Price, Volatility and the Second-Order Economic Theory

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Abstract

This paper models price volatility through description of the second-degree transactions and expectations averaged by time interval Δ. We call it - the second-order economic theory. First two price statistical moments define volatility. To model volatility one needs description of the squares of trades aggregated during interval Δ. To describe price probability one should model all *n-th* price statistical moments but that seems almost implausible. Hence the desire to find out the price probability will remain unfeasible. We assume that it is possible to accomplish risk assessment for almost all economic agents. Agents risk ratings distribute agents, their transactions and expectations in economic space determined by numerical continuous risk grades. We define aggregate squares of agents transactions and introduce expectations those approve the second-degree transactions executed during interval Δ. We derive equations on the second-degree transactions and expectations in economic space. In the linear approximation we derive mean square price and volatility disturbances as functions of the first and second-degree trades disturbances. In simple approximation numerous expectations and their perturbations can cause small harmonic oscillations of the second-degree trades disturbances and induce harmonic oscillations of price and volatility perturbations.

Keywords: volatility, economic theory, market transactions, expectations, price probability

JEL : C1, D4, E3, E4, G1, G2

1. Introduction

"In everyday language, volatility refers to the fluctuations observed in some phenomenon over time" (Andersen et.al., 2005). Price and returns volatility and their impact on economy are studied for decades: Hall and Hitch (1939), Fama (1965), Stigler and Kindahl (1970), Pearce (1983), Tauchen and Pitts (1983), Schwert, (1988), Mankiw, Romer and Shapiro (1991), Campbell et.al. (1993), Heston (1993), Brock and LeBaron (1995), Bernanke and Gertler (1999), Andersen et.al. (2001), Engle and Patton (2001), Poon and Granger (2003), Andersen et.al. (2005), Daly (2008), Christiansen, Schmeling and Schrimpf (2012), Padungsaksawasdi and Daigler (2018), Bogousslavsky and Collin-Dufresne (2019). We refer a small part of infinite number of volatility studies to outline importance and complexity of the volatility problem.

As usual volatility is defined as standard deviations of certain probability distribution. This definition of volatility is the starting point for most studies. Each new volatility model considers properties of the proposed probability distribution and properties of the standard deviations. Description of volatility is concentrated on the properties of price probability that assumed to reflect properties of the market in the best way. This approach separates studies of the price uncertainties from studies of the financial markets itself. Nevertheless a lot of papers investigate correlations and mutual dependence between volatility, volume and number of market trades (Tauchen and Pitts 1983; Campbell et.al. 1993; Ito and Lin 1993; Avramov et.al. 2006; Ciner and Sackley 2007; Miloudi et.al. 2016; Takaishi and Chen 2017). Impact of market trades on price and volatility plays important role for volatility forecasting.

Volatility modelling uncovers important problem of economic theory. Each economic model describes economic and financial variables aggregated during certain time interval Δ. Interval Δ can be equal minute, hour, day, month and etc. Any description of price volatility uses financial data aggregated during certain interval Δ. The choice of interval Δ and the method for aggregation the economic and financial data, price series, assets, investment, credits, profits and etc., plays core role for adequacy of the macroeconomic modelling. The duration of interval Δ determines main properties of the economic model – its accuracy, sustainability, scales of possible fluctuations and etc. Choice of time interval Δ and choice of data aggregation method are the most important initial steps of economic forecasting.

Modern macro theory describes evolution of first-degree macro variables, transactions and expectations aggregated during certain interval Δ. For example, macro investment, credits, assets are determined as sum of first-degree investment, credits and assets of economic

agents (without doubling) during time interval Δ. Macro trades equal the sum of first-degree trades during interval Δ. These macro variables and trades are defined as sum of first-degree variables and trades. We call description of these first-degree variables and trades as the first-order economic theory. Almost all macroeconomic and financial variables are the first-degree variables. Price volatility is nearly only exception. Indeed, price volatility as the standard deviation describes difference between mean square price and square of mean price during certain interval Δ. This simple notion – mean square price - leads to numerous problems. During last decades mean price $p(1;t)$ is determined as volume weighted average price (VWAP) (Berkowitz et.al 1988; Buryak and Guo, 2014; Guéant and Royer, 2014; Busseti and Boyd, 2015; Padungsaksawasdi and Daigler, 2018, CME Group, 2020). VWAP mean price $p(1;t)$ equals the ratio of sum of the values C_i to sum of the volumes U_i of all transactions i during interval Δ. We call the pair (U,C) of the volume U and the value C of the transaction as the first-degree transaction. We call the pair (U^2,C^2) of squares of the volume U^2 and squares of the value C^2 as the second-degree transaction. We follow the economic meaning of VWAP and define the mean square price $p(2;t)$ as the ratio of the sum of squares of the values C_i^2 to the sum of squares of the volumes U_i^2 of trades i performed during interval Δ. We give formal definitions in Sec.2 but underline that description of mean square price $p(2;t)$ requires modelling sums of squares of the values and the volumes – sums of second-degree trades performed during interval Δ.

It is obvious that price volatility is not the only variable that requires description of the second-degree variables and trades. Bank rates, inflation, currency exchange rates, return on investment, GDP growth rates and etc., are subject to fluctuations. Investment and profits, supply and demand, production function and almost all macro variables fluctuate. Description of volatility of macro variables requires model collective second-degree economic and financial variables, trades and expectations aggregated during interval Δ. We call this the second-order economic theory. Development of the second-order economic theory doubles the complexity of the current macroeconomic theory.

In this paper we only start explore the particular case of the second-order economic theory. We model price volatility and describe the second-degree trades collected during interval Δ. We introduce main notions and derive equations that describe evolution of squares of market trades. It is generally accepted that agents perform transactions under impact of expectations. We describe expectations those approve the second-degree trades and derive equations that govern evolution of the second-degree expectations. We show how equations on the second-

degree trades cause equations on mean square price $p(2;t)$. We believe that math complexity of our methods and equations corresponds the complexity of this economic problem.

In this paper we develop results on price volatility presented in (Olkhov, 2020). We introduce price statistical moments for certain averaging time interval $\varDelta$. First and second price statistical moments determine price volatility. We describe price volatility as function of the number of trades, first and second-degree values and volumes of trades performed during interval $\varDelta$.

In Sec.2 we present brief description of price volatility due to (Olkhov, 2020). In Sec.3 we describe second-degree macro trades and their flows. In Sec. 4 we introduce collective expectations, collective expected trades and their flows. In Sec 5 we describe disturbances of second-degree trades and model small price and volatility disturbances. Sec.6 - Conclusion. In Appendix A we derive relations on second-degree macro trades and their flows. In Appendix B we introduce math for collective expectations. In Appendix C we derive equations on second-degree trades. Appendix D describes perturbations of second-degree trades, price and volatility.

Equations (3.4) denote Section 3 and equation 4. We use roman letters *A, B, d* to denote scalars and bold ***B, P, v*** – to denote vectors.

2. Price and volatility

We take the price *p* definition mentioned at least by Fetter among 117 others (1912): "Ratio-of-exchange definitions of price in terms of value in the sense of a mere ratio of exchange" – price *p* is a coefficient between the value *C* and the volume *U* of market trade:

$$C = pU \tag{2.1}$$

We call the pair *(U,C)* as the market trade or the transaction. This paper studies price properties, mean price, price volatility and price probability distribution on ground of definition (2.1) only. We don't consider here any other price treatment. However economic agents go into market transactions (2.1) under action of their personal expectations. Agents can develop their personal price expectations and believes on base of any of 117 price definitions mentioned by Fetter (1912), any possible price assumptions, prospects and forecasts. Agents can use any arbitrary price models and reasons that lead them to execution of the transaction with particular price value (2.1). For sure agents price expectations impact the properties of the price value (2.1). We take the price (2.1) as the result of the performed transaction (2.1) and describe random properties of the price determined by (2.1). Relation (2.1) defines the price *p* of a single trade at moment *t*. Meanwhile most studies treat price

averaged during certain time interval Δ. Interval Δ can equal minute, hour, day, month and etc. Let's chose interval Δ and define number $N(t)$ of market trades performed during interval Δ near moment t. For continuous time we define the number $N=N(t)$ of trades during interval Δ as:

$$N(t) = \int_{-\Delta/2}^{+\Delta/2} N_{tr}(t+\tau)d\tau \tag{2.2}$$

Here $N_{tr}(t+\tau)$ - number of trades at $t+\tau$. For discrete time model number $N=N(t)$ of trades performed during interval Δ near moment t can be determined as:

$$N(t) = \sum_i \theta\left(t_i - \left(t - \frac{\Delta}{2}\right)\right)\theta\left(\left(\frac{\Delta}{2} + t\right) - t_i\right) \tag{2.3}$$

$$\theta(t) = 1 \,, \; if \; t \geq 0 \,; \; \theta(t) = 0 \,, \; if \; t < 0 \tag{2.4}$$

Here t_i denotes the moment of transaction. Relations (2.3, 2.4) collect all moment t_i of transactions during interval [t-$\Delta/2$; t+$\Delta/2$]. Now let's sum the *n-th* degree of the value $C^n(t_i)$ and *n-th* degree of the volume $U^n(t_i)$ for all $N(t)$ (2.2-2.4) transactions during interval Δ:

$$C(n;t) = \sum_{i=1}^{N(t)} C^n(t_i) \; ; \quad U(n;t) = \sum_{i=1}^{N(t)} U^n(t_i) \tag{2.5}$$

Relations (2.5) define the sum $C(n;t)$ of *n-th* of the values $C^n(t_i)$ and the sum $V(n;t)$ of *n-th* of the volumes $V^n(t_i)$ of trades during interval Δ. For interval Δ and $C(n;t)$ and $U(n;t)$ we define mean *n-th* degree price $p(n;t)$ (2.1, 2.6) as:

$$C(n;t) = p(n;t)U(n;t) \tag{2.6}$$

For *n=1* relations (2.1, 2.5, 2.6) are identical to VWAP definition (Berkowitz et.al 1988; Buryak and Guo, 2014; Guéant and Royer, 2014; Busseti and Boyd, 2015; Padungsaksawasdi and Daigler, 2018; CME Group, 2020):

$$p(1;t) = \frac{1}{V(1;t)} \sum_{i=1}^{N(t)} p(t_i) \, U(t_i) = \frac{C(1;t)}{V(1;t)} \tag{2.7}$$

Price volatility $\sigma_p^2(t)$ as deviation of price probability during interval Δ takes form (2.6, 2.7):

$$\sigma_p^2(t) = <\delta p^2(t)> = p(2;t) - p^2(1,t) \tag{2.8}$$

We use $<...>$ to denote averaging procedure. To define price probability distribution one should describe all *n-th* statistical moments of random price process $p(t)$ those match relations (2.1, 2.6). For time set $t_1,...t_n$, we take the *n-th* price statistical moment $p(n;t_1,...t_n)$:

$$p(n;t_1,..t_n) = <\prod_{j=1}^{n} p(\tau_j)> \tag{2.9}$$

$$C(n;t_1,..t_n) = p(n;t_1,..t_n)U(n;t_1,..t_n) \tag{2.10}$$

$$C(n;t_1,..t_n) = \sum \prod_{j=1}^{n} C(t_j;\tau_{ji}) \; ; \quad U(n;t_1,..t_n) = \sum \prod_{j=1}^{n} U(t_j;\tau_{ji}) \tag{2.11}$$

Here $C(t_j; \tau_{ji})$ denote the value and $U(t_j; \tau_{ji})$ denote the volume of trade performed at moment τ_{ji} during time intervals [t$_j$-$\Delta/2$; t$_j$+$\Delta/2$] for *j=1,..n*. Sums in (2.11) collect the values C and the volumes U of all trades performed at moments τ_{ji} for all *i* for each interval [t$_j$-$\Delta/2$; t$_j$+$\Delta/2$],

j=1,..n. Relations (2.9, 2.10) define the *n-th* price statistical moments $p(n;t_1,...t_n)$ of price random process *p(t)* averaged during intervals [t_j-$\Delta/2$; t_j+$\Delta/2$],], *j=1,..n.* The set of statistical moments $p(n;t_1,...t_n)$ (2.9, 2.10) determine characteristic functional and gives complete description of all statistical properties of random process *p(t)*. We refer (Klyatskin, 2005; 2015) for usage of characteristic functional and functional calculus for description of random processes and stochastic dynamic systems. Characteristic functional *F(x(t))* of random process *p(t)* takes form:

$$F(x(t)) = \sum_{i=1}^{\infty} \frac{i^n}{n!} \int dt_1 \dots dt_n \ p(n; t_1, \dots t_n) \ x(t_1)..x(t_n) \tag{2.12}$$

We underline, that (2.12) describes all statistical properties of the price random process *p(t)* and (2.12) depends on choice of averaging interval *Δ*. Thus to identify price probability by (2.12) one should describe evolution of collective values *C(n; t_1,...t_n)* and volumes *U(n; t_1,...t_n)* (2.11) for all *n=1,2...* aggregated during interval *Δ*. In other words for each n=1,2,... one should develop *n-th* order macro theory that describe evolution of *n-th* degrees of trades, economic variables and expectations. This problem seems a little challenging to be solved ever and hence exact description of market price probability distribution will remain unattainable.

Forecasting of price volatility (2.8) requires description of *p(1;t)* and *p(2;t)* (2.6). Expression (2.8) describes price volatility $\sigma_p^2(t)$ as function of first and second moments. Relations (2.6) for *n=2* define price second moment *p(2;t)* (2.6) as function of sums of squares of the values and the volumes of the trades collected during interval *Δ*. To forecast price volatility one should describe the second-degree values *C(2;t)* and volumes *U(2;t)* (2.5) aggregated during interval *Δ*.

