Long-run dynamics of exchange rates: A multi-frequency investigation

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Abstract

The empirical observation that purchasing power parity (PPP) holds in the long run but not in the short run has enjoyed a near-consensus status in international finance literature. However, a similar degree of agreement has not been reached with respect to the exact horizon of this "long run" aspect. To shed light on this matter, a novel approach is adopted in this paper to combine conventional time series methodology with insights from multi-frequency analyses. In particular, we simultaneously explore price-exchange-rate dynamics not only through time, but also at various horizons via a wavelet decomposition. Unit root tests applied to wavelet-based decomposed real exchange rates indicates that PPP holds at horizons consistent with the literature. With respect to the predictive value of our approach, we show that our decomposed measures provide guidance to future movements of real change rates. Additionally, we find that nominal exchange-rate dynamics are dominated by activities corresponding to low frequencies. Results from this study thus enable researchers and practitioners to establish an exchange-rate modelling framework with increased efficiency.

Keywords: Purchasing Power Parity; Wavelets; Multi-frequency Analysis

JEL classifications: F31, F39, C58

1. Introduction

The law of one price (LOP) states that the price of a good converted to the same currency is the same in any country. This law is generalised by the purchasing power parity (PPP) which established the relationship between nominal exchange rates and country-specific price levels. For nearly a century since its formation, PPP served as the building block upon which exchange rates are determined. Numerous researches seeking to empirically validate the effect of PPP generally arrived at a consensus: PPP tends to hold in the long run while it is violated in the short run (see e.g., Lothian 2016; Marsh et al., 2012). This idea resonates with the paradigm of the efficient market hypothesis (EMH), which supports long run market equilibrium while it is ambiguous on short run fluctuations in prices. However, the degree to which this line of though is accepted varies widely among academics. This is because of the limitation of conventional time series methodology in exploring multiple time horizons. A direct consequence of this dispute is varying performance of economic models aiming to forecast exchange rates, much to the dismay of both investing practitioners and policy makers (Cheung et al., 2019).

Previous research indicates that the real exchange rate time series, $\{r_t\}$, would constitute a stationary process for the PPP to hold. If this condition is violated, i.e. $\{r_t\}$ is non-stationary, the fundamental implications of monetary policies based on PPP may be misplaced. The notion regarding PPP is similar to the debate on the validity of the (more general) EMH, which basically argues that price discovery is a completely random process in an (informative) efficient market. This is to say that, theoretically, the price process should constitute a Brownian motion thus the prices changes (or returns) follow a white noise or martingale process, with zero autocorrelation among increments. This ensures that price movements cannot be predicted given information about past movements, thus preventing possible arbitrage activities. Equivalently speaking, the best forecast (conditioned on historical information) of tomorrow's price would be today's price, so that the impact of the one-period lagged observations would persist indefinitely.[1]

The multi-scale relationship is important in economics and finance because each

[1] The EMH is more often associated with stock markets than exchange-rate markets. However, if we think of exchange rates in place of randomly generated stock price then we can relate the two markets. This assumption is similar to (real) exchange rates are the prices of currencies, which is plausible. The common principle governing the two sectors is that the more efficient the market is, the stronger the unpredictability of future prices/exchange rates will be. Should this condition be violated, arbitrage trading would take effect. For example, an accurate prediction (if possible) of a fall in price would induce immediate short selling and price drop, resulting in a "self-fulfilled prophecy". As a result, the deviations of the real exchange rates from PPP is expected to be stationary over time.

economic agent/investor has a different investment horizon. Consider the large number of investors participating in the foreign exchange market and operate over different time scales. Market participants are a diverse group that include intraday traders, hedging strategists, international portfolio managers, commercial banks, large multinational corporations and national central banks (or the "market makers"). Market makers mostly interested in the long-run trends of foreign exchange market: They typically trade very infrequently but with large volume to ensure currency values are kept stable over the medium to longer terms. On the contrary, intraday traders and hedgers trade at a much higher frequency, are more interested in the highly volatile movements of currency values to exploit possible market inefficiencies and obtain short-run abnormal returns. In combination, the activities of independent market participants constitute the observed patterns of market indicators, i.e., prices and volumes. Due to the different decision-making time scales among traders, the true dynamic structure of the relationship between relative prices and exchange rates will vary over different horizons but is hidden under the observed data (which is often recorded at a fixed frequency, i.e., daily, monthly, annually). If the PPP dynamics change not only over time, but also over trading horizons, existing approaches are silent on the question "how long is the long run?", because most previous studies focus on a stylised two-scale analysis – short- and long-run. Recently, wavelet analysis has attracted attention in the field of financial economics as a mean of filling this gap (In and Kim, 2013).

Following this introduction, Section 2 discusses related literature on the topic. In Section 3, we present a novel wavelet-based decomposition framework to explore time series dynamics at multiple horizons. Section 4 describes our data and presents preliminary analyses based on various conventional stationary tests and unit root tests. In Section 5, we reapply these tests to the decompositions obtained via the wavelet transform and discuss the new implications. This discussion is followed by an investigation of the forecasting power of wavelet-based measures of real exchange rates and further analyses on power structure of nominal exchange rate time series. Section 6 provides concluding comments.

2. Related literature

It is well established that PPP holds in the long run. Because an international arbitrage forces the market to correct for any price misalignment, real exchange rate is expected to be mean reverting. Amongst others, a study of real exchange rate data in 20 countries by Taylor (2002) strongly confirmed this expectation. More importantly, his results were insensitive to the base currency examined and the development state of these countries. On the other hand, many researches argue that cross-border price differentials are large, persistent and generally violate the law of one price for a wide range of goods. Contributing to the deviation from PPP are numerous transportation and insurance costs, as well as non-traded expense such as retail and wholesale distribution costs (Engel and Rogers, 1996). Further differences in price come through the presence of tariff and non-tariff barriers (Knetter, 1994). It is found that in the context of international trade, these factors significantly reduce the validity of PPP, at least in the short run.

A consensus is reached stating that in reality, it is possible to exploit real exchange rate differentials between two countries due to market inefficiencies. Delay in price adjustments are captured by Dornbusch (1980) who, among others, questioned the validity of PPP in the short run due to "overshot" exchange-rate volatility. To validate for PPP, the most popular techniques involve some test for the stationary of real exchange rates. Although the specifications of such tests vary, in general if the null hypothesis of stationarity is rejected (or in many cases, unit root is detected) then $\{r_t\}$ is said to follow a random walk/Brownian motion, and the PPP is also rejected. A weakness of conventional test for stationarity such as Augmented-Dickey Fuller (ADF) or Phillip-Perron (PP) is that they tend to confirm full integration when the order of integration (denoted as d) is very close to one. In other words, these tests were criticised for lacking the power to distinguish between unit root and near-unit root behaviour. It has also been observed that alternative tests of unit root such as KPSS have more power than ADF and PP when the underlying process exhibits unit root close to non-stationary boundary. Abuaf and Jorion (1990, p. 157) indicated that *"(...) the negative results obtained in previous empirical research reflect the poor power of the test rather than evidence against PPP"*.

In practice, financial time series with a close-to-one value of d may still be stationary and at the same time exhibit long-range dependence/long memory (instead of infinite dependence as suggested by the random walk model). If exchange rate time series have long memory, we need a different class of process to model it. This class includes, but is not limited to, the fractionally difference process (FDP). The FDP allows the time series to have a non-integer order of integration. The process is considered stationary (in the strong sense, i.e., both mean and covariance stationary) if $-1/2 < d < 0$. Granger and Joyeux (1980) added that as long as $0 \leq d < 1/2$ a time series is still mean stationary (with long memory) even though it no longer has a finite covariance matrix. Generally, we can classify financial time series based on their order of integration (expressed via the Hurst exponent and/or the fractional differenced parameter d) as follows:

Table 1. Categorizing stochastic processes based on their long-memory property

Hurst exponent	Fractional difference parameter	The behavior of the process
$H \leq 0$	$d \leq -1/2$	Non stationary
$0 < H < 1/2$	$-1/2 < d < 0$	Anti-persistent, mean reversing
$H = 1/2$	$d = 0$	Random, no trend, Brownian motion
$1/2 < H < 1$	$0 < d < 1/2$	Long range dependence
$H \geq 1$	$d \geq 1/2$	Non stationary

Source: Vo and Vo (2019).

Hurst (1951) was the first to propose a method to detect and estimate the widely observed empirical long-memory in the form of the 'rescaled range' statistic, denoted as $R/S(n)$, with n the sample size. His method aims to infer the Hurst index H as implied by the relationship $E[R/S(n)] \sim Cn^H$ when $n \to \infty$ and $C > 0$ is independent of n. This empirical law is referred to as the 'Hurst effect'. The parameter H takes on value in the interval $[0, 1]$ and if observations are generated from a short-range dependent process then $H = 0.5$. In this case the process is said to be "self-determining". As there is no long-range dependence, time series generated by such process cannot be forecast from past information. This is analogous to the case when stock prices follow a Brownian motion, with discrete realizations following a random walk model.

