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Abstract.

To understand the impact of temporal aggregation on the properties of a seasonal long-memory process, the effects of skip and cumulation sampling on both stationary and nonstationary processes with poles at several potential frequencies are analyzed. By allowing for several poles in the disaggregated process, their interaction in the aggregated series is investigated. Further, by defining the process according to the truncated Type II definition, the proposed approach encompasses both stationary and nonstationary processes without requiring prior knowledge of the case. The frequencies in the aggregated series to which the poles in the disaggregated series are mapped can be directly deduced. Specifically, unlike cumulation sampling, skip sampling can impact on non-seasonal memory properties. Moreover, with cumulation sampling, seasonal long-memory can vanish in some cases. Using simulations, the mapping of the frequencies implied by temporal aggregation is illustrated and the estimation of the memory at the different frequencies is analyzed.

JEL Classification: C12, C22.

Keywords: Aggregation, cumulation sampling, skip sampling, seasonal long memory.

1 Introduction

Macroeconomic and financial time series often exhibit high correlations. Both stationary and nonstationary long-memory models have proven to be successful in modeling these correlations. Further, many empirical time series show seasonal or cyclical features. The use of temporally aggregated data is common practice in empirical applications, including macroeconomic and financial ones. Aggregated data are obtained by means of skip sampling or cumulation sampling. While the results apply to both seasonal and cyclical series, most of the present paper focuses

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on seasonal series. For such series, we define $Q \equiv S/S_A$, where S denotes the number of seasons per year and S_A denotes the number of seasons per year after temporal aggregation. While skip sampling is typically used for stock variables, such as the interest rate, cumulation sampling is used for flow variables, such as GDP. (As Hassler (2013) shows for flow variables, the proper way to aggregate data is cumulation sampling. In fact, many papers, including Pons (2006) and del Barrio Castro, Rodrigues, and Taylor (2019), refer to cumulation sampling when they speak about average sampling.) Skip sampling takes each Q-th observation and cumulation sampling aggregates Q observations into a new aggregated observation. Both types of temporal aggregation can have important implications on the properties of the resulting time series.

In the time series analysis literature, the study of the effects of aggregation has largely focused on how aggregation methods modify the data generating process of the time series being considered (basically within the context of mixed autoregressive-moving average processes), see for example, Wei (2006) and Silvestrini and Veredas (2008) for detailed reviews. In a recent paper, del Barrio Castro, Rodrigues, and Taylor (2019) studied the effects of aggregation in seasonal near-integrated processes, in terms of their frequency allocation, exploiting the properties of the demodulator operator and using a partial fraction decomposition.

This paper aims to model a $(\lfloor S/2 \rfloor + 1)$-factor seasonal long-memory process with both stationary and nonstationary memory, and to analyze the impact of temporal aggregation on the properties of the series. $\lfloor . \rfloor$ denotes the integer part. Hence, when S is even, $\lfloor S/2 \rfloor = S/2$, and when S is odd, $\lfloor S/2 \rfloor = (S-1)/2$.

The contribution of this paper is threefold. First, the process allows for multiple poles, and secondly, it covers both stationary and nonstationary long memory. In particular, using an approximation by Giraitis and Leipus (1995), we extend the analysis for stationary one-factor seasonal long-memory processes by Tsai and Chan (2005), Sun and Shi (2014), and Hassler (2011) to the case of multiple poles of stationary and/or nonstationary memory. Thirdly, as Hassler (2011) does, we reach similar conclusions about the effect of aggregation on the poles of the aggregated series, however, we use a different approach and unlike him, work in the time domain. In particular, we obtain immediate knowledge of the location of the poles after aggregation without the need to calculate the spectra. Simulations illustrate the mapping of the frequencies implied by temporal aggregation. Finally, the recently proposed semi-parametric seasonal exact local Whittle estimator (Arteche, 2020) works satisfactorily, and estimated memory parameters are in line with theoretical predictions.

2 Seasonal long-memory processes

2.1 The complex-valued long-memory process

The generalized long-memory (LM) process, proposed by Hosking (1981) and Gray et al. (1989) is defined as:

$$\left(1 - 2\cos\omega L + L^2\right)^d y_t = \varepsilon_t, \tag{1}$$

where ε_t is the innovation and ω is the frequency where the generalized long-memory process or Gegenbauer process has long-memory. Hence, if $\omega \in (0, \pi)$, the process (1) has seasonal/cyclical long-memory behavior at frequency ω with a period $2\pi/\omega$. The special frequencies $\omega = \omega_k = 2\pi k/S$, with $k \in \{1, 2, ..., \lfloor (S-1)/2 \rfloor\}$ and S being the number of seasons per year, lead to seasonal behavior. Yet, the results hold for more general frequencies, and that leads to other types of cyclical behavior. For simplicity, throughout the paper we are going to assume that the innovation ε_t is identically and independently distributed with an expected value of zero and constant variance, that is $\varepsilon_t \sim iid\left(0, \sigma_\varepsilon^2\right)$. But the main conclusions of the paper should also be valid for more general assumptions for the innovation ε_t, such as infinite moving average processes with absolute summable weights (see Assumption 6.1 in Hassler (2019)).

Noting that $\left(1 - 2\cos\omega L + L^2\right)^d = \left(1 - e^{-i\omega}L\right)^d \left(1 - e^{i\omega}L\right)^d$, as in Giraitis and Leipus (1995), we illustrate the role played by the demodulator operator in our approach by focusing on the complex-valued long-memory process,

$$\left(1 - e^{-i\omega}L\right)^d y_t = \varepsilon_t. \tag{2}$$

As in the standard long-memory process $(1-L)^d y_t = \varepsilon_t$, it is possible to write the following moving average representation:

$$\begin{aligned} y_t &= \varepsilon_t + e^{-i\omega} d\varepsilon_{t-1} + e^{-i2\omega}\frac{d(d+1)}{2!}\varepsilon_{t-2} + e^{-i3\omega}\frac{d(d+1)(d+2)}{3!}\varepsilon_{t-3} \\ &= \sum_{j=0}^{\infty} e^{-i\omega j}\psi_j \varepsilon_{t-j}, \end{aligned}$$

with $\psi_j = \frac{(j+d-1)!}{j!(d-1)!} = \frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)}$. These weights correspond to the weights of a process which is fractionally integrated of order d, $\sum_{j=0}^{\infty}\psi_j\varepsilon_{t-j}$. It is well known that the coefficient ψ_j, as $j \to \infty$, $\psi_j \sim j^{d-1}/\Gamma(d)$ (see, for example, Beran (1994)). If we truncate the process to the initial observation of ε_t (that is ε_1), then

$$y_t = \sum_{j=0}^{t-1} e^{-i\omega j}\psi_j \varepsilon_{t-j},$$

which can be written as

$$y_t = e^{-i\omega t}\sum_{j=1}^{t} \psi_{t-j} e^{i\omega j}\varepsilon_j. \tag{3}$$

Hence, in (3) we have a situation equivalent to the one in Gregoir (1999, 2006 and 2010) and del Barrio Castro, Rodrigues, and Taylor (2019), where a complex-valued I(1) process associated with frequency ω, $y_t = e^{-i\omega}y_{t-1} + \varepsilon_t$ can be decomposed into two parts, a complex-valued I(1) process in the zero frequency $y_0 + \sum_{j=1}^{t} e^{i\omega j}\varepsilon_j$ and the demodulator operator $e^{-i\omega t}$ that shifts the former process from the zero frequency to the frequency ω. Process (3) is a complex-valued Type-II fractionally integrated process in the zero frequency $\sum_{j=1}^{t} \psi_{t-j}e^{i\omega j}\varepsilon_j$ with innovation $e^{i\omega j}\varepsilon_j$ together with the demodulator operator, $e^{-i\omega t}$, which shifts the complex-valued zero frequency long-memory process to frequency ω.

