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15 August 2020

Online at <https://mpra.ub.uni-muenchen.de/103358/>  
MPRA Paper No. 103358, posted 09 Oct 2020 11:22 UTC

# Volatility Depend on Market Trades and Macro Theory

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## Abstract

This paper presents probability distributions for price and returns random processes for averaging time interval  $\Delta$ . These probabilities determine properties of price and returns volatility. We define statistical moments for price and returns random processes as functions of the costs and the volumes of market trades aggregated during interval  $\Delta$ . These sets of statistical moments determine characteristic functionals for price and returns probability distributions. Volatilities are described by first two statistical moments. Second statistical moments are described by functions of second degree of the cost and the volumes of market trades aggregated during interval  $\Delta$ . We present price and returns volatilities as functions of number of trades and second degree costs and volumes of market trades aggregated during interval  $\Delta$ . These expressions support numerous results on correlations between returns volatility, number of trades and the volume of market transactions. Forecasting the price and returns volatilities depend on modeling the second degree of the costs and the volumes of market trades aggregated during interval  $\Delta$ . Second degree market trades impact second degree of macro variables and expectations. Description of the second degree market trades, macro variables and expectations doubles the complexity of the current macroeconomic and financial theory.

Keywords: price and returns volatility, price-volume relations, macro theory

JEL : D4, E3, E44, G1, G2

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This research did not receive any assistance, specific grant or financial support from TVEL or funding agencies in the public, commercial, or not-for-profit sectors. We appreciate proposals to fund our studies.

## 1. Introduction

In this paper we present price and returns probability distributions and derive simple expressions for price and returns volatilities as functions of properties of market trades. We argue that forecasting the volatility requires methods and models that describe evolution of second order market trades.

Price and returns uncertainties govern multibillion investment decisions and economic growth. Volatilities of price and returns are the core economic issues under research for decades and centuries. Price as one of most common notions till now hides a lot of mysteries. Indeed, price is treated in too different manners. Fetter (1912) mentions 117 price definitions stating with one made by A. Smith in his "The Wealth of Nations" published in 1776. Fetter (1912): "With the purpose of determining not only what definitions of price have been used, but also what, if any, trend of thought in the subject could be discovered, the writer consulted many texts and found some 117 definition". Wide range of price definitions implies a great variety for price and returns modeling methods. The same time price notion determines major econometrics and National Account data. We refer Hall and Hitch (1939), Heflebower (1955), Diewert (1995) and Fox, et.al. (2019) as a great source of knowledge on key macroeconomic accounts, price definitions and measurements methodologies. Price behavior is described in numerous studies by Muth (1961), Fama (1965), Stigler and Kindahl (1970), Friedman (1990), Cochrane (2001), Cochrane and Culp (2003), Nakamura and Steinsson (2008), Borovička and Hansen (2012), Weyl (2019) and we refer only small part.

We avoid discuss variety of price treatments but choose one simple and standard definition and use it as a base to introduce volatility measures for price and returns uncertainties. Let's take well-known price definition presented long ago by Fetter (1912): "Ratio-of-exchange definitions of price in terms of value in the sense of a mere ratio of exchange". In simple words we consider price  $p$  as coefficient between cost  $C$  and volume  $V$  of performed market transaction:

$$C = pV \tag{1.1}$$

Relations (1.1) are trivial and price definition is absolutely standard. We consistently use (1.1) to introduce volatility measures for price and returns uncertainties and describe dependence of volatility on properties of market transactions.

Price and return volatility modeling and correlations with volume of market trades are studied for decades: Tauchen and Pitts (1983), Mankiw, Romer and Shapiro (1991), Campbell, Grossman and Wang (1993), Ito and Lin, (1993), Brock and LeBaron (1995), Bernanke and

Gertler (1999), Shiryaev (1999), Andersen et.al. (2001), Plerou et.al. (2001), Poon and Granger (2003), Andersen et.al. (2005), Avramov et.al. (2006), Ciner and Sackley (2007), Daly (2008), Christiansen, Schmeling and Schrimpf (2012), Miloudi, Bouattour and Benkraiem (2016), Takaishi and Chen (2017), Bogousslavsky and Collin-Dufresne (2019). These references are only small part of the multiple research of this important problem. Numerous of references indicate that description of the price dispersion and return volatility and their dependence on trading volume are well studied and any new contribution should be reasonably substantiated.