Many recent studies of financial data adopt the approach from a 'time domain' perspective, that is, the data are analyzed as time series which are commonly recorded at a pre-determined frequency(s) (i.e., daily, weekly, monthly etc.). This approach, no matter how effective, implicitly limits the recorded frequency as the sole frequency to be considered when studying realizations of a time varying variable. Problems emerge when this assumption turns out to be insufficient. Specifically, what will the situation be when there are many, not one, frequencies that dictate the underlying generating process of the variable of interest? To address this concern, a new approach taking into account the frequency aspect was proposed. A well-established methodology representing this branch of 'frequency-domain' analysis is the Fourier transform/spectral analysis. In general, this method is a very powerful tool specifically designed to study cyclical behavior of stationary variables, such as those frequently observed in financial data. Based on this fundamental idea, an advanced technique was developed to simultaneously incorporate both aspects, i.e., time and frequency, of a data sequence. This relatively novel methodology is known as the wavelet transform. It is worth noting that though wavelet analysis is well-established in the fields of physics and engineering (in particular, signal processing), its application in economics and finance is only recently becoming popular thanks to the effort of pioneers such as Gencay et al. (2002) and In and Kim (2013).

3. A primer on wavelet methodology

In the following discussion, we present a brief revision of the wavelet theory with focus on the intuition and most direct application rather than technical discussion. The technical details are simplified to emphasize only the most essential aspects of the theory.[2] Wavelet-based methodology has its origin traced back to from Haar (1910), although developments in modern wavelet theory date back to the 1980s. To illustrate the approach, we directly describe two interrelated procedures: (i) Wavelet decomposition and (ii) its inverse process, wavelet reconstruction, the two of which are crucial for analysing our time series. The 'wavelet' at its core is simply a function, which creates a representation of the original data series by convolving with it. Academics often describe this method as a way to 'project' the data on a function that oscillates on a short time interval. This function, termed the 'mother' wavelet, satisfies two basic conditions, as noted by Baqaee (2010):

$$\int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1 \; ; \;\; \int_{-\infty}^{\infty} \psi(t) dt = 0.$$

The first condition implies that the energy of the function, expressed by a sum of squares, equals one (we say it has unit energy). The second shows that the sum of oscillations is zero. Taken together, we have a 'small wave' whose (non-zero) fluctuations die out (or cancel out) quickly. This is in contrast to the infinite persistence of the sine/cosine functions which forms the basis of the Fourier transform (Masset, 2008). In addition, these two conditions are complemented by a fundamental rule that all wavelet functions $\psi(t)$ must satisfy, known as the "admissibility rule" (Goupillaud et al., 1984):

$$C_\psi = \int_0^{\infty} \frac{|\Psi(f)|}{f} df < \infty \, ,$$

where $\Psi(f)$ is the Fourier transform of $\psi(t)$.

To begin our analysis, the mother wavelet must be transformed into a scaled (dilated) version and then shifted (translated) to a recursive form: $\psi_{u,s} = \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right)$, where s and u are the dilated and translated parameter, respectively. The continuous wavelet transform (CWT) can now be derived as the projection of function $f(t)$ on the

[2] Interested readers are referred to detailed accounts of the methodology from the theoretical works of Daubechies (1992), Percival and Walden (2000) and Mallat (2009), among others.

wavelet $\psi_{u,s}(t)$, that is:

$$W(u,s) = \int_{-\infty}^{\infty} f(t)\psi_{u,s}(t)dt.$$

By continuously applying this operator to an infinite range of u and s we are able to break the original function down to its simpler components, a process referred to as the 'decomposition'.[3]

Since our interest mainly lies in the decomposition of a finite time series, weadopt the discrete approach [called the Discrete Wavelet Transform (DWT)] in the form of the pyramid algorithm developed by (Mallat, 2009). Gencay et al. (2002) refers to the DWT as some type of critical sampling of the CWT. This means the DWT contains the minimum number of CWT coefficients sufficient to preserve information of the original signal in a parsimonious way. In the DWT, we have the translated and dilated wavelet function in discrete form:

$$\psi_{j,k}(t) = 2^{j/2}\psi(2^{j}t - k),$$

where j and k are scale and location parameters, respectively. These parameters determine the length and location of the wavelet over time. One of Daubechies (1992)'s propositions in her book is that a family of all $\psi_{j,k}$ forms an orthonormal basis for the space of all square integrable functions [denoted as $L_2(\mathbb{R})$]. This implication is the cornerstone of wavelet theory which allows us to express any function in this space as a "linear combination of multiple wavelets" in a similar manner as decomposing a time series into sinusoids.

The mother wavelet is the key component to our analysis; however, to completely decompose any function f in the space $L_2(R)$ we need a complementary component called the 'father' wavelet function (denoted as φ). According to Baqaee (2010), with this addition we can represent a function f as follows:

$$f(t) = \sum_{l\in Z} < f, \phi_l > \phi_l(t) + \sum_{j=0}^{\infty}\sum_{k\in Z} < f, \psi_{j,k} > \psi_{j,k}(t),$$

where $<.,.>$ denotes the convolution or inner product between the signal and the filters. The components of the output of DWT are called the wavelet coefficients: $W(j,k) = 2^{j/2}\sum_t x_t\psi(2^{j}t - k)$, and the scaling coefficients: $V(j,k) = 2^{j/2}\sum_t x_t\phi(2^{j}t - k)$ $\left(j = 1,\dots,J; k = 1,\dots,n/2^{j}\right)$ in which ψ and ϕ are discrete functions. From the output of the DWT we can derive the "detail" coefficient vector (D_j) and "smooth" coefficient vector (S_j) following the procedure outlined in Gencay et. al (2010). It can further be shown that

[3] It is also possible to reconstruct the function by an inverse operation (Gencay et al., 2002). Because $W(u,s)$ is redundant, it is impractical and computationally costly to decompose a discrete signal with all wavelet coefficients from the CWT.

the original signal (X_t) can be decomposed into these vectors, and its variance can be decomposed into the component variances as follows:

$$X_t = \sum_{j=1}^{J} D_{j,t} + S_{J,t};\ \|X\|^2 = \sum_{j=1}^{J} \|D_j\|^2 + \|S_J\|^2\ .$$

At level j, where the detail vector corresponds to all fluctuations associated with the frequency band $\left(\frac{1}{2^{j+1}}, \frac{1}{2^j}\right)$, the smooth vector corresponds to all activities associated with frequencies lower than $\frac{1}{2^{j+1}}$ (or with periods longer than 2^{j+1}).[4] As a result, by examining these coefficients at individual frequencies we can identify underlying local fluctuations in time series.

To sum up, the discrete wavelet transform provides us a tool to look at our data at different horizons by mean of an additive decomposition, (that is, we can recover the original signal by summing the detail and smooth components) hence this framework is often referred to as the multi-resolution analysis (MRA).

4. Data description and preliminary analyses

To implement the unit root test for real exchange rate time series, we first need to gather the nominal exchange rate and the country-specific inflation rates proxied by the logarithmic changes in CPI. For each country, we collect CPI for all items and for the total population for consistency. Also, since the highest data release frequency in most countries is monthly, we shall use this as our primary frequency whenever possible, in order to get the highest number of observations. We focus on the currencies comprising the current Special Drawing Right basket of the IMF (the USD, EUR, JPY, GBP and CNY) and two currencies that are heavily traded: CAD and CHF. All data were obtained from U.S. Federal Reserve Bank of St. Louis.[5] Since the Euro was in effect only after 1999, we study the sample spanning the period 01/1999 - 02/2019, covering 242 months. The real exchange rate

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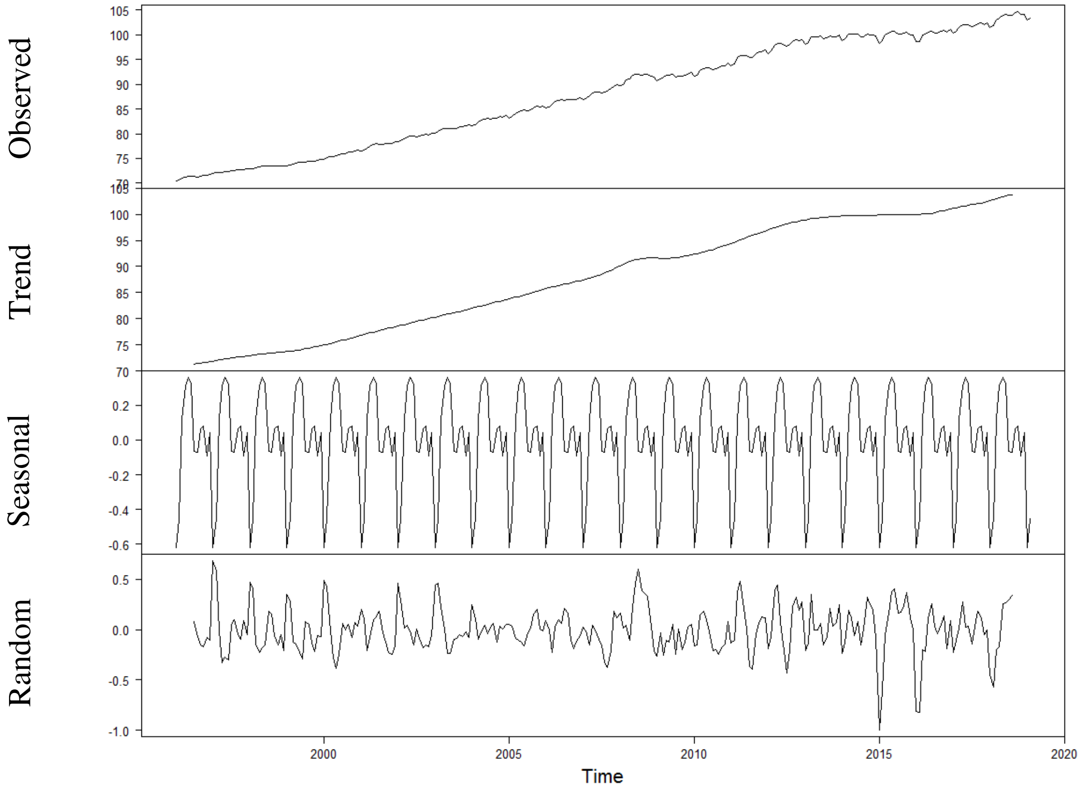
[4] These definitions imply that the wavelet is localised in time and frequency, and the shape of the wavelet coefficient time series will resemble that of the original series. As Kim and In (2010, p. 3) pointed out, *"coefficients over rough sections or over jumps [...] will be large relative to [...] smooth sections"*. In particular, we have larger wavelet coefficients whenever the wavelet function resembles the signal more closely. In a similar manner, when a sine wave at a particular frequency resembles the signal, its periodogram at that frequency spikes up. A recent empirical investigation of the periodogram of long-memory time series can be found in Vo and Vo (2019).