In the rest of this section, we present the vector of seasons representation of process (3), which we will use later, together with the result of Giraitis and Leipus (1995), to derive our main result regarding the effect of aggregation on seasonal long-memory processes. When we work with seasonal data, it is convenient to replace the subscript "t" with "$S\tau + s$" where S is the number of total seasons per year, τ is the year ($\tau = 1, 2, \ldots, N$; N is the total number of years) and $s = -(S-1), -(S-2), \ldots, -1, 0$. For simplicity, we assume a total number of observation of SN. In the case of cyclical behavior, $S = 2\pi/\omega$. That is, the cyclical behavior completes a full cycle every S observations.

Hence (with $\mathbf{1}_{(S\tau+s>0)}(S\tau + s)$ being the usual indicator function used for Type II processes),

$$\begin{aligned}
\left(1 - e^{-i\omega}L\right)^d y_{S\tau+s} &= \mathbf{1}_{(S\tau+s>0)}(S\tau + s)\,\varepsilon_{S\tau+s}, \\
y_{S\tau+s} &= e^{-i\omega(S\tau+s)}\sum_{j=1}^{S\tau+s} \psi_{S\tau+s-j}e^{i\omega j}\varepsilon_j. \qquad (4)
\end{aligned}$$

The vector of seasons representation of (4) with $Y_\tau = \left[y_{S\tau-(S-1)}, y_{S\tau-(S-2)}, \cdots, y_{S\tau-1}, y_{S\tau}\right]'$; $E_\tau = \left[\varepsilon_{S\tau-(S-1)}, \varepsilon_{S\tau-(S-2)}, \cdots, \varepsilon_{S\tau-1}, \varepsilon_{S\tau}\right]'$; and defining $(v^-)' = \left[\begin{array}{cccc} e^{-i\omega} & e^{-i\omega 2} & \cdots & e^{-i\omega S}\end{array}\right]$, its complex conjugate $(v^+)' = \left[\begin{array}{cccc} e^{+i\omega} & e^{+i\omega 2} & \cdots & e^{+i\omega S}\end{array}\right]$, and the diagonal matrix $\Psi_j = diag$ $\left(\psi_{S(\tau-j)+(S-1)},\ \psi_{S(\tau-j)+(S-2)}, \ldots, \psi_{S(\tau-j)}\right)$, will be as follows:

$$\begin{aligned}
Y_\tau &= \left(v^-\right)\left(v^+\right)'\sum_{j=1}^{\tau}\Psi_j E_j \qquad (5) \\
&\quad + terms\ fractionally\ integrated\ of\ order\ (d-1).
\end{aligned}$$

The proof of (5) can be found in the appendix. In the appendix we also show that the additional terms of fractional order $d-1$ are affected by the demodulator operator $e^{-i\omega(S\tau+s)}$ (see expression (25) in the appendix), which equals 1 for $\omega = 0$ and $(-1)^{S\tau+s}$ for $\omega = \pi$.

A standard long-memory process $(1-L)^d y_{S\tau+s} = \mathbf{1}_{(S\tau+s>0)}(S\tau + s)\,\varepsilon_{S\tau+s}$ associated with the zero frequency and a long-memory process $(1+L)^d y_{S\tau+s} = \mathbf{1}_{(S\tau+s>0)}(S\tau + s)\,\varepsilon_{S\tau+s}$ associated with the Nyquist frequency are particular cases of (4)-(5) where $\omega = 0$ and $\omega = \pi$, respectively. In the following, for the ease of exposition, we use "t" instead of "$S\tau + s$", unless needed.

2.2 $(\lfloor S/2 \rfloor + 1)$-factor seasonal long-memory process

In this section, we extend the one-factor seasonal long-memory process from the previous section to a $(\lfloor S/2 \rfloor + 1)$-factor seasonal LM process with potentially $\lfloor S/2 \rfloor + 1$ poles and different memory parameters governing the behavior at these poles,

$$\Pi_{k=0}^{\lfloor S/2 \rfloor} \omega_k (L)^{d_k} y_t = \varepsilon_t. \tag{6}$$

$\omega_0 (L)^{d_0} := (1 - L)^{d_0}$ denotes the factor associated with the zero frequency, $\omega_0 = 0$, with d_0 being the corresponding fractional integration order. $\omega_k(L)^{d_k} := \left(1 - 2cos\left(\omega_k\right) L + L^2\right)^{d_k}$ corresponds to the conjugate (harmonic) seasonal frequencies $(\omega_k, 2\pi - \omega_k)$ with $\omega_k = 2\pi k/S$, $k = 1, ..., \lfloor (S-1)/2 \rfloor$, and d_k being the corresponding fractional integration order. For even-numbered S, $\omega_{S/2}(L)^{d_{S/2}} := (1 + L)^{d_{S/2}}$ associated with the Nyquist frequency $\omega_{S/2} := \pi$, with $d_{S/2}$ being the corresponding fractionally integrated order. In what follows, it is understood that terms relating to frequency π are to be omitted when S is odd,

$$(1 - L)^{d_0} \Pi_{k=1}^{S/2-1} \left(1 - 2\cos\omega_k L + L^2\right)^{d_k} y_t = \varepsilon_t, \tag{7}$$

and that references to the Nyquist frequency only apply when S is even,

$$(1 - L)^{d_0} (1 + L)^{d_{S/2}} \Pi_{k=1}^{(S-1)/2} \left(1 - 2\cos\omega_k L + L^2\right)^{d_k} y_t = \varepsilon_t. \tag{8}$$

Further, both have poles at the zero frequency and at (several) harmonic seasonal frequencies. As mentioned before, for simplicity, we assume the innovation $\varepsilon_t \sim iid\left(0, \sigma_\varepsilon^2\right)$.

The $(\lfloor S/2 \rfloor + 1)$-factor seasonal long-memory process (6) is general enough to allow for processes with long memories with different orders d_k for each frequency $\omega_k = 2\pi k/S$, $k = 0, ..., \lfloor S/2 \rfloor$. Furthermore, the process allows one or more d_k to be equal to zero.

First, we consider stationary seasonal Type I LM processes by assuming $-1/2 < d_k < 1/2$, $k = 0, ..., \lfloor S/2 \rfloor$. Giraitis and Leipus (1995) show that the process can then be written as

$$y_t = \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \tag{9}$$

where the coefficients ψ_j have the asymptotic expansion

$$\begin{aligned}
\psi_j \quad \sim \quad & 2 \sum_{k=1}^{S/2-1} D(k) \frac{\Gamma(j + d_k)}{\Gamma(j+1)\Gamma(d_k)} \cos(\omega_k j + \nu_k) \\
& + D(0) \frac{\Gamma(j + d_0)}{\Gamma(j+1)\Gamma(d_0)} \\
& + D(S/2) \frac{\Gamma\left(j + d_{S/2}\right)}{\Gamma(j+1)\Gamma\left(d_{S/2}\right)} \cos(\pi j) + O\left(j^{-2+\max\{d_0,...,d_{\lfloor S/2 \rfloor}\}}\right),
\end{aligned}$$

as $j \to \infty$. Note that with odd-numbered S the term $D(S/2)\frac{\Gamma(j+d_{S/2})}{\Gamma(j+1)\Gamma(d_{S/2})}\cos(\pi j)$ is not present. And

$$D(k) = \begin{cases} |2\sin\omega_k|^{-d_k}\Pi_{j\neq k}|2(\cos\omega_k - \cos\omega_j)|^{-d_j} & \text{if } 0 < \omega_k < \pi \\ \Pi_{j\neq k}|2(\cos\omega_k - \cos\omega_j)|^{-d_j} & \text{if } \omega_k = 0 \text{ or } \pi \end{cases}$$

is a constant depending on ω_k and d_k, $k = 0, \ldots, \lfloor S/2 \rfloor$. Next,

$$\nu_k = \omega_k \sum_{j=0}^{\lfloor S/2 \rfloor} d_j - \pi \sum_{j=0}^{k-1} d_j - d_k \frac{\pi}{2}$$

is a constant depending on ω_k and d_k, $k = 0, \ldots, \lfloor S/2 \rfloor$ with neither $D(k)$ nor ν_k depending on j. Note that

$$\begin{aligned} \frac{\Gamma(j+d_k)}{\Gamma(j+1)} &\sim j^{d_k-1} \text{ as } j \to \infty \text{ and} \\ O\left(j^{-2+\max\left\{d_1^*,\ldots,d_{\lfloor S/2\rfloor}^*\right\}}\right) &\to 0 \text{ as } j \to \infty. \end{aligned}$$

