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Woźny, Łukasz and Growiec, Jakub

Warsaw School of Economics, Poland

20 July 2008

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Łukasz Woźny[†] Jakub Growiec[‡]

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Abstract

We analyze an economy populated by a sequence of generations who decide over their consumption levels and the levels of investment in human capital of their immediate descendants. The objective of the paper is to identify the impact of strategic interactions between consecutive generations on the time path of human capital accumulation. To this end, we characterize the Markov perfect equilibrium (MPE) in such an economy and derive the sufficient conditions for its existence and uniqueness. The equilibrium path is computed using a novel constructive approach: extending Reffett and Woźny (2008), we put forward an iterative procedure which converges to the MPE as its limit.

To benchmark our results, we also calculate the optimal human capital accumulation paths for (i) a Ramsey-type model with dynastic optimization, and (ii) a model with joy-of-giving altruism. We prove analytically that human capital accumulation is unambiguously lower in the "strategic" model than in the Ramsey-type dynastic model. We complement our results with a series of numerical exercises.

Keywords: human capital, intergenerational interactions, Markov perfect equilibrium, stochastic transition, constructive approach
JEL codes: C73, I20, J22

*Financial support from the Foundation for Polish Science (Professor Szapiro's subsidy "Mistrz") is gratefully acknowledged. We are grateful to Kevin Reffett and the participants of the Macromodels 2007 conference for helpful comments. All errors are our responsibility.

†Warsaw School of Economics, Theoretical and Applied Economics Department, Warsaw, Poland. Address: al. Niepodległości 162, 02-554 Warszawa, Poland. E-mail: lukasz.wozny@sgh.waw.pl.

‡Warsaw School of Economics, Institute of Econometrics, Warsaw, Poland. Address: al. Niepodległości 162, 02-554 Warszawa, Poland. E-mail: jakub.growiec@sgh.waw.pl.

1 Introduction

It is nowadays both intuitively clear and widely acknowledged in the literature that individuals, when making long-term economic decisions, frequently transcend the narrow notion of self-interest (Arrondel and Masson, 2006). One particular reason for this is the fact that apart from their own well-being, people usually care about the well-being of their children as well. Intergenerational altruism is thus a natural source of linkages between consecutive generations. More precisely, strategic interactions of this sort should be especially apparent in relation to schooling: on the one hand, a substantial fraction of investment in accumulating human capital of an individual is made by her parents, while on the other hand, these parents cannot fully anticipate what use will be eventually made of these personal assets (Becker and Tomes, 1986; Galor and Tsiddon, 1997; Lochner, 2008; Loury, 1981; Orazem and Tesfatsion, 1997).[1]

Since upon growing up, the children will themselves become independent decision makers caring also for their own children, the parents who derive utility from (among other things) the *utility* of their children should logically find themselves facing an infinite-horizon planning problem (i.e. parents care for children who care for grandchildren who care for great-grandchildren, etc.). This leads to the standard approach of constructing Ramsey-type models with dynastic optimization where the benevolent dynastic head finds the optimal allocation once and for all. Obviosuly, this approach is extremely useful, well-documented in the literature, and it has found many interesting applications. There is however an important reason why we believe that alternative approaches should be studied with great attention as well: in fact, there is convincing empirical evidence from a variety of research fields – ranging from dynamics of natural resource extraction to market bubbles – that actual decision makers tend to be quite myopic, especially when the actual planning horizon exceeds their lifetime.

Myopia, translated into the language of intergenerational linkages, means that even if the next consecutive generation's *consumption* is fully taken into account in the original generation's decision problem, there must exist some finite cut-off point beyond which the consumption of further generations does not matter. And it is precisely the inclusion of such a finite cut-off which

[1]The classic works within the human capital accumulation literature, such as Mincer (1958) or Ben-Porath (1967), focus primarily on the other component of investment in education which is individuals' own purposeful educational spending motivated by the expected increases in their future earnings. The Ben-Porath's model specification is however already flexible enough to allow for intergenerational transmission of human capital as well.

takes us from the standard Ramsey-type dynastic optimization frameworks to models where the planning problem becomes *strategic*. As the natural first step in this procedure, this article considers strategic interactions between *two* generations only: the original generation cares for the consumption of its children, but not of its grandchildren. Pushing the cut-off point further would be a natural extension which we leave for further work.

The point of departure of the current article is the following. The original generation (i.e. the parents) would like to choose their consumption level and the level of investment in human capital of their children *optimally* which requires considering the possible options the children will face in the subsequent period – when they will themselves become independent utility maximizers. The parents would therefore like to embed their children's optimization problems in their own and thus become "leaders" of such an intergenerational strategic game. Unfortunately, this procedure cannot be carried out directly: since the children's optimization problem embeds the optimization problem of their own children, and so forth *ad infinitum*, we end up with an infinite series of embedded strategic games. The problem with applying usual fixed-point arguments here is that the strategic component of the embedded games creates a "vicious circle" of strategy space (Leininger, 1986) which has obstructed the development of economic theories in this vein for many years (see e.g. Strotz (1955) and Phelps and Pollak (1968)). This issue has been resolved only recently, thanks to the crucial technical developments of Amir (1996a,c) and Nowak (2006). The current article applies these developments to the case of intergenerational interactions in human capital accumulation.

The contribution of this paper is twofold. First, we identify the impact of strategic interactions between consecutive generations on the time path of human capital accumulation in an economy populated by a sequence of generations allowed to decide over their consumption levels as well as over the levels of investment in human capital of their immediate descendants. We are able to obtain clear-cut results here thanks to a novel constructive approach to computing Markov perfect equilibria in games of intergenerational altruism which has recently been put forward by Reffett and Woźny (2008). Second, we benchmark our results against two standard models, both of which assume away strategic interactions across generations: (i) the usual Ramsey-type model with dynastic optimization, and (ii) a model with joy-of-giving altruism (used by, among numerous others, Artige, Camacho, and de la Croix (2004)). Our most important result here is an analytical proof that, other things equal, human capital accumulation is unambiguously smaller in the strategic model than in the dynastic model. We also run a series of numerical exercises quantifying how large the differences between the optimal human capital accumulation decisions could be whether strategic interactions are present or not. Additionally, we also show numerically that the joy-of-giving altruism model differs markedly from the strategic model, insofar the implied optimal decisions cannot be unambiguously compared against each other: for most parameter values, joy-of-giving altruism implies more human capital investment than the strategic model, but for a range of specific parametric choices, this relationship is reversed. At the same time, our numerical examples show how the constructive approach to computing Markov perfect equilibria could be used in computational practice.

The two crucial contributions of this paper are, therefore, purely theoretical. The lack of immediate empirical applications of our theory comes from the fact that the model developed herein, though based on sound microeconomic foundations, is admittedly simplified. We are therefore convinced that it would be a stark exaggeration to calibrate it in its current form in order to draw quantitative implications aimed at discriminating between competing theories of human capital accumulation based on empirical evidence. Instead, our model should be considered as an important first step: it is the first model of human capital accumulation which integrates fully-specified strategic interactions between consecutive generations into an (otherwise standard) overlapping-generations framework.

Technically, to show existence and uniqueness of Markov perfect equilibrium (MPE) in an economy with strategic interactions we use the method of monotone operators pioneered by Coleman (1991) and developed further by Datta and Reffett (see a survey of these methods in Datta and Reffett (2006)). One important caveat is that in our current case, the appropriate operator whose fixed points are the Markov perfect equilibria of the considered economy turns out to be decreasing while the relevant theorems summarized by Datta and Reffett (2006) deal with increasing operators only. Hence, we cannot apply the established methods directly: the appropriate theorems must be generalized first.[2] Several of these generalizations have already been made by Reffett and Woźny (2008) in their constructive study of Markov perfect equilibria in stochastic games of intergenerational altruism, but even those results have to be generalized further in order to accomodate the problem we are dealing with here. Hence, we have included complete proofs for all theorems presented in the text.

The remainder of the article is structured as follows. In Section 2 we lay out our basic model with strategic interactions and derive the principal

[2]Even more generally, the geometrical characteristics of monotone operators in abstract cones and their fixed point properties have been described by Guo and Lakshmikantham (1988) as well as Guo, Cho, and Zhu (2004).

theoretical results. In Section 3 we compare this model with two benchmark models where no strategic interactions are allowed. Section 4 provides an illustrative numerical example for our calculations of the preceding chapters. Section 5 discusses the role of strategic interactions in shaping human capital investment decisions. Section 6 concludes. Definitions and proofs of theorems have been relegated to the appendix.

2 The model and main results

2.1 Setup

Our model economy is populated by an infinite sequence of generations whose sizes are equal and normalized to unity. Each generation $t = 0, 1, 2, \ldots$ is characterized by the common utility function U, taking values $U(c_t, c_{t+1})$, where c_t is the total consumption of generation t. We assume U to be time-separable[3] and take the form: $U(c_t, c_{t+1}) = u(c_t) + v(c_{t+1})$. The consumption set is $Y = [0, \bar{Y}]$ where $\bar{Y} \in \mathbb{R}_+$. The unique consumption good is produced using technology f which requires two kinds of inputs: (i) time devoted to work $\hat{l}_t$, and (ii) human capital h_t. The set $H = [0, \bar{H}]$, where $\bar{H} \in \mathbb{R}_+$, represents all possible levels of human capital. We assume away all physical capital accumulation in our basic model. Human capital, on the other hand, is accumulated using technology $\tilde{g}$ taking as inputs: (i) the current level of human capital h_t, and (ii) time devoted to human capital accumulation $1 - \hat{l}_t$.