[5] Available at: https://fred.stlouisfed.org/series/. The data mnemonics for exchange rates are as follows: Eurozone ("DEXUSEU"), Japan ("DEXJPUS"), U.K. ("DEXUSUK"), China ("DEXCHUS"), Canada ("DEXCAUS"), Switzerland ("DEXSZUS"). The data mnemonics for CPIs are as follows: U.S. ("CPIAUCSL"), Eurozone ("CPHPTT01EZM661N"), Japan ("JPNCPIALLMINMEI"), U.K. ("GBRCPIALLMINMEI"), China ("CHNCPIALLMINMEI"), Canada ("CPALCY01CAM661N"), Switzerland ("CHECPIALLMINMEI"). To download the data, simply combine the listed URL with these mnemonics.

series, which is computed as $r_t = \log P - \log P^* - \log S$, where P and P^* denote the price levels in corresponding countries and in the US and S denotes the domestic currency unit prices of one USD. Because most of the available data were not seasonally adjusted (the only exception being the U.S.'s CPI), we perform this crucial step using the method proposed by Kendall et al. (1983). Specifically, since a CPI series tends to exhibit monthly seasonality and spans over a uniform number of periods (months), it is appropriate to decompose the CPI time series using an additive model such as: $P_t = T_t + SE_t + e_t$, where P_t denotes the original CPI series, T_t is its trend component (which is often estimated using a moving average operator), SE_t represents the seasonal component (which is computed by averaging, for each time unit, over all periods) and e_t denotes a noise component. As an example, Figure 1 shows the seasonality adjustment of the Eurozone's CPI series.

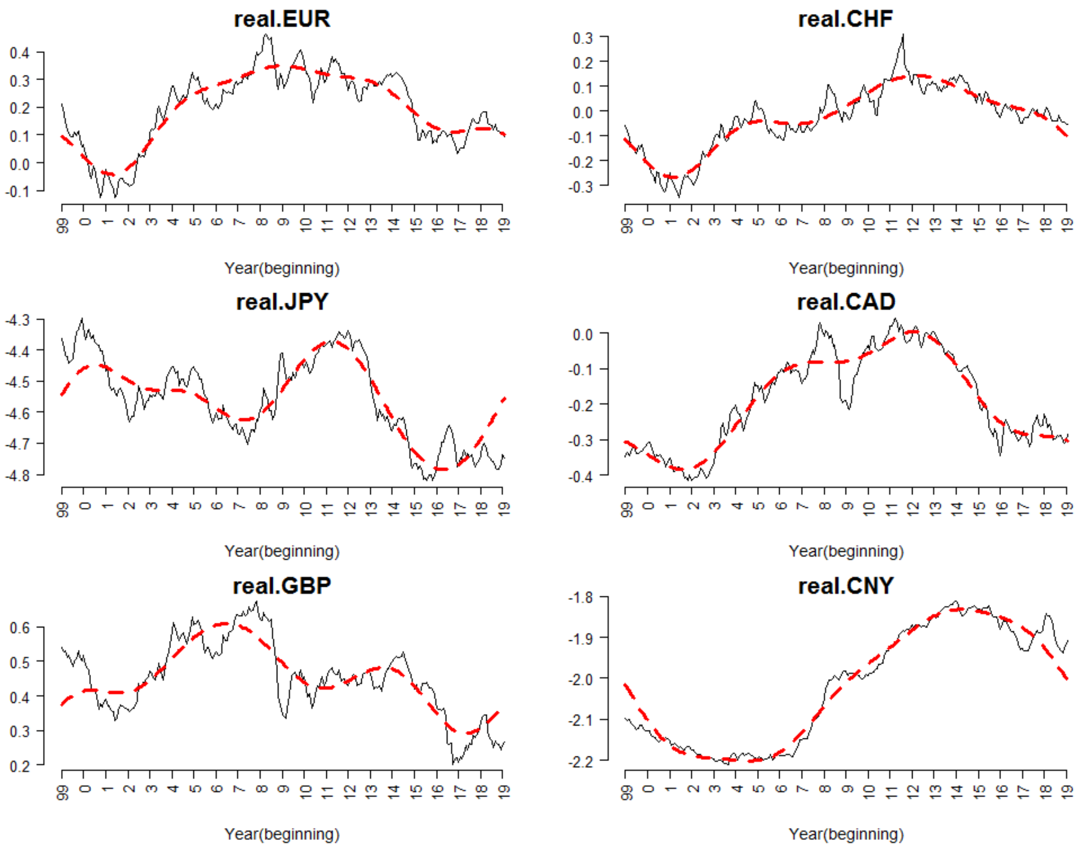
The summary statistics of the seasonally-adjusted CPI and real exchange-rate time series are presented in Table 2. On the basis of the Jarque-Beta tests for the null of normality, the real exchange rates appear to have non-normal distribution (except for USD/JPY) and exhibit negative skewness and kurtosis. Additionally, among these series there exist a strong degree of autocorrelation (up to lag 21), as indicated by the Ljung-Box statistics. The real exchange rate series are plotted in Figure 2 (solid lines). We can see that these currencies (except for the EUR and the GBP) exhibit negative values for most of the sample period, which implies that the currencies are undervalued with respect to the USD. Indeed, from the first row of Table 2, we can see that on average, over the sample period, the EUR appreciated by $100 \times (e^{0.19} - 1) = 21\%$ and the GBP appreciated by 57%. The Yuan's undervaluation/misalignment is significantly reduced when China floats its currency. The USD/JPY real exchange rate seems to exhibit more random-walk like behaviour. In contrast, the other four series all seem to follow a trend stationary pattern. Another notable feature is the sharp depreciation of the USD against all four currencies around the beginning of 2001. This could reflect the impact of the 2000 Dotcom bubble burst. Subsequently, the USD regained its value and appreciated (except for USD/JPY) up until the financial crisis in 2008. The dashed lines indicate the fitted smooth components derived from our wavelet decomposition, which will be discussed in Section 5.

Figure 1. Additive decomposition of the Eurozone's CPI time series, 1999 – 2019



Notes: The decomposition is computed using Kendall et al. (1983).

Figure 2. Real exchange rate time series



Notes: In each panel, the solid line presents the values of $r_t = \log P - \log P^* - \log S$, where P and P* denote the price levels in corresponding countries and in the US, and S denotes the local currency cost of 1 USD. This implies that positive (negative) values indicate over (under)-valuation of the corresponding currencies. The dashed line indicates a smoothed trend (to be discussed in Section 5) derived from a wavelet decomposition.

Table 2. Summary statistics of real exchange-rate time series

	US	Eurozone	Japan	UK	Switzerland	Canada	China
Mean		0.19	-4.56	0.45	-0.03	-0.19	-2.02
Median		0.22	-4.55	0.45	-0.02	-0.20	-1.99
Variance		0.02	0.02	0.01	0.02	0.02	0.02
Skewness		-0.45	-0.09	-0.15	-0.45	0.01	0.00
Kurtosis		-0.59	-1.07	-0.46	-0.07	-1.33	-1.66
JB		11.46	11.58	2.79	8.33	17.60	27.45
		(0.00)	(0.00)	(0.25)	(0.02)	(0.00)	(0.00)
LB(21)		3073.63	2592.19	2262.34	3337.66	3421.57	4412.81
		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
No. of observations	242	242	242	242	242	242	242

Notes: The real exchange rate is computed as $r_t = \log P_t - \log P_t^* - \log S_t$, where P_t and P_t^* denote the price levels at time t, measured by seasonally adjusted CPIs, of the corresponding countries and of the US. S_t denotes the nominal exchange rates at t. JB and LB (21) represent the Jarque-Bera and Ljung-Box statistics (with 21 lags), respectively. Corresponding p-values are in parentheses.