Therefore, for large values of j (i.e., as $j \to \infty$), the behavior of the weights,

$$\psi_j \sim \left[D(0)\frac{j^{d_0-1}}{\Gamma(d_0)} + D(S/2)(-1)^j \frac{j^{d_{S/2}-1}}{\Gamma\left(d_{S/2}\right)} + \sum_{k=1}^{(S-1)/2} D(k)\frac{j^{d_k-1}}{\Gamma(d_k)}\cos(\omega_k j + \nu_k) \right],$$

is periodic, since their periodicity depends on j. As previously mentioned for odd-numbered S, the term $D(S/2)(-1)^j j^{d_{S/2}-1}$ is not present. Note that, for one pole at a harmonic frequency, this approximation corresponds to expression (9) in Chung (1996). Then defining

$$\psi_j^* = \left[D(0)\frac{j^{d_0-1}}{\Gamma(d_0)} + D(S/2)(-1)^j \frac{j^{d_{S/2}-1}}{\Gamma\left(d_{S/2}\right)} + \sum_{k=1}^{S/2-1} D(k)\frac{j^{d_k-1}}{\Gamma(d_k)}\cos(\omega_k j + \nu_k) \right],$$

with $\psi_j^* \sim \psi_j$ for large j, and $y_t^* = \sum \psi_j^* \varepsilon_{t-j}$, hence,

$$y_t^* = \sum_{j=0}^{\infty}\psi_j^*\varepsilon_{t-j} = \sum_{j=0}^{\infty}\left[D(0)\frac{j^{d_0-1}}{\Gamma(d_0)} + D(S/2)(-1)^j \frac{j^{d_{S/2}-1}}{\Gamma\left(d_{S/2}\right)} + \sum_{k=1}^{S/2-1} D(k)\frac{j^{d_k-1}}{\Gamma(d_k)}\cos(\omega_k j + \nu_k) \right]\varepsilon_{t-j}. \tag{10}$$

Then, we can reorganize (10) as

$$\underbrace{D(0)\sum_{j=0}^{\infty}\frac{j^{d_0-1}}{\Gamma(d_0)}\varepsilon_{t-j}}_{I(d_0)} + \underbrace{D(S/2)\sum_{j=0}^{\infty}(-1)^j\frac{j^{d_{S/2}-1}}{\Gamma\left(d_{S/2}\right)}\varepsilon_{t-j}}_{I\left(d_{S/2}\right) \text{ at frequency } \pi} + \sum_{k=1}^{S/2-1} D(k)\underbrace{\sum_{j=0}^{\infty}\cos(\omega_k j + \nu_k)\frac{j^{d_k-1}}{\Gamma(d_k)}\varepsilon_{t-j}}_{I(d_k) \text{ at frequency } \omega_k}, \tag{11}$$

where the first summand behaves as $I(d_0)$ at frequency 0, the second one (which is present only when S is even) as $I(d_{S/2})$ at frequency π, and the final ones as $I(d_k)$ at frequency ω_k, for $k = 1, ..., \lfloor (S-1)/2 \rfloor$, the latter, all with complex weights.

In order to deal with both stationary and nonstationary seasonal LM processes, we truncate expansion (10) at 0,

$$y_t^* = \sum_{j=0}^{t-1} \psi_j^* \varepsilon_{t-j}, \tag{12}$$

which corresponds to the Type II definition of long memory (see Marinucci and Robinson (1999) and Hassler (2019) for a detailed treatment in the non-seasonal case). For $d_k < 1/2$, the truncated process is asymptotically stationary, and for $d_k \geq 1/2$, the process is non-stationary. In both cases, its (pseudo)spectral density $f_y(\omega_k + \lambda) \sim C_h(\mathbf{d}) |\lambda|^{-2d_k}$, with $C_h(\mathbf{d})$ being a function of all memory parameters (see Section 2, expressions (2) and (3) in Arteche (2020)). See Arteche (2020) for further details and for a recent discussion of the resulting $(\lfloor S/2 \rfloor + 1)$-factor seasonal LM process, and more specifically, the estimation of the $\lfloor S/2 \rfloor + 1$ memory parameters by an extension of the exact local Whittle (ELW) estimator (Shimotsu and Phillips, 2005).

3 The Impact of Temporal Aggregation

We use the framework introduced by del Barrio, Rodrigues, and Taylor (2019) to analyze the consequences of aggregation in both skip sampling and cumulation sampling. In particular, we express the seasonal LM process in its vector of seasons representation,

$$Y_\tau^* = \left[y^*_{S\tau-(S-1)}, y^*_{S\tau-(S-2)}, \cdots, y^*_{S\tau-1}, y^*_{S\tau} \right]'.$$

Aggregation results from pre-multiplying Y_τ^* by A, an $S_A \times S$ full-row rank matrix, with S_A denoting the number of seasons after aggregation,

$$A = \begin{bmatrix} a_Q & 0_Q & \dots & 0_Q \\ 0_Q & a_Q & \dots & 0_Q \\ \vdots & \vdots & \ddots & 0_Q \\ 0_Q & 0_Q & 0_Q & a_Q \end{bmatrix}, \tag{13}$$

where a_Q is a $1 \times Q$ vector, with $Q = S/S_A$. For skip sampling, $a_Q = (0, 0, ..., 1)$, and for cumulation sampling, $a_Q = 1_Q$, a $1 \times Q$ vector of ones.

3.1 Vector of seasons representation

In order to obtain the vector of seasons representation associated with (6), we only need to combine (11) and (5). From (11), we can split the coefficients ψ_j^* in (12) into the contributions from the different frequencies. Therefore, it is sufficient to analyze them separately. Using $\cos(\omega_k n + \nu_k) =$

$\frac{1}{2}\left(e^{-i(\omega_k n+\nu_k)}+e^{i(\omega_k n+\nu_k)}\right)$ and considering the role played by the demodulator, (11) can be re-written as

$$\begin{aligned}
y_t^* &= D(0)\sum_{j=0}^{t-1}\frac{j^{d_0-1}}{\Gamma(d_0)}\varepsilon_{t-j}+D(S/2)\sum_{j=0}^{t-1}(-1)^j\frac{j^{d_{S/2}-1}}{\Gamma\left(d_{S/2}\right)}\varepsilon_{t-j} \\
&+\sum_{k=1}^{(S-1)/2}D(k)\sum_{j=0}^{t-1}\frac{1}{2}\left(e^{-i\omega_k j}e^{-i\nu_k}+e^{i\omega_k j}e^{i\nu_k}\right)\frac{j^{d_k-1}}{\Gamma(d_k)}\varepsilon_{t-j} \\
&= D(0)\sum_{j=0}^{t-1}\frac{j^{d_0-1}}{\Gamma(d_0)}\varepsilon_{t-j}+D(S/2)\sum_{j=0}^{S\tau+s-1}(-1)^j\frac{j^{d_{S/2}-1}}{\Gamma\left(d_{S/2}\right)}\varepsilon_{t-j} \qquad (14)\\
&+\sum_{k=1}^{(S-1)/2}\frac{D(k)}{2}\sum_{j=0}^{t-1}e^{-i\omega_k t}\ e^{i\omega_k(t-j)}e^{i\nu_k}\frac{j^{d_k-1}}{\Gamma(d_k)}\varepsilon_{t-j} \qquad (15)\\
&+\sum_{k=1}^{(S-1)/2}\frac{D(k)}{2}\sum_{j=0}^{t-1}e^{i\omega_k t}\ e^{-i\omega_k(t-j)}e^{-i\nu_k}\frac{j^{d_k-1}}{\Gamma(d_k)}\varepsilon_{t-j}. \qquad (16)
\end{aligned}$$

