Multi-Unit Auctions: A Survey of Theoretical Literature

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Multi-unit Auctions: A Survey of Theoretical Literature

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Abstract

This article aims to provide a comprehensive survey of the theoretical research on multi-unit auctions to help identify the gap in this literature. Multi-unit auctions have been extensively used in practise and account for significant amount of transactions in some real-world markets. However, the theoretical research on these auctions has attract less attention compared to single unit auctions. The focus of this article is on those research that study multi-unit auctions for the sale of multiple units of homogeneous objects to potential buyers with more than one unit demand. The articles are categorized based on the assumptions of their models regarding bidders' values and the type of auction. Further the gap in this literature is identified with those areas that require further theoretical research.

Keywords: Auctions; multi-unit; uniform; discriminatory.

JEL Classification: D44.

1 Introductions

Multi-unit auctions or multiple object auctions refer to those auctions where multiple units are available for sale in a single auction. They are commonly used in many real-world

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markets to sell variety of goods such as treasury bills, emissions permits, electricity and spectrum licenses. The common feature of these markets is that there are multiple units of a homogeneous good available for sale to a group of potential buyers. Buyers usually demand multiple units and therefore submit a schedule of bids in the auction. Despite the massive use of these auctions in real-world markets, they are significantly understudied compared to single-unit auctions. This is the first article that specifically considers the theoretical work on multi-unit auctions and provide a survey of the theoretical literature related to these auctions.[1] The aim is to identify the gap in this literature and to recognise those areas that require further theoretical analysis.

Single unit auctions are well understood and surveyed (Klemperer, 1999; Krishna, 2009). For instance, two preliminary studies are Vickrey (1961) and Milgrom and Weber (1982) where authors provide results regarding to the comparison of efficiency and revenue of standard single-unit auctions. Unlike single-unit auctions where bidders compete through price bids, in multi-unit auctions bidders compete in both price and quantities which greatly increases the bidders' strategy space. Therefore, as Vickrey (1961) suggests most of the results for single-unit auctions are not extendible to the case with multi units. As a result, the strategic nature of bidding in multi-unit auctions is still an area of active research. To date, the literature suggests ambiguous theoretical results and mixed empirical results regarding the comparison of efficiency and revenue ranking of famous multi-unit auction formats. Therefore, there is no strong revenue equivalence theorem for multi-unit auctions. Also, there is no clear revenue and efficiency rankings of auction formats.

This paper is a survey of the current theoretical literature on multi-unit auctions with homogeneous goods. Section 1 gives an overview of the components of multi-unit auctions, Section 2 describes examines common value auctions and Section 3 examines private value models.

2 Overview of multi-unit auctions

This section identifies the fundamental components of auction models and describes the different assumptions used in multi-unit auction literature. Theoretical papers define a stylised model of an auction in either discrete or continuous space, and establish Nash or Bayesian-Nash equilibrium for bidding behaviour. The final allocation of goods and the bidder payments are of interest in determining the auction's allocative and revenue efficiency. Multi-unit auctions have the following characteristics:

1. A seller offers multiple units of a good to a number of bidders.

[1] Kwasnica and Sherstyuk (2013) survey the experimental research on multi-unit auctions.

2. Bidders have a vector of valuations for the units.

3. Bidders submit bids for the units they demand.

4. Based on an allocation rule the units are allocated among bidders.

5. The payment rule determines the payment of each buyers.

2.1 Items for sale

The good for sale can be either an infinitely divisible quantity of a good, or an integer number of units of an indivisible good. The former lends itself to analysis in continuous space whereas the latter is discrete. In the majority of papers the quantity of the good is fixed and known (Krishna, 2009), however, some papers examine the case where supply is uncertain. Supply uncertainty can be exogenous, as is the case when a random non-competitive bid wins some units with certainty (Kremer and Nyborg, 2004b; Wang and Zender, 2002). In this instance, bidders consider the supply to be random but with a known distribution. Supply can be determined endogenous when the seller strategically chooses quantity offered to increase revenue. The seller may choose the supply after observing the bids (Back and Zender 2001) or pre-commit to a known supply curve prior to bidding (Khezr and MacKenzie, 2018b; LiCalzia and Pavan, 2005). In either case, the seller's behaviour is taken into account by the bidders when constructing their bidding strategies.

2.2 Valuations

Auctions are commonly used when a seller is uncertain about the value of goods. Bidders may have a common value, private values or a combination of common and private values. Also it is possible that the values of bidders are affiliated as described in, Milgrom and Weber (1982). Furthermore, the bidders' certainty of the value may vary based on the known distribution of the value of the item and/or whether they have received a signal indicating the item's value. Thus, bidders may have common knowledge and play a game of complete information, or have private knowledge and play a game of incomplete information. In this paper we study the multi-unit auction literature on common value models in Section 3 and then investigate private value models in Section 4.