As mentioned in the first section, tests designed for detecting non-stationarity are useful for validating PPP. Table 3 presents the results for the three tests (all specifications include a drift and a trend term, and the lag selection is based on the Akaike Information Criteria). As we can see, there is strong agreement among test results: The tests for unit root (ADF [Dickey and Fuller, 1979] and PP [Phillips and Perron, 1988]) and stationary (KPSS [Kwiatkowski et al., 1992]) all imply non-stationarity of the real exchange rate series. Most notably, the KPSS test rejects the null hypothesis of stationarity at 1% for all series. According to these univariate tests, for the sample period, all six real exchange-rate series do not follow patterns determined by the PPP theory. However, as unit-root and stationary tests for individual time series suffer from lower power (see e.g., Abuaf and Jorion, 1990; O'Connell, 1998), we complement these tests with the panel unit root tests developed by Choi (2001), Hadri (2000), Im, Pesaran and Shin (2003) (IPS) and Maddala and Wu (1999) (MW).[6] As can be seen from Table 4, even the increased power of the panel tests does not help alleviate the concern that the real exchange rate series are non-stationary, since we cannot reject the null of unit root for the panel as a whole.

Table 3. Univariate test results for real exchange rate time series

	USD/EUR	USD/JPY	USD/GBP	USD/CHF	USD/CAD	USD/CNY
ADF	-1.56	-1.74	-2.14	-1.45	-1.34	-1.53
	(0.76)	(0.68)	(0.52)	(0.81)	(0.85)	(0.78)
PP	-4.97	-8.19	-8.35	-8.15	-2.87	-3.81
	(0.83)	(0.65)	(0.64)	(0.65)	(0.94)	(0.90)
KPSS	1.54	2.23	1.91	3.50	1.91	5.30
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)

Notes: The null for the ADF and PP tests is that the times series contain a unit root, while the null for the KPSS test is that the time series is stationary. All specifications contain a constant and a trend term. Lag order selection is based on the Akaike Information Criteria. Corresponding p-values are in parentheses.

Table 4. Panel test results for real exchange rate time series

	Choi	Hadri	IPS	MW
Statistic	1.14	196.7	0.98	5.77
	(0.87)	(0.00)	(0.84)	(0.93)

Notes: The null hypothesis for all tests (except for Hadri's) is that all series contain a unit root, against the alternative of at least one series is stationary. The null of Hadri test is that all series are stationary,

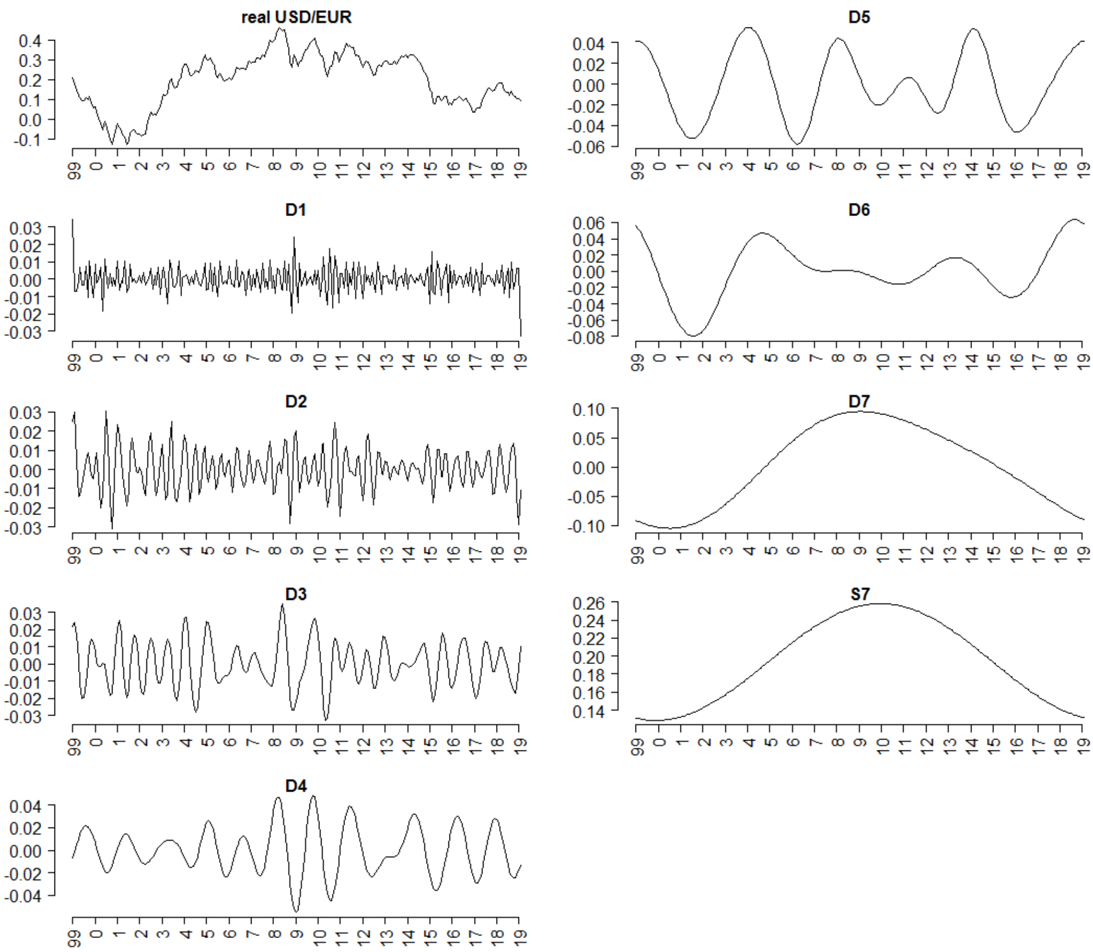
[6] All tests except for Hadri (2000)'s are based on a combination of estimations from ADF regressions for each time series. The Hadri residual-based LM statistic is a cross-sectional average of the univariate KPSS statistics, normalised by their asymptotic mean and standard deviation (Croissant and Millo, 2008). That is, Hadri's can be considered as a panel KPSS test.

against the alternative of at least one series contain a unit root. All specifications contain a constant and a trend term. Lag selection is based on AIC. Corresponding p-values are in parentheses.

5. Analyses of wavelet-based decomposition

To begin this section, Figure 3 presents the illustrative wavelet decomposition of the real USD/EUR exchange rate, at multiple horizons corresponding to multiple frequencies, using the MRA framework described in Section 3. It is noted how the MRA allows us to decompose the time series of interest into multiple time horizons. The detail series at high frequencies (D1 to D5) capture noisy fluctuations and are quite different from the underlying smooth trend component (S7). In contrast, the pattern of detail series at low frequencies (D6 and D7) are closer to that of the smooth component, because at these frequencies, most noises are filtered out. The MRAs for other currencies are not qualitative different, as are available upon request.

Figure 3. Multi-resolution decomposition for the real USD/EUR exchange rate



Notes: This figure presents the MRA for the monthly real USD/EUR series. Data range is Jan 1999 – Feb 2019. Note that the series in the top left panel (the original data) equals the sum of the 7 detail components (D1 to D7) and the smooth component (S7).

Tests for PPP revisited

It is now time to answer the question raised in the introduction, i.e., at which particular horizons this violation of PPP does/does not persist. To help clarify the relationship between details and smooths, in Table 5 we provide guidance with respect to the data used in our tests.

Table 6 presents iterations of the same tests in Section 4 applied to seven components of the wavelet-decomposed exchange rate real series. Our wavelet decomposition procedure (described in Section 3) provides numerical representations of the original series in the form of "detailed" components (which capture transient fluctuations within specific consecutive frequency band) and a "smooth" trend component. According to Table 6, for detail series, unit root and stationary tests confirm stationarity at lower frequencies. This is perhaps not surprising, considering that detail series from D1 to D5 essentially capture the high-frequency fluctuations of the data (as deviations from the trend) and thus tend to be stationary (see Figure 3).

On the other hand, D6 and D7 are "denoised" and are non-stationary. Therefore, a more intuitive and reliable approach to PPP testing is to examine the corresponding smoothed series constructed at each level, rather than the detail series, as these reflect the real exchange-rate trend better. Table 7 present analogous results for the tests for smoothed series' stationarity. In this case, the tests only provide supportive evidence for PPP at very low frequencies, corresponding to scales higher than level 5 (i.e, for either S6 or S7), and only for ADF tests. PP and KPSS tests generally indicate that unit-roots exist. The disagreement among the univariate test results could be a result of low power.

Table 5. Decomposition components used in unit-root tests

Decomposition level								
1	D1	**S1**						
2	D1	D2	**S2**					
3	D1	D2	D3	**S3**				
4	D1	D2	D3	D4	**S4**			
5	D1	D2	D3	D4	D5	**S5**		
6	D1	D2	D3	D4	D5	D6	**S6**	
7	**D1**	**D2**	**D3**	**D4**	**D5**	**D6**	**D7**	**S7**

Notes: This table presents the relationship between detail and smooth components derived from the MRA. The summation of these components in each row equals the original real exchange-rate series. The emboldened components indicate those used in univariate tests (Tables 6 and 7) and panel tests (Table 8).