For a generic term in (15),

$$y_t^{*(k)}=\frac{D(k)}{2}e^{i\nu_k}\sum_{j=0}^{t-1}e^{-i\omega_k t}\ e^{i\omega_k(t-j)}\frac{j^{d_k-1}}{\Gamma(d_k)}\varepsilon_{t-j}.$$

Hence, following the lines of (5), it is possible to write the vector of seasons representation,

$$Y_\tau^{*(k)}=\left[\begin{array}{cccc} y_{S\tau-(S-1)}^{*(k)} & y_{S\tau-(S-2)}^{*(k)} & \cdots & y_{S\tau}^{*(k)}\end{array}\right]',$$

as

$$\begin{aligned}
Y_\tau^{*(k)} &= \frac{D(k)}{2}e^{i\nu_k}\left(v_k^-\right)\left(v_k^+\right)'\sum_{j=1}^{\tau}\Gamma_j E_j \qquad (17)\\
&\quad +terms\ fractionally\ integrated\ of\ order\ (d_k-1),
\end{aligned}$$

with $\left(v_k^-\right)'$ and $\left(v_k^+\right)'$ corresponding to $(v^-)'$ and $(v^+)'$ for ω_k rather than ω, and Γ_j being a diagonal matrix such that $\Gamma_j=1/\Gamma(d_k)\,diag\left([S(\tau-j)+(S-1)]^{d_k-1},[S(\tau-j)+(S-2)]^{d_k-1},\ldots,[S(\tau-j)]^{d_k-1}\right)$.

(14) is a special case of (15) with $\omega_k=0$ for the first term and with $\omega_k=\pi$ for the second term. (16) is the complex conjugate of (15) and thus behaves equivalently.

Note that (17) corresponds to (5). Similarly, it is possible to arrive from the first and second terms of (14) to corresponding expressions for $\omega_k=0$ and $\omega_k=\pi$, respectively. Finally, for (16), an equivalent result is obtained by interchanging the order of v_k^- and v_k^+ in (17) and replacing $e^{i\nu_k}$ by $e^{-i\nu_k}$.

3.2 Aggregation

Aggregation results from pre-multiplying the vector of seasons representation associated with (14)-(16) (i.e., (17)) by matrix A, (13). The process, in its vector of seasons representation, and more specifically the demodulation operator, behaves as in del Barrio, Rodrigues, and Taylor (2019) when being pre-multiplied by matrix A, (13). As in del Barrio, Rodrigues, and Taylor (2019), the demodulator operator collected in vectors v_k^- and v_k^+ plays a key role. Recalling that $Q \equiv S/S_A$, Proposition 1 summarizes the impact of temporal aggregation under the following assumption, repeated here for clarity:

A1 The innovation ε_t is identically and independently distributed with zero expected value and constant variance, that is, $\varepsilon_t \sim iid\left(0, \sigma_\varepsilon^2\right)$.

Proposition 1 *A temporally aggregated ($\lfloor S/2 \rfloor$+1)-factor seasonal LM process, as defined in (6), with poles at ($\lfloor S/2 \rfloor$+1) frequencies with any degree of long-memory has the following properties:*

a) a pole at ω_k is mapped from ω_k to $\omega_k Q$, , $k = 1, ..., \lfloor S/2 \rfloor$,

b) if $\omega_k Q = \omega_l Q$, for $k \neq l$, the memory at this pole is $\max\{d_k, d_l\}$, and

c) for cumulation sampling, if $\omega_k Q$ is a multiple of 2π, the long-memory vanishes.

Proposition 1 (whose proof can be found in the appendix) generalizes Theorem 2 of Sun and Shi (2014) and Proposition 4 of Hassler (2011) by allowing for multiple factors and by covering both stationary and nonstationary memory at the different poles. Part a) corresponds to the results in Hassler (2011), who derived this property in the frequency domain. Part b) extends Theorem 2 of Sun and Shi (2014) to the case of multiple factors. Part c) is, in spirit, similar to Remark F of Hassler (2011), which discusses the closedness under aggregation for this case.

Remark 1. For simplicity, in Proposition 1 we assume that the innovation ε_t follows Assumption A1, but as can be seen in the Appendix and in (17), the behavior of the innovation ε_t is not relevant in terms of the allocation of the long-memory behavior at the specific frequencies. That is, $\left(v_k^+\right)' \sum_{j=1}^{\tau} \Gamma_j E_j$ in (17) collects the long-memory behavior, v_k^+ determines at which frequency the long-memory behavior is allocated, and Av_k^+ determines at which frequency the long-memory behavior is re-allocated after aggregation. Hence, the results of Proposition 1 should also hold for more general assumptions, such as infinite moving average processes with absolute summable weights (see Assumption 6.1 in Hassler (2019)).

Table 1 a) illustrates the effects for skip sampling from monthly (S=12) observations, to quarterly (S_A=12), Q=3, or annual (S_A=1) data, Q=12; Table 1 b) illustrates the effects for cumulation sampling. In both cases, the frequency is mapped from ω_k to $\omega_k Q$. In the latter case, the long-memory vanishes if $\omega_k Q$ is a multiple of 2π, or equivalently (k/S_A) is an integer value.

Finally, our analysis can also be used to explore the effect of aggregation in

$$(1-L)^{d_0}\left(1-2\cos\lambda L+L^2\right)^{d_\lambda} y_t = \varepsilon_t, \tag{18}$$

Table 1: **Summary of frequency allocations under aggregation of monthly data**

a) Skip sampling

	k	0	1	2	3	4	5	6
Original monthly frequency	ω_k	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
Allocation in quarterly data	ω_k	0	$\frac{\pi}{2}$	π	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π
Allocation in annual data	ω_k	0	0	0	0	0	0	0

b) Cumulation sampling

	k	0	1	2	3	4	5	6
Original monthly frequency	ω_k	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
Allocation in quarterly data	ω_k	0	$\frac{\pi}{2}$	π	$\frac{\pi}{2}$	-	$\frac{\pi}{2}$	π
Allocation in annual data	ω_k	0	-	-	-	-	-	-

a time series with long-memory associated with the zero frequency $(1-L)^{d_0}$ and with cyclical behavior $\left(1-2\cos\lambda L+L^2\right)^{d_\lambda}$, where a full cycle is completed every $2\pi/\lambda$, or in the simpler case in which the time series only has cyclical long-memory behavior, $\left(1-2\cos\lambda L+L^2\right)^{d_\lambda} y_t=\varepsilon_t$. As pointed out by Hassler (2019), there is no need for λ in the Gegenbauer polynomial to be restricted to seasonal frequencies. Thus, by appropriately defining the vector of seasons representation, our results apply.

