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TVEL

15 August 2020

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Abstract

We show that the price and returns volatilities depend on the first and the second degree of the total values and the total volumes of the transactions aggregated during averaging time interval Δ. We derive expressions that describe price volatility via volatilities of the value and the volume and the number of trades during interval Δ. We introduce notions of the value and the volume returns and describe price returns volatility through volatilities of the volume and the value returns and number of trades during Δ. We describe price and returns random processes probability distributions by the complete set of statistical moments determined by corresponding *n-th* degrees products of the values and the volumes of the executed market transactions. Adequate model of volatility requires macroeconomic theory that describes second-degree value and volume of transactions, the second-degree macro variables and expectations. This problem doubles the complexity of the current macroeconomic and financial theory.

Keywords: price and returns volatility, price-volume relations, macro theory

JEL : D4, E3, E44, G1, G2

This research did not receive any assistance, specific grant or financial support from TVEL or funding agencies in the public, commercial, or not-for-profit sectors. We appreciate proposals to fund our studies.

1. Introduction

This paper describes price and returns volatilities through description of the first and second-degree of the values and the volumes of market transactions.

Volatilities of price and returns are the core economic issues under research for decades and centuries. Price as one of core notion till now hides a lot of mysteries. Indeed, Fetter (1912) mentioned 117 price definitions stating with one made by A. Smith in his “The Wealth of Nations" published in 1776. Fetter (1912): ”With the purpose of determining not only what definitions of price have been used, but also what, if any, trend of thought in the subject could be discovered, the writer consulted many texts and found some 117 definition”. Wide range of price definitions implies a great variety of price and returns modeling methods. The same time price notion determines major econometrics and National Account data. We refer Hall and Hitch (1939), Heflebower (1955), Diewert (1995) and Fox, et.al. (2019) as a source of knowledge about key macroeconomic accounts, price definitions and measurements methodologies. Price behavior is described in numerous studies by Muth (1961), Fama (1965), Stigler and Kindahl (1970), Friedman (1990), Cochrane (2001), Cochrane and Culp (2003), Nakamura and Steinsson (2008), Borovička and Hansen (2012), Weyl (2019) and we refer only small part.

We avoid discuss variety of price treatments but choose one simple and standard definition and use it to model price and returns volatility. The well-known price definition was presented long ago by Fetter (1912): “Ratio-of-exchange definitions of price in terms of value in the sense of a mere ratio of exchange”. In simple words we consider price p as coefficient between the total value C and the total volume V of the single transaction:

$$C = pV \tag{1.1}$$

Relations (1.1) are trivial and price definition is absolutely standard. We consistently use (1.1) to model price and returns volatility and its dependence on market transactions.

Price and returns volatility modeling and price correlations with volume of market trades are studied for decades: Tauchen and Pitts (1983), Mankiw, Romer and Shapiro (1991), Campbell, Grossman and Wang (1993), Ito and Lin, (1993), Brock and LeBaron (1995), Bernanke and Gertler (1999), Shiryaev (1999), Andersen et.al. (2001), Plerou et.al. (2001), Poon and Granger (2003), Andersen et.al. (2005), Avramov et.al. (2006), Ciner and Sackley (2007), Daly (2008), Christiansen, Schmeling and Schrimpf (2012), Miloudi, Bouattour and Benkraiem (2016), Takaishi and Chen (2017), Bogousslavsky and Collin-Dufresne (2019). These references are only small part of the countless research of this important problem.

Numerous references indicate that description of the price and returns volatility and their dependence on trading volume are well studied and any new contribution should be reasonably substantiated.

Our contribution is based on consistent use of simple relations (1.1). Markets almost never follow the price of single transaction (1.1) but follow the price trends, mean price dynamics, price volatility and etc. Relations (1.1) define the price of individual deal and the same time determine the way for evaluation of price statistical moments. We underline that any price averaging procedure requires certain time interval *Δ*. Price (1.1) is a result of singular deal. To calculate mean price *p(1;t)* of during time interval *Δ* one should collect total values and total volumes of the transactions performed during this interval *Δ*. This mean price is well known at least since 1988 (Berkowitz et.al 1988; Buryak and Guo, 2014; Guéant and Royer, 2014; Busseti and Boyd, 2015; Padungsaksawasdi and Daigler, 2018) and denoted as volume waited average price (VWAP). We denote VWAP as *p(1;t)*:

$$p(1;t) = \frac{C(t)}{V(t)} = \frac{1}{V(t)} \sum_{i=1}^{N} p(t_i)V(t_i)\,;\;\; C(t) = \sum_{i=1}^{N} C(t_i)\,;\;\; V(t) = \sum_{i=1}^{N} V(t_i) \qquad (1.2)$$

In (1.2) the sum is taken by all *N(t)* deals performed at moments t_i during interval *Δ* at time *t:*

$$N(t) = \sum_i \theta\left(t_i - \left(t - \frac{\Delta}{2}\right)\right)\theta\left(\left(\frac{\Delta}{2} + t\right) - t_i\right) \qquad (1.3)$$

$$\theta(t) = 1\,,\; if\; t \geq 0\,;\; \theta(t) = 0\,,\; if\; t < 0 \qquad (1.4)$$

