I_0(1)$ process $x_t^{(0)-}$.

[2] The demodulator operator it is also used in Theorem 4 of Johansen and Schaumburg (1999) in a multivariate setting.

$\sum_{\ell=1}^{t}\cos(\omega\,\ell)\,\nu_\ell$ and $\sum_{\ell=1}^{t}\sin(\omega\,\ell)\,\nu_\ell$ converge to the independent Brownian motions $w_R^\nu(r)$ and $w_I^\nu(r)$ respectively of Lemma 1; see also Gregoir (2006) and del Barrio Castro, Osborn and Taylor (2012, Remark 7). In Lemma 1 and in all the lemmas in the appendix we assume for simplicity and to focus on the main ideas of the paper that $\nu_t \sim iid\left(0,\sigma^2\right)$.

3 Cointegration for processes integrated at different frequencies

In this section we initially focus on the long run relationships between complex-valued processes with unit roots at different frequencies, showing that long-run (cointegrating) relationships between such processes can exist. Cointegration is then discussed for real-valued processes, with these long-run relationships generally polynomial in form with periodically (seasonally or cyclically) varying coefficients. The final subsection considers econometric approaches to testing for cointegration. To ensure distinct frequencies, we consider $x_t \sim I_{\omega_j}(1)$ and $y_t \sim I_{\omega_k}(1)$ where $\omega_j = 2\pi j/N$ and $\omega_k = 2\pi k/N$ with $j \neq k$.

3.1 Cointegration between complex-valued processes

Based on the results of the previous section, define the following triangular system with a long-run relationship (cointegration) at the zero frequency between two complex-valued processes:

$$\begin{aligned} y_t^{(0)-} &= \beta x_t^{(0)-} + u_t \\ x_t^{(0)-} &= x_{t-1}^{(0)-} + e^{i\omega_j t}\nu_t \end{aligned} \tag{10}$$

where both the cointegrating coefficient β and the process $u_t \sim I(0)$ are generally complex-valued. Since $x_t^{(0)-} = e^{i\omega_j t}x_t^-$ from (6), (10) can also be written as

$$\begin{aligned} y_t^{(0)-} &= \beta e^{i\omega_j t}x_t^- + u_t \\ x_t^- &= e^{-i\omega_j}x_{t-1}^- + \nu_t. \end{aligned} \tag{11}$$

Note that multiplying the first line of (11) by $e^{-i\omega_j t}$ leads to the triangular system of Gregoir (2010, p. 1499).

The system (11) exhibits a long-run (cointegrating) relationship between $y_t^{(0)-} \sim I_0(1)$, a complex-valued process integrated at the zero frequency, and $x_t^- \sim I_{\omega_j}(1)$, a complex-valued process integrated at the frequency ω_j. It is also important to note that the relationship between $y_t^{(0)-}$ and x_t^- is a form of periodic cointegration, since the cointegrating coefficient $\beta e^{i\omega_j t} = \beta e^{i\omega_j(N(\tau-1)+n)} = \beta e^{i\omega_j n}$ is cyclically varying[3]. Lemma 5[4] in the appendix summarizes the stochastic behaviour of the triangular system (11) using the vector of seasons notation and, in particular, (48) shows that $y_t^{(0)-}$ and x_t^- share a single common complex-valued stochastic trend; hence there are $N-1$ cointegrating relationships between the two series across the N observations of a complete cycle.

Premultiplying (11) by the demodulator operator $e^{-i\omega_k t}$ shifts the complex valued process $y_t^{(0)-}$ from the zero frequency to frequency $\omega_k = 2\pi k/N$ with $k = 0,1,\ldots,\lfloor N/2\rfloor$, where $\lfloor N/2\rfloor$ is the integer part of $N/2$. With $j \neq k$ we then have:

$$e^{-i\omega_k t}y_t^{(0)-} = \beta e^{i[\omega_j-\omega_k]t}x_t^- + e^{-i\omega_k t}u_t.$$

Since, analogously to (6), $y_t^- = e^{-i\omega_k t}y_t^{(0)-}$, clearly y_t^- is a complex-valued process integrated at frequency ω_k that shares a single common stochastic trend with x_t^-, a complex valued process integrated at frequency ω_j. Hence the bivariate system

$$\begin{aligned} y_t^- &= \beta e^{-i[\omega_k-\omega_j]t}x_t^- + e^{-i\omega_k t}u_t \\ x_t^- &= e^{-i\omega_j}x_{t-1}^- + \nu_t. \end{aligned} \tag{12}$$

links $y_t^- \sim I_{\omega_k}(1)$ and $x_t^- \sim I_{\omega_j}(1)$ through the periodic cointegration relationship $\left[1, -\beta e^{i[\omega_j-\omega_k]t}\right]'$ with a different coefficient for each observation within the cycle of N observations. Lemma 6 in the appendix

[3] As $\omega_j = 2\pi j/N$ it is evident that $\omega_j\,(N(\tau-1)+n)$ is periodic and hence the identity $\beta e^{i\omega_j(N(\tau-1)+n)} = \beta e^{i\omega_j n}$ holds.

[4] In all the lemmas in the appendix we asume that the innovations on the cointegration relationships are iid with zero expected value and constant variance.

formalizes this result and summarizes the stochastic behaviour of the triangular system (12) using the vector of seasons notation, with equation (57) writing this system in terms of the same complex-valued common trend as for the system (11).

To summarize, (11) and (12), together with (48) and (57) of the Appendix, show that a unique long-run cointegrating relationship can exist between complex-valued processes integrated at different frequencies.

The next subsection discusses the implications of such cointegration for real-valued processes. For this purpose, it will be convenient to explicitly indicate the complex-valued nature of the cointegration by writing $\beta = \beta_R + i\beta_I$ and $u_t = \mathrm{Re}(u_t) + i\,\mathrm{Im}(u_t)$, so that (10) becomes

$$\begin{aligned} y_t^{(0)-} &= [\beta_R + i\beta_I]\, x_t^{(0)-} + \mathrm{Re}(u_t) + i\,\mathrm{Im}(u_t) \\ x_t^{(0)-} &= x_{t-1}^{(0)-} + e^{i\omega_j t}\nu_t. \end{aligned} \tag{13}$$

Further, there also exists the system

$$\begin{aligned} y_t^{(0)+} &= [\beta_R - i\beta_I]\, x_t^{(0)+} + \mathrm{Re}(u_t) - i\,\mathrm{Im}(u_t) \\ x_t^{(0)+} &= x_{t-1}^{(0)+} + e^{-i\omega_j t}\nu_t \end{aligned} \tag{14}$$

which forms the complex conjugate system to (10). Since, clearly,

$$\mathrm{Re}(x_t^{(0)-}) = \mathrm{Re}(x_t^{(0)+}),\ \mathrm{Im}(x_t^{(0)-}) = -\,\mathrm{Im}(x_t^{(0)+}) \tag{15}$$

and

$$\mathrm{Re}(y_t^{(0)-}) = \mathrm{Re}(y_t^{(0)+}),\ \mathrm{Im}(y_t^{(0)-}) = -\,\mathrm{Im}(y_t^{(0)+}), \tag{16}$$

cointegration can be equivalently considered using either $y_t^{(0)-}$ and $x_t^{(0)-}$ or $y_t^{(0)+}$ and $x_t^{(0)+}$.

Finally, note also that $\omega_k = \omega_{N/2} = \pi$ in (12) leads to a long-run relationship between a process integrated at a harmonic frequency $\omega_j = 2\pi j/N$ and a process integrated at the Nyquist frequency (π). Similarly, $\omega_k = 0$ leads to cointegration between a process integrated at a harmonic frequency and a zero frequency unit root process. These special cases and their implications are also discussed in the next subsection.

3.2 Cointegration between real-valued processes

This subsection extends the analysis to examine the implications of cointegration at different frequencies for real-valued processes, which enables inference to be applied to observed time series. Since slightly different considerations arise when one process is integrated at the zero or Nyquist frequency, four cases are considered: namely $\omega_k, \omega_j \in (0,\pi)$; $\omega_k = 0$, $\omega_j \in (0,\pi)$; $\omega_k = \pi$, $\omega_j \in (0,\pi)$; $\omega_k = 0$, $\omega_j = \pi$.

3.2.1 $I_{\omega_k}(1)$ and $I_{\omega_j}(1)$ processes with $\omega_k, \omega_j \in (0,\pi)$

One approach to cointegration is to transform one variable (say $x_t \sim I_{\omega_j}(1)$) so that its unit root is shifted to the frequency of the unit root in the other ($y_t \sim I_{\omega_k}(1)$). This is achieved in the cointegrating relation of the first line of (12) in which $e^{i(\omega_j-\omega_k)t}x_t^- \sim I_{\omega_k}(1)$ and $e^{-i\omega_k t}u_t \sim I_{\omega_k}(0)$. It is tedious but not difficult to see that taking the real and imaginary parts of this equation leads to

$$y_t = \cos(\omega_k)y_{t-1} + \beta_{0,t}x_t + [\beta_{1,t}\sin(\omega_j) - \beta_{0,t}\cos(\omega_j)]x_{t-1} + z_{R,t} \tag{17}$$

$$\sin(\omega_k)y_{t-1} = -\beta_{1,t}x_t + [\beta_{1,t}\cos(\omega_j) + \beta_{0,t}\sin(\omega_j)]x_{t-1} - z_{I,t} \tag{18}$$

where

$$\begin{aligned} \beta_{0,t} &= \beta_R\cos[(\omega_j-\omega_k)t] - \beta_I\sin[(\omega_j-\omega_k)t)], \\ \beta_{1,t} &= \beta_R\sin[(\omega_j-\omega_k)t] + \beta_I\cos[(\omega_j-\omega_k)t] \end{aligned} \tag{19}$$

and

$$\begin{aligned} z_{R,t} &= \mathrm{Re}(e^{-i\omega_k t}u_t) = \cos(\omega_k t)\,\mathrm{Re}(u_t) + \sin(\omega_k t)\,\mathrm{Im}(u_t) \\ z_{I,t} &= \mathrm{Im}(e^{-i\omega_k t}u_t) = \cos(\omega_k t)\,\mathrm{Im}(u_t) - \sin(\omega_k t)\,\mathrm{Re}(u_t) \end{aligned}$$

Given that $z_{R,t}, z_{I,t} \sim I_{\omega_k}(0)$, the system (17)-(18) provides two cointegrating relationships at frequency ω_k between time series that are integrated at different harmonic frequencies, both of which are implied by the single complex-valued cointegration relationship of (12). The relationships of (17)-(18) are polynomial, because the lags of y_t and x_t are involved, and time-evolving, because their coefficients vary in a periodic fashion. For the special case of $\omega_k = \omega_j$, the system (17)-(18) reduces to the result reported in Gregoir (2010). Lemma 9 of the appendix, specifically, (73) summarizes the stochastic behaviour of the triangular system (17)-(18) with $x_t^- = e^{-i\omega_j} x_{t-1}^- + \nu_t$ and establishes that the combined $2N \times 1$ vector of processes for y_t and x_t over a cycle of N observations is driven by two common trends.

Also note that (17) and (18) can be summarized in a single expression involving the observed series by taking (17)+ $\cos(\omega_k)/\sin(\omega_k)$(18), yielding

$$\begin{aligned} y_t &= \left[\beta_{0,t} - \frac{\cos(\omega_k)}{\sin(\omega_k)}\beta_{1,t}\right] x_t \\ &+ \left[\beta_{1,t}\left(\sin(\omega_j) + \frac{\cos(\omega_k)}{\sin(\omega_k)}\cos(\omega_j)\right) - \beta_{0,t}\left(\cos(\omega_j) - \frac{\cos(\omega_k)}{\sin(\omega_k)}\sin(\omega_j)\right)\right] x_{t-1} \\ &+ z_{R,t} - \frac{\cos(\omega_k)}{\sin(\omega_k)} z_{I,t}. \end{aligned} \tag{20}$$

Using (19) and trigonometric identities for the cosine and sine of the sum of two angles, (20) can also be expressed

$$\begin{aligned} y_t = & \left\{\beta_R\left[\cos\left([\omega_j - \omega_k]\, t\right) - \frac{\cos(\omega_k)}{\sin(\omega_k)}\sin\left([\omega_j - \omega_k]\, t\right)\right] - \right. \\ & \left. -\beta_I\left[\sin\left([\omega_j - \omega_k]\, t\right) + \frac{\cos(\omega_k)}{\sin(\omega_k)}\cos\left([\omega_j - \omega_k]\, t\right)\right]\right\} x_t \\ & - \left\{\beta_R\left[\cos\left(\omega_j\,[t+1] - \omega_k t\right) - \frac{\cos(\omega_k)}{\sin(\omega_k)}\sin\left(\omega_j\,[t+1] - \omega_k t\right)\right] - \right. \\ & \left. -\beta_I\left[\sin\left(\omega_j\,[t+1] - \omega_k t\right) + \frac{\cos(\omega_k)}{\sin(\omega_k)}\cos\left(\omega_j\,[t+1] - \omega_k t\right)\right]\right\} x_{t-1} \\ & + z_{R,t} - \frac{\cos(\omega_k)}{\sin(\omega_k)} z_{I,t}. \end{aligned} \tag{21}$$

The relationship of (21) is used in Lemma 9 and also for generating simulated series in the Monte Carlo analysis of Section 4 below.

To obtain an alternative representation to (17)-(18) that is more suitable for inference purposes, note that the complex-valued cointegrating relationship of (13) can be written as

$$\begin{aligned} \mathrm{Re}(y_t^{(0)-}) + i\,\mathrm{Im}(y_t^{(0)-}) &= \beta_R\,\mathrm{Re}(x_t^{(0)-}) - \beta_I\,\mathrm{Im}(x_t^{(0)-}) + i\left[\beta_I\,\mathrm{Re}(x_t^{(0)-}) + \beta_R\,\mathrm{Im}(x_t^{(0)-})\right] \\ &+ \mathrm{Re}(u_t) + i\,\mathrm{Im}(u_t). \end{aligned} \tag{22}$$

Equating the respective real and imaginery parts on both sides, (22) immediately leads to

$$\mathrm{Re}(y_t^{(0)-}) = \beta_R\,\mathrm{Re}(x_t^{(0)-}) - \beta_I\,\mathrm{Im}(x_t^{(0)-}) + \mathrm{Re}(u_t) \tag{23}$$

$$\mathrm{Im}(y_t^{(0)-}) = \beta_R\,\mathrm{Im}(x_t^{(0)-}) + \beta_I\,\mathrm{Re}(x_t^{(0)-}) + \mathrm{Im}(u_t) \tag{24}$$

where $\mathrm{Re}(y_t^{(0)-}), \mathrm{Im}(y_t^{(0)-}) \sim I_0(1)$, and $\mathrm{Re}(u_t), \mathrm{Im}(u_t) \sim I_0(0)$. The system (23)-(24) suggests a straightforward way to make inference on the presence of a long-run relationship between series y_t and x_t is to search for two cointegrating relationships among the real and imaginary parts of the demodulated series $y_t^{(0)-}$ and $x_t^{(0)-}$. Notice that, in order to achieve full statistical efficiency, the cross-equation restrictions between (23) and (24) should be imposed in estimation.

Equivalently, for the complex conjugate system (14), we have

$$\begin{aligned} \mathrm{Re}(y_t^{(0)+}) - i\,\mathrm{Im}(y_t^{(0)+}) &= \beta_R\,\mathrm{Re}(x_t^{(0)+}) - \beta_I\,\mathrm{Im}(x_t^{(0)+}) - i\left[\beta_I\,\mathrm{Re}(x_t^{(0)+}) + \beta_R\,\mathrm{Im}(x_t^{(0)+})\right] \\ &+ \mathrm{Re}\,[u_t] - i\,\mathrm{Im}\,[u_t] \end{aligned} \tag{25}$$

which also leads to (23-24), noting that (16) and (15) apply.

Therefore, cointegration analysis can be carried out using either the demodulated series $y_t^{(0)-}$ and $x_t^{(0)-}$ in the complex-valued representation (10) or the real-valued series $\mathrm{Re}(y_t^{(0)-})$, $\mathrm{Im}(y_t^{(0)-})$, $\mathrm{Re}(x_t^{(0)-})$ and

$\mathrm{Im}(x_t^{(0)-})$ in the system (23)-(24). Comparing the results given in Lemma 9 and Lemma 6 of the Appendix, namely (73) and (57) respectively, it can be seen that the nonstationary behaviour in each case is driven by the pair of complex-valued conjugate Brownian motions $w_R^v(r) \pm i w_I^v(r)$, together with the demodulator operators $e^{\mp i\omega_j}$ and $e^{\mp i\omega_k}$ in the vectors $\mathbf{v}_j^{\mp}$ and $\mathbf{v}_k^{\mp}$ respectively. Hence the econometric strategy exposited in subsection 3.3 to test for cointegration based on either the complex-valued demodulated time series or the real-valued real and imaginary parts make sense.

3.2.2 $I_0(1)$ and $I_{\omega_j}(1)$ processes with $\omega_j \in (0, \pi)$

Since $y_t \sim I_0(1)$, only the process $x_t \sim I_{\omega_j}(1)$ needs to be demodulated and the triangular system of (10) becomes

$$
\begin{aligned}
y_t &= \beta x_t^{(0)-} + u_t = \beta e^{i\omega_j t} x_t^- + u_t \\
x_t^- &= e^{-i\omega_j} x_{t-1}^- + \nu_t.
\end{aligned} \tag{26}
$$

Noting that y_t is real, taking the real part of the right-hand side of the cointegrating relationship in (26) leads to the following representation in terms of the original variables:

$$
y_t = \beta_{0,t} x_t + [\beta_{1,t} \sin(\omega_j) - \beta_{0,t} \cos(\omega_j)] x_{t-1} + z_{R,t} \tag{27}
$$

where

$$
\begin{aligned}
\beta_{0,t} &= [\beta_R \cos(\omega_j t) - \beta_I \sin(\omega_j t)], \\
\beta_{1,t} &= [\beta_R \sin(\omega_j t) + \beta_I \cos(\omega_j t)]
\end{aligned} \tag{28}
$$

and $z_{R,t} = \mathrm{Re}(u_t)$. Clearly (27) is a polynomial and periodic cointegrating relationship between the processes $y_t \sim I_0(1)$ and $x_t \sim I_{\omega_j}(1)$ for $\omega_j \in (0, \pi)$. The stochastic behaviour of the system given by (27) with $x_t^- = e^{-i\omega_j} x_{t-1}^- + \nu_t$ is summarized in Lemma 8 of the Appendix, where (66) shows that the $2N \times 1$ combined vector for y_t and x_t over a cycle is driven by two common trends, as in the case where both processes are integrated at different harmonic frequencies $\omega_k, \omega_j \in (0, \pi)$. However, unlike this previous case, (27) shows there is one cointegrating relationship involving the two observed time series and their lags.