4 Simulations

In this section, we report Monte Carlo results confirming the theoretical findings of the previous section. As previously mentioned, our results not only apply to seasonal but also to cyclical long-memory processes. In the first subsection, we consider seasonal long-memory processes with two poles at different frequencies. Our results allow us to explain how the long-memory shifts from one frequency to another and how, in some cases, the long-memory behavior vanishes after cumulation sampling, without the need to calculate the spectra of the aggregated process. In the second subsection, we focus on process (18), a process with long-memory behavior at the zero frequency and at a cyclical harmonic frequency, which corresponds to the one in Sun and Shi (2014). The Monte Carlo results are based on 10,000 replications. For seasonal processes, the sample size $T=2040$, which for monthly data ($S=12$) corresponds to a total number of years of $N=170$; for cyclical processes, $T=2048$, as in Sun and Shi (2014).

4.1 Two-factor seasonal long-memory processes

In this subsection, we explore the effects of aggregation on processes with non-zero memory at two poles. These results confirm the predictions of Proposition 1, specifically the case where two poles move to the same frequency and show that the memory at this frequency is the maximum of the two memories in the disaggregated series. In particular, we deal with the following two processes:

$$(1-L)^{0.3}\left(1-2\cos\left(\frac{2\pi}{3}\right)L+L^2\right)^{0.8} y_t = \varepsilon_t, \tag{19}$$

$$\left(1-2\cos\left(\frac{\pi}{6}\right)L+L^2\right)^{0.8}\left(1-2\cos\left(\frac{5\pi}{6}\right)L+L^2\right)^{0.3} y_t = \varepsilon_t. \tag{20}$$

In both cases, we work with $Q = 2$, 3, and 12. All frequencies in (19) and (20) are compatible with monthly data, $S = 12$. In particular, the factor $(1-L)$ is associated with the zero frequency and is the only non-seasonal factor. Finally, the factors $\left(1-2\cos\left(\frac{2\pi}{3}\right)L+L^2\right)$, $\left(1-2\cos\left(\frac{\pi}{6}\right)L+L^2\right)$, and $\left(1-2\cos\left(\frac{5\pi}{6}\right)L+L^2\right)$ are associated with frequencies $\frac{2\pi}{3}$, $\frac{\pi}{6}$, and $\frac{5\pi}{6}$, respectively, and to oscillations that complete a full cycle every 3, 12, and 12/5 periods, respectively. For $Q = 2$, we move from monthly to bimonthly data, with $Q = 3$, we move from monthly to quarterly data, and finally $Q = 12$ corresponds to the common case of moving from monthly data to annual data.

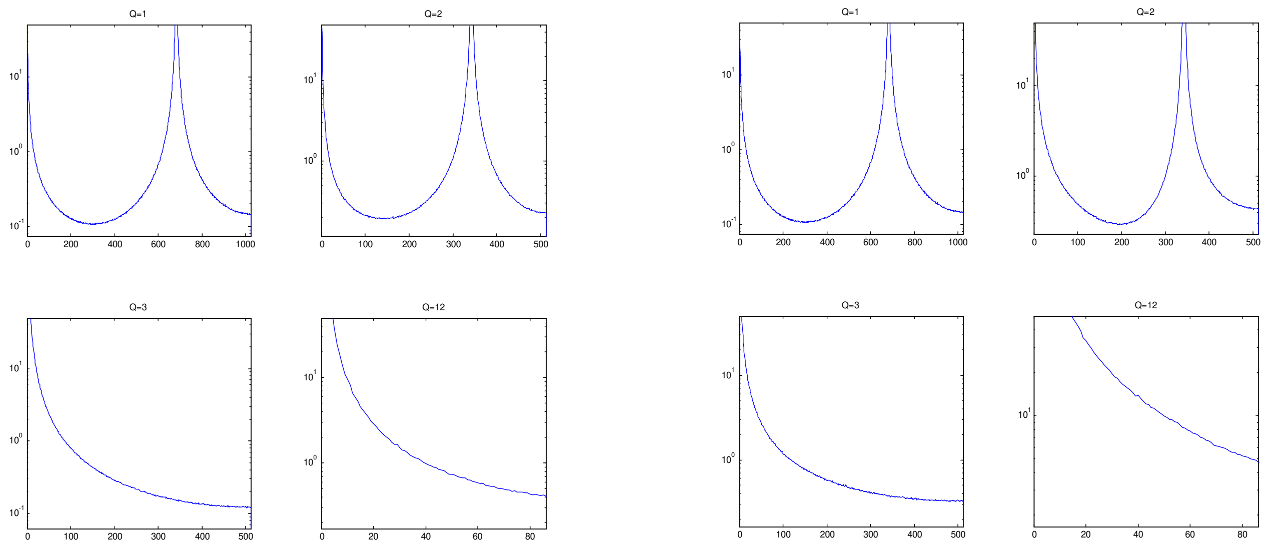
From Proposition 1, with skip sampling, the long-memory behavior associated to frequency $\omega_k = 2\pi k/S$ shifts to frequency $Q\omega_k = 2\pi k/S_A$.

Figure 1 shows the average periodograms for the simulated process (19), with long-memory behavior at the zero frequency and at $2\pi/3$ for skip sampling (panel a), and cumulation sampling (panel b). It is illustrative of the effects that skip and cumulation sampling have on seasonal long-memory processes as shown in Proposition 1. Table 2 shows the results of applying Arteche's (2020) ELW estimator with a bandwidth choice of $m = (SN)^{1/2}$ to estimate the memory for the simulated processes before and after aggregation. Note the results might be somehow sensitive to the bandwidth choice (see, for example, Arteche and Orbe (2017) for the optimal bandwidth choice in a non-seasonal LW estimator). Based on the results of the previous section, the long-memory behavior associated with $2\pi/3$ will remain at frequency $2\pi/3$ for $Q = 2$ (since both $4\pi/3$ and $8\pi/3$ are multiples of $2\pi/3$) and move to frequency zero for $Q = 3$ and 12. With cumulation sampling, the long memory originally associated with $2\pi/3$ remains at frequency $2\pi/3$ for $Q = 2$ and vanishes for $Q = 3$ and 12. The long-memory behavior associated with the zero frequency remains at this frequency for skip and cumulation sampling, but with skip sampling and $Q = 3$ and 12, the long-memory behavior $d_1 = 0.3$ is dominated by $d_2 = 0.8$ as seen in Table 2.

Figure 1 and Table 2 illustrate these results. In particular, in Table 2 with Q=12 and skip sampling, the long-memory behavior originally associated with frequency $2\pi/3$ moves to the zero frequency, and, as predicted in part b of Proposition 1, the higher memory, $d_1 = 0.8$, dominates.

Figure 1: **Average over simulated periodograms for process (19)**

a) Skip sampling b) Cumulation sampling



Sample size T=2040. Based on 10,000 replications.