Technically, our assumptions on the considered economy are the following:

Assumption 1 *Let:*

- $u, v : Y \to \mathbb{R}$ *be increasing, continuously differentiable, and satisfying* $\lim_{c \to 0} u'(c) = \lim_{c \to 0} v'(c) = \infty$*;* $(\forall c \in Y, c > 0) \quad u'(c) < \infty$ *and* $(\forall c \in Y, c > 0) \quad v'(c) < \infty$*. Moreover, let u and v be strictly concave and such that* $u(0) = v(0) = 0$*,*
- $f : H \times [0, 1] \to Y$ *be strictly concave with respect to the second argument, twice continuously differentiable with finite partial derivatives, and satisfying* $(\forall \hat{l} \in [0, 1]) \quad f(0, \hat{l}) = 0$*,* $(\forall h \in H) \quad \lim_{\hat{l} \to 0} f_2'(h, \hat{l}) = \infty$*.*

[3] We analyze the case of time-separable utility functions only because the monotone methods used in Theorem 2 rely on this assumption heavily and because this assumption has been extensively used in literature. The case of non-time separable utility functions could also be analyzed nonetheless. This would require the use of results on mixed monotone operators. See Guo, Cho, and Zhu (2004) and the applications in Reffett and Woźny (2008).

Furthermore, assume that $(\forall h \in (0, \bar{H}])\, f(h, \cdot)$ *and* $(\forall \hat{l} \in (0, 1])\, f(\cdot, \hat{l})$ *are strictly increasing functions.*

Within each generation, the household chooses its consumption level c_t to maximize utility U, that is:

$$\max_{c_t} u(c_t) + v(c_{t+1}). \tag{2.1}$$

Assuming out physical capital accumulation amounts to assuming full depreciation as well. All output is thus immediately consumed: $c_t = f(h_t, \hat{l}_t)$, where $\hat{l}_t \in [0, 1]$. Human capital is accumulated according to the equation: $h_{t+1} = \tilde{g}(h_t, 1 - \hat{l}_t)$, where $\tilde{g} : H \times [0, 1]$ is a continuous, strictly positive function. Substituting the relations specified above into (2.1) and ignoring time subscripts we obtain the following household maximization problem:

$$\max_{\hat{l} \in [0,1]} u(f(h, \hat{l})) + v(f(\tilde{g}(h, 1 - \hat{l}), \tilde{l})). \tag{2.2}$$

The problem (2.2) features two endogenously determined variables which are taken as given by the original generation: their own human capital level $h \in H$ and the labor choice of the next generation $\tilde{l} \in [0, 1]$.

We propose two alternative economic interpretations for our modeling approach summarized by the maximization problem (2.2):

- each household lives for one period and derives utility from its own consumption, $u(c_t)$, and the consumption of its immediate successor, $v(c_{t+1})$;
- each household lives for two periods but chooses the fraction of time devoted to the production of consumption goods and the fraction of time devoted to the accumulation of human capital of the subsequent generation in the first period only. Its consumption in the second period is chosen by the next generation, and thus is only indirectly influenced by the level of human capital left to the next generation.

2.2 The concept of Markov perfect equilibrium

The primary objective of this paper is to analyze closed-loop Markov perfect equilibria (MPE) of the economy specified above. To this end, we must now introduce some new notation. Namely, by $l' \in L$, where $L = \{l : (0, \overline{H}] \to [0, 1], l \in \mathscr{C}\}$, we will denote the Markov strategy of the next generation. Moreover, we shall let $\mathbf{0} \in L$ denote the constant zero function, and let

$\mathbf{1} \in L$ denote a constant function whose values are always equal to 1. We shall also introduce the correspondence $D : L \times H \to [0,1]$ defined by

$$D(l', h) = \arg\max_{\hat{l}\in[0,1]} u(f(h, \hat{l})) + v(f(\tilde{g}(h, 1-\hat{l}), l'(\tilde{g}(h, 1-\hat{l})))). \tag{2.3}$$

The best response of the current generation for next generation's strategy $l' \in L$ is therefore a selection $l(\cdot)$ from $D(l'|\cdot)$.

We adopt the following definition of MPE:

Definition 1 *A Markov perfect equilibrium (MPE) of the economy is a selection*[4] $l^* : (0, \bar{H}] \to [0,1]$ *from* $D(l^*|\cdot)$.

The MPE can be interpreted either as a subgame perfect Nash equilibrium of an sequential intergenerational game or as a time-consistent policy which is suited for each generation. Since the time horizon of the economy is infinite, we concentrate on *stationary* Markov policies, i.e. such that in each period, the same function of the state variable h is applied.[5]

2.3 Introducing stochastic transition

Unfortunately, as discussed by Leininger (1986) and others, the standard way of obtaining results on the existence and uniqueness of MPE in similar setups – as fixed points of some self maps – is obstructed by the so-called "vicious circle" of strategy space. The problem occurs when trying to construct appropriate sets of admissible strategies/policies. Even very strong assumptions made on the strategy/policy of the subsequent generation cannot guarantee that the best response to that strategy would belong the the same set of strategies/policies.

The crucial step required to solve this problem is to break the deterministic links between subsequent generations/periods (see Amir, 1996c; Nowak, 2003). In our case, this would correspond to assuming that the transition (human capital accumulation function) $\tilde{g}$ be stochastic. We shall let $G(\cdot; h, 1-l)$ be the distribution of human capital in the subsequent period, parametrized by the current human capital level h and the time investment $1 - l$.

The introduction of stochastic factors in human capital accumulation is thus motivated primarily by technical reasons. Such factors have sound economic motivation, though. Indeed, (i) heredity involves randomness: the

[4] We are leaving $l^*(0)$ undefined here, since under assumptions 1 and 2, as we shall show later, it is not single-valued. The economic justification is the following: having no human capital one produces, consumes and invests nothing but since there is a no disutility of work, any level of l could be optimal.

[5] If the horizon of the economy were finite, we could solve for non-stationary policies by backward induction.

unobservable skill levels are not inherited from one's parents deterministically; (ii) human capital is not homogenous: it is technology-specific and thus up-front investment in it might (but might not) be ineffective (Chari and Hopenhayn, 1991), depending on the future pattern of technological progress; (iii) the motivation of children to learn is endogenous (Orazem and Tesfatsion, 1997). All these factors taken together make it clear that treating investment in education as a lottery where future payoffs depend on stochastic factors is quite reasonable.[6]

The following assumption on the stochastic transition follows Amir (1996c) and Nowak (2006).

Assumption 2 (Technology) *The distribution G satisfies the following conditions:*

- $\forall h \in H, \quad G(0|h, 0) = 1,$
- $\forall h \in H, l \in [0, 1),$

$$G(\cdot|h, 1-l) = (1 - g(h, 1-l))\delta_0(\cdot) + g(h, 1-l)\lambda(\cdot),$$

where

- $g : H \times [0, 1] \to [0, 1]$ *is strictly concave with respect to the second argument, twice continuously differentiable, satisfies the condition:* $(\forall l \in [0, 1]), g(0, 1-l) = 0,$
- $(\forall l \in [0, 1))\, g(\cdot, 1-l)$ *and* $(\forall h \in (0, \bar{H}])\, g(h, \cdot)$ *are strictly increasing functions,*
- $(\forall h \in H) \lim_{l \to 1} g_2'(h, 1-l) = \infty$ *and* $(\forall h \in H, l < 1), 0 < g_2'(h, 1-l) < \infty,$
- λ *is a Borel transition probability on* $(0, \bar{H}]$*,*
- δ_0 *is a probability measure concentrated at zero.*

The crucial implications of this specification are as follows: with probability $1-g(h, 1-l)$, the next generation's human capital will be zero, indicating that the investment in it has been completely ineffective. This relates to the argument that skills be technology-specific, and that technology might change fast enough to make all previously acquired skills obsolete. With

[6] It should be noted that we rule out all systematic human capital externalities from non-relatives (Ben-Porath, 1967; Rangazas, 2000) and assume that children's human capital is created from parental human capital, education effort, and stochastic factors only.

probability $g(h, 1-l)$, however, human capital is drawn from a distribution λ which does not depend on h or l. This relates to the stochastic heredity assumption, coupled with the random motivation of children to learn.

Assuming that the next generation follows a Markov strategy $l' \in L$, the maximization problem (2.2) augmented by stochastic transition takes the form:

$$\max_{\hat{l}\in[0,1]} u(f(h,\hat{l})) + \int_H v(f(y,l'(y)))G(dy; h, 1-\hat{l}). \tag{2.4}$$

Under assumptions 1 and 2, the maximand of (2.4) (for a given $h \in (0, \bar{H}]$) is strictly concave and differentiable with respect to $\hat{l}$ on $(0,1)$. Furthermore, the unique optimal labor supply level l^* solves $\zeta(l^*, h, l') = 0$ whenever interior, where ζ is defined as:

$$\zeta(l, h, l') := u'(f(h,l))f_2'(h,l) - g_2'(h, 1-l)\int_H v(f(y,l'(y)))\lambda(dy). \tag{2.5}$$

A MPE of the economy with stochastic transition is then a function l which solves $\zeta(l(h), h, l) = 0$ for all $h \in (0, \bar{H}]$.

2.4 Characteristics of the closed-loop MPE

Let us now comment on the possibilities of showing existence of a MPE in the given class of functions. In a paper most closely related to this one, Reffett and Woźny (2008) have constructed an operator whose fixed points are MPE of an economy with intergenerational altruism (see also Bernheim and Ray (1987)). The operator is defined implicitly on the set of Lipschitz continuous functions belonging to L by an appropriate first order condition. The authors find that it suffices to show continuity of such an operator, and existence of a MPE follows by the Brouwer fixed point theorem. In our particular case, however, their method fails due to the non-uniqueness of the maximizer in equation (2.5) for $h = 0$. Specifically, for any $l' \in L$ the optimal $l^*(0) \in [0,1]$. Notice also that $(\forall h \in H, h > 0)$, $l^*(h) = 1$ is the best response to $l' = \mathbf{0}$. Hence, we cannot apply those results directly. Fortunately, there exists a way to circumvent this problem and we will present it below.