To obtain an alternate representation for inference purposes, sum (22) and (25), to yield

$$
\mathrm{Re}(y_t^{(0)-}) + \mathrm{Re}(y_t^{(0)+}) = \beta_R \left[\mathrm{Re}(x_t^{(0)-}) + \mathrm{Re}(x_t^{(0)+})\right] - \beta_I \left[\mathrm{Im}(x_t^{(0)-}) + \mathrm{Im}(x_t^{(0)+})\right] + 2\,\mathrm{Re}(u_t).
$$

Since $\mathrm{Re}(x_t^{(0)-}) = \mathrm{Re}(x_t^{(0)+})$ and noting that $\mathrm{Re}(y_t^{(0)-}) = \mathrm{Re}(y_t^{(0)+}) = y_t$, this implies that the single cointegrating relationship between the two variables can be represented as

$$
y_t = \beta_R \,\mathrm{Re}(x_t^{(0)-}) - \beta_I \,\mathrm{Im}(x_t^{(0)-}) + \mathrm{Re}(u_t). \tag{29}
$$

Therefore, cointegration between $y_t \sim I_0(1)$ and $x_t \sim I_{\omega_j}(1)$ implies the existence of a single cointegrating relationship between y_t, $\mathrm{Re}(x_t^{(0)-})$ and $\mathrm{Im}(x_t^{(0)-})$. Clearly, this cointegrating relationship can also be equivalently expressed in terms of y_t, $\mathrm{Re}(x_t^{(0)+})$ and $\mathrm{Im}(x_t^{(0)+})$.

3.2.3 $I_\pi(1)$ and $I_{\omega_j}(1)$ processes with $\omega_j \in (0, \pi)$

The case $y_t \sim I_\pi(1)$ is essentially analogous to the previous one, hence the main results are briefly reported. By premultiplying both sides of the first equation of (10) by $e^{-i\pi t} = \cos(\pi t)$ yields the triangular syste,

$$
\begin{aligned}
y_t &= \beta e^{i(\omega_j - \pi)t} x_t^- + \cos(\pi t) u_t \\
x_t^- &= e^{-i\omega_j} x_{t-1}^- + \nu_t
\end{aligned} \tag{30}
$$

where $\cos(\pi t) u_t \sim I_\pi(0)$. Taking the real part of the cointegrating relationship leads to

$$
y_t = \beta_{0,t} x_t + [\beta_{1,t} \sin(\omega_j) - \beta_{0,t} \cos(\omega_j)] x_{t-1} + z_{R,t} \tag{31}
$$

which provides a representation in terms of the original variables, where

$$
\begin{aligned}
\beta_{0,t} &= [\beta_R \cos((\omega_j - \pi)t) - \beta_I \sin((\omega_j - \pi)t)], \\
\beta_{1,t} &= [\beta_R \sin((\omega_j - \pi)t) + \beta_I \cos((\omega_j - \pi)t)]
\end{aligned} \tag{32}
$$

and $z_{R,t} = \cos(\pi t)\,\mathrm{Re}(u_t)$. Since $z_{R,t} \sim I_0(0)$, (31) represents a single polynomial and periodic cointegrating relationship between the processes $y_t \sim I_\pi(0)$ and $x_t \sim I_{\omega_j}(1)$ with $\omega_j \in (0, \pi)$. It can be noted that although (31) has the same form as (27), their time-varying coefficients of (32) and (28) respectively differ, reflecting the frequency at which y_t is integrated in each case. Lemma 10 of the appendix summarizes the stochastic behaviour of the system (31) or (32) with $x_t^- = e^{-i\omega_j} x_{t-1}^- + \nu_t$, with (80) showing that the series y_t and x_t share two common trends.

Noting that $y_t^{(0)-} = y_t^{(0)+} = \cos(\pi t) y_t$, summing (22) and (25) leads to

$$\cos(\pi t) y_t = \beta_R\,\mathrm{Re}(x_t^{(0)-}) - \beta_I\,\mathrm{Im}(x_t^{(0)-}) + \mathrm{Re}(u_t) \tag{33}$$

which represents the single cointegrating relationship between y_t and x_t in terms of the series $\cos(\pi t) y_t$, $\mathrm{Re}(x_t^{(0)-})$, and $\mathrm{Im}(x_t^{(0)-})$. Once again, the relationship can be equivalently expressed in terms of $\cos(\pi t) y_t$, $\mathrm{Re}(x_t^{(0)+})$ and $\mathrm{Im}(x_t^{(0)+})$.

3.2.4 $I_0(1)$ and $I_\pi(1)$ processes

Finally, $y_t \sim I_0(1)$ and $x_t \sim I_\pi(1)$ is the simplest case, because the demodulated process $x_t^{(0)-}$, in addition to y_t, is real-valued. Indeed, (10) for this case reduces to

$$y_t = \beta \cos(\pi t) x_t + u_t \tag{34}$$

where β is real-valued and $u_t \sim I_0(0)$. It is then clear that (34) represents the unique cointegrating relationship between the series. Lemma 11 of the appendix summarizes the stochastic behaviour of (34) when $x_t^- = -x_{t-1}^- + \nu_t$, with (83) establishing that y_t and x_t share a single common trend over the cycle of N observations.

3.3 Econometric strategies

Having shown that long-run relationships can exist between processes integrated at different frequencies, an econometric strategy is required to detect such cointegration. Here we discuss possible approaches to such inference.

The first and perhaps simplest approach consists of testing the cointegration rank in a VAR model applied to a system of $2N$ variables formed from the vector of seasons representation for each of the two individual time series[5], which treats the intra-cycle observations $n = 1, ..., N$ as distinct time series. This approach is discussed by, for example, Ghysels and Osborn (2001, Chapter 6) in the context of contemporaneous periodic cointegration. For two processes integrated at different harmonic frequencies (that is, $y_t \sim I_{\omega_k}(1)$, $x_t \sim I_{\omega_j}(1)$, with $\omega_k, \omega_j \in (0, \pi)$ and $\omega_k \neq \omega_j$), each series has two common trends across its N intra-cycle series and hence no cointegration implies four common trends or $2N - 4$ cointegrating relationships in the $2N \times 1$ vector of seasons for the two time series. However, as discussed in subsection 3.2.1, the presence of cross-series cointegration implies two cointegrating relationships of the periodic polynomial form (17)-(18); hence, overall, the $2N \times 1$ vector contains two common trends or $2N - 2$ cointegrating relationships.

When either $y_t \sim I_0(1)$ or $y_t \sim I_\pi(1)$ while $x_t \sim I_{\omega_j}(1)$, $\omega_j \in (0, \pi)$, the N observations for y_t over a cycle share one common trend while those for x_t have two, so that no cointegration implies the presence of three common trends and $2N - 3$ cointegrating relationships. However, the $2N$ observations share two common trends when the processes are cointegrated, implying that the combined $2N \times 1$ vector contains $2N - 2$ cointegrating relationships. Finally, for $y_t \sim I_0(1)$ and $x_t \sim I_\pi(1)$ then (because the N intra-cycle series for each of y_t and x_t is driven by its own common trend), no cointegration implies the presence of two common trends or $2N - 2$ cointegrating relationships between the elements of the $2N \times 1$ vector formed from the vector of seasons of the two time series. On the other hand, cointegration between the series implies that the same common trend drives both series and hence there are $2N - 1$ cointegrating relationships between the elements of the combined $2N \times 1$ system.

This approach has the obvious disadvantage of typically requiring the use of high dimensional systems. For example, with quarterly seasonal data, the combined vector of seasons for the two variables has 8 elements, while it has 24 elements with monthly data. As shown by Franses (1994), the vector of seasons approach lacks power even for the analysis of seasonal unit roots in a univariate quarterly time series.

This dimensionality problem can be avoided by transforming one or both of the original series so that the unit roots under examination apply at the same frequency. An intuitively straightforward method is to

[5] In the case of k variables the system will be of kN where each of the time series is treated as $N \times 1$ vector of seasons.

apply testing after any necessary transformation so that both are $I_0(1)$. For processes integrated at different harmonic frequencies ($\omega_k, \omega_j \in (0,\pi), \omega_k \neq \omega_j$), and as noted above, the complex reduced-rank regression approach by Cubadda (2001) can be applied to the demodulated time series $y_t^{(0)-}$ and $x_t^{(0)-}$ in search of a single cointegration vector.

An alternative to dealing with complex-valued processes is to use the real-valued representations discussed in the preceding section. For the case of two series integrated at different harmonic frequencies ω_k and ω_j, the usual Johansen (1996) method can be applied to the 4×1 vector consisting of $\text{Re}(y_t^{(0)-})$, $\text{Im}(y_t^{(0)-})$, $\text{Re}(x_t^{(0)-})$, and $\text{Im}(x_t^{(0)-})$, with cointegration requiring two cointegrating relationships, which are given by(23)-(24). For processes $y_t \sim I_0(1)$ and $x_t \sim I_{\omega_j}(1)$ having, respectively, a zero-frequency unit real and complex unit root, the Johansen (1996) method can be applied to test for the existence of a single cointegrating relationship among y_t and the real and imaginary parts of $x_t^{(0)-}$, as indicated by (27). Finally, in the case of processes $y_t \sim I_0(1)$ and $x_t \sim I_\pi(1)$, the Johansen method can be applied to test for the existence of cointegration between y_t and $\cos(\pi t)x_t$ in (34).

For all cases except $y_t \sim I_0(1)$ and $x_t \sim I_\pi(1)$, applying the tests just described requires computing the demodulated from the observed series. For $y_t \sim I_{\omega_k}(1)$, $x_t \sim I_{\omega_j}(1)$, with $\omega_k, \omega_j \in (0,\pi)$, using definitions corresponding to (3) and (6) it is easily seen that

$$x_t^{(0)-} = e^{i\omega_j t} x_t^- = e^{i\omega_j t}(1 - e^{i\omega_j} L)x_t \tag{35}$$

$$y_t^{(0)-} = e^{i\omega_k t} y_t^- = e^{i\omega_k t}(1 - e^{i\omega_k} L)y_t \tag{36}$$

where L is the conventional lag operator. The real and imaginery parts of $x_t^{(0)-}$ and $y_t^{(0)-}$ can then be obtained using the identity $e^{i\omega} = \cos(\omega) + i\sin(\omega)$. As noted in the discussion above, and due to the relationship between complex conjugate pairs, numerically identical results will be obtained using $x_t^{(0)+}$ and $y_t^{(0)+}$ as those from employing $x_t^{(0)-}$ and $y_t^{(0)-}$.

Finally, note that the effect of the demodulator operator $e^{-i\omega_j t}$ on the deterministic part of the relationship can be seen following the lines of chapter 7 in Bloomfield (2000). For example, seasonal dummy variables have a one-to-one correspondence with their trigonometric representations written in terms of $\cos(\omega_j t)$ and $\sin(\omega_j t)$. Since $\cos(\omega_j t) = \left(e^{-i\omega_j t} + e^{i\omega_j t}\right)/2$, hence $(1 - e^{-i\omega_k} L)\cos(\omega_j t) = \left(e^{i\omega_j t} - e^{i\omega_j (t-2)}\right)/2$ and $e^{-i\omega_k t}(1 - e^{-i\omega_k} L)\cos(\omega_j t) = \left(1 - e^{-i\omega_j 2}\right)/2$ are constants once demodulated.

4 Monte Carlo analysis

This section provides the results of a Monte Carlo experiment to illustrate the nature of long-run relationships between processes that are integrated at different frequencies and to examine the performance of the approaches to testing discussed in subsection 3.3. We consider an overall cycle of $N = 6$ observations (the smallest N which has two distinct harmonic frequencies) and sample sizes with 200, 100 and 50 complete cycles; hence the total number of observations considered is $T = 1200$, 600 and 300. Cointegration is examined using three approaches, namely applying the Johansen (1996) rank test to the combined $2N \times 1 = 12 \times 1$ vector of seasons for the two series, applying the Johansen procedure to the appropriate real-valued system and applying the complex-valued test of Cubadda (2001) to the demodulated time series $y_t^{(0)-}$ and $x_t^{(0)-}$. In each case, the procedure is applied using a VAR(1) specification with the inclusion of a constant.

The Monte Carlo results are based on 5,000 replications and all tests are conducted at a nominal 5% level of significance using asymptotic critical values. Those provided by Hamilton (1994, Table B.2, Case 2) are employed when testing for cointegration for up to five series, with critical values obtained by simulation using the same conditions as Hamilton (1994) when testing involves a larger number of series (namely up to 12 for the vector of seasons). The complex-valued test uses the critical values of Cubadda (2001, Table 1). The first subsection considers situations where cointegration applies, followed by case where there is no cointegration.

4.1 Cointegrated processes

Corresponding to the cases discussed in subsection 3.2, four cases are considered for cointegrated processes[6]:

[6] We also calculated results for $y_t \sim I_{2\pi/3}(1)$, $x_t \sim I_{\pi/3}(1)$ for Case I, for Cases II and III with $x_t \sim I_{\pi/3}(1)$ in place of $x_t \sim I_{2\pi/3}(1)$ and for Case IV with $y_t \sim I_\pi(1)$, $x_t \sim I_0(1)$. For all of these, the results are effectively the same as those reported in Tables 1 to 4, respectively.

- Case I: $y_t \sim I_{\pi/3}(1)$, $x_t \sim I_{2\pi/3}(1)$. Values of y_t are generated using the cointegrating relationship expressed in terms of the real-valued series as (21).
- Case II: $y_t \sim I_0(1)$, $x_t \sim I_{2\pi/3}(1)$. From (27) and (28), the cointegrating relationship used between the real-valued series is

$$\begin{aligned} y_t &= [\beta_R \cos(\omega_j t) - \beta_I \sin(\omega_j t)]\, x_t \\ &\quad - [\beta_R \cos(\omega_j [t+1]) - \beta_I \sin(\omega_j [t+1])]\, x_{t-1} + u_t. \end{aligned} \tag{37}$$

- Case III: $y_t \sim I_\pi(1)$, $x_t \sim I_{2\pi/3}(1)$, for which the cointegrating relationship is given by (33).
- Case IV: $y_t \sim I_0(1)$, $x_t \sim I_\pi(1)$. The cointegrating relationship here is as given in (34), with the coefficient β real-valued.

In addition

$$x_t = \begin{cases} 2\cos(2\pi/3)\, x_{t-1} - x_{t-2} + \varepsilon_t^{(x)} & \text{Cases I, II, III} \\ -x_{t-1} + \varepsilon_t^{(x)} & \text{Case IV} \end{cases} \tag{38}$$

For all cases, $u_t, \varepsilon_t^{(x)} \sim Niid(0,1)$ and are mutually independent, while pre-sample starting values for all series are set to zero.

In order to investigate the performance of the testing procedures discussed above, a range of five values[7] are considered for the complex-valued cointegrating coefficient, with real and imaginery parts (β_R and β_I) as shown below:

DGP	β_R	β_I
1	$\cos\left(\frac{\pi}{3}\right) = 0.5$	$\sin\left(\frac{\pi}{3}\right) = 0.866$
2	$\cos\left(\frac{2\pi}{3}\right) = -0.5$	$\sin\left(\frac{2\pi}{3}\right) = 0.866$
3	$\cos\left(\frac{\pi}{6}\right) = 0.866$	$\sin\left(\frac{\pi}{6}\right) = 0.5$
4	$\cos\left(\frac{5\pi}{6}\right) = -0.866$	$\sin\left(\frac{5\pi}{6}\right) = 0.5$
5	$\cos\left(\frac{\pi}{8}\right) = 0.924$	$\sin\left(\frac{\pi}{8}\right) = 0.383$

Only the coefficient β_R is used in Case IV.

In line with the analysis of the previous section, we first consider cointegration between two processes each integrated at different harmonic frequencies, with $x_t \sim I_{2\pi/3}(1)$ and $y_t \sim I_{\pi/3}(1)$; hence x_t and y_t have 3- and 6-period cycles, respectively[8]. Results are shown in Table 1 for each cointegrated process considered, with Panel A providing those obtained from applying the Johansen (1996) procedure to the 12×1 vector of observations formed from the two variables over a cycle of 6 observations, Panel B applies the Johansen procedure to the 4×1 vector consisting of the real and imaginery parts of $x_t^{(0)-}$ and $y_t^{(0)-}$, and, finally, Panel C applies the complex-valued cointegration test of Cubadda (2001) to $x_t^{(0)-}$ and $y_t^{(0)-}$.

As discussed in subsection 3.3, cointegration implies the existence of $2N - 2 = 10$ cointegrating relations and two common trends in the 12×1 vector of seasons. Although highly parameterized, the Johansen (1996) procedure performs well in detecting the correct number of cointegrating vectors for the largest sample size of $T = 1200$ (namely 200 complete cycles of 6 observations), it is less satisfactory for smaller sample sizes. Perhaps not surprisingly, for $T = 300$ (50 cycles), ten cointegrating vectors are relatively rarely detected. Even with $T = 600$ (100 cycles) observations, the correct number of cointegrating vectors is detected in only about three-quarters of the replications.

However, the much more parsiminous method that applies the Johansen (1996) procedure to the 4×1 consisting of $\mathrm{Re}(y_t^{(0)-})$, $\mathrm{Im}(y_t^{(0)-})$, $\mathrm{Re}(x_t^{(0)-})$ and $\mathrm{Im}(x_t^{(0)-})$ performs well in Panel B in detecting the presence of two cointegrating vectors for all values of T considered. Even for $T = 600$, one cointegrating relation is always rejected against two or more, while the test for the null hypothesis of two relations shows an empirical rejection rate only modestly greater than the nominal 5%. Finally, in Panel C, the Cubadda (2001) test reliably detects the presence of a single cointegrating vector in $x_t^{(0)-}$ and $y_t^{(0)-}$ for all three sample sizes.