Table 6. Univariate test results for 7 detail components

	D1	D2	D3	D4	D5	D6	D7
				A. USD/EUR			
ADF	-18.25	-9.93	-6.91	-5.50	-5.92	-2.48	-3.22
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.37)	(0.09)
PP	-284.61	-68.66	-78.92	-43.65	-8.34	-5.56	1.13
	(0.01)	(0.01)	(0.01)	(0.01)	(0.64)	(0.80)	(0.99)
KPSS	0.17	0.05	0.02	0.03	0.09	0.58	1.91
	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.02)	(0.01)
				B. USD/JPY			
ADF	-17.94	-8.76	-6.15	-4.08	-6.87	-2.66	-3.48
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.30)	(0.05)
PP	-290.25	-75.49	-77.65	-35.95	-10.48	-1.35	-3.13
	(0.01)	(0.01)	(0.01)	(0.01)	(0.52)	(0.98)	(0.93)
KPSS	0.32	0.16	0.09	0.20	0.44	0.99	2.46
	(0.10)	(0.10)	(0.10)	(0.10)	(0.06)	(0.01)	(0.01)
				C. USD/GBP			
ADF	-17.55	-8.87	-6.71	-4.15	-5.69	-2.40	-2.92
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.41)	(0.19)
PP	-291.47	-74.61	-71.03	-38.76	-9.63	-3.28	-2.29
	(0.01)	(0.01)	(0.01)	(0.01)	(0.57)	(0.92)	(0.96)
KPSS	0.29	0.17	0.09	0.14	0.28	0.34	2.18
	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.01)
				D. USD/CHF			
ADF	-20.39	-10.04	-6.34	-5.03	-4.98	-2.08	-3.00
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.54)	(0.15)
PP	-311.12	-70.21	-81.97	-43.84	-10.23	-4.40	3.71
	(0.01)	(0.01)	(0.01)	(0.01)	(0.54)	(0.86)	(0.99)
KPSS	0.01	0.01	0.01	0.02	0.12	0.78	3.72
	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.01)	(0.01)
				E. USD/CAD			
ADF	-20.54	-9.00	-8.10	-6.08	-4.58	-0.92	-1.96
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.95)	(0.59)
PP	-301.27	-62.83	-80.62	-38.39	-12.66	-3.54	1.64
	(0.01)	(0.01)	(0.01)	(0.01)	(0.40)	(0.91)	(0.99)
KPSS	0.12	0.02	0.01	0.02	0.07	0.18	2.18
	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.10)	(0.01)
				F. USD/CNY			
ADF	-19.45	-10.34	-6.67	-5.76	-4.11	-4.22	-2.29
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.45)
PP	-246.30	-73.51	-71.82	-26.80	-12.14	-6.42	-1.35
	(0.01)	(0.01)	(0.01)	(0.02)	(0.43)	(0.75)	(0.98)
KPSS	0.36	0.22	0.15	0.27	0.47	1.95	4.75

(0.10)	(0.10)	(0.10)	(0.10)	(0.05)	(0.01)	(0.01)

Notes: The null for the ADF and PP tests is that the times series contain a unit root, while the null for the KPSS test is that the time series is stationary. p-values are in parentheses. See Section 3 for details on wavelet-based decomposition.

Table 7. Univariate test results for 7 smooth components

	S1	S2	S3	S4	S5	S6	S7
				A. USD/EUR			
ADF	-1.39	-1.39	-1.36	-0.63	-1.36	-3.21	-4.30
	(0.83)	(0.83)	(0.84)	(0.98)	(0.84)	(0.09)	(0.01)
PP	-4.74	-4.11	-2.51	-1.20	0.24	1.06	0.93
	(0.84)	(0.88)	(0.95)	(0.98)	(0.99)	(0.99)	(0.99)
KPSS	1.55	1.56	1.60	1.68	1.90	1.87	1.80
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
				B. USD/JPY			
ADF	-1.36	-2.58	-1.48	-3.69	-2.58	-3.39	-3.74
	(0.84)	(0.33)	(0.79)	(0.03)	(0.33)	(0.06)	(0.02)
PP	-8.45	-8.23	-6.31	-4.45	-2.01	-3.52	-2.51
	(0.64)	(0.65)	(0.76)	(0.86)	(0.97)	(0.91)	(0.95)
KPSS	2.20	2.17	2.14	2.09	2.00	2.86	2.59
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
				C. USD/GBP			
ADF	-1.97	-2.32	-1.87	-2.77	-3.41	-2.84	-2.92
	(0.59)	(0.44)	(0.63)	(0.25)	(0.05)	(0.22)	(0.19)
PP	-9.17	-9.54	-7.09	-4.05	-2.69	-2.35	-2.37
	(0.60)	(0.57)	(0.71)	(0.88)	(0.95)	(0.96)	(0.96)
KPSS	1.89	1.86	1.86	1.88	2.05	2.32	2.48
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
				D. USD/CHF			
ADF	-1.01	-0.88	-0.57	0.50	-0.19	-2.90	-3.26
	(0.94)	(0.95)	(0.98)	(0.99)	(0.99)	(0.20)	(0.08)
PP	-7.32	-5.83	-2.72	-0.74	2.09	3.30	2.62
	(0.70)	(0.78)	(0.95)	(0.99)	(0.99)	(0.99)	(0.99)
KPSS	3.50	3.53	3.57	3.67	3.89	3.67	3.54
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
				E. USD/CAD			
ADF	-0.22	-0.76	-0.33	-1.37	-1.94	-2.03	-3.71
	(0.99)	(0.96)	(0.99)	(0.84)	(0.60)	(0.56)	(0.02)
PP	-2.36	-1.89	-0.81	0.33	0.71	1.60	1.53
	(0.96)	(0.97)	(0.99)	(0.99)	(0.99)	(0.99)	(0.99)
KPSS	1.90	1.90	1.92	1.98	2.05	2.17	2.15
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
				F. USD/CNY			
ADF	-0.14	-0.69	-0.85	-0.22	-3.28	-2.26	-8.26
	(0.99)	(0.97)	(0.96)	(0.99)	(0.07)	(0.46)	(0.01)

PP	-4.52	-4.74	-3.50	-3.40	-1.81	-1.27	-1.15
	(0.86)	(0.84)	(0.91)	(0.92)	(0.97)	(0.98)	(0.98)
KPSS	5.28	5.25	5.21	5.14	4.96	4.69	4.60
	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)	(0.01)

Notes: The null for the ADF and PP tests is that the times series contain a unit root, while the null for the KPSS test is that the time series is stationary. p-values are in parentheses. See Section 3 for details on wavelet-based decomposition.

Does our conclusion change when using the panel tests? Table 8 contains the results for the detail components (panel A) and smooth components (panel B). From panel A of Table 8, in agreement with ADF and PP test results in univariate cases (Table 6), PPP in holds in the short run using the Choi, Hadri, IPS and MW tests. For smooth series, PPP generally holds only in the long run, and this is consistent with Table 7. Overall, though not completely conclusive, we obtain evidence that PPP holds at trading horizons longer than 2^6 = 64 months, or approximately 5.3 years, which is consistent with most conventional estimates in the PPP literature (see, e.g., Rogoff, 1996).

Based on these results, following Baqaee (2010), we perform a full "denoising" process to our real exchange-rate series and discard all detailed information corresponding to frequencies equal to or longer than 2^6 = 64 months. Then, for comparison purpose, we superimpose the residual series, i.e., the "S5" series, on actual data in Figure 2, Section 4. It is clear that the S5 series provides a trend line with few turning points. The usefulness of such a denoised measure is clear: It picks up movements in the real exchange rate that may require the attention of policy-makers while filters out short-term changes.

Table 8. Panel test results for 7 levels of decomposition

				A. Details			
	D1	D2	D3	D4	D5	D6	D7
Choi	-22.41	-18.65	-16.12	-9.12	-11.05	-1.05	-2.26
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.15)	(0.01)
Hadri	-2.86	-2.16	-1.19	3.83	19.89	59.34	267.22
	(1.00)	(0.99)	(0.88)	(6.50)	(0.00)	(0.00)	(0.00)
IPS	-52.51	-22.98	-18.79	-9.38	-11.93	-1.20	-2.28
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.12)	(0.01)
MW	540.71	384.38	295.72	114.88	163.97	20.20	23.09
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)	(0.06)	(0.03)
				B. Smooths			
	S1	S2	S3	S4	S5	S6	S7
Choi	3.58	2.36	3.47	2.13	-0.06	-2.12	-5.96
	(1.00)	(0.99)	(1.00)	(0.98)	(0.48)	(0.02)	(1.24)
Hadri	197.54	199.43	203.93	213.34	2.20	280.80	296.61

	(0.00)	(0.00)	(0.00)	(0.00)	(1.00)	(0.00)	(0.00)
IPS	3.27	2.04	3.08	2.20	-0.17	-2.15	-6.03
	(1.00)	(0.98)	(1.00)	(0.99)	(0.43)	(0.02)	(8.10)
MW	1.79	4.98	1.68	11.66	15.93	21.78	67.12
	(1.00)	(0.96)	(1.00)	(0.47)	(0.19)	(0.04)	(1.10)

Notes: The null hypothesis for all tests (except for Hadri's) is that all series contain a unit root, against the alternative of at least one series is stationary. The null of Hadri test is that all series are stationary, against the alternative of at least one series contain a unit root. All specifications contain a constant and a trend term. Lag selection is based on AIC. Corresponding p-values are in parentheses.