In common value auctions, the good has the same value to all bidders and it is usual to assume constant marginal valuations for all units. Examples of common value goods are treasury bills that can be traded in a liquid secondary market. When bidders do not know the value of the good but receive a signal related to the actual common value the good is a 'risky assets' and bidders form an expectation for the value of the goods. When bidders receive the same signal they are symmetrically informed and form the same expected value for the goods. If they receive different signals they are asymmetrically informed and form a conditional expectation for the value of the good.

In a single unit auction, winners' curse reflects the fact that a bidder's expected value of the item, conditional on winning, is less than the unconditional expected value. In the multi-unit case the generalised winners' curse takes into account that winning more units reveals even worse news about the true value of the object. For instance, as Ausubel et al. (2014) show, bidders behave strategically and shade their bids to avoid winners' curse.

In private value auctions, the value a bidder assigns to the items is bidder's private information. Unlike the complete information case where bidders know each others' valuations, in incomplete information case, bidders only know the distribution of the other valuations but do not know the realisations. Bidders may be symmetric or asymmetric. In asymmetric auctions, the distribution of bidders' values are asymmetric and one may stochastically dominate the others Baisa and Burkett (2018). Bidders may have constant or diminishing marginal valuations for the items are usually capacity constrained.

Some papers consider circumstances when items may be substitutes or complements and where some bidders prefer specific units, or value combinations differently from their parts (McMillan, 1994). This is relevant when goods are similar but heterogeneous such as spectrum licenses. Heterogeneous items can be auctioned using combination auctions (or menu auctions) where bidders submit different bids for different combinations of items (Bernheim and Whinston, 1986). These types of auctions are beyond the scope of this paper, which focuses on homogeneous goods for which valuations are weakly diminishing.

2.3 Bidding

One common way of bidding in multi-unit auctions is the static bid or the seald-bid. In static auctions, bidders submit their bids once without knowing other bids. However, in dynamic auctions, there are multiple rounds of bidding and bidders revise their information in each round to reflect the bidding history. In static auctions, most models assume bidders submit a demand function across all or a fraction of the units, however, some models impose a quantity or tick restrictions on submitted bids (Kastl, 2011; Kremer and Nyborg, 2004a,b). In some cases a continuum of equilibrium outcomes are possible and a common approach in the literature is to identify the Pareto dominant equilibrium from the bidders perspective Back and Zender (1993).

In dynamic multi-unit auctions such as open ascending auctions the common way to model is to incrementally raise the auction price whilst bidders indicate the quantity they demand. When the aggregate demanded is equal to the supply the auction stops. Conceptually, a descending price auction is possible, but not explored in the literature. Multiple simultaneous single-unit English auctions are another form of dynamic multi-unit auctions. One of the advantages of dynamic auctions is the case where buyers have common values. The dynamic bidding would allow bidders to revise their valuations and reduces the risk of winners curse Ausubel and Cramton (2004).

2.4 Allocation rule

All conventional multi-unit auctions allocate units to bidders who submit the highest bids. In the case of infinitely divisible goods and continuous demand schedules. In the case where the aggregate demand is flat at the point of total supply of units there may be excess demand and a rationing rule is required to ration the units at margin. Most papers ration the marginal units using pro-rata or random allocation. If there are a large number of units, such as in models of treasury bills, then a pro-rata allocation of marginal units amongst marginal bids is often used (Back and Zender, 1993; Kastl, 2011; LiCalzia and Pavan, 2005). When there are a smaller number of units, a random allocation rule is more common (Bresky, 2013; Noussair, 1995). Rationing is not only applied to marginal units, for instance, Kremer and Nyborg (2004a) introduces the rationing rule where the total supply is allocated pro-rata amongst all winning bids.

2.5 Payment rule

In the multi-unit auction literature, the three most most common payment rules are the uniform-price, the discriminatory price and the Vickrey. The uniform-price auction, first introduced by Milton Friedman (Friedman, 1960), is the auction that utilise a uniform payment for all bidders. Most uniform-price auctions use the first rejected bid (the highest losing bid) as the market-clearing price, however, the last accepted bid (or the lowest winning bid) is also used (Burkett and Woodward, 2020).

The discriminatory price auction, also known as, pay-as-bid auction utilises a similar payment rule to the first-price auction for single units. This auction was initially used to sell treasury bills in the US during mid 1970s (Back and Zender, 1993). In the discriminatory auction each winning bidder pays their own bids for each unit won. The two most commonly used payment rules in practice are the uniform payment rule and discriminatory payment rule. The Vickrey payment rule is rarely used in practice but is a useful theoretical benchmark. Papers that compare the equilibria of the three payment rules often compare them on grounds of revenue and efficiency. The Vickrey payment rule requires bidders to pay the social opportunity cost of the units they purchase. This amount is calculated as the sum of highest losing bids placed by competing bidders (Krishna, 2009). One of the important properties of the Vickrey payment rule is that bidders' payments are independent of their bids. Thus it results in truthful bidding and efficient allocation of units.