Before we actually prove the existence of a MPE, however, let us present some of its basic properties. They will be helpful in our further analysis.

Theorem 1 (Characteristics of MPE) *Suppose that a MPE exists. Then:*

- *the set of Markov perfect equilibria of the economy has no ordered (in a pointwise order) elements in L.*

- *If* $f''_{12}(\cdot,\cdot) \leq 0$ *and* $g''_{12}(\cdot,\cdot) \geq 0$*, then* l^* *is strictly decreasing on* $(0, \bar{H})$ *wherever interior.*

The first assertion results from the fact that an operator defined on the first order conditions whose fixed points are MPE of the economy is decreasing. The second assertion follows from established theorems on strict monotone comparative statics (Amir (1996b); Edlin and Shannon (1998)) of optimal solutions to maximization problems featuring a submodular function on a lattice. Please observe that the reverse to the second assertion does not have to hold. Generally, even if $f''_{12}(\cdot,\cdot) \geq 0$ and $g''_{12}(\cdot,\cdot) \leq 0$, the optimal labor supply policy l^* does not need to increase with h due to the strictly decreasing marginal utility.

2.5 The main result: Existence and uniqueness

To state our main results on existence and uniqueness, we have to rearrange the first order condition of maximization in (2.4) for $h \in (0, \bar{H}]$:

$$\xi_h(\hat{l}) := \frac{u'(f(h,\hat{l}))f'_2(h,\hat{l})}{g'_2(h, 1-\hat{l})} = \int_H v(f(y, l(y)))\lambda(dy). \tag{2.6}$$

The function $\xi_h(0,1] \to \mathbb{R}_+$, with $\xi_h(1) = 0$, introduced just above, captures the marginal utility of consumption coupled with marginal labor productivities in both sectors. Using this function, we can substitute the MPE condition specified in the previous subsection with the following functional equation:

$$\xi_h(l(h)) = \int_H v(f(y, l(y)))\lambda(dy). \tag{2.7}$$

The next lemma summarizes the main properties of function ξ_h.

Lemma 1 *For all* $h \in (0, \bar{H}]$ *the function* $\xi_h : (0,1] \to \mathbb{R}_+$ *is continuously differentiable, strictly decreasing, and invertible with continuously differentiable inverse. The inverse function* $\xi_h^{-1} : \mathbb{R}_+ \to (0,1)$ *is in addition strictly decreasing.*

Proof. See Appendix.

Similarly to the method used in Coleman (2000), let us now consider the functional equation:

$$\bar{l}(h) = \int_H v(f(y, \xi_h^{-1}(\bar{l}(y))))\lambda(dy). \tag{2.8}$$

The great usefulness of equation (2.8) stems from the fact that any function $l \in L$ is a solution to the functional equation (2.7) if and only if $\bar{l} : (0, \bar{H}] \to [0, \infty)$ defined by $\bar{l}(y) := \xi_h(l(y))$ is a solution to (2.8).

Let us also define an operator B on $P = \{\bar{l} : (0, \bar{H}] \to [0, \infty)\}$ such that for any $h \in (0, \bar{H}]$, B satisfies:

$$B\bar{l}(h) = \int_H v(f(y, \xi_h^{-1}(\bar{l}(y))))\lambda(dy). \tag{2.9}$$

Furthermore, by E_x^f we will denote the elasticity of a single-variable function f, measured in point x in its domain.

The next theorem gives the conditions under which B has a unique fixed point in P. This finding is equivalent to showing under which conditions the MPE of the considered economy exists and is unique.

Theorem 2 (Existence and uniqueness) *Let Assumptions 1 and 2 be satisfied. Assume in addition that* $(\exists r, 0 < r < 1)$ *such that* $(\forall h \in H)$ *the following holds:*

$$(\forall h \in (0, \bar{H}])(\forall x > 0) \quad r \geq \left[-E^v_{f(h,\xi^{-1}(x))} E^{f,2}_{\xi^{-1}(x)} E^{\xi^{-1}}_x \right]. \tag{2.10}$$

Then B has a unique fixed point $\bar{l}^*$ in P° and repeated iteration of B from any $p_0 \in P^\circ$ assures uniform convergence to $\bar{l}^*$ (that is, the condition (7.28) holds).

Proof. See Appendix.

Theorem 2 is probably the most important result in this paper. It gives us sufficient conditions for the existence and uniqueness of a fixed point of an operator B (and hence, of a MPE of the considered economy) and allows to compute it using a straightforward iterative procedure. Having calculated the unique fixed point $\bar{l}^*$ of the operator B in P, we can pin down the unique MPE of the analyzed economy by $l^*(y) = \xi_h^{-1}(\bar{l}^*(y))$.

The mathematical intuition behind Theorem 2 is the following: since the operator B is decreasing, it may have multiple, unordered fixed points. The condition in Theorem 2 asserts, however, that this operator is "convex" (see Guo and Lakshmikantham (1988) for details) or – in other words – is a "local contraction". This property is sufficient for existence of a unique fixed point. Economically, the condition (2.10) ("convexity" or "local contraction") could be interpreted in terms of partial elasticities: it requires that the product of elasticities of v, f and ξ^{-1} cannot exceed unity, i.e. that the percentage change in next-period utility v resulting from a one per-cent change in labor supply $\bar{l}$ cannot be "too high". Otherwise, it could be profitable to deviate from the given policy – the loss in instantaneous consumption sub-utility u would be more than compensated by the gain in next-period consumption sub-utility v – indicating that the given policy could not be an equilibrium any more.

We leave the questions on existence and number of equilibria when condition (2.10) is not satisfied for further work. Instead, we shall now present our workhorse example which will be used in subsequent numerical exercises.

Example 1 *Let* $U(c_1, c_2) = c_1^{\gamma_1} + \delta c_2^{\gamma_2}$*,* $f(h, l) = h^{\alpha_1} l^{\beta_1}$*. Furthermore, take any* g *satisfying Assumption 2 with* $\alpha_1, \beta_1, \gamma_1, \gamma_2 \in (0, 1)$ *and* $\delta \in (0, 1]$*. If* $1 > \beta_1(\gamma_1 + \gamma_2)$ *then there exists a unique MPE in* L*.*

Proof of Example 1: Observe that is this case elasticities of the utilities u and v as well as f are constant. Hence we may apply the Guo, Cho, and Zhu (2004) theorem (see Theorem 4 in the Appendix) directly to the (decreasing) operator B which can be calculated explicitly for the given functions. ■

3 Human capital dynamics with and without strategic interactions

The theoretical results presented above will now be compared to the results obtained within similar setups which do not allow for strategic interactions between generations. Specifically, we shall focus on two of such models:

- a Ramsey-type model where generations do not derive utility directly from their successors' consumption, but from their *utility*. The utility function is assumed to be constant across generations with a discount factor $\delta \in (0, 1)$. Hence, all generations' choices can be embedded in the first generation's optimization problem, ultimately yielding a "dynastic" model with infinite-horizon planning where each generation t maximizes $\sum_{\tau=t}^{\infty} \delta^{\tau-t} u(c_\tau)$.[7] Observe that such a setup rules out all strategic aspects of the decision process.

- a model of "joy-of-giving" altruism where generations do not derive utility directly from their successors' consumption, but are instead interested in providing them with the *means* allowing for consumption.

[7]Provided that the transversality condition holds: $\lim_{\tau\to\infty} \Lambda_\tau h_\tau = 0$ (where Λ is the shadow price of human capital). If the set of admissible human capital levels H is bounded, as it is in our case, this transversality condition holds for sure.

In the context of human capital accumulation it means that their utility function is $u(c_t) + v(h_{t+1})$. Hence, the decisions made by the next generation do not matter for the utility of the current generation.

3.1 Dynastic optimization

Consider an economy populated by a sequence of generations each represented by a single household with preferences $U(c_t, V_{t+1})$ over its consumption c_t and its immediate descendants' utility V_{t+1}. Since all generations' utility functions are the same, their choices can be embedded in the first generation's optimization problem. The solution to this maximization problem corresponds to a stationary solution to an infinite-horizon Ramsey-type model with stochastic transition in human capital levels: $\max_{\{c_\tau\}} \sum_{\tau=t}^{\infty} \delta^{\tau-t} u(c_\tau)$, where $\delta \in (0,1)$ is a discount factor.

The first order condition reads:

$$u'(f(h,l(h)))f_2'(h,l(h)) = \delta g_2'(h, 1-l(h)) \int_H V(y)\lambda(dy), \tag{3.11}$$

where $V(h)$ is the Bellman's value function defined as

$$V(h) = \max_{\hat{l}\in[0,1]} \left\{ u(f(h,\hat{l})) + \delta \int_H V(y) G(dy; h, 1-\hat{l}) \right\}. \tag{3.12}$$

Standard arguments of dynamic programming (Stokey, Lucas, and Prescott (1989)) guarantee that under our assumptions the functional equation (3.12) has a unique solution V and that the solution to the Ramsey-type optimization problem corresponds to a function $l(h)$ which solves $V(h) = u(f(h,l(h))) + \delta \int_H V(y)G(dy, h, 1-l(h))$.

The first order condition (3.11) guarantees that the marginal utility of consumption of the current generation, acquired thanks to an extra unit of time devoted to work, is exactly equal to the expected marginal cost in terms of utility lost by the next generation because of having marginally less human capital.

Note that such a setup rules out all strategic aspects of the decision process and therefore the (full-commitment) Markov policy for a dynastic optimization economy is (generally) not a MPE of an economy with strategic interactions.