Forecasting results

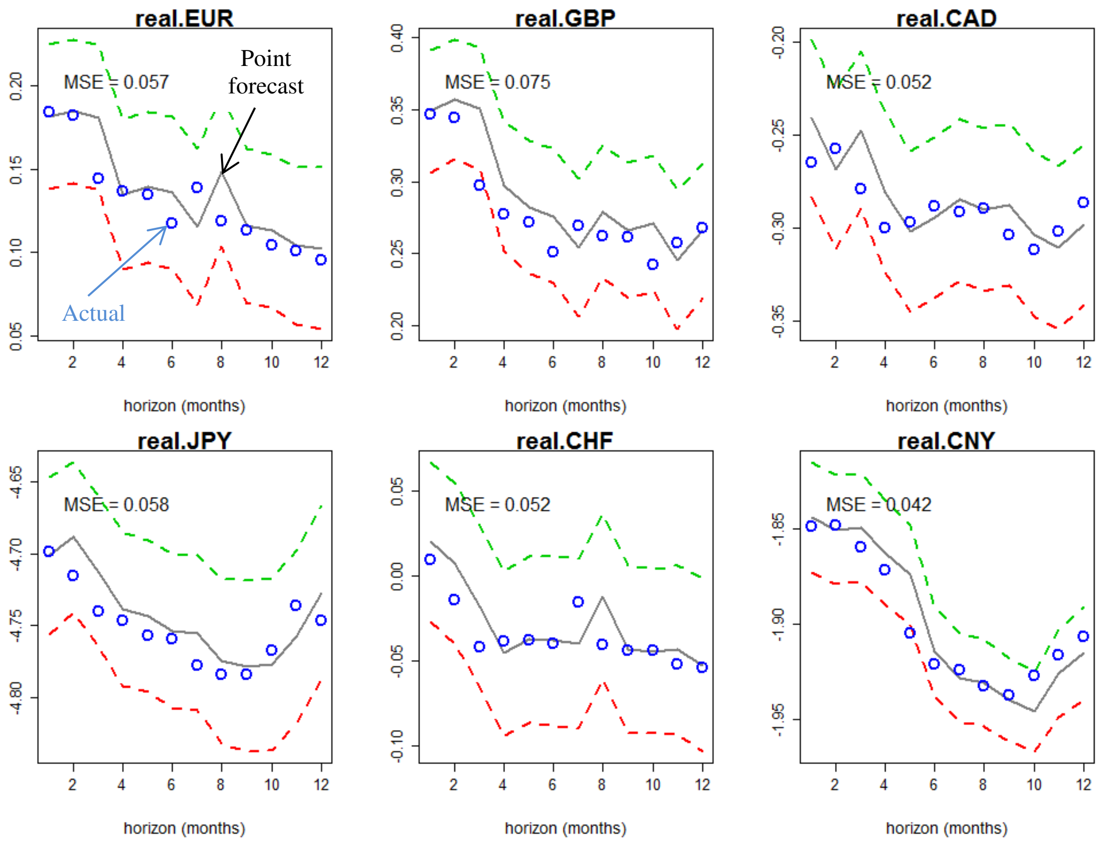
In this section, we present results of a forecasting experiment with the wavelet-based measure of real exchange rates discussed above. To begin, we select as our training sample (in-sample) data set covering the period 01/1999 – 02/2018 (230 observations). The most recent year, 03/2018 – 02/2019 (12 observations), is used as our testing sample (out-of-sample). Then, we fit a simple AR(p) model to each of the real exchange rate series with the order p determined by the Akaike Information Criteria. The model is:

$$r_t = c + \sum_{i=1}^{p} \phi_i r_{t-i} + \varepsilon_t \quad (t = 1, \ldots, 230),$$

where ε_t is a white noise process. We estimate this model using a maximum likelihood estimator, following Hyndman and Koehler (2006).[7] Based on estimated results, we compute the one-step ahead (t+1) forecast as: $\hat{r}_{t+1} = \hat{c} + \sum_{i=1}^{p} \hat{\phi}_i r_{t-i+1}$. Subsequently, we roll the training sample forward by 1 month and repeat this computation 12 times to obtain a series of forecasts, denoted as $\{\hat{r}_{t+h}\}$ $(h = 1, \ldots, 12)$. A measure of forecast performance is the root mean squared errors (RMSE), computed as: $\sqrt{\sum_{h=1}^{12} (\hat{r}_{t+h} - r_{t+h})^2}$. Figure 4 presents these 1-month forecast results and compare them with actual data. As can be seen, the rolling window approach gives highly accurate forecasts for all real exchange rate series, with the best performance observed for CNY, which yields a RMSE of 0.042 and the smallest confidence interval.

[7] Model-selection diagnostic test results are available upon request.

Figure 4. Out-of-sample forecasts of real exchange rates



Notes: In each panel, the solid line indicates forecasts of the real exchange rates derived from an AR(p) model fitted over the period 01/1999 – 02/2019, using monthly data. The dashed lines indicate the 95% confidence interval for these estimates, computed using heteroscedasticity-robust standard errors. The dots represent actual data.

Next, denote the S5 series as r_S. Following Cogley (2002), we are interested in seeing whether the current deviation of actual data from its trend (the S5 series) correctly measures the magnitude of the forecasted changes, using the following regression:

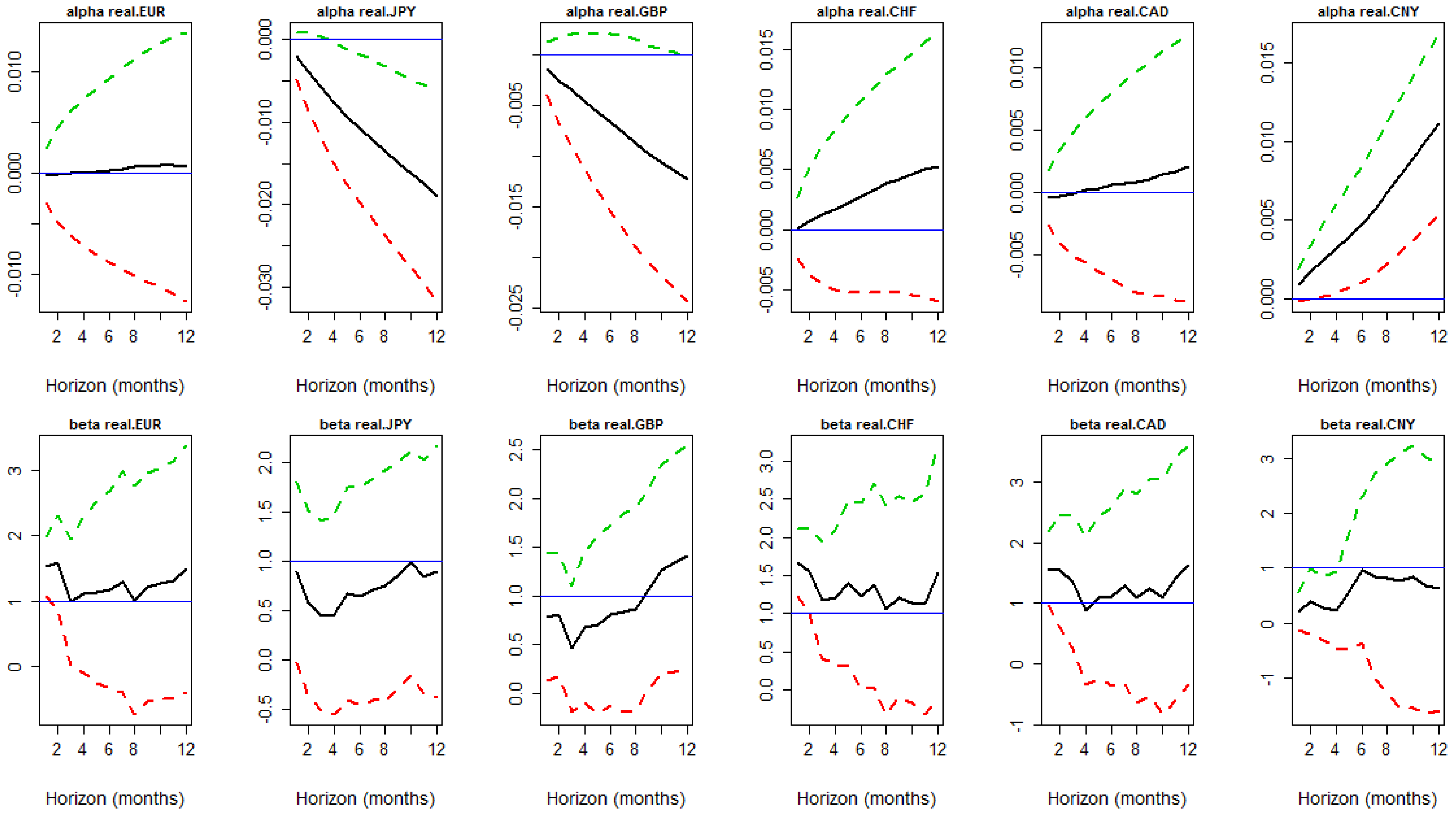
$$\hat{r}_{t+h} - r_t = \alpha_h + \beta_h(r_t - r_S) + u_{t+h},$$

As the means of $\hat{r}_{t+h} - r_t$ and $r_t - r_S$ are approximately zero, α_h should be close to zero. Additionally, $\beta_h = -1$ indicates a correct prediction of current deviation on future changes.[8] Therefore, a joint test for $\alpha_h = 0$ and $\beta_h = -1$ can be interpreted as a test for the forecasting power of the S5 series. As can be seen in Figure 5, estimates of β_h are not statistically significantly different from -1 and those of α_h not different from zero at most horizons, except for CNY and JPY.[9] This exercise shows that measured trend deviations

[8] If $-1 < \beta_h < 0$ ($\beta_h < -1$), the measured deviation overstates (understates) future changes (Cogley, 2002).
[9] The low forecast performance of wavelet smooths in these cases could be a result of sustained period of deflation in Japan and fixed exchange rates in China.

are negatively correlated with subsequent changes in real exchange rates. It can be concluded that over horizons corresponding to 1 year, the wavelet-based measure, which preserves the most important information from the observed data and filters out transient variations, provides useful guidance on future movements of real exchange rates.