Turning to the case where cointegration exists between a series integrated at the zero frequency and another at a harmonic frequency, Table 2 provides the results for $y_t \sim I_0(1)$, $x_t \sim I_{2\pi/3}(1)$. As noted in subsection 3.3, and in common with the case where the series are cointegrated at different harmonic frequencies, there are $2N - 2 = 10$ cointegrating relationships and hence two common trends across the

[7] Further cases were also examined, with results very close to those shown in Tables 1 to 4.

[8] We also computed results for $x_t \sim I_{\pi/3}(1)$ and $y_t \sim I_{2\pi/3}(1)$, with results effectively the same results as those shown in Table 1.

12×1 combined vector of seasons. Once again, the Johansen (1996) procedure applied to the 12×1 vector provides reliable results only for largest sample size of $T = 1200$; indeed, the results in Panel A of Table 2 are very similar to those in Table 1. The test for cointegration between y_t, $\mathrm{Re}(x_t^{(0)-})$ and $\mathrm{Im}(x_t^{(0)-})$, however, works well in Panel B; the null hypothesis of no cointegration is always rejected, while the presence of one vector is rejected against two with a rejection frequency very close to the nominal 5% level across all three value of T. However, the complex-valued reduced-rank regression procedure applied at the zero frequency using y_t and $x_t^{(0)-}$ does not perform well in Panel C. In particular, the initial null hypothesis of no cointegration is rejected in only 30% (or fewer) of the replications across all DGPs and different values of T. This finding is not surprising since the procedure is designed by Cubadda (2001) to test for complex-valued cointegration between series at a harmonic frequency. Therefore, the procedure is seeking cointegration involving the real and imaginery parts of both series, whereas in this case y_t is real.

The third situation, for which results are provided in Table 3, examines $y_t \sim I_\pi(1)$, $x_t \sim I_{2\pi/3}(1)$[9], again using the five sets of values for β_R and β_I as above. These results are very similar to those where $y_t \sim I_0(1)$ in Table 2, and hence each of the methods used to detect the cointegration between y_t and x_t performs in a equivalent way for the cases where one series is integrated at the zero or Nyquist frequencies. Note that, for the testing of Panels B and C of Table 3, y_t is demodulated to $\cos(\pi t)y_t$. These results also confirm that the complex-valued approach of Cubadda is not appropriate because y_t and $\cos(\pi t)y_t$ are real-valued variables.

Finally, Table 4 considers cointegration between to an $I_0(1)$ process and an $I_\pi(1)$ process, as in (34). As mentioned in subsection 3.3, the vector of seasons approach implies 11 cointegrating relationships, while the other methods test for a single long-run relationship between the appropriate series. The results are in line with those of earlier tables. Both the vector of seasons approach and the direct test applied to y_t and $\cos(\pi t)x_t$ work well, with the qualification that the vector of seasons approach is less satisfartory for smaller sample sizes. The approach of Cubadda (2001) is again not satisfactory (or, more accurately, not applicable) here, as both y_t and $\cos(\pi t)x_t$ are real-valued time series.

4.2 No cointegration

In order to complete the picture, the performance is examined of the above tests in situations when there is no cointegration between the processes integrated at different frequencies. For this purpose, we consider the four cases as in the last subsection[10], namely: $y_t \sim I_{\pi/3}(1)$, $x_t \sim I_{2\pi/3}(1)$; $y_t \sim I_0(1)$, $x_t \sim I_{2\pi/3}(1)$; $y_t \sim I_\pi(1)$, $x_t \sim I_{2\pi/3}(1)$; $y_t \sim I_0(1)$, $x_t \sim I_\pi(1)$. The data generating processes for x_t are as in (38), while those for y_t are:

$$y_t = \begin{cases} 2\cos(\pi/3)\, y_{t-1} - y_{t-2} + \varepsilon_t^{(y)} & \text{Case I} \\ y_{t-1} + \varepsilon_t^{(y)} & \text{Cases II, IV} \\ -y_{t-1} + \varepsilon_t^{(y)} & \text{Case III} \end{cases} \tag{39}$$

where $\varepsilon_t^{(x)}, \varepsilon_t^{(y)} \sim Niid(0,1)$ and mutually independent, with pre-sample starting values for both series set to zero.

The results are shown in Table 5. It can be seen that the Johansen (1996) approach applied to the 12×1 vector of seasons, for which the results are presented in Panel A, reliably (and correctly) detects the presence of four common trends between the series in Case I (that is, two separate common trends in each of x_t and y_t) only for the largest sample size, with $T = 1200$ (200 complete cycles of $N = 6$ observations). For smaller samples the procedure lacks power. In particular, with $T = 300$, the null hypothesis of six common trends is rejected against the alternative of fewer in only around a quarter of the replications. On the other hand, for this case, the Johansen (1996) procedure applied to the four series formed from the real and imaginery parts of $y_t^{(0)-}$ and $x_t^{(0)-}$ in Panel B correctly finds little evidence of cointegration: the initial null hypothesis of no cointegration (four common trends) is rejected with a frequency only modestly above the nominal 5% level. Finally, the Cubadda (2001) procedure in Panel C performs very well for the Case, rejecting the presence of a single complex cointegrating vector at close to the 5% level.

Once again, Table 5 reveals similar results overall for Cases II and III, when one series is integrated at a harmonic frequency and the other at the zero (Case II) or Nyquist frequency (Case III). The vector of seasons approach works well in detecting the presence of three common trends across the 12×1 vector, implying no cross-series cointegration, only for the largest sample size. However, testing for cointegration using y_t, $\mathrm{Re}(x_t^{(0)-})$, and $\mathrm{Im}(x_t^{(0)-})$ works well overall, rejecting the initial (and correct) null hypothesis of

[9] Using $x_t \sim I_{\pi/3}(1)$ again yields very similar results.

[10] We also calculated results for $y_t \sim I_{2\pi/3}(1)$, $x_t \sim I_{\pi/3}(1)$ for Case I, for Cases II and III with $x_t \sim I_{\pi/3}(1)$ in place of $x_t \sim I_{2\pi/3}(1)$ and for Case IV with $y_t \sim I_\pi(1)$, $x_t \sim I_0(1)$. For all of these, the results are effectively the same as those reported in Table 5.

three common trends with a size modestly larger than the nominal 5%. The rejection rate for (true) null of two common trends is higher when the Cubadda complex cointegration approach is employed, but as already noted this procedure is not designed for the situation where one of the two series is real-valued. The pattern of results is largely repeated for Case IV, where both series are real and not cointegrated, and hence there are two common trends across the 12×1 vector of seasons, which is reliably detected only for the largest sample size (Panel A). The performance of the usual Johansen (1996) approach applied to y_t and $\cos(\pi t)x_t$ is good in Panel B, while the complex-valued cointegration approach of Panel C is not appropriate to this case.

Overall, therefore, we conclude that the best approach in general is to employ the approach using the appropriate real and imaginery parts of the demodulated time series. The Cubadda (2001) approach works very well when both time series are integrated at harmonic frequencies, but is not appropriate when one or both series are real-valued. However, the Johansen (1996) test applied to the entire vector of seasons requires long time series, with a large number of years of data for seasonal time series (or, more generally, complete cycles). Hence, from a practical point of view, we recommend applying the Cubadda (2001) complex-valued test to the demodulated time series when both time series are integrated at harmonic frequencies, or forming the real and imaginary parts of the complex demodulated time series and then employing the Johansen (1996) approach.

5 Empirical application

This section explores the presence of cointegrating relationship between processes integrated at different frequencies using quarterly data from the Balearic Islands. In particular we analyze the relationship between tourist arrivals (arr_t) and total employment (emp_t) from the first quarter of 1979 to the fourth quarter of 2015[11]. Figures 1.a and 2.a show the two time series after taking logarithms, while the sample spectra can be found in Figures 1.b and 2.b. It is evident from Figure 1 that tourist arrivals exhibit a clear seasonal pattern which Figure 1.b shows to be is associated particularly with the annual frequency $\pi/2$. On the other hand, employment in Figure 2 shows relatively little seasonal pattern and its spectrum is dominated by a zero frequency peak.

Table 6 provides results for the HEGY test (Hylleberg, Engle, Granger and Yoo, 1990) for seasonal unit roots, which are obtained from the regression

$$\begin{aligned}\Delta_4 y_t &= \alpha_q + \beta_q t + \pi_0 y_{t-1}^{(0)} + \pi_1^{\alpha} y_{t-1}^{(1\alpha)} + \pi_1^{\beta} y_{t-1}^{(1\beta)} + \pi_2 y_{t-1}^{(2)} \\ &\quad + \sum_{j=1}^{p} \gamma_j \Delta_4 y_{t-j} + \varepsilon_t,\end{aligned} \tag{40}$$

where α_q and β_q ($q = 1, 2, 3, 4$) are understood to be the coefficients of quarterly dummy variables for an intercept and trend, respectively, $y_{t-1}^{(0)} = y_t + y_{t-1} + y_{t-2} + y_{t-3}$, $y_{t-1}^{(1\alpha)} = -y_{t-1} + y_{t-3}$, $y_{t-1}^{(1\beta)} = -y_{t-2} + y_{t-4}$, $y_{t-1}^{(2)} = -y_{t-1} + y_{t-2} - y_{t-3} + y_{t-4}$, p is the order of augmentation and ε_t is uncorrelated over time. Results are obtained using both OLS and GLS detrending and the MAIC criteria is used to determine the order of augmentation; see del Barrio Castro, Osborn and Taylor (2016) for details. Asymptotic critical values are employed, with these obtained from the quantile functions in del Barrio Castro, Bodnar and Sansó (2017). As usual, one-sided t-type tests are employed for the null hypotheses $\pi_0 = 0$ and $\pi_2 = 0$ (associated with the zero and Nyquist frequencies, respectively), together with a joint F-type test for $\pi_1^{\alpha} = \pi_1^{\beta} = 0$ (associated with the frequency $\pi/2$).

The tests indicate that $\ln(arr_t)$ is a seasonally integrated process, with unit roots at the zero and both seasonal frequencies, $\pi/2$ and π. Although the sample spectrum of Figure 1.b does not indicate substantial power associated with the Nyquist frequency π, the null hypothesis of a unit root at this frequency is not rejected at any conventional level of significance. On the other hand, $\ln(emp_t)$ apparently has only a zero frequency unit root, with a unit root at the Nyquist frequency and a pair of complex unit roots at frequency $\pi/2$ both rejected at the 1% level of significance.

Based on these results, Table 7 explores the possibility of a long-run cointegrating relationship between $\ln(emp_t) \sim I_0(1)$ and $\ln(arr_t) \sim I_{\pi/2}(1)$. To be specific, In Panel A, the Johansen (1996) procedure is applied to $\ln(emp_t)$ and the real and imaginery parts of $(1-L)\ln(arr_t)$ while Panel B examines $\ln(emp_t)$

[11] The data was obtained from the web page of the IBESTAT (Regional Statistical office of the Balearic Islands), in the case of the quarterly employment data the source is the EPA (Encuesta de Población Activa) of the INE (National Statistical office of Spain). And in the case of the passenger arrivals the source is the IBESTAT based on the data provided by AENA.

and the real and imaginery parts of $(1-L)(1+L)\ln(arr_t) = \left(1-L^2\right)\ln(arr_t)$. In both cases, a constant is included in each equation of the VAR model employed for testing, while significance is judged using the critical values of Hamilton (1994, Table B.10, case 2). The first difference transformation is applied to $\ln(arr_t)$ in both panels to remove the zero frequency unit root in the series, while $(1-L)(1+L) = 1-L^2$ removes unit roots at both the zero and Nyquist frequencies and leaves only the unit root at the frequency $\pi/2$ if $\ln(arr_t)$ contains unit roots at both the zero and Nyquist frequencies (see Table 6). In other words, we test whether the nonstationary behavior at each of the annual frequency $\pi/2$ in tourism arrivals is related with the zero frequency nonstationary behavior observed in employment.

Panel A of Table 7 indicates the presence of one cointegrating relationship (and hence two common trends) among $\ln(emp_t)$, $\mathrm{Re}[e^{i\frac{\pi}{2}t}\left(1-e^{i\frac{\pi}{2}}L\right)(1-L)\ln(arr_t)]$ and $\mathrm{Im}[e^{i\frac{\pi}{2}t}\left(1-e^{i\frac{\pi}{2}}L\right)(1-L)\ln(arr_t)]$, implying the existence of cointegration between the zero frequency unit root of $\ln(emp_t)$ and the pair of annual frequency unit roots in $\ln(arr_t)$. The evidence in Panel B, using the variables $\ln(emp_t)$, $\mathrm{Re}[e^{i\frac{\pi}{2}t}\left(1-e^{i\frac{\pi}{2}}L\right)\left(1-L^2\right)\ln(arr_t)]$ and $\mathrm{Im}[e^{i\frac{\pi}{2}t}\left(1-e^{i\frac{\pi}{2}}L\right)\left(1-L^2\right)\ln(arr_t)]$ is a little less clear, in that the initial null hypothesis of no cointegration is not rejected at even the 10% level with VAR orders of 3 and 5, although it is rejected at the 5% level (or below) using orders of 2 or 4. However, as noted above, the sample spectrum of $\ln(arr_t)$ in Figure 1.b does not have substantial power at the Nyquist frequency and hence the transformation employed in Panel B may represent over-differencing. Overall, the results in Table 7 support cross-frequency cointegration between the toursim arrivals and employment series for the Balearic Islands.

6 Conclusions

A stochastic process that is $I(1)$ at given frequency is characterized by having an unbounded spectrum at that frequency. Hence, it is clear that if two stochastic processes are $I(1)$ at different frequencies, no time-invariant linear combinations of them can remove the unit roots at those frequencies. However, a transformation known as complex demodulation is capable of shifting a unit root at a non-zero frequency to a unit root at frequency zero. Hence, it is possible that a common (complex-valued) stochastic trend can exist between demodulated stochastic processes that are not $I(1)$ at the same frequency.

In terms of the original variables, the form of cointegration under consideration is periodic (that is, has cyclically varying coefficients) and generally polynomial. This notwithstanding, statistical inference may be easily conducted by already available methods for cointegration analysis. Using simulations and an empirical example, the present paper both examines the theory underlying this form of cointegration and documents the practical value of the proposed approach.

7 References

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8 Appendix

For the analysis it is useful to employ a double subscript notation $x_{n\tau}$ where the subscripts $n\tau$ indicate the n^{th} observation within the τ^{th} cycle, where the total number of observations per complete cycle is N. The spectral frequencies associated with $x_{n\tau}$are then $\omega_j = 2\pi j/N$ where $j = 0, 1, \ldots, \lfloor N/2 \rfloor$ and $\lfloor . \rfloor$ denotes the integer part. Hence for example $\omega_1 = 2\pi/N$ completes a full cycle every N observations. Using the double subscript notation for an $I_{\omega_j}(1)$ process, (2) is written as

$$x_{n\tau} = (2\cos\omega_j)\, x_{n-1,\tau} - x_{n-2,\tau} + v_{n\tau}, \quad n = 1,\, 2, \cdots, N. \tag{41}$$

Also note that when using the double subscript notation, it is understood that $x_{n-k,\tau} = x_{N-n+k,\tau-1}$ for $n - k \le 0$. Adopting the convention that $t = 1$ corresponds to $n = \tau = 1$, then $t = N(\tau - 1) + n$ provides the one-to-one mapping between the notations x_t and $x_{n\tau}$.

Defining the $N \times 1$ vector of seasons $X_\tau^- = \left[x_{1\tau}^-, x_{2\tau}^-, x_{3\tau}^-, \ldots, x_{N\tau}^-\right]'$ for the process of (4), the following lemma summarizes the stochastic characteristics of this process:

Lemma 1 *For $X_\tau^- = \left[x_{1\tau}^-, x_{2\tau}^-, x_{3\tau}^-, \ldots, x_{N\tau}^-\right]'$ with $x_{n\tau}^-$ $n = 1,\, 2, \cdots, N$ defined in (4) and with $\nu_{n\tau} \sim iid\left(0, \sigma^2\right)$ then*

$$\begin{aligned} \frac{1}{\sqrt{T}} X_{\lfloor Tr \rfloor}^- &\Rightarrow \sigma \mathbf{C}_j^- W(r) = \sigma \mathbf{v}_j^- \mathbf{v}_j^{+'} W^{v}(r) \\ &= \sigma\,(N/2)^{1/2}\, \mathbf{v}_j^- \left(w_R(r) + i w_I(r)\right) \end{aligned} \tag{42}$$

where $\mathbf{C}_j^-$ is the circulant matrix of rank one $\mathbf{C}_j^- = \mathrm{Ci}\,rc\left[1, e^{-i(N-1)\omega_j}, e^{-i(N-2)\omega_j}, \cdots, e^{-i\omega_j}\right]$, the vectors $\mathbf{v}_j^-$ and $\mathbf{v}_j^+$ are defined as $\mathbf{v}_j^- = \left[\begin{array}{ccccc} e^{-i\omega_j} & e^{-i2\omega_j} & e^{-i3\omega_j} & \cdots & e^{-iN\omega_j}\end{array}\right]'$ and $\mathbf{v}_j^+ = \left[\begin{array}{cccccc} e^{i\omega_j} & e^{i2\omega_j} & e^{i3\omega_j} & \cdots & e^{iN\omega_j}\end{array}\right]$, $W^\nu(r) = \left[\begin{array}{cccc} W_1^\nu(r) & W_2^\nu(r) & \cdots & W_N^\nu(r)\end{array}\right]'$ is $N \times 1$ vector Brownian motion with $w_R^\nu(r)$ and $w_I^\nu(r)$ two scalar Brownian motions defined as $w_R^\nu(r) = (N/2)^{-1/2} \sum_{k=1}^N \cos(k\omega_j)\, W_k^\nu(r)$ and $w_I^\nu(r) = (N/2)^{-1/2} \sum_{k=1}^N \sin(k\omega_j)\, W_k^\nu(r)$ respectively.