3.2 Joy-of-giving altruism

The model with joy-of-giving altruism (and, to guarantee direct comparability, with a stochastic transition in human capital levels) can be generally specified as:

$$\max_{\hat{l}\in[0,1]} u(f(h,\hat{l})) + \int_H v(y)G(dy;h,1-\hat{l}). \tag{3.13}$$

Concentrating on Markovian policies, the first order condition for optimal labor supply $l(h)$ is given by:

$$u'(f(h,l(h)))f_2'(h,l(h)) = g_2'(h,1-l(h))\int_H v(y)\lambda(dy), \tag{3.14}$$

guaranteeing that the marginal utility of consumption acquired thanks to an extra unit of time devoted to work is exactly equal to the expected marginal cost in terms of lost human capital of the next generation.

3.3 Comparing equilibrium policies

It turns out that equilibrium policies for our basic model with strategic interactions and for the Ramsey-type dynastic optimization model economy abstracting from such interactions can be directly compared:

Theorem 3 (On comparing equilibria) *Let l_{MPE} be a MPE of an economy with strategic interactions with $v(\cdot) = \delta u(\cdot)$, and l_R be the optimal stationary policy of a dynastic economy with utility u. Then $l_{MPE}(h) > l_R(h)$ for all $h \in (0, \bar{H}]$.*

Proof. See Appendix.

Theorem 3 asserts that equilibrium human capital investment is *unambiguously lower* in an economy with strategic interactions than in a economy with dynastic optimization. The intuition behind this result is straightforward: the optimal investment policy under full commitment must exceed the equilibrium investment policy when only partial commitment between consecutive generations is possible. Indeed, in the Ramsey-type model, the dynastic head from generation t will take into account not only the consumption of the following generation $t+1$, but of all generations from t onwards. She will therefore be willing to save more for the future than a generation t member of the strategic model: the latter person is myopic and wishes to save for her children but not for her grandchildren.

Theorem 3 is the second main result in this paper. It provides a formal argument determining the direction of the bias incurred when a baseline model with strategic interactions is replaced with its non-strategic counterpart.

Unfortunately, a similar clear-cut relationship does not exist between the strategic model and the model with joy-of-giving altruism. As will be shown in a numerical example in the following section, equilibrium policies of those two setups cannot be unambiguously compared. Even though each of them has been prepared so that direct comparisons could be possible, we find that for different parameter configurations, different results are possible. Usually it is the strategic model which puts more weight on immediate consumption and less on human capital accumulation; sometimes the result is reversed, though.

4 Numerical example

4.1 Numerical computation of the MPE

The objective of the current section is to compute numerically the equilibrium policy l^* for an economy with strategic interactions and to analyze the equilibrium dynamics of human capital accumulation given certain functional assumptions on u, v, f and G. To facilitate economic interpretation, we will concentrate on iso-elastic utility and Cobb-Douglas production functions here. We will then benchmark these numerical results against the corresponding ones obtained for non-strategic models discussed in the previous section.

Example 2 *Extending Example 1, let us additionally assume that* $g(h, 1-l) = \frac{1}{\bar{H}^{\alpha_2}} h^{\alpha_2}(1-l)^{\beta_2}$ *where* $\alpha_2, \beta_2 \in (0,1)$*. The function* ξ_h *is then given by:*

$$\xi_h(l) = \frac{\beta_1\gamma_1}{\beta_2}\bar{H}^{\alpha_2} h^{\alpha_1\gamma_1-\alpha_2}\frac{l^{\beta_1\gamma_1-1}}{(1-l)^{\beta_2-1}}. \tag{4.15}$$

Furthermore, we assume that $\beta_2 = \beta_1\gamma_1$.

The last equality assumption has been made for the sole purpose of analytical tractability: it is only when $\beta_2 = \beta_1\gamma_1$ that the ξ_h mapping is analytically invertible. Relaxing it increases the computational burden but does not overturn any of our results. If $\beta_2 = \beta_1\gamma_1$, we obtain:

$$\xi_h^{-1}(\bar{l}) = \frac{\bar{l}^{\frac{1}{\beta_2-1}} h^{\frac{\alpha_1\gamma_1-\alpha_2}{1-\beta_2}} \bar{H}^{\frac{\alpha_2}{1-\beta_2}}}{1+\bar{l}^{\frac{1}{\beta_2-1}} h^{\frac{\alpha_1\gamma_1-\alpha_2}{1-\beta_2}} \bar{H}^{\frac{\alpha_2}{1-\beta_2}}}. \tag{4.16}$$

Assuming furthermore that the distribution λ is uniform on H, the MPE policy can be found as $l^*(y) = \xi_h^{-1}(\bar{l}(y))$ where $\bar{l}$ is found as the fixed point of the operator B given by

$$B\bar{l}(h) = \frac{\delta}{\bar{H}}\int_0^{\bar{H}} y^{\alpha_1\gamma_2}\left(\frac{\bar{l}(y)^{\frac{1}{\beta_2-1}} h^{\frac{\alpha_1\gamma_1-\alpha_2}{1-\beta_2}} \bar{H}^{\frac{\alpha_2}{1-\beta_2}}}{1+\bar{l}(y)^{\frac{1}{\beta_2-1}} h^{\frac{\alpha_1\gamma_1-\alpha_2}{1-\beta_2}} \bar{H}^{\frac{\alpha_2}{1-\beta_2}}}\right)^{\beta_1\gamma_2} dy. \tag{4.17}$$

As stated in Theorem 2, repeated iteration of B guarantees convergence to the MPE (see Figure 1).[8]

Proposition 1 *The MPE policy l^* is monotone. It is everywhere decreasing iff $\alpha_1\gamma_1 < \alpha_2$, everywhere increasing iff $\alpha_1\gamma_1 > \alpha_2$, and constant iff $\alpha_1\gamma_1 = \alpha_2$.*

Proof of Proposition 1: In equilibrium, $\bar{l}(h) = \xi_h(l(h))$ can by defined as the right-hand side of (4.17).

We will now differentiate $l(h) = \xi_h^{-1}(\bar{l}(h))$ with respect to h. Observe that it is justified since lemma 1 states that ξ_h^{-1} is differentiable while from equations (4.16) and (4.17) we also have that functions η_z (where, for given $z \in [0, \infty)$, $\eta_z(h) := \xi_h^{-1}(z)$) and $\bar{l}$ are differentiable with respect to h on $(0, \bar{H})$. It is obtained that:

$$\begin{aligned} \frac{dl(h)}{dh} &= \frac{\partial \xi_h^{-1}(\bar{l}(h))}{\partial \bar{l}(h)}\frac{\partial \bar{l}(h)}{\partial h} + \frac{\partial \xi_h^{-1}(\bar{l}(h))}{\partial h} = \\ &= \frac{1}{(1+\Xi(h))^2}\left(\frac{l(h)^{\frac{1}{\beta_2 - 1}} h^{\frac{\alpha_1\gamma_1 - \alpha_2}{1-\beta_1}-1}}{1-\beta_2}\right)(\alpha_1\gamma_1 - \alpha_2) \times \\ &\times \left(1 - \frac{\frac{\beta_1\gamma_2}{1-\beta_2}\frac{\delta}{\bar{H}}\int_0^{\bar{H}} y^{\alpha_1\gamma_2}\left(\frac{\Xi(y)}{1+\Xi(y)}\right)^{\beta_1\gamma_2}\frac{1}{1+\Xi(y)}dy}{\frac{\delta}{\bar{H}}\int_0^{\bar{H}} y^{\alpha_1\gamma_2}\left(\frac{\Xi(y)}{1+\Xi(y)}\right)^{\beta_1\gamma_2} dy}\right), \end{aligned} \tag{4.18}$$

with $\Xi(y) \equiv \bar{l}(y)^{\frac{1}{\beta_2-1}} h^{\frac{\alpha_1\gamma_1-\alpha_2}{1-\beta_2}} \bar{H}^{\frac{\alpha_2}{1-\beta_2}}$. Since $\beta_1\gamma_1 = \beta_2$, and by assumption, $1 > \beta_1(\gamma_1 + \gamma_2)$, it follows that $\frac{\beta_1\gamma_2}{1-\beta_2} < 1$ and thus the ratio of two integrals in the last parenthesis is smaller than one, we find the expression in the last parenthesis to be positive. In conclusion, $\frac{dl(h)}{dh} > 0$ and thus $l(h)$ is increasing in its domain iff $\alpha_1\gamma_1 > \alpha_2$, $\frac{dl(h)}{dh} < 0$ and thus $l(h)$ is decreasing in its domain iff $\alpha_1\gamma_1 < \alpha_2$, and $l(h)$ is constant iff $\alpha_1\gamma_1 = \alpha_2$. ■

Having specified the three cases in which the optimal labor supply policy is increasing, decreasing, or constant in the human capital endowment, let us discuss the empirical plausibility of each of the cases. The results are somewhat reassuring here. Namely, the case where $\alpha_2 > \alpha_1\gamma_1$, guaranteed

[8] To calculate the equilibrium policies of any of the three models numerically, we have used the discretization method discussed by Judd (1998). MATLAB codes used to compute the numerical results quoted throughout the paper as well as to produce Table 1 are available from the authors upon request.