Relations (1.2) define VWAP as ratio of sum of the values to sum of the volumes of all transactions performed during interval *Δ*. We extend VWAP approach and state that any definition of mean price, mean square price and price statistical moments should fit (1.1). In simple words – we require that mean *n-th* degree of price *p(n;t)* be equal the ratio of sum of *n-th* degree of the values *C(n;t)* to sum of *n-th* degree of the volumes *V(n;t)* of all deals performed during interval *Δ* near moment *t*. Indeed, the *n-th* degree of (1.1) gives obvious relations:

$$C^n = p^n V^n \qquad (1.5)$$

To define mean *n-th* degree of price *p(n;t)* in a way that fit (1.1) and similar to (1.2-1.4) we collect the sums of *n-th* degree of the values *C(n;t)* and *n-th* degree of the volumes *V(n;t)* of all transactions performed during interval *Δ* (1.3; 1.4) and define:

$$C(n;t) = \sum_{i=1}^{N(t)} C^n(t_i) \quad ; \quad V(n;t) = \sum_{i=1}^{N(t)} V^n(t_i) \qquad (1.6)$$

Sums *C(n;t)* and *V(n;t)* define mean *n-th* degree of price *p(n,t)* as

$$C(n;t) = p(n;t)V(n;t) \qquad (1.7)$$

Rationale in favor of the definitions (1.6, 1.7) is simple. Relations (1.5) indicate that *n-th* degree of price p^n has "weight" proportional to *n-th* degree of the volume V^n of the single deal. The sum of *N(t)* (1.3, 1.4) of *n-th* degree of values of transactions equals sum of product of *n-th* degree of price p^n of each transaction weighted by the *n-th* degree of the volume of transaction

$$C(n;t) = \sum_{i=1}^{N(t)} C^n(t_i) = \sum_{i=1}^{N(t)} p^n(t_i)\, V^n(t_i) \tag{1.8}$$

Thus (1.7) simply define mean *n-th* degree price weighted by *n-th* degree volumes

$$p(n;t) = \frac{1}{V(n;t)} \sum_{i=1}^{N(t)} p^n(t_i)\, V^n(t_i) \tag{1.9}$$

For *n=1* relations (1.5-1.9) are identical to VWAP (Berkowitz et.al 1988; Buryak and Guo, 2014; Guéant and Royer, 2014; Busseti and Boyd, 2015; Padungsaksawasdi and Daigler, 2018). We state that mean *n-th* degree price definitions (1.6-1.9) give the only economically correct treatment of price statistical moments. Any exogenous treatment of price random process violates its economic meaning and disturbs relations (1.1, 1.5) between random properties of the value and volume of transactions and the price. However, any hypothesis on price random properties, any exogenous price models and forecasts may remain as essential part of agents price expectations. Agents take trade decisions under their personal price expectations and for sure agents expectations impact price dynamics and fluctuations. We suggest definitions (1.6-1.9) as a general tool for studies of stochastic properties of the price determined by executed transactions.

Below we show how market transactions and simple relations (1.1-1.9) lead to the price and returns probability distributions. We derive expressions those describe volatilities as functions of the second order transactions. We regard description and forecasting of the price and returns volatilities as important but small piece of more general problem of macroeconomic and macro financial modeling of the 2-*nd* order macro transactions, macro variables and expectations.

In Sec. 2 we remind standard treatments of volatility and argue problems that are hidden by variety of averaging procedures. In Sec. 3 and 4 we derive price and returns volatilities in the forms that have certain similarities between themself. We derive expressions that present dependence of price and returns volatilities on properties of market deals. In the same Sec.2 and 3 we describe price and returns statistical moments and determine characteristic functionals of price and returns probability distributions. In the Sec. 5 we argue that forecasting volatility requires development of macroeconomic theory that describe second order transactions, variables and expectations. Conclusions are in Sec.6. Appendix presents a brief treatment of characteristic functional as a tool that describe probability distributions of random processes.

2. Volatility

"In everyday language, volatility refers to the fluctuations observed in some phenomenon over time" (Andersen et.al., 2005). Due to current approach the returns volatility is standard deviations of the returns. For continuous time financial model, the price *p(t)* define the returns *r(t;d)* at time *t* regarding the moment *t-d* as

$$r(t;d) = \frac{p(t)-p(t-d)}{p(t-d)} \tag{2.1}$$

and for discrete time model the market price $p(t_i)$ at moment t_i define the returns $r(t_i;m)$ with respect to the price $p(t_{i-m})$ at moment t_{i-m} as

$$r(t_i;m) = \frac{p(t_i)-p(t_{i-m})}{p(t_{i-m})} \tag{2.2}$$

Often $m=1$ and returns $r(t_i;1)$ describe the price change with respect to the "previous moment" t_{i-1}. The returns (2.1, 2.2) may be described in the log-form

$$R(t;d) = ln\frac{p(t)}{p(t-d)} \tag{2.3}$$

$$R(t_i;m) = ln\frac{p(t_i)}{p(t_{i-m})} \tag{2.4}$$

Relations between *r* and *R* are trivial.