Let's clarify once more the main issue of our treatment of price statistical properties. As usual, researchers study price *p(1;t)* aggregated during interval *Δ* that equals minute, hour, day and etc., and assume that this function *p(1;t)* is random. Indeed, price *p(1;t)* averaged during minute or hour behaves absolutely unpredictable during long time terms – days or weeks. Mean price *p(1;t)* averaged during one hour behaves like a random variable on scales of days, weeks and months. Hence researchers propose that *p(1;t)* obeys certain stochastic distribution and calculate *p(1;t)* volatility and other price stochastic properties. However, we demonstrate that aggregation of price *p(t)* during any interval *Δ* carries properties of the mean price *p(1;t)* (2.6, 2.7) only. It is obvious, that during time terms *T >> Δ* function *p(1;t)* can behaves irregular. But mean price *p(1;t)* (2.6, 2.7) carries information about properties of the mean price only. Moreover, as we use VWAP procedure (2.5-2.7), thus one can derive mean

price $p(1;t;T)$ aggregated during time interval $T > \Delta$ with help of corresponding aggregated first-degree volume $U(1;t)$ only. To do that in continuous approximation one should use first-degree values $C(1;t;T)$ and first-degree volumes $U(1;t;T)$ (2.13, 2.14) aggregated during time interval $T > \Delta$ as:

$$C(1;t;T) = \int_{t-T/2}^{t+T/2} p(1;\tau)U(1;\tau)\, d\tau \; ; \quad U(1;t;T) = \int_{t-T/2}^{t+T/2} U(1;\tau)\, d\tau \qquad (2.13)$$

$$C(1;t;T) = p(1;t;T)U(1;t;T) \; ; \quad T > \Delta \qquad (2.14)$$

Mean price $p(1;t;T)$ can irregularly behave on time terms $T_2 > T$ and one can recursively apply same procedure (2.13, 2.14) for $T_2 > T$ to obtain mean price $p(1;t;T_2)$ and etc. Thus relations (2.6, 2.7, 2.13, 2.14) for $n=1$ define VWAP procedures and determine mean fist-degree price $p(1;t;T)$ only. To assess mean squares or *n-th* degree price statistical moments one must use (2.6, 2.7) complemented by relations (2.15, 2.16) similar to (2.13, 2.14):

$$C(n;t;T) = \int_{t-T/2}^{t+T/2} p(n;\tau)U(n;\tau)\, d\tau \; ; \quad U(n;t;T) = \int_{t-T/2}^{t+T/2} U(n;\tau)\, d\tau \qquad (2.15)$$

$$C(n;t;T) = p(n;t;T)U(n;t;T) \; ; \quad T > \Delta \qquad (2.16)$$

Transition of price $p(1;t;T_1)$, .. $p(n;t;T_1)$ from one averaging time interval T_1 to another interval $T_2 > T_1$ should follow (2.15, 2.16) for each $n=1, 2,...$ independently. This procedure completely differs from common description of price random properties. We propose that procedures (2.5-2.16) should be treated as the only correct because they describe price statistical moments that correspond to economic meaning of price (2.1). Transition from averaging interval T_1 to interval $T_2 > T_1$ establishes certain relations between $p(1;t;T_1)$ and $p(1;t;T_2)$, $p(2;t;T_1)$ and $p(2;t;T_2)$ and etc., but we leave this for the future studies. For brevity below we omit interval T in $p(n;t;T)$, $C(n;t;T)$, $U(n;t;T)$ and denote these functions as $p(n;t)$, $C(n;t)$, $U(n;t)$.

Current macroeconomic and financial models describe interdependence of macroeconomic and financial variables of the first-degree. Macro variables are composed as sum of corresponding variables of all economic agents in economy during interval Δ. For example, macro investment is determined as investment made by all agents in economy during interval Δ. Macro credit, taxes, consumption, output, profits and etc., are determined as sum (without doubling) of credits, taxes, consumption, output, profits and etc., of all agents in economy during interval Δ. To describe price volatility (2.8) one should complement model of the first-degree macro variables and transactions with description of squares of trades aggregated during interval Δ. To begin with in the next section we consider methods that help describe the second-degree macro trades.

3. Second-degree transactions

We describe the second-degree macro trades using the methods (Olkhov, 2016-2019) and refer these papers for details. For reader's convenience we present brief explanations. We base our theory of the second-degree macro transactions on several assumptions.

First, we assume that it is possible develop and accept the unified methodology for agents risk assessments based on numerical continuous risk scoring. Up now risk rating noted by letters. Each agency - S&P, (2014, 2016), Moody's, (2010, 2018), Fitch, (2018) - introduces its own ratings methodology. However numerical credit scoring was suggested eighty years ago by Durand (1941) and sixty years ago by Myers and Forgy (1963). Last years researchers use numerical risk rating to compare risk assessment provided by different rating agencies (Beaver et.al., 2006; Poon and Shen, 2020). We assume that no internal econometric obstacles prevent the transfer from letters to numerical risk scoring. This transition depends mostly on business interests of major ratings companies. Development of unified methodology that will be adopted by major business players should deliver benefits to rating agencies, business and economic authorities. Unified numerical continuous risk grades methodology will significantly improve methods for modelling and forecasting of macroeconomic and financial processes. Introduction of continuous risk grades is a natural development of numerical grades methodology. We don't specify any particular risks like credit or inflation risk but regard all set of economic, financial, political and etc., risks those impact economic and financial development.

Second, we assume that it is possible to assess risk ratings for almost all economic agents, for international banks, corporations and for local firms, companies and even households. That will distribute all agents of the economy by their risk ratings as coordinates of risk space.

Third, agents in economy always act under pressure of numerous risks. Assessment of agents ratings for one risk distribute agents over 1-dimensional risk space. Simultaneous assessment of agents ratings for two or three risks distribute agents by two or three dimensional risk space. Absolute numerical values of risk ratings have no sense. For certainty, for any risk we take most secure rating to be equal zero and the most risky rating to be equal 1. Thus agents continuous numerical ratings for a single risk fill unit interval [0,1] and agents ratings for m risks fill unit cube in space R^m.

Forth. Explicit description of numerous separate agents of the entire economy is almost impossible. It requires too much exact data of economic and financial variables for millions separate agents, data about all particular market trades between agents, risk assessment of all

agents and etc. It is impossible to collect precise data for all agents and small perturbations could move the forecasts in the wrong directions. To solve the problem we reduce the accuracy of economic description. We aggregate description of agents in the risk economic space and derive the continuous media approximation (Olkhov, 2016a, 2018; 2019a; 2019b) that describes collective transactions. Below we explain meaning of the continuous media approximation for transactions in more detail.

Let's explain the benefits of our approach to economic modelling. First, transition from letters to continuous numerical risk scoring uncovers the hidden motion of economic agents. Indeed, rating agencies for decades use transition matrix to describe probability of agent's transfer from particular risk grade to a different one (Belkin, 1998; Schuermann and Jafry, 2003; Ho et.al, 2017; S&P, 2018). Introduction of numerical continuous scoring defines the numerical distance between risk ratings and hence introduces the mean velocity of agent's motion during risk transition. Hence one describes the motion of agents in the risk economic space and describes flows of macro variables, trades and expectations induced by risk motion of agents. Below we use this approach to develop description of second-degree trades and refer to (Olkhov, 2016-2019) for all further details and definitions.

3.1. Second-degree transactions

We describe second-degree trades between economic agents by methods developed in Olkhov (2018, 2019b). We regard economy as a system of numerous economic agents those perform economic and financial transactions. We assume that agents perform transactions under action of *m* risks and it is possible to assess risk ratings for all agents in the economy. Each economic agent *i* is determined by its risk coordinates - numerical continuous risk ratings $\boldsymbol{x}$

$$\boldsymbol{x} = (x_1, \dots x_m) \; ; \quad 0 \le x_i \le 1 \; ; \quad i = 1, \dots m \tag{3.1}$$

Due to (3.1) coordinates $\boldsymbol{x}$ of agents fill economic domain of risk space R^m. Further we note risk space as economic space. We treat risk assessments of agents as procedures similar to measurements of coordinates of physical particles and call agents numerical continuous risk ratings $\boldsymbol{x}$ (3.1) as agents coordinates in economic domain

$$0 \le x_i \le 1 \; ; \quad i = 1, \dots m \tag{3.2}$$

Let's take that agent *i* with risk coordinates $\boldsymbol{x}$ executes transaction with agent *j* with risk coordinates $\boldsymbol{y}$. To simplify the model we assume that at moment *t* agent *i* at point $\boldsymbol{x}$ sells the volume U_{ij} of variable *E* to agent *j* at point $\boldsymbol{y}$. As *E* one can consider any commodities, goods, service, credits, investment, assets and etc. Let's assume that the volume U_{ij} of this particular

transaction between agents i and j has the total value C_{ij}. We define the transaction at moment t between agents i and j as two component function $\boldsymbol{b}_{ij}$:

$$\boldsymbol{b}_{ij}(1; t, \boldsymbol{z}) = \left(U_{ij}(t, \boldsymbol{z}); C_{ij}(t, \boldsymbol{z})\right) ; \; \boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \tag{3.3}$$

Coordinates $\boldsymbol{x}$ and $\boldsymbol{y}$ of agents i and j involved into this trade establish economic domain of double dimension (3.1; 3.2; 3.4; 3.5) with coordinates $\boldsymbol{z}=(\boldsymbol{x},\boldsymbol{y})$:

$$\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \; ; \quad \boldsymbol{x} = (x_1 \dots x_m) \; ; \quad \boldsymbol{y} = (y_1 \dots y_m) \tag{3.4}$$

$$0 \leq x_i \leq 1 \; ; \; 0 \leq y_j \leq 1; \quad i = 1, \dots m \; ; \; j = 1, \dots m \tag{3.5}$$

We define the price p_{ij} of the trade $\boldsymbol{b}_{ij}$ (3.3) between agents i and j similar to (2.1):

$$C_{ij}(t, \boldsymbol{z}) = p_{ij}(t, \boldsymbol{z})U_{ij}(t, \boldsymbol{z}) \tag{3.6}$$

One need certain time dt to perform the trade $\boldsymbol{b}_{ij}$ (3.3). The volume U_{ij} of the trade $\boldsymbol{b}_{ij}$ (3.3) change the amount of economic or financial variable E of agents i and j involved into the trade during time dt and the value C_{ij} change the amount of funds. We consider trades as a rate of change of economic and financial variables of economic agents. We define the second-degree transactions $\boldsymbol{b}_{ij}(2;t,z)$ as:

$$\boldsymbol{b}_{ij}(2; t, \boldsymbol{z}) = \left(U_{ij}^2(t, \boldsymbol{z}); C_{ij}^2(t, \boldsymbol{z})\right) \tag{3.7}$$

We define transactions $\boldsymbol{b}_{ij}(n;t,z)$ of degree n as

$$\boldsymbol{b}_{ij}(n; t, \boldsymbol{z}) = \left(U_{ij}^n(t, \boldsymbol{z}); C_{ij}^n(t, \boldsymbol{z})\right) \tag{3.8}$$

We determine the n-th degree price $p^n{}_{ij}(t,z)$ of trade $\boldsymbol{b}_{ij}(n;t,z)$ (3.8) as:

$$C_{ij}^n(t, \boldsymbol{z}) = p_{ij}^n(t, \boldsymbol{z})U_{ij}^n(t, \boldsymbol{z}) \tag{3.9}$$

To avoid excess accuracy and to derive the economic continuous media approximation for the second-degree trades we replace description of the second-degree trades $\boldsymbol{b}_{ij}(2;t,z)$ between individual agents i at $\boldsymbol{x}$ and j at $\boldsymbol{y}$ by description of collective second-degree trades $\boldsymbol{B}(2;t,z)$, $z=(\boldsymbol{x},\boldsymbol{y})$ between points $\boldsymbol{x}$ and $\boldsymbol{y}$. To do that we introduce small scale d and a unit volume $dV(z)$ in economic domain (3.4, 3.5)

$$dV(\boldsymbol{z}) = dV(\boldsymbol{x})dV(\boldsymbol{y}) ; \; dV(\boldsymbol{x}) = d^n ; \; dV(\boldsymbol{y}) = d^n ; \; \boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y}) \tag{3.10}$$

We assume that scale $d \ll 1$ but each unit volume $dV(\boldsymbol{x})$ and $dV(\boldsymbol{y})$ contains a lot of agents with risk coordinates inside $dV(\boldsymbol{x})$ and $dV(\boldsymbol{y})$. Let's assume that during time Δ agents inside $dV(\boldsymbol{x})$ and $dV(\boldsymbol{y})$ perform a lot of mutual transactions. We define the collective second-degree trade $\boldsymbol{B}(2;t,z)$ between points $\boldsymbol{x}$ and $\boldsymbol{y}$ as a sum of all second-degree trades $\boldsymbol{b}_{ij}(2;t,z)$ of agents i with coordinates in a unit volume $dV(\boldsymbol{x})$ and agents j with coordinates in a unit volume $dV(\boldsymbol{y})$ (3.10) and then average this sum during the interval Δ as:

$$\boldsymbol{B}(2; t, \boldsymbol{z}) = \sum_{i \in dV(\boldsymbol{x}); j \in dV(\boldsymbol{y}); \Delta} \boldsymbol{b}_{i,j}(2; t, \boldsymbol{z}) \tag{3.11}$$

$$\sum_{i\in dV(x);j\in dV(y);\Delta} b_{i,j}(2;t,z) = \frac{1}{\Delta}\int_{t-\Delta/2}^{t+\Delta/2} d\tau \sum_{i\in dV(x);j\in dV(y)} b_{i,j}(2;\tau,z) \quad (3.12)$$

$$B(2;t,z) = \big(U(2;t,z);C(2;t,z)\big)\,;\ z = (x,y) \quad (3.13)$$

$$U(2;t,z) = \sum_{i\in dV(x);j\in dV(y);\Delta} U_{ij}^2(t,z) \quad (3.14)$$

$$C(2;t,z) = \sum_{i\in dV(x);j\in dV(y);\Delta} C_{ij}^2(t,z) \quad (3.15)$$

The second-degree collective trades $B(2;t,z)$ (3.11-3.15) define the second-degree price $p(2;t,z)$ of trades between agents at points x and agents at point y in domain (3.4, 3.5) averaged during interval Δ.

$$C(2;t,z) = p(2;t,z)U(2;t,z) \quad (3.16)$$

Relation (3.16) describes mean second-degree price $p(2;t,z)$ between points x and y averaged during interval Δ. Relation (2.6) describes mean second-degree price $p(2;t)$ of all trades in the economy averaged during interval Δ. To derive (2.6) one should take integral by collective second-degree transactions $B(2;t,z)$ over economic domain (3.4, 3.5) or calculate the sum over all agents in economy involved into trades during time interval Δ:

$$B(2;t) = \big(U(2;t);C(2;t)\big) \quad (3.17)$$

$$U(2;t) = \int dz\, U(2;t,z)\ \ ;\ \ C(2;t) = \int dz\, C(2;t,z) \quad (3.18)$$

$$C(2;t) = p(2;t)U(2;t) \quad (3.19)$$

Relations (3.19) define price $p(2;t)$ in the same way as (2.6) for $n=2$. Description of price (2.6) requires models of the second-degree trades $B(2;t)$ and $B(2;t,z)$.

