Fixed and Flexible Exchange-rates in Two Matching Models: Non-equivalence Results

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Abstract

There is a presumption that fixed and flexible (floating or market-determined) exchange-rate systems are equivalent if prices are flexible. We show that the presumption does not hold in two matching models of money. In both models, (i) currencies are the only assets and all trade is spot trade; (ii) the trades that directly determine welfare occur in pairwise meetings between buyers and sellers; and (iii) imperfect substitutability (including, as a special case, no substitutability) among currencies is a consequence of the trading protocol in those meetings. The two models are variants of the Lagos-Wright (2005) model and differ regarding the timing of the shock realizations relative to the centralized trade opportunities. One version has a *speculative fringe*. In it, the unique stationary (monetary) equilibrium under the fixed exchange-rate regime is one of a continuum of equilibria under a flexible exchange-rate regime. The other version has no *speculative fringe*. In it, there is a unique (monetary) stationary equilibrium under each exchange-rate regime and they differ.

Key words: Matching models of money; exchange-rate regimes

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1 Introduction

There is a presumption that fixed and flexible (floating or market-determined) exchange-rate systems are equivalent if prices are flexible. Friedman's influential essay, The Case for Flexible Exchange Rates, rests entirely on the assumption that domestic prices are sticky, while exchange rates are not (see [2]). A similar view is the foundation of the literature on optimum currency areas (see Mundell [11]). A formal statement of equivalence under flexible prices is in Lucas [8]. Implicit in Friedman and Mundell and explicit in Lucas is a classical-dichotomy, flexible-price model in which the quantities of currencies and their rates of return do not affect anything real and in which, less significantly, any kind of currency substitution is ruled out. Here, in contrast, (i) currencies are the only assets and all trade is spot trade; (ii) the trades that directly determine welfare occur in pairwise meetings between buyers and sellers; and (iii) following Zhu and Wallace (ZW) [14], imperfect substitutability (including, as a special case, no substitutability) among currencies is a consequence of the trading protocol in those meetings. We show that fixed and flexible exchange-rate regimes are not equivalent in our models.

The two models are variants of Lagos-Wright (LW) [7]. They differ regarding the timing of the shock realizations relative to the centralized trade opportunities. In one variant, the *no-speculative-fringe* version, the timing is such that the only holders of money are consumers who plan to spend all their money holdings. In it, there is a unique (monetary) stationary equilibrium under each exchange-rate regime and they differ. In the other version, half of all money holdings are held by producers, who will not be spending their money holdings immediately. Their holdings constitute a large *speculative fringe* and its existence implies that the unique stationary equilibrium under the fixed exchange-rate regime is one of a continuum of equilibria under a flexible exchange-rate regime.

As will be demonstrated, the assumption that the decentralized stage in both models has pairwise meetings serves only one purpose. It allows us to reconcile imperfect substitution between the two monies in the decentralized stage with the requirement that trade be in the core in the meetings between buyers and sellers. If, instead, that decentralized stage is replaced by a second centralized stage (by, in effect, unlimited replication of the buyer-seller pairs in meetings) and if country-specific cash-in-advance constraints are imposed in that stage—constraints which are *not* consistent with trade being in the core—then the same results hold.[1] There is, though, one special case of the *no-speculative-fringe* version under centralized trade and country-specific cash-in-advance constraints that does imply equivalence; namely, when the model is converted into an endowment economy. In that case, the model becomes a classical-dichotomy model.

So far as we know, there is no precursor of our results for the *no-speculative-fringe*

[1] Under replication, the core converges to competitive equilibrium. That implies that the two monies must be perfect substitutes at the terms of trade given by their prices in the next market.

version—perhaps, because of the dominance of classical-dichotomy formulations in open-economy monetary models. A precursor of our results for the *speculative-fringe* version is King *et. al.* [6], who obtain a similar result, but with a different basis for imperfect substitution between the two monies. In focusing on trade in two-person meetings, our work bears some resemblance to earlier efforts to study the role of multiple monies in such models—in particular, Matsuyama *et. al.* [9]. However, that work and some follow-up work used models with indivisible money—individual money holdings in the set $\{0, 1\}$. In such settings, it is awkward to model a foreign-exchange market and there is a troublesome dependence of real outcomes on the exogenous quantities of monies.

We start by setting out the structures of both versions of LW. Then we analyze each version; we prove existence of equilibrium and the above nonequivalence claims. Then we study a simple example for which we obtain closed-form solutions for equilibria. Finally, we discuss the consequences of abandoning the matching structure with centralized trade under country-specific cash-in-advance constraints.

2 The models

We start by describing the *no-speculative-fringe* version. After describing it, the *speculative-fringe* version can be described easily as a variant of it.

There are two symmetric countries, each populated by a nonatomic unit measure of people. Each country has its own divisible money. There are two stages at each discrete date. Stage 1 has a centralized market for everyone, residents of both countries, where the objects traded are the two currencies and the single stage-1 good, a good that is perishable and gives additively separable linear period utility. Stage 2 has random pairwise meetings between producers and consumers in each of the two countries. Producers produce only in their home country; some consumers, the *tourists*, visit the foreign country for consumption and then return home. Real international trade consists solely of tourism.

At the start of stage 1, prior to stage-1 trade, each person has a portfolio of amounts of the two monies and sees the realization of three preference shocks: two idiosyncratic shocks and one aggregate shock. The first idiosyncratic shock determines whether the person will be a producer or a consumer of the stage-2 perishable good. With probability one-half, the person becomes a consumer and, otherwise, becomes a producer. Producers stay in their home countries; a producer at stage 2 who produces y amount of the stage-2 good experiences disutility $c(y)$. The function c is strictly increasing, convex, twice differentiable, and satisfies $c(0) = 0$. The second idiosyncratic shock determines where a consumer consumes at the next stage 2. In each country, a constant fraction become tourists for one date and will consume in the other country at the next stage-2. The fraction is the same for both countries and its magnitude does not matter. The utility for a consumer who consumes y at stage-2 is $\theta u(y)$, where θ is the aggregate preference shock which is described below. The function u is strictly increasing, strictly concave, twice differentiable, and satisfies

$u(0) = 0$. Later, we let $U(\cdot) \equiv u(c^{-1}(\cdot))$ and assume that $\beta U'(0) > 2 - \beta$.

The aggregate shock θ depends on where stage-2 consumption will occur and follows an I-state Markov chain. We denote by $\theta_i = (\theta_i^1, \theta_i^2)$ a state in the chain, $i \in \{1, ..., I\}$, where θ_i^k is the preference shock for those who will consume in country-k, both tourists who visit country-k and non-tourist residents of country-k. We denote by π_{ij} the transition probability from state θ_i to state θ_j, $i, j \in \{1, ..., I\}$. The Markov chain is symmetric in that there exists a bijection $i \mapsto \sigma(i)$ on $\{1, ..., I\}$ such that $(\theta^1_{\sigma(i)}, \theta^2_{\sigma(i)}) = (\theta_i^2, \theta_i^1)$ for all i and $\pi_{ij} = \pi_{\sigma(i)\sigma(j)} > 0$ for all i, j.[2]

We assume that (θ_i^1, θ_i^2) is in an ϵ-neighborhood of $(1, 1)$ for all i. This small-shock assumption is used in the existence proofs below to insure that the expected rate of return on either money in any state does not exceed $1/\beta$, a necessary condition for equilibrium in this model. We need something like this because the equilibrium conditions are I non-linear simultaneous equations under fixed exchange rates and $2I$ non-linear simultaneous equations under flexible exchange rates. Each proof of existence of an equilibrium applies the implicit function theorem and rests on some mild additional regularity conditions. Those conditions are stated in the respective proofs, all of which appear in the appendix.

To summarize, the realized period utility in a period of a person who becomes a consumer and who consumes x amount of the stage-1 good and y amount of the stage-2 good is $x + \theta u(y)$, while that of a person who becomes a producer and who consumes x amount of the stage-1 good and produces y of the stage-2 good is $x - c(y)$, where $y \geq 0$. (We interpret $x < 0$ as production.)