Let's argue our approach to the volatility problem. We regard price and returns as consequences of market transactions (1.1). Relations (1.1) define price  $p$  of particular transaction with cost  $C$  and volume  $V$ . Thus single transaction define  $n$ -th degree price  $p^n$  as:

$$C^n = p^n V^n \quad (1.2)$$

Relations (1.2) allow introduce price and returns statistical moments for averaging time interval  $\Delta$ . Price and returns volatility are examples of uncertainty measures taken as dispersions of the proposed probability distributions. Such probabilities distributions are not unique. The choice of price probability distribution determines the dispersion properties and thus impact volatility characteristics. Hence reasons that explain choice of probability distribution become crucial for volatility modeling. In this paper we introduce price and returns probabilities distributions that are determined by the properties (1.1, 1.2) of the market transactions for certain averaging time interval  $\Delta$ . We use (1.2) and define price  $n$ -th statistical moment  $p(n,t)$

$$p(n; t) = \langle p^n(t) \rangle \quad (1.3)$$

averaged during time interval  $\Delta$  ( $\langle \dots \rangle$  means averaging procedure). To do that at moment  $t$  we sum  $n$ -th degree of cost  $C^n$  and volume  $V^n$  of all  $N(t)$  transactions performed during time interval  $\Delta$  and define

$$C(n; t) = \sum_{i=1}^{N(t)} C^n(t_i) ; \quad V(n; t) = \sum_{i=1}^{N(t)} V^n(t_i) \quad (1.4)$$

Sums  $C(n;t)$  and  $V(n;t)$  define  $p(n,t)$  (1.2, 1.3) as

$$C(n; t) = p(n; t)V(n; t) \quad (1.5)$$

For  $n=1$  relations (1.3-1.5) are identical to well-known volume weighed average price (VWAP) (Berkowitz et.al 1988; Buryak and Guo, 2014; Guéant and Royer, 2014; Busseti and Boyd, 2015; Padungsaksawasdi and Daigler, 2018). We show how market transactions and simple relations (1.1-1.5) lead to the price and returns probability distributions. We derive expressions those describe volatilities as functions of the second order properties of

market trades. We regard description and forecasting of the price and returns volatilities as important but particular piece of more general problem of macroeconomic and macro financial modeling of the 2-nd order macro transactions, macro variables and expectations.

In Sec. 2 we remind standard treatments of volatility and argue problems that are hidden by variety of averaging procedures. In Sec. 3 and 4 we derive price and returns volatilities in the forms that have certain similarities between themselves. We derive expressions that present dependence of price and returns volatilities on properties of market trades. In the same Sec.2 and 3 we describe price and returns statistical moments. We argue how price and returns statistical moments determine characteristic functionals and corresponding probability distributions. In the Sec. 5 we argue that forecasting volatility requires development of macroeconomic theory that can describe evolution of second order market transactions, variables and expectations. Conclusions are in Sec.6. Appendix presents a brief treatment of characteristic functional as a tool for description probability distributions of random processes.

## 2. Volatility

“In everyday language, volatility refers to the fluctuations observed in some phenomenon over time” (Andersen et.al., 2005). Due to current approach the returns volatility is standard deviations of the returns. For continuous time financial model, the price  $p(t)$  define the returns  $r(t;d)$  at time  $t$  regarding the moment  $t-d$  as

$$r(t; d) = \frac{p(t)-p(t-d)}{p(t-d)} \quad (2.1)$$

and for discrete time model the market price  $p(t_i)$  at moment  $t_i$  define the returns  $r(t_i;m)$  with respect to the price  $p(t_{i-m})$  at moment  $t_{i-m}$  as

$$r(t_i; m) = \frac{p(t_i)-p(t_{i-m})}{p(t_{i-m})} \quad (2.2)$$

Often  $m=1$  and returns  $r(t_i;1)$  describe the price change with respect to the “previous moment”  $t_{i-1}$ . The returns (2.1, 2.2) may be described in the log-form

$$R(t; d) = \ln \frac{p(t)}{p(t-d)} \quad (2.3)$$

$$R(t_i; m) = \ln \frac{p(t_i)}{p(t_{i-m})} \quad (2.4)$$

Relations between  $r$  and  $R$  are trivial.

$$\exp R(t; d) = 1 + R(t; d) + \dots = \frac{p(t)}{p(t-d)} = 1 + r(t; d) \quad (2.5)$$

Thus with accuracy  $R^2$  definitions (2.1; 2.3) and definitions (2.2; 2.4) can be treated as identical. Due to current definition returns volatility is defined as the standard deviations  $\sigma_r(t;d)$  of the returns:

$$\sigma_r^2(t;d) = \langle [r(t;d) - \bar{r}(t;d)]^2 \rangle = \langle r^2(t;d) \rangle - \bar{r}^2(t;d) \quad (2.6)$$

$$\bar{r}(t;d) = \langle r(t;d) \rangle \quad (2.7)$$

Here we use  $\langle \dots \rangle$  to denote the averaging procedure. The similar is valid for the returns volatility based on relations (2.2-2.4).