Remark 2 *Note that the result in Lemma 1 above also applies to x_t^+ in (8), as it is straightforward to see that for $X_\tau^+ = \left[x_{1\tau}^+, x_{2\tau}^+, x_{3\tau}^+, \ldots, x_{N\tau}^+\right]'$ it follows that $T^{1/2} X_{\lfloor Tr \rfloor}^+ \Rightarrow \sigma \mathbf{C}_j^+ W^\nu(r) = \sigma \mathbf{v}_j^+ \mathbf{v}_j^{-'} W^\nu(r) = \sigma\,(N/2)^{1/2}\, \mathbf{v}_j^- \left(w_R^\nu(r) - i w_I^\nu(r)\right)$ with $\mathbf{C}_j^+ = \mathrm{Ci}\,rc\left[1, e^{i(N-1)\omega_j}, e^{i(N-2)\omega_j}, \cdots, e^{i\omega_j}\right]$. Hence we have a pair of complex-valued scalar Brownian motions $w_R^\nu(r) \pm i w_I^\nu(r)$ as in Gregoir (2010, p.1494). Note also that del Barrio Castro, Rodrigues and Taylor (2018, equations (3.12) and (3.13)) prove a similar result but consider complex-valued Near-Integrated processes and also allow serial correlation in the innovations.*

Remark 3 *From (42) and (5) it is clear that $\left(w_R^\nu(r) + i w_I^\nu(r)\right)$ provides the behaviour. of the stochastic trend $\left[x_0^- + \sum_{k=1}^t e_k^{i\omega_j k} \nu_k\right]$, the vector $\mathbf{v}_j^- = \left[\begin{array}{ccccc} e^{-i\omega_j} & e^{-i2\omega_j} & e^{-i3\omega_j} & \cdots & e^{-iN\omega_j}\end{array}\right]'$ and the effect of the demodulator operator $e^{-it\omega_j}$. Another interesting point from (42) is that it shows that N processes comprising the elements of X_τ^- share a common stochastic trend, or equivalently that there are $N - 1$ cointegration relationships between the elements of X_τ^-.*

Remark 4 *For the process $x_{n\tau}$ of (41), Smith, Taylor and del Barrio Castro (2009, p 540, Lemma 1 and Remark) show that for the circulant matrix of rank 2 $\mathbf{C}_j = \mathrm{Ci}\,rc\left[\frac{\sin(\omega_j)}{\sin(\omega_j)}, \frac{\sin(N\omega_j)}{\sin(\omega_j)}, \frac{\sin([N-1]\omega_j)}{\sin(\omega_j)}, \ldots, \frac{\sin(2\omega_j)}{\sin(\omega_j)}\right]$ then $\mathbf{C}_j$, $\mathbf{C}_j^-$ and $\mathbf{C}_j^+$ satisfy $\mathbf{C}_j = \frac{e^{-i\omega_j}}{e^{-i\omega_j} - e^{i\omega_j}} \mathbf{C}_j^- + \frac{e^{i\omega_j}}{e^{i\omega_j} - e^{-i\omega_j}} \mathbf{C}_j^+$.*

Proof of Lemma 1: First note that the process (4) admits the vector of seasons representation $X_\tau^- = \left[x_{1\tau}^-, x_{2\tau}^-, ..., x_{N\tau}^-\right]'$, then

$$\Phi_0^- X_\tau^- = \Phi_1^- X_{\tau-1}^- + V_\tau \tag{43}$$

where, $V_\tau = [\nu_{1\tau}, \nu_{2\tau}, ..., \nu_{N\tau}]'$ and Φ_0^- and $\Phi_1^-$are $N \times N$ matrices with generic element

$$\begin{aligned} \phi_{0(h,j)}^- &= \begin{cases} 1 & h = j,\ j = 1, ..., N \\ -e^{-i\omega} & h = j + 1,\ j = 1, ..., N - 1 \\ 0 & \text{otherwise} \end{cases} \\ \phi_{1(h,j)}^- &= \begin{cases} e^{-i\omega} & h = 1,\ j = N \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

in which the subscript (h,j) indicates the $(h,j)^{th}$ element of the respective matrix. As in the quarterly $PI(1)$ model studied by Paap and Franses (1999), successively substituting in (43) yields

$$\begin{aligned} X_\tau^- &= [(\Phi_0^-)^{-1}\Phi_1^-]^\tau X_0^- + (\Phi_0^-)^{-1}V_\tau + \sum_{j=1}^{\tau-1}[(\Phi_0^-)^{-1}\Phi_1^-]^j(\Phi_0^-)^{-1}V_{\tau-j} \\ &= [(\Phi_0^-)^{-1}\Phi_1^-]^\tau X_0^- + (\Phi_0^-)^{-1}V_\tau + [(\Phi_0^-)^{-1}\Phi_1^-](\Phi_0^-)^{-1}\sum_{j=1}^{\tau-1}V_{\tau-j}. \end{aligned} \tag{44}$$

Note that this result follows because $(\Phi_0^-)^{-1}\Phi_1^-$ is idempotent, which can be seen by generalizing the form of $(\Phi_0^-)^{-1}\Phi_1^-$ presented by Paap and Franses (1999) and noting that $e^{-i\omega N} = 1$ for $x_{n\tau}^- \sim I_\omega(1)$, and hence[12] $[(\Phi_0^-)^{-1}\Phi_1^-]^j = [(\Phi_0^-)^{-1}\Phi_1^-]$ for $j = 2, 3, ...$ Clearly, (44) provides an alternative representation of (5), but now expressed in terms of the vector of (complex-valued) observations over an entire cycle, where $[(\Phi_0^-)^{-1}\Phi_1^-](\Phi_0^-)^{-1}$ gives the effect of the accumulated vector of shocks $\sum_{j=1}^{\tau-1}V_{\tau-j}$ (see for example Boswijk and Franses (1996), Paap and Franses (1999), del Barrio Castro and Osborn (2008), and for complex-valued processes in the context of seasonally integrated processes, del Barrio Castro, Rodrigues and Taylor (2018)). The matrix $[(\Phi_0^-)^{-1}\Phi_1^-](\Phi_0^-)^{-1}$ has rank one and hence can be written as

$$\mathbf{C}_j^- = [(\Phi_0^-)^{-1}\Phi_1^-](\Phi_0^-)^{-1} = (\mathbf{a}_j^-)(\mathbf{b}_j^-)' \tag{45}$$

where, for (45),

$$\begin{aligned} \mathbf{a}_j^- &= \begin{bmatrix} 1 & e^{-i\omega_j} & e^{-i2\omega_j} & \cdots & e^{-i(N-1)\omega_j} \end{bmatrix}' \\ \mathbf{b}_j^- &= \begin{bmatrix} 1 & e^{-i(N-1)\omega_j} & e^{-i(N-2)\omega_j} & \cdots & e^{-i\omega_j} \end{bmatrix}'. \end{aligned} \tag{46}$$

Therefore, the scalar partial sum $(\mathbf{b}_j^-)'\sum_{j=1}^{\tau-1}V_{\tau-j}$ in (44) is integrated at the zero frequency, while $\mathbf{a}_j^-$ allocates this across the N observations of the cycle at frequency ω_j, yielding an $I_{\omega_j}(1)$ process. This is, of course, the same as (5)-(6), but the cyclicality of the resulting process is now made clear by representing the demodulation operator through the vector $\mathbf{a}_j^-$. Note also that $\frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor}V_j \Rightarrow \sigma W^\nu(r)$ where $W^\nu(r)$ is $N\times 1$ vector Brownian motion. Also, the circulant matrix $\mathbf{C}_j^-$ can be written as

$$\begin{aligned} \mathbf{C}_j^- &= (\mathbf{a}_j^-)(\mathbf{b}_j^-)' = (\mathbf{a}_j^-)e^{-i\omega_j}e^{i\omega_j}(\mathbf{b}_j^-)' = \mathbf{v}_j^-\mathbf{v}_j^{+\prime} \\ \mathbf{v}_j^- &= \begin{bmatrix} e^{-i\omega_j} & e^{-i2\omega_j} & e^{-i3\omega_j} & \cdots & e^{-iN\omega_j} \end{bmatrix}' \\ \mathbf{v}_j^+ &= \begin{bmatrix} e^{i\omega_j} & e^{i2\omega_j} & e^{i3\omega_j} & \cdots & e^{iN\omega_j} \end{bmatrix}'. \end{aligned} \tag{47}$$

Finally note that $\mathbf{v}_j^{+\prime}W^\nu(r) = \sum_{k=1}^N e^{ik\omega_j}W_k^v(r) = \sum_{k=1}^N\cos(k\omega_j)W_k^v(r) + i\sum_{k=1}^N\sin(k\omega_j)W_k^v(r)$, and that $w_R(r) = (N/2)^{-1/2}\sum_{k=1}^N\cos(k\omega_j)W_k^v(r)$ and $w_I(r) = (N/2)^{-1/2}\sum_{k=1}^N\sin(k\omega_j)W_k^v(r)$.■

The following lemmas provide the asymptotic results that underpin the discussion of Section 3.

Lemma 5 *For* $Z_\tau^{(0,\omega_j)-} = \left[y_{1\tau}^{(0)-}, y_{2\tau}^{(0)-}, \ldots, y_{N\tau}^{(0)-}, x_{1\tau}^-, x_{2\tau}^-, \ldots, x_{N\tau}^-\right]'$, *with* $x_{n\tau}^-$ *and* $y_{n\tau}^{(0)-}$ $n = 1, 2, \cdots, N$ *defined in (11),* $\nu_{n\tau}\sim iid\left(0,\sigma^2\right)$ *and* $u_{n\tau}\sim iid\left(0,\sigma_u^2\right)$, *then*

$$\frac{1}{\sqrt{T}}Z_{\lfloor Tr\rfloor}^{(0,\omega_j)-} \Rightarrow \sigma(N/2)^{1/2}\begin{bmatrix}\beta\mathbf{1}\\ \mathbf{v}_j^-\end{bmatrix}(w_R^v(r) + iw_I^v(r)) \tag{48}$$

with $(w_R^\nu(r) + iw_I^\nu(r))$ *and* $\mathbf{v}_j^-$ *as in Lemma 1, while* $\mathbf{1}$ *is an* $N\times 1$ *vector of ones.*

Proof of Lemma 5: As in Lemma 1, define $V_\tau = [\nu_{1\tau}, \nu_{2\tau}, \cdots, \nu_{N\tau}]'$ as in Lemma 1, and also $V_\tau^u = [u_{1\tau}, u_{2\tau}, \cdots, u_{N\tau}]'$ and $V_\tau^Z = [V_\tau^{u\prime}V_\tau']'$. Using the same line of argument as in the proof of Lemma 1 we can write

$$\Phi_0^- Z_\tau^{(0,\omega_j)-} = \Phi_1^- Z_{\tau-1}^{(0,\omega_j)-} + V_\tau^Z \tag{49}$$

[12] Note that matrix Φ_0^- (see chapter 2 pp 45-48 of Pollock (1999)) is an $N\times N$ lower-triangular Toeplitz matrix associated with the polynomial $\left(1-e^{-i\omega_j}L\right)$. Hence the matrix $\left(\Phi_0^-\right)^{-1}$ collects the coefficients of the expansion of the inverse polynomial associated with $\left(1-e^{-i\omega_j}L\right)$. Based on the form of the matrices $(\Phi_0^-)^{-1}$ and Φ_1^-, it is clear that he resulting matrix $(\Phi_0^-)^{-1}\Phi_1^-$ is an $N\times N$ matrix with the first $N-1$ columns having elements equal to zero and the last column equal to the column vector $\mathbf{v}_j^- = \begin{bmatrix} e^{-i\omega_j} & e^{-i2\omega_j} & e^{-i3\omega_j} & \cdots & e^{-iN\omega_j}\end{bmatrix}'$. Finally note that the last element of $\mathbf{v}_j^-$, that is, $e^{-iN\omega_j}$, is equal to 1 as $\omega_j = 2\pi j/N$ and hence the idempotency of $(\Phi_0^-)^{-1}\Phi_1^-$ can be deduced easily.

where

$$\Phi_0^- = \begin{bmatrix} I_{N\times N} & \Phi_0^{-12} \\ 0_{N\times N} & \Phi_0^{-22} \end{bmatrix}, \quad \Phi_1^- = \begin{bmatrix} 0_{N\times N} & 0_{N\times N} \\ 0_{N\times N} & \Phi_1^{-22} \end{bmatrix}$$

in which all sub-matrix are $N \times N$, with

$$\begin{aligned}
\Phi_0^{-12} &= Diag\left[-\beta_1, -\beta_2, \cdots, -\beta_N\right] \\
&= Diag\left[-e^{i\omega_j}\beta, -e^{i\omega_j 2}\beta, ..., -e^{i\omega_j N}\beta\right] \\
&= Diag\left[-e^{-i\omega_j[N-1]}\beta, -e^{-i\omega_j[N-2]}\beta, ..., \beta\right] \\
\phi_{0(h,k)}^{-22} &= \begin{cases} 1 & h = k,\ k = 1, ..., N \\ -e^{-i\omega_j} & h = k+1,\ k = 1, ..., N \\ 0 & \text{otherwise} \end{cases} \\
\phi_{1(h,k)}^{-22} &= \begin{cases} e^{-i\omega_j} & h = 1,\ k = N \\ 0 & \text{otherwise} \end{cases},
\end{aligned}$$

where β is the zero frequency cointegration coefficient of (10) and the subscripts (h, j) refer to the $(h, j)^{th}$ element of the respective sub-matrix, Φ_0^{-12} or Φ_1^{-22}. Using results for the inverse of partitioned matrix,

$$\left(\Phi_0^-\right)^{-1} = \begin{bmatrix} I_{N\times N} & -\Phi_0^{-12}\left(\Phi_0^{-22}\right)^{-1} \\ 0_{N\times N} & \left(\Phi_0^{-22}\right)^{-1} \end{bmatrix}.$$

Note also that, as previously stated, the sub matrix $\left(\Phi_0^{-22}\right)^{-1}$ is the inverse of an $N \times N$ lower-triangular Toeplitz matrix associated with the polynomial $\left(1 - e^{-i\omega_j}L\right)$. Hence $\left(\Phi_0^-\right)^{-1}$ collects the coefficients in the expansion of the inverse polynomial associated with $\left(1 - e^{-i\omega_j}L\right)$. Using the form of $\left(\Phi_0^-\right)^{-1}$, it is easy to check that $\left(\Phi_0^-\right)^{-1}\Phi_1^-$ is an $2N \times 2N$ matrix with all elements in its first $N-1$ columns equal to zero and the last column collecting the elements of the first column of $-\Phi_0^{-12}\left(\Phi_0^{-22}\right)^{-1}$ multiplied by[13] $e^{-i\omega_j}$, followed by the first column of $\left(\Phi_0^{-22}\right)^{-1}$ multiplied by $e^{-i\omega_j}$ as well[14]; hence as the final element of the last column of $\left(\Phi_0^-\right)^{-1}\Phi_1^-$ is equal to $e^{-iN\omega_j} = 1$. It is also clear that $\left(\Phi_0^-\right)^{-1}\Phi_1^-$ is idempotent.

Recursive substitution from (49), yields

$$Z_\tau^{(0,\omega_j)-} = \left(\Phi_0^-\right)^{-1}\Phi_1^- Z_0^{(0,\omega_j)-} + \left(\Phi_0^-\right)^{-1}V_\tau^Z + \left(\Phi_0^-\right)^{-1}\Phi_1^-\left(\Phi_0^-\right)^{-1}\sum_{j=1}^{\tau-1}V_{\tau-j}^Z. \tag{50}$$

We can write

$$\left(\Phi_0^-\right)^{-1}\Phi_1^-\left(\Phi_0^-\right)^{-1} = \begin{bmatrix} 0_{N\times N} & \mathbf{C}_y^{(0)-} \\ 0_{N\times N} & \mathbf{C}_x^- \end{bmatrix} \tag{51}$$

where the $N \times N$ sub-matrices $\mathbf{C}_y^{(0)-}$ and $\mathbf{C}_x^-$ each have rank one, so that

$$\begin{aligned}
\mathbf{C}_x^- &= a_x^- b_x^{-\prime} \\
\mathbf{C}_y^{(0)-} &= a_y^{(0)-} b_x^{-\prime}
\end{aligned} \tag{52}$$

with

$$\begin{aligned}
a_x^- &= \begin{bmatrix} 1 & e^{-i\omega_j} & e^{-i2\omega_j} & \cdots & e^{-i(N-2)\omega_j} & e^{-i(N-1)\omega_j} \end{bmatrix}' \\
a_y^{(0)-} &= e^{i\omega_j}\beta\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix}' \\
b_x^- &= b_y^- = \begin{bmatrix} 1 & e^{-i(N-1)\omega_j} & e^{-i(N-2)\omega_j} & \cdots & e^{-i2\omega_j} & e^{-i\omega_j} \end{bmatrix}'.
\end{aligned} \tag{53}$$

It is clear from (50) to (53) that $\left(\Phi_0^-\right)^{-1}\Phi_1^-\left(\Phi_0^-\right)^{-1}$, $\mathbf{C}_y^{(0)-}$ and $\mathbf{C}_x^-$ have rank 1, and hence there is one common stochastic trend shared by the seasons of both time series $y_{n\tau}^{(0)-}$ and $x_{n\tau}^-$, or equivalently we have

[13] That is $e^{-i\omega_j}\left[\beta_1, e^{-i\omega_j}\beta_2, \cdots, e^{-i\omega_j[N-1]}\beta_N\right]' = e^{-i\omega_j}\left[e^{-i\omega_j[N-1]}\beta, e^{-i\omega_j[N-2]}e^{-i\omega_j}\beta, \cdots, e^{-i\omega_j[N-1]}\beta\right]' = \beta\left[1, 1, \cdots, 1\right]$.

[14] That is $e^{-i\omega_j}\left[1, e^{-i\omega_j}, \cdots, e^{-i\omega_j[N-1]}\right]' = \left[e^{-i\omega_j}, e^{-i2\omega_j}, \cdots, e^{-iN\omega_j}\right]'$.