As hinted at above, a crucial aspect of our model depends on how stage-2 trade in a meeting between a producer and consumer is modeled. We want the trade to satisfy three conditions: it is in the pairwise core in meetings, both monies are valued, the two monies are not perfect substitutes. One way to accomplish that is to have the trade in a stage-2 meeting be the trade that maximizes the Nash product with a larger weight on the consumer the larger is the share of the payment that is in the form of the producer's home money. A simpler way, which we adopt, is to use the scheme in ZW: the *favored* asset is money of the producer's home country and the trade in a meeting is the result of the following two-step protocol: first, the buyer makes a *take-it-or-leave-it* offer subject to the offer consisting solely of the *favored* asset; then, the seller makes a *take-it-or-leave-it* offer for any remaining assets that the buyer may hold. As demonstrated in ZW, this produces a trade that is in the pairwise core for the buyer-seller meeting.

The *speculative-fringe* version is identical except for the specification of stage 1 and the timing of the realization of the shocks. In it, stage 1 has two substages. Before any shock realizations, there is a centralized market in which the objects traded are the two monies and the linear good. Then, all the shocks are realized and there is a

[2] A simple example that satisfies all our assumptions has two equally likely states with $\theta_1 = (\theta_-, \theta_+)$ and $\theta_2 = (\theta_+, \theta_-)$, where $\theta_- \leq 1 \leq \theta_+$ and $(\theta_- + \theta_+)/2 = 1$. This is the example studied in section 5.

centralized market in which only the two monies are traded. Although this version may seem contrived, it is designed to have a speculative fringe, while maintaining the LW property that the distribution of money holdings at the end of a date is not a state variable of the model. A model with the timing of the *no-speculative fringe* version but in which stage 1 is limited to trade only between the two monies would have the same kind of speculative fringe. In it, however, the state of the economy entering stage-1 would be the distribution of money holdings at the end of the previous period and the current aggregate shock.[3]

Stage-1 trade in both versions of the model is price-taking trade and the exchange rate is either flexible or fixed. When it is flexible, the quantity of each money is fixed and constant and the stage-1 market has to clear, meaning that the exogenous quantities of both monies are held in equilibrium at the end of stage 1 (or at the end of each substage of stage 1). When the exchange rate is fixed, each country participates at stage 1 by supplying unlimited amounts of its own money at the fixed exchange rate which is constant. Such intervention, which is the standard way of modeling a fixed exchange-rate regime, does not affect the *total* quantity of money and that total must be held in equilibrium. Of course, the model under fixed exchange rates is equivalent to the same model with a single money.

3 The no-speculative-fringe version

We start by defining a stationary equilibrium under each exchange-rate regime and establishing existence and uniqueness. Then we prove that the equilibria differ.

3.1 Equilibrium

In order to define equilibrium, let $m = (m_1, m_2) \in \mathbb{R}^2_+$ be an individual portfolio, where m_l is the amount of country-l money. Let $\phi_i = (\phi_i^1, \phi_i^2)$ be the price vector of monies in terms of the stage-1 good and let $v_i(m)$ be the value for a person of ending stage 2 of the period with portfolio m. Using a well-known feature of linear utility of the stage-1 good, we can express $m \mapsto v_i(m)$ as an affine function of the expected value of the portfolio m at the next stage 1. Limiting consideration to stationary equilibria in which the current stage-1 price of each money depends at most on the current realization of the aggregate shock, it follows that

$$v_i(m) = \beta(A + \sum_{j=1}^{I} \pi_{ij}\phi_j m), \tag{1}$$

where A is a constant and $\phi_j m$ is the inner product of the vectors ϕ_j and m.

[3] See Molico [10] and Zhu [12] and [13] for the study of such models without any stage-1 and without any aggregate shock. ZW describe and analyze settings with both stages, where stage 1 is restricted to asset trade. However, none of these have aggregate shocks.

Consider a consumer with m who consumes in country k at stage 2. The trade between that consumer and a producer residing in country k holding m' is determined by the following problem that is adapted from ZW.

Problem 1 *Let* $y \geq 0$ *(output),* $d = (d_1, d_2) \in [0, m_1] \times [0, m_2]$ *(payment), and* $\bar{\phi}_i = \sum_j \pi_{ij}\phi_j$*. First, let*

$$(\tilde{y}, \tilde{d}) = \arg\max_{(y,d)} [\theta_i^k u(y) - \beta\bar{\phi}_i d], \tag{2}$$

subject to $-y + \beta\bar{\phi}_i d \geq 0$*,* $d_k \in [0, m_k]$*, and* $d_{\text{-}k} = 0$*; next, let*

$$(y_i^k(m), d_i^k(m)) = \arg\max_{(y,d)} [-c(y) + \beta\bar{\phi}_i d] \tag{3}$$

subject to $\theta_i^k u(y) - \beta\bar{\phi}_i d \geq \theta_i^k u(\tilde{y}) - \beta\bar{\phi}_i\tilde{d}$*.*

At stage 1, a consumer who consumes in country k at stage 2 chooses

$$m_i^k = \arg\max_{m\geq(0,0)} [-\phi_i m + \theta_i^k u(y_i^k(m)) + v_i(m - d_i^k(m))]. \tag{4}$$

As noted above, existence of an equilibrium requires that following any realization of the aggregate state, any asset has a gross expected return that does not exceed $1/\beta$. Given this, it is without loss of generality to assume that producers choose $m = (0, 0)$ in the stage-1 market, a condition we impose as part of the definition of equilibrium. (It also follows that consumers specialize in the *favored* money, but we do not build that conclusion into the definition of equilibrium.) Therefore, the market-clearing condition is

$$\sum_{k=1}^{2} m_i^k = (1, 1) \tag{5}$$

under flexible exchange rates and is

$$\sum_{k=1}^{2}\sum_{l=1}^{2} m_{il}^k = 2, \tag{6}$$

under fixed rates. Here, we normalize the quantity of each money to be unity per consumer per country-of-residence and in (6), we normalize the fixed exchange rate to be unity.

Definition 1 *A positive price vector* $\phi = \{\phi_i\}_{i=1}^I$ *with* $\max_{i,j}(\phi_i^k/\phi_j^k) < 1/\beta$*,* $k \in \{1, 2\}$*, is a stationary flexible exchange-rate equilibrium if (5) holds for all* i *and is a stationary fixed exchange-rate equilibrium if* $\phi_i^1 = \phi_i^2$ *and (6) hold for all* i*.*

In order to establish existence of an equilibrium, we describe a stationary equilibrium by a system of equations in ϕ.[4] In order to show that this system has a solution when ϵ is sufficiently close to zero, we use the implicit function theorem and the knowledge that the system has a solution when $\epsilon = 0$.

[4]Readers who want to see both equilibria for a simple example should turn to section 5.

In order to describe the system of equations, we let ϕ be such that $\max_{i,j}(\phi_i^k/\phi_j^k) < 1/\beta$. Then, it is optimal for a producer to leave stage 1 with $m = (0,0)$ and for a consumer who consumes in country k at stage 2 to leave stage 1 only with country-k money.[5] Therefore, the consumer's problem in (4) can be simplified to

$$m_{ik}^k = \arg\max_{m_k}[\theta_i^k U(\beta\bar{\phi}_i^k m_k) - \phi_i^k m_k]. \tag{7}$$

Under flexible exchange-rates, it follows from (7) and the stage-1 market-clearing condition (5) that

$$F_i^k(\phi, \epsilon) \equiv \beta\bar{\phi}_i^k \theta_i^k U'(\beta\bar{\phi}_i^k) - \phi_i^k = 0. \tag{8}$$

When $\epsilon = 0$, the system of equations $F_i^k(\phi, \epsilon) = 0$, has a unique solution,

$$\phi(0) = ((\eta, \eta), ..., (\eta, \eta)) \in \mathbb{R}^{2I},$$

where η satisfies

$$\beta U'(\beta\eta) = 1.$$

Under fixed rates, it follows from (7) that $\phi_i^k = \beta\bar{\phi}_i^k \theta_i^k v'(\beta\bar{\phi}_i^k m_{ik}^k)$. Letting W be the inverse of U', we have

$$m_{ik}^k = \frac{1}{\beta\bar{\phi}_i^k} W(\frac{\phi_i^k}{\beta\bar{\phi}_i^k \theta_i^k}). \tag{9}$$