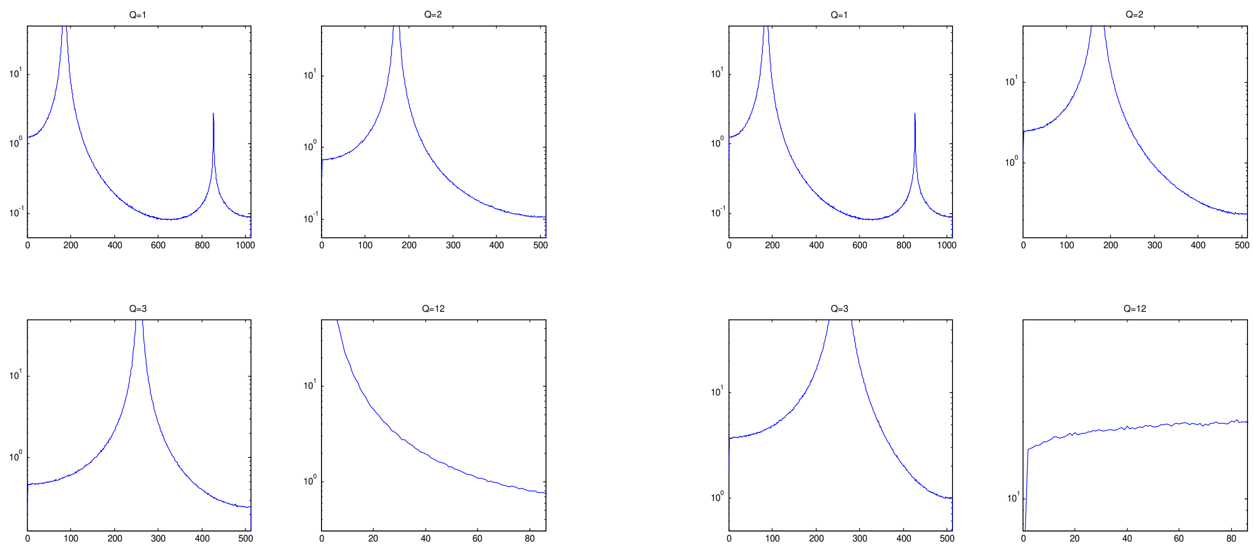
Table 2: **Estimation by seasonal ELW for process (19)**

	pole 1							pole 2						
Q		skip sampling			cumulation sampling				skip sampling			cumulation sampling		
	$Q\omega_1$	d_1	$\hat{d}_1$	std($\hat{d}_1$)	d_1	$\hat{d}_1$	std($\hat{d}_1$)	$Q\omega_2$	d_2	$\hat{d}_2$	std($\hat{d}_2$)	d_2	$\hat{d}_2$	std($\hat{d}_2$)
1		0.3	0.296	0.044	0.3			$\frac{2\pi}{3}$	0.8	0.798	0.062			
2	0	0.3	0.284	0.056	0.3	0.295	0.056	$\frac{2\pi}{3}$	0.8	0.795	0.079	0.8	0.786	0.079
3	0	0.3	dominated		0.3	0.291	0.064	0	0.8	0.770	0.063		vanishes	
12	0	0.3	dominated		0.3	0.290	0.106	0	0.8	0.754	0.105		vanishes	

Sample size T=2040. Based on 10,000 replications.

Figure 2: **Average over simulated periodograms for process (20)**

a) Skip sampling b) Cumulation sampling



Sample size T=2040. Based on 10,000 replications.

Table 3: **Estimation by seasonal ELW for process (20)**

Q	pole 1							pole 2						
Q		skip sampling			cumulation sampling				skip sampling			cumulation sampling		
	$Q\omega_1$	d_1	$\hat{d}_1$	std($\hat{d}_1$)	d_1	$\hat{d}_1$	std($\hat{d}_1$)	$Q\omega_2$	d_2	$\hat{d}_2$	std($\hat{d}_2$)	d_2	$\hat{d}_2$	std($\hat{d}_2$)
1	$\frac{\pi}{6}$	0.8	0.798	0.062				$\frac{5\pi}{6}$	0.3	0.300	0.061			
2	$\frac{\pi}{3}$	0.8	0.791	0.078	0.8	0.795	0.078	$\frac{\pi}{3}$	0.3	dominated		0.3	dominated	
3	$\frac{\pi}{2}$	0.8	0.789	0.087	0.8	0.794	0.087	$\frac{\pi}{2}$	0.3	dominated		0.3	dominated	
12	0	0.8	0.762	0.105	0	-0.03	0.103	0	0.3	dominated		0	vanishes	

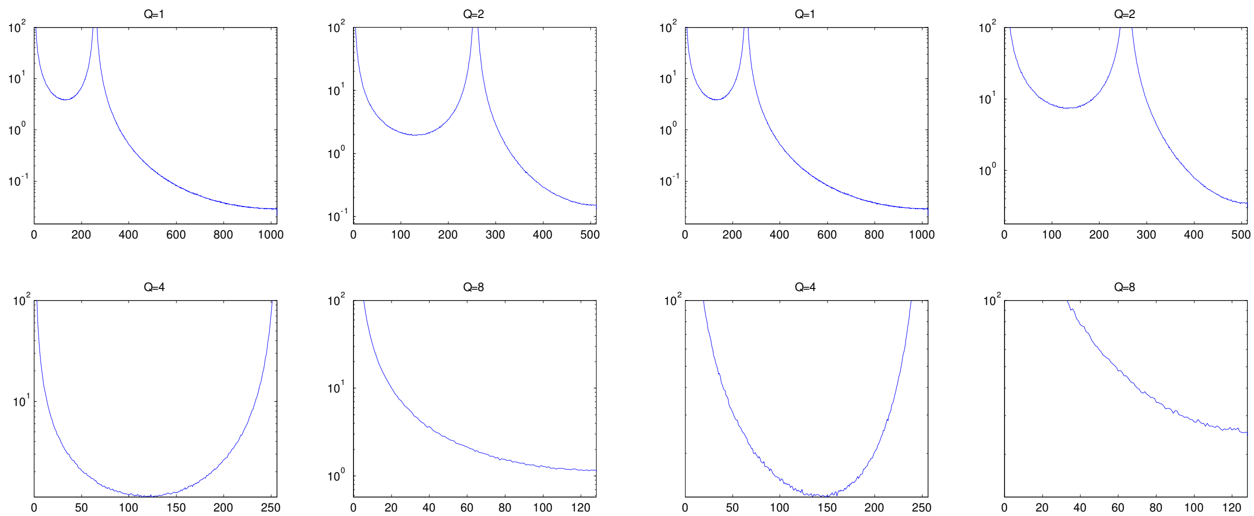
Sample size T=2040. Based on 10,000 replications.

The results for process (20), a process with poles at $\frac{\pi}{6}$ with $d_1 = 0.8$ and at $\frac{5\pi}{6}$ with $d_2 = 0.3$, are shown in Figure 2 and Table 3. Figure 2 shows that with skip and cumulation sampling, the poles at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ move to frequencies $Q\frac{\pi}{6}$ and $Q\frac{5\pi}{6}$ for skip sampling and to $Q\frac{\pi}{6}$ and $Q\frac{5\pi}{6}$ or vanish if $Q\frac{\pi}{6}$ and $Q\frac{5\pi}{6}$ are multiples of 2π. Table 3 shows the estimation of the memory at the poles for this case, and they both move to the same frequency, as $Q\frac{\pi}{6}$ and $Q\frac{5\pi}{6}$ correspond to the same frequency. In all the cases, $d_1 = 0.8$ dominates $d_2 = 0.3$, except for $Q = 12$ with cumulation sampling, where both poles vanish.

Overall, the periodograms of the aggregated series behave as predicted, and the estimation of the memory parameter in both aggregated and disaggregated series behaves satisfactorily and in

Figure 3: **Average over simulated periodograms for process (21)**

a) Skip sampling b) Cumulation sampling



Sample size T=2048. Based on 10,000 replications.

line with the predictions.

4.2 A process with long memory at a cyclical frequency and at the zero frequency

Finally, we consider cyclical behavior with two poles, at $\frac{\pi}{4}$ with $d = 0.8$ and at 0 with $d = 0.3$,

$$(1 - L)^{0.3} \left(1 - 2\cos\left(\frac{\pi}{4}\right) L + L^2\right)^{0.8} y_t = \varepsilon_t. \tag{21}$$

The pole at $\frac{\pi}{4}$ corresponds to the one in Sun and Shi (2014), and as in their paper, $T = 2048$ observations and $Q = 2$, 4, and 8. The results are shown in Figure 3 and Table 4, and are consistent with the predictions of Proposition 1. With skip sampling, the pole at $\frac{\pi}{4}$ moves to $\frac{\pi}{2}$, π, and 0 for $Q = 2$, 4, and 8, respectively; and, for $Q = 8$, $d_2 = 0.8$ dominates $d_1 = 0.3$. With cumulation sampling, the pole at $\frac{\pi}{4}$ moves to $\frac{\pi}{2}$, π, and vanishes for $Q = 2$, 4, and 8, respectively. Finally, Table 4 shows that the estimation of the memory at the poles in the aggregated series to which the poles in the disaggregated series are mapped works well.