$$exp\,R(t;d) = 1 + R(t;d) + .. = \frac{p(t)}{p(t-d)} = 1 + \ r(t;d) \tag{2.5}$$

Thus with accuracy R^2 definitions (2.1; 2.3) and definitions (2.2; 2.4) can be treated as identical. Due to current definition returns volatility is defined as the standard deviations $\sigma_r(t;d)$ of the returns:

$$\sigma_r^2(t;d) = < [r(t;d) - \bar{r}(t;d)]^2 > = < r^2(t;d) > - \bar{r}^2(t;d) \tag{2.6}$$

$$\bar{r}(t;d) =< r(t;d) > \tag{2.7}$$

Here we use <…> to denote the averaging procedure. The similar is valid for the returns volatility based on relations (2.2-2.4).

And now it is time to remind: *the devil is in the details*. The averaging procedure <…> in (2.6; 2.7) is not unique and varieties of averaging procedures hide many options. As usual (Goldsmith and Lipsey, 1963; Stigler and Kindahl, 1970; Tauchen and Pitts, 1983; Plerou et.al., 2001; Daly, 2008; Weyl, 2019) concepts of the averaging procedures of financial time series are not discussed. For price $p(t_i)$ time series simple averaging $<p>$ is applied as:

$$< p > = \frac{1}{N}\sum_{i=1}^{N} p(t_i) \tag{2.8}$$

However at least since Berkowitz et.al (1988) the volume weighted average price (VWAP) (1.2) is in use and exactly matches relations (1.1). Models on base of VWAP now are widely used in research, investment strategies and trading (Buryak and Guo, 2014; Guéant and Royer, 2014; Busseti and Boyd, 2015; Padungsaksawasdi and Daigler, 2018). Chicago Exchange (CME Group, 2020) use VWAP averaging and provide VWAP market data on a regular daily basis - VWAP is a common and well-know price averaging tool. We underline that price averaging or averaging of any other economic or financial variable have sense only for definite time term Δ. All economic models operate with variables or processes averaged during some interval Δ. Time interval Δ can be equal minutes, days, months and etc. The choice of interval Δ determines the internal time scale of the economic model and defines the accuracy and time variations of the model. Indeed, the economic model with internal averaging time term Δ can describe time variations of variables and processes with time scales $l > \Delta$. Time variations of price or economic variables on scales $l > \Delta$ can demonstrate irregular and random behavior. For example the VWAP $p(1;t)$ (1.2) averaged during minutes, hours or days can behave like random process during months and ears. We underline that any description of irregular behavior of VWAP $p(1;t)$ (1.2) as random function on time scales $l > \Delta$ gives properties of *mean price* $p(1;t)$, but not the properties of price random process. This misuse and substitution of price statistical moments by mean price $p(1;t)$ statistical properties may be the source of numerous mistakes. Below we explain that description of price random process for selected averaging term Δ should follow parallels to the VWAP (1.2-1.9). This simple thesis gives straight and reasonable way to describe price and returns volatility. On other hand it is the source of hidden and unremovable obstacle that makes any attempt to exact description of price random processes impossible.

3. Price Volatility

Fifty years ago Stigler and Kindahl (1970) started their article "The Dispersion of Price Movements" with the statement: "The Unique Price, as we observed, is a myth. Differences among prices paid or received are almost universal." We agree with Stigler and Kindahl but outline that the price differences strongly depend on the averaging time scale Δ. If time scale Δ is so small and precise that it resolves singular particular deals, then prices fluctuate with each new transaction. However, as usual averaging time scales Δ equal hours, days or event months and hence information about the deals, their values, volumes and prices are collected and averaged during these time terms. Such aggregation smooth and average price fluctuations during time scale Δ and aggregate value and volume of transactions performed during this time Δ. Thus description of price volatility should directly depend on the time term Δ. Relations (1.3-1.9) define $C(n;t)$ as the sum of *n-th* degree of the value $C(t_i)$ of deals and $V(n;t)$ as the sum of *n-th* degree of the volumes $V(t_i)$ of deals performed during Δ. The average price $p(n;t)$ of transactions performed during the time Δ takes form (1.7, 1.9).

Usual definition of price volatility as price dispersion $\sigma_p{}^2$ gives dependence on first two price statistical moments $p(1;t)$ and $p(2;t)$ and takes form:

$$\sigma_p^2(t) \ = \ p(2;t) - \ p^2(1;t) \tag{3.1}$$

It is obvious that price volatility $\sigma_p{}^2(t)$ and price probability distribution depend on time t and on interval Δ. To derive price probability distribution in an exact form one should describe all price statistical moments. Price $p(t)$ is a random process and description of price probability distribution requires usage of price characteristic functional. We refer (Klyatskin, 2005; 2015) for all technical details on methods and operations with characteristic functional. In Appendix we briefly explain how the value and the volume of transactions determine all price statistical moments (A.4-A.6) and thus determine price characteristic functional (A.1).