Let φ_i be the common value of ϕ_i^1 and ϕ_i^2, $\varphi = (\varphi_1, \varphi_2, ..., \varphi_I)$, and $\bar{\varphi}_i = \sum_j \pi_{ij}\varphi_j$; then the stage-1 market-clearing condition (6) becomes

$$G_i(\varphi, \epsilon) \equiv W(\frac{\varphi_i}{\beta\bar{\varphi}_i\theta_i^1}) + W(\frac{\varphi_i}{\beta\bar{\varphi}_i\theta_i^2}) - 2\beta\bar{\varphi}_i = 0. \tag{10}$$

Again, when $\epsilon = 0$, the system of equations $G_i(\varphi, \epsilon) = 0$, $1 \le i \le I$, has a unique solution

$$\varphi(0) = (\eta, \eta, ..., \eta) \in \mathbb{R}^I$$

As described in the proof in the appendix, the relevant Jacobian matrix evaluated at $(\varphi, \epsilon) = (\varphi(0), 0)$ is nonsingular provided that some mild regularity conditions hold.

Proposition 1 *Let the ϵ-neighborhood of (1,1) that contains θ_i be sufficiently small. Then under mild regularity conditions, there exists a unique stationary flexible exchange-rate equilibrium and a unique stationary fixed-rate equilibrium which are in a neighborhood of $\phi(0)$.*

[5] Suppose the consumer has some assets after step-1 of the trading protocol. Any step-2 trade in the protocol leaves the consumer indifferent between that trade and not trading. However, having some assets and not trading is costly because the rate of return is less than $1/\beta$.

This implies that the equilibrium is identical to that for a version in which consumers make *take-it-or-leave-it* offers and are subject to country-specific cash-in-advance constraints.

3.2 Nonequivalence

Fix $\epsilon > 0$ and consider the two stationary equilibria in Proposition 1. We say that the two equilibria are *equivalent* if they support the same stage-2 output in each state. We focus on stage-2 output, because a welfare criterion that weights people equally at the start of any period before the shocks are realized depends only on stage-2 output.

Now we show that the two equilibria are non equivalent. The main idea in the proof which follows is that equality between the stage-2 outputs in the two exchange-rate regimes requires that the two monies have the same expected rates-of-return under flexible exchange rates. In the case of *i.i.d.* shocks, the expected values of the two monies at the next stage 1 are equal and do not depend on the current stage-1 shock. But, the current aggregate shock implies that the two monies have different stage-1 values. Therefore, the expected rates of return cannot be equal. The argument in the case of non *i.i.d.* shocks is similar, but a bit more complicated.

Proposition 2 *The two equilibria in Proposition 1 are not equivalent unless there are no aggregate shocks; i.e., unless* $(\theta_i^k, \theta_i^k) = (1, 1)$ *for all* i.

Proof. Suppose, by way of contradiction, that they are equivalent. Let state i be the current aggregate state. Let δ_i be the (gross) one-period expected rate of return on money in the Proposition-1 fixed-rate equilibrium. Let γ_i^1 and γ_i^2 be those rates for monies 1 and 2, respectively, in the Proposition-1 flexible-rate equilibrium.

Under fixed rates, the problem in (7) stated in terms of real balances is

$$x_i^k = \arg\max_{x \geq 0} [-x + \theta_i^k U(\beta \delta_i x)] \tag{11}$$

The analogous problem under flexible rates is

$$z_i^k = \arg\max_{z \geq 0} [-z + \theta_i^k U(\beta \gamma_i^k z)] \tag{12}$$

By (11), it follows that x_i^k satisfies

$$1 = \theta_i^k \beta \delta_i U'(\beta \delta_i x_i^k). \tag{13}$$

Therefore,

$$\theta_i^1 \gtreqless \theta_i^2 \Leftrightarrow x_i^1 \gtreqless x_i^2. \tag{14}$$

By (12), z_i^k satisfies

$$1 = \theta_i^k \beta \gamma_i^k U'(\beta \gamma_i^k z_i^k). \tag{15}$$

By equivalence, we have $\delta_i x_i^k = \gamma_i^k z_i^k$. This, together with (13) and (15), implies $\gamma_i^1 = \gamma_i^2 = \delta_i$ and $x_i^k = z_i^k$.

Without loss of generality, suppose that $2\hat{\imath} = I$ and that $\theta_i^1 > \theta_i^2$ and $\sigma(i) = \hat{\imath} + i$ for $1 \leq i \leq \hat{\imath}$. By symmetry, $(z_{\sigma(i)}^1, z_{\sigma(i)}^2) = (z_i^2, z_i^1)$. By the definition of equilibrium, $z_i^k = \phi_i^k$. Therefore, $\gamma_i^k = \sum_{j=1}^{I} \pi_{ij} z_j^k / z_i^k$. It follows that

$$\gamma_i^1 = \sum\nolimits_{j=1}^{\hat{\imath}} [\pi_{ij} z_j^1 + \pi_{i\sigma(j)} z_{\sigma(j)}^1] / z_i^1$$

and

$$\gamma_i^2 = \sum_{j=1}^{\hat{\imath}} [\pi_{ij} z_j^2 + \pi_{i\sigma(j)} z_{\sigma(j)}^2]/z_i^2 = \sum_{j=1}^{\hat{\imath}} [\pi_{ij} z_{\sigma(j)}^1 + \pi_{i\sigma(j)} z_j^1]/z_{\sigma(i)}^1.$$

So by $\gamma_i^1 = \gamma_i^2$,

$$\sum_{j=1}^{\hat{\imath}} [\pi_{ij} z_j^1 + \pi_{i\sigma(j)} z_{\sigma(j)}^1]/z_i^1 = \sum_{j=1}^{\hat{\imath}} [\pi_{ij} z_{\sigma(j)}^1 + \pi_{i\sigma(j)} z_j^1]/z_{\sigma(i)}^1.$$

Rearranging, we have

$$\sum_{j=1}^{\hat{\imath}} \pi_{ij} [z_j^1 z_{\sigma(i)}^1 - z_{\sigma(j)}^1 z_i^1] = \sum_{j=1}^{\hat{\imath}} \pi_{i\sigma(j)} [z_j^1 z_i^1 - z_{\sigma(j)}^1 z_{\sigma(i)}^1]. \tag{16}$$

By (14), $x_i^k = z_i^k$, and $(z_{\sigma(i)}^1, z_{\sigma(i)}^2) = (z_i^2, z_i^1)$, the right-hand side of (16) is positive. But the left-hand side of (16) is nonpositive when

$$i \in \arg\max_{1 \leq j \leq \hat{\imath}} z_j^1 / z_{\sigma(j)}^1.$$

Hence, the two equilibria are not equivalent so long as there are aggregate shocks. ■

4 The speculative-fringe version

Now all current shocks are realized after the first market of stage-1 trade and before the second stage-1 market at which only the currencies are traded in a competitive market. We call the second stage-1 market a foreign-exchange market. In this version, the equilibrium conditions are a set of simultaneous equations if and only if the aggregate shock is serially correlated. Our formulation allows for serial correlation.

Let m and $v_i(m)$ be the same as in section 3, but let $\phi_i = (\phi_i^1, \phi_i^2)$ be redefined as the price vector of monies in terms of the stage-1 good on the first stage-1 market when θ_i is the aggregate state of the *previous* period. In this notation, the presence of the linear good in the first stage-1 market implies

$$v_i(m) = \beta(A' + \phi_i m) \tag{17}$$

for some constant A'. Consequently, Problem 1 can be adapted to this setting by replacing $\bar{\phi}_i$ with ϕ_i. Let e_i be the price of currency 2 in units of currency 1 on the present foreign-exchange market when θ_i is the *current* realized aggregate state; let $e = (e_1, e_2, ..., e_I)$. As in section 3, we use the normalization that the nominal exchange rate; *i.e.*, ϕ_i^2/ϕ_i^1 and e_i, is fixed at unity for all i under fixed rates.