There are other payment rules except the ones mentioned above. For instance, Ausubel and Cramton (2004) suggests a dynamic auction with a payment rule equivalent to the Vickrey auction that is efficient and induces truthful bidding. We further explain this payment rule in Section 5.

3 Common values

In common value auctions, all bidders have the same value for the units. They may know the value with certainty or they may receive a private or common signal of the actual value. All allocations of common goods are efficient, however, different auctions can have very different expected revenues. This section first assesses how allocation and payment rules affect bidders' strategies and resulting equilibira. It then examines how alternations to the auction design can eliminate low revenue equilibria in uniform-price auctions.

3.1 Uniform-price auction with common values

In the uniform-price auction, all bidders pay the same price for the units sold. In a sealed-bid uniform-price auction, bidders submit a demand schedule specifying the quantity demanded at each price. The demand schedules are aggregated and the price at which demand equals supply is the market price. Bidders know that the price is determined by the marginal bid, and they face a trade off between bidding higher to increase the chance to win more units and paying more for inframarginal units. In equilibrium, bidders face a residual supply curve and behave like monopsonists. They underbid on subsequent units to lower the price. Alternative terms used in the literature to describe the underbidding on subsequent units are bid shading and demand reduction (Wilson, 1979; Back and Zender, 1993).

Wilson (1979) was the first study to consider low revenue equilibria in the uniform-price auction. The model consisted of a perfectly divisible unit being auctioned to a fixed and known number of bidders who submit a sealed schedule of prices they were willing to pay for different sized shares of the unit. The paper finds that when bidders are risk-neutral and symmetric, there are equilibria in which the multi-unit auction (the share auction) results in significantly lower prices compared to a case where the whole share is sold in a unit-auction. This is the first evidence of bid shading and demand reduction in the uniform-price auction.

Back and Zender (1993) extended Wilson (1979) by allowing discontinuous bid schedules and found that uniform-price auctions can support equilibria such that the seller is indifferent to a case where all units are sold at a price equal to the reserve price. In the set of equilibria they suggest, each bidder submits a demand schedule such that the demand is equal to zero at prices above a threshold, downward sloping at price lower than that threshold but above the expected clearing price, and flat for prices below the clearing price. The demand schedules are such that no bidder has an individual incentive to deviate from the bidding strategy. Because winning additional units means raising the price of all the units and the marginal increase in price outweighs the marginal benefits. Also there is no incentives to win less items in order to reduce the total price given the slope of the demand schedules. Back and Zender (1993) show that all market prices between the reserve price and the common value are supportable as an equilibrium clearing price, but the lowest price, that is, the reserve price is the Pareto optimal outcome from the bidders' perspective. Therefore, the seller is indifferent to a case where all units are sold at a fixed price equal to the reserve price.

3.2 Discriminatory price auctions with common values

Milgrom and Weber (1982) study the equilibrium bidding of the first-price auction for a single unit. As suggested by Bukhchandani and Huang (1989) a similar characterization is applicable to the multi-unit discriminatory auction with common values. Back and Zender (1993) also study a discriminatory auction with common values. They suggest it is intuitive or rather obvious that bidders bid flatter demands than their actual demand. This is mainly because if they bid truthfully a zero payoff is guaranteed. They show in a multi-unit discriminatory auction, it is an equilibrium for bidders to bid equal to the optimal bid in a single unit first-price auction.

Some single-unit auctions allow bidders to submit joint bids in sealed-bid auctions because it facilitates information sharing, reduces the risk of winners' curse and increases bidder aggressiveness. It can also reduce the number of bidders, which reduces competition, aggressiveness and revenues. However, as long as the first effect outweighs the second then joint bidding can increase revenue. Levin (2004) recognises that when joint bidding is extended to multi-unit auctions a third effect occurs in which having fewer bidders encourages greater demand reduction as each bidding group has more monopsony power.

Holmberg (2009) and Anderson et al. (2013) are two studies that analyse the supply function equilibria of the pay-as-bid (discriminatory) auction. Holmberg (2009) show that in a case where cost function are non-decreasing there exist a single equilibrium for the pay-as-bid auction. Anderson et al. (2013) consider the case where costs are common knowledge and demand is uncertain. Given that in this case a pure strategy Nash equilibrium does not generally exist, they characterise the mixed strategy equilibrium in discriminatory auctions.