5 Conclusion

In this paper, we have analyzed the impact of the temporal aggregation of a $(\lfloor S/2 \rfloor + 1)$-factor seasonal/cyclical long-memory process regardless of the memory at each pole, considering both

Table 4: **Estimation by seasonal ELW for process (21)**

	pole 1							pole 2						
Q		skip sampling			cumulation sampling				skip sampling			cumulation sampling		
	$Q\omega_1$	d_1	$\hat{d}_1$	std($\hat{d}_1$)	d_1	$\hat{d}_1$	std($\hat{d}_1$)	$Q\omega_2$	d_2	$\hat{d}_2$	std($\hat{d}_2$)	d_2	$\hat{d}_2$	std($\hat{d}_2$)
1	0	0.3	0.297	0.044				$\frac{\pi}{4}$	0.8	0.805	0.060			
2	0	0.3	0.297	0.055	0.3	0.298	0.055	$\frac{\pi}{2}$	0.8	0.802	0.075	0.8	0.802	0.075
4	0	0.3	0.292	0.072	0.3	0.294	0.072	π	0.8	0.798	0.070	0.3	0.782	0.069
8	0	0.3	dominated		0.3	0.289	0.089	0	0.8	0.734	0.088	0.3	vanishes	

Sample size T=2048. Based on 10,000 replications

stationary and nonstationary long-memory processes. By using the approach suggested by Giraitis and Leipus (1995), we extended the analysis for stationary one-factor seasonal long-memory processes by Tsai and Chan (2005), Hassler (2011), and Sun and Shi (2014) to the case of multiple poles of stationary and/or nonstationary memory. We have expanded a corresponding analysis in the frequency domain (Hassler, 2011) to the time domain, and thus facilitated immediate knowledge of the location of the poles after aggregation without the need to calculate all spectra. Further, we have shown that the estimation of memory of the aggregated processes by Arteche's (2020) seasonal exact local Whittle estimator works satisfactorily and that the estimated memory parameters are in line with the theoretical predictions.

We assumed that we know the frequencies of the poles of the disaggregated series and analyzed the memory of the aggregated series at the mapped frequencies. An extension with the disaggregated series with memory associated to unknown frequencies could be conducted along the lines of Hidalgo and Soulier (2004) but is beyond the scope of this paper.

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Appendix

Proof of (5)

Note that in (4) for season $s = 0$,

$$
\begin{aligned}
y_{S\tau} &= e^{-i\omega S\tau} \sum_{j=1}^{S\tau} \psi_{S\tau+s-j} e^{i\omega j} \varepsilon_j \\
&= e^{-i\omega S\tau} \left(e^{i\omega S\tau} \varepsilon_{S\tau} + \psi_1 e^{i\omega(S\tau-1)} \varepsilon_{S\tau-1} + \psi_2 e^{i\omega(S\tau-2)} \varepsilon_{S\tau-2} + \ldots \right) \\
&= e^{-i\omega S} \left(e^{i\omega S} \varepsilon_{S\tau} + \psi_1 e^{i\omega(S-1)} \varepsilon_{S\tau-1} + \psi_2 e^{i\omega(S-2)} \varepsilon_{S\tau-2} + \ldots \right),
\end{aligned}
$$

where we use the fact that $e^{-i\omega(S\tau+s)} = e^{-i\omega s}$ as $S = 2\pi/\omega$ and hence we have periodic behavior in the general case of a cyclical long-memory process. Clearly, this is also the case for $\omega_k = 2\pi k/S$. Next, it is possible to write:

$$
\begin{aligned}
y_{S\tau} &= e^{-i\omega S} \left(e^{i\omega S} \varepsilon_{S\tau} + \psi_1 e^{i\omega(S-1)} \varepsilon_{S\tau-1} + \psi_2 e^{i\omega(S-2)} \varepsilon_{S\tau-2} + \ldots \right) \\
&= e^{-i\omega S} \left(v_k^+ \right)' \sum_{j=1}^{\tau} \sum_{j=1}^{\tau} \Psi_j E_j, \qquad (22)
\end{aligned}
$$

with $\left(v_k^+ \right)' = \left[\begin{array}{cccc} e^{+i\omega} & e^{+i\omega 2} & \cdots & e^{+i\omega S} \end{array} \right]$, $E_\tau = \left[\varepsilon_{S\tau-(S-1)}, \varepsilon_{S\tau-(S-2)}, \cdots, \varepsilon_{S\tau-1}, \varepsilon_{S\tau} \right]'$, and $\Psi_j = diag \left(\psi_{S(\tau-j)+(S-1)}, \psi_{S(\tau-j)+(S-2),}, \ldots, \psi_{S(\tau-j)} \right)$. Note that (22) corresponds to (5). For season $s = -1$,

$$
\begin{aligned}
y_{S\tau-1} &= e^{-i\omega(S\tau-1)} \sum_{j=1}^{S\tau-1} \psi_{S\tau+s-j} e^{i\omega j} \varepsilon_j \\
&= e^{-i\omega(S\tau-1)} \left(e^{i\omega(S\tau-1)} \varepsilon_{S\tau-1} + \psi_1 e^{i\omega(S\tau-2)} \varepsilon_{S\tau-2} + \psi_2 e^{i\omega(S\tau-3)} \varepsilon_{S\tau-3} + \ldots \right) \\
&= e^{-i\omega(S-1)} \left(e^{i\omega(S-1)} \varepsilon_{S\tau-1} + \psi_1 e^{i\omega(S-2)} \varepsilon_{S\tau-2} + \psi_2 e^{i\omega(S-3)} \varepsilon_{S\tau-3} + \ldots \right) \\
&= e^{-i\omega(S-1)} \left(e^{i\omega(S-1)} \varepsilon_{S\tau-1} + \psi_1 e^{i\omega(S-2)} \varepsilon_{S\tau-2} + \psi_2 e^{i\omega(S-3)} \varepsilon_{S\tau-3} + \ldots \right) \\
&\quad \pm e^{-i\omega(S-1)} \left(e^{i\omega S} \varepsilon_{S\tau} + \psi_1 e^{i\omega(S-1)} \varepsilon_{S\tau-1} + \psi_2 e^{i\omega(S-2)} \varepsilon_{S\tau-2} + \ldots \right) \\
&= e^{-i\omega(S-1)} \left(e^{i\omega S} \varepsilon_{S\tau} + \psi_1 e^{i\omega(S-1)} \varepsilon_{S\tau-1} + \psi_2 e^{i\omega(S-2)} \varepsilon_{S\tau-2} + \ldots \right) \\
&\quad - \left(e^{i\omega} y_{S\tau} - y_{S\tau-1} \right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
y_{S\tau-1} &= e^{-i\omega(S-1)} \left(e^{i\omega S} \varepsilon_{S\tau} + \psi_1 e^{i\omega(S-1)} \varepsilon_{S\tau-1} + \psi_2 e^{i\omega(S-2)} \varepsilon_{S\tau-2} + \ldots \right) \qquad (23) \\
&\quad - \left(e^{i\omega} y_{S\tau} - y_{S\tau-1} \right),
\end{aligned}
$$

which corresponds to (5).