3.2. Flows of the second-degree transactions

Let's reproduce briefly the meaning of transaction's flows due to (Olkhov, 2018-2019b). We start with the definition of agent's velocities in the economic domain (3.1, 3.2). As we mentioned above any agents economic activity is the source of risks and agents always act under the pressure of risks – credit, inflation, market, political and etc. Rating agencies for decades provide assessments of risk transition matrices (Belkin, 1998; Schuermann and Jafry, 2003; Ho et.al, 2017; S&P, 2018). Up now risk grades are determined by letters and elements a_{ij} of transition matrices mean probabilities of transition from grade i to j during time interval T. As usual interval T equals half year, one, two three years. If and when ratings methodology use numerical continuous grades then transition matrices get completely different and much more important economic meaning. For numerical continuous grades the transition from rating x_i to x_j defines numerical interval l_{ij} :

$$l_{ij} = x_j - x_i \quad (3.20)$$

Transition from x_i to x_j takes time T. Hence element a_{ij} of transition matrices defines the probability of agent's motion in the economic domain (3.2) during time T with velocity v_{ij} :

$$v_{ij} = \frac{l_{ij}}{T} \ with\ probability\ a_{ij}; \ \sum_j a_{ij} = 1 \tag{3.21}$$

As we mentioned above, introduction of agents velocities in the economic domain is important contribution of the transfer from letters to continuous numerical risk grades. Hence transition matrices define mean velocity of agent in point x_i during time interval T as:

$$v(t,x_i) = < v_{ij} >_j = \sum_{j=1}^{K} v_{ij} a_{ij} = \frac{1}{T} \sum_{j=1}^{K} l_{ij} a_{ij} \tag{3.22}$$

Here K means the number of different numerical risk grades that defines the degree KxK of the transition matrix. Thus the transition matrices can define mean velocities of agents in the economic domain. Risk motion of agents induces flows of economic and financial variables and flows of market trades as well (Olkhov, 2018-2019b). Indeed, the motion of agent in economic domain with velocity v cause that this agent carries its economic and financial variables through domain (3.1). Collective effect of such transport of economic variables by agents in a small volume $dV(x)$ is described by flow of variables in (3.2). Similar effect defines the flow of collective trades in economic domain with double dimension (3.4,3.5).

To describe the flows of the second-degree transactions $B(2;t,z)$ let's assume that in economic domain (3.2) agent i at a moment t have risk coordinates $x=(x_1,...x_m)$ and velocities $v_x=(v_{x1},...v_{xm})$. Let's take that agent i at point x executes transaction $b_{ij}(2;t,z)$ with agent j at point $y=(y_1,...y_m)$ and velocities $v_y=(v_{y1},...v_{ym})$. Similar to (Olkhov, 2018-2019b) let's define the flows $p_{ij}(2;t,z)$ of the transactions $b_{ij}(2;t,z)$ (3.7) between agents i and j as:

$$\boldsymbol{p}_{ij}(2;t,\boldsymbol{z}) = \left(\boldsymbol{p}_{Uij}(2;t,\boldsymbol{z}), \boldsymbol{p}_{Cij}(2;t,\boldsymbol{z})\right) \tag{3.23}$$

$$\boldsymbol{p}_{Uij}(2;t,\boldsymbol{z}) = \left(\boldsymbol{p}_{Uxij}(2;t,\boldsymbol{z}); \boldsymbol{p}_{Uyij}(2;t,\boldsymbol{z})\right) ; \ \boldsymbol{p}_{Cij}(2;t,\boldsymbol{z}) = \left(\boldsymbol{p}_{Cxij}(2;t,\boldsymbol{z}); \boldsymbol{p}_{Cyij}(2;t,\boldsymbol{z})\right) \tag{3.24}$$

$$\boldsymbol{p}_{Uxij}(2;t,\boldsymbol{z}) = U_{ij}^2(t,\boldsymbol{z})\boldsymbol{v}_{xi}(t,\boldsymbol{x}) \ ; \ \ \boldsymbol{p}_{Uyij}(t,\boldsymbol{z}) = U_{ij}^2(t,\boldsymbol{z})\boldsymbol{v}_{yj}(t,\boldsymbol{y}) \tag{3.25}$$

$$\boldsymbol{p}_{Cxij}(t,\boldsymbol{z}) = C_{ij}^2(t,\boldsymbol{z})\boldsymbol{v}_{xi}(t,\boldsymbol{x}) \ \ ; \ \ \boldsymbol{p}_{Cijy}(t,\boldsymbol{z}) = C_{ij}^2(t,\boldsymbol{z})\boldsymbol{v}_{yj}(t,\boldsymbol{y}) \tag{3.26}$$

The flows $\boldsymbol{p}_{ij}(2;t,z)$ (3.23) define the flows $\boldsymbol{p}_{Uij}(2;t,z)$ (3.25) that carry the square of volume U_{ij}^2 of the trade $b_{ij}(2;t,z)$ (3.7). The flows $\boldsymbol{p}_{Cij}(t,z)$ (3.26) carry the square of the value C_{ij}^2 of the trade $b_{ij}(2;t,z)$ (3.7). We determine the cumulative flows $P(2;t,z)$ of the macro second-degree transaction $B(2;t,z)$, $z=(x,y)$ between points x and y similar to (3.11-3.15) as aggregation of flows $p_{ij}(2;t,z)$ (3.23) of all second-degree transactions $b_{ij}(2;t,z)$ (3.7) between agents in small volumes $dV(x)$ and $dV(y)$ (3.10) in economic domain (3.4, 3.5) and averaging during interval $\varDelta$. Due to (3.11-3.15) we introduce flows $P(2;t,z)$ and velocities $v(2;t,z)$ as:

$$\boldsymbol{P}(2;t,\boldsymbol{z}) = \left(\boldsymbol{P}_U(2;t,\boldsymbol{z}), \boldsymbol{P}_C(2;t,\boldsymbol{z})\right); \ \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \tag{3.27}$$

$$\boldsymbol{P}_U(2;t,z) = \sum_{i \in dV(x); j \in dV(y)\,\Delta} \boldsymbol{p}_{Uij}(2;t,z) \tag{3.28}$$

$$\boldsymbol{P}_C(t,z) = \sum_{i \in dV(x); j \in dV(y)\,\Delta} \boldsymbol{p}_{Cij}(2;t,z) \tag{3.29}$$

$$\boldsymbol{P}_U(2;t,z) = \Big(\boldsymbol{P}_{Ux}(t,z); \boldsymbol{P}_{Uy}(t,z)\Big) \;;\; \boldsymbol{P}_C(t,z) = \Big(\boldsymbol{P}_{Cx}(t,z); \boldsymbol{P}_{Cy}(t,z)\Big) \tag{3.30}$$

$$\boldsymbol{P}_{Ux}(2;t,z) = \sum_{i \in dV(x); j \in dV(y)\,\Delta} U_{ij}^2(t,z)\boldsymbol{v}_i(t,x) = U(2;t,z)\boldsymbol{v}_{Ux}(t,z) \tag{3.31}$$

$$\boldsymbol{P}_{Uy}(2;t,z) = \sum_{i \in dV(x); j \in dV(y)\,\Delta} U_{ij}^2(t,z)\boldsymbol{v}_j(t,y) = U(2;t,z)\boldsymbol{v}_{Uy}(t,z) \tag{3.32}$$

$$\boldsymbol{P}_{Cx}(2;t,z) = \sum_{i \in dV(x); j \in dV(y)\,\Delta} C_{ij}^2(t,z)\boldsymbol{v}_i(t,x) = C(2;t,z)\boldsymbol{v}_{Cx}(t,z) \tag{3.33}$$

$$\boldsymbol{P}_{Cy}(2;t,z) = \sum_{i \in dV(x); j \in dV(y)\,\Delta} C_{ij}^2(t,z)\boldsymbol{v}_j(t,y) = C(2;t,z)\boldsymbol{v}_{Cy}(t,z) \tag{3.34}$$

$$\boldsymbol{v}(2;t,z) = \Big(\boldsymbol{v}_U(2;t,z); \boldsymbol{v}_C(2;t,z)\Big) \tag{3.35}$$

$$\boldsymbol{v}_U(2;t,z) = \Big(\boldsymbol{v}_{Ux}(2;t,z); \boldsymbol{v}_{Uy}(2;t,z)\Big) \;;\; \boldsymbol{v}_C(2;t,z) = \Big(\boldsymbol{v}_{Cx}(2;t,z); \boldsymbol{v}_{Cy}(2;t,z)\Big) \tag{3.36}$$

The flows $\boldsymbol{P}(2;t,z)$ (3.27-3.34) of the second-degree transaction $\boldsymbol{B}(2;t,z)$ between points $\boldsymbol{x}$ and $\boldsymbol{y}$ describe the amounts of second-degree volume $U(2;t,z)$ (3.14) and the value $C(2;t,z)$ (3.15) of trades $\boldsymbol{B}(2;t,z)$ (3.11) carried by transactions velocities $\boldsymbol{v}(2;t,z)$ (3.35, 3.36) through $2m$-dimensional economic domain (3.4, 3.5). Velocity $\boldsymbol{v}_U(2;t,z)$ (3.36) defines the motion of second-degree volume $U(2;t,z)$ (3.14) and may be different from velocity $\boldsymbol{v}_C(2;t,z)$ (3.36) that describes the motion of second-degree value $C(2;t,z)$ (3.15).

Integral of transactions $\boldsymbol{B}(2;t,z)$ and their flows $\boldsymbol{P}(2;t,z)$ by $d\boldsymbol{y}$ over entire economic domain (3.4, 3.5) defines the second-degree sell-transactions $\boldsymbol{B}_s(2;t,\boldsymbol{x})$ from point $\boldsymbol{x}$. Integral by $d\boldsymbol{x}$ defines buy-transactions $\boldsymbol{B}_s(2;t,\boldsymbol{y})$ at point $\boldsymbol{y}$. Integral by $d\boldsymbol{z}=d\boldsymbol{x}d\boldsymbol{y}$ defines total transactions $\boldsymbol{B}(2;t)$ in the entire economy at moment t and their flows (see Appendix A). Long definitions (Appendix A) demonstrate the hidden complexities of the second-degree economic processes in economy and are useful for description of macroeconomic evolution.

Introduction of the second-degree transactions and their flows as functions of risk in economic domain (3.4,3.5) outlines the main contribution of our approach to macroeconomics, finance, and price volatility modelling. Relations on the second-degree transactions define different forms of the price. Relations (3.6, 3.9) define price $p_{ij}(t,z)$ and $p_{ij}^n(t,z)$ of particular transaction between agents i and j at points $\boldsymbol{x}$ and $\boldsymbol{y}$, $z=(\boldsymbol{x},\boldsymbol{y})$. Relations (3.16) define average square price $p(2;t,z)$ at moment t of all transactions between agents in small volumes near points $\boldsymbol{x}$ and $\boldsymbol{y}$, averaged during interval $\varDelta$. Relations (3.19) introduce average squared price $p(2;t)$ of all transactions in the economy averaged during time interval $\varDelta$. Relations (A.1-A.4) in Appendix A allow introduce average square price $p_s(2;t,\boldsymbol{x})$ for all sell-trades from point $\boldsymbol{x}$ and average square price $p_b(2;t,\boldsymbol{y})$ for all buy-trades at point $\boldsymbol{y}$ at moment t averaged during time $\varDelta$ as:

$$C_s(2; t, \boldsymbol{x}) = p_s(2; t, \boldsymbol{x}) U_s(2; t, \boldsymbol{x}) \qquad (3.37)$$

$$C_b(2; t, \boldsymbol{y}) = p_b(2; t, \boldsymbol{y}) U_b(2; t, \boldsymbol{y}) \qquad (3.38)$$

Relations (3.6, 3.9, 3.16, 3.19, 3.37, 3.38) describe different forms of price. Key contribution of our approach to macroeconomic modelling: evolution of transactions depends on transactions flows. Description of price must take into account transactions flows – these new economic factors impact price dynamics. Below we derive equations that govern the second-degree transactions and their flows.

4. Collective expectations

It is generally considered that agents take trade decisions under press of their expectations. Expectations play the crucial role for market trade performance. Decades of research describe impact of expectations on financial markets, price dynamics and economic evolution (Muth, 1961; Lucas, 1972; Blume and Easley, 1984; Brunnermeier and Parker, 2005; Dominitz and Greenwood and Shleifer, 2014). Observations and measurements of expectations is a difficult problem (Manski, 2004; Dominitz and Manski, 2005; Janžek and Ziherl, 2013; Manski, 2017). A lot of efforts are spent to observe, identify, estimate and measure the expectations. These problems are very complex. We suggest simplify the description of expectations and their impact on performance of market trades. To do that we regard expectations as agents assumptions and forecasts of numerous economic and financial variables, price and inflation trends, market trade activity, expectations of other agents, forecasts of any factors that can impact economy and etc. We assume that agents execute transactions under action of multiple expectations. Each agent may have multiple expectations and agent can perform similar transaction under different expectations. Below we model action of multiple agents expectations on market trades.