And now it is time to remind: *the devil is in the details*. The averaging procedure  $\langle \dots \rangle$  in (2.6; 2.7) is not unique and varieties of averaging procedures hide many options. As usual (Goldsmith and Lipsey, 1963; Stigler and Kindahl, 1970; Tauchen and Pitts, 1983; Plerou et.al., 2001; Daly, 2008; Weyl, 2019) concepts of the averaging procedures of financial time series are not discussed. For price  $p(t_i)$  time series simple averaging  $\langle p \rangle$  is applied as:

$$\langle p \rangle = \frac{1}{N} \sum_{i=1}^N p(t_i) \quad (2.8)$$

However at least since Berkowitz et.al (1988) the volume weighted average price (VWAP) was introduced

$$\langle p \rangle = \frac{1}{V} \sum_{i=1}^N p(t_i)V(t_i) \quad ; \quad V = \sum_{i=1}^N V(t_i) \quad (2.9)$$

Relations (2.9) define mean price  $\langle p \rangle$  VWAP of transactions  $i$  at moment  $t_i$  with volume  $V(t_i)$  and price  $p(t_i)$  averaged over total volume of transactions  $V = \sum_{i=1}^N V(t_i)$  during certain time term  $\Delta$ . It is obvious that VWAP  $\langle p \rangle$  (2.9; 2.10) exactly matches relations (1):

$$C(t) = \sum_{i=1}^N C(t_i) = \sum_{i=1}^N p(t_i)V(t_i) \quad ; \quad V(t) = \sum_{i=1}^N V(t_i) \quad ; \quad C = \langle p \rangle V \quad (2.10)$$

Here  $N$  – number of trades performed during the averaging time term  $\Delta$ .  $C$  – is the total cost of all  $N$  transactions and  $V$  is the total volume of all  $N$  transactions during the time term  $\Delta$ . Price  $\langle p \rangle$  is the average VWAP price that match relations (1; 2.9; 2.10). It is important to underline that any averaging procedure with price, volume, cost or other economic or financial variables is performed during definite time term  $\Delta$ . Models on base of VWAP now are widely used in research, investment strategies and trading (Buryak and Guo, 2014; Guéant and Royer, 2014; Busseti and Boyd, 2015; Padungsaksawasdi and Daigler, 2018). Chicago Exchange (CME Group, 2020) use VWAP averaging and provide VWAP market data on a regular daily basis - VWAP is a common and well-know price averaging tool.

We don't argue preferences between averaging (2.8) and VWAP (2.9; 2.10). Preferences in financial and investment decisions are the special issue and at least sometimes the market decisions are governed by psychology factors (Barberis, 2018).

In this paper we consider price definition (1) as the base requirement on all price statistical moments. In other words we study only those price averaging procedure, like VWAP, that present mean price, mean square price and etc., in the form that match the relations (1). This simple thesis leads to interesting conclusions on financial volatility and economic modeling.

### 3. Price Volatility

Let's start our treatment of volatility with modeling the price dispersion. Fifty years ago Stigler and Kindahl (1970) started their article "The Dispersion of Price Movements" with the statement: "The Unique Price, as we observed, is a myth. Differences among prices paid or received are almost universal." We agree with Stigler and Kindahl but outline that the price differences strongly depend on the averaging time scale  $\Delta$ . If time scale  $\Delta$  is so small and precise that it resolves singular particular transactions, then prices fluctuate with each new transaction. However, as usual averaging time scales  $\Delta$  equal minutes, hours, days or event months and hence information about the transactions, their costs, volumes and prices are collected and averaged during these time terms. Such aggregation smooth and average price fluctuations during time scale  $\Delta$  and aggregate cost and volume of transactions during this time scale  $\Delta$ . Thus description of price uncertainty and price dispersion in particular should directly depend on the time term  $\Delta$ .

Let's denote as  $C(n;t)$  the total sum of  $n$ -th degree of the cost  $C(t_i)$  of market trade and denote as  $V(n;t)$  the sum of  $n$ -th degree of the volume  $V(t_i)$  of market trades at moment  $t_i$  during the time term  $\Delta$ . The average price  $p(n;t)$  of market trades during the time term  $\Delta$  takes form:

$$C(n;t) = \sum_{i=1}^N C^n(t_i) ; V(n;t) = \sum_{i=1}^N V^n(t_i) ; C(n;t) = p(n;t)V(n;t) \quad (3.1)$$

For  $n=1$  relations (3.1) equal VWAP (2.8; 2.9). Relations (3.1) define  $n$ -th statistical moments of price  $p(n;t)$  as coefficient between sum of  $n$ -th degree cost  $C(n;t)$  and sum of  $n$ -th degree volume  $V(n;t)$  for all  $N=N(t)$  (3.2; 3.4) transactions performed during interval  $\Delta$ . In other words for certain  $n$  (3.1) define  $n$ -th degree volume weighted average price of degree  $n$  similar to VWAP procedure. Here  $N$  – denote number of trades performed during the averaging time scale  $\Delta$ . For continuous time model let's define the number  $N=N(t)$  of trades during  $\Delta$  as:

$$N(t) = \int_{-\Delta/2}^{+\Delta/2} N_{tr}(t + \tau) d\tau \quad (3.2)$$

Here  $N_{tr}(t + \tau)$  - number of trades at  $t+\tau$ . For discrete time model number  $N=N(t)$  of trades can be determined as:

$$N(t) = \sum_i \theta \left( t_i - \left( t - \frac{\Delta}{2} \right) \right) \theta \left( \left( \frac{\Delta}{2} + t \right) - t_i \right) \quad (3.3)$$

$$\theta(t) = 1, \text{ if } t \geq 0; \theta(t) = 0, \text{ if } t < 0 \quad (3.4)$$