Figure 5. Prediction regression coefficients



Notes: This figure plots the intercept (alpha) and slope (beta) coefficients of the regression: $\hat{r}_{t+h} - r_t = \alpha_h + \beta_h(r_t - r_{S5}) + u_{t+h}$ where r_t denotes the (log) real exchange rates, $\hat{r}_{t+h}$ is the h-month horizon forecast, and r_{S5} is the smooth component derived from a wavelet decomposition. In each panel, the solid line illustrates point estimates, and dashed lines represent boundaries of the 95% confidence bands, computed using heteroscedasticity-robust standard errors.

Wavelet-based power decomposition

The empirical studies above failed to confirm the governing power of PPP dynamics for real exchange rates at shorter horizons. It is useful to revisit and analyse the daily nominal exchange rate to explore the multi-scale composition of power (or variation) of these series. In this final discussion, we adopt the wavelet representation in the form of a "heat map" proposed by Torrence and Compo (1998). This is a visualisation tool that allows us to gauge how much each frequency-specific component contributes to the data generating process's total variance, thus determining the relative importance of these components. Essentially, this graph indicates the power or magnitude of the signal/series through both time and frequencies. The power meter exhibits a color indicator similar to that of traditional heat map. If "hot" colors (i.e., reddish) are observed at a particular time and frequency, this indicates that the series exhibit cyclical pattern corresponding to that frequency during that time. We plot a heat maps for each of the six exchange rate series in Figure 6.[10]

The maps reveal features that are in close conjunction with the cyclical behaviour of these series. For example, for the USD/EUR series, power is concentrated at the frequency corresponding to the periods of 1 to 2 years (or 256 to 512 days). It appears that there are some cross-frequency spillover effects between these frequencies during the earlier years in the sample. The concentration of power at the low frequencies is observed after the Internet bubble, leading to the GFC (2002-mid 2007), and then continues from GFC to around 2011. Similar patterns are observed in the USD/JPY series, although the distinction between 1 and two-year cycles is clearer for the 2002-2007 period. For the USD/JPY series, the one-year cycle seems to be dominant and it starts much earlier (since 2000). The USD/GBP series also exhibit comparable features to those of the above two series, however, its one-year cycle starts later (in 2004). Also, its two-year cycle starts sooner (from 2006 to 2012) and is more profound (as evident by the peaks and troughs roughly every two years during this period). For USD/CNY, the period corresponding to the fixed exchange rate regime is dominated by high-frequency changes.

A common feature of the six series is that most of the power is concentrated at lower frequencies while we observed virtually no strong cyclical behaviour at high

[10] The computation is performed via a continuous Morlet continuous wavelet transform. Due to some issues with this transform at the beginning and ending of the data sample, we ignore uncertain information in the cross-hatched areas in Figure 7 (which is also known as the "cone of influence" [Torrence and Compo, 1998; Bunn et al., 2019]).

frequencies (as indicated by the “colder” colored regions). This is in line with the conventional wisdom of market dynamics: While there are large swings at the longer horizons, at shorter horizons we only observed noises. As a result, traders need to factor in preferred trading horizons before designing their strategies: They may be more interested about the long-term cycles since they contain more information about the trend of the series, as opposed to noisy short- term (daily or intra-daily) fluctuations.

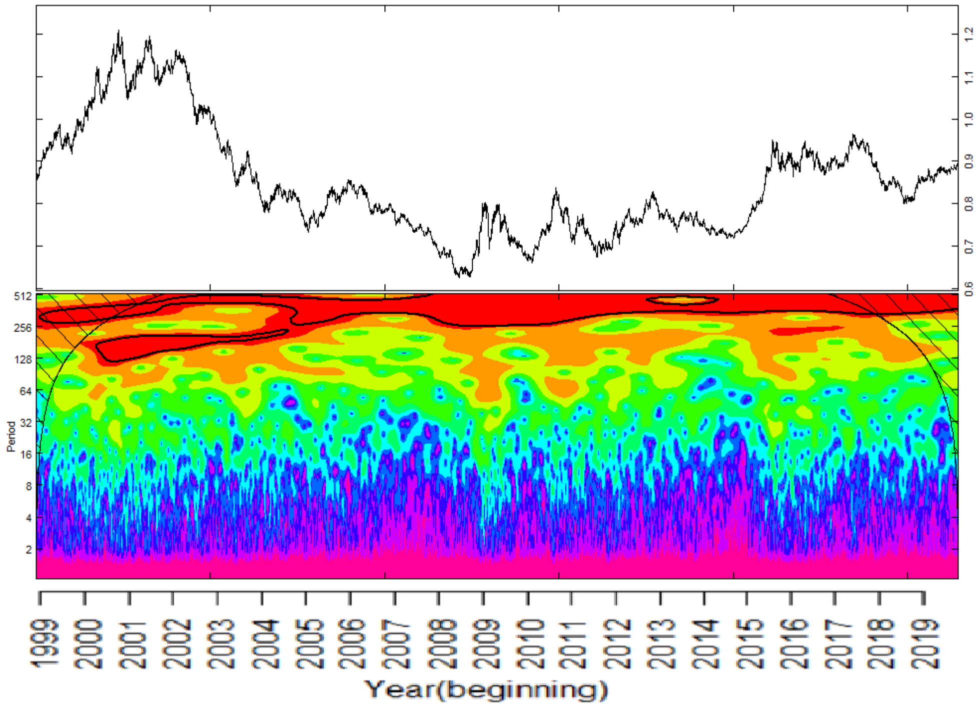
6. Conclusions

This study investigates the long-run interrelation of prices and exchange rates by means of a multi-resolution analysis based on several tools of wavelet methodology. We offer three important insights with respect to the understanding of such dynamics: First, in agreement with previous research, the purchasing power parity tend to hold at horizon longer than 5 years. This result is obtained by applying conventional univariate and panel unit root tests to the different multi-frequency representations of the real exchange-rate data. Second, deviations from trend estimates derived from wavelet-based decomposition provide solid guidance to future movements of real exchange rates. Third, a large portion of variation in exchange rate is attributable to trading activities at low frequencies.

We concluded that our wavelet approach has the credibility and performance needed to be a useful tool for policy makers and traders to gain insight into how financial variables behave at different frequencies, in addition to the passage of time. For those markets participants who are operating in the international exchange-rate markets on longer time horizons, such as central banks, our findings may lead to better informed views on the behaviour of real exchange rates.

Figure 6. Wavelet heat maps of six daily nominal exchange rate series

A. USD/EUR



B. USD/JPY

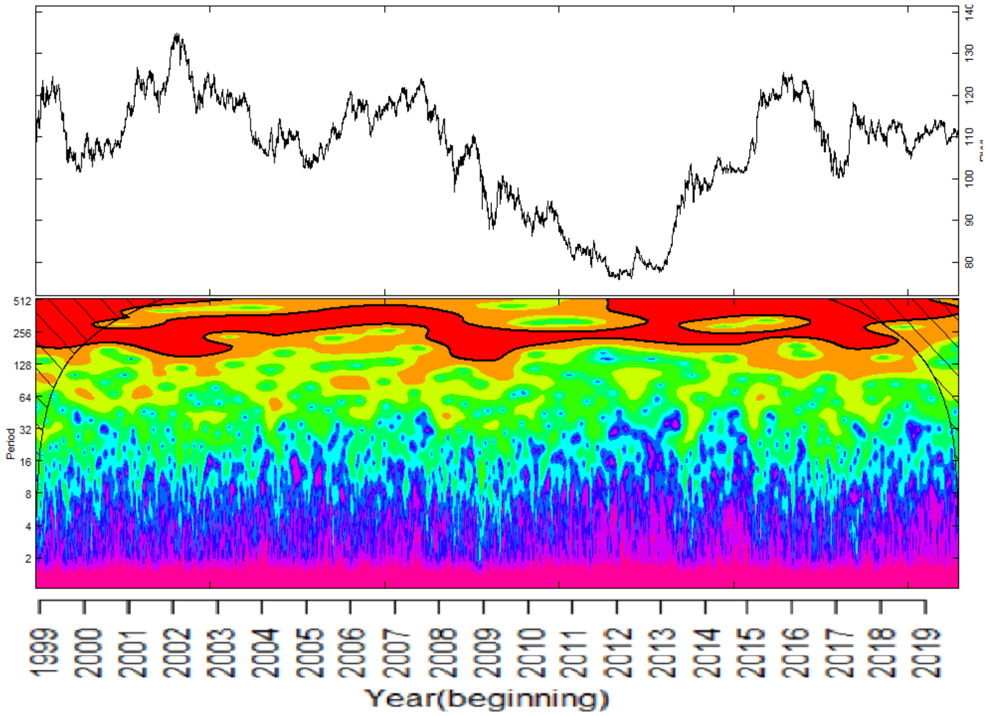
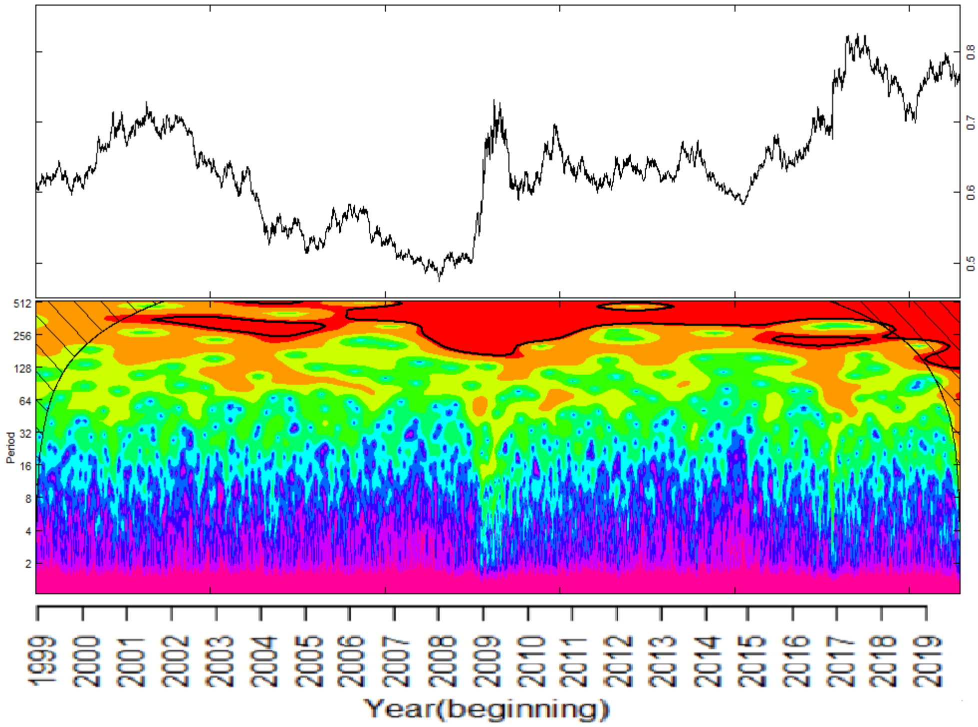