Existence of equilibrium here requires not only $\bar{\phi}_i^k/\phi_i^k \leq 1/\beta$, but also $\phi_i^2/\phi_i^1 = e_i$. The second condition eliminates the possibility of arbitrage gains for producers between the current foreign-exchange market and the first stage-1 market of the next period. It must hold in any equilibrium in which producers as a group leave the foreign exchange market willing to hold both monies. In what follows, we restrict attention to ϕ and e satisfying these two conditions.

Consider an agent who enters the foreign-exchange market with the portfolio m when θ_j is the aggregate state of the present period. If the agent is a consumer who consumes in country k, their problem is

$$w_j^{c,k}(m) = \max_{n \geq (0,0)} [\theta_j^k u(y_j^k(n)) + v_j(m - d_j^k(n))] \tag{18}$$

subject to $n_1 + e_i n_2 = m_1 + e_i m_2$, where $(y_j^k(n), d_j^k(n))$ is the trade determined by Problem 1 when the consumer carries the portfolio n into a meeting. If the agent is a producer, their problem is

$$w_j^p(m) = \max_{n \geq (0,0)} v_j(n) \text{ subject to } n_1 + e_i n_2 = m_1 + e_i m_2. \tag{19}$$

Here we use the fact that the producer gets zero surplus from meeting an on-path consumer at stage 2.[6] Therefore, in the first stage-1 market when θ_i is the aggregate state of the last period, each agent from country-k chooses

$$m_i^k = \arg\max_{m \geq 0} \{-m\phi_i + 0.5 \sum\nolimits_j \pi_{ij}[\alpha w_j^{c,k}(m) + (1-\alpha) w_j^{c,-k}(m) + w_j^p(m)]\}, \tag{20}$$

where α and $1-\alpha$ are the agent's probability of consuming at home and in the other country, respectively, conditional on the agent being a consumer.

Under flexible exchange-rates, the first stage-1 market-clearing condition is

$$m_i^1 = m_i^2 = (0.5, 0.5) \tag{21}$$

and the foreign-exchange market-clearing condition is

$$\frac{\alpha n_j^{c,1}(m_i^1) + (1-\alpha) n_j^{c,1}(m_i^2)}{4} + \frac{\alpha n_j^{c,2}(m_i^2) + (1-\alpha) n_j^{c,2}(m_i^1)}{4} + \frac{n_j^p(m_i)}{2} = (1,1), \tag{22}$$

where $n_{jl}^p(m)$ is a solution to the problem in (19) and $n_j^{ck}(m)$ is a solution to the problem in (18). Under fixed rates, the stage-1 market-clearing condition is

$$\sum_{k=1}^{2} \sum_{l=1}^{2} m_{il}^k = 2 \tag{23}$$

and the foreign-exchange market-clearing condition is

$$\sum_{l=1}^{2} [\frac{n_{jl}^{c,1}(m_i)}{4} + \frac{n_{jl}^{c,2}(m_i)}{4} + \frac{n_{jl}^p(m_i)}{2}] = 2. \tag{24}$$

Then, we have the following definition.

Definition 2 *A pair of positive price vectors $\phi = \{\phi_i\}_{i=1}^I$ and $e = \{e_i\}_{i=1}^I$ with $\max_{i,j}(\phi_i^k/\phi_j^k) < 1/\beta$, $k \in \{1,2\}$, is a stationary flexible exchange-rate equilibrium if $\phi_i^1/\phi_i^2 = e_i$, (21) and (22) hold for all i and is a stationary fixed exchange-rate equilibrium if $\phi_i^1/\phi_i^2 = e_i = 1$, (23) and (24) hold for all i.*

The agent's problem in (20) can be simplified to

$$\max_{m \geq 0} \{-m\phi_i + 0.5 \sum\nolimits_j \pi_{ij}[\alpha \theta_j^k U(\beta m \phi_j) + (1-\alpha)\theta_j^{-k} U(\beta m \phi_j) + \beta m \phi_j]\}.$$

[6]This follows because the preference shocks are in a neighborhood of unity. This implies that in any stationary equilibrium, all rates of return are close to unity. That implies that people acquire money at the first-stage-1 market only because they may become consumers. And although they face uncertainty about their preferences as consumers, that uncertainty is small enough to make them choose an amount that implies that they leave the foreign exchange market only with the *favored* money and that they offer all of it in the first step of the stage-2 trading protocol.

Under fixed rates, this problem and the market-clearing condition (23) give rise to

$$\varphi_i = 0.5 \sum\nolimits_j \pi_{ij} \beta \varphi_j [\alpha \theta_j^k U'(\beta \varphi_j) + (1-\alpha)\theta_j^{-k} U'(\beta \varphi_j) + 1], \tag{25}$$

where φ_i, as above, stands for the common value of $\phi_i^1 = \phi_i^2$. Notice that (25) defines a system of equations in φ. When $\epsilon = 0$, the system has a unique solution $\varphi' = (\eta', \eta', ..., \eta') \in \mathbb{R}^I$, where η' is determined by

$$2 = \beta[U'(\beta\eta') + 1]. \tag{26}$$

As is well known, setting the nominal exchange rate equal to unity under fixed rates is a normalization. Indeed, under fixed rates, if $\phi_i^2/\phi_i^1 = e_i = \lambda > 0$, then the equilibrium ϕ_i, denoted $\phi_i(\lambda)$, can be recovered from the relationship,

$$\phi_i^1 + \phi_i^2 = (1+\lambda)\phi_i^1 = 2\varphi_i. \tag{27}$$

The equilibria for different λ have the same real allocations in stages 1 and 2. Now, however, if λ is sufficiently close to unity, then (ϕ, e) with $\phi_i = \phi_i(\lambda)$ and $e_i = \lambda$ is a flexible-rate equilibrium. The reason is that producers can and are willing to absorb or support the extra supply or demand of any money on the part of consumers in the foreign-exchange market. Therefore, we have the following existence result.

Proposition 3 *Let the ϵ-neighborhood of (1,1) that contains θ_i be sufficiently small. Then, under mild regularity conditions, there exists a unique stationary fixed exchange -rate equilibrium such that the price vector ϕ is in a neighborhood of (φ', φ'). In addition, there exists a continuum of stationary flexible exchange-rate equilibria, each of which has the price vector ϕ in the neighborhood of (φ', φ') and supports the same real allocation as the fixed exchange -rate equilibrium.*

The range of indeterminacy of the flexible exchange rate is easy to describe. It is implied by the different possible equilibrium portfolios of producers after the foreign-exchange market. The consumers who will consume in country-1 hold 1/4 of total wealth measured in units of the stage-1 good, as do those who will consume in country 2. In equilibrium, each consumer uses the foreign-exchange market so that all their wealth is in the form of the *favored* money. Producers hold half of wealth. Those fractions of wealth do not depend on last period's preference shock, which can affect total wealth, but not those fractions. The fraction of producer wealth in the form of country-1 money after the foreign-exchange market can range from zero to one. Hence, as a fraction of total wealth of everyone after the foreign-exchange market, holdings of country-1 money could range from 1/4 to 3/4. That implies that there is a flexible exchange-rate equilibrium for any constant exchange rate in the interval $[1/3, 3]$.[7]

As is well-known, given a continuum of constant exchange-rate equilibria under flexible exchange rates, it is easy to construct equilibria with random paths. One example is a path with one-time uncertainty of the following sort. Let λ_1, and λ_2

[7] Bounds arise because all trades are spot *quid pro quo* trades; there are no short sales.

with $\lambda_1 < \lambda_2$ be alternative constant equilibrium exchange-rates and let $\lambda' = \mu\lambda_1 + (1-\mu)\lambda_2$ for any $\mu \in [0,1]$. Then, for any fixed future date T, a path with $\lambda_t = \lambda'$ for $t < T$ and $\lambda_t = \lambda_1$ for $t \geq T$ with probability μ and $\lambda_t = \lambda_2$ for $t \geq T$ with probability $1-\mu$ is also an equilibrium. In this model, all such equilibria have the same real allocations because those who turn out to be producers are indifferent about all portfolios of the two monies chosen in the foreign-exchange market of stage-1 that have the same expected return. In a model in which stage-1 only has a foreign-exchange market, a similar kind of multiplicity is also present, but holders of money would not be indifferent among all portfolios with the same expected return and the uncertainty would affect welfare.