It seems that our assumption on multiplicity of expectations increases the complexity of the model: the difficulties of identification and measurement of expectations are increased by huge amount of different expectations of each agent. To overcome that we aggregate expectations of different agents alike to aggregation of transactions (3.11-3.15) performed by agents with risk coordinates inside the unit volume $dV(\boldsymbol{z})$ (3.10) in economic domain (3.4, 3.5). We rough the description of expectations of separate agents and introduce smooth collective expectations. This approach to expectations modelling is similar to description of macro variables and transactions developed in (Olkhov, 2016-2019) via the continuous economic approximation.

Multiplicity of expectations arises the problem of their comparative effect on market. Agents expectations differ from agents economic and financial variables. Agents variables like investment, assets, consumption – are additive. Sum of assets of group of agents (without overlapping) define assets of the group of agents. But expectations are not additive. Sum of expectations have no economic sense. Economic impact of the expectation that approve 1 bln.\$ deal must be a little more valuable than impact of the expectation that approve 1\$ deal. To collect agents expectations those approve the second-degree trades (3.7) one must take into account the economic value of the trades approved by these expectations. One can't simply sum agent's expectations to derive collective expectations of numerous agents. We state that market value of the collective expectation with regard to definite second-degree transaction should be proportional to the second-degree transaction (3.7) made under selected expectations. We assume that agents take decisions on squares of the volume U^2 and the value C^2 of the transaction under different expectations. Thus expectation that approve U^2 of the second-degree transaction (3.7) should be weighted by square volume U^2. Expectations those approve the C^2 of the second-degree transaction (3.7) must be weighted by square value C^2. We refer to (Olkhov, 2019b) for details and apply this method for description of collective expectations of the second-degree transactions $b_{ij}(2;t,z)$ (3.7). Let's regard description of collective expectations in a more formal manner.

Let's assume that agents in economy have $j=1,...K$ different expectations. Each agent involved into transaction $b_{ij}(2;t,z)$ (3.7) between agents i and j should take decisions on the volume U_{ij} and the value C_{ij}. Thus any trade is performed under action of four expectation – two expectations of agent i on the volume U_{ij} and the value C_{ij} and two expectations of agent j. Let's propose that agent i take decision on the volume U_{ij} under expectation $ex_{Ui}(k;t,x)$ of type k, and the value C_{ij} under expectation $ex_{Ci}(l;t,x)$ of type l, k, $l=1,..K$. We propose that expectations $ex_{Ui}(k;t,x)$ that approve the volume $U_{ij}(k;t,z)$ of trade may depend on expectations $ex_{Ci}(l;t,x)$ that approve the value $C_{ij}(l;t,z)$ and vice versa. Hence we propose that the volume U_{ij} and the value C_{ij} of the trade performed by the seller can depend on both expectations k and l: $ex_{Ui}(k,l;t,x)$ and $ex_{Ci}(k,l;t,x)$. Let's call as second-degree trade $b_{ij}(2;k;t,z)$ the pair (U_{ij}^2, C_{ij}^2) of squares of the volume and the value made under sellers expectations of type $\textbf{\textit{k}}=(k,l)$ from seller i at point $\textbf{\textit{x}}$ with buyer j at point $\textbf{\textit{y}}$ as:

$$\boldsymbol{bs}_{ij}(2;\boldsymbol{k};t,\boldsymbol{z}) = \left(U_{ij}^2(\boldsymbol{k};t,\boldsymbol{z}); C_{ij}^2(\boldsymbol{k};t,\boldsymbol{z})\right); \boldsymbol{k} = (k,l); k,l = 1,\dots K; \boldsymbol{z} = (\boldsymbol{x},\boldsymbol{y}) \quad (4.1)$$

and expectations $\boldsymbol{ex}(\boldsymbol{k};t,\boldsymbol{x})$ those approve transactions (4.1) :

$$\boldsymbol{ex}(\boldsymbol{k};t,\boldsymbol{x}) = \left(ex_{Ui}(\boldsymbol{k};t,\boldsymbol{x}); ex_{Ci}(\boldsymbol{k};t,\boldsymbol{x})\right) \quad (4.2)$$

We propose that all expectations $ex(k;t,x)$ have same measure and sum of two expectations has meaning of expectation. Thus we propose that one can sum different expectations and derive sum of expectations weighted by the volume or the value. Relations (4.1, 4.2) define the second-degree transactions and expectations of individual agent i as a seller at point x. Similar relations define the second-degree transactions $b_{ij}(2;t,z;l)$ and expectations $ex(t,y;l)$ of a buyer at point y.

$$bs_{ij}(2;t,z;l) = \left(U_{ij}^2(t,z;l); C_{ij}^2(t,z;l) \right); l = (k,l); \ k,l = 1,\dots K; \ z = (x,y) \quad (4.3)$$

and expectations $ex(t,y;l)$ those approve transactions (4.2) :

$$ex(t,y;l) = \left(ex_{Ui}(t,y;l); ex_{Ci}(t,y;l) \right); l = (k,l); \ k,l = 1,\dots K \quad (4.4)$$

To derive collected expectations of agents with coordinates inside (3.10) we define new additive factor - expected trades. We weight sellers $ex(k;t,x)$ (4.2) and buyers $ex(t,y,l)$ (4.4) expectations by squares of the volumes and squares of the values of trades and define the second-degree sellers expected trades $et_{ij}(k;t,z)$ as:

$$et_{ij}(k;t,z) = \left(et_{Uij}(k;t,z); \ et_{Cij}(k;t,z) \right) \quad (4.5)$$

$$et_{Uij}(k;t,z) = ex_{Ui}(k;t,x) U_{ij}^2(k;t,z) \quad (4.6)$$

$$et_{Cij}(k;t,z) = ex_{Cj}(k;t,y) C_{ij}^2(k;t,z) \quad (4.7)$$

The similar relations define the second-degree buyers expected trades $et_{ij}(t,z;l)$. The second-degree sellers expected trades of seller $et_{ij}(k;t,z;l)$ and buyer $et_{ij}(t,z;l)$ are additive functions and we use the procedure similar to (3.11-3.15) and (3.27-3.36) to define aggregate second-degree expected trades and their flows (Appendix B). We define collective sellers $Ex(k;t,z)$ (B.12-B14) and buyers $Ex(t,z;l)$ (B.15-B.17) expectations. Integral by economic domain (3.4, 3.5) introduce collective sellers $Ex(k;t)$ (B.20, B.210) and buyers $Ex(t;l)$ (B.22, B.23) expectations as function of time t and type of expectation (4.3, 4.4). Below we use these expectations to model oscillations of price $p(2;t)$ (2.6).

5. Price and volatility oscillations

In Appendix C we derive the system of equations on the second-degree transactions $B(2;t,z)$ (3.11-3.15) and their flows $P(2;t,z)$ (3.28-3.37) and on expected trades according to (Olkhov, 2019b). The system of equation (C.1-C.5) in economic domain (3.4, 3.5) and aggregated form of the equations (C.6-C.9) that describe collective transactions of the entire economy as functions of type of expectations k and time t only are complex. We present the simplest consequence of the equations (C.5-C.9) to demonstrate possible origin of price fluctuations and present equations that describe price and volatility evolution.

In Olkhov (2019b) we show how simple equations on the first-degree transactions imply equations on the price $p(1,t)$ (2.7). Here we derive the similar equations on the price $p(2,t)$ (2.6) for $n=2$. To do that we neglect the action of the flows in (C.6, C.7) and take equations on trades as:

$$\frac{d}{dt}U(2,\boldsymbol{k};t) = F_U(\boldsymbol{k};t) \quad ; \quad \frac{d}{dt}C(2,\boldsymbol{k};t) = F_C(\boldsymbol{k};t) \tag{5.1}$$

Let's remind the form of the second-degree price $p(2,\boldsymbol{k};t)$ as

$$C(2,\boldsymbol{k};t) = p(2,\boldsymbol{k};t)U(2,\boldsymbol{k};t) \tag{5.2}$$

Equations (5.1, 5.2) on $U(2;\boldsymbol{k},t)$ and $C(2;\boldsymbol{k},t)$ allow present equations on $p(2;\boldsymbol{k},t)$ as:

$$U(2,\boldsymbol{k};t)\frac{d}{dt}p(2,\boldsymbol{k};t) + p(2,\boldsymbol{k};t)F_U(\boldsymbol{k};t) = F_C(\boldsymbol{k};t) \tag{5.3}$$

Equations (5.1, 5.3) show that dynamics of price $p(2,\boldsymbol{k};t)$ or similar equations (5.4) for price $p(2;t)$ and volume $U(2;t)$ (3.18, 3.19):

$$\frac{d}{dt}U(2;t) = F_U(t) \quad ; \quad U(2;t)\frac{d}{dt}p(2;t) + p(2;t)F_U(t) = F_C(t) \tag{5.4}$$

are determined by functions F_U, F_C. As we discussed above, F_U, F_C can depend on expectations, expected transactions and their flows. Even in the simplest case if one neglects impact of transactions flows P_U, P_C (A.7-A.10) and flows of expected transactions (B.28, B.29) and their equations (C.3, C.5) the equations (5.3, 5.4) on price $p(2,\boldsymbol{k};t)$ or price $p(2;t)$ depend on functions F_U, F_C that model dynamics of collective the second-degree trades. Equations on $p(2,\boldsymbol{k};t)$ or $p(2;t)$ and similar equations on $p(1;t)$ (Olkhov, 2019) describe evolution of price volatility σ_p^2 (2.8) determined by the first and the second-degree transactions.

Equations (5.1) allow describe fluctuations of the price $p(2,\boldsymbol{k};t)$. To do that assume that in linear approximation by perturbations functions F_U and F_C in (5.1) depend on perturbations of collective expected trades $Et_U(\boldsymbol{k};t)$ and $Et_C(\boldsymbol{k};t)$ (B.19) as (D.9). Let's take equations on expected trades $Et_U(\boldsymbol{k};t)$ and $Et_C(\boldsymbol{k};t)$ as (C.8) :

$$\frac{d}{dt}Et_U(\boldsymbol{k};t) = W_U(\boldsymbol{k};t) \quad ; \quad \frac{d}{dt}Et_C(\boldsymbol{k};t) = W_C(\boldsymbol{k};t) \tag{5.5}$$

and assume that in linear approximation by perturbations functions W_U and W_C in (5.5) depend on perturbations of the volume $U(2,\boldsymbol{k};t)$ and the value $C(2,\boldsymbol{k};t)$ as (D.10). Assumptions (5.1, 5.5) describe small oscillations of collective transactions $U(2,k;t)$ and $C(2,k;t)$ and hence describe fluctuations of price $p(2,\boldsymbol{k};t)$ (5.2). We model fluctuations in Appendix D.

Relations (D.4) express the mean square price $p(2;t)$ (3.19) disturbances though disturbances of the squares volumes $u(2,\boldsymbol{k};t)$ (D.1) and the squares values $c(2,\boldsymbol{k};t)$ (D.1). Relations (D.7,

D.8) describe *p(2;t)* (3.19) dependence on partial price disturbances $\pi(2,\boldsymbol{k};t)$ (D.6) and volume disturbances *u(2,**k**;t)* (D.1). Disturbances of *p(2;t)* (D.4, D.8) and disturbances of *p(1;t)* (D.26) describe price volatility σ_p^2 (2.8) in the linear approximation by disturbances (D.27):

$$\sigma_p^2(t) = \sigma_{p0}^2 \{ 1 + \sum_{k,l=1}^{K} \sigma_2 [\mu_{2k} c(2,\boldsymbol{k};t) - \lambda_{2k} u(2,\boldsymbol{k};t)] - \sigma_1 [\mu_{1k} c(1,\boldsymbol{k};t) - \lambda_{1k} u(1,\boldsymbol{k};t)] \}$$

We leave further development of these problems for future.

6. Conclusion

Macroeconomics describes mutual dependence of macro variables and market transactions aggregated during time interval *Δ*. All macro variables and trades fluctuate. To describe volatility of these fluctuations one should describe the second-degree variables and transactions aggregated during interval *Δ*. Methods for aggregating variables during interval *Δ* must have economic sense. We call model of the second-degree variables and trades as the second-order economic theory. Development of the second-order economic theory is a though problem and this paper only starts study price volatility and the second-degree trades.

We take VWAP as the only example of the price aggregation procedure with economic sense. VWAP defines mean price *p(1;t)* as ratio of sum of the values and sum of the volumes of trades performed during interval *Δ*. We generalize the VWAP procedure and use the similar approach to define mean *n-th* degree price *p(n;t)* (2.5, 2.6) and price statistical moments *p(n;t₁,..tₙ)* (2.10, 2.11). To define price probability one should model all price statistical moments. Thus price probability depends on description of collective *n-th* degree products of trades (2.5). It seems almost impossible describe *n-th* degree trades, economic variables and expectations for all *n=1,2,… .* Hence the bright dreams to find out the price probability distribution fortunately or unfortunately unattainable.

We underline that we study the price random properties, mean price, price volatility and etc., on base of the definitions (2.1, 2.6) only. Relations (2.1) is only one in 117 price definitions mentioned by Fetter (1912). We don't reject the usefulness and usage of the remaining 116 price definitions. However we suppose that any other price definitions, notions and models, those agents or researchers have in their mind can impact agents expectations. Agents are free in their choice of price notions, forecasts and projections. Any possible or impossible price expectations can be established. These price expectations approve decisions for execution of market transactions and for sure impact price properties determined by (2.1, 2.6). Nevertheless (2.1) is the only price of the executed market transaction that collects impacts of

all possible price expectations of agents involved. This is the reason we use (2.1) as the only ground of the price theory.