$2N-1$ cointegration relationships between the elements of the vector Z_τ^-. Note also that $\mathbf{C}_x^-$ in (52) has the same form as (47) and it is possible to write:

$$\begin{array}{rcl} \mathbf{C}_x^- & = & (\mathbf{a}_x^-)(\mathbf{b}_x^-)' = (\mathbf{a}_x^-)e^{-i\omega_j}e^{i\omega_j}(\mathbf{b}_x^-)' = \mathbf{v}_x^-\mathbf{v}_x^{+\prime} \\ \mathbf{v}_x^- & = & \left[\begin{array}{ccccc} e^{-i\omega_j} & e^{-i2\omega_j} & e^{-i3\omega_j} & \cdots & e^{-iN\omega_j} \end{array}\right]' \\ \mathbf{v}_x^+ & = & \left[\begin{array}{ccccc} e^{i\omega_j} & e^{i2\omega_j} & e^{i3\omega_j} & \cdots & e^{iN\omega_j} \end{array}\right]'. \end{array} \tag{54}$$

In the case of $\mathbf{C}_y^{(0)-}$, note that:

$$\mathbf{C}_y^{(0)-} = a_y^{(0)-}b_x^{-\prime} = \beta\mathbf{1}e^{i\omega_j}b_x^{-\prime} = \beta\mathbf{1}\mathbf{v}_x^{+\prime} \tag{55}$$

with $\mathbf{1}$ an $N\times 1$ vector of ones. Based on

$$\frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor}V_j^Z = \left[\begin{array}{c} \frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor}V_j^u \\ \frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor}V_j \end{array}\right] \Rightarrow \left[\begin{array}{c} \sigma_u W^u(r) \\ \sigma W^v(r) \end{array}\right]$$

where $W^u(r)$ and $W^v(r)$ are two $N\times 1$ vector Brownian motions. It is possible to see that for Z_τ^- in (50) we have:

$$\frac{1}{\sqrt{T}}Z_{\lfloor Tr\rfloor}^{(0,\omega_j)-} \Rightarrow \left[\begin{array}{cc} 0_{N\times N} & \mathbf{C}_y^{(0)-} \\ 0_{N\times N} & \mathbf{C}_x^- \end{array}\right]\left[\begin{array}{c} \sigma_u W^u(r) \\ \sigma W^v(r) \end{array}\right] \tag{56}$$

and the result in Lemma 5 comes from (56),(54) and (55).■

Lemma 6 *For* $Z_\tau^{(\omega_k,\omega_j)-} = \left[y_{1\tau}^-, y_{2\tau}^-, \ldots, y_{N\tau}^-, x_{1\tau}^-, x_{2\tau}^-, \ldots, x_{N\tau}^-\right]'$ *with* $x_{n\tau}^-$ *and* $y_{n\tau}^-$, $n = 1, 2, \cdots, N$ *defined in (12) and with* $\nu_{n\tau} \sim iid\left(0,\sigma^2\right)$ *and* $u_{n\tau} \sim iid\left(0,\sigma_u^2\right)$, *then*

$$\frac{1}{\sqrt{T}}Z_{\lfloor Tr\rfloor}^{(\omega_k,\omega_j)-} \Rightarrow \sigma\left(N/2\right)^{1/2}\left[\begin{array}{c} \beta\mathbf{v}_k^- \\ \mathbf{v}_j^- \end{array}\right]\left(w_R^v(r) + iw_I^v(r)\right) \tag{57}$$

with $\left(w_R^v(r) + iw_I^v(r)\right)$ *and* $\mathbf{v}_j^-$ *as in Lemma 1 and finally* $\mathbf{v}_k^- = \left[\begin{array}{ccccc} e^{-i\omega_k} & e^{-i2\omega_k} & e^{-i3\omega_k} & \cdots & e^{-iN\omega_k} \end{array}\right]'$.

Remark 7 *As a particular case of (12) and (57) we can define a triangular system between two complex-valued integrated processes, one associated with the Nyquist frequency* (π) *and the other to a harmonic frequency* ω_j, *by multiplying (11) by* $e^{-i\pi(N(\tau-1)+n)}$.

Proof of Lemma 6: First note that $e^{-i\omega_k t}u_t = e^{-i\omega_k(N[\tau-1]+n)}u_{n\tau} = e^{-i\omega_k n}u_{n\tau}$. Define $V_\tau^Z = [V_\tau^{u\prime}V_\tau^\prime]'$ with $V_\tau = [v_{1\tau}, v_{2\tau}, \cdots, v_{N\tau}]'$ as in Lemma 1 and $V_\tau^u = \left[-e^{i\omega_k}u_{1\tau}, -e^{i2\omega_k}u_{2\tau}, \cdots, -e^{iN\omega_k}u_{N\tau}\right]'$. Using the same line of argument as in the proof of Lemma 5 we can write

$$\Phi_0^- Z_\tau^{(\omega_k,\omega_j)-} = \Phi_1^- Z_{\tau-1}^{(\omega_k,\omega_j)-} + V_\tau^Z \tag{58}$$

where

$$\Phi_0^- = \left[\begin{array}{cc} I_{N\times N} & \Phi_0^{-12} \\ 0_{N\times N} & \Phi_0^{-22} \end{array}\right], \quad \Phi_1^- = \left[\begin{array}{cc} 0_{N\times N} & 0_{N\times N} \\ 0_{N\times N} & \Phi_1^{-22} \end{array}\right]$$

in which all sub-matrices are $N\times N$, with

$$\begin{array}{rcl} \Phi_0^{-12} & = & Diag\left[-\beta_1, -\beta_2, \cdots, -\beta_N\right] \\ & = & Diag\left[-e^{i(\omega_j-\omega_k)}\beta, -e^{i(\omega_j-\omega_k)2}\beta, ..., -e^{i(\omega_j-\omega_k)N}\beta\right] \\ & & Diag\left[-e^{-i(\omega_j-\omega_k)[N-1]}\beta, -e^{-i(\omega_j-\omega_k)[N-1]}\beta, ..., -e^{-i(\omega_j-\omega_k)}\beta\right] \\ \phi_{0(h,k)}^{-22} & = & \left\{\begin{array}{cl} 1 & h=k,\ k=1,...,N \\ -e^{-i\omega_j} & h=k+1,\ k=1,...,N \\ 0 & \text{otherwise} \end{array}\right. \\ \phi_{1(h,k)}^{-22} & = & \left\{\begin{array}{cl} e^{-i\omega_j} & h=1,\ k=N \\ 0 & \text{otherwise} \end{array}\right. , \end{array}$$

Recursively substituting from (58), and recognizing that $\left(\Phi_0^-\right)^{-1}\Phi_1^-$ is idempotent[15], yields

$$Z_\tau^{(\omega_k,\omega_j)-} = \left(\Phi_0^-\right)^{-1}\Phi_1^- Z_0^{(\omega_k,\omega_j)-} + \left(\Phi_0^-\right)^{-1} V_\tau^Z + \left(\Phi_0^-\right)^{-1}\Phi_1^- \left(\Phi_0^-\right)^{-1}\sum_{j=1}^{\tau-1} V_{\tau-j}^Z. \tag{59}$$

Also

$$\left(\Phi_0^-\right)^{-1}\Phi_1^- \left(\Phi_0^-\right)^{-1} = \begin{bmatrix} 0_{N\times N} & \mathbf{C}_y^- \\ 0_{N\times N} & \mathbf{C}_x^- \end{bmatrix} \tag{60}$$

where the $N\times N$ sub-matrices $\mathbf{C}_y$ and $\mathbf{C}_x^-$ each have rank one, so

$$\begin{aligned} \mathbf{C}_x^- &= a_x^- b_x^{-\prime} \\ \mathbf{C}_y^- &= a_y^- b_x^{-\prime} \end{aligned} \tag{61}$$

with

$$\begin{aligned} a_x^- &= \left[\begin{array}{cccccc} 1 & e^{-i\omega_j} & e^{-i2\omega_j} & \cdots & e^{-i(N-2)\omega_j} & e^{-i(N-1)\omega_j} \end{array}\right]' \\ a_y^- &= e^{-i\omega_k}e^{i\omega_j}\beta\left[\begin{array}{cccccc} 1 & e^{-i\omega_k} & e^{-i2\omega_k} & \cdots & e^{-i(N-2)\omega_k} & e^{-i(N-1)\omega_k} \end{array}\right]' \\ b_x^- &= b_y^- = \left[\begin{array}{cccccc} 1 & e^{-i(N-1)\omega_j} & e^{-i(N-2)\omega_j} & \cdots & e^{-i2\omega_j} & e^{-i\omega_j} \end{array}\right]'. \end{aligned} \tag{62}$$

It is clear from (60) to (61) that $\left(\Phi_0^-\right)^{-1}\Phi_1^-\left(\Phi_0^-\right)^{-1}$, $\mathbf{C}_y^-$ and $\mathbf{C}_x^-$ have rank 1, and hence there is one common stochastic trend shared by the seasons of both time series $y_{n\tau}^-$and $x_{n\tau}^-$ or equivalently we have $2N-1$ cointegration relationships between the seasons of the vector $Z_\tau^{(\omega_k,\omega_j)-}$. Note also that $\mathbf{C}_x^-$ in (61) has the same form as (47) and it is possible to write:

$$\begin{aligned} \mathbf{C}_x^- &= \mathbf{v}_x^-\mathbf{v}_x^{+\prime} \\ \mathbf{v}_x^- &= \left[\begin{array}{ccccc} e^{-i\omega_j} & e^{-i2\omega_j} & e^{-i3\omega_j} & \cdots & e^{-iN\omega_j} \end{array}\right]' \\ \mathbf{v}_x^{-+} &= \left[\begin{array}{ccccc} e^{i\omega_j} & e^{i2\omega_j} & e^{i3\omega_j} & \cdots & e^{iN\omega_j} \end{array}\right]'. \end{aligned} \tag{63}$$

In the case of $\mathbf{C}_y^-$ note that

$$\begin{aligned} \mathbf{C}_y^- &= a_y^- b_x^{-\prime} = \beta\mathbf{v}_y^-\mathbf{v}_x^{+\prime} \\ \mathbf{v}_y^- &= \left[\begin{array}{ccccc} e^{-i\omega_k} & e^{-i2\omega_k} & e^{-i3\omega_k} & \cdots & e^{-iN\omega_k} \end{array}\right]'. \end{aligned} \tag{64}$$

Based of the fact that

$$\frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor} V_j^Z = \begin{bmatrix} \frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor} V_j^u \\ \frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr\rfloor} V_j \end{bmatrix} \Rightarrow \begin{bmatrix} \sigma_u W^u(r) \\ \sigma W^v(r) \end{bmatrix}$$

where $W^\nu(r)$ is an $N\times 1$ vector Brownian motion as in Lemma 2 and $W^u(r)$ is an $N\times 1$ complex-valued vector Brownian motion. Hence for $Z_\tau^{(\omega_k,\omega_j)-}$ in (59) we have:

$$\frac{1}{\sqrt{T}} Z_{\lfloor Tr\rfloor}^{(0,\omega_j)-} \Rightarrow \begin{bmatrix} 0_{N\times N} & \mathbf{C}_y^- \\ 0_{N\times N} & \mathbf{C}_x^- \end{bmatrix}\begin{bmatrix} \sigma_u W^u(r) \\ \sigma W^\nu(r) \end{bmatrix} \tag{65}$$

and the result in Lemma 6 comes from (65),(63) and (64).■

[15] First note that the for the inverse of Φ_0^- we have:

$$\left(\Phi_0^-\right)^{-1} = \begin{bmatrix} I_{N\times N} & -\Phi_0^{-12}\left(\Phi_0^{-22}\right)^{-1} \\ 0_{N\times N} & \left(\Phi_0^{-22}\right)^{-1} \end{bmatrix}.$$

Note also that the resulting matrix $\left(\Phi_0^-\right)^{-1}\Phi_1^-$ is a $2N\times 2N$ matrix with its first colunmns with all the elements equal to zero and the last column collecting in first place the elements of the first column of $-\Phi_0^{-12}\left(\Phi_0^{-22}\right)^{-1}$ multiplied by $e^{-i\omega_j}$, followed by the first column of matrix $\left(\Phi_0^{-22}\right)^{-1}$ multiplied by $e^{-i\omega_j}$ as well, in this case also the last element of the last column of $\left(\Phi_0^-\right)^{-1}\Phi_1^-$ is equal to $e^{-iN\omega_j}=1$. it is clear that matrix $\left(\Phi_0^-\right)^{-1}\Phi_1^-$ is idempotent.

Lemma 8 *For* $Z_\tau^{(0,\omega_j)} = [y_{1\tau}, y_{2\tau}, \ldots, y_{N\tau}, x_{1\tau}, x_{2\tau}, \ldots, x_{N\tau}]'$ *with* $y_{n\tau}$ *and* $x_{n\tau}^-$ $(n = 1, 2, \cdots, N)$ *defined by (26) with* $v_{n\tau} \sim iid\left(0, \sigma^2\right)$, $u_{n\tau} \sim iid\left(0, \sigma_u^2\right)$, *and* $z_{Rn\tau} = \mathrm{Re}\left[u_{n\tau}\right]$, *then*

$$\frac{1}{\sqrt{T}} Z_{\lfloor Tr \rfloor}^{(0,\omega_j)} \Rightarrow \sigma\left(N/2\right)^{1/2} \left[\begin{array}{c} \frac{1}{2}\left[\left(\beta_R + i\beta_I\right)\mathbf{1}\left(w_R^v(r) + iw_I^v(r)\right) + \left(\beta_R - i\beta_I\right)\mathbf{1}\left(w_R^v(r) - iw_I^v(r)\right)\right] \\ \frac{e^{-i\omega_j}}{-2i\sin(\omega_j)}\mathbf{v}_j^-\left(w_R^v(r) + iw_I^v(r)\right) + \frac{e^{i\omega_j}}{2i\sin(\omega_j)}\mathbf{v}_j^+\left(w_R^v(r) - iw_I^v(r)\right) \end{array} \right] \tag{66}$$

with $\left(w_R^v(r) + iw_I^v(r)\right)$, $\mathbf{v}_j^-$ *and* $\mathbf{v}_j^+$ *as in Lemma 1,* $\left(w_R^v(r) - iw_I^v(r)\right)$ *the complex conjugate of* $\left(w_R^v(r) + iw_I^v(r)\right)$ *and* $\mathbf{1}$ *is an* $N \times 1$ *vector of ones.*

Proof of Lemma 8: For $V_\tau^Z = [z_{R1\tau}, z_{R2\tau}, \cdots, z_{RN\tau}, v_{1\tau}, v_{2\tau}, \cdots, v_{N\tau}'] = [V_\tau^{u\prime} V_\tau']'$ with $V_\tau = [v_{1\tau}, v_{2\tau}, \cdots, v_{N\tau}]'$ and $V_\tau^u = [z_{R1\tau}, z_{R2\tau}, \cdots, z_{RN\tau}]'$, using the same line of argument as in the proof of previous lemmas we can write

$$\Phi_0 Z_\tau^{(0,\omega_j)} = \Phi_1 Z_\tau^{(0,\omega_j)} + V_\tau^Z \tag{67}$$

where

$$\Phi_0 = \left[\begin{array}{cc} I_{N\times N} & \Phi_0^{12} \\ 0_{N\times N} & \Phi_0^{22} \end{array} \right], \quad \Phi_1 = \left[\begin{array}{cc} 0_{N\times N} & \Phi_1^{12} \\ 0_{N\times N} & \Phi_1^{22} \end{array} \right]$$

in which all sub-matrices are $N \times N$, with

$$\Phi_0^{12} = \left[\begin{array}{cccccc} -\beta_{11} & 0 & 0 & 0 & \cdots & 0 \\ -\beta_{22} & -\beta_{12} & 0 & 0 & \cdots & 0 \\ 0 & -\beta_{23} & -\beta_{13} & 0 & \cdots & 0 \\ 0 & 0 & -\beta_{24} & -\beta_{14} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\beta_{2N} & -\beta_{1N} \end{array} \right]$$

with β_{1n} and β_{2n} the coefficients associated with $x_{n\tau}$ and $x_{n-1,\tau}$ in (27). Hence we can see from (27) and (28) that

$$\begin{array}{rcl} \beta_{1n} & = & \left[\beta_R \cos\left(\omega_j n\right) - \beta_I \sin\left(\omega_j n\right)\right] \\ \beta_{2n} & = & -\left[\beta_R \cos\left(\omega_j\left[n+1\right]\right) - \beta_I \sin\left(\omega_j\left[n+1\right]\right)\right] \end{array}$$

and the remaining submatrices are

$$\Phi_0^{22} = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & \cdots & 0 \\ -2\cos\left(\omega_j\right) & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2\cos\left(\omega_j\right) & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2\cos\left(\omega_j\right) & 1 & \cdots & 0 \\ 0 & 0 & 1 & -2\cos\left(\omega_j\right) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -2\cos\left(\omega_j\right) & 1 \end{array} \right]$$

$$\Phi_1^{12} = \left[\begin{array}{cccccc} 0 & 0 & 0 & \cdots & 0 & \beta_{12} \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -0 \end{array} \right]$$

$$\Phi_1^{22} = \left[\begin{array}{cccccc} 0 & 0 & \cdots & 0 & -1 & 2\cos\left(\omega_j\right) \\ 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & -0 \end{array} \right].$$

Using results form the inverse of a partitioned matrix,

$$\left(\Phi_0\right)^{-1} = \left[\begin{array}{cc} I_{N\times N} & -\Phi_0^{12}\left(\Phi_0^{22}\right)^{-1} \\ 0_{N\times N} & \left(\Phi_0^{22}\right)^{-1} \end{array} \right].$$

The matrix Φ_0^{22} is a lower-triangular Toeplitz matrix associated with the polynomial $\left(1-2\cos(\omega_j)L+L^2\right)$, hence $\left(\Phi_0^{22}\right)^{-1}$ collects the coefficients of the expansion of the inverse of $\left(1-2\cos(\omega_j)L+L^2\right)$, that is:

$$\left(\Phi_0^{22}\right)^{-1}=\frac{1}{\sin(\omega_j)}\begin{bmatrix} \sin(\omega_j) & 0 & 0 & 0 & \cdots & 0\\ \sin(2\omega_j) & \sin(\omega_j) & 0 & 0 & \cdots & 0\\ \sin(3\omega_j) & \sin(2\omega_j) & \sin(\omega_j) & 0 & \cdots & 0\\ \sin(4\omega_j) & \sin(3\omega_j) & \sin(2\omega_j) & \sin(\omega_j) & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ \sin(N\omega_j) & \sin([N-1]\omega_j) & \sin([N-2]\omega_j) & \sin([N-3]\omega_j) & \cdots & \sin(\omega_j) \end{bmatrix}$$