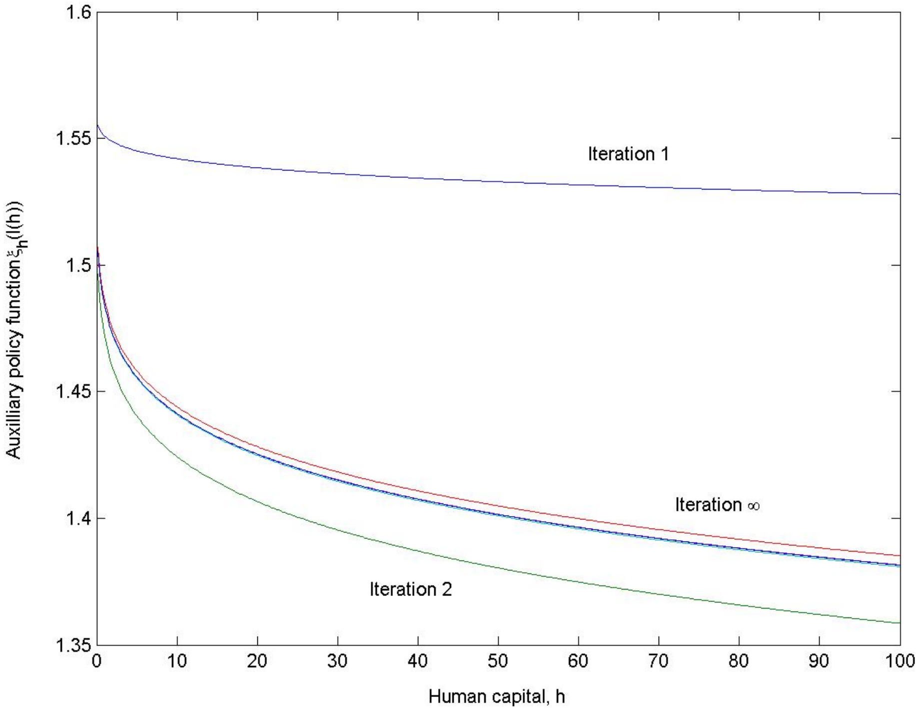

Figure 1: Convergence to the fixed point of operator B. The fixed point is the auxilliary policy function $\bar{l}(h) = \xi_h(l(h))$. Assumed parameter values: $\alpha_1 = .3; \beta_1 = .7; \alpha_2 = .3; \gamma_1 = .6; \gamma_2 = .5; \beta_2 = \beta_1\gamma_1 = .42; \bar{H} = 100; \delta = .9$.

to hold e.g. if $\alpha_1 \approx \alpha_2$ (i.e. if the shares of human capital in production of the consumption good and of human capital, respectively, are approximately equal), turns out to be significantly more plausible empirically than any of the other cases.[9] This case, implying that labor supply decreases (and human capital accumulation increases) with the stock of human capital, is thus going to be our benchmark case.

4.2 Dynamics

The dynamic properties of the economy are as follows. If all generations play the MPE strategy, then in the limit as $t \to \infty$, average human capital tends

[9] Becker and Tomes (1986), Lochner (2008), among numerous others, discuss the empirical evidence that the educational effort and children's school attainments are unambiguously positively related to the parental human capital level.

to $\bar{h}$ solving the implicit equation:

$$\bar{h} = 2^{\frac{1}{\alpha_2 - 1}} \bar{H} (1 - l(\bar{h}))^{\frac{\beta_2}{1-\alpha_2}}. \tag{4.19}$$

This result has been confirmed numerically.[10]

The distribution of human capital will also evolve over time as consecutive generations will invest different fractions of time to work and education. By definition, however, the distribution of human capital over H will have a constant density $\frac{1}{\bar{H}} g(\bar{h}, 1 - l(\bar{h})) = \frac{1}{\bar{H}^{\alpha_2+1}} \bar{h}^{\alpha_2} (1 - l(\bar{h}))^{\beta_2}$ and a probability mass $1 - g(\bar{h}, 1 - l(\bar{h})) = 1 - \frac{1}{\bar{H}^{\alpha_2}} \bar{h}^{\alpha_2} (1 - l(\bar{h}))^{\beta_2}$ concentrated at zero.

4.3 Role of the transition distribution λ

The MPE policy $l^*(h)$ depends on the underlying transition distribution λ but this impact turns out to be rather modest. As a robustness check of our earlier numerical results, we have substituted the uniform distribution λ with two alternatives:

- a triangular distribution with density

$$\varphi(h) = \begin{cases} \frac{4}{\bar{H}^2} h, & h \in (0, \frac{\bar{H}}{2}), \\ \frac{4}{\bar{H}} - \frac{4}{\bar{H}^2} h, & h \in (\frac{\bar{H}}{2}, \bar{H}); \end{cases} \tag{4.20}$$

- a one-point distribution[11] with all probability mass concentrated in $\bar{H}/2$: $P(h = \bar{H}/2) = 1$.

As we have confirmed numerically,[12] the greatest labor supply is obtained when the distribution is uniform, and the least labor is supplied when the probability mass is concentrated at the mean human capital level. The policy for the triangular distribution falls in between these two extreme cases (uniform and one-point). The interpretation of this result is straightforward: the more risk remains that human capital of the successive generation would be low despite substantial investment, the less willing the decision maker would be to invest in human capital. Since individuals are risk-averse in this model, additional risk lowers education effort and increases labor supply which guarantees a certain payoff.

[10] The results are available from the authors upon request.

[11] Note that even when λ is one-point, there remains a probability that the next generation's human capital will be zero. Hence, the assumptions and interpretations of the economy with strategic interactions studied in Section 2 are still satisfied.

[12] These results are available from the authors upon request.

5 Numerical assessment of the role of strategic interactions

Let us now compare the equilibrium dynamics obtained in the numerical example presented above to the ones generated by a Ramsey-type dynastic model as well as by a model with "joy-of-giving" altruism. In both of these alternatives, no strategic intergenerational interactions are present.

Example 3 (Dynastic optimization) *Let* $u(c) = c^\gamma$*,* $f(h,l) = h^{\alpha_3} l^{\beta_3}$*,* $g(h, 1-l) = \frac{1}{\bar{H}^{\alpha_4}} h^{\alpha_4}(1-l)^{\beta_4}$*. Let the decision maker born at* t *maximize* $u(c_t) + \delta u(c_{t+1})$*. From (3.11), we obtain the first order condition for the optimal policy function* $l(h)$*. It is given as an implicit solution to the equation:*

$$\frac{l^{1-\beta_3\gamma}}{(1-l)^{1-\beta_4}} = \frac{\bar{H}^{\alpha_4}}{\delta I} h^{\alpha_3\gamma - \alpha_4}, \tag{5.21}$$

where $I \equiv \int_H V(y)\lambda(dy)$ *is a predetermined constant.*

Using the implicit function theorem, it can again be easily shown that $l(h)$ is everywhere decreasing whenever $\alpha_4 > \alpha_3\gamma$ and everywhere increasing whenever $\alpha_4 < \alpha_3\gamma$. In the special case where $\alpha_3\gamma = \alpha_4$, (5.21) implies that $l(h)$ is constant, independent of h. This finding paralells Proposition 1 precisely: there are absolutely no qualitative differences in the optimal policy behavior between the strategic and the non-strategic model. *Quantitative* differences are substantial, though, as we shall see shortly.

Moreover, just like in the strategic case, the first order condition (5.21) can be solved for $l^*(h)$ explicitly in the special case $\beta_3\gamma = \beta_4$. In such case,

$$l^*(h) = \frac{\left(\frac{\bar{H}^{\alpha_4}}{\delta I}\right)^{\frac{1}{1-\beta_4}} h^{\frac{\alpha_3\gamma-\alpha_4}{1-\beta_4}}}{1 + \left(\frac{\bar{H}^{\alpha_4}}{\delta I}\right)^{\frac{1}{1-\beta_4}} h^{\frac{\alpha_3\gamma-\alpha_4}{1-\beta_4}}}. \tag{5.22}$$

What remains to be derived is the constant $I = \int_H V(y)\lambda(dy)$. It can be found as an implicit solution of the following equation:

$$I = \frac{\int_H y^{\alpha_3\gamma} l^*(y)^{\beta_1\gamma}\lambda(dy)}{1 - \delta\int_H \left(\frac{y}{\bar{H}}\right)^{\alpha_4}(1-l^*(y))^{\beta_4}\lambda(dy)}, \tag{5.23}$$

with l^* defined as in (5.22) and thus containing I. The approximate solution to this equation can be easily computed numerically. Please note that knowing I, we can also obtain an explicit formula for the value function:

$$\begin{aligned} V(h) &= h^{\alpha_3\gamma} l^*(h)^{\beta_3\gamma} + \\ &+ \left(\frac{\delta \int_H y^{\alpha_3\gamma} l^*(y)^{\beta_1\gamma} \lambda(dy)}{1 - \delta \int_H \left(\frac{y}{\bar{H}}\right)^{\alpha_4} (1 - l^*(y))^{\beta_4} \lambda(dy)} \right) \left(\frac{h}{\bar{H}}\right)^{\alpha_4} (1 - l^*(h))^{\beta_4}. \end{aligned} \tag{5.24}$$

The direct computation of I would not have been possible if not for the introduction of stochastic transition in human capital levels. Thanks to that step, the infinite series expansion of $V(h)$ can be computed as a simple geometric series which has a closed-form sum. It also enables us to use the law of iterated expectations to convert an n-tuple integral into a product of n simple integrals.

Example 4 (Joy-of-giving altruism) *Let* $u(c) = c^{\gamma_5}$*,* $v(h') = (h')^{\gamma_6}$*,* $f(h, l) = h^{\alpha_5} l^{\beta_5}$*,* $g(h, 1 - l) = \frac{1}{\bar{H}^{\alpha_6}} h^{\alpha_6} (1 - l)^{\beta_6}$*. From* (3.14)*, we obtain the first order condition for the optimal policy* $l(h)$*. It is given as an implicit solution to the equation:*

$$\frac{l^{1-\beta_5\gamma_5}}{(1-l)^{1-\beta_6}} = \frac{\beta_5\gamma_5}{\delta\beta_6} (1 + \gamma_6) \bar{H}^{\alpha_6 - \gamma_6} h^{\alpha_5\gamma_5 - \alpha_6}. \tag{5.25}$$

Using the implicit function theorem, it is straightforward to show that $l(h)$ is everywhere decreasing whenever $\alpha_6 > \alpha_5\gamma_5$ and everywhere increasing whenever $\alpha_6 < \alpha_5\gamma_5$. In the special case where $\alpha_3\gamma_3 = \alpha_4$, (5.25) implies that $l(h)$ is constant, independent of h. This finding is crucial here because it is an exact analogue to Proposition 1 and an equivalent proposition which holds for the dynastic model: whenever the MPE labor supply policy of the model with strategic interactions is decreasing/increasing, it is also decreasing/increasing in both alternative models.