The *n-th* degree price *p(n;t)* (2.5, 2.6) and price statistical moments $p(n;t_1,..t_n)$ (2.10, 2.11) depend on the time interval Δ. For time terms $T >> \Delta$ the *n-th* degree price *p(n;t)* (2.5, 2.6) and statistical moments can behave irregular. One can recursively perform transitions from one averaging time interval Δ to another averaging interval $T_1 > \Delta$ and further from T_1 to $T_2 > T_1$. These transitions imply change of the *n-th* degree price *p(n;t)* (2.5, 2.6) and price statistical moments $p(n;t_1,..t_n)$ (2.10, 2.11) determined by averaging time interval T_1, T_2 and etc. Each change of averaging interval cause change of *n-th* degree price *p(n;t;T)* according to (2.15, 2.16) and corresponding change of characteristic functional (2.12). Relations between *n-th* degree prices *p(n;t;T)* and characteristic functionals (2.12) for averaging intervals $T_1, T_2...$ should be studied further.

We underline that for each *n* price *p(n;t)* (2.6) caries information about the *n-th* price degree only. In simple words: mean price *p(1;t)* (2.5, 2.6) for *n=1* carries information about the mean price only. Function *p(1;t)* can behave irregular, "like random" function at time $T > \Delta$ but nevertheless averaging procedure over time $T > \Delta$ give the mean price for time interval *T* only. To assess mean square price *p(2;t)* or mean *n-th* degree price *p(n;t)* for time interval $T > \Delta$ one must use *p(2;t)* or *p(n;t)*. Usage of irregular "random" behavior of *p(1;t)* during time $T > \Delta$ to assess mean square price, price volatility and etc., has no economic sense.

Price volatility is determined by first two statistical moments *n=1,2* (2.6) and depends on sums of squares of the volumes and the values of trades. All sums are taken during interval Δ and the choice of interval Δ determines properties of second-degree trades and price volatility.

To describe skewness of price probability one should model *3-rd* statistical moments. Price skewness depends on description on *3-rd* degree trades and needs development of the third-order macro theory – theory that takes into account the *3-rd* degree economic and financial variables and *3-rd* degree trades aggregated during interval Δ.

We begin our study of price volatility with model of the second-degree trades. To do that we introduce numerical continuous risk grades and suggest that modern econometric data provide sufficient info to assess risks of almost all economic agents. 80 years ago Durand, (1941) and 60 years ago Myers and Forgy, (1963) already introduced numerical risk grades. We regard usage letters risk grades as a way to protect the business of major rating agencies – S&P, Moody's, Fitch. However we believe that development and usage of unified numerical continuous rating methodology should boost market value of major rating companies by

several orders. Introduction of numerical continuous risk grades enlarge the meaning of risk transition matrices. Transition matrices could describe not only the probability to change risk rating from current state to another one during certain time term, but also determine mean velocities of economic agents in the risk economic space. Exact description of numerous separate agents and their economic and financial variables is too difficult problem. We rough accuracy of the description and aggregate variables and transactions of agents with coordinates inside small volume dV in economic space and average during time interval Δ. This replace description of economic variables and trades those belong to separate agents by description of economic variables and trades as functions of coordinates in economic space. Motion of agents in the risk economic space induces collective motion of aggregated economic variables and trades. We show that agent's motion cause flows of economic variables and flows of transactions in the risk economic space. Flows of trades and variables are new issues in economic theory. We develop the continuous economic media approximation that describes densities of economic variables, trades and their flows in economic space as functions of time and risk coordinates. Aggregation over economic space describes macroeconomic variables and trades as functions of time only.

To describe the second-degree trades we define transactions performed by separate agents and collect transactions and their flows performed by agents in small volume dV as functions of coordinates in economic space. We follow the general economic consensus that agents execute transactions under action of their expectations. We propose that the second-degree trades economic "weight" of agents expectations should be proportional to squares of the volumes and the values of transactions performed under expectations. We define aggregated expectations and derive equations that govern macro expectations and their flows.

Introduction of macro second-degree trades defines mean square price as function of risk coordinates. Aggregation of trades over economic space defines mean square price as function of time only.

For the simple model in the linear approximation by perturbations we derive equations on perturbations of the second-degree transactions and expectations. We derive mean square price as function of perturbations of the second-degree values and the volumes of trades. Simples assumptions allow describe the second-degree trade perturbations as set of harmonic oscillations induced by perturbations of multiple expectations. We describe similar perturbations of the VWAP mean price $p(1;t)$ in (Olkhov, 2019b). Both results give price volatility perturbations through perturbations of the first and the second-degree trades (D.27).

These expressions demonstrate dependence of price volatility on trade volume – the problem studied for a long time on base of market econometric data.

We hope that further development of the second-order economic theory may bring interesting economic problems and useful financial solutions.

Appendix A. Transactions and their flows

Integral of the second-degree trade $B(2;t,z)$ (3.11-3.15) by dy over economic domain (3.2) determines all sell-trades $B_s(2;t,x)$ from point x :

$$B_s(2; t, x) = \int dy \ B(2; t, x, y) = \big(U_s(2; t, x); C_s(2; t, x)\big) \tag{A.1}$$

$$U_s(2; t, x) = \int dy \ U(2; t, x, y) \ ; \ C_s(2; t, x) = \int dy \ C(2; t, x, y) \tag{A.2}$$

Integral $B(2;t,z)$ by dx over economic domain (3.2) determines all buy-trades $B_b(2;t,y)$ at y

$$B_b(2; t, y) = \int dx \ B(2; t, x, y) = \big(U_b(2; t, y); C_b(2; t, y)\big) \tag{A.3}$$

$$U_b(2; t, y) = \int dx \ U(2; t, x, y) \ ; \ C_b(2; t, y) = \int dx \ C(2; t, x, y) \tag{A.4}$$

Integral of the second-degree trade $B(2;t,z)$ by $dz=dxdy$ over economic domain (3.2) define all macro second-degree trades $B(2;t)$ as function of time t only:

$$B(2; t) = \int dz \ B(2; t, z) = \big(U(2; t); C(2; t)\big) \tag{A.5}$$

$$U(2; t) = \int dz \ U(2; t, z) \ ; \ C(2; t) = \int dz \ C(2; t, z) \tag{A.6}$$

Relations (A.1-A.6) introduce flows and velocities for corresponding second-degree trades. Due to (3.31-3.32) obtain:

$$\int dy \ U(2; t, z) \ \boldsymbol{v}_{Ux}(2; t, z)$$

$$\boldsymbol{P}_{Ux}(2; t, x) = \int dy \ \boldsymbol{P}_{Ux}(2; t, x, y) = \sum_{i \in dV(x); j \in dV(y)} \Delta \int dy \ U_{ij}^2(t, x, y) v_i(t, x) \tag{A.7}$$

$$\boldsymbol{P}_{Ux}(2; t, x) = \int dy \ U(2; t, z) \ \boldsymbol{v}_{Ux}(2; t, z) = U_s(2; t, x) \boldsymbol{v}_{Usx}(2; t, x) \tag{A.8}$$

$$\boldsymbol{P}_{Uy}(2; t, x) = \int dy \ \boldsymbol{P}_{Uy}(2; t, x, y) = \sum_{i \in dV(x); j \in dV(y)} \Delta \int dy \ U_{ij}^2(t, x, y) v_j(t, y) \tag{A.9}$$

$$\boldsymbol{P}_{Uy}(2; t, x) = \int dy \ U(2; t, z) \ \boldsymbol{v}_{Uy}(2; t, z) = U_s(2; t, x) \boldsymbol{v}_{Uby}(2; t, x) \tag{A.10}$$

Relations (A.7, A.8) introduce aggregate flow of sales $\boldsymbol{P}_{Ux}(2;t,x)$ of the second-degree volumes $U_s(2;t,x)$ and collective velocity $\boldsymbol{v}_{Usx}(2; t, x)$ of sellers at point x along axis X to all buyers in economy. Relations (A.9, A.10) introduce aggregate flow of sales $\boldsymbol{P}_{Uy}(2;t,x)$ of second-degree volumes $U_s(2;t,x)$ and collective velocity $\boldsymbol{v}_{Usy}(2; t, x)$ of all buyers in economy along axis Y from collective sellers at point x. Corresponding relations for the second-degree values flows $\boldsymbol{P}_{Csx}(2;t,x)$ and $\boldsymbol{P}_{Csy}(2;t,x)$ take form:

$$\boldsymbol{P}_{Csx}(2; t, x) = \int dy \ \boldsymbol{P}_{Cx}(2; t, x, y) = \sum_{i \in dV(x); j \in dV(y)} \Delta \int dy \ C_{ij}^2(t, x, y) v_i(t, x) \tag{A.11}$$

$$\boldsymbol{P}_{Csx}(2; t, x) = C_s(2; t, x) \boldsymbol{v}_{Csx}(2; t, x) \tag{A.12}$$

$$\boldsymbol{P}_{Csy}(2; t, x) = \int dy \ \boldsymbol{P}_{Cy}(2; t, x, y) = \sum_{i \in dV(x); j \in dV(y)} \Delta \int dy \ C_{ij}^2(t, x, y) v_j(t, y) \tag{A.13}$$

$$\boldsymbol{P}_{Csy}(2; t, x) = C_s(2; t, x) \boldsymbol{v}_{Csy}(2; t, x) \tag{A.14}$$

The similar relations define the buyer's flows. To derive buyer's the second-degree volume and the value flows $\boldsymbol{P}_{Ubx}(2;t,y)$ and $\boldsymbol{P}_{Uby}(2;t,y)$, $\boldsymbol{P}_{Cbx}(2;t,y)$ and $\boldsymbol{P}_{Cby}(2;t,y)$ one should take integral for (3.32-3.35) by dx over economic domain (3.4, 3.5). For brevity we omit it here.

Integrals by *dxdy* for (3.31-3.35) over entire economy – over economic domain (3.4,3.5) introduce macro flows of second-degree transactions as functions of time *t* only:

$$\boldsymbol{P}_{Ux}(2;t) = \int d\boldsymbol{z}\, \boldsymbol{P}_{Ux}(2;t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x}); j\in dV(\boldsymbol{y})} {}_{\Delta}\int dxdy\, U_{ij}^2(t,\boldsymbol{x},\boldsymbol{y})\boldsymbol{v}_i(t,\boldsymbol{x}) \quad (A.15)$$

$$\boldsymbol{P}_{Ux}(2;t) = U(2;t)\boldsymbol{v}_{Ux}(2;t) \quad (A.16)$$

$$\boldsymbol{P}_{Uy}(2;t) = \int d\boldsymbol{z}\, \boldsymbol{P}_{Uy}(2;t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x}); j\in dV(\boldsymbol{y})} {}_{\Delta}\int dxdy\, U_{ij}^2(t,\boldsymbol{x},\boldsymbol{y})\boldsymbol{v}_j(t,\boldsymbol{y}) \quad (A.17)$$

$$\boldsymbol{P}_{Uy}(2;t) = U(2;t)\boldsymbol{v}_{Uy}(2;t) \quad (A.18)$$

$$\boldsymbol{P}_{Cx}(2;t) = \int d\boldsymbol{z}\, \boldsymbol{P}_{Cx}(2;t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x}); j\in dV(\boldsymbol{y})} {}_{\Delta}\int dxdy\, C_{ij}^2(t,\boldsymbol{x},\boldsymbol{y})\boldsymbol{v}_i(t,\boldsymbol{x}) \quad (A.19)$$

$$\boldsymbol{P}_{Cx}(2;t) = C(2;t)\boldsymbol{v}_{Cx}(2;t) \quad (A.20)$$

$$\boldsymbol{P}_{Cy}(2;t) = \int d\boldsymbol{z}\, \boldsymbol{P}_{Cy}(2;t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x}); j\in dV(\boldsymbol{y})} {}_{\Delta}\int dxdy\, C_{ij}^2(t,\boldsymbol{x},\boldsymbol{y})\boldsymbol{v}_j(t,\boldsymbol{y}) \quad (A.21)$$

$$\boldsymbol{P}_{Cy}(2;t) = C(2;t)\boldsymbol{v}_{Cy}(2;t) \quad (A.22)$$

Relations (A.15-A.22) introduce new economic concepts – collective flows of all agents of the economy involved into the second-degree trades in economic domain (3.4, 3.5). Collective flows of the second-degree volumes ($\boldsymbol{P}_{Ux}(2;t)$; $\boldsymbol{P}_{Uy}(2;t)$), and flows of the second-degree values ($\boldsymbol{P}_{Cx}(2;t)$; $\boldsymbol{P}_{Cy}(2;t)$) define corresponding collective velocities of the volume ($\boldsymbol{v}_{Ux}$, $\boldsymbol{v}_{Uy}$) and the value ($\boldsymbol{v}_{Cx}$, $\boldsymbol{v}_{Cy}$). Velocities $\boldsymbol{v}_{Ux}$ define collective motion of all sellers in economy weighted by volumes of trades during interval Δ. Velocities $\boldsymbol{v}_{Uy}$ define collective motion of all buyers in economy weighted by volumes of trades during interval Δ. Velocities ($\boldsymbol{v}_{Cx}$, $\boldsymbol{v}_{Cy}$) define collective motion of all sellers and all buyers weighted by value of all trades during interval Δ. These velocities describe collective motion of the entire economy in the economic domain (3.4, 3.5) reduced by its borders at 0 and 1 by all axes. Hence these collective velocities can't have the constant direction and must oscillate. Collective velocities of the volume ($\boldsymbol{v}_{Ux}$, $\boldsymbol{v}_{Uy}$) and the value ($\boldsymbol{v}_{Cx}$, $\boldsymbol{v}_{Cy}$) occasionally change their direction and describe collective motion of sellers and buyers from the secure area of economic domain to the risky one and back. We suggest that such fluctuations of the second-degree trades accompanied with rise and fall of market activity, production functions, demand and supply, price variations and etc., establish ground for economic fluctuations of business cycles. We describe motion of collective trades in the economic domain (3.4, 3.5) as hidden properties of the business cycles (Olkhov, 2017b, 2019a).