In the case of the case of $-\Phi_0^{12}\left(\Phi_0^{22}\right)^{-1}$ it is easy to check that:

$$\begin{aligned} -\Phi_0^{12}\left(\Phi_0^{22}\right)^{-1} &= \frac{1}{\sin(\omega_j)}\times\\ &\begin{bmatrix} \beta_{11}\sin(\omega_j) & 0 & \cdots & 0\\ \beta_{12}\sin(2\omega_j)+\beta_{22}\sin(\omega_j) & \beta_{12}\sin(\omega_j) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \beta_{1N}\sin(N\omega_j)+\beta_{2N}\sin([N-1]\omega_j) & \beta_{1N}\sin([N-1]\omega_j)+\beta_{2N}\sin([N-2]\omega_j) & \cdots & \beta_{1N}\sin(\omega_j \end{bmatrix} \end{aligned}$$

The matrix $(\Phi_0)^{-1}\Phi_1$ is $2N\times 2N$, but as the matrix Φ_1 has oly 4 elements different from zero (see the definitions Φ_1^{12} and Φ_1^{22} above), then

$$(\Phi_0)^{-1}\Phi_1=\begin{bmatrix} 0_{2N\times 2(N-1)} & \mathbf{V}_y\\ 0_{2N\times 2(N-1)} & \mathbf{V}_x \end{bmatrix} \tag{68}$$

where $\mathbf{V}_y$ and $\mathbf{V}_x$ are $N\times 2$ matrices with the following form[16]:

$$\mathbf{V}_y=\frac{1}{\sin(\omega_j)}\begin{bmatrix} -\left[\beta_{11}\sin(\omega_j)\right] & \beta_{11}\sin(2\omega_j)+\beta_{12}\sin(\omega_j)\\ -\left[\beta_{12}\sin(2\omega_j)+\beta_{22}\sin(\omega_j)\right] & \beta_{12}\sin(3\omega_j)+\beta_{22}\sin(2\omega_j)\\ -\left[\beta_{13}\sin(3\omega_j)+\beta_{23}\sin(2\omega_j)\right] & \beta_{13}\sin(4\omega_j)+\beta_{23}\sin(3\omega_j)\\ -\left[\beta_{14}\sin(4\omega_j)+\beta_{24}\sin(3\omega_j)\right] & \beta_{14}\sin(5\omega_j)+\beta_{24}\sin(4\omega_j)\\ \vdots & \vdots\\ -\left[\beta_{1N}\sin(N\omega_j)+\beta_{2N}\sin([N-1]\omega_j)\right] & \beta_{1N}\sin([N+1]\omega_j)+\beta_{2N}\sin(N\omega_j) \end{bmatrix} \tag{69}$$

$$\mathbf{V}_x=\frac{1}{\sin(\omega_j)}\begin{bmatrix} -\sin(\omega_j) & \sin(2\omega_j)\\ -\sin(2\omega_j) & \sin(3\omega_j)\\ -\sin(3\omega_j) & \sin(4\omega_j)\\ -\sin(4\omega_j) & \sin(5\omega_j)\\ \vdots & \vdots\\ -\sin(N\omega_j) & \sin([N+1]\omega_j) \end{bmatrix}. \tag{70}$$

It is easy to check that the matrix $\Phi_0^{-1}\Phi_1$ is idempotent[17]. Recursive substitution from (67) yields:

$$Z_\tau^{(0,\omega_j)}=\Phi_0^{-1}\Phi_1^{-}Z_o^{(0,\omega_j)}+\Phi_0^{-1}V_\tau^Z+\Phi_0^{-1}\Phi_1\Phi_0^{-1}\sum_{j=1}^{\tau-1}V_{\tau-j}^Z.$$

[16] Based on the form of Φ_1 it is possible to check that the fist column of $\mathbf{V}_y$ and $\mathbf{V}_x$ is going to be equal to minus th first column of $\left(\Phi_0^{22}\right)^{-1}$ and $-\Phi_0^{12}\left(\Phi_0^{22}\right)^{-1}$ respectively. Finally in the case of the second column of $\mathbf{V}_y$ and $\mathbf{V}_x$ is the weigthed sum of the first and second column of $\left(\Phi_0^{22}\right)^{-1}$ and $-\Phi_0^{12}\left(\Phi_0^{22}\right)^{-1}$ with weigths $2\cos(\omega_k)$ and -1 respectively. Finally note to obtain the second column for (69) and (70) we use the following recurrent expression for multiple angle:

$$\begin{aligned} \sin(j\omega_k) &= 2\cos(\omega_k)\sin([j-1]\omega_k)-\sin([j-2]\omega_k)\\ \cos(j\omega_k) &= 2\cos(\omega_k)\cos([j-1]\omega_k)-\cos([j-2]\omega_k) \end{aligned}$$

[17] The lower 2×2 submatrix of $\mathbf{V}_x$ is equal to:

$$\begin{bmatrix} \frac{-\sin([N-1]\omega_j)}{\sin(\omega_j)} & \frac{\sin(N\omega_j)}{\sin(\omega_j)}\\ \frac{-\sin(N\omega_j)}{\sin(\omega_j)} & \frac{\sin([N+1]\omega_k)}{\sin(\omega_j)} \end{bmatrix}$$

and it is equal to $I_{2\times 2}$. Hence $\left[(\Phi_0)^{-1}\Phi_1\right]^h=(\Phi_0)^{-1}\Phi_1$ for $h=2,3,\ldots$.

The matrix $\Phi_0^{-1}\Phi_1\Phi_0^{-1}$ displays the impact of the accumulation shocks $\sum_{j=1}^{\tau-1} V_{\tau-j}^Z$ and has the form:

$$\begin{aligned}
\Phi_0^{-1}\Phi_1\Phi_0^{-1} &= \begin{bmatrix} 0_{N\times N} & \mathbf{V}_y\mathbf{U}_x' \\ 0_{N\times N} & \mathbf{V}_x\mathbf{U}_x' \end{bmatrix} \\
\mathbf{U}_x' &= \frac{1}{\sin(\omega_j)}\begin{bmatrix} \sin([N-1]\omega_j) & \sin([N-2]\omega_j) & \sin([N-3]\omega_j) & \sin([N-4]\omega_j) & \cdots & 0 \\ \sin(N\omega_j) & \sin([N-1]\omega_j) & \sin([N-2]\omega_j) & \sin([N-3]\omega_j) & \cdots & \sin(\omega_j) \end{bmatrix}.
\end{aligned}$$

In this case

$$\begin{aligned}
\mathbf{V}_x\mathbf{U}_x' &= \mathbf{C}_j = \mathrm{Ci}\,rc\left[\frac{\sin(\omega_j)}{\sin(\omega_j)}, \frac{\sin(N\omega_j)}{\sin(\omega_j)}, \cdots, \frac{\sin(2\omega_j)}{\sin(\omega_j)}\right] \\
&= \frac{e^{-i\omega_j}}{-2i\sin(\omega_j)}\mathbf{C}_j^- + \frac{e^{i\omega_j}}{2i\sin(\omega_j)}\mathbf{C}_j^+ \qquad (71)
\end{aligned}$$

where $\mathbf{C}_j$ is defined in Remark 2 and $\mathbf{C}_j^-$and $\mathbf{C}_j^+$ in Lemma 1. Also, $\mathbf{V}_y$ of (69) can be simplified as

$$\mathbf{V}_y = \begin{bmatrix} -[\beta_R\cos(\omega_j) - \beta_I\sin(\omega_j)] & \beta_R \\ -[\beta_R\cos(\omega_j) - \beta_I\sin(\omega_j)] & \beta_R \\ -[\beta_R\cos(\omega_j) - \beta_I\sin(\omega_j)] & \beta_R \\ -[\beta_R\cos(\omega_j) - \beta_I\sin(\omega_j)] & \beta_R \\ \vdots & \vdots \\ -[\beta_R\cos(\omega_j) - \beta_I\sin(\omega_j)] & \beta_R \end{bmatrix}. \qquad (72)$$

To obtain (72) from (69), we use the expressions for β_{1n} and β_{2n} in (27) and the formula connecting products of sin and cos to sums[18] jointly with the expressions for the double angle sin and cos[19]. Clearly the two columns of $\mathbf{V}_y$ are linearly dependent. Note also that it is possible to check that $\mathbf{V}_y\mathbf{U}_x'$ is a matrix with generic row element $\beta_R\cos(\omega_j h) - \beta_I\sin(\omega_j h)$ with $h = 1, 2, \ldots, N$, that is, all the elements of the first column are equal to $\beta_R\cos(\omega_j) - \beta_I\sin(\omega_j)$, the elements of the second column to $\beta_R\cos(\omega_j 2) - \beta_I\sin(\omega_j 2)$, and so on and so forth. Finally it is possible to check that for $\mathbf{V}_y\mathbf{U}_x'$

$$\mathbf{V}_y\mathbf{U}_x' = \frac{1}{2}\left[(\beta_R + i\beta_I)\,\mathbf{1}\mathbf{v}_j^{+\prime} + (\beta_R - i\beta_I)\,\mathbf{1}\mathbf{v}_j^{-\prime}\right]$$

where $\mathbf{1}$ is a $N\times 1$ vector of ones and $\mathbf{v}_j^- = \begin{bmatrix} e^{-i\omega_j} & e^{-i2\omega_j} & e^{-i3\omega_j} & \cdots & e^{-iN\omega_j} \end{bmatrix}'$ $\mathbf{v}_j^+ = \begin{bmatrix} e^{i\omega_j} & e^{i2\omega_j} & e^{i3\omega_j} & \cdots & e^{iN\omega_j} \end{bmatrix}'$. Hence the result in (66) it is easily obtained, following the lines of the proofs of the previous lemmas.■

Lemma 9 *For $Z_\tau^{(\omega_k,\omega_j)} = [y_{1\tau}, y_{2\tau}, \ldots, y_{N\tau}, x_{1\tau}, x_{2\tau}, \ldots, x_{N\tau}]'$ with $x_{n\tau}$ and $y_{n\tau}$ $n = 1, 2, \cdots, N$ satisying (17)-(18) with $x_t^- = e^{-i\omega_j}x_{t-1}^- + \nu_t$, $v_{n\tau} \sim iid\left(0, \sigma^2\right)$ and $\varpi_{n\tau} = z_{R,n\tau} - \cos(\omega_k)/\sin(\omega_k)\,z_{I,n\tau}$ it is possible to write:*

$$\frac{1}{\sqrt{T}}Z_{\lfloor Tr\rfloor}^{(\omega_k,\omega_j)} \Rightarrow \sigma(N/2)^{1/2}\begin{bmatrix} \left\{\frac{(\beta_R+i\beta_I)}{2}\mathbf{v}_k^-\left(w_R^v(r) + iw_I^v(r)\right) + \frac{(\beta_R-i\beta_I)}{2}\mathbf{v}_k^+\left(w_R^v(r) - iw_I^v(r)\right) + \right. \\ \left. +\frac{\cos(\omega_k)}{\sin(\omega_k)}\left[\frac{(\beta_R+i\beta_I)}{-2i}\mathbf{v}_k^-\left(w_R^v(r) + iw_I^v(r)\right) + \frac{(\beta_R-i\beta_I)}{2i}\mathbf{v}_k^+\left(w_R^v(r) - iw_I^v(r)\right)\right]\right\} \\ \left\{\frac{e^{-i\omega_j}}{-2i\sin(\omega_j)}\mathbf{v}_j^-\left(w_R^v(r) + iw_I^v(r)\right) + \frac{e^{i\omega_j}}{2i\sin(\omega_j)}\mathbf{v}_j^+\left(w_R^v(r) - iw_I^v(r)\right)\right\} \end{bmatrix} \qquad (73)$$

with $\left(w_R^v(r) \pm iw_I^v(r)\right)$, $\mathbf{v}_j^-$ and $\mathbf{v}_j^+$ as in Lemma 8, and $\mathbf{v}_k^- = \begin{bmatrix} e^{-i\omega_k} & e^{-i2\omega_k} & e^{-i3\omega_k} & \cdots & e^{-iN\omega_k} \end{bmatrix}'$ and $\mathbf{v}_k^+ = \begin{bmatrix} e^{i\omega_k} & e^{i2\omega_k} & e^{i3\omega_k} & \cdots & e^{iN\omega_k} \end{bmatrix}'$.

Proof of Lemma 9: Define $V_\tau^Z = [\varpi_{1\tau}, \varpi_{2\tau}, \cdots, \varpi_{N\tau}, v_{1\tau}, v_{2\tau}, \cdots, v_{N\tau}]' = [V_\tau^{\varpi\prime}V_\tau']'$ with $V_\tau = [v_{1\tau}, v_{2\tau}, \cdots, v_{N\tau}]'$ and $V_\tau^\varpi = [\varpi_{1\tau}, \varpi_{2\tau}, \cdots, \varpi_{N\tau}]'$. Using the same line of argument as in the proofs of previous lemmas we can write

$$\Phi_0 Z_\tau^{(\omega_k,\omega_j)} = \Phi_1 Z_\tau^{(\omega_k,\omega_j)} + V_\tau^Z \qquad (74)$$

[18] $\cos(\omega)\sin(\varphi) = \frac{\sin(\omega+\varphi)-\sin(\omega-\varphi)}{2}$ and $\sin(\omega)\sin(\varphi) = \frac{\cos(\omega-\varphi)-\cos(\omega+\varphi)}{2}$

[19] $\sin(2\omega) = 2\cos(\omega)\sin(\omega)$ and $\frac{1-\cos(2\omega)}{2\sin(\omega)} = \sin(\omega)$

where Φ_0 and Φ_1 in (74) have the same expressions as in Lemma 8 but in this case β_{1n} and β_{2n} are defined as:

$$\begin{aligned}
\beta_{1n} &= \left[\beta_R\left(\cos\left(\left[\omega_k-\omega_j\right]n\right)+\frac{\cos\left(\omega_k\right)}{\sin\left(\omega_k\right)}\sin\left(\left[\omega_k-\omega_j\right]n\right)\right)\right. \\
&\left.+\beta_I\left(\sin\left(\left[\omega_k-\omega_j\right]n\right)-\frac{\cos\left(\omega_k\right)}{\sin\left(\omega_k\right)}\cos\left(\left[\omega_k-\omega_j\right]n\right)\right)\right] \\
\beta_{2n} &= -\left[\beta_R\left(\cos\left(\omega_k n-\omega_j\left[n+1\right]\right)+\frac{\cos\left(\omega_k\right)}{\sin\left(\omega_k\right)}\sin\left(\omega_k n-\omega_j\left[n+1\right]\right)\right)\right. \\
&\left.+\beta_I\left(\sin\left(\omega_k n-\omega_j\left[n+1\right]\right)-\frac{\cos\left(\omega_k\right)}{\sin\left(\omega_k\right)}\cos\left(\omega_k n-\omega_j\left[n+1\right]\right)\right)\right].
\end{aligned} \tag{75}$$

The expressions for β_{1n} and β_{2n} in (75) are obtained from (21).

Hence Φ_0^{12} and Φ_1^{12} are defined as in Lemma 8 but with β_{1n} and β_{2n} following the expression collected in (75), and Φ_0^{22} and Φ_1^{22} with exactly the same expression as in Lemma 8. Similarly, $(\Phi_0)^{-1}$ and $(\Phi_0)^{-1}\Phi_1$ have equivalent expression as in Lemma 8. Clearly $(\Phi_0)^{-1}\Phi_1$ is idempotent and has the form reported in (68) with $\mathbf{V}_x$ defined in (70) and $\mathbf{V}_y$ as in (69) but with β_{1n} and β_{2n} defined in (75) by recursive substitution from (74) yields

$$Z_\tau^{(\omega_k,\omega_j)} = \Phi_0^{-1}\Phi_1^{-}Z_o^{(\omega_k,\omega_j)} + \Phi_0^{-1}V_\tau^Z + \Phi_0^{-1}\Phi_1\Phi_0^{-1}\sum_{j=1}^{\tau-1}V_{\tau-j}^Z.$$

The matrix $\Phi_0^{-1}\Phi_1\Phi_0^{-1}$ is also equal to

$$\Phi_0^{-1}\Phi_1\Phi_0^{-1} = \begin{bmatrix} 0_{N\times N} & \mathbf{V}_y\mathbf{U}_x' \\ 0_{N\times N} & \mathbf{C}_j \end{bmatrix}$$

with $\mathbf{U}_x$ and $\mathbf{C}_j$ exactly as in Lemma 8 (71) and $\mathbf{V}_y$ is equal to (69) with β_{1n} and β_{2n} defined in (75). Replacing β_{1n} and β_{2n} defined in (75) in (69) and using the expressions $\cos(\omega)\sin(\varphi) = \frac{\sin(\omega+\varphi)-\sin(\omega-\varphi)}{2}$, $\sin(\omega)\sin(\varphi) = \frac{\cos(\omega-\varphi)-\cos(\omega+\varphi)}{2}$, $\sin(2\omega) = 2\cos(\omega)\sin(\omega)$ and $\frac{1-\cos(2\omega)}{2\sin(\omega)} = \sin(\omega)$ the generic element in the first column of $\mathbf{V}_y$ is seen to be

$$\begin{aligned}
&-\beta_R\left[\cos\left(\omega_k h-\omega_j\right)+\frac{\cos\left(\omega_k\right)}{\sin\left(\omega_k\right)}\sin\left(\omega_k h-\omega_j\right)\right]- \\
&-\beta_I\left[\cos\left(\omega_k h-\omega_j\right)-\frac{\cos\left(\omega_k\right)}{\sin\left(\omega_k\right)}\sin\left(\omega_k h-\omega_j\right)\right]
\end{aligned} \tag{76}$$

and the generic element in the second column of $\mathbf{V}_y$ is

$$\begin{aligned}
&\beta_R\left[\cos\left(\omega_k h\right)+\frac{\cos\left(\omega_k\right)}{\sin\left(\omega_k\right)}\sin\left(\omega_k h\right)\right]+ \\
&+\beta_I\left[\cos\left(\omega_k h\right)-\frac{\cos\left(\omega_k\right)}{\sin\left(\omega_k\right)}\sin\left(\omega_k h\right)\right],
\end{aligned} \tag{77}$$

with h in (76) and (77) being the row position in each column, that is $h = 1, \ldots, N$.