3.3 Uniform versus discriminatory auctions

The two most common auctions used in practice are the uniform and discriminatory auctions. The revenue ranking lists auction formats in terms of their expected revenue. The overall consensus in the literature is that the revenue ranking between the two auctions is ambiguous in general. However, different articles try to show different conditions under which each auctions result in higher expected revenue.

Back and Zender (1993) shows the low price equilibria of the uniform-price auction could result in the revenue supremacy of the discriminatory auction. Although the overall revenue ranking of the two auctions could be ambiguous because of the existence of multiple equilibria, they suggest the low price equilibrium is Pareto optimal for buyers. Therefore if buyers play their Pareto optimum strategies, then the discriminatory auction could result in larger expected revenues.

Wang and Zender (2002) compare a uniform and discriminatory auction when a common value asset is risky, bidders are asymmetrically informed and submit a sealed bid demand schedule. In the discriminatory auction, risk neutral bidders shade their bids and submit a flat demand schedule. The auction generates revenue equal to the expected value of the item. In contrast, a uniform-price auction generates the low revenue equilibria of Back and Zender (1993) and has lower expected revenue than the discriminatory auction. Wang and Zender (2002) show if bidders are risk averse and the supply is uncertain, then bidders in the discriminatory auction submit downward sloping demand schedules and the expected revenues are less than in the risk neutral case but are strictly greater than zero. In the uniform-price auction, the expected revenue increases compared to the risk neutral case and depends on the number of bidders, the degree of risk aversion, and the asset's expected value and variance. Depending on these variables it is possible for the uniform auction to have higher expected revenue.

Nyborg and Strebulaev (2004) consider the common value model where bidder marginal valuations are endogenously determined. Players enter the auction holding either a long or a short position in the underlying asset. After the auction, players may trade the good but short players must close their position. The paper aims to model the market for treasury bills however the described market could also be relevant to the market for carbon permits where firms surrender permits to offset their emissions on an annual basis. The model used in Nyborg and Strebulaev (2004) utilises a sealed-bid uniform-price auction where bidders submit a bid schedule. After the auction, a long player is said to hold monopoly power if the short players cannot close their position without purchasing units from the long player. If there is no monopolist then the asset trades at the common value. If a long player holds a monopoly position then they implement a squeeze and short players must purchase all the units they need to cover their short for price. The more units a long player holds the more likely they are to be able to implement a squeeze and so the marginal valuation schedule for long players is upward sloping. In contrast, short players place a higher valuation on their first units and less on later units so their marginal valuation schedules are downward sloping.

Further Nyborg and Strebulaev (2004) show in the uniform-price auction, the pure strategy equilibrium is for short players to bid for the units they are short and for other units. Long players cannot implement a squeeze, therefore bid for all units resulting in the equilibrium price being and a squeeze does not occur. In discriminatory auctions, players play mixed strategies and a squeeze occurs with positive probability. Short players bid a price somewhere between and for the units they are short and for additional units. A long player occasionally bids a price higher than for all units but most of the time bids for all units. The discriminatory auction raises greater expected revenue but has higher price volatility than the uniform price auction. The policy implication of the paper is when a squeeze is possible the auction designer faces a choice between lower expected revenue with lower price volatility or higher expected revenue and higher price volatility.

3.4 Eliminating low-price equilibria in uniform-price auctions

Revenue maximisation is one of the major goals of the seller in an auction. However, the existence of low and possibly zero revenue equilibria in uniform-price auctions raised several concerns for the researchers. Therefore, researchers have been motivated to investigate alternative auction rules that could possibly eliminate low-price equilibria. We divide the literature into two major categories: first, those that study alternative allocation rules, and second, those that study alternative supply adjustments.

3.4.1 Alternative allocation rules

For instance, Kremer and Nyborg (2004a) show that using an alternative allocation rule can eliminate under-pricing and Kremer and Nyborg (2004b) show that restricting bids to a finite number of price-quantity pairs can reduce under-pricing to an arbitrarily small amount.

Kremer and Nyborg (2004a) propose an alternative allocation rule to be used in auctions where ties occur with positive probability. Typically, if there is excess demand at the clearing price, bids above the clearing price are allocated in full and marginal bids are rationed. Low-revenue equilibria occur because, in equilibrium, each bidder submits a steep demand schedule and faces a steep residual supply curve. A bidder who increases his marginal bid expects to pay more for all inframarginal units, but only expects to receive a small number of additional marginal units. Consequently, it is not rational to deviate from their steep demand schedule.