Appendix B. Collective expectations

To define collective expectations we introduce new issue: collective expected trades. To do that we use (4.1) to define collective second-degree trades $U(2,\boldsymbol{k};t,\boldsymbol{z})$, $C(2,\boldsymbol{k};t,\boldsymbol{z})$ approved by sellers expectations of type $\boldsymbol{k}=(k,l)$ as:

$$U(2,\boldsymbol{k};t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x}); j\in dV(\boldsymbol{y})\,\Delta} U_{ij}^2(\boldsymbol{k};t,\boldsymbol{z}) \tag{B.1}$$

$$C(2,\boldsymbol{k};t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x}); j\in dV(\boldsymbol{y})\,\Delta} C_{ij}^2(\boldsymbol{k};t,\boldsymbol{z}) \tag{B.2}$$

Definitions of collective flows $P_U(2,\boldsymbol{k};t,\boldsymbol{z})$ and $P_C(2,\boldsymbol{k};t,\boldsymbol{z})$ of trades with the volume $U(2,\boldsymbol{k};t,\boldsymbol{z})$ and the value $C(2,\boldsymbol{k};t,\boldsymbol{z})$ (B.1,B.2) are similar to (A.7-A.10) and we omit it here for brevity. Trades $U(2,\boldsymbol{k};t,\boldsymbol{z})$, $C(2,\boldsymbol{k};t,\boldsymbol{z})$ (B.1, B.2) obey obvious relations (3.13-3.15):

$$U(2;t,\boldsymbol{z}) = \sum_{k,l=1}^{K} U(2,\boldsymbol{k};t,\boldsymbol{z}) \;\; ; \;\; C(2;t,\boldsymbol{z}) = \sum_{k,l=1}^{K} C(2,\boldsymbol{k};t,\boldsymbol{z}) \tag{B.3}$$

Collective buyers second-degree trades $U(2;t,\boldsymbol{z};\boldsymbol{l})$, $C(2;t,\boldsymbol{z};\boldsymbol{l})$ approved by buyers expectations take the similar form:

$$U(2;t,\boldsymbol{z};\boldsymbol{l}) = \sum_{i\in dV(\boldsymbol{x}); j\in dV(\boldsymbol{y})\,\Delta} U_{ij}^2(t,\boldsymbol{z};\boldsymbol{l}) \tag{B.4}$$

$$C(2;t,\boldsymbol{z};\boldsymbol{l}) = \sum_{i\in dV(\boldsymbol{x}); j\in dV(\boldsymbol{y})\,\Delta} C_{ij}^2(t,\boldsymbol{z};\boldsymbol{l}) \tag{B.5}$$

Buyer's trades obey same relations (B.3). Similar to (3.11-3.15) we use (4.5-4.7) and define sellers collective expected trades $\boldsymbol{Et}(k;t,\boldsymbol{z})$ as:

$$\boldsymbol{Et}(k;t,\boldsymbol{z}) = \big(Et_U(\boldsymbol{k};t,\boldsymbol{z}); Et_C(\boldsymbol{k};t,\boldsymbol{z})\big) \tag{B.6}$$

$$Et_U(\boldsymbol{k};t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x}); j\in dV(\boldsymbol{y});\,\Delta} ex_{Ui}(\boldsymbol{k};t,\boldsymbol{x}) U_{ij}^2(\boldsymbol{k};t,\boldsymbol{z}) \tag{B.7}$$

$$Et_C(\boldsymbol{k};t,\boldsymbol{z}) = \sum_{i\in dV(\boldsymbol{x}); j\in dV(\boldsymbol{y});\,\Delta} ex_{Ci}(\boldsymbol{k};t,\boldsymbol{x}) C_{ij}^2(\boldsymbol{k};t,\boldsymbol{z}) \tag{B.8}$$

Buyer's collective second-degree expected trades $\boldsymbol{Et}(t,\boldsymbol{z};\boldsymbol{l})$,

$$\boldsymbol{Et}(t,\boldsymbol{z};\boldsymbol{l}) = \big(Et_U(t,\boldsymbol{z};\boldsymbol{l}); Et_C(t,\boldsymbol{z};\boldsymbol{l})\big) \tag{B.9}$$

$$Et_U(t,\boldsymbol{z};\boldsymbol{l}) = \sum_{i\in dV(\boldsymbol{x}); j\in dV(\boldsymbol{y});\,\Delta} ex_{Uj}(t,\boldsymbol{y};\boldsymbol{l}) U_{ij}^2(t,\boldsymbol{z};\boldsymbol{l}) \tag{B.10}$$

$$Et_C(t,\boldsymbol{z};\boldsymbol{l}) = \sum_{i\in dV(\boldsymbol{x}); j\in dV(\boldsymbol{y});\,\Delta} ex_{Cj}(t,\boldsymbol{y};\boldsymbol{l}) C_{ij}^2(t,\boldsymbol{z};\boldsymbol{l}) \tag{B.11}$$

Relations (B.6-B.8) and (B.1,B.2) introduce collective sellers expectations of the second-degree $\boldsymbol{Ex}(k;t,\boldsymbol{z})$, as:

$$Ex(\boldsymbol{k};t,\boldsymbol{z}) = \big(Ex_U(\boldsymbol{k};t,\boldsymbol{z}), Ex_C(\boldsymbol{k};t,\boldsymbol{z})\big) \tag{B.12}$$

$$Ex_U(\boldsymbol{k};t,\boldsymbol{z}) U(2,\boldsymbol{k};t,\boldsymbol{z}) = Et_U(\boldsymbol{k};t,\boldsymbol{z}) \tag{B.13}$$

$$Ex_C(\boldsymbol{k};t,\boldsymbol{z}) C(2,\boldsymbol{k};t,\boldsymbol{z}) = Et_C(\boldsymbol{k};t,\boldsymbol{z}) \tag{B.14}$$

Collective buyers expectations take similar form:

$$Ex(t,\boldsymbol{z};\boldsymbol{l}) = \big(Ex_U(t,\boldsymbol{z};\boldsymbol{l}), Ex_C(t,\boldsymbol{z};\boldsymbol{l})\big) \tag{B.15}$$

$$Ex_U(t,\boldsymbol{z};\boldsymbol{l}) U(2;t,\boldsymbol{z};\boldsymbol{l}) = Et_U(t,\boldsymbol{z};\boldsymbol{l}) \tag{B.16}$$

$$Ex_C(t,\boldsymbol{z};\boldsymbol{l}) C(2;t,\boldsymbol{z};\boldsymbol{l}) = Et_C(t,\boldsymbol{z};\boldsymbol{l}) \tag{B.17}$$

Let's take integrals over economic domain (3.4,3.5) of *U(2,k;t,z)*, *C(2,k;t,z)* (B.1, B.2):

$$U(2, \boldsymbol{k}; t) = \int d\boldsymbol{z} \; U(2, \boldsymbol{k}; t, \boldsymbol{z}) \; ; \; C(2, \boldsymbol{k}; t) = \int d\boldsymbol{z} \; C(2, \boldsymbol{k}; t, \boldsymbol{z}) \tag{B.18}$$

Relations (B.18) define collective sellers second-degree trades *U(2,k;t)*, *C(2,k;t)* performed under expectations of type *k=(k,l)*. Then integral by economic domain (3.4, 3.5) of expected trades (B.13, B.14) define collective sellers expected trades $Et_U(\boldsymbol{k};t)$, $Et_C(\boldsymbol{k};t)$ of the entire economy

$$Et_U(\boldsymbol{k}; t) = \int d\boldsymbol{z} \; Et_U(\boldsymbol{k}; t, \boldsymbol{z}) \; ; \; Et_C(\boldsymbol{k}; t) = \int d\boldsymbol{z} \; Et_C(\boldsymbol{k}; t, \boldsymbol{z}) \tag{B.19}$$

(B.18, B.19) define collective sellers expectations $Ex_U(\boldsymbol{k};t)$, $Ex_C(\boldsymbol{k};t)$ of the entire economy

$$Ex_U(\boldsymbol{k}; t)U(2, \boldsymbol{k}; t) = Et_U(\boldsymbol{k}; t) \tag{B.20}$$

$$Ex_C(\boldsymbol{k}; t)C(2, \boldsymbol{k}; t) = Et_C(\boldsymbol{k}; t) \tag{B.21}$$

The similar relations define collective buyers expected trades $Et_U(t;\boldsymbol{l})$, $Et_C(t;\boldsymbol{l})$ and expectations $Ex_U(t;\boldsymbol{l})$, $Ex_C(t;\boldsymbol{l})$ of the entire economy:

$$Ex_U(t; \boldsymbol{l})U(2; t; \boldsymbol{l}) = Et_U(t; \boldsymbol{l}) \tag{B.22}$$

$$Ex_C(t; \boldsymbol{l})C(2; t; \boldsymbol{l}) = Et_C(t; \boldsymbol{l}) \tag{B.23}$$

Relations (B.20) introduce collective seller expected trades $Et_{Us}(t)$, $Et_{Cs}(t)$:

$$Et_{Us}(t) = \sum_{k,l=1}^{K} Et_U(\boldsymbol{k}; t) \; ; \; Et_{Cs}(t) = \sum_{k,l=1}^{K} Et_C(\boldsymbol{k}; t) \tag{B.24}$$

Relations (B.24) define collective sellers expectations $Ex_{Us}(t)$, $Ex_{Cs}(t)$:

$$Ex_{Us}(t)U(2; t) = Et_{Us}(t) \; ; \; Ex_{Cs}(t)C(2; t) = Et_{Cs}(t) \tag{B.25}$$

The similar relations define collective buyers expectations of the entire economy. We omit them for brevity. It is clear that collective sellers expectations those approve squares of the volume U^2 and squares of the value C^2 of transactions may be different. Collective seller's expectations may be different from buyer's expectations. Such diversity uncovers hidden complexity of the mutual interactions between trades and expectations. We present above set of long definitions to outline multiplicity of different states of collective trades and expectations. One easy defines collective flows of sellers and buyers expected trades similar to (Olkhov, 2019b). For brevity we present only definition of sellers volume flow of expected trades $\boldsymbol{Pe}_{Ux}$, $\boldsymbol{Pe}_{Uy}$ in (B.26-B2.9):

$$\boldsymbol{pe}_{Uxij}(\boldsymbol{k}; t, \boldsymbol{z}) = et_{Uij}(\boldsymbol{k}; t, \boldsymbol{z})\boldsymbol{v}_i(t, \boldsymbol{x}) = ex_i(\boldsymbol{k}; t, \boldsymbol{x})U_{ij}^2(\boldsymbol{k}; t, \boldsymbol{z})\boldsymbol{v}_i(t, \boldsymbol{x}) \tag{B.26}$$

$$\boldsymbol{pe}_{Uyij}(\boldsymbol{k}; t, \boldsymbol{z}) = et_{Uij}(\boldsymbol{k}; t, \boldsymbol{z})\boldsymbol{v}_j(t, \boldsymbol{y}) = ex_i(\boldsymbol{k}; t, \boldsymbol{x})U_{ij}^2(\boldsymbol{k}; t, \boldsymbol{z})\boldsymbol{v}_j(t, \boldsymbol{y}) \tag{B.27}$$

$$\boldsymbol{Pe}_{Ux}(\boldsymbol{k}; t, \boldsymbol{z}) = \sum_{i \in dV(\boldsymbol{x}); j \in dV(\boldsymbol{y})} \Delta \, et_{Uij}(\boldsymbol{k}; t, \boldsymbol{z})\boldsymbol{v}_i(t, \boldsymbol{x}) = Et_U(\boldsymbol{k}; t, \boldsymbol{z})\boldsymbol{ve}_{Ux}(\boldsymbol{k}; t, \boldsymbol{z}) \tag{B.28}$$

$$\boldsymbol{Pe}_{Uy}(\boldsymbol{k}; t, \boldsymbol{z}) = \sum_{i \in dV(\boldsymbol{x}); j \in dV(\boldsymbol{y})} \Delta \, et_{Uij}(\boldsymbol{k}; t, \boldsymbol{z})\boldsymbol{v}_j(t, \boldsymbol{y}) = Et_U(\boldsymbol{k}; t, \boldsymbol{z})\boldsymbol{ve}_{Uy}(\boldsymbol{k}; t, \boldsymbol{z}) \tag{B.29}$$

For brevity we don't' define full set of flows and refer (Olkhov, 2019b) for further detail.