Just like in Example 2, the above equation (5.25) can be solved for $l^*(h)$ explicitly in the special case $\beta_5\gamma_5 = \beta_6$. In such case,

$$l^*(h) = \frac{\left(\frac{\gamma_6+1}{\delta}\right)^{\frac{1}{1-\beta_6}} \bar{H}^{\frac{\alpha_6-\gamma_6}{1-\beta_6}} h^{\frac{\alpha_5\gamma_5-\alpha_6}{1-\beta_6}}}{1 + \left(\frac{\gamma_6+1}{\delta}\right)^{\frac{1}{1-\beta_6}} \bar{H}^{\frac{\alpha_6-\gamma_6}{1-\beta_6}} h^{\frac{\alpha_5\gamma_5-\alpha_6}{1-\beta_6}}}. \tag{5.26}$$

We are now in the position to compare the equilibrium labor supply policy function derived from the model with strategic intergenerational interactions with the two alternative non-strategic scenarios. To attain direct comparability of all three setups, we must assure $\gamma = \gamma_1 = \gamma_2 = \gamma_5 = \gamma_6/\beta_1$ – in the dynastic model, the shape parameters of utility functions u and v must be equal while for the joy-of-giving altruism model, one has to impose $\gamma_6 = \beta_1\gamma_2$

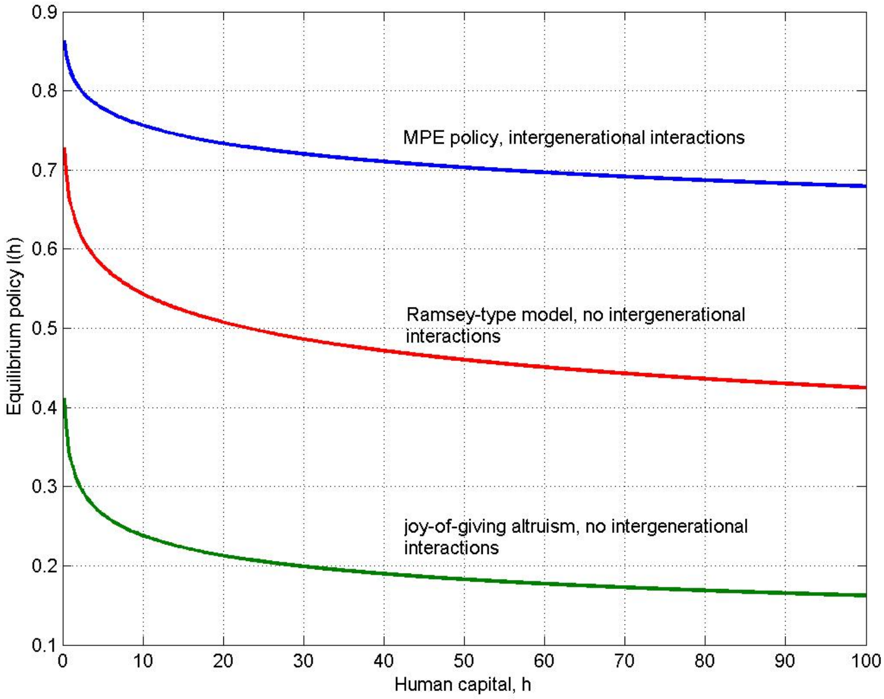

Figure 2: The difference between equilibrium policy functions $l^*(h)$ in the baseline model, compared with the Ramsey-type model and the model with joy-of-giving altruism. Assumed parameter values: $\alpha_1 = .3; \beta_1 = .7; \alpha_2 = .3; \gamma = .6; \beta_2 = \beta_1\gamma_1 = .42; \bar{H} = 100; \delta = .9$.

in order to equalize the elasticities of h' in both utility functions. We shall also fix our other parameters at equal levels, $\alpha_1 = \alpha_3 = \alpha_5, \beta_1 = \beta_3 = \beta_5, \alpha_2 = \alpha_4 = \alpha_6, \beta_2 = \beta_4 = \beta_6$.

The results are apparent in Figure 2. Most labor is supplied (and thus, least human capital is accumulated) in the case of the MPE policy in our baseline model with strategic interactions between subsequent generations, the second position is taken by the Ramsey-type model with dynastic optimization while the last position obtains to the model with joy-of-giving altruism.

While the ordering of the strategic model and the Ramsey-type dynastic model is certain (by Theorem 3, labor supply is always greater in the strategic model than in the dynastic model), the joy-of-giving model cannot be unambiguously ranked and thus the result presented in Figure 2 is not completely generic. There exist certain cases (though arguably unusual) in which joy-of-giving altruism could give rise to less human capital accumulation (and more labor supply) than dynastic optimization, possibly even more than the strategic intergenerational game.

Furthermore, even though there is a marked difference in the levels of human capital investment between the models, the *shapes* of the three policy functions are remarkably similar. With iso-elastic utility and Cobb-Douglas production functions, and under our benchmark parametrization, labor supply functions $l^*(h)$ always decrease with h, indicating that human capital and education effort are positively related, in line with empirical observations (e.g. Becker and Tomes (1986)).

5.1 Equilibrium investment in human capital: an interpretation

The uniform ordering of labor supply functions obtained from the three models under consideration (the policy curves such as the ones depicted in Figure 2 never intersect) offers an intuitive and convincing explanation. In simple words: the more directly does child's human capital enter parent's utility function, the more willing will she be to invest in it. With joy-of-giving altruism, utility is derived from child's human capital directly; consequently, investment in human capital will be the highest in such case, unless β_1 and β_2 are very low, indicating that current production as well as human capital accumulation react to changes in labor supply with a small elasticity. The rationale is that with strategic interactions, utility acquired from second period consumption is conditional on the strategy chosen by the subsequent generation while with joy-of-giving altruism, it is certain. Bernheim and Ray (1987) identify, however, another force at work here: since in the strategic model, each generation views the investment made by their children, $(1-l')$, as pure waste, it must invest more to obtain the same effect (compared to the joy-of-giving altruism model where investment in grandchildren is not thought of as waste). The latter force turns out to have a relatively smaller impact on our results in the benchmark parametrization, but it could become dominant if β's are sufficiently small.

With dynastic optimization, utility is derived from children's *utility* which is a function of their human capital. In such case, the parents know exactly what would eventually be optimal for their children; because of that knowledge, they can anticipate their children's choices and solve for the social planner's first best which involves substantial human capital investment (once you care for your children's utility, you also care for your grandchildren's, great-grandchildren's, etc.). Perfect anticipation across generations is not possible in our baseline model with intergenerational interactions, though. In such a model, utility is derived from children's *consumption* which is decided endogenously by them in a process of utility maximization which takes into account also the grandchildren's consumption, for which the original generation does not care. This gives one more intermediate step of embeddedness: human capital $\rightarrow$ children's utility $\rightarrow$ children's consumption. In result, the interest in investing in children's human capital is the least under this scenario. The unambiguous ordering of the strategic and the dynastic model, proved formally in Theorem 3, leads to the conclusion that strategic interactions across generations are an important source of underinvestment in human capital as compared to the intergenerational first best.[13]

5.2 Sensitivity analysis

In order to obtain a rough approximation of the magnitude of difference between equilibrium policies in the three considered models, we have carried out a numerical sensitivity analysis exercise: we have manipulated the parameters of the three models under study and compared the resultant equilibrium policy functions $l^*(h)$. For each parameter configuration, we calculated four measures of distance between the three functions. Since our initial presumption was that in principle, $l_{MPE} > l_R > l_{JOG}$ (where MPE stands for the Markov perfect equilibrium of our baseline strategic model, R stands for "Ramsey", i.e. the model featuring dynastic optimization, and JOG denotes joy-of-giving altruism), our proposed distance measures have been defined as follows:

1. <u>The area between l_{MPE} and l_R:</u> $D_1 = \int_H (l_{MPE}(h) - l_R(h))dh$.
2. <u>The area between l_R and l_{JOG}:</u> $D_2 = \int_H (l_R(h) - l_{JOG}(h))dh$. In the case this integral is negative, it follows that $l_{JOG}(h) > l_R(h)$ for all $h \in H$ (policy functions never intersect). If it is negative and $D_1 + D_2 = \int_H (l_{MPE}(h) - l_{JOG}(h))dh < 0$, it is also true that $l_{JOG}(h) > l_{MPE}(h)$ for all $h \in H$.
3. <u>The minimum distance between l_{MPE} and l_R:</u> $D_3 = \inf_{h\in H} |l_{MPE}(h) - l_R(h)|$.
4. <u>The minimum distance between l_R and l_{JOG}:</u> $D_4 = \inf_{h\in H} |l_R(h) - l_{JOG}(h)|$.

[13]This is of course just an intuitive argument, not a formal statement: no meaningful welfare comparisons can be made when welfare is measured differently across cases.

One crucial finding which facilitates the subsequent analysis and justifies the above definitions is that the policy functions never intersect.

For simplicity of computations, we have maintained the assumption $\beta_2 = \beta_1\gamma_1$; for comparability of our results, we have also retained the condition $\gamma_1 = \gamma_2$. This limits the scope of this sensitivity analysis exercise markedly, but our intention was not to search through the whole parameter space anyway. Even under these restrictions, we find both important departures from the baseline parametrization illustrated in Figure 2 and potentially large distances between the three policy functions.