The alternative allocation rule suggested by Kremer and Nyborg (2004a) is a pro-rata allocation of all bids that exceed the clearing price. It rewards aggressive marginal bids because bidders can receive a higher proportion of all units, not just the marginal units. When bidders are risk neutral, the unique equilibrium bidding strategy is to submit a flat demand schedule with a price equal to the expected common value. Equilibrium bidding strategies are independent of irrelevant demand/uncertain supply and expected revenue is equal to the expected value. If bidders are risk averse and the value of the common good is uncertain then using the alternative allocation rule reduces, but does not eliminate, under-pricing. If bids are discrete and restricted to tick sizes then the clearing price may be slightly lower but will be within where is the number of bidders and is the minimum price tick.

Returning to a model where only marginal bids are rationed, Kremer and Nyborg (2004b) find that under-pricing can be reduced by requiring bidders to submit a finite number of price and quantity pairs. The model includes risk neutral bidders competing for an uncertain supply of a perfectly divisible common value good with known value sold in a uniform price auction. The bidders' problem is to maximise their payoff, which is the value of the good minus the price paid, multiplied by the fraction of the good won. As in most real world auctions, bidders may only submit a finite number of price-quantity pairs. The paper finds that the equilibrium price is equal to the common value almost surely. The explanation for the finding is that once inframarginal bids are satisfied the residual supply being competed for has positive mass and so a bidder can increase their payoff by increasing their marginal bid negligibly. If bidders are restricted to tick sizes and quantity multiples then under-pricing is bound by a function that is decreasing in the number of bidders and the quantity multiple, and increasing in the tick size and quantity being sold. Removing either the tick size or the quantity multiple will eliminate under-pricing. If there is uncertainly in supply then under-pricing is eliminated. All results are robust to capacity constrained bidders.

Kastl (2011) examines the same auction format as Kremer and Nyborg (2004b) but with bidders who each receive a private signal of the value of the good. The paper finds that when bidders are restricted to a limited number of bids, bidders may find it optimal to bid above their marginal valuation for some units. Consequently, the Vickrey auction revenue is not the upper bound for uniform auction revenue. The rational is bidders face a trade off between paying above their valuation for some units, if their bundle is the marginal bid and they receive their full bundle, and receiving a greater number of higher value inframarginal units in the case of rationing at the margin.

3.4.2 Alternative supply adjustments

In some of the previously examined models the quantity supplied was exogenously determined and low-revenue equilibria existed. Some authors consider models where the supply is no longer fixed and exogenous to see the effects on low-price equilibria. For instance, Back and Zender (2001) consider a model where the seller chooses the quantity supplied to maximise revenue after bids are submitted. LiCalzia and Pavan (2005) and Khezr and MacKenzie (2018b) also examine models where the quantity supplied is determined after bids are submitted, however, sellers pre-commit to a supply function. All the papers found that allowing the quantity supplied to be endogenously determined can eliminate low revenue equilibria.

Back and Zender (2001) show that if a seller has the right to decide about the quantity of supply after bids are submitted then the equilibrium price will be significantly larger compared to previously identified low-price equilibria. The intuition is that low-price equilibria exist because bidders can submit lower demand curves and reduce competition without changing the total number of units won. However, when the seller can reduce supply ex-post to maximise revenue, any marginal demand reduction will be punished by less supply and larger prices. Therefore, the incentives to reduce demands would be significantly lower and in some cases close to zero.

LiCalzia and Pavan (2005) examines the case where the seller is required to pre-commit to a supply curve and finds that an increasing supply function creates a positive quantity effect that encourages price competition amongst bidders. The paper examines a linear supply function and finds that for any reserve, common value and number of bidders the seller can find a slope that eliminates under-pricing equilibria. When the supply is sufficiently elastic the under-pricing equilibria are no longer self-enforcing. The paper finds that neither a perfectly inelastic, nor a perfectly elastic supply curve is optimal and the results are robust to an increasing supply. Damianov (2005) is another study that analyses the uniform-price auction with endogenous supply. They show the low-price equilibria no longer exist when the supply is endogenous. Also there exist an equilibrium in which the Walrasian quantity and price is attained.

McAdams (2007) also study a case when the auctioneer decides about the actual number of units to sell after the bids are submitted. McAdams (2007) suggests in this case both the expected revenue and the social welfare could be higher than a case where the auctioneer pre-commits to a fixed quantity of units.

Khezr and MacKenzie (2018b) model the supply using a step function where either a low quantity or a high quantity is offered. There is a reserve price, below which no units sell. The first step occurs between the reserve price and a trigger price which is a price above which the extra units will be available. Their model of supply captures the effect of the cost containment reserve used in the US Regional Greenhouse Gas Initiative (RGGI) auction for carbon permits. In their model bidders have common value for the goods. However, they show the results are robust to a case of discrete private values. The auction is a sealed-bid uniform-price auction where bidders submit a full demand schedule of price quantity pairs and excess demand is rationed pro-rata on the margin. The paper finds it is still possible that a low-price equilibrium exists where the common value is below the trigger price. However, if the trigger price is chosen below the common value the low revenue equilibrium may be replaced by a high price equilibrium where all the units are sold at the trigger price. By using a step function the seller can gain greater revenues and firms may have lower payoffs.