Appendix C. Equations of the second-degree trades

For brevity we present here only equations on second-degree trades $B(2;t,z)$ (3.11-3.15) and refer for details (Olkhov, 2018-2019b). Equations on the second-degree volume $U(2,k;t,z)$ and the value $C(2,k;t,z)$ (B.1,B.2) take form:

$$\frac{\partial}{\partial t} U(2,\boldsymbol{k};t,\boldsymbol{z}) + \nabla \cdot \left(U(2,\boldsymbol{k};t,\boldsymbol{z})\, \boldsymbol{v}_U(2,\boldsymbol{k};t,\boldsymbol{z}) \right) = F_U(\boldsymbol{k};t,\boldsymbol{z}) \tag{C.1}$$

$$\frac{\partial}{\partial t} C(2,\boldsymbol{k};t,\boldsymbol{z}) + \nabla \cdot \left(C(2,\boldsymbol{k};t,\boldsymbol{z})\, \boldsymbol{v}_C(2,\boldsymbol{k};t,\boldsymbol{z}) \right) = F_C(\boldsymbol{k};t,\boldsymbol{z}) \tag{C.2}$$

Equations on flows $\boldsymbol{P}_{Ux}(2;t,z)$ (3.32) of second-degree volume $U(2;t,z)$ take form:

$$\frac{\partial}{\partial t} \boldsymbol{P}_{Ux}(\boldsymbol{k};t,\boldsymbol{z}) + \nabla \cdot \left(\boldsymbol{P}_{Ux}(\boldsymbol{k};t,\boldsymbol{z})\, \boldsymbol{v}_{Ux}(\boldsymbol{k};t,\boldsymbol{z}) \right) = \boldsymbol{G}_{Ux}(\boldsymbol{k};t,\boldsymbol{z}) \tag{C.3}$$

Equations on flows (3.33-3.35) have similar form. Similar equations valid for expected trades $Ex_U(\boldsymbol{k};t,z)$ (B.6-B.8) and their flows $\boldsymbol{Pe}_{Ux}(\boldsymbol{k};t,\boldsymbol{z})$:

$$\frac{\partial}{\partial t} Et_U(\boldsymbol{k};t,\boldsymbol{z}) + \nabla \cdot \left(Et_U(\boldsymbol{k};t,\boldsymbol{z})\, \boldsymbol{ve}_{Ux}(\boldsymbol{k};t,\boldsymbol{z}) \right) = W_U(\boldsymbol{k};t,\boldsymbol{z}) \tag{C.4}$$

$$\frac{\partial}{\partial t} \boldsymbol{Pe}_{Ux}(\boldsymbol{k};t,\boldsymbol{z}) + \nabla \cdot \left(\boldsymbol{Pe}_{Ux}(\boldsymbol{k};t,\boldsymbol{z})\, \boldsymbol{ve}_{Ux}(\boldsymbol{k};t,\boldsymbol{z}) \right) = \boldsymbol{R}_{Ux}(\boldsymbol{k};t,\boldsymbol{z}) \tag{C.5}$$

Integrals of divergence by dz over domain (3.4,3.5) equal zero as no economic agents and no fluxes exist outside (3.4,3.5). Hence integrals of (C.1-C.5) by dz over domain (3.4,3.5) give:

$$\int d\boldsymbol{z} \left[\frac{\partial}{\partial t} U(2,\boldsymbol{k};t,\boldsymbol{z}) + \nabla \cdot \left(U(2,\boldsymbol{k};t,\boldsymbol{z})\, \boldsymbol{v}_U(2,\boldsymbol{k};t,\boldsymbol{z}) \right) \right] = \frac{d}{dt} U(2,\boldsymbol{k};t) = F_U(\boldsymbol{k};t) = \int d\boldsymbol{z}\, F_U(t,\boldsymbol{z}) \tag{C.6}$$

$$\frac{d}{dt} C(2,\boldsymbol{k};t) = F_C(\boldsymbol{k};t) \quad ; \quad \frac{d}{dt} \boldsymbol{P}_{Ux}(\boldsymbol{k};t) = \boldsymbol{G}_{Ux}(\boldsymbol{k};t) \tag{C.7}$$

$$\frac{d}{dt} Et_U(\boldsymbol{k};t) = W_U(\boldsymbol{k};t) \quad ; \quad \frac{d}{dt} \boldsymbol{Pe}_{Ux}(\boldsymbol{k};t) = \boldsymbol{R}_{Ux}(\boldsymbol{k};t) \tag{C.8}$$

(C.6-C.8) describe equations on collective second-degree trades, expectations and their flows as functions of time t and type of expectations $\boldsymbol{k}$. Formal simplicity of (C.6-C.8) hides complexity of the functions $F_U(t)$, $F_U(t)$, $\boldsymbol{G}_{Ux}$, $\boldsymbol{W}_{Ux}$, $\boldsymbol{R}_{Ux}$, and etc. These functions describe impact of expectations of the second-degree trades and their flows and impact of trades on the expected trades and their flows. These functions model the economic and financial processes, social expectations, technology forecast, and flows of these parameters those impact the second-degree trades $B(2,t)$ (3.17) and flows $P(2,t)$ and back action of trades on expected trades and their flows. Integrals in right side of (C.6-C.8) can describe integrals by flows and their products. Such factors are completely new for macroeconomic theory and were never taken into account before.

Appendix D. Price fluctuations

Let's follow (Olkhov, 2019b) and neglect the impact of trades and expected trades flows.
Let's study small perturbations of $U(2,\boldsymbol{k};t)$, $C(2,\boldsymbol{k};t)$ and $Ex_U(\boldsymbol{k};t)$ and $Ex_C(\boldsymbol{k};t)$ as:

$$U(2,\boldsymbol{k};t) = U_{\boldsymbol{k}}^2\big(1 + u(2,\boldsymbol{k};t)\big) \; ; \; C(2,\boldsymbol{k};t) = C_{\boldsymbol{k}}^2\big(1 + c(2,\boldsymbol{k};t)\big) \qquad (D.1)$$

$$Et_U(\boldsymbol{k};t) = Et_{U\boldsymbol{k}}\big(1 + et_u(\boldsymbol{k};t)\big); \; Et_C(\boldsymbol{k};t) = Et_{C\boldsymbol{k}}\big(1 + et_c(\boldsymbol{k};t)\big) \qquad (D.2)$$

Relations (D.1, D.2) describe small dimensionless disturbances u, c, et_u et_c. Relations (3.19, 5.2, D.1) present the second-degree volume $U(2;t)$, the value $C(2;t)$ and price $p(2,t)$ (3.19) as:

$$U(2;t) = \sum_{k,l=1}^K U(2,\boldsymbol{k};t) \; ; \; C(2;t) = \sum_{k,l=1}^K C(2,\boldsymbol{k};t) \; ; \; C(2;t) = p(2;t)U(2;t) \quad (D.3)$$

$$p(2;t) = \frac{C(2;t)}{U(2;t)} = \frac{\sum_{k,l} C_{\boldsymbol{k}}^2\big(1 + c(2,\boldsymbol{k};t)\big)}{\sum_{k,l} U_{\boldsymbol{k}}^2\big(1 + u(2,\boldsymbol{k};t)\big)}$$

In linear approximation by the disturbances $u(2,\boldsymbol{k};t)$ and $c(2,\boldsymbol{k};t)$, the price $p(2;t)$ takes form:

$$p(2;t) = p_{20}[1 + \pi(2;t)] = p_{20}\big[1 + \sum_{k,l=1}^K (\mu_{2\boldsymbol{k}}c(2,\boldsymbol{k};t) - \lambda_{2\boldsymbol{k}}u(2,\boldsymbol{k};t))\big] \qquad (D.4)$$

$$U_{20} = \sum_{k,l} U_{\boldsymbol{k}}^2 \; ; \; C_{20} = \sum_{k,l} C_{\boldsymbol{k}}^2 \; ; \; C_{20} = p_{20}U_{20}$$

$$\lambda_{2\boldsymbol{k}} = \frac{U_{\boldsymbol{k}}^2}{U_{20}} \; ; \quad \mu_{2\boldsymbol{k}} = \frac{C_{\boldsymbol{k}}^2}{C_{20}} \; ; \quad \sum \lambda_{2\boldsymbol{k}} = \sum \mu_{2\boldsymbol{k}} = 1$$

We use index *2* to underline that p_{20} and other constants describe properties of the price $p(2;t)$. Let's take into account (D.1) and present price $p(2;\boldsymbol{k},t)$ (5.2) disturbances as

$$C_{\boldsymbol{k}}^2[1 + c(2,\boldsymbol{k};t)] = p_{2\boldsymbol{k}}[1 + \pi(2,\boldsymbol{k};t)]U_{\boldsymbol{k}}^2[1 + u(2,\boldsymbol{k};t)] \qquad (D.5)$$

Then in the linear approximation by disturbances obtain:

$$C_{\boldsymbol{k}}^2 = p_{2\boldsymbol{k}}U_{\boldsymbol{k}}^2 \; ; \; \pi(2;\boldsymbol{k};t) = c(2,\boldsymbol{k};t) - u(2,\boldsymbol{k};t) \qquad (D.6)$$

Now substitute (D.6) into (D.4) and obtain dependence of price disturbances $\pi(2;t)$ on the volume disturbances $u(2,\boldsymbol{k};t)$:

$$\pi(2;t) = \sum_{k,l} \mu_{2\boldsymbol{k}}\pi(2,\boldsymbol{k};t) + \sum_{k,l}(\mu_{2\boldsymbol{k}} - \lambda_{2\boldsymbol{k}})\,u(2,\boldsymbol{k};t) \qquad (D.7)$$

$$p(2;t) = p_{20}[1 + \pi(2;t)] = p_{20}\big[1 + \sum_{k,l} \mu_{2\boldsymbol{k}}\pi(2,\boldsymbol{k};t) + (\mu_{2\boldsymbol{k}} - \lambda_{2\boldsymbol{k}})u(2,\boldsymbol{k};t)\big] (D.8)$$

(D.7) describes fluctuations $\pi(2;t)$ (D.4) of mean square price $p(2;t)$ as function of partial square price disturbances $\pi(2,\boldsymbol{k};t)$ (D.6) and disturbances of squares volume $u(2,\boldsymbol{k};t)$ (D.1).

Now let's study simple model that can describe volume, value and price disturbances (D.1, D.4, D.8). Assume that $U_k{}^2$, $C_k{}^2$, Et_{Uk} and Et_{Ck} in (D.1, D.2) are constant or their changes are slow to compare with changes of the small dimensionless disturbances $u(2,\boldsymbol{k};t)$, $c(2,\boldsymbol{k};t)$, $et_q(\boldsymbol{k};t)$ and $et_c(\boldsymbol{k};t)$. We assume that in the linear approximation by perturbations F_U, F_C in (5.1) and W_U, W_C in (5.5) take form:

$$F_U(\boldsymbol{k};t) = a_{uk}Et_{U\boldsymbol{k}}\big(1 + et_u(\boldsymbol{k};t)\big); \; F_C(\boldsymbol{k};t) = a_{ck}Et_{C\boldsymbol{k}}\big(1 + et_c(\boldsymbol{k};t)\big) \qquad (D.9)$$

$$W_U(\boldsymbol{k};t) = b_{uk}U_{\boldsymbol{k}}^2\big(1 + u(2,\boldsymbol{k};t)\big) \quad ; \quad W_C(\boldsymbol{k};t) = b_{ck}C_{\boldsymbol{k}}^2\big(1 + c(2,\boldsymbol{k};t)\big) \tag{D.10}$$

Taking into account (B.20, B.21) and (D.1, D.2) equations (5.1, 5.6) in linear approximation by disturbances take form:

$$U_{\boldsymbol{k}}^2 \frac{d}{dt}u(2,\boldsymbol{k};t) = a_{uk}Et_{Uk}et_u(\boldsymbol{k};t) \quad ; \quad C_{\boldsymbol{k}}^2 \frac{d}{dt}c(2,\boldsymbol{k};t) = a_{ck}Et_{Ck}et_c(\boldsymbol{k};t) \tag{D.11}$$

$$Et_{Uk}\frac{d}{dt}et_U(\boldsymbol{k};t) = b_{uk}U_{\boldsymbol{k}}^2 u(2,\boldsymbol{k};t) \quad ; \quad Et_{Ck}\frac{d}{dt}et_C(\boldsymbol{k};t) = b_{ck}C_{\boldsymbol{k}}^2 c(2,\boldsymbol{k};t) \tag{D.12}$$

For

$$\omega_{uk}^2 = -a_{uk}\,b_{uk} > 0 \quad ; \quad \omega_{ck}^2 = -a_{ck}\,b_{ck} > 0 \tag{D.13}$$

simple solutions for disturbances $u(2,\boldsymbol{k};t)$, $c(2,\boldsymbol{k};t)$, $et_u(\boldsymbol{k};t)$, $et_c(\boldsymbol{k};t)$ take harmonic oscillations:

$$u(2,\boldsymbol{k};t) = u_{1k}sin\omega_{uk}t + u_{2k}cos\omega_{uk}t \quad ; \quad u_{1k}, u_{2k} \ll 1 \tag{D.14}$$

$$c(2,\boldsymbol{k};t) = c_{1k}sin\omega_{ck}t + c_{2k}cos\omega_{ck}t \quad ; \quad c_{1k}, c_{2k} \ll 1 \tag{D.15}$$

Solutions (D.8, D.9) are similar to relations derived for the first-degree trades disturbances (Olkhov, 2019b). For the case

$$\gamma_{uk}^2 = -a_{uk}\,b_{uk} < 0 \quad ; \quad \gamma_{ck}^2 = -a_{ck}\,b_{ck} < 0 \tag{D.16}$$

equations (D.11, D.12) describe exponential growth or dissipations of perturbations

$$u(2,\boldsymbol{k};t) \sim \exp\gamma_{uk}t \quad ; c(2,\boldsymbol{k};t) \sim \exp\gamma_{ck}t \tag{D.17}$$

and perturbations $\pi(2,\boldsymbol{k};t)$ (D.6) also may follow exponential growth or decline. For different expectations $\boldsymbol{k}=(k,l)$ (4.1, B.7, B.8) solutions for disturbances $u(2,\boldsymbol{k};t)$, $c(2,\boldsymbol{k};t)$ can take (D.14, D.15) or (D.17). Combinations of perturbations determined by (D.14, D.15) or (D.17) define perturbations of price $p(2;t)$ (D.4). Thus simple equations (5.1, 5.5) on the second-degree volume and value may describe unexpected exponential growth of price $p(2;t)$ perturbations $\pi(2;t)$ and hence exponential growth of price volatility σ_p^2 (2.8).