It is possible to check (using trigonometric identities) that the generic element of $\mathbf{V}_y\mathbf{U}_x'$ can be expressed as

$$\begin{aligned}
&\beta_R\left[\cos\left(\omega_k h-\omega_j f\right)+\frac{\cos\left(\omega_k\right)}{\sin\left(\omega_k\right)}\sin\left(\omega_k h-\omega_j f\right)\right]+ \\
&+\beta_I\left[\sin\left(\omega_k h-\omega_j f\right)-\frac{\cos\left(\omega_k\right)}{\sin\left(\omega_k\right)}\cos\left(\omega_k h-\omega_j f\right)\right] \\
h &= 1, \ldots, N; j = 1, \ldots, N
\end{aligned} \tag{78}$$

where h refers to the rows position and f to the column position in $\mathbf{V}_y\mathbf{U}_x'$. Finally it is possible to write:

$$\begin{aligned}
\mathbf{V}_y\mathbf{U}_x' &= \left\{\frac{1}{2}\left[\left(\beta_R+i\beta_I\right)\mathbf{v}_k^-\mathbf{v}_j^{+\prime}+\left(\beta_R-i\beta_I\right)\mathbf{v}_k^+\mathbf{v}_j^{-\prime}\right]+\right. \\
&\left.+\frac{\cos\left(\omega_k\right)}{\sin\left(\omega_k\right)}\left[\frac{\left(\beta_R+i\beta_I\right)}{-2i}\mathbf{v}_k^-\mathbf{v}_j^{+\prime}+\frac{\left(\beta_R+i\beta_I\right)}{2i}\mathbf{v}_k^+\mathbf{v}_j^{-\prime}\right]\right\}.
\end{aligned} \tag{79}$$

Using (79) and (71) and following the lines of the proofs of the previous lemmas, the result reported in (73) is easily obtained.

Lemma 10 *For* $Z_\tau^{(\pi,\omega_j)} = [y_{1\tau}, y_{2\tau}, \ldots, y_{N\tau}, x_{1\tau}, x_{2\tau}, \ldots, x_{N\tau}]'$ *with* $x_{n\tau}$ *and* $y_{n\tau}$ $n = 1, 2, \cdots, N$ *defined in (31) and with* $v_{n\tau} \sim iid\left(0, \sigma^2\right)$,$u_{n\tau} \sim iid\left(0, \sigma_u^2\right)$ *and* $z_{Rn\tau} = \cos(\pi n) \operatorname{Re}[u_{n\tau}]$ *it is possible to write:*

$$\frac{1}{\sqrt{T}} Z_{\lfloor Tr \rfloor}^{(\pi,\omega_j)} \Rightarrow \sigma (N/2)^{1/2} \left[\begin{array}{c} \frac{1}{2}\left[(\beta_R + i\beta_I)\, \mathbf{v}_{N/2} \left(w_R^v(r) + iw_I^v(r)\right) + (\beta_R - i\beta_I)\, \mathbf{v}_{N/2} \left(w_R^v(r) - iw_I^v(r)\right) \right] \\ \frac{e^{-i\omega_j}}{-2i\sin(\omega_j)} \mathbf{v}_j^- \left(w_R^v(r) + iw_I^v(r)\right) + \frac{e^{i\omega_j}}{2i\sin(\omega_j)} \mathbf{v}_j^+ \left(w_R^v(r) - iw_I^v(r)\right) \end{array} \right] \tag{80}$$

with $\left(w_R^v(r) + iw_I^v(r)\right)$, $\mathbf{v}_j^-$ *and* $\mathbf{v}_j^+$ *as in Lemma 1,* $\left(w_R^v(r) - iw_I^v(r)\right)$ *the complex conjugate of* $\left(w_R^v(r) + iw_I^v(r)\right)$ *and* $\mathbf{v}_{N/2}$ *the* $N \times 1$ *vector* $\mathbf{v}_{N/2} = \left[-1, 1, -1, \cdots, -1^N\right]$.

Proof of Lemma 10: Defining $V_\tau^Z = [z_{R1\tau}, z_{R2\tau}, \cdots,\ z_{RN\tau}, v_{1\tau}, v_{2\tau}, \cdots, v_{N\tau}'] = [V_\tau^{u\prime} V_\tau']'$ with $V_\tau = [v_{1\tau}, v_{2\tau}, \cdots, v_{N\tau}]'$ and $V_\tau^u = [z_{R1\tau}, z_{R1\tau},\ \cdots, z_{R1\tau}]'$, then using the same line of argument as in the proof of previous lemmas we can write

$$\Phi_0 Z_\tau^{(\pi,\omega_j)} = \Phi_1 Z_\tau^{(\pi,\omega_j)} + V_\tau^Z$$

where

$$\Phi_0 = \left[\begin{array}{cc} I_{N\times N} & \Phi_0^{12} \\ 0_{N\times N} & \Phi_0^{22} \end{array} \right], \quad \Phi_1 = \left[\begin{array}{cc} 0_{N\times N} & \Phi_1^{12} \\ 0_{N\times N} & \Phi_1^{22} \end{array} \right]$$

in which all sub-matrices are $N \times N$. These are exactly as in Lemma 8 above, with the only difference being, rather than β_{1n} and β_{2n} from (31) and (32), we have

$$\begin{aligned} \beta_{1n} &= [\beta_R \cos([\omega_j - \pi]\, n) - \beta_I \sin([\omega_j - \pi]\, n)] \\ &= \cos(\pi n) [\beta_R \cos(\omega_j n) - \beta_I \sin(\omega_j n)] \\ \beta_{2n} &= -[\beta_R \cos(\omega_j [n+1] - \pi n) - \beta_I \sin(\omega_j [n+1] - \pi n)] \\ &= -\cos(\pi n) [\beta_R \cos(\omega_j [n+1]) - \beta_I \sin(\omega_j [n+1])]\,. \end{aligned} \tag{81}$$

To obtain (81), the trigonometric identities relating to the *cos* and *sin* of a difference are used. Hence we could obtain the same results as in Lemma 8 but with the β_{1n} and β_{2n} defined in (81). Therefore, we have the same $\mathbf{V}_y$, $\mathbf{V}_x$ and $\mathbf{U}_x$ as in Lemma 8 but with β_{1n} and β_{2n} as defined in (81). It is possible to to rewrite $\mathbf{V}_y$ using the formulas connecting the products of *cos* and *sin* with sums (see footnote 13) as

$$\mathbf{V}_y = \left[\begin{array}{cc} -\cos(\pi) [\beta_R \cos(\omega_j) - \beta_I \sin(\omega_j)] & \cos(\pi) \beta_R \\ -\cos(2\pi) [\beta_R \cos(\omega_j) - \beta_I \sin(\omega_j)] & \cos(2\pi) \beta_R \\ -\cos(3\pi) [\beta_R \cos(\omega_j) - \beta_I \sin(\omega_j)] & \cos(3\pi) \beta_R \\ -\cos(4\pi) [\beta_R \cos(\omega_j) - \beta_I \sin(\omega_j)] & \cos(4\pi) \beta_R \\ \vdots & \vdots \\ -\cos(N\pi) [\beta_R \cos(\omega_j) - \beta_I \sin(\omega_j)] & \cos(N\pi) \beta_R \end{array} \right]. \tag{82}$$

Hence the result in (80) can be easily obtained.■

Lemma 11 *For* $Z_\tau^{(0,\pi)} = [y_{1\tau}, y_{2\tau}, \ldots, y_{N\tau}, x_{1\tau}, x_{2\tau}, \ldots, x_{N\tau}]'$ *with* $x_{n\tau} = -x_{n-1,\tau} + \nu_{n\tau}$ *and* $y_{n\tau}$ *defined in (31),* $n = 1, 2, \cdots, N$ *, and with* $\nu_{n\tau} \sim iid\left(0, \sigma^2\right)$*, and* $u_t \sim I_0(0)$ *it is possible to write:*

$$\frac{1}{\sqrt{T}} Z_{\lfloor Tr \rfloor}^{(0,\pi)} \Rightarrow \sigma (N/2)^{1/2} \left[\begin{array}{c} \beta_R \mathbf{1} w^v(r) \\ \mathbf{v}_{N/2} w^v(r) \end{array} \right] \tag{83}$$

with $w^\nu(r) = (N/2)^{-1/2} \sum_{k=1}^N (-1)^k W_k^v(r)$*. being a scalar Brownian motion defined as in Lemma 10 and* **1** *as in Lemma 8.*

Proof of Lemma 11: Defining $V_\tau^Z = [u_{1\tau}, u_{2\tau}, \cdots,\ u_{N\tau}, \nu_{1\tau}, \nu_{2\tau}, \cdots, \nu_{N\tau}] = [V_\tau^{u\prime} V_\tau']'$ with $V_\tau = [\nu_{1\tau}, \nu_{2\tau}, \cdots, \nu_{N\tau}]'$ and $V_\tau^u = [u_{1\tau}, u_{2\tau},\ \cdots, u_{N\tau}]'$, and then using the same line of argument as in the proof of previous lemmas we can write

$$\Phi_0 Z_\tau^{(0,\pi)} = \Phi_1 Z_\tau^{(0,\pi)} + V_\tau^Z \tag{84}$$

where

$$\Phi_0 = \left[\begin{array}{cc} I_{N\times N} & \Phi_0^{12} \\ 0_{N\times N} & \Phi_0^{22} \end{array} \right], \quad \Phi_1 = \left[\begin{array}{cc} 0_{N\times N} & 0_{N\times N} \\ 0_{N\times N} & \Phi_1^{22} \end{array} \right]$$

in which all sub-matrix are $N \times N$, with

$$\begin{aligned} \Phi_0^{12} &= -\beta \times diag\left[\cos(\pi), \cos(2\pi), \ldots, \cos(N\pi)\right] \\ &= -\beta \times diag\left[-1, 1, \ldots, -1^N\right] \end{aligned}$$

and the remaing submatrices are

$$\Phi_0^{22} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$$

$$\Phi_1^{22} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Using results form the inverse of a partitioned matrix, we have:

$$(\Phi_0)^{-1} = \begin{bmatrix} I_{N\times N} & -\Phi_0^{12}\left(\Phi_0^{22}\right)^{-1} \\ 0_{N\times N} & \left(\Phi_0^{22}\right)^{-1} \end{bmatrix}.$$

The matrix Φ_0^{22} is a lower-triangular Toeplitz matrix associated with the polynomial $(1-L)$, hence $\left(\Phi_0^{22}\right)^{-1}$ collects the coefficients of the expansion of the inverse of $(1-L)$, that is:

$$\left(\Phi_0^{22}\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 1 & 0 & \cdots & 0 \\ -1 & 1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1^{(N-1)} & -1^{(N-2)} & -1^{(N-3)} & -1^{(N-4)} & \cdots & 1 \end{bmatrix}.$$

In the case of the case of $-\Phi_0^{12}\left(\Phi_0^{22}\right)^{-1}$ it is easy to check that:

$$\begin{aligned} -\Phi_0^{12}\left(\Phi_0^{22}\right)^{-1} &= \\ & \begin{bmatrix} -\beta & 0 & 0 & 0 & \cdots & 0 \\ -\beta & \beta & 0 & 0 & \cdots & 0 \\ -\beta & \beta & -\beta & 0 & \cdots & 0 \\ -\beta & \beta & -\beta & \beta & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\beta & \beta & -\beta & \beta & \cdots & (-1)^N \beta \end{bmatrix}. \end{aligned}$$

The matrix $(\Phi_0)^{-1}\Phi_1$ is a $2N \times 2N$ matrix, but as Φ_1 has only one non-zero element, it can be verified that

$$(\Phi_0)^{-1}\Phi_1 = \begin{bmatrix} 0_{2N\times 2(N-1)} & \beta \times \mathbf{1} \\ 0_{2N\times 2(N-1)} & \mathbf{v}_{N/2} \end{bmatrix}.$$

It is easy to check that $\Phi_0^{-1}\Phi_1$ is idempotent[20]. Recursive substitution from (84) yields:

$$Z_\tau^{(0,\pi)} = \Phi_0^{-1}\Phi_1 Z_o^{(0,\pi)} + \Phi_0^{-1}V_\tau^Z + \Phi_0^{-1}\Phi_1\Phi_0^{-1}\sum_{j=1}^{\tau-1} V_{\tau-j}^Z$$

and matrix $\Phi_0^{-1}\Phi_1\Phi_0^{-1}$ displays the impact of the accumulation shocks of $\sum_{j=1}^{\tau-1} V_{\tau-j}^Z$ and has the form:

$$\Phi_0^{-1}\Phi_1\Phi_0^{-1} = \begin{bmatrix} 0_{N\times N} & \beta \times \mathbf{1}\mathbf{v}_{N/2}' \\ 0_{N\times N} & \mathbf{v}_{N/2}\mathbf{v}_{N/2}' \end{bmatrix}.$$

Finally note that $w^\nu(r) = (N/2)^{-1/2}\mathbf{v}_{N/2}' W^\nu(r) = (N/2)^{-1/2}\sum_{k=1}^N \cos(k\pi) W_k^\nu(r) = (N/2)^{-1/2}\sum_{k=1}^N (-1)^k W_k^\nu(r)$.■

[20] The last element of $\mathbf{v}_{N/2}$ is equal to one as frequency π only it is relevant for even number of seasons N.

Table 1: Test Results for Cointegrated Processes, Case I: $y_t \sim I_{\pi/3}(1)$, $x_t \sim I_{2\pi/3}(1)$

Panel A:Johansen Test Vector of Seasons Approach

DGP	r_0= 0	r_0= 1	r_0= 2	r_0= 3	r_0= 4	r_0= 5	r_0= 6	r_0= 7	r_0= 8	r_0= 9	r_0= 10	r_0= 11
$T = 1200$												
1	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0542	0,0036
2	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0576	0,0042
3	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0514	0,0042
4	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0536	0,0044
5	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0574	0,0068
$T = 600$												
1	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9960	0,7548	0,0510	0,0062
2	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9964	0,7416	0,0520	0,0042
3	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9968	0,7522	0,0486	0,0040
4	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9958	0,7494	0,0514	0,0046
5	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9970	0,7612	0,0504	0,0036
$T = 300$												
1	1,0000	1,0000	0,9948	0,9506	0,8020	0,5506	0,3126	0,2246	0,0914	0,0290	0,0084	0,0026
2	1,0000	0,9998	0,9934	0,9408	0,7856	0,5324	0,2946	0,2186	0,0828	0,0242	0,0078	0,0024
3	1,0000	0,9996	0,9950	0,9508	0,8052	0,5548	0,3094	0,2234	0,1002	0,0318	0,0080	0,0026
4	1,0000	0,9998	0,9928	0,9484	0,7960	0,5452	0,3048	0,2218	0,0924	0,0268	0,0074	0,0018
5	1,0000	0,9998	0,9932	0,9496	0,8152	0,5742	0,3164	0,2290	0,0942	0,0294	0,0076	0,0026

Panel B:Johansen Test Real and Imaginary Parts Approach

	$T = 1200$				$T = 600$				$T = 300$			
DGP	r_0= 0	r_0= 1	r_0= 2	r_0= 3	r_0= 0	r_0= 1	r_0= 2	r_0= 3	r_0= 0	r_0= 1	r_0= 2	r_0= 3
1	1,0000	1,0000	0,0498	0,0032	1,0000	1,0000	0,0510	0,0066	1,0000	1,0000	0,0540	0,0048
2	1,0000	1,0000	0,0572	0,0044	1,0000	1,0000	0,0584	0,0034	1,0000	1,0000	0,0622	0,0044
3	1,0000	1,0000	0,0512	0,0040	1,0000	1,0000	0,0594	0,0038	1,0000	1,0000	0,0586	0,0064
4	1,0000	1,0000	0,0470	0,0038	1,0000	1,0000	0,0608	0,0038	1,0000	1,0000	0,0542	0,0060
5	1,0000	1,0000	0,0518	0,0044	1,0000	1,0000	0,0528	0,0026	1,0000	1,0000	0,0550	0,0044

Panel C: Cubadda Complex-Valued Test

	$T = 1200$		$T = 600$		$T = 300$	
DGP	r_0= 0	r_0= 1	r_0= 0	r_0= 1	r_0= 0	r_0= 1
1	1,0000	0,0558	1,0000	0,0532	1,0000	0,0552
2	1,0000	0,0544	1,0000	0,0538	1,0000	0,0560
3	1,0000	0,0480	1,0000	0,0552	1,0000	0,0558
4	1,0000	0,0490	1,0000	0,0552	1,0000	0,0574
5	1,0000	0,0506	1,0000	0,0480	1,0000	0,0490

Notes: The DGPs are defined in subsection 4.1 while the tests are described in subsection 3.3; r_0 is the number of cointegrating vectors under the null hypothesis. All tests are conducted at a nominal 5% level of significance; for further details see subsection 4.1. The true number of cointegrating vectors is 10 in Panel A, 2 in Panel B and 1 in Panel C.