Due to (3.1)  $n$ -th statistical moments of price that equal mean  $n$ -th degree of price match relations (1, 3.1) and:

$$\langle p^n(t) \rangle = p(n; t) \quad (3.5)$$

Price volatility as the measure of price uncertainty treated as dispersion  $\sigma_p^2$  for the price probability distribution determined by price statistical moments (3.1; 3.1) takes form:

$$\sigma_p^2(t) = \langle (\delta p)^2 \rangle = p(2, t) - p^2(1, t) \quad (3.6)$$

It is obvious that price statistical moments (3.1, 3.6) and corresponding price probability distribution depends on time  $t$  and on averaging time interval  $\Delta$ . To derive price probability distribution in an exact form one should describe all price statistical moments. Price  $p(t)$  behave as a random process and description of its stochastic properties requires usage of price characteristic functional. We refer (Klyatskin, 2005; 2015) for all technical details on methods and operations with characteristic functional. In Appendix we briefly explain how the cost and the volume of market transactions determine all price statistical moments (A.4-A.7) and thus determine price characteristic functional (A.3). Hence description of price random properties and price volatility in particular is determined by random properties of market transactions.

Relations (3.1-3.6) show direct dependence of prices uncertainty  $\sigma_p^2(t)$  and properties of the cost and the volume of transactions performed during time term  $\Delta$ . Relations between market volatility and volume of the transactions, number of the trades are studied in many papers (Campbell et.al., 1993; Ito and Lin, 1993; Brock and LeBaron, 1995; Plerou et.al., 2001; Avramov et.al., 2006; Ciner and Sackley, 2007; Takaishi, and Chen, 2017; Bogousslavsky and Collin-Dufresne, 2019). As we show below definition (3.6) opens the way for direct description of price and returns uncertainty as functions of number of trades, their volume and cost.

As a first step let's show how  $\sigma_p^2(t)$  (3.6) establishes relations between price uncertainty measure and properties of the market transactions. Due to (3.1) let's present (3.6) as:

$$\sigma_p^2(t) = p(2; t) - p^2(1; t) = \frac{C(2, t)}{V(2, t)} - \frac{C^2(1, t)}{V^2(1, t)} \quad (3.7)$$

Let's introduce fluctuations of the cost  $\delta C(t_i)$  and fluctuations of the volume  $\delta V(t_i)$  for the transaction  $i$  at time  $t_i$  as:

$$\delta C(t_i) = C(t_i) - \frac{1}{N} C(1; t) = C(t_i) - \frac{1}{N} \sum_{i=1}^N C(t_i) \quad (3.8)$$

$$\delta V(t_i) = V(t_i) - \frac{1}{N} V(1; t) = V(t_i) - \frac{1}{N} \sum_{i=1}^N V(t_i) \quad (3.9)$$



Mean squares fluctuations of the cost and the volume equal dispersions of the cost  $\sigma_C^2(t)$  and dispersion of the volume  $\sigma_V^2(t)$  of  $N=N(t)$  (3.2-3.4) transactions performed during time interval  $\Delta$  take form:

$$\sigma_C^2(t) = \frac{1}{N(t)} \sum_{i=1}^N \delta C^2(t_i) = \frac{1}{N(t)} \sum_{i=1}^N C^2(t_i) - C_1^2(t) = C_2(t) - C_1^2(t) \quad (3.10)$$

$$\sigma_V^2(t) = \frac{1}{N(t)} \sum_{i=1}^N \delta V^2(t_i) = \frac{1}{N(t)} \sum_{i=1}^N V^2(t_i) - V_1^2(t) = V_2(t) - V_1^2(t) \quad (3.11)$$

We remind that here  $N=N(t)$  (3.2-3.4) –is the total number of transactions performed during the time term  $\Delta$ . For convenience let's introduce functions  $\phi_C^2(t)$  and  $\phi_V^2(t)$  as

$$\phi_C^2(t) = C_2(t) + C_1^2(t) \quad (3.12)$$

$$\phi_V^2(t) = V_2(t) + V_1^2(t) \quad (3.13)$$

Then it is easy to show that price volatility  $\sigma_p^2(t)$  (3.6, 3.7) equals:

$$\sigma_p^2(t) = p_2(t) - p_1^2(t) = 2 \frac{\phi_V^2 \sigma_C^2 - \phi_C^2 \sigma_V^2}{\phi_V^4 - \sigma_V^4} \quad (3.14)$$

Relations (3.14) describe dependence of price volatility (3.6; 3.7) on dispersions of the cost  $\sigma_C^2(t)$  (3.10) and the volume  $\sigma_V^2(t)$  (3.11) and on number of trades  $N(t)$  (3.2-3.4) performed during time  $\Delta$  as well as on functions  $\phi_C^2(t)$  and  $\phi_V^2(t)$  (3.12; 3.13).