Figure 6. (continued)

C. USD/GBP



D. USD/CHF

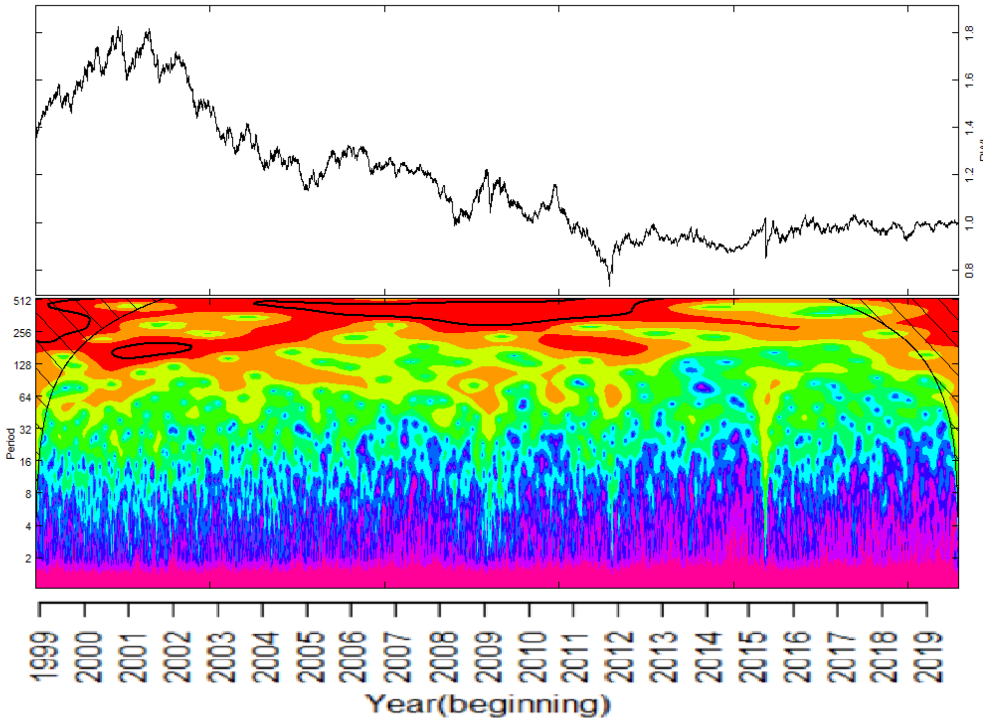
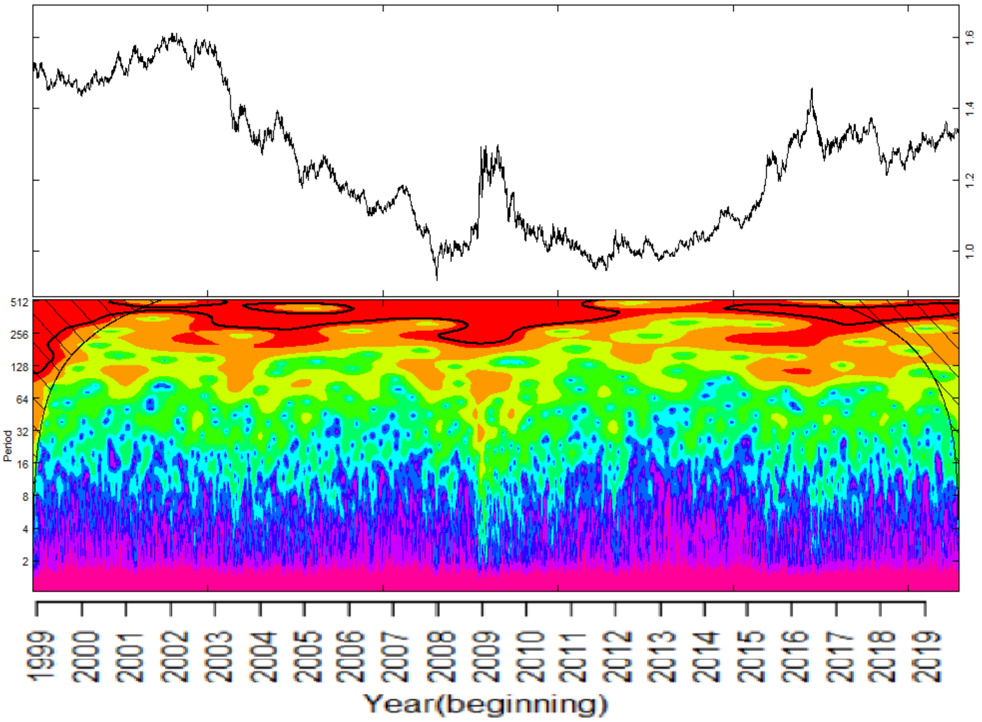
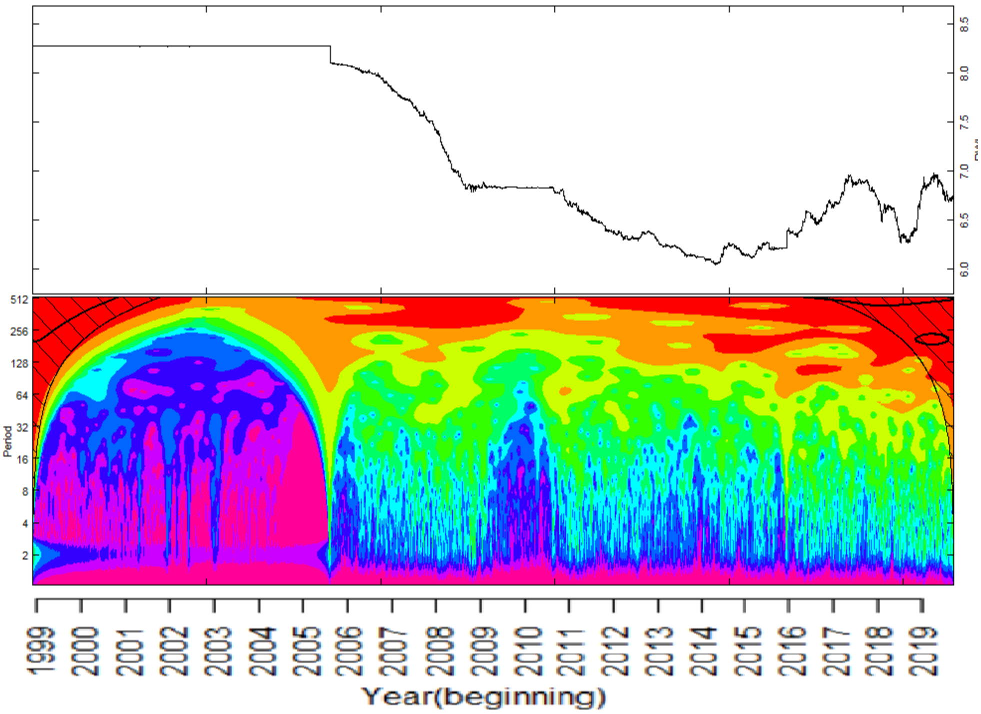


Figure 6. (continued)

E. USD/CAD



F. USD/CNY



Notes: Horizontal axis ranges from 01/01/1999 to 12/04/2019, a total of trading 5291 days. Vertical axis indicates the horizons/periods (in days) corresponding to the frequencies at which the underlying time series fluctuates. Cross-hatched regions on either upper corners indicate the "cone of influence" where edge effects become important (for details, see Torrence and Compo, 1998). These plots are drawn using functions provided in the R package dplR (Bunn, 2008; Bunn et al., 2019).

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