First of all, our numerical exercise confirms that equilibrium policy functions l^* from different models indeed never intersect ($D_3, D_4 > 0$). Their benchmark ordering ($l_{MPE} > l_R > l_{JOG}$) is not robust across all cases, however. In the cases where β's are very low, joy-of-giving altruism has *less* human capital investment in equilibrium than any other case ($l_{JOG} > l_{MPE} > l_R$), and when they are slightly greater, we get $l_{MPE} > l_{JOG} > l_R$. This result has to deal with the strategic effect described by Bernheim and Ray (1987) – in our baseline model, each generation views the investment made by their children as pure waste and thus it must invest more to obtain the same utility gains – outweighing the direct effect of giving more when the joy is in giving itself. Indeed, when β's are low, human capital investment is inefficient, and thus in the baseline model, any investment in grandchildren must look like enormous waste. In the Ramsey-type model, on the other hand, greater investment in human capital is necessary to counteract the very low returns to human capital accumulation so that further generations of the dynasty would still get access to non-negligible utility levels: $u'(c) \to +\infty$ as $c \to 0$.

The numerical results on the ordering of policy functions obtained from the strategic model and from the Ramsey-type model ($l_{MPE} > l_R$) are, of course, consistent with implications of Theorem 3. The distance between these two policy functions can vary considerably, though: under some parametrizations (such as the baseline parametrization), it is large, while under others, in particular those involving radically low δ's, it may even be close to zero.

The results of our sensitivity analysis exercise have been summarized in Table 1. The baseline parametrization is: $\alpha_1 = 0.3; \beta_1 = 0.7; \alpha_2 = 0.3; \gamma = 0.6; \beta_2 = \beta_1\gamma_1 = 0.42; \bar{H} = 100; \delta = 0.9$, just like in the previous section. Unless indicated otherwise, these parameter choices are maintained throughout the table.

In conclusion, when comparing the joy-of-giving altruism model against any of the two other models, parametrization matters. Moreover, for quantitative results, parametrization matters a lot as well.

Case	D_1	D_2	D_3	D_4
Close to Baseline				
Baseline	23.7462	28.1911	0.1353	0.2624
$\beta_1 = 0.5$	25.9257	0.2803	0.1884	0.0027
$\alpha_1 = 0.6$	24.2728	3.8213	0.2336	0.0335
$\alpha_1 = \alpha_2 = 0.6$	13.0828	10.3587	0.0215	0.0292
$\alpha_2 = 0.6$	13.2903	35.1804	0.0043	0.0430
$\beta_1 = 0.6; \gamma = 0.8$	22.3790	36.1964	0.1617	0.3471
$l_{MPE} \approx l_R$: low δ				
$\alpha_1 = \alpha_2 = 0.6; \delta = 0.6$	4.0759	22.5245	0.0044	0.0469
$\alpha_1 = \alpha_2 = 0.6; \delta = 0.3$	0.4628	25.7418	0.0004	0.0416
$\delta = 0.6$	7.7896	50.7985	0.0296	0.4245
$\beta_1 = 0.6.\gamma = 0.8; \delta = 0.6$	6.4581	60.0304	0.0361	0.5829
$\delta = 0.3$	0.9392	61.0867	0.0026	0.4023
$\beta_1 = 0.6; \gamma = 0.8; \delta = 0.3$	0.5958	70.7015	0.0027	0.6077
$l_{MPE} > l_{JOG} > l_R$				
$\beta_1 = 0.25$	27.4195	-26.5800	0.2481	0.2482
$\beta_1 = 0.4; \gamma = 0.8$	25.7481	-3.2412	0.2237	0.0320
$\beta_1 = 0.45$	25.7630	-0.2971	0.2044	0.0029
$l_{JOG} > l_{MPE} > l_R$				
$\beta_1 = 0.1, \gamma = 0.8$	28.1443	-40.7932	0.2770	0.3888
$\beta_1 = 0.1; \alpha_1 = \alpha_2 = 0.6$	19.3906	-39.0093	0.1091	0.1863

Table 1: Sensitivity analysis results.

6 Conclusion

The purpose of the current paper has been to accomplish the two principal tasks: (i) to show how a Markov perfect equilibrium (MPE) policy function can be computed in a model with fully-specified intergenerational interactions in human capital accumulation, within an otherwise standard overlapping-generations framework; (ii) to compare the outcomes of the strategic model with two benchmark models which assume away intergenerational interactions. To this end, we have proven analytically that when compared to a model with dynastic optimization, our strategic model predicts *unambiguously lower* equilibrium investment in human capital accumulation. When compared to a model with joy-of-giving altruism, the results are ambiguous (though with a tentative indication towards the strategic model having less human capital accumulation as well).

We believe that finding a constructive algorithm for computing MPE policies in models of intergenerational altruism is a significant step forward in modeling strategic linkages across generations. In this paper, we have shown that this novel tool, developed by Reffett and Woźny (2008), can be generalized to capture intergenerational linkages in human capital accumulation. We have shown under which conditions the MPE policy exists and is unique, we have proven its monotonicity, and also presented a workhorse example for which most calculations could be done analytically, and for which the numerical convergence of our iterative procedure to the MPE is quick and easy.

We have also presented the conditions under which the MPE labor supply policy is increasing or decreasing. These conditions are the same for our strategic model, a model with joy-of-giving altruism, and a Ramsey-type model with dynastic optimization. Under a wide spectrum of "reasonable" parametrizations, all three models predict the equilibrium labor supply function to decrease with human capital levels.

What remains to be done is, first and foremost, a generalization of the constructive algorithm for computing MPE policies into higher dimensions. This is enforced by the fact that most economic models featuring intergenerational altruism are set up with multiple choice and state variables. Another issue which ought to be dealt with is the generalization of the stochastic transition function which was admittedly simplified in the current paper. We feel that these two steps are necessary in order to bring models with strategic interactions in human capital accumulation to the level of sophistication which is now common with models lacking such strategic interactions.

7 Appendix

Definition 2 *Let E be a real Banach space and $P \subseteq E$ be a nonempty, closed, convex set. Then:*

- *P is called a cone if it satisfies two conditions: (i) $x \in P, \epsilon > 0 \Rightarrow \epsilon x \in P$ and (ii) $x \in P, -x \in P \Rightarrow x = \theta$, where θ is a zero element of P,*
- *suppose P is a cone in E and $P^\circ \neq \emptyset$, where P° denotes the set of interior points of P, we say that P is a solid cone,*
- *every cone P in E defines an order relation $\leq$ in E as follows:*

$$x \leq y \text{ if } y - x \in P,$$

- *a cone P is said to be normal if there exists a constant $N > 0$ such that:*

$$(\forall x, y \in P) \quad \underline{\theta} \leq x \leq y \Rightarrow \|x\| \leq N\|y\|.$$

Theorem 4 (Guo, Cho, and Zhu (2004)) *Let P be a normal solid cone in a real Banach space with partial ordering $\leq$ and $B : P \to P$ be a decreasing operator (i.e. if $l_1 < l_2 \in P$ then $Bl_2 \leq Bl_1$) satisfying:*

$$(\exists r, 0 < r < 1)(\forall l \in P^\circ), (\forall t, 0 < t < 1) \quad t^r B(tl) \leq Bl, \tag{7.27}$$

then B has a unique fixed point in P° and the following holds:

$$(\forall l_0 \in P^\circ) \quad \lim_{n\to\infty} \|l_n - l^*\| \to 0, \tag{7.28}$$

where $(\forall n \geq 1)\, l_n = B(l_{n-1})$.

Proof of Theorem 1: Define an operator $A : L \to L$ in the following way:

$$\begin{aligned} (\forall h \in (0, \bar{H}]) \quad & A\mathbf{0}(h) = 1 \\ (\forall l \in L, l \neq \mathbf{0})(\forall h \in (0, \bar{H}]) \quad & \zeta(Al(h), h, l) = 0, \end{aligned} \tag{7.29}$$

and observe that the MPE of the economy are the fixed points of A on L. Note that under Assumptions 1 and 2, operator A is well-defined.

Let us first show by contradiction that A is strictly decreasing, i.e. $(\forall l_1, l_2 \in L)$ with $(\forall h \in (0, \bar{H}])\, l_2(h) > l_1(h)$ we have $(\forall h \in (0, \bar{H}])\, Al_1(h) > Al_2(h)$.

Let $l_1, l_2 \in L$ with $(\forall h \in (0, \bar{H}])\, l_2(h) > l_1(h)$ and suppose that $(\exists \tilde{h} \in (0, \bar{H}])$ such that $Al_2(\tilde{h}) \geq Al_1(\tilde{h})$. We consider two cases: $l_1 \neq \mathbf{0}$ and $l_1 = \mathbf{0}$.

Let us start with the former one. By definition of Al_1 and monotonicity of ζ we get for the given $\tilde{h}$:

$$0 = u'(f(\tilde{h}, Al_1(\tilde{h})))f_2'(\tilde{h}, Al_1(\tilde{h})) - g_2'(\tilde{h}, 1 - Al_1(\tilde{h})) \int_H v(f(y, l_1(y)))\lambda(dy) > \\ u'(f(\tilde{h}, Al_1(\tilde{h})))f_2'(\tilde{h}, Al_1(\tilde{h})) - g_2'(\tilde{h}, 1 - Al_1(\tilde{h})) \int_H v(f(y, l_2(y)))\lambda(dy) \geq \\ u'(f(\tilde{h}, Al_2(\tilde{h})))f_2'(\tilde{h}, Al_2(\tilde{h})) - g_2'(\tilde{h}, 1 - Al_2(\tilde{h})) \int_H v(f(y, l_2(y)))\lambda(dy),$$

which is a contradiction to the definition of Al_2 at $\tilde{h} \in (0, \bar{H}]$. If $l_1 = \mathbf{0}$ then by definition and hypothesis $1 = Al_1(\tilde{h}) \leq Al_2(\tilde{h}) = 1$. This contradicts the observation that $(\forall h \in (0, \bar{H}])\, Al_2(h) < 1$ when $l_2 \neq \mathbf{0}$ resulting from the Inada conditions stated in Assumption 2. We conclude therefore that A is strictly decreasing.