#### 4. Returns Volatility

Numerous studies (Engle and Patton, 2001; Andersen et.al., 2002; Poon and Granger, 2003; Andersen et.al., 2005; Daly, 2008 ; Padungsaksawasdi and Daigler, 2018) describe returns volatility as dispersion of returns (2.6;2.7). As we mentioned above, the crucial issue for such volatility modeling is the choice of probability distribution that determine corresponding dispersion. In this Section we introduce returns probability distribution that define returns volatility in a way alike to (3.1, 3.6). Let's take returns  $r(t_i; m)$  (2.2) and define function  $q_p(t_i; m)$ :

$$q_p(t_i; m) = 1 + r(t_i; m) = \frac{p(t_i)}{p(t_{i-m})} \quad (4.1)$$

Let's introduce returns  $q_C$  of the cost  $C(t_i)$  and returns  $q_V$  of the volume  $V(t_i)$  of trade  $i$  at moment  $t_i$  with respect to moment  $t_{i-m}$  as:

$$q_C(t_i; m) = \frac{C(t_i)}{C(t_{i-m})} \quad ; \quad q_V(t_i; m) = \frac{V(t_i)}{V(t_{i-m})} \quad (4.2)$$

The cost returns  $q_C(t_i; m)$  and the volume returns  $q_V(t_i; m)$  describe properties of the transactions at moment  $t_i$  relative to moment  $t_{i-m}$ . Functions  $q_C(t_i; m)$  and  $q_V(t_i; m)$  allow present (4.1) in the form (4.3) that is alike to (1):

$$q_C(t_i; m) = q_p(t_i; m) q_V(t_i; m) \quad (4.3)$$

Similar to (3.1) let's introduce sum of  $n$ -th degree of functions  $q_C(t_i; m)$  and  $q_V(t_i; m)$ :

$$Q_C(n; t; m) = \sum_{i=1}^N q_C^n(t_i; m); \quad Q_V(n; t; m) = \sum_{i=1}^N q_V^n(t_i; m) \quad (4.4)$$

$$Q_C(n; t; m) = q_p(n; t; m)Q_V(t; m) \quad (4.5)$$

Relations (4.4) define sum of  $n$ -th degree of cost returns  $Q_C(n; t; m)$  and sum of  $n$ -th degree of volume returns  $Q_V(n; t; m)$  during time interval  $\Delta$ . Thus (4.5) defines price returns statistical moment of  $n$ -th degree  $q_p(n; t; m)$  as coefficient between sum of  $n$ -th degree cost returns  $Q_C(n; t; m)$  and sum of  $n$ -th degree volume returns  $Q_V(n; t; m)$  during time interval  $\Delta$ . For  $n=1$  (4.5) defines first statistical moment - mean returns  $q_p(1; t; m)$  as

$$Q_C(1; t; m) = q_p(1; t; m)Q_V(1; t; m) \quad (4.6)$$

Due to (4.1) mean returns  $\langle r(t; m) \rangle$  take form:

$$\langle r(t; m) \rangle = q_{p1}(t; m) - 1 \quad (4.7)$$

Due to (4.3-4.5) mean returns  $q_p(1; t; m)$  can be treated as volume returns weighted average (VRWA):

$$q_p(1; t; m) = \frac{1}{Q_V(1; t; m)} \sum q_p(t_i, m)q_V(t_i, m) \quad (4.8)$$

Second returns statistical moment – mean squares of returns  $q_p(2; t; m)$  take form:

$$Q_C(2; t; m) = q_p(2; t; m)Q_V(2; t; m) \quad (4.9)$$

Due to (4.3-4.5) mean squares of returns  $q_p(2; t; m)$  can be treated as squares volume returns weighted average (SVRWA):

$$q_p(2; t; m) = \frac{1}{Q_V(2; t; m)} \sum q_p^2(t_i; m)q_V^2(t_i; m) \quad (4.10)$$

Similar to the price volatility (3.1) we introduce volatility of returns  $\Sigma_q^2(t; m)$  as:

$$\Sigma_q^2(t; m) = q_p(2; t; m) - q_p^2(1; t; m) \quad (4.11)$$

$$\Sigma_p^2(t, m) = r_{22} - r_{11}^2 + 2(r_{21} - r_{11}) \quad (4.12)$$

$$r_{22}Q_V(2; t, m) = \sum_{i=1}^N r^2(t_i, m)q_V^2(t_i, m) \quad (4.13)$$

$$r_{11}Q_V(1; t, m) = \sum_{i=1}^N r(t_i, m)q_V(t_i, m) \quad (4.14)$$

$$r_{21}Q_V(2; t, m) = \sum_{i=1}^N r(t_i, m)q_V^2(t_i, m) \quad (4.15)$$

It is obvious that price returns volatility measure  $\Sigma_q^2$  (4.11) looks alike to price volatility measure  $\sigma_p^2$  (3.6). To define price returns  $q_p$  probability distribution that match returns statistical moments (4.5, 4.8, 4.10) and price returns volatility  $\Sigma_q^2$  as dispersion (4.11) one should follow the same way we use to determine price  $p$  probability distribution and price characteristic functional  $F(x(t))$  (A1, A.3; Appendix). To avoid excess formulas we refer to (Klyatskin, 2005; 2015) or (Appendix) for all details on characteristic functionals and introduce price returns  $q_p$  characteristic functional  $D(y(t))$  as:

$$D(y(t)) = \sum_{i=1}^{\infty} \frac{i^n}{n!} \int dt_1 \dots dt_n q_p(n; t_1, \dots, t_n; m) y(t_1) \dots y(t_n) \quad (4.16)$$

and determine price returns statistical moments similar to (A.5,A.7). Let's define sum of products of cost returns  $q_c(n; t_1, \dots, t_n; m)$  and volumes returns  $q_v(n; t_1, \dots, t_n; m)$  over all different combinations  $i=\{t_1, \dots, t_n\}$  with total number  $N=N(n, \Delta; t_1, \dots, t_n)$  during averaging time interval  $\Delta$

$$Q_c(n; t_1, \dots, t_n; m) = \sum_{i=\{1, n\}}^{N(n, \Delta; t_1, \dots, t_n)} q_c(n; t_1, \dots, t_n; m) = \sum_{i=\{1, n\}}^{N(n, \Delta; t_1, \dots, t_n)} \prod_{j=1}^n \frac{C(t_j)}{C(t_{j-m})} \quad (4.17)$$

$$Q_v(n; t_1, \dots, t_n; m) = \sum_{i=\{1, n\}}^{N(n, \Delta; t_1, \dots, t_n)} q_v(n; t_1, \dots, t_n; m) = \sum_{i=\{1, n\}}^{N(n, \Delta; t_1, \dots, t_n)} \prod_{j=1}^n \frac{V(t_j)}{V(t_{j-m})} \quad (4.18)$$

We define  $n$ -th statistical moments of price returns  $q_p(n; t_1, \dots, t_n; m)$  as:

$$Q_c(n; t_1, \dots, t_n; m) = q_p(n; t_1, \dots, t_n; m) Q_v(n; t_1, \dots, t_n; m) \quad (4.19)$$

$$q_p(n; t_1, \dots, t_n; m) = \langle \prod_{j=1}^n q_p(t_j; m) \rangle = \langle \prod_{j=1}^n \frac{p(t_j)}{p(t_{j-m})} \rangle \quad (4.20)$$

For averaging time interval  $\Delta$  relations (4.19, 4.20) describe price returns statistical moments  $q_p(n; t_1, \dots, t_n; m)$  and hence define price returns characteristic functional  $D(y(t))$  (4.16) through factors determined by cost and volume of market transactions (4.17, 4.18).

Returns volatility (4.11) depends on corresponding properties of cost and volume returns and follows relations similar to (3.14). To show this let's define volatilities of the cost returns  $\Omega_c^2(t)$  and volatilities of the volume returns  $\Omega_v^2(t)$  of trades as:

$$\Omega_c^2(t, m) = Q_c(2; t; m) - Q_c^2(1; t; m) \quad (4.21)$$

$$\Omega_v^2(t, m) = Q_v(2; t; m) - Q_v^2(1; t; m) \quad (4.22)$$

Similar to (3.12-3.13) let's define functions (4.23, 4.24)

$$\Phi_c^2(t, m) = Q_c(2; t; m) + Q_c^2(1; t; m) \quad (4.23)$$

$$\Phi_v^2(t, m) = Q_v(2; t; m) + Q_v^2(1; t; m) \quad (4.24)$$

Relations (4.21-4.24) present the returns volatility  $\Sigma_q^2(t, m)$  (4.11) in the form similar to (3.14) as function of the cost returns volatilities  $\Omega_c^2(t)$  (4.20), the volume returns volatilities  $\Omega_v^2(t)$  (4.22) number of trades  $N(t)$  (3.2-3.4) and functions (4.23; 4.24):

$$\Sigma_q^2(t; m) = 2 \frac{\Phi_v^2 \Omega_c^2 - \Phi_c^2 \Omega_v^2}{\Phi_v^4 - \Omega_v^4} \quad (4.25)$$

The main advantages for presenting returns volatility measure (4.11) as (4.25) concern the direct dependence of returns volatility (4.11) on volatilities of the cost returns (4.21) and the volume returns (4.22) of the market trades and number of the trades  $N(t)$  (3.2-3.4) during the averaging time term  $\Delta$ . Many researchers describe correlations between volatilities, volumes and number of market transactions (Tauchen and Pitts, 1983; Campbell, et.al., 1993; Ito and Lin, 1993; Brock and LeBaron, 1995; Plerou et.al., 2001; Avramov, 2006; Ciner and Sackley, 2007; Miloudi et.al., 2016; Takaishi and Chen, 2017; Bogousslavsky and Collin-Dufresne,

2019). Thus relations (3.14; 4.24) give certain support for the results presented in the above studies. Expression (4.24) establishes the relations between price returns volatility (4.11) that reflect price returns uncertainties and evolution of mean squares  $Q_{C2}(t, m), Q_{V2}(t, m)$  of market trades.