Relations between price volatility and the volume of the deals, number of the trades are studied in many papers (Campbell et.al., 1993; Ito and Lin, 1993; Brock and LeBaron, 1995; Plerou et.al., 2001; Avramov et.al., 2006; Ciner and Sackley, 2007; Takaishi, and Chen, 2017; Bogousslavsky and Collin-Dufresne, 2019). As we show below definition (3.1) opens the way for direct description of price and returns volatilities as functions of number of trades, their volume and value. As a first step let's show that $\sigma_p^2(t)$ (3.1) depend on properties of the transactions. Due to (1.7) let's present (3.1) as:

$$\sigma_p^2(t) = p(2;t) - p^2(1;t) = \frac{C(2;t)}{V(2;t)} - \frac{C^2(1;t)}{V^2(1;t)} \tag{3.2}$$

Let's introduce fluctuations of the value $\delta C(t_i)$ and fluctuations of the volume $\delta V(t_i)$ of the deal i at time t_i as:

$$\delta C(t_i) = C(t_i) - \frac{1}{N} C(1;t) = C(t_i) - \frac{1}{N}\sum_{i=1}^{N} C(t_i) \tag{3.3}$$

$$\delta V(t_i) = V(t_i) - \frac{1}{N} V(1;t) = V(t_i) - \frac{1}{N}\sum_{i=1}^{N} V(t_i) \tag{3.4}$$

Mean squares fluctuations (3.3, 3.4) define volatility of the value $\sigma_C^2(t)$ and volatility of the volume $\sigma_V^2(t)$ of N=$N(t)$ (1.3, 1.4) transactions performed during Δ.

$$\sigma_C^2(t) = \frac{1}{N(t)}\sum_{i=1}^{N} \delta C^2(t_i) = \frac{1}{N} C(2;t) - \frac{1}{N^2} C^2(1;t) \tag{3.5}$$

$$\sigma_V^2(t) = \frac{1}{N(t)}\sum_{i=1}^{N} \delta V^2(t_i) = \frac{1}{N} V(2;t) - \frac{1}{N^2} V^2(1;t) \tag{3.6}$$

We remind that here $N=N(t)$ (1.3, 1.4) – is the total number of deals performed during the time Δ. For convenience let's introduce functions $\phi_C^2(t)$ and $\phi_V^2(t)$ as

$$\phi_C^2(t) = \frac{1}{N}C(2;t) + \frac{1}{N^2}C^2(1;t) \tag{3.7}$$

$$\phi_V^2(t) = \frac{1}{N}V(2;t) - \frac{1}{N^2}V^2(1;t) \tag{3.8}$$

Then it is easy to show that price volatility $\sigma_p^2(t)$ (3.2) equals:

$$\sigma_p^2(t) = p_2(t) - p_1^2(t) = 2\,\frac{\phi_V^2\,\sigma_C^2\ \ -\phi_C^2\,\sigma_V^2}{\phi_V^4 - \sigma_V^4} \tag{3.9}$$

Relations (3.9) describe dependence of price volatility (3.1, 3.2) on volatilitites of the value $\sigma_C^2(t)$ (3.5) and the volume $\sigma_V^2(t)$ (3.6) and on number of trades $N(t)$ (1.3) performed during time Δ as well as on functions $\phi_C^2(t)$ and $\phi_V^2(t)$ (3.7; 3.8).

4. Returns Volatility

Numerous studies (Engle and Patton, 2001; Andersen et.al., 2002; Poon and Granger, 2003; Andersen et.al., 2005; Daly, 2008 ; Padungsaksawasdi and Daigler, 2018) describe returns volatility as dispersion of returns (2.6, 2.7). As we mentioned above, the crucial issue for modeling volatility is the choice of probability distribution. In this Section we introduce returns volatility in a way similar to (3.1, 3.2). Let's take returns $r(t_i;m)$ (2.2) and define price returns function $q_p(t_i;m)$:

$$q_p(t_i;m) = 1 + r(t_i;m) = \frac{p(t_i)}{p(t_{i-m})} \tag{4.1}$$

We introduce returns q_C of the value $C(t_i)$ and returns q_V of the volume $V(t_i)$ of the deal i at moment t_i with respect to moment t_{i-m} similar to (4.1) as:

$$q_C(t_i;m) = \frac{C(t_i)}{C(t_{i-m})} \quad ; \quad q_V(t_i;m) = \frac{V(t_i)}{V(t_{i-m})} \tag{4.2}$$