Khezr and MacKenzie (2018a) study a special supply regime motivated by the California's market for emissions, where all players are endowed by fractions of total units and must consign these units into the auction. They show there exist e class of equilibria where firms are essentially indifferent between equilibrium prices since they will receive their own endowed units.

4 Private values

This section reviews those articles that allow bidders to have private values for the items being sold. Private valuations may be constant for additional units or diminishing and players may be symmetric or asymmetric. Because bidders have different values for the units, inefficient allocations may arise. The distributions of valuations are usually common knowledge and known by all parties. This section will examine uniform-price auctions with private values, discriminatory auctions with private values and will then compare the two in terms of revenue and efficiency.

4.1 Uniform auctions with private values

In the common value setting low (including zero revenue) equilibria are possible because it is rational for bidders to submit symmetric steep downward sloping demand schedules. Low revenue equilibria can also occur in the private value setting, however because of asymmetric valuations, a bidder may choose to outbid other bidders rather than share units.

Noussair (1995) considers a model where a fixed number of bidders are bidding for a fixed number of units and each bidder has weakly diminishing values for two units. All valuations are drawn from a common and known distribution and each bidder knows his own valuations. Bidders submit sealed bids for two units and the uniform-price auction uses the first-rejected-bid pricing rule. The paper finds that the dominant bidding strategy for a bidder is to bid their value for the first unit and underbid for the second unit. The bidders are symmetric and risk-neutral but because they have private values and shade their bids differently across the first and second units inefficient allocations occurs with positive probability.

The reason for the underbidding is the same as in the common value model. The bid for subsequent units represents the trade off between winning more units and paying a higher price for inframarginal units. Bidding the true value for the first unit is similar to bidding ones value in a second-price single unit auction. It increases the probability you win the unit at the price, but does not affect your payment. The bid for the second unit influences both the probability of winning a second unit and the expected price paid for the first unit. Bidders have an incentive to shade their second bid but do not have an incentive to shade their first bid.

Engelbrecht-Wiggans and Kahn (1998) uses the same auction model as Noussair (1995) and assumes that there aren't enough goods for all bidders to receive two units and that probability of ties is zero. The paper defines necessary conditions for pooling equilibria and raises the possibility of a zero-revenue equilibrium in which all bidders win at least one unit. This equilibrium exists even if one bidder's second valuation exceeded another bidder's first valuation meaning that in equilibrium allocations are inefficient. The existence of multiple equilibria, low-revenue equilibria and inefficient equilibria arises not because multiple units are being sold, but because bidders demand multiple units.[2]

Ausubel and Cramton (1995) study the uniform-price auction with interdependent values. They show in a flat demand setting the uniform-price auction is always inefficient. The only exception is a case where bidders are symmetric with similar capacities and the total capacity divided by individual capacity is an integer. Further they extend the model

[2] Khezr and Menezes (2017) provide an alternative characterisation of the bidding function that applies to Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998) models.

to a case with downward sloping demand and show the uniform-price continues to result in inefficient allocation of goods.

4.2 Discriminatory auctions with private values

In the discriminatory auction, bidders pay their bid for each item they buy and winning a marginal unit does not affect the price paid for inframarginal units. Bid shading can occur for every unit because bidders are motivated to make a surplus on the items they win. Ausubel and Cramton (1995) show that risk-neutral bidders who have flat marginal valuations submit flat bid strategies, but when marginal valuations are decreasing bid shading is greater for higher value items; resulting in a bid schedule that is flatter than the marginal valuation schedule. The paper finds that if bidders' valuations are interdependent and capacities are asymmetric, in general, discriminatory auctions are not efficient.

McAdams (2006) characterises monotone pure-strategy equilibrium of the discriminatory and the uniform-price auction with bidders that have multi-dimensional independent types and interdependent values. They show all mixed strategy equilibria have similar interim expected payment as well as ex post similar allocation as the pure-strategy equilibrium.

Anwar (2007) extends the result in Milgrom and Weber (1982) to a case of discriminatory auction with multi-units. In a two-bidder constant marginal valuations model, they show that there exists a unique equilibrium for the discriminatory auction that corresponds to the result in Milgrom and Weber (1982) for the single-unit first price auction.