5 A two-state i.i.d. example

Here is an example in which we have closed-form solutions for the equilibria for each version of the model. There are two aggregate states: $\theta_1 = (\theta_+, \theta_-)$, $\theta_2 = (\theta_-, \theta_+)$ where $\theta_+ \geq 1 \geq \theta_-$ and $(\theta_- + \theta_+)/2 = 1$. Moreover, $\pi_{ij} = 1/2$ for all i, j.

5.1 The no-speculative fringe version

In this special case, the value of money under fixed exchange rates does not depend on the state. (The easiest way to see this is to use the equivalence between fixed exchange rates and a single money and to notice that the total demand for money does not depend on the state in this two-state example.) Therefore, under fixed rates the expected rate of return is constant at unity; i.e., $\delta_i = 1$ in (11). Letting $x_1^1 = x_+$ and $x_1^2 = x_-$, the fixed-rate equilibrium is given by

$$\theta_+ \beta U'(\beta x_+) = \theta_- \beta U'(\beta x_-) = 1. \tag{28}$$

It follows that $\beta x_+ > \beta x_-$; that is, output in the country with the high preference shock is higher than that in the country with the low preference shock.

Under flexible exchange-rates, the equilibrium rates of return satisfy $\gamma_i^k = 0.5(z_i^k + z_i^{-k})/z_i^k$. Letting $z_1^1 = z_+$ and $z_1^2 = z_-$, the equilibrium conditions are

$$z_+ = 0.5\beta\theta_+(z_+ + z_-)U'[0.5\beta(z_+ + z_-)] \tag{29}$$

and

$$z_- = 0.5\beta\theta_-(z_+ + z_-)U'[0.5\beta(z_+ + z_-)]. \tag{30}$$

Summing (29) and (30), we have

$$1 = 0.5\beta(\theta_+ + \theta_-)U'[0.5\beta(z_+ + z_-)]. \tag{31}$$

Let $\bar{z} = 0.5(z_+ + z_-)$. It follows from (31) that $\beta U'(\beta\bar{z}) = 1$ or that $\bar{z} = \eta$ (where $\beta\eta$ is the output when there are no shocks). So $z_+ = \theta_+\eta$ and $z_- = \theta_-\eta$. But, from $\gamma_i^k = 0.5(z_i^k + z_i^{-k})/z_i^k$, it follows that the rate of return in the country with the θ_+ realization is $1/\theta_+$ and that in the other country is $1/\theta_-$. Therefore, output in each country is always $\beta\eta$, which, as we have seen, does not happen under fixed rates.

Although not a main focus of our analysis, this example lends itself to a simple welfare analysis. Given that the welfare criterion assigns equal weights to people entering stage-1, welfare depends only on stage-2 output and is proportional to the expected value of $[\theta u(y(\theta)) - c(y(\theta))]$, the expected value of the gains from trade in stage-2 meetings. In this example under flexible exchange rates, that expected value is $u(\beta\eta) - c(\beta\eta)$. This is welfare under both regimes when there are no shocks. Hence, the welfare comparison between the two regimes in this two-state example comes down to the following. In a fixed exchange-rate regime (or one-money world), would a person rather be born into the economy without aggregate preference shocks or into the world with a mean preserving spread on the aggregate preference shocks. If the former, then the flexible exchange-rate regime is better; if the latter, then the fixed exchange-rate regime is better.

It follows that the welfare ranking of the two regimes in this example depends on the curvature of the function,

$$H(\theta) = [\theta u(y(\theta)) - c(y(\theta))], \tag{32}$$

where $y(\theta) = \beta x(\theta)$ and $x(\theta)$ satisfies $\theta\beta U'(\beta x(\theta)) = 1$ (see (28)). If H is convex, then fixed rates are better; if H is concave, then flexible rates are better.

In order to explore in some examples, let $c(x) = x$. Then W is the same as w (i.e., the inverse of the function u') and, by (28), $y(\theta) = \beta x(\theta) = w(\frac{1}{\beta\theta})$ and

$$H(\theta) = \theta u[w(\frac{1}{\beta\theta})] - w(\frac{1}{\beta\theta}).$$

Suppose $u(x) = 1 - ae^{-x}$ with $a > 1/\beta$. Then $u'(x) = ae^{-x}$ so $w(x) = -\ln(x/a)$ and

$$H(\theta) = 1 - \theta a e^{\ln(\frac{1}{a\beta\theta})} + \ln(\frac{1}{a\beta\theta}) = 1 - \frac{1}{\beta} - \ln a\beta\theta,$$

which is convex. (The function H is also convex in some others examples we tried: $u(x) = a\ln(x+1)$ with $a > 1/\beta$ and $u(x) = x^{1/2}$.) Therefore, for this example, the fixed exchange-rate regime is better.

Not too much should be made of this finding because it depends on the example and on every feature of the model, including the trading protocol which was chosen to get imperfect substitutability between home and foreign currency in a simple way. One reason the welfare analysis is complicated is because all the equilibria have stage-2 outputs that are too low—outputs in the neighborhood of $\beta\eta$, where $U'(\beta\eta) = 1/\beta$. An optimum would require that outputs be in a neighborhood of y^*, where $U'(y^*) = 1$.[8]

5.2 The speculative-fringe version

In this version, if the aggregate shock is *i.i.d.*, then a stationary equilibrium under fixed exchange rates is given by the solution to a single equation in one unknown. In the two-state example, that equation, (25), is

$$x/4 = \beta x[\alpha\theta_+ U'(\beta x) + (1-\alpha)\theta_- U'(\beta x) + 1] + \beta x[\alpha\theta_- U'(\beta x) + (1-\alpha)\theta_+ U'(\beta x) + 1],$$

[8] For an analysis of trading protocols and their welfare consequences in the LW model, see Hu *et. al.* [3] and Hu and Rocheteau [4].

where x is the unit price of each money in terms of the stage-1 good. It follows that x satisfies $1 = 0.5\beta[U(\beta x) + 1]$. Therefore, $x = \eta'$, the unit price of money when there is no aggregate uncertainty (see (26)). And, in this two-state example, welfare, the expected value of $[\theta u(y(\theta)) - c(y(\theta))]$, is $u(\beta\eta') - c(\beta\eta')$, which does not depend on how much aggregate uncertainty there is. And, of course, all of these conclusions about allocations hold for any one of the continuum of equilibria under flexible exchange rates.

6 Centralized stage-2-trade with country-specific cash-in-advance constraints

In order to convert our setting into one that is closer to Lucas [8], we here consider the no speculative-fringe version, but with price-taking trade at stage-2 and with country-specific cash-in advance (CIA) constraints. A quick way to show that this does not produce equivalence is to note that if the c function, the function that describes the stage-2 disutility of production, is linear, then the matching model and this centralized version are identical. After all, the trading protocol we used in the matching model leads consumers to hold only the *favored* money, the money that satisfies the country-specific CIA constraint. And, although the trading protocol in the matching model gives consumers monopsony power in meetings, that power is, of course, ineffective if the c

function is linear. Although we could stop there, we want a more systematic approach in order to show that only a c function that converts stage-2 into an endowment economy produces equivalence.