The value returns $q_C(t_i;m)$ and the volume returns $q_V(t_i;m)$ describe the transactions at moment t_i with respect to moment t_{i-m}. Functions $q_C(t_i;m)$ and $q_V(t_i;m)$ allow present (4.1) in the form (4.3) that is alike to (1.1):

$$q_C(t_i;m) = q_p(t_i;m)q_V(t_i;m) \tag{4.3}$$

Relations (4.3) define q_p (4.1) as coefficient between q_C and q_V. That help use the approach similar to price volatility (3.1-3.9). Let's introduce sum of *n-th* degree of functions $q_C(t_i,m)$ and $q_V(t_i,m)$:

$$Q_C(n;t;m) = \sum_{i=1}^{N} q_C^n(t_i;m)\,; \quad Q_V(n;t;m) = \sum_{i=1}^{N} q_V^n(t_i;m) \tag{4.4}$$

$$Q_C(n;t;m) = q_p(n;t;m)Q_V(t;m) \tag{4.5}$$

Relations (4.4) define sum of *n-th* degree of the value returns $Q_C(n;t,m)$ and sum of *n-th* degree of the volume returns $Q_V(n;t;m)$ during interval Δ. The sum in (4.4) is taken by all

$N=N(t)$ trades (1.3, 1.4). Thus (4.5) defines the mean *n-th* degree price returns $q_p(n;t;m)$ as coefficient between sum of *n-th* degree value returns $Q_C(n;t;m)$ and sum of *n-th* degree volume returns $Q_V(n;t;m)$ during interval Δ. For $n=1$ (4.5) defines mean price returns $q_p(1;t;m)$ as

$$Q_C(1;t;m) = q_p(1;t;m)Q_V(1;t;m) \tag{4.6}$$

Due to (4.1) mean returns $<r(t;m)>$ take form:

$$< r(t;m) >= q_p(1;t;m) - 1 \tag{4.7}$$

Due to (4.3-4.5) mean price returns $q_p(1;t,m)$ can be treated as the volume returns weighted average (VRWA):

$$q_p(1;t,m) = \frac{1}{Q_V(1;t,m)} \sum q_p(t_i,m)q_V(t_i,m) \tag{4.8}$$

Mean squares of returns $q_p(2;t,m)$ take form:

$$Q_C(2;t,m) = q_p(2;t,m)Q_V(2;t,m) \tag{4.9}$$

Due to (4.3-4.5) mean squares of returns $q_p(2;t;m)$ can be treated as squares volume returns weighted average (SVRWA):

$$q_p(2;t;m) = \frac{1}{Q_V(2;t,m)} \sum q_p^2(t_i;m)q_V^2(t_i;m) \tag{4.10}$$

Similar to the price volatility (3.1) we introduce volatility of returns $\Sigma_q^2(t;m)$ as:

$$\Sigma_q^2(t;m) = q_p(2;t;m) - q_p^2(1;t;m) \tag{4.11}$$

Returns volatility (4.11) can be presented through returns $r(t_i;m)$ (2.2, 4.1) as

$$\Sigma_q^2(t,m) = r_{22} - r_{11}^2 + 2(r_{21} - r_{11}) \tag{4.12}$$

Relations (4.12) describe dependence of returns volatility $\Sigma_q^2(t;m)$ on returns $r(t_i;m)$ (2.2, 4.1) averaged by the volume returns $q_V(t_i;m)$ (4.2) and volume square returns $q^2_V(t_i;m)$:

$$r_{11} = \frac{1}{Q_V(1;t,m)} \sum_{i=1}^N r(t_i,m)q_V(t_i,m) \tag{4.13}$$

$$r_{21} = \frac{1}{Q_V(2;t,m)} \sum_{i=1}^N r(t_i,m){q_V}^2(t_i,m) \tag{4.14}$$

$$r_{22} = \frac{1}{Q_V(2;t,m)} \sum_{i=1}^N r^2(t_i,m){q_V}^2(t_i,m) \tag{4.15}$$

Functions $Q_V(1;t;m)$ and $Q_V(2;t;m)$ are determined by (4.4). To define price returns q_p probability distribution that match returns statistical moments $q_p(1;t;m)$, $q_p(2;t;m)$, ..$q_p(n;t;m)$ and volatility Σ_q^2 (4.11) one should follow the same approach we use to determine price p characteristic functional $F(x(t))$ (Appendix A.1-A.6). To avoid excess formulas we refer to (Klyatskin, 2005; 2015) and introduce price returns q_p characteristic functional $D(y(t))$ as:

$$D(y(t)) = \sum_{i=1}^{\infty} \frac{i^n}{n!} \int dt_1 \dots dt_n \; q_p(n;t_1,\dots t_n;m) \; y(t_1)\dots y(t_n) \tag{4.16}$$

Characteristic functional $D(y(t))$ describes all statistical properties of price returns random process $q_p(t;m)$ and defines moments similar to (A.4-A.6). Let's define sum of products of the value returns $q_c(n;t_1,\ldots t_n;m)$ and the returns $q_V(n;t_1,\ldots t_n;m)$ over all different combinations $i=\{t_1,\ldots t_n\}$ with total number $N=N(n,\Delta;t_1,\ldots t_n)$ during averaging interval Δ

$$Q_c(n;t_1,\ldots t_n;m) = \sum_{i=(1,n\}}^{N(n,\Delta;t_1,\ldots t_n)} q_c(n;\tau_1,\ldots \tau_n;m) = \sum_{i=(1,n\}}^{N(n,\Delta;t_1,\ldots t_n)} \prod_{j=1}^{n} \frac{C(\tau_j)}{C(\tau_{j-m})} \quad (4.17)$$

$$Q_V(n;t_1,\ldots t_n;m) = \sum_{i=(1,n\}}^{N(n,\Delta;t_1,\ldots t_n)} q_V(n;\tau_1,\ldots \tau_n;m) = \sum_{i=(1,n\}}^{N(n,\Delta;t_1,\ldots t_n)} \prod_{j=1}^{n} \frac{V(\tau_j)}{V(\tau_{j-m})} \quad (4.18)$$