4.3 Uniform-prices versus discriminatory auction

Ausubel and Cramton (1995) show that the revenue and efficiency rankings of the uniform, discriminatory and Vickrey auctions are in general ambiguous because they depend on the distributions of bidder valuations. In the case with flat demand and capacity constrained bidders, the Vickrey auction is efficient and revenue superior to the other two formats. Further, the uniform-price and the discriminatory auctions are not efficient and their revenue ranking is ambiguous. Additionally, when bidders' marginal values are smoothly decreasing, setting an optimal reserve price can increase revenue but there is a conflict between revenue maximisation and efficiency. Ausubel et al. (2014) builds on Ausubel and Cramton (1995) and presents a detailed comparison of the uniform-price and discriminatory auctions.

Tenorio (1999) shows there is no strong revenue equivalence theory for multi-unit auctions because the different auction formats result in different allocations. A weaker revenue equivalence concept exists that proposes that auctions which have the same equilibrium quantity profiles have the same expected revenues.

Baisa and Burkett (2018) study a case where a large bidder with multi-unit demand competes with many small bidders with single-unit demand. They show that the revenue ranking of the uniform-price and discriminatory auctions are ambiguous in general. However, the large bidder favours the discriminatory auction. They also provide the conditions such that small bidders prefer the uniform-price auction.

4.4 Eliminating low-revenue equilibria in private value uniform auctions

Bresky (2013) uses the same two-unit demand model as Engelbrecht-Wiggans and Kahn (1998) with symmetric risk-neutral bidders who have independent private values. They find that increasing the reserve price slightly above zero will increase both revenue and efficiency of the uniform-price auction. The paper considers two forms of inefficiency: 'misallocation efficiency' which occurs when lower valuation bidders receive units and the 'supply restriction' effect, which occurs when items fail to sell. A higher reserve price reduces the probability of misallocation efficiency, but increases the probability of supply restriction inefficiency. The two effects work in opposite directions, implying there is an optimal reserve price. An optimal reserve price will reduce the amount of bid shading on the second unit, prevent some inefficient allocations and increases the seller's revenue.

Tenorio (1999) uses a two-bidder, three-unit model and finds that imposing a quantity floor also increases expected revenue in a uniform price auction. In case bidders are required to bid for three units, they face full rationing and bid more aggressively compared to the case when they only bid for two units and were only partially rationed. Tenorio (1999) does not examine a more general case comparing two quantity floors that are both smaller than the total supply.

Burkett and Woodward (2020) with a relatively general model of private values, show that the undesirable properties of the uniform-price auction are closely related to the first losing bid rule as the payment rule. They suggest the alternative last accepted bid pricing rule results in an equilibrium that has relatively flat bids compared to the full information case. The equilibrium bids are a generalisation of the bids in a first-price auction with a single unit. Therefore price selection could be an additional tool to avoid undesirable low-price equilibria of the uniform-price auction.

Khezr and Menezes (2020) show if only the marginal bidder pays a different price than other bidders, then many zero-price equilibria of the uniform-price auction will be eliminated. In particular, if the marginal bidder who sets the price for all units does not pay her own bid, but pays the second-highest losing bid, then zero-price equilibria of the uniform-price would break because bidders who used to bid zero now have incentives to raise their bids without the cost of raising their own equilibrium price.

5 The effect of resale on revenue

When items can be resold after an auction, there is a possibility that a bidder with a low valuation, or even a pure speculator, increase their willing to pay because of an expectation of reselling to a buyer with a higher valuation after the auction. Conversely, when a buyer with a high valuation has demands for more than one unit, she may be willing to forgo the opportunity to bid higher for some items in the auction to keep the auction price low. This would result in paying less for the items she win in the auction and still have the chance to buy the extra units demanded after the auction. For instance, Pagnozzi (2009) examines the effects of resale and bundling in auctions and Pagnozzi (2010) examines the effects of allowing resale in the presence of speculators. Both papers conclude that the distribution of valuations is important to determine whether resale increases or decreases revenue.

Pagnozzi (2009) finds that allowing resale alone, or allowing bundling alone can have an ambiguous effect on seller revenue, but combining the two is a complementary strategy that can increase the seller's revenue as long as the valuations of weaker bidders and their bargaining power are not too much lower than the stronger bidder's. The model is a two-bidder, two-unit sealed bid auction. Each bidder has weakly diminishing marginal values and valuations are common knowledge amongst the bidders. If a trade occurs in the secondary market then gains from trade are shared in a pre-determined ratio.

Further Pagnozzi (2009) show that resale alone will reduce the seller's revenue if bidders have very different values and the strong bidder adopts an accommodation strategy by reducing their willingness to outbid a low value bidder. For example, consider the situation where a strong bidder values their second unit more than twice as much as a weak bidder values their first unit. If resale is not allowed then a strong bidder will outbid the weak bidder, but if resale is allowed then the goods will be split between the bidders in a zero price equilibrium and the stronger bidder will buy their second unit from the weaker bidder in the secondary market. If resale is inefficient (i.e. trades do not occur with positive probability) then allowing resale alone can induce inefficient allocations. If the no-resale equilibrium was a zero-revenue equilibrium then allowing resale does not change the outcome.