Here, let m, ϕ_i, and $v_i(m)$ be the same as in section 3. Now, however, those objects are not sufficient to determine the stage-2 trades because individual decisions at stage-2 also depend on the price in the stage-2 market. Let ψ_i^k denote the real value of money k in the stage-2 spot market in country k and let $\psi_i = \{\psi_i^1, \psi_i^2\}$. As in the matching models, existence of an equilibrium requires that following any realization of the aggregate state, any asset has a gross expected return that does not exceed $1/\beta$. Therefore, it is again without loss of generality to assume that producers choose $m = (0, 0)$ in the stage-1 market, a condition we impose as part of the definition of equilibrium, and that consumers do not leave stage-2 with unspent money. Then the stage-1 problem of a consumer who will consume in country-k at stage-2 is to choose

$$m_i^k = \arg\max_{m^k \geq 0}[-m^k\phi_i^k + \theta_i^k u(m^k\psi_i^k)]. \tag{33}$$

The stage-1 market-clearing condition is

$$(m_i^1, m_i^2) = (1, 1) \tag{34}$$

under flexible exchange-rates and

$$m_i^1 + m_i^2 = 2, \tag{35}$$

under fixed rates. For the stage-2 market to clear in each country, we must also have

$$\psi_i^k c'(m_i^k \psi_i^k) = \beta \bar{\phi}_i^k, \tag{36}$$

the condition that equates the marginal cost of production to the relevant price.

Definition 3 *A pair of positive price vectors $\phi = \{\phi_i\}_{i=1}^I$ and $\psi = \{\psi_i\}_{i=1}^I$ with $\max_{i,j}(\phi_i^k/\phi_j^k) \leq 1/\beta$, $k \in \{1,2\}$, is a stationary flexible exchange-rate equilibrium if (34) and (36) hold for all i and is a stationary fixed exchange-rate equilibrium if $\phi_i^1 = \phi_i^2$, (35) and (36) hold for all i.*

Under flexible exchange rates, it follows from (33) and (34) that

$$\phi_i^k = \theta_i^k \psi_i^k u'(\psi_i^k). \tag{37}$$

By (36) and (34), we have

$$\psi_i^k c'(\psi_i^k) = \beta \bar{\phi}_i^k. \tag{38}$$

Equations (37) and (38) are a system of $4I$ equations in $4I$ unknowns, (ϕ, ψ). When $\epsilon = 0$, the solution is $(\phi, \psi) = (\phi^\circ, \psi^\circ)$, where

$$\phi^\circ = ((\rho u'(\rho), \rho u'(\rho)), ..., (\rho u'(\rho), \rho u'(\rho))) \in \mathbb{R}^{2I}$$

and

$$\psi^\circ = ((\rho, \rho), ..., (\rho, \rho)) \in \mathbb{R}^{2I}$$

with $\rho = c^{-1}(\beta\eta)$. (When $\epsilon = 0$, (37) and (38) imply $u'(x)/c'(x) = 1/\beta$), which, by the definition of η, implies that $x = \rho$.) Under fixed rates, let φ_i be the common value of ϕ_i^1 and ϕ_i^2. It follows from (33) that

$$\varphi_i = \theta_i^k \psi_i^k u'(m_i^k \psi_i^k), \tag{39}$$

or

$$m_i^k \psi_i^k = w(\frac{\varphi_i}{\theta_i^k \psi_i^k}), \tag{40}$$

where w is the inverse of u'. Substituting (40) into (35) and (36), we have

$$w(\frac{\varphi_i}{\theta_i^1 \psi_i^1})\psi_i^2 + w(\frac{\varphi_i}{\theta_i^2 \psi_i^2})\psi_i^1 = 2\psi_i^1 \psi_i^2, \tag{41}$$

and

$$\psi_i^k c'(w(\frac{\varphi_i}{\theta_i^k \psi_i^k})) = \beta \bar{\varphi}_i. \tag{42}$$

Equations (41) and (42) are a system of $3I$ equations in $3I$ unknowns, (φ, ψ). When $\epsilon = 0$, the solution is $(\varphi, \psi) = (\varphi^\circ, \psi^\circ)$, where $(\varphi^\circ, \varphi^\circ) = \phi^\circ$.

Proposition 4 *Let the ϵ-neighborhood of (1,1) that contains θ_i be sufficiently small. Then, under mild regularity conditions, there exist a unique stationary flexible exchange-rate equilibrium and a unique stationary fixed exchange-rate equilibrium that are in a neighborhood of (ϕ°, ψ°).*

Now fix $\epsilon > 0$ and consider the two equilibria in Proposition 4. As in section 4, we say that the two equilibria are *equivalent* if they support the same stage-2 output in each state.

Proposition 5 *The two equilibria in Proposition 4 are not equivalent.*

Proof. In this proof, we use tildes ($\sim$) to denote objects in the flexible exchange-rate equilibrium. In that equilibrium,

$$\theta_i^k \tilde{\psi}_i^k u'(\tilde{\psi}_i^k) = \tilde{\phi}_i^k \tag{43}$$

and

$$\tilde{\psi}_i^k c'(\tilde{\psi}_i^k) = \beta \sum_j \pi_{ij} \tilde{\phi}_j^k. \tag{44}$$

In the fixed exchange-rate equilibrium,

$$\theta_i^k \psi_i^k u'(m_i^k \psi_i^k) = \varphi_i \tag{45}$$

and

$$\psi_i^k c'(m_i^k \psi_i^k) = \beta \sum_j \pi_{ij} \varphi_j, \tag{46}$$

where φ_i denotes the common value of ϕ_i^1 and ϕ_i^2.

We first claim that the two equilibria are equivalent only if $m_i^1 = m_i^2 = 1$ in the fixed exchange-rate equilibrium. Because the two equilibria are equivalent,

$$\tilde{\psi}_i^k = m_i^k \psi_i^k. \tag{47}$$

Then (47), (43) and (45) imply $\tilde{\psi}_i^k / \psi_i^k = \tilde{\phi}_i^k / \varphi_i$. Using (47) again, this implies

$$\tilde{\phi}_i^k = m_i^k \varphi_i. \tag{48}$$

By (44) and (47), $m_i^k \psi_i^k c'(m_i^k \psi_i^k) = \beta \sum_j \pi_{ij} \tilde{\phi}_j^k$. This and (46) imply

$$\sum_j \pi_{ij} \tilde{\phi}_j^k = m_i^k \sum_j \pi_{ij} \varphi_i. \tag{49}$$

Substituting (48) (which holds for all i) and into (49), we have $\sum_j \pi_{ij} m_j^k \phi_j^k = m_i^k \sum_j \pi_{ij} \phi_j^k$ or

$$\sum_j \pi_{ij} (m_j^k - m_i^k) \phi_j^k = 0. \tag{50}$$

Notice that (49) can hold only if m_i^k is constant in i. For, otherwise, (49) cannot hold if $i \in \arg\max_{1 \leq j \leq I} m_j^k$. Because $m_i^k + m_i^{-k} = 2$ all i, it follows that $m_i^1 = m_i^2 = 1$.

Now, suppose by contradiction that the two equilibria are equivalent. By the above claim, $m_i^1 = m_i^2 = 1$. So by (45),

$$\theta_i^1 \gtrless \theta_i^2 \Leftrightarrow \psi_i^1 \gtrless \psi_i^2.$$

But by (46), $\psi_i^1 = \psi_i^2$, a contradiction. ■

Why, then, is there a presumption that flexible prices implies equivalence between the two exchange-rate regimes? We suspect that it arises from the use of models in which nothing real depends on the rates of return of the two monies. In our model, consumers are not directly affected by the future exchange rate because they spend all their money holdings in current transactions. However, producers are directly affected because they carry money into the next period. Therefore, flexibility of prices is not sufficient to maintain the optimizing conditions for both consumers and producers. That is, if we let the current real exchange rates be the same in the two regimes in order to obtain the same consumption, then the expected future exchange rates cannot be arranged to satisfy the conditions for the choice of production. Indeed, this is how the contradiction is derived in the proof of Proposition 5.[9]

We could get equivalence by adopting a special case of our model that eliminates the influence of future exchange rates on the producer's stage-2 decision. The specification is $c(y) = 0$ for $y \in [0, \omega]$ and $c(y) = \infty$ for $y > \omega$, which, eliminates any decision by producers.[10] In this specification, (42) is no longer an equilibrium condition and consumption is ω, the endowment of producers. Then, we can equate $\tilde{\psi}_i^k$ in (43) to ψ_i^k in (45) and set $\tilde{\phi}_i^k$ equal to φ_i. Of course, if we adopt this form for c, then we convert the model into a classical-dichotomy model because we fix exogenously the consumptions that determine welfare.