Relations (3.6; 3.7; 3.10-3.14; 4.4; 4.5; 4.11-4.15; 4.21-4.25) describe economic and financial variables and properties of the market transactions of the second order. These relations indicate that price and returns volatilities depend on squares of cost and volume of market transactions. Thus forecasting of financial markets and volatility requires development of methods and models for description of the second order market trades. And this is a new and a tough problem.

As usual introduction of any new treatment or new definition of economic or financial variables, like volatility, should be accompanied by the comparisons of current and proposed versions. We avoid comparison but present certain reasons in favor of our approach.

## **5. Volatility as a Piece of Macro Financial Puzzle**

Everyone always prefers simple solutions. This is probably one of the reasons why Black and Scholes (1973) and Merton (1973) options pricing model with constant volatility becomes the classical theory. Certain simplicity helps Heston (1993) develop reasonable stochastic volatility model that is widely adopted for option pricing. Further studies of volatility mostly follow the same way – authors guess and model certain random properties of volatility that helps solve particular problem or match certain amount of econometric data. But times change. Available simple solutions are over. Moreover, now it is clear that simple solutions don't solve the financial problems but transfer them to the next day. Macroeconomics and macro finance are extremely complex systems with huge amount of economic agents those perform multiple market transactions on all available markets. Mutual interdependence of all involved entities and market properties impact macro financial processes and nonlinear backward linkages between all markets, trades and expectations establish a real tough challenge for researchers. It is assumed that market transactions are performed under agents expectations. Impact of agents expectations formed by economic and financial forecasts or by individual mental or emotional reasons add surrealistic complexity for financial markets modeling. It seems clear that attempts to make a simple and correct guess on volatility evolution or suggest simple hypothesis on probability distribution that match price and returns uncertainties have no chances for success. For sure one may argue pros and cons of proposed measures for price (3.6) and returns (4.11) uncertainty. However we outline that

relations between (3.6; 4.11) and properties of market transactions (3.14) and (4.25) uncover important macro financial links. In simple words: (3.14; 4.25) describe direct dependence of volatility on evolution of mean squares of cost and volume of market transactions.

Sum of squares of cost  $C(2;t)$  and volume  $V(2;t)$  (3.1) of all trades performed during time interval  $\Delta$  indicate existence of huge hidden complexity for development of adequate economic and financial theory. Current macroeconomic and financial theories describe evolution and mutual interdependence between numerous economic and financial variables that are formed as sum of corresponding agents variables of the first order. In other words – current macro financial theories describe macro variables determined similar to  $C(1;t)$  or  $V(1;t)$  (3.1). Indeed, macro investment, credits, profits, demand and supply, taxes and GDP are formed as sum of investment or credits made by all economic agents, demand and supply of economic agents, GDP as sum of value added (Fox, 2019) of all economic agents of the entire economy. Most macroeconomic and financial variables are first order variables similar to (3.1). Price and returns volatility are almost the only financial variables that depend on second order variables  $C(2;t)$  or  $V(2;t)$  (3.1; 3.7). It seems obvious that modeling and forecasting of second order variables like  $C(2;t)$  or  $V(2;t)$  (3.1) can't be based on first order variables. Description of second order variables and transactions requires theory that can model and forecast trades dynamics of the first and the second order. Description of sum of squares of the cost and the volume of market trades determine market uncertainties and volatilities. Market transactions are drivers and indicators of economic and financial development and growth. Relations between macro variables of the first and second order determine macro uncertainties alike to price and returns volatilities (3.6; 4.11). Market volatility modeling is particular and a small piece of entire macro financial puzzle that should include description of second order transactions and macro variables.

Description of the second order macroeconomic and financial variables requires significant change in the general approach to description of economic and financial processes. In (Olkhov, 2016-2019) we develop methods for macroeconomic and financial modeling based on treatment of agents risk ratings as parallel to coordinates of agents. This approach allows rougher description that aggregates economic and financial variables or their squares in the risk rating space. Such aggregation presents an intermediate approximation between precise description of all economic agents as separate entities and description of sum of economic and financial variables of all agents as functions of time only – standard macroeconomic description. Intermediate approximation in the risk rating space allows describe first order and second order macroeconomic and macro financial variables and market transactions alike

to flows of economic and financial densities. Methods of this description have certain similarities with methods of continuous media. These methods can incorporate description of second order financial variables and market transactions in very natural way. Moreover, extension of macroeconomic models by description of second order variables and market transactions enhance and emphasize surprising parallels between methods of theoretical economic and theoretical physics. As is well known description of physical phenomena is evaluated by variables of the first and the second order. In very simple words - energy of the system as variable of the second order and Hamiltonian models allows match experimental data with high accuracy. We don't argue here deep justifications for this statement but underline – most part of the observed physical phenomena are described by no more than second order variables. Definitions of price (3.6) and returns (4.11) volatilities and relations (3.14; 4.24) establish dependence of volatilities on the second order properties of market transactions. Thus theoretical economics as well as theoretical physics should describe processes of the second order. Nevertheless the nature of economic and physical phenomena is completely different such parallels between them seems exiting. We hope present further results on macro financial processes that describe volatility and second order variables in the forthcoming papers.

## **6. Conclusion**

“Return volatility is, of course, central to financial economics” (Andersen et.al., 2005).