Pagnozzi (2009) also show that bundling without resale can have an ambiguous effect on revenue. Bundling without resale is equivalent to a single unit second-price auction where the winner pays the sum of the losers' valuations. If the unbundled equilibrium is a zero revenue equilibrium then bundling increases revenue. But if the bidders have very different values and the stronger bidder would have outbid the weaker bidder in an unbundled auction then the stronger player pays less in a bundled auction.

Combining bundling and resale can increase the seller's revenue. Allowing resale induces the weaker bidder to bid more aggressively and bundling prevents the stronger bidder participating in demand reduction. Pagnozzi (2009) concludes that if resale cannot be prevented then the seller can increase revenue by bundling the units on sale. Overall, if a seller is choosing between no bundling, bundling only, or bundling with resale, then bundling with resale yields the higher revenue as long as bidders are not too asymmetric.

Pagnozzi (2010) extends the resale model to allow units and bidders with constant private marginal valuations, which are common knowledge. The paper finds the dispersion of bidder valuations is relevant to whether resale will increase or decrease revenue. The paper includes speculators, which are bidders with zero value for the good but who may wish to participate in bidding when there is a resale market. Bidders submit bids, there is no reserve price and gains from resale are split equally. The paper finds that if bidders have similar valuations (clustered values) then allowing resale increases revenue. Speculators cause a competition effect during the bidding and allowing them to participate weakly increases revenue. If the highest bidders value the good sufficiently more than other bidders (dispersed top values) then allowing resale induces accommodating behaviour and reduces revenue.

6 Other multi-unit auctions

In this Section we introduce two more commonly known types of multi-unit auctions. First is a dynamic open ascending auction that results in truthful bidding and efficient allocation of units. Second, we discuss the generalized second-price auction which has been used for the allocation of online advertisement spots.

6.1 Ausubel auction

Ausubel (2004) introduces a dynamic auction which is now commonly known as the *Ausubel auction*. In the Ausubel auction the auctioneer announces a low price and bidders indicate how many units they want at that price. The auctioneer continuously raises the price until the residual demand faced by a bidder is less than the total supply. When this happens the bidder is said to have 'clinched' an item and is awarded the item at that price. The auction continues until all units are awarded to bidders. The price paid by the winner is independent of their bid and so the mechanism induces sincere bidding. In a model of pure private values, the dynamic auction results in the same outcome as the static Vickrey auction, but if there are affiliated or common values and private information then the information sharing aspect of the Ausubel auction increases revenue and efficiency because it reduces the generalised winners' curse.

If bidders in the Ausubel auction have budget constraints, the allocation process can relax binding budget constrains and be more efficient than the static Vickrey auction, however, overall the effect is ambiguous (Ausubel, 2004). Perry and Reny (2005) propose a variation to the Ausubel auction where bidders may submit different quantity bids against different bidders at each price. The alternative designs facilitates information discovery when bidders are heterogeneous and valuations are interdependent.

6.2 Generalized second-price auction

Edelman et al. (2007) is a preliminary study that analyses a new format of multi-unit auctions which is called the generalized second price auction. The generalised second price auction is a relatively new format which has been used for the auctions that allocate online advertisement spots. The auction has a similar payment rule to the single unit second-price auction, however with one major difference. There are more than one units available. Thus each winning bidder pays the next highest bid. Edelman et al. (2007) show that unlike the Vickrey auction, the generalized second price auction does not result in truthful bidding and does not have a dominant strategy equilibrium.

7 Conclusions and future works

This paper summarised the main theoretical results for multi-unit auctions. It has largely focused on uniform and discriminatory auctions and found that in general these two popular auction formats are not efficient. Additionally, their revenue rankings are ambiguous and require empirical research. In many circumstances uniform auctions admit seemingly collusive low or zero revenue equilibriums, however, endogenously varying supply or restricting bidders to thick and quantity increments can eliminate them.

Given the importance of these auctions in real-world markets the need for further theoretical research is evident. The main gap in the literature is with regards to the models where bidders have both private and diminishing marginal values for units. Although there may be several technical difficulties for studying such models, they could provide the best representation of real-world situations in some markets.

Furthermore, note that the main reason for popularity of the uniform-price and the discriminatory auctions is the simplicity of the payment rule. Uniform pricing has the advantage of price discovery as well. Therefore, further studies that can provide alternative allocation rules or supply adjustments are essential to improve the performance of these two auction by keeping their advantages.

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