7 Concluding remarks

Our models are special in at least three respects. First, there is no direct substitutability between home and foreign goods. Such substitutability could be introduced by letting the heterogeneous shock that determines *tourism* status be less extreme. Instead of directly determining tourism status, it could determine a degree of relative preference between home and foreign goods. Then, tourist status would be endogenous and would interact with returns on the two monies under flexible exchange rates.

Second, in our model all trade is spot *quid pro quo* trade for money. Such trade is rationalized by assuming an extreme version of imperfect monitoring; namely, that the past actions of each person are private information to the person. If we want a model in which spot trade for money is *essential*, then some imperfect monitoring is necessary, but it need not be so extreme.[11]

Third, our trading protocol produces an extreme version of imperfect substitutability between monies: all trades of money for goods involve only the home money of the producers. For the *no-speculative-fringe* version, we are confident that there are trading protocols that produce trades in the pairwise core and give rise to a

[9]What happens if we drop the country-specific CIA constraints in this model of centralized trade? That depends on what we put in its place. We would favor having the monies be perfect substitutes at the exchange rate in the next stage-1 market. If so, then under flexible exchange rates there is exchange-rate indeterminacy of the kind displayed in Kareken and Wallace [5].

[10]This kind of producer is just like the worker from the shopper-worker family who sells the family endowment in Lucas [8]. In this case, our optimizing conditions for consumers imply a condition that resembles the condition in Lucas [8] on the marginal rate of substitution for his representative consumer.

[11]See Araujo and Hu [1] for an analysis of credit in a version of the LW model.

less extreme version of imperfect substitutability—one in which some trades involve the money that is not the home money of the producer. However, we decided that the benefit of achieving a less extreme pattern of transactions was more than offset by the added complexity of using such protocols.

In any case, we suspect that our results will survive various generalizations in those and other directions. The results are negative in the sense that they show nonequivalence between the sets of equilibria under fixed and flexible exchange-rate regimes. Because they are negative in that sense, it is unlikely that generalizations of the model will overturn them.

Appendix

Proof of Proposition 1

Let $B = \beta[U'(\beta\eta) + \beta\eta U''(\beta\eta)]$ and assume that the following regularity condition holds:

Condition 1 $(\pi_{ii}B - 1)(\pi_{jj}B - 1) \neq \pi_{ij}\pi_{ji}B^2$, *all* $1 \leq i < j \leq I$.

The partial derivative of $F_i^k(.)$ with respect to ϕ_j^k is

$$\beta\pi_{ij}\theta_i^k[U'(\beta\bar{\phi}_i^k) + \beta\bar{\phi}_i^k U''(\beta\bar{\phi}_i^k)]$$

if $j \neq i$ and is

$$\beta\pi_{ii}\theta_i^k[U'(\beta\bar{\phi}_i^k) + \beta\bar{\phi}_i^k U''(\beta\bar{\phi}_i^k)] - 1$$

if $j = i$. Evaluated at $(\phi, \epsilon) = (\phi(0), 0)$, this derivative is $\pi_{ij}B$ if $j \neq i$ and $\pi_{ii}B - 1$ if $j = i$. Therefore, the Jacobian matrix composed of the partial derivatives of $(F_1^k(.), F_2^k(.), ..., F_I^k(.))$ with respect to $(\phi_1^k, .., \phi_I^k)$, evaluated at $(\phi, \epsilon) = (\phi(0), 0)$, is

$$B\mathbf{\Pi} - \mathbf{I}, \tag{51}$$

where $\mathbf{\Pi} \equiv [\pi_{ij}]$ is the transition matrix of aggregate states and $\mathbf{I}$ is the $I \times I$ identity matrix. The partial derivative of $G_i(.)$ with respect to φ_j is

$$\frac{\varphi_i\pi_{ij}}{\beta(\bar{\varphi}_i)^2\theta_i^1}W'(\frac{\varphi_i}{\beta\bar{\varphi}_i\theta_i^1}) + \frac{\varphi_i\pi_{ij}}{\beta(\bar{\varphi}_i)^2\theta_i^2}W'(\frac{\varphi_i}{\beta\bar{\varphi}_i\theta_i^2}) - 2\beta\pi_{ij}$$

if $j \neq i$ and

$$\frac{\varphi_i\pi_{ij} + \bar{\varphi}_i}{\beta(\bar{\varphi}_i)^2\theta_i^1}W'(\frac{\varphi_i}{\beta\bar{\varphi}_i\theta_i^1}) + \frac{\varphi_i\pi_{ij} + \bar{\varphi}_i}{\beta(\bar{\varphi}_i)^2\theta_i^2}W'(\frac{\varphi_i}{\beta\bar{\varphi}_i\theta_i^2}) - 2\beta\pi_{ii}$$

if $j = i$, where, as in the text, W is the inverse of U'. Evaluated at $(\varphi, \epsilon) = (\varphi(0), 0)$, this derivative is $\pi_{ij}2[\frac{1}{\beta\eta}W'(\frac{1}{\beta}) - \beta]$ if $j \neq i$ and $\pi_{ii}2[\frac{1}{\beta\eta}W'(\frac{1}{\beta}) - \beta] + 2\frac{1}{\beta\eta}W'(\frac{1}{\beta})$ if $j = i$. Therefore, because $W'(\frac{1}{\beta})U''(\beta\eta) = 1$ and $\beta U'(\beta\eta) = 1$, the Jacobian matrix composed of the partial derivatives of $(G_1(.), G_2(.), ..., G_I(.))$, with respect to φ is also $B\mathbf{\Pi} - \mathbf{I}$.

If $B = 0$ then $B \cdot \Pi - Id$ is invertible. So suppose $B \neq 0$ and consider the ith and jth columns of $B \cdot \Pi - Id$ with $j > i$. If either $\pi_{ii}B - 1 = 0$ or $\pi_{j,j}B - 1 =$

0, then these two columns are obviously linearly independent. So, suppose that $(\pi_{ii}B-1)(\pi_{jj}B-1) \neq 0$. It suffices to show that the ith and jth rows of these two columns are linearly independent or that

$$\begin{bmatrix} \pi_{ii}B-1 & \pi_{ij}B \\ \pi_{ji}B & \pi_{jj}B-1 \end{bmatrix}$$

is invertible. This is ensured by condition 1. Hence, $B\mathbf{\Pi}-\mathbf{I}$ is invertible and the proposition follows from the implicit function theorem.

Proof of Proposition 3

Let $B' = 0.5\beta[U'(\beta\eta') + \beta\eta' U''(\beta\eta') + 1]$ and assume that the following regularity condition holds:

Condition 2 $(\pi_{ii}B'-1)(\pi_{jj}B'-1) \neq \pi_{ij}\pi_{ji}B'B'$, *all* $1 \leq i < j \leq I$.

Referring to (25), let

$$\bar{G}_i(\varphi,\epsilon) = 0.5\sum\nolimits_j \pi_{ij}\beta\varphi_j[\alpha\theta_j^k U'(\beta\varphi_j) + (1-\alpha)\theta_j^{-k}U'(\beta\varphi_j) + 1] - \varphi_i.$$

Notice that $\bar{G}_i(\eta',0) = 0$. Let

$$\begin{aligned} \chi_{ij} &= 0.5\pi_{ij}\beta[\alpha\theta_j^k U'(\beta\varphi_j) + (1-\alpha)\theta_j^{-k}U'(\beta\varphi_j) + 1] \\ &\quad +0.5\pi_{ij}\beta^2\varphi_j[\alpha\theta_j^k U''(\beta\varphi_j) + (1-\alpha)\theta_j^{-k}U'(\beta\varphi_j)]. \end{aligned}$$

The partial derivative of $\bar{G}_i(.)$ with respect to φ_j is χ_{ij} if $j \neq i$ and is $1+\chi_{ii}$ if $j = i$. The Jacobian matrix composed of the partial derivatives of $(\bar{G}_1(.), \bar{G}_2(.), ..., \bar{G}_i(.))$ with respect to $(\varphi_1, .., \varphi_I)$, evaluated at $(\varphi,\epsilon) = (\eta',0)$, is $B'\mathbf{\Pi}-\mathbf{I}$, where $\mathbf{\Pi}$ and $\mathbf{I}$ are the same as in (51). By the argument in the proof of Proposition 1, Condition 2 implies that $B'\mathbf{\Pi}-\mathbf{I}$ is invertible. Therefore, the implicit function theorem applies.