Table 2: Test Results for Cointegrated Processes, Case II: $y_t \sim I_0(1)$, $x_t \sim I_{2\pi/3}(1)$

Panel A: Johansen Test Vector of Seasons Approach

DGP	r_0= 0	r_0= 1	r_0= 2	r_0= 3	r_0= 4	r_0= 5	r_0= 6	r_0= 7	r_0= 8	r_0= 9	r_0= 10	r_0= 11
$T = 1200$												
1	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0520	0,0056
2	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0584	0,0044
3	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0500	0,0052
4	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0588	0,0048
5	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0554	0,0040
$T = 600$												
1	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9960	0,7468	0,0540	0,0058
2	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9968	0,7468	0,0536	0,0036
3	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9964	0,7466	0,0490	0,0046
4	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9958	0,7458	0,0468	0,0040
5	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9952	0,7410	0,0496	0,0050
$T = 300$												
1	1,0000	1,0000	0,9952	0,9468	0,7932	0,5468	0,3026	0,2246	0,0928	0,0282	0,0052	0,0022
2	1,0000	1,0000	0,9944	0,9490	0,8082	0,5638	0,3162	0,2308	0,1016	0,0314	0,0088	0,0022
3	1,0000	1,0000	0,9926	0,9390	0,7880	0,5560	0,3072	0,2212	0,0950	0,0254	0,0068	0,0008
4	1,0000	1,0000	0,9924	0,9476	0,7972	0,5606	0,3102	0,2336	0,0978	0,0270	0,0062	0,0018
5	1,0000	1,0000	0,9926	0,9478	0,7998	0,5488	0,3018	0,2204	0,0968	0,0274	0,0068	0,0014

Panel B: Johansen Test Real and Imaginary Parts Approach

	$T = 1200$			$T = 600$			$T = 300$		
DGP	r_0= 0	r_0= 1	r_0= 2	r_0= 0	r_0= 1	r_0= 2	r_0= 0	r_0= 1	r_0= 2
1	1,0000	0,0500	0,0030	1,0000	0,0602	0,0056	1,0000	0,0616	0,0044
2	1,0000	0,0536	0,0038	1,0000	0,0554	0,0034	1,0000	0,0578	0,0054
3	1,0000	0,0500	0,0052	1,0000	0,0550	0,0030	1,0000	0,0624	0,0060
4	1,0000	0,0526	0,0066	1,0000	0,0566	0,0038	1,0000	0,0538	0,0026
5	1,0000	0,0530	0,0042	1,0000	0,0530	0,0048	1,0000	0,0528	0,0046

Panel C: Cubadda Complex-Valued Test

	$T = 1200$		$T = 600$		$T = 300$	
DGP	r_0= 0	r_0= 1	r_0= 0	r_0= 1	r_0= 0	r_0= 1
1	0,2840	0,0234	0,2942	0,0308	0,2786	0,0264
2	0,2992	0,0246	0,2978	0,0256	0,2802	0,0276
3	0,2956	0,0244	0,2820	0,0248	0,2866	0,0242
4	0,3004	0,0290	0,2840	0,0250	0,2836	0,0240
5	0,2948	0,0234	0,2956	0,0216	0,2874	0,0240

Notes: As for Table 1, except that the true number of cointegrating vectors is 1 in Panel B.

Table 3: Test Results for Cointegrated Processes, Case III: $y_t \sim I_{\pi}(1)$, $x_t \sim I_{2\pi/3}(1)$

Panel A: Johansen Test Vector of Seasons Approach

DGP	r_0= 0	r_0= 1	r_0= 2	r_0= 3	r_0= 4	r_0= 5	r_0= 6	r_0= 7	r_0= 8	r_0= 9	r_0= 10	r_0= 11
$T = 1200$												
1	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0612	0,0042
2	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0576	0,0048
3	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0624	0,0048
4	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0574	0,0046
5	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0480	0,0048
$T = 600$												
1	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9998	0,9970	0,7430	0,0508	0,0042
2	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9966	0,7610	0,0498	0,0040
3	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9972	0,7478	0,0492	0,0042
4	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9974	0,7550	0,0494	0,0048
5	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9966	0,7562	0,0434	0,0038
$T = 300$												
1	1,0000	1,0000	0,9932	0,9488	0,8000	0,5438	0,2970	0,2186	0,0950	0,0264	0,0082	0,0024
2	1,0000	0,9998	0,9948	0,9552	0,8118	0,5676	0,3184	0,2284	0,0936	0,0270	0,0068	0,0014
3	1,0000	0,9996	0,9942	0,9438	0,7886	0,5418	0,2950	0,2156	0,0934	0,0282	0,0090	0,0020
4	1,0000	0,9998	0,9968	0,9520	0,8088	0,5622	0,3044	0,2188	0,0896	0,0258	0,0070	0,0016
5	1,0000	1,0000	0,9942	0,9428	0,7952	0,5448	0,2948	0,2150	0,0924	0,0304	0,0074	0,0024

Panel B: Johansen Test Real and Imaginary Parts Approach

	$T = 1200$			$T = 600$			$T = 300$		
DGP	r_0= 0	r_0= 1	r_0= 2	r_0= 0	r_0= 1	r_0= 2	r_0= 0	r_0= 1	r_0= 2
1	1,0000	0,0600	0,0054	1,0000	0,0534	0,0040	1,0000	0,0610	0,0050
2	1,0000	0,0552	0,0048	1,0000	0,0554	0,0046	1,0000	0,0526	0,0038
3	1,0000	0,0640	0,0040	1,0000	0,0538	0,0044	1,0000	0,0622	0,0060
4	1,0000	0,0534	0,0048	1,0000	0,0580	0,0038	1,0000	0,0568	0,0048
5	1,0000	0,0494	0,0048	1,0000	0,0518	0,0028	1,0000	0,0574	0,0036

Panel C: Cubadda Complex-Valued Test

	$T = 1200$		$T = 600$		$T = 300$	
DGP	r_0= 0	r_0= 1	r_0= 0	r_0= 1	r_0= 0	r_0= 1
1	1,0000	0,0510	0,9996	0,0492	0,9716	0,0496
2	1,0000	0,0476	0,9902	0,0426	0,9242	0,0430
3	0,9858	0,0454	0,9144	0,0416	0,8002	0,0442
4	0,9526	0,0422	0,8498	0,0390	0,6854	0,0348
5	0,9298	0,0398	0,8228	0,0354	0,6830	0,0320

Notes: As for Table 1, except that the true number of cointegrating vectors is 1 in Panel B.

Table 4: Test Results for Cointegrated Processes, Case IV: $y_t \sim I_0(1)$, $x_t \sim I_\pi(1)$

Panel A: Johansen Test Vector of Seasons Approach												
$T = 1200$												
DGP	r_0= 0	r_0= 1	r_0= 2	r_0= 3	r_0= 4	r_0= 5	r_0= 6	r_0= 7	r_0= 8	r_0= 9	r_0= 10	r_0= 11
1	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0560
2	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0528
3	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0510
4	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0546
5	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0496
$T = 600$												
1	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9998	0,9602	0,0492
2	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9608	0,0474
3	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9634	0,0494
4	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9570	0,0508
5	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9618	0,0514
$T = 300$												
1	1,0000	1,0000	0,9948	0,9568	0,8346	0,6250	0,4104	0,3426	0,1948	0,0846	0,0328	0,0092
2	1,0000	1,0000	0,9944	0,9512	0,8274	0,6246	0,4064	0,3552	0,2070	0,0944	0,0318	0,0104
3	1,0000	1,0000	0,9950	0,9522	0,8264	0,6198	0,3884	0,3402	0,1914	0,0888	0,0300	0,0086
4	1,0000	1,0000	0,9946	0,9532	0,8276	0,6200	0,4110	0,3552	0,2012	0,0928	0,0320	0,0118
5	1,0000	0,9996	0,9958	0,9532	0,8314	0,6254	0,4132	0,3624	0,2118	0,0918	0,0316	0,0096

Panel B: Johansen Test, y_t & $\cos(\pi t)x_t$						
	$T = 1200$		$T = 600$		$T = 300$	
DGP	r_0= 0	r_0= 1	r_0= 0	r_0= 1	r_0= 0	r_0= 1
1	1,0000	0,0518	1,0000	0,0514	1,0000	0,0518
2	1,0000	0,0522	1,0000	0,0562	1,0000	0,0428
3	1,0000	0,0522	1,0000	0,0546	1,0000	0,0546
4	1,0000	0,0556	1,0000	0,0506	1,0000	0,0496
5	1,0000	0,0478	1,0000	0,0544	1,0000	0,0520

Panel C: Cubadda Complex-Valued Test						
	$T = 1200$		$T = 600$		$T = 300$	
DGP	r_0= 0	r_0= 1	r_0= 0	r_0= 1	r_0= 0	r_0= 1
1	1,0000	0,1040	1,0000	0,1012	1,0000	0,0998
2	1,0000	0,0972	1,0000	0,1070	1,0000	0,0942
3	1,0000	0,1010	1,0000	0,1004	1,0000	0,1006
4	1,0000	0,1032	1,0000	0,1036	1,0000	0,1004
5	1,0000	0,0888	1,0000	0,1008	1,0000	0,0982

Notes: As for Table 1, except that the true number of cointegrating vectors is 11 in Panel A and 1 in Panel B.

Table 5: Test Results for Processes with No Cointegration

	Panel A. Johansen Test Vector of Seasons Approach											
T	$r_0=0$	$r_0=1$	$r_0=2$	$r_0=3$	$r_0=4$	$r_0=5$	$r_0=6$	$r_0=7$	$r_0=8$	$r_0=9$	$r_0=10$	$r_0=11$
Case I: $x_t \sim I_{2\pi/3}(1), y_t \sim I_{\pi/3}(1)$												
1200	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9998	0,0912	0,0076	0,0012	0,0000
600	1,0000	1,0000	1,0000	1,0000	1,0000	0,9982	0,9290	0,5898	0,0940	0,0118	0,0006	0,0004
300	1,0000	0,9998	0,9968	0,9626	0,8246	0,5328	0,2512	0,1486	0,0518	0,0128	0,0032	0,0012
Case II: $x_t \sim I_{2\pi/3}(1), y_t \sim I_0(1)$												
1200	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0650	0,0044	0,0006
600	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9984	0,9822	0,6208	0,0582	0,0062	0,0004
300	1,0000	1,0000	0,9966	0,9586	0,8206	0,5646	0,2888	0,1866	0,0636	0,0152	0,0040	0,0018
Case III: $x_t \sim I_{2\pi/3}(1), y_t \sim I_\pi(1)$												
1200	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0654	0,0058	0,0008
600	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9994	0,9870	0,6510	0,0656	0,0056	0,0010
300	1,0000	1,0000	0,9966	0,9540	0,8096	0,5478	0,2862	0,1870	0,0664	0,0184	0,0030	0,0010
Case IV: $x_t \sim I_\pi(1), y_t \sim I_0(1)$												
1200	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,0580	0,0054
600	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	1,0000	0,9976	0,7998	0,0468	0,0048
300	1,0000	1,0000	0,9960	0,9568	0,8416	0,6224	0,3650	0,2730	0,1196	0,0362	0,0098	0,0022

	Panel B: Johansen Test Real and Imaginary Parts Approach											
	Case I				Case II			Case III			Case IV	
T	$r_0=0$	$r_0=1$	$r_0=2$	$r_0=3$	$r_0=0$	$r_0=1$	$r_0=2$	$r_0=0$	$r_0=1$	$r_0=2$	$r_0=0$	$r_0=1$
1200	0,0670	0,0056	0,0004	0,0002	0,0600	0,0030	0,0008	0,0594	0,0056	0,0000	0,0528	0,0050
600	0,0786	0,0056	0,0004	0,0000	0,0644	0,0028	0,0006	0,0712	0,0038	0,0008	0,0570	0,0038
300	0,0780	0,0054	0,0010	0,0004	0,0644	0,0030	0,0006	0,0682	0,0060	0,0000	0,0518	0,0040

	Panel C: Cubadda Complex-Valued Test							
	Case I		Case II		Case III		Case IV	
T	$r_0=0$	$r_0=1$	$r_0=0$	$r_0=1$	$r_0=0$	$r_0=1$	$r_0=0$	$r_0=1$
1200	0,0512	0,0032	0,0724	0,0048	0,0698	0,0050	0,1134	0,0154
600	0,0546	0,0054	0,0726	0,0062	0,0774	0,0042	0,1172	0,0160
300	0,0564	0,0046	0,0766	0,0068	0,0714	0,0070	0,1170	0,0162

Notes: As for Table 1, except that the DGPs are described in subsection 4.2 and the processes are not cointegrated.

Figure 1. Quarterly Tourist Arrivals in Balearic Islands

Figure 1.a

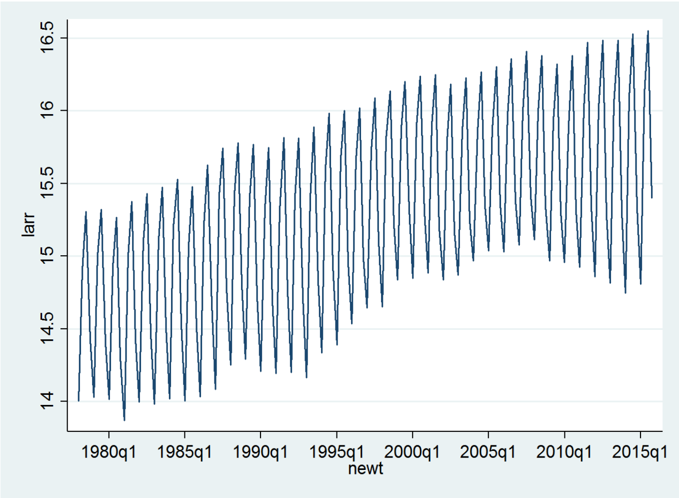
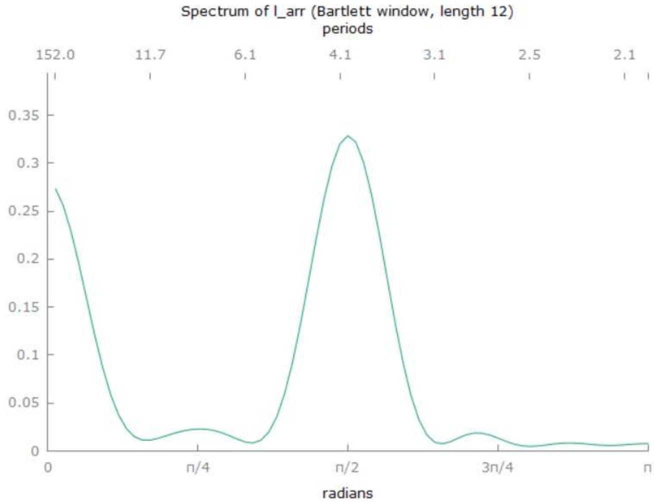


Figure 1.b



Notes: Data are quarterly, 1979Q1-2015Q4. Figure 1.a shows the values after taking natural logarithms, while Figure 1.b shows the sample spectrum of the log values obained using the Bartlett window.

Figure 2. Quarterly Employment in Balearic Islands

Figure 2.a

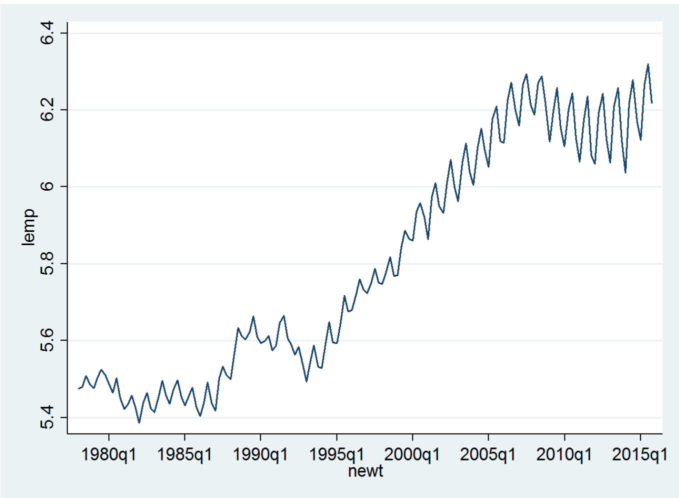
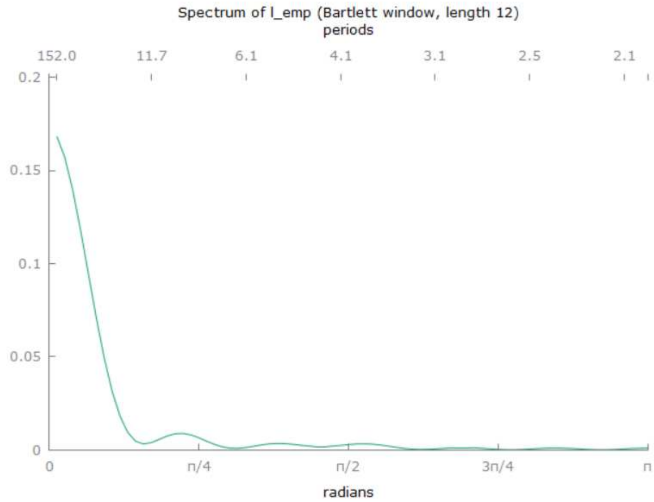


Figure 2.b



Notes: As for Figure 1.

Table 6: HEGY Test Results for Balearic Islands Series

	$\ln(arr_t)$		$\ln(emp_t)$	
	GLS detrending	OLS detrending	GLS detrending	OLS detrending
t_{π_0}	-1.776	-1.620	-2.053	-2.511
t_{π_2}	-1.975	-2.183	-4.769 ***	-4.858 ***
$F_{\pi_1^\alpha \pi_1^\beta}$	1.958	2.170	17.577 ***	18.236 ***

Notes: The HEGY test regression is (40), for which t_{π_0} and t_{π_2} are t-type statistics for one-sided unit root tests at the zero and Nyquist frequencies, respectively, while $F_{\pi_1^\alpha \pi_1^\beta}$ is a two-sided F-type test for a pair of complex unit roots at the seasonal frequency $\pi/2$; the order of augmentation p is 4 for $\ln(arr_t)$ and 1 for $\ln(emp_t)$. In principle, *, ** and *** indicate significance at the 10%, 5% and 1% levels, respectively. For further details see Section 5.

Table 7: Tests for Cointegration across Frequencies between Balearic Islands Series

	Panel A. $(1-L)\ln(arr_t)$			Panel B. $(1-L)(1+L)\ln(arr_t)$		
VAR order	$r_0 = 0$	$r_0 = 1$	$r_0 = 2$	$r_0 = 0$	$r_0 = 1$	$r_0 = 2$
2	58,2273***	6,2268	1,1521	39,8339***	2,8350	0,8059
3	42,6551***	2,4235	0,4932	31,0149	3,1387	0,8589
4	32,7465**	3,5900	1,0814	35,7733**	4,8300	1,3913
5	31,8937**	5,1240	0,3816	19,4901	4,7642	0,8576

Notes: The Johansen trace test is applied to a VAR consisting of $\ln(emp_t)$ and the real and imaginary parts of either $e^{i\frac{\pi}{2}t}\left(1-e^{i\frac{\pi}{2}}L\right)(1-L)\ln(arr_t)$ (Panel A) or $e^{i\frac{\pi}{2}t}\left(1-e^{i\frac{\pi}{2}}L\right)(1-L)(1+L)\ln(arr_t)$ (Panel B); r_0 is the number of cointegrating vectors under the null hypothesis. Significance is indicated as in Table 6. For further details see Section 5.