We define *n-th* statistical moments of price returns $q_p(n;t_1,\ldots t_n;m)$ as:

$$Q_C(n;t_1,\ldots t_n;m) = q_p(n;t_1,\ldots t_n;m)Q_V(n;t_1,\ldots t_n;m) \quad (4.19)$$

$$q_p(n;t_1,\ldots t_n;m) =< \prod_{j=1}^{n} q_p(t_j;m) >=< \prod_{j=1}^{n} \frac{p(t_j)}{p(t_{j-m})} > \quad (4.20)$$

For averaging time interval Δ relations (4.19, 4.20) describe price returns statistical moments $q_p(n;t_1,\ldots t_n;m)$ and price returns characteristic functional $D(y(t))$ (4.16) through factors determined by the value and the volume of transactions (4.17, 4.18).

Returns volatility (4.11) depends on corresponding properties of the value and the volume returns and follows relations similar to (3.9). To show this let's define volatilities of the value returns $\Omega_C^2(t)$ and volatilities of the volume returns $\Omega_V^2(t)$ of trades as:

$$\Omega_C^2(t;m) = \frac{1}{N} Q_C(2;t;m) - \frac{1}{N^2} Q_C^2(1;t;m) \quad (4.21)$$

$$\Omega_V^2(t;m) = \frac{1}{N} Q_V(2;t;m) - \frac{1}{N^2} Q_V^2(1;t;m) \quad (4.22)$$

Similar to (3.7,3.8) we define functions (4.23, 4.24)

$$\Phi_C^2(t;m) = \frac{1}{N} Q_C(2;t;m) + \frac{1}{N^2} Q_C^2(1;t;m) \quad (4.23)$$

$$\Phi_V^2(t;m) = \frac{1}{N} Q_V(2;t;m) + \frac{1}{N^2} Q_V^2(1;t;m) \quad (4.24)$$

Relations (4.21-4.24) present the returns volatility $\Sigma_q^2(t;m)$ (4.11) in the form similar to (3.9) as function of the value returns volatilities $\Omega_C^2(t)$ (4.21), the volume returns volatilities $\Omega_V^2(t)$ (4.22) number of trades $N(t)$ (1.3, 1.4) and functions (4.23; 4.24):

$$\Sigma_q^2(t;m) = 2\,\frac{\Phi_V^2\,\Omega_C^2 - \Phi_C^2\,\Omega_V^2}{\Phi_V^4 - \Omega_V^4} \quad (4.25)$$

The main advantages for presenting returns volatility (4.11) as (4.25) concern the direct dependence of returns volatility (4.11) on volatilities of the value returns (4.21) and the volume returns (4.22) of the transactions and number of the trades $N(t)$ (1.3, 1.4) performed during the averaging interval Δ. Many researchers describe correlations between volatilities, volumes and number of market trades (Tauchen and Pitts, 1983; Campbell, et.al., 1993; Ito and Lin, 1993; Brock and LeBaron, 1995; Plerou et.al., 2001; Avramov, 2006; Ciner and

Sackley, 2007; Miloudi et.al., 2016; Takaishi and Chen, 2017; Bogousslavsky and Collin-Dufresne, 2019). Thus relations (3.9; 4.25) give certain support for the results presented in the above studies.

Relations (3.1; 3.2; 3.9, 4.9; 4.11, 4.25) describe properties of the transactions of the second order. These relations indicate that price and returns volatilities depend on squares of the value and of the volume of deals. Thus forecasting of financial markets, price and returns volatility need methods and models for description of the second order transactions. And this is a new and a tough problem.

As usual introduction of any new treatment of economic or financial variables, like volatility, should be accompanied by the comparisons of current and proposed versions. We avoid here any comparisons but present certain reasons in favor of our approach.