Proof of Proposition 4

Recall that $\rho = c^{-1}(\beta\eta)$ where η satisfies $\beta U'(\beta\eta) = 1$, and that w is the inverse of u. Let $C_0 = c''(w(u'(\rho)))w'(u'(\rho))$ and $C_1 = c'(w(u'(\rho))) - u'(\rho)C_0$. We assume that the following regularity conditions hold:

Condition 3 $[c'(\rho) + \rho c''(\rho)] \neq \beta\pi_{ii}[u'(\rho) + \rho u''(\rho)]$, *all i;*

Condition 4 $C_1 \neq 0$.

Referring to (37) and (38), let $\hat{F}_i^k(\phi,\psi,\epsilon) = (\hat{F}_i^{k,0}(\phi,\psi,\epsilon), \hat{F}_i^{k,1}(\phi,\psi,\epsilon))$ be defined by

$$\begin{aligned} \hat{F}_i^{k,0}(\phi,\psi,\epsilon) &= \phi_i^k - \theta_i^k\psi_i^k u'(\psi_i^k), \\ \hat{F}_i^{k,1}(\phi,\psi,\epsilon) &= \psi_i^k c'(\psi_i^k) - \beta\bar{\phi}_i^k. \end{aligned}$$

Notice that $\hat{F}_i^k(\phi^\circ, \psi^\circ, 0) = (0, 0)$. The partial derivative of $\hat{F}_i^{k,0}(.)$ with respect to ϕ_j^k is zero if $j \neq i$ and is one if $j = i$. The partial derivative of $\hat{F}_i^{k,1}(.)$ with respect to ϕ_j^k is $-\beta\pi_{ij}$. The partial derivative of $\hat{F}_i^{k,0}(.)$ with respect to ψ_j^k is zero if $j \neq i$ and is $-\theta_i^k[u'(\psi_i^k) + \psi_i^k u''(\psi_i^k)]$ if $j = i$. The partial derivative of $\hat{F}_i^{k,1}(.)$ with respect to ψ_j^k is zero if $j \neq i$ and is $c'(\psi_i^k) + \psi_i^k c''(\psi_i^k)$ if $j = i$. Hence, the Jacobian matrix composed of the partial derivatives of $(\hat{F}_1^k(.), \hat{F}_2^k(.), ..., \hat{F}_I^k)$ with respect to (ϕ, ψ), evaluated at $(\phi, \psi, \epsilon) = (\phi^\circ, \psi^\circ, 0)$ is

$$\begin{bmatrix} \mathbf{I} & \vdots & -[u'(\rho) + \rho u''(\rho)]\mathbf{I} \\ \cdots & \cdots & \cdots \\ -\beta\mathbf{\Pi} & \vdots & [c'(\rho) + \rho c''(\rho)]\mathbf{I} \end{bmatrix}, \tag{52}$$

where $\mathbf{\Pi}$ and $\mathbf{I}$ are the same as in (51). By its structure, the matrix in (52) is invertible if its ith and $(i + I)$th columns are linearly independent for $1 \leq i \leq I$. This is the case if the ith and $(i+1)$th rows of these two columns constitute an invertible matrix. That, in turn, follows from Condition 3.

Referring to (41) and (42), let $\hat{G}_i(\varphi, \psi, \epsilon) = (\hat{G}_i^0(\varphi, \psi, \epsilon), \hat{G}_i^1(\varphi, \psi, \epsilon), \hat{G}_i^2(\varphi, \psi, \epsilon))$ be defined by

$$\begin{aligned} \hat{G}_i^0(\varphi, \psi, \epsilon) &= w(\frac{\varphi_i}{\theta_i^1 \psi_i^1})\psi_i^2 + w(\frac{\varphi_i}{\theta_i^2 \psi_i^2})\psi_i^1 - 2\psi_i^1\psi_i^2, \\ \hat{G}_i^1(\varphi, \psi, \epsilon) &= \psi_i^1 c'(w(\frac{\varphi_i}{\theta_i^1 \psi_i^1})) - \beta\bar{\varphi}_i, \\ \hat{G}_i^2(\varphi, \psi, \epsilon) &= \psi_i^2 c'(w(\frac{\varphi_i}{\theta_i^2 \psi_i^2})) - \beta\bar{\varphi}_i. \end{aligned}$$

Notice that $\hat{G}_i(\varphi^\circ, \psi^\circ, 0) = (0, 0, 0)$. The partial derivative of $\hat{G}_i^0(.)$ with respect to φ_i is zero if $j \neq i$ and is $w'(\frac{\varphi_i}{\theta_i^1\psi_i^1})\frac{\psi_i^2}{\theta_i^1\psi_i^1} + w'(\frac{\varphi_i}{\theta_i^2\psi_i^2})\frac{\psi_i^1}{\theta_i^2\psi_i^2}$ if $j = i$. The partial derivative of $\hat{G}_i^k(.)$ with respect to φ_i is $-\beta\pi_{ij}$ if $j \neq i$ and is $\frac{\psi_i^k}{\theta_i^k\psi_i^k}c''(w(\frac{\varphi_i}{\theta_i^k\psi_i^k}))w'(\frac{\varphi_i}{\theta_i^k\psi_i^k}) - \beta\pi_{ii}$ if $j = i$ for $k = 1, 2$. The partial derivative of $\hat{G}_i^0(.)$ with respect to ψ_j^1 is zero if $j \neq i$ and is $-w'(\frac{\varphi_i}{\theta_i^1\psi_i^1})\frac{\psi_i^2}{\theta_i^1(\psi_i^1)^2} + w(\frac{\varphi_i}{\theta_i^2\psi_i^2}) - 2\psi_i^2$ if $j = i$. The partial derivative of $\hat{G}_i^0(.)$ with respect to ψ_j^2 is zero if $j \neq i$ and is $w(\frac{\varphi_i}{\theta_i^1\psi_i^1}) - w'(\frac{\varphi_i}{\theta_i^2\psi_i^2})\frac{\psi_i^1}{\theta_i^2(\psi_i^2)^2} - 2\psi_i^1$ if $j = i$. The partial derivative of $\hat{G}_i^k(.)$ with respect to ψ_j^k is zero if $j \neq i$ and is $c'(w(\frac{\varphi_i}{\theta_i^k\psi_i^k})) - \frac{\varphi_i\psi_i^k}{\theta_i^k(\psi_i^k)^2}c''(w(\frac{\varphi_i}{\theta_i^k\psi_i^k}))w'(\frac{\varphi_i}{\theta_i^k\psi_i^k})$ if $j = i$; and the partial derivative of $\hat{G}_i^k(.)$ with respect to ψ_j^{-k} is zero. Hence, the Jacobian matrix composed of the partial derivatives of $(\hat{G}_1(.), \hat{G}_2(.), ..., \hat{G}_I(.))$ with respect to (φ, ψ), evaluated at $(\varphi, \psi, \epsilon) = (\varphi^\circ, \psi^\circ, 0)$ is

$$\begin{bmatrix} 2w'(u'(\rho))\mathbf{I} & \vdots & C_2\mathbf{I} & \vdots & C_2\mathbf{I} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\beta\Pi + C_0\mathbf{I} & \vdots & C_1\mathbf{I} & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\beta\Pi + C_0\mathbf{I} & \vdots & \mathbf{0} & \vdots & C_1\mathbf{I} \end{bmatrix}, \tag{53}$$

where $\mathbf{\Pi}$ and $\mathbf{I}$ are the same as in (51), $C_2 = -w'(u'(\rho))/\rho + w(u'(\rho)) - 2\rho$, and $\mathbf{0}$ is the $I \times I$ zero matrix. By Condition 4, the matrix in (53) is invertible if its ith, $(i + I)$th, and $(i + 2I)$th columns are linearly independent for $1 \leq i \leq I$, which is clearly the case.

Therefore, we obtain the proposition from the implicit function theorem.

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