In this paper we introduce price and returns probability distributions and derive expressions that describe price and returns volatilities as functions of second order properties of cost and volume of market trades and their number during averaging time interval  $\Delta$ .

We outline two interrelated problems. Price and returns probabilities that match volatilities (3.6, 4.11) can be determined by corresponding characteristic functionals (A.3, 4.16). To derive characteristic functionals one should obtain all statistical moments for price (A.7) and returns (4.19) for all time terms. It is a rather difficult problem, as the future still remains unknown. Predictions of price and returns volatility require modeling and forecasting aggregate properties of market transactions (3.1, 4.4). To do that one should develop methods that can model evolution of aggregated second order market transactions, expectations and macroeconomic and financial variables. This interesting problem duplicates the complexity of current macroeconomic and macro financial theory and econometrics.

### Characteristic functional of price probability distribution

Price  $p(t)$  of transactions is a random process. Stochastic dynamic systems and in radiophysics describe random processes through characteristic functionals. We refer (Klyatskin, 2005; 2015) for all technical details on characteristic functional and functional calculus and present here only brief treatment of this problem. Characteristic functional  $F(x(t))$  of random process  $p(t)$  is determined as:

$$F(x(t)) = \langle \exp\{i \int dt p(t)x(t)\} \rangle = \int d\mu_p \exp\{i \int dt p(t)x(t)\} \quad (\text{A.1})$$

Here  $\langle \dots \rangle$  denote averaging by measure  $d\mu_p$  of random process  $p(t)$ . Functional derivatives of characteristic functional  $F(x(t))$  determine price statistical moments  $p(n; t_1, \dots, t_n)$  as:

$$\frac{\delta^n}{\delta x(t_1) \dots \delta x(t_n)} F(x(t))|_{x=0} = p(n; t_1, \dots, t_n) = \langle \prod_{i=1}^n p(t_i) \rangle \quad (\text{A.2})$$

That allow present characteristic functional  $F(x(t))$  as:

$$F(x(t)) = \sum_{i=1}^{\infty} \frac{i^n}{n!} \int dt_1 \dots dt_n p(n; t_1, \dots, t_n) x(t_1) \dots x(t_n) \quad (\text{A.3})$$

To derive price statistical moments  $p(n; t_1, \dots, t_n)$  from properties of market transactions one should aggregate  $n$ -th degree of cost and  $n$ -th degree of volume of all market transactions alike to (3.1) during time averaging interval  $\Delta$ . For the given set of time moments  $(t_1, \dots, t_n)$  let's define product of costs  $c(n; t_1, \dots, t_n)$  and product of volumes  $v(n; t_1, \dots, t_n)$  of transactions performed at moments  $(t_1, \dots, t_n)$ :

$$c(n; t_1, \dots, t_n) = \prod_{i=1}^n C(t_i) ; \quad v(n; t_1, \dots, t_n) = \prod_{i=1}^n V(t_i) \quad (\text{A.4})$$

Total number  $N=N(n, \Delta; t_1, \dots, t_n)$  of all combinations of transactions performed during time interval  $\Delta$  near each moment  $t_i$  depends upon the distance between moments  $t_i$  and  $t_j$  and duration of time interval  $\Delta$ . If all moments are the same  $t_1 = \dots = t_n = t$  then  $N=N(n, \Delta; t)$  equals number of transactions  $N(t)$  (3.2-3.4) during interval  $\Delta$  at moment  $t$ . Let's define sum of products of costs  $c(n; t_1, \dots, t_n)$  and products of volumes  $v(n; t_1, \dots, t_n)$  (A.4) over all different combinations  $i=\{t_1, \dots, t_n\}$  with total number  $N=N(n, \Delta; t_1, \dots, t_n)$

$$C(n; t_1, \dots, t_n) = \sum_{i=\{1, n\}}^{N(n, \Delta; t_1, \dots, t_n)} c(n; t_1, \dots, t_n) = \sum_{i=\{1, n\}}^{N(n, \Delta; t_1, \dots, t_n)} \prod_{j=1}^n C(t_j) \quad (\text{A.5})$$

$$V(n; t_1, \dots, t_n) = \sum_{i=\{1, n\}}^{N(n, \Delta; t_1, \dots, t_n)} v(n; t_1, \dots, t_n) = \sum_{i=\{1, n\}}^{N(n, \Delta; t_1, \dots, t_n)} \prod_{j=1}^n V(t_j) \quad (\text{A.6})$$

Due to general rule (1; 3.1) we define  $n$ -th degree price statistical moment  $p(n; t_1, \dots, t_n)$  as:

$$C(n; t_1, \dots, t_n) = p(n; t_1, \dots, t_n) V(n; t_1, \dots, t_n) \quad (\text{A.7})$$

Relations (A.7) match (3.1) for identical  $t_1 = \dots = t_n = t$ . Relations (A4-A.7) express price statistical moments  $p(n; t_1, \dots, t_n)$  through factors determined by cost and volume of market transactions and hence determine price characteristic functional (A.3).

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