5. Volatility as a Piece of Macro Financial Puzzle

Everyone always prefers simple solutions. This is probably one of the reasons why Black and Scholes (1973) and Merton (1973) options pricing model with constant volatility becomes the classical theory. Certain simplicity helps Heston (1993) develop reasonable stochastic volatility model that is widely adopted for option pricing. Further studies of volatility mostly follow the same way. Researchers make a guess on price random properties and model volatility that match certain amount of econometric data. It becomes common to state exogenous scenarios on price random behavior and adopt it to fit observed econometric data. But times change. Available simple solutions are over. Moreover, now it is clear that simple solutions don't solve the financial problems but transfer them to the next day. Macroeconomics and finance are extremely complex systems with huge amount of economic agents those perform multiple transactions on all available markets. Mutual interdependence of all involved entities and market processes impact macro financial relations and nonlinear backward linkages between markets, trades and expectations establish a real tough challenge for researchers. It is assumed that market deals are performed under agents expectations. Impact of expectations formed by agents economic and financial forecasts or by individual mental or emotional reasons add surrealistic complexity for financial modeling. It seems clear that attempts to make a simple and correct guess on price volatility evolution or suggest simple hypothesis on price probability distribution that match price and returns uncertainties have no chances for success. For sure one may argue pros and cons of proposed price (3.1) and returns (4.11) volatilities. However we outline that relations between (3.1; 4.11) and properties of market transactions (3.9) and (4.25) uncover important macro financial links. In simple words: (3.9; 4.25) describe direct dependence of volatilities on evolution of aggregate squares of the values and the volumes of transactions during certain time period *Δ*.

Sum of squares of the value *C(2;t)* and the volume *V(2;t)* (1.6) of all trades performed during time interval *Δ* indicate existence of huge hidden complexity which prevents the development of adequate economic and financial theory. Indeed, current macroeconomic and financial theories describe evolution and interdependence between numerous economic and financial variables that are formed as sum of corresponding agents variables of *the first order*. In other words – current macro financial theories describe macro variables determined similar to *C(1;t)* or *V(1;t)* (1.2, 1.6). Really, macro investment, credits, profits, demand and supply, taxes and GDP are formed as sum of investment or credits made by all economic agents, demand and supply of economic agents, GDP as sum of value added (Fox, 2019) of all economic agents of the entire economy during some time period *Δ*. Most macroeconomic and financial variables are first order variables similar to (1.2). Price and returns volatility are almost the only financial variables that depend on *second order variables C(2;t)* or *V(2;t)* (1.6; 3.1, 3.2, 4.11). It seems obvious that modeling and forecasting of second order variables like *C(2;t)* or *V(2;t)* can't be based on first order variables. Description of second order variables and trades requires theory that models and forecasts trade dynamics of the first and the second order. Description of sum of squares of the values and the volumes of transactions determines market uncertainties and volatilities. Market deals are drivers and indicators of economic and financial development and growth. Description of price and returns volatility is a small part of entire macro financial puzzle that should include description of interdependence between second order transactions, macro variables and expectations.

Description of the second order macroeconomic and financial variables requires significant change in the general approach to economic and financial processes. Moreover, extension of macroeconomic models to description of the second order variables and transactions enhance and emphasize surprising parallels between theoretical economic and theoretical physics. It is well known that description of physical phenomena is evaluated by variables of the first and the second order. In very simple words - energy of the system as variable of the second order and Hamiltonian models allows match almost all experimental data with very high accuracy. We don't discuss here deep justifications of this statement but underline – most part of the observed physical phenomena is described by no more then second order variables. Definitions of price (3.2) and returns (4.11) volatilities and relations (3.9; 4.25) demonstrate dependence of volatilities on the second order properties of market transactions. Thus theoretical economics as well as theoretical physics should describe processes of the second order. Nevertheless the nature of economic and physical phenomena is completely different such similarity seems exiting. But results of our paper give formal proof that theoretical macroeconomics should be much more complex than physics. And the origin of this complexity is hidden in the price random nature. Price forecasting is the most powerful driver of market research and the Holy Grail of all investors. However, our results prove that even imaginary description of price random properties will remain impossible. Indeed, as we show, to describe price characteristic functional (A.1), that determine all stochastic properties of price random process, one must describe all price statistical moments (A.6) for all n=1,2… As we explained above, current macro theory describes fist order macro variables, trades and expectations. To describe price and returns volatility one should develop second-order macro theory that describes second-degree trades, second degree macro variables and expectations. The attempts to model price skewness and kurtosis require description of third and forth price statistical moments (1.7, A.6) and hence need description of third and forth degrees trades (1.6, A.4, A.5) macro variables and expectations. Imaginary definition of price characteristic functional equal description of macroeconomic theory that describes macro variables, trades and expectations of degree *n=1,2,…*. Development of such powerful theory seems impossible and hence even imaginary precise price forecasting remains a myth.

However the first step ahead and the development of the second-order economic theory that can describe the price and returns volatility seems possible and interesting. We hope present further results on second order macro theory in the forthcoming papers.

6. Conclusion

"Return volatility is, of course, central to financial economics" (Andersen et.al., 2005).

Actually, all economic and financial variables are subject to volatility. All economic and financial environments fluctuate. Randomness of economic evolution makes studies of volatility nature the essential part of theoretical economics. Above we mentioned that price and returns volatilities are almost the only characteristics that should be described by second-degree transactions, macro variables and expectations. Of coarse this is simplification. All economic parameters fluctuate and description of volatility should be the necessary part of economic model of any variable, market processes or expectations. All economic models need description of volatility and hence need description of second-degree variables, transactions and expectations.