Suppose now that A has two ordered fixed points in L i.e. $(\forall h \in (0, \bar{H}])$, $l_2(x) \geq l_1(x)$ and $(\exists \tilde{h} \in (0, \bar{H}])$ such that $l_2(\tilde{h}) > l_1(\tilde{h})$. From the fixed point property at $\tilde{h}$ we get $Al_2(\tilde{h}) = l_2(\tilde{h}) > l_1(\tilde{h}) = Al_1(\tilde{h})$, which contradicts that A is strictly decreasing. Hence the set of fixed points of A, i.e. the set of the MPE of the economy, has no ordered elements.

The second statement of the theorem follows from the observation that for given assumptions, the objective function in (2.4) has strictly increasing marginal returns. An application of the theorem due to Amir (1996b) and Edlin and Shannon (1998) on strict comparative statics completes the proof.
■

Proof of Lemma 1: Let $h \in (0, \bar{H}]$ be given. From the definition, $\xi_h(\hat{l}) = \frac{u'(f(h,\hat{l}))f_2'(h,\hat{l})}{g_2'(h,1-\hat{l})}$. Note that $g_2' > 0$ for all arguments. Hence, $\xi_h(\hat{l})$ is well defined and ξ_h is continuous at the point $\hat{l} = 1$. Moreover, $\lim_{\hat{l} \to 0} \xi_h(l) = \infty$ and $\xi_h(1) = 0$. As u' and f_2' are strictly decreasing (with the second argument) and $g_2'(h, 1 - \hat{l})$ is strictly increasing as a function of l we conclude that ξ_h is strictly decreasing on $(0, 1]$. As a result ξ_h is invertible and its inverse is strictly decreasing. It is straightforward to verify (by strict monotonicity, strict concavity and continuous differentiability of u', f_2' and g_2') that ξ_h is continuously differentiable and $\xi_h' \neq 0$. Function ξ_h is also proper because ξ_h is continuous and $\xi_h^{-1}(\mathbb{R}_+) \subset [0, 1]$ which is compact in the standard topology on $\mathbb{R}$. Finally recalling the global implicit function theorem we get that function ξ_h^{-1} is also continuously differentiable. ■

Proof of Theorem 2: We will apply theorem 4 (see appendix) due to Guo, Cho, and Zhu (2004). Observe that P is a normal solid cone. The fact that B is a decreasing operator can be shown analogously to the proof of Theorem 1.

We will now show that the condition (7.27) in Theorem 4 (see Appendix) is satisfied.

Let $\bar{l} \in P, \bar{l} \neq 0$, $h \in (0, \bar{H}]$ and t such that $0 < t < 1$ be given. For a given r define a function $\phi_r : [0,1] \to \mathbb{R}_+$, $\phi_r(t) = t^r B(t\bar{l})(h)$. We will now show that there exists an $r, 0 < r < 1$, such that ϕ_r is increasing with t on $(0,1)$. By monotonicity and continuity of ϕ_r from the left at 1 we will conclude that there exists $r, 0 < r < 1$, for which the inequality $\phi_r(t) \leq \phi_r(1)$ is satisfied and so is $t^r B(t\bar{l}) \leq B\bar{l}$.

From the definition of $B(t\bar{l})$ we get: $\phi_r(t) = t^r \int_H v(f(y, \xi_h^{-1}(t\bar{l}(y))))\lambda(dy)$. Note also that $\bar{l}(y)v'(f(y,\xi_h^{-1}(tp(y))))f_2'(y,\xi_h^{-1}(t\bar{l}(y)))(\xi_h^{-1})'(t\bar{l}(y))$ is bounded for a given $\bar{l}$ and h. As a result, the function ϕ_r is continuously differentiable and

$$\begin{aligned}\phi_r'(t) = t^{r-1}\big[r \textstyle\int_H v(f(y,\xi_h^{-1}(t\bar{l}(y))))\lambda(dy) + \\ +t \textstyle\int_H \bar{l}(y)v'(f(y,\xi_h^{-1}(t\bar{l}(y))))f_2'(y,\xi_h^{-1}(t\bar{l}(y)))(\xi_h^{-1})'(t\bar{l}(y))\lambda(dy)\big].\end{aligned}$$

Denoting by E_x^f the elasticity of function f at point x in its domain observe that the second integral in the above expression can be reformulated to:

$$\begin{aligned}& t\textstyle\int_H \bar{l}(y)v'(f(y,\xi_h^{-1}(t\bar{l}(y))))f_2'(y,\xi_h^{-1}(t\bar{l}(y)))(\xi_h^{-1})'(t\bar{l}(y))\lambda(dy) = \\ & \textstyle\int_H v(f(y,\xi_h^{-1}(t\bar{l}(y))))\left[\frac{v'(f(y,\xi_h^{-1}(t\bar{l}(y))))}{v(f(y,\xi_h^{-1}(t\bar{l}(y))))}f(y,\xi_h^{-1}(t\bar{l}(y)))\right]\cdot \\ & \left[\frac{f_2'(y,\xi_x^{-1}(t\bar{l}(y)))}{f(y,\xi_h^{-1}(t\bar{l}(y)))}\xi_h^{-1}(t\bar{l}(y))\right]\left[\frac{(\xi_h^{-1})'(t\bar{l}(y))}{\xi_h^{-1}(t\bar{l}(y))}t\bar{l}(y)\right]\lambda(dy) = \\ & \textstyle\int_H v(\xi_h^{-1}(t\bar{l}(y)))E^v_{f(y,\xi_h^{-1}(t\bar{l}(y)))}E^{f,2}_{\xi_h^{-1}(t\bar{l}(y))}E^{\xi_h^{-1}}_{t\bar{l}(y)}\lambda(dy).\end{aligned}$$

Using the above reformulations and condition (2.10) we conclude that there exists an $r, 0 < r < 1$, such that $r \geq -E^v_{f(y,\xi_h^{-1}(t\bar{l}(y)))}E^{f,2}_{\xi_h^{-1}(t\bar{l}(y))}E^{\xi_h^{-1}}_{t\bar{l}(y)}$ holds for any $h \in (0, \bar{H}]$ and $t, 0 < t < 1$, and $\bar{l} \in P, \bar{l} \neq \mathbf{0}$. Adding non-negativity of v, we obtain:

$$\begin{aligned}\int_H rv(f(y,\xi_x^{-1}(t\bar{l}(y))))\lambda(dy) \geq \\ -\int_H v(f(y,\xi_x^{-1}(t\bar{l}(y))))E^v_{f(y,\xi_h^{-1}(t\bar{l}(y)))}E^{f,2}_{\xi_h^{-1}}(t\bar{l}(y))E^{\xi_h^{-1}}_{t\bar{l}(y)}\lambda(dy).\end{aligned} \tag{7.30}$$

It follows that $\phi_r'(t) \geq 0$ for $t \in (0,1)$ (since r does not depend on $\bar{l}$ or h). Hence, ϕ_r is increasing on $(0,1)$ for this r. Adding continuity of ϕ_r from the left at 1 we have: $t^r B(t\bar{l}) \leq B\bar{l}$ for any $t \in (0,1]$ and any $\bar{l} \in P$. We conclude therefore, that for all $h \in (0, \bar{H}]$ the inequality $t^r B(t\bar{l}) \leq B\bar{l}$ holds for any $t, \bar{l}$ as required by Theorem 4. ■

Proof of Theorem 3: Consider two families of functions parametrized by $h \in (0, \bar{H}]$, denoted as $S_h, Z_h : [0,1] \to \mathbb{R}_+$, such that for a given $h \in (0, \bar{H}]$, $S_h(l) = u(f(h,l)) + \delta g(h, 1-l) \int_H u(f(y, l_{MPE}(y)))\lambda(dy)$ and $Z_h(l) = u(f(h,l)) + \delta g(h, 1-l) \int_H V(y)\lambda(dy)$, where V is the value function corresponding to the Bellman equation (3.12).

We would like to show that for any given h, $S'_h(l) > Z'_h(l)$ in their whole domain. To this end, first note that

$$u(f(h, l_{MPE}(h))) \leq \max_{l\in[0,1]} u(f(h,l)) <$$
$$< \max_{l\in[0,1]} \{u(f(h,l)) + \delta g(h, 1-l) \int_H V(y)\lambda(dy)\} = V(h). \tag{7.31}$$

From the above reasoning, it immediately follows that

$$\int_H u(f(y, l_{MPE}(y)))\lambda(dy) < \int_H V(y)\lambda(dy) \tag{7.32}$$

and hence:

$$S'_h(l) = u'(f(h,l))f'_2(h,l) - \delta g'_2(h, 1-l) \int_H u(f(y, l_{MPE}(y)))\lambda(dy) >$$
$$u'(f(h,l))f'_2(h,l) - \delta g'_2(h, 1-l) \int_H V(y)\lambda(dy) = Z'_h(l), \tag{7.33}$$

which completes the first part of the proof.

Now let us superimpose another function $T : \{1,2\} \times [0,1] \to \mathbb{R}_+$ on top of that, such that $T(1,l) = Z(l)$ and $T(2,l) = S(l)$. From inequality (7.33) we have that $T'_2(2,l) > T'_2(1,l)$, and thus T has increasing marginal returns with $i = 1, 2$. For $i = 1, 2$, the function $T(i, \cdot)$ defined on the lattice $[0,1]$ is thus supermodular. Hence, by the theorem due to Amir (1996b) and Edlin and Shannon (1998), we obtain that $(\forall h \in (0, \overline{H}])\, l_{MPE}(h) = \arg\max_{l\in[0,1]} T(2,l) > \arg\max_{l\in[0,1]} T(1,l) = l_R(h)$. ∎

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