Now let's assume that $u(2,\boldsymbol{k};t)$ and $c(2,\boldsymbol{k};t)$ disturbances depend on disturbances of expectations $ex_u(\boldsymbol{k};t)$, $ex_c(\boldsymbol{k};t)$ (see B.20, B.21):

$$Ex_U(\boldsymbol{k};t) = Ex_{Uk}\big(1 + ex_u(\boldsymbol{k};t)\big); \; Ex_C(\boldsymbol{k};t) = Ex_{Ck}\big(1 + ex_c(\boldsymbol{k};t)\big) \tag{D.18}$$

Let's take into account (D.1) and in the linear approximation by disturbances obtain:

$$Et_U(\boldsymbol{k};t) = Ex_{Uk}U_{\boldsymbol{k}}^2\big(1 + ex_u(\boldsymbol{k};t) + u(2,\boldsymbol{k};t)\big) \tag{D.19}$$

$$Et_C(\boldsymbol{k};t) = Ex_{Ck}U_{\boldsymbol{k}}^2\big(1 + ex_c(\boldsymbol{k};t) + c(2,\boldsymbol{k};t)\big) \tag{D.20}$$

Now use (D.19, D.20) and replace (D.9) by (D.21, D.22):

$$F_U(\boldsymbol{k};t) = a_{uk}Et_{Uk}U_{\boldsymbol{k}}^2\big(1 + ex_u(\boldsymbol{k};t) + u(2,\boldsymbol{k};t)\big) \tag{D.21}$$

$$F_C(\boldsymbol{k};t) = a_{ck}Et_{Ck}C_{\boldsymbol{k}}^2\big(1 + ex_c(\boldsymbol{k};t) + c(2,\boldsymbol{k};t)\big) \tag{D.22}$$

It is easy to show that for

$$\omega_{uk}^2 = -\frac{a_{uk}\,b_{uk}}{U_{\boldsymbol{k}}} > 0 \quad ; \quad \omega_{ck}^2 = -\frac{a_{ck}\,b_{ck}}{C_{\boldsymbol{k}}} > 0 \tag{D.23}$$

$$\gamma_{uk} = a_{uk}\frac{Ex_{Uk}}{U_k} \; ; \quad \gamma_{ck} = a_{ck}\frac{Ex_{Ck}}{C_k} \tag{D.24}$$

equations (5.1, 5.5, D.10, D.21, D.22) describe damped harmonic oscillators:

$$\left[\frac{d^2}{dt^2} + \gamma_{uk}\frac{d}{dt} + \omega_{uk}^2\right]u(\boldsymbol{k};t) = 0 \; ; \quad \left[\frac{d^2}{dt^2} + \gamma_{ck}\frac{d}{dt} + \omega_{ck}^2\right]c(\boldsymbol{k};t) = 0$$

$$u(\boldsymbol{k};t) \sim \sin\varphi_{uk}t \exp-\frac{\gamma_{uk}}{2}t \; ; \quad c(\boldsymbol{k};t) \sim \sin\varphi_{ck}t \exp-\frac{\gamma_{ck}}{2}t \tag{D.25}$$

$$\varphi_{uk}^2 = \omega_{uk}^2 - \frac{\gamma_{uk}^2}{4} \; ; \quad \varphi_{ck}^2 = \omega_{ck}^2 - \frac{\gamma_{ck}^2}{4}$$

If $\gamma_{uk} > 0$ and $\gamma_{ck} > 0$ than (D.25) and (D.4) describe trade and price *p(2;t)* disturbances as harmonic oscillations with dissipating amplitudes. But if $\gamma_{uk} < 0$ or $\gamma_{ck} < 0$ than (D.25) and *p(2;t)* in (D.4) and volatility (5.6) have harmonic modes with exponentially growing amplitudes. It is interesting that perturbations of expectations $Ex_u(\boldsymbol{k};t)$, $Ex_c(\boldsymbol{k};t)$ (D.18) can generate disturbances of trades (D.25), price *p(2;t)* (D.4) and volatility σ_p^2 (5.6) like harmonic oscillations with amplitude growth by exponent in time. It is obvious that such exponential rise of perturbations *π(2;t)* and price volatility σ_p^2 (2.8) is reduced by applicability of model equations (5.1, 5.5) and relations (D.1, D.2, D.4, D.14, D.15, D.17). This approximation has sense till perturbations remain small to compare with 1.

Relations (D.4, D.8) describe mean square price *p(2;t)* (3.19) disturbances. We derive similar relations on mean price *p(1;t)* disturbances (Olkhov, 2019b):

$$p(1;t) = p_{10}\left[1 + \sum_{k,l=1}^{K}(\mu_{1k}c(1,\boldsymbol{k};t) - \lambda_{1k}u(1,\boldsymbol{k};t))\right] \tag{D.26}$$

Coefficients p_{10}, μ_{1k}, λ_{1k} and functions *c(1,**k**;t)*, *u(1,**k**;t)* (D.26) describe price *p(1;t)* and the first-degree value and volume in a way similar to p_{10}, μ_{1k}, λ_{1k} (D.4) and functions *c(1,**k**;t)*, *u(1,**k**;t)* (D.5) for *p(2;t)* (D.4) and the second-degree value and volume (D.1). Disturbances of *p(2;t)* (D.4) and *p(1;t)* (D.26) describe price volatility σ_p^2 (2.8) in linear approximation by disturbances as:

$$\sigma_p^2(t) = \sigma_{p0}^2\left\{1 + \sum_{k,l=1}^{K}\sigma_2[\mu_{2k}c(2,\boldsymbol{k};t) - \lambda_{2k}u(2,\boldsymbol{k};t)] - \sigma_1[\mu_{1k}c(1,\boldsymbol{k};t) - \lambda_{1k}u(1,\boldsymbol{k};t)]\right\}$$

$$\sigma_{p0}^2 = p_{20} - p_{10}^2 \; ; \quad \sigma_2 = \frac{p_{20}}{\sigma_{p0}^2} \; ; \quad \sigma_1 = 2\frac{p_{10}^2}{\sigma_{p0}^2} \tag{D.27}$$

In (D.27) constants and functions with index 2 describe mean square price *p(2;t)* and its disturbances and index 1 describes mean VWAP price *p(1;t)* (Olkhov, 2019b) and its disturbances.

References

Andersen, T., Bollerslev, T., Diebold, F.X. and Ebens, H., 2001. The Distribution of Realized Stock Return Volatility, Journal of Financial Economics, 61, 43-76.

Andersen, T.G., Bollerslev, T., Christoffersen, P.F. and F.X. Diebold, 2005. Volatility Forecasting, CFS WP 2005/08, 1-116.

Avramov, D., Chordia, T., Goyal, A., 2006. The Impact of Trades on Daily Volatility, The Review of Financial Studies, 19, (4), 1241- 1277.

Beaver, W. H., Shakespeare, C. and M. T. Soliman, 2006. Differential properties in the ratings of certified versus non-certified bond-rating agencies, Journal of Accounting and Economics 42, 303–334

Belkin, B., Suchower, S. and L. R. Forest, 1998. A one-parameter representation of credit risk and transition matrices. JP Morgan, CreditMetrics® Monitor, 3-d Q, 46-56

Berkowitz, S.A., Dennis E. Logue, D.E. and E. A. Noser, Jr., 1988. The Total Cost of Transactions on the NYSE, The Journal Of Finance, 43, (1), 97-112.

Bernanke, B. and M. Gertler, 1999. Monetary Policy and Asset Price Volatility. FRB of Kansas City, Economic Review, 4Q, 1-36.

Blume, L.E., Easley, D., 1984. Rational Expectations Equilibrium: An Alternative Approach. Journal Of Economic Theory, 34, 116-129

Bogousslavsky,V. and P. Collin-Dufresne, 2019. Liquidity, Volume, and Volatility, Swiss Finance Institute, Research Paper Series 19-69, 1-57.

Brock, W.A. and B.D. LeBaron, 1995. A Dynamic structural model for stock return volatility and trading volume. NBER WP 4988, 1-46.

Brunnermeier, M.K., Parker, J.A., 2005. Optimal Expectations. American Economic Review, 95, (4), 1092-1118

Buryak, A. and I. Guo, 2014. Effective And Simple VWAP Options Pricing Model, Intern. J. Theor. Applied Finance, 17, (6), 1450036, https://doi.org/10.1142/S0219024914500356

Busseti, E. and S. Boyd, 2015. Volume Weighted Average Price Optimal Execution, 1-34, arXiv:1509.08503v1

Campbell, J.Y., Grossman, S.J., Wang, J., 1993. Trading Volume And Serial Correlation In Stock Returns. The Quarterly Journal of Economics, 905-939.

Christiansen, C., Schmeling, M. and A. Schrimpf, 2012. A Comprehensive Look at Financial Volatility Prediction by Economic Variables. BIS WP 374, 1-46.

Ciner, C. and W. H. Sackley, 2007. Transactions, volume and volatility: evidence from an

emerging market, Applied Financial Economics Letters, 3, 161-164.

CME Group, 2020. www.cmegroup.com/confluence/display/EPICSANDBOX/GovPX+Historical+Data ;
www.cmegroup.com/confluence/display/EPICSANDBOX/Standard+and+Poors+500+Futures

Daly, K., 2008. Financial volatility: Issues and measuring techniques, Physica A 387, 2377–2393.

Dominitz, J., Manski, C.F., 2005. Measuring And Interpreting Expectations Of Equity Returns. NBER, WP 11313, Cambridge, MA

Durand, D., 1941. Volume Title: Risk Elements in Consumer Instalment Financing, Technical Edition, NBER, http://www.nber.org/books/dura41-1

Engle, R.F. and A.J. Patton, 2001. What good is a volatility model? Quantitative Finance, 1, 237–245.

Fama, E.F., 1965. The Behavior of Stock-Market Prices. The Journal of Business, 38, (1), 34-105.

Fetter, F.A., 1912. The Definition of Price. The American Economic Review, 2 (4), 783-813.

Fitch, 2018. Procedures and Methodologies for Determining Credit Ratings. Fitch Ratings, Inc., Form 25-101F1

Greenwood, R., Shleifer, A., 2014. Expectations of Returns and Expected Returns.
The Review of Financial Studies, 27 (3), 714–746

Guéant, O. and G. Royer, 2014. VWAP execution and guaranteed VWAP, SIAM J. Finan. Math., 5(1), 445–471.

Hall, R.L. and C.J. Hitch, 1939. Price Theory and Business Behaviour, Oxford Economic Papers, 2. Reprinted in T. Wilson and P. W. S. Andrews (eds.), Oxford Studies in the Price Mechanism (Oxford, 1951).

Heston, S.L., 1993. A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, The Rev. Financial Studies, 6, (2), 327-343.

Ho, A., et.al., 2017. Asia-Pacific Structured Finance 2016 Transition and Default Study, FitchRatings Structured Finance, 1-14

Ito, T. and W-L.Lin, 1993. Price Volatility And Volume. Spillovers Between The Tokyo And New York Stock Markets, NBER WP 4592, 1-33.

Janžek, T., Ziherl, P., 2013. Overview of models and methods for measuring economic agent's expectations. BIS, IFC Bulletin 36, 172-179. https://www.bis.org/ifc/publ/ifcb36.htm

Lucas, R.E., 1972. Expectations and the Neutrality of Money. J. Econ. Theory 4, 103-124

Mankiw, N.G., Romer, D. and M.D. Shapiro, 1991. Stock Market Forecastability and
Volatility: A Statistical Appraisal, Rev.Economic Studies, 58,455-477.

Manski, C., 2004. Measuring Expectations. Econometrica, **72**, 1329-1376

Manski, C.F., 2017. Survey Measurement Of Probabilistic Macroeconomic Expectations: Progress and Promise. NBER, WP 23418

Miloudi, A., Bouattour, M., Benkraiem, R., 2016. Relationships between Trading Volume, Stock Returns and Volatility: Evidence from the French Stock Market, Bankers, Markets & Investors, 144, 1-15.

Moody's, 2010. Rating Symbols and Definitions. Moody's Investors Service

Moody's, 2018. Procedures and Methodologies Used to Determine Credit Ratings. Moody's Investors Service

Muth, J.F., 1961. Rational Expectations and the Theory of Price Movements, Econometrica, 29, (3) 315-335.

Myers, J.H. and E.W. Forgy, 1963. The Development of Numerical Credit Evaluation Systems, Jour. American Statistical Association, 58, 303, 799-806

Olkhov, V., 2016a. On Economic space Notion. Intern. Rev. Financial Analysis, 47, 372–81.

Olkhov, V., 2016b. Finance, Risk and Economic space. ACRN Oxford Journal of Finance and Risk Perspectives 5, 209–21.

Olkhov, V., 2017a. Quantitative Wave Model of Macro-Finance. Intern. Rev. Financial Analysis 50, 143–50.

Olkhov, V., 2017b. Econophysics of Business Cycles: Aggregate Economic Fluctuations, Mean Risks and Mean Square Risks, 1-31, http://arxiv.org/abs/1709.00282.

Olkhov, V., 2018. How Macro Transactions Describe the Evolution and Fluctuation of Financial Variables. Intern. Jour. Financial Studies 6, 38, 1-19.

Olkhov, V., 2019a. Economic Transactions Govern Business Cycles. ACRN Oxford Journal of Finance and Risk Perspectives 7, 102–22.

Olkhov, V., 2019b. Financial Variables, Market Transactions, and Expectations as Functions of Risk. Int. Jour. Financial Stud., 7, 66, 1-27.

Olkhov, V., 2020. Volatility depend on Market Trades and Macro Theory. MPRA WP 102434

Padungsaksawasdi, C., and R. T. Daigler, 2018. Volume weighted volatility: empirical evidence for a new realized volatility measure, Int. J. Banking, Accounting and Finance, 9, (1), 61-87.

Pearce, D.K., 1983. Stock Prices and the Economy. FRB Kansas, Economic Review, 7-22

Poon, S-H., and C.W.J. Granger, 2003. Forecasting Volatility in Financial Markets: A Review, J. of Economic Literature, 41, 478–539.

Poon, W.P.H. and J. Shen, 2020. The roles of rating outlooks: the predictor of creditworthiness and the monitor of recovery efforts. Rev.Quant.Finan.Acc., (2020). https://doi.org/10.1007/s11156-019-00868-7

Stigler, G.J., and J.K. Kindahl, 1970. The Dispersion of Price Movements, NBER, 88 - 94 in Ed. Stigler,G.J., and J.K. Kindahl ,The Behavior of Industrial Prices

Schuermann, T. and Y. Jafry, 2003. Measurement and Estimation of Credit Migration Matrices, The Wharton Financial Institutions Center, 1-44

Schwert, G.W., 1988. Why Does Stock Market Volatility Change Over Time? NBER WP 2798.

S&P, 2014. Guide To Credit Rating Essentials. What are credit ratings and how do they work? McGraw Hill Financial

S&P&, 2016. S&P Global Ratings Definitions. S&P Global Ratings

S&P, 2018. 2018 Annual Global Corporate Default And Rating Transition Study

Takaishi, T., and T. T. Chen, 2017, The relationship between trading volumes, number of transactions, and stock volatility in GARCH models, J. Phys., Conf. Ser. 738 012097, 1-4, doi:10.1088/1742-6596/738/1/012097.

Tauchen, G.E. and M. Pitts, 1983, The Price Variability-Volume Relationship On Speculative Markets, Econometrica, 51, (2), 485-505.