Obviously, price and returns volatilities are the most "beneficial" subjects. Thus gambling, divination and fortune-telling of price random properties will remain in demand for economic modeling and forecasting for a long time. We remind Fetter (1912) and his 117 price definitions. This paper discusses the only one price definition – (1.1). Readers should not mistake this one with many others price notions they may have in their minds. We treat the price *p* as the result of market transaction and trivial relations (1.1) underline that it have sense for the performed transactions only. However agents may take trading decisions and execute transactions according to their price expectations, forecasts and models. The results of completed transactions will determine the price (1.1) and the influence of agents price expectations will depend on the values and the volumes of executed transactions.

The relations (1.1) outline the impact of the value and the volume of the transactions executed at moment *t* on the price *p(t)*. Markets almost never follow the price of the single transaction but relay on price trends and price forecasts, mean price dynamics, price volatility and etc. Price trends are determined by time averaging interval *Δ*. The price averaged during interval *Δ* depends on the values and volumes of transactions aggregated during interval *Δ*. To describe price and returns volatility one should describe second-degree transactions aggregated during averaging interval *Δ*. Description of the second-degree transactions is completely different from current economic theory, methods and models.

Trivial relations (1.1) establishes firm ground for description of first, second, *n-th* degree of the values and the volumes of transactions aggregated during interval *Δ* as the only tool for description of price statistical moments and price probability distribution. Any exogenous hypothesis on price random properties may have sense for agents price expectations only. Collisions between price properties determined by performed transactions and price expectations in agents minds those determine agents decisions on transactions add extreme complexity to economic modeling.

Modeling of the observed volatility of price, returns, volatility of macroeconomic and financial variables means description of the second-degree variables, transactions and expectations. This interesting problem duplicates the complexity of current macroeconomic and macro financial theory and econometrics.

Appendix

Characteristic functional of price probability distribution

Usage of characteristic functional is the common tool to describe random processes. We refer (Klyatskin, 2005; 2015) for all technical details on characteristic functional and functional calculus and present here only brief treatment of this problem. Characteristic functional *F(x(t))* of random process *p(t)* can be presented as:

$$F(x(t)) = \int d\mu_p \ \exp\{i \int dt \ p(t)x(t)\} = \sum_{i=1}^{\infty} \frac{i^n}{n!} \int dt_1 \dots dt_n \ p(n; t_1, \dots t_n) \, x(t_1) \dots x(t_n) \quad (A.1)$$

Here $d\mu_p$ denote probability measure of random price process *p(t)*. Functional derivatives of characteristic functional *F(x(t))* determine price statistical moments $p(n; t_1, \dots t_n)$ as:

$$\frac{\delta^n}{\delta x(t_1) \dots \delta x(t_n)} F(x(t))|_{x=0} = p(n; t_1, \dots t_n) \quad (A.2)$$

To derive price statistical moments $p(n;t_1,\dots t_n)$ one should aggregate *n-th* degree of the values and *n-th* degree of volumes of transactions performed during averaging interval Δ. Let's define product of the values $c(n;\tau_1,\dots\tau_n)$ and product of the volumes $\upsilon(n;\tau_1,\dots\tau_n)$ of the deals performed at moments $(\tau_1,\dots\tau_n)$ as:

$$c(n; \tau_1, \dots \tau_n) = \prod_{i=1}^{n} C(\tau_i) \quad ; \quad v(n; \tau_1, \dots \tau_n) = \prod_{i=1}^{n} V(\tau_i) \quad (A.3)$$

Total number $N=N(n;\Delta;t_1,\dots t_n)$ of all combinations of transactions performed during interval Δ near each moment t_i depends upon the distance between moments t_i and t_j and duration of interval Δ. If all moments are the same $t_1=\dots=t_n=t$ then $N=N(n,\Delta;t)$ equals number of transactions *N(t)* (1.3,1.4) during interval Δ at moment *t*. Let's define sum $C(n; t_1,\dots t_n)$ of products of the values (A.4) and sum $V(n; t_1,\dots t_n)$ of products of the volumes (A.5) over all different combinations $i=\{\tau_1,\dots\tau_n\}$ with total number $N=N(n,\Delta;t_1,\dots t_n)$

$$C(n; t_1, \dots t_n) = \sum_{i=(1,n\}}^{N(n,\Delta;t_1,\dots t_n)} c(n; \tau_1, \dots \tau_n) \quad (A.4)$$

$$V(n; t_1, \dots t_n) = \sum_{i=(1,n\}}^{N(n,\Delta;t_1,\dots t_n)} v(n; \tau_1, \dots \tau_n) \quad (A.5)$$

Similar to (1.7, 1.9) we define the *n-th* degree price statistical moment $p(n;t_1,\dots t_n)$ as:

$$C(n; t_1, \dots t_n) = p(n; t_1, \dots t_n) V(n; t_1, \dots t_n) \quad (A.6)$$

Relations (A.6) match (1.1, 1.7, 1.9) for identical $t_1=\dots=t_n=t$. Relations (A3-A.6) express price statistical moments $p(n;t_1,\dots t_n)$ through factors determined by the value and the volume of transactions and determine price characteristic functional (A.1).

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