For season $s=-m$, $m=2,...,(S-1)$

$$
\begin{aligned}
y_{S\tau-m} &= e^{-i\omega(S\tau-m)}\sum_{j=1}^{S\tau-m}\psi_{S\tau+s-j}e^{i\omega j}\varepsilon_j \\
&= e^{-i\omega(S\tau-m)}\left(e^{i\omega(S\tau-m)}\varepsilon_{S\tau-m}+\psi_1 e^{i\omega(S\tau-m-1)}\varepsilon_{S\tau-(m+1)}+\psi_2 e^{i\omega(S\tau-m-2)}\varepsilon_{S\tau-(m+2)}+\dots\right) \\
&= e^{-i\omega(S-m)}\left(e^{i\omega(S-m)}\varepsilon_{S\tau-m}+\psi_1 e^{i\omega(S-m-1)}\varepsilon_{S\tau-(m+1)}+\psi_2 e^{i\omega(S-m-2)}\varepsilon_{S\tau-(m+2)}+\dots\right) \\
&= e^{-i\omega(S-m)}\left(e^{i\omega S}\varepsilon_{S\tau}+\psi_1 e^{i\omega(S-1)}\varepsilon_{S\tau-1}+\psi_2 e^{i\omega(S-2)}\varepsilon_{S\tau-2}+\dots\right) \\
&\quad -\left(e^{i\omega m}y_{S\tau}-y_{S\tau-m}\right).
\end{aligned}
$$

We can write for the vector $Y_\tau=\left[y_{S\tau-(S-1)},y_{S\tau-(S-2)},\cdots,y_{S\tau-1},y_{S\tau}\right]'$:

$$
\begin{aligned}
Y_\tau &= \left(v^-\right)\left(v^+\right)'\sum_{j=1}^{\tau}\Psi_j E_j + \\
&\quad +\begin{bmatrix} -\left(e^{i\omega(S-1)}y_{S\tau}-y_{S\tau-(S-1)}\right) \\ -\left(e^{i\omega(S-2)}y_{S\tau}-y_{S\tau-(S-2)}\right) \\ \vdots \\ 0 \end{bmatrix}
\end{aligned}
\tag{24}
$$

with $(v^-)'$ being the complex conjugate of $(v^+)'$, i.e. $(v^-)'=\begin{bmatrix} e^{-i\omega} & e^{-i\omega 2} & \cdots & e^{-i\omega S}\end{bmatrix}$. The last term on the right-hand side of (24) can be written as

$$
\begin{aligned}
\begin{bmatrix} -\left(e^{i\omega(S-1)}y_{S\tau}-y_{S\tau-(S-1)}\right) \\ -\left(e^{i\omega(S-2)}y_{S\tau}-y_{S\tau-(S-2)}\right) \\ \vdots \\ -\left(e^{i\omega 2}y_{S\tau}-y_{S\tau-2}\right) \\ -\left(e^{i\omega}y_{S\tau}-y_{S\tau-1}\right) \\ 0 \end{bmatrix}
&= \begin{bmatrix} -e^{i\omega(S-1)}\left[\left(1-e^{-i\omega}L\right)\sum_{j=0}^{(S-1)-1}e^{-ij\omega}L^j\right]y_{S\tau} \\ -e^{i\omega(S-2)}\left[\left(1-e^{-i\omega}L\right)\sum_{j=0}^{(S-2)-1}e^{-ij\omega}L^j\right]y_{S\tau} \\ \vdots \\ -e^{i\omega 2}\left[\left(1-e^{-i\omega}L\right)+e^{-i\omega}L\left(1-e^{-i\omega}L\right)\right]y_{S\tau} \\ -e^{i\omega}\left(1-e^{-i\omega}L\right)y_{S\tau} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -e^{-i\omega}\left[\left(1-e^{-i\omega}L\right)\sum_{j=0}^{(S-1)-1}e^{-ij\omega}L^j\right]y_{S\tau} \\ -e^{-i\omega 2}\left[\left(1-e^{-i\omega}L\right)\sum_{j=0}^{(S-2)-1}e^{-ij\omega}L^j\right]y_{S\tau} \\ \vdots \\ -e^{-i\omega(S-2)}\left[\left(1-e^{-i\omega}L\right)+e^{-i\omega}L\left(1-e^{-i\omega}L\right)\right]y_{S\tau} \\ -e^{-i\omega(S-1)}\left(1-e^{-i\omega}L\right)y_{S\tau} \\ 0 \end{bmatrix}
\end{aligned}
\tag{25}
$$

Note that $y_{S\tau}$ is fractionally integrated of order d, that is, $I(d)$. Hence $\left(1-e^{-i\omega}L\right)y_{S\tau}$ will be an $I(d-1)$ process. The element at position $S-1$ in (25), $e^{-i\omega(S-1)}\left(1-e^{-i\omega}L\right)y_{S\tau}$, is an $I(d-1)$ process multiplied by the demodulator operator associated with season $S-1$. It is possible to show that the remaining elements in the $S \times 1$ vector in (25) are weighted sums of $I(d-1)$ affected by the demodulator operator associated with the corresponding season.

Proof of Proposition 1

Note first that (6) admits representation (9). Since we are interested in the long-memory behavior in (9), we can focus on (10)-(11) and finally on its truncated version (12), which could be expressed as the sum of three terms (14), (15), and (16). It is clear that (15) and (16) are complex conjugates. Then, (12) admits a vector of seasons representation which is a sum of terms, such as (17). The effects of aggregation can then be evaluated by pre-multiplying (17) by A, as defined in (13):

$$\begin{aligned} AY_\tau^{*(k)} &= \frac{D(k)}{2} e^{i\nu_k} A\left(v_k^-\right)\left(v_k^+\right)' \sum_{j=1}^{\tau} \Gamma_j E_j \\ & \quad + terms\ fractionally\ integrated\ of\ order\ (d_k - 1). \end{aligned}$$

Clearly, $\left(v_k^+\right)' \sum_{j=1}^{\tau} \Gamma_j E_j$ is the vector of seasons representation of a complex-valued fractionally integrated process of order d_k at the zero frequency. The effect of aggregation on $Y_\tau^{*(k)}$, (17), in terms of the frequency allocation of the fractionally integrated process, can be seen by evaluating the effect of A in the demodulation operator collected in $\left(v_k^-\right)$. Hence part a) is a consequence of the fact that, for skip sampling,

$$A\left(v_k^-\right) = (e^{-i\omega_k Q}, e^{-i\omega_k 2Q}, e^{-i\omega_k 3Q}, \ldots, e^{-i\omega_k S_A Q})'. \tag{26}$$

Then, the demodulator moves from $e^{-i\omega_k t}$ to $e^{-i\omega_k Qt}$, and part a) is proved for skip sampling. In the case of cumulation sampling, it is possible to check that

$$A\left(v_k^-\right) = \frac{\sin\left(Q\omega_k/2\right)}{\sin\left(\omega_k/2\right)} e^{\mathrm{i}(Q-1)\frac{\omega_k}{2}} ((e^{-i\omega_k Q}, e^{-i\omega_k 2Q}, e^{-i\omega_k 3Q}, \ldots, e^{-i\omega_k S_A Q}))', \tag{27}$$

where we use the following result about the sum of a complex exponential expression:

$$\sum_{j=1}^{Q} e^{-i\omega_k j} = e^{-i\omega_k [Q+1]/2} \frac{\sin\left(Q\omega_k/2\right)}{\sin\left(Q\omega_k\right)}.$$

Hence, for (26) and (27), we clearly obtain the same effect on the demodulator; so, part a) is proved. Part c) is obtained when $\sin\left(Q\omega_k/2\right) = 0$, which happens when $Q\omega_k/2 = Q\pi k/S$ is a multiple of 2π. Finally, part b) is a straightforward consequence of part a), noting that the sum of two long-memory processes with potentially different memory parameters at frequency 0 corresponds to a long-memory series with memory corresponding to the higher memory parameter.