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THE STRUCTURE OF (LOCAL) ORDINAL BAYESIAN INCENTIVE COMPATIBLE RANDOM RULES*

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Abstract

We explore the structure of local ordinal Bayesian incentive compatible (LOBIC) random Bayesian rules (RBRs). We show that under lower contour monotonicity, almost all (with Lebesgue measure 1) LOBIC RBRs are local dominant strategy incentive compatible (LDSIC). We also provide conditions on domains so that unanimity implies lower contour monotonicity for almost all LOBIC RBRs. We provide sufficient conditions on a domain so that almost all unanimous RBRs on it (i) are Pareto optimal, (ii) are tops-only, and (iii) are only-topset. Finally, we provide a wide range of applications of our results on the unrestricted, single-peaked (on graphs), hybrid, multiple single-peaked, single-dipped, single-crossing, multidimensional separable, lexicographic, and domains under partitioning. We additionally establish the marginal decomposability property for both random social choice functions and almost all RBRs on multi-dimensional domains, and thereby generalize Breton and Sen (1999). Since OBIC implies LOBIC by definition, all our results hold for OBIC RBRs.

KEYWORDS. random Bayesian rules; random social choice functions; (local) ordinal Bayesian incentive compatibility; (local) dominant strategy incentive compatibility

JEL CLASSIFICATION CODES. D71; D82

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1. Introduction

We consider social choice problems where a random social choice function (RSCF) selects a probability distribution over a finite set of alternatives at every collection of preferences of the agents in a society. It is incentive compatible (IC) if no agent can increase the probability of any upper contour set by misreporting her preference. A random Bayesian rule (RBR) consists of an RSCF and a prior belief of each agent about the preferences of the others. We assume that the prior of an agent is "partially correlated": her belief about the preference of one agent may depend on that about another agent, but it does not depend on her own preference. Ordinal Bayesian incentive compatibility (OBIC) is the natural extension of the notion of IC for RBRs. This notion is introduced in d'Aspremont and Peleg (1988) and it captures the idea of Bayes-Nash equilibrium in the context of incomplete information game. An RBR is OBIC if no agent can increase the expected probability (with respect to her belief) of any upper contour set by misreporting her preference.

Local DSIC (LDSIC) or local OBIC (LOBIC) are weaker versions of the corresponding notions. As the name suggests, they apply to deviations/misreports to only "local" preferences (the notion of which is fixed a priori). The importance of these local notions is well-established in the literature: on one hand, they are useful in modeling behavioral agents (see Carroll (2012)), on the other hand, on many domains they turn out to be equivalent to their corresponding global versions and thereby used as a simpler way to check whether a given RSCF is DSIC (see Carroll (2012), Kumar et al. (2020), Sato (2013), Cho (2016), etc.).

The structure of DSIC RSCFs is well-explored in the literature. On the unrestricted domain, they turn out to be random dictatorial, and on restricted domains such as single-peaked or single-crossing or single-dipped, they are some versions of probabilistic fixed ballot rules. However, to the best of our knowledge, the structure of LOBIC (or OBIC) RBRs is not at all explored. Even for *deterministic* Bayesian rules (DBRs), not much is known: on the unrestricted domain they are dictatorial for almost all priors (with Lebesgue measure 1) (see Majumdar and Sen (2004) and Mishra (2016), and on single-peaked domains, they are Pareto efficient (see Mishra (2016)).

The main objective of this paper is to explore the structure of LOBIC RBRs on different domains. The importance of Bayesian rules is well-established in the literature: on one hand, they model real life situations where agents behave according to their beliefs, on the other hand, they are significant weakening of the seemingly too demanding requirement of DSIC that leads to dictatorship (or random dictatorships) unless the domain is restricted. Moreover, randomization has long been recognized as a useful device to achieve fairness in allocation problems. This comprises our motivation to study RBRs.

We consider arbitrary notion of localness formulated by a graph over preferences. We introduce the notion of lower contour monotonicity for an RBR and establish the equivalence between LOBIC and the much stronger (and well-studied) notion LDSIC on *any* domain for RBRs satisfying this property. The deterministic version of this result for the special case of swap-local domains is proved in Mishra (2016).[1] However, Mishra (2016) considers "totally independent" priors: belief of an agent, apart from being independent of her own preferences, is also independent over other agents' preferences.

We provide conditions on swap-local domains so that under LOBIC, unanimity implies lower contour monotonicity, and thereby making the equivalence of LDSIC and LOBIC hold under unanimity. It turns out that the said equivalence does not hold on most well-known restricted domains. Therefore, we provide conditions on arbitrary graph-connected domains so that almost all unanimous and LOBIC RBRs on it (i) are tops-only, (ii) are Pareto optimal, and (iii) are only-topset.[2] Finally, we establish our main equivalence result for weak preferences and provide a discussion explaining why none of these results can be extended for fully correlated priors (that is, when the prior of an agent depends on her own preference). It is worth emphasizing that all the existing results for LOBIC DBRs (Majumdar and Sen (2004) and Mishra (2016)) follow from our results. Furthermore, since every OBIC rule is LOBIC by definition, all our results hold for OBIC rules in particular.

Majumdar and Sen (2004) introduce the notion of generic priors, the particularity

[1]A graph on a domain is swap-local if any two local preferences differ by a swap of consecutively ranked alternatives.

[2]An RSCF is only-topset if it gives positive probabilities to only the alternatives that appear as a top-ranked alternative in the domain.

of which is that they have Lebesgue measure 1. It is well-known that a unanimous and OBIC RBR with respect to a generic prior need not be random dictatorial, and therefore, it was believed for long that the dictatorial result does not extend (almost surely) for OBIC RBRs. However, it follows from our results that in fact it does, only thing is that one needs to construct the right class of priors ensuring the Lebesgue measure to be 1.

We provide a wide range of applications of our results. We introduce the notion of betweenness domains and establish the structure of almost all LOBIC RBRs on these domains. Well-known restricted domains such as single-peaked on arbitrary graphs, hybrid, multiple single-peaked, single-dipped, single-crossing, and domains under partitioning are important examples of betweenness domains. We introduce a weaker version of lower contour monotonicity and obtain a characterization of almost all unanimous and LOBIC RSCFs or DSCFs (depending on what is known in the literature regarding the equivalence of LDSIC and DSIC) on these domains under that condition. Furthermore, we explain with the help of an example how our results can be utilized to construct the remaining RBRs (that is, the ones that do not satisfy lower contour monotonicity).

Our consideration of arbitrary notion of localness allows us to provide the structure of LOBIC RBRs on full separable multi-dimensional domains and lexicographically separable multi-dimensional domains when the marginal domains satisfy the betweenness property, that is, when the marginal domains are unrestricted or single-peaked on graphs or hybrid or multiple single-peaked or single-dipped or single-crossing. Additionally, we prove an important property, called marginal decomposability, of almost all RBRs on multidimensional separable domains. The deterministic version of it, namely decomposability, is proved for DSCFs in Breton and Sen (1999) under DSIC. To the best of our knowledge, this property is not established for RSCFs (even under DSIC), which now follows from our general result about the same for RBRs.

The rest of the paper is organized as follows. Sections 2, 3, and 4 introduce the notions of domains, RSCFs, RBRs, and their relevant properties. Sections 5 and 6 present our results for graph-connected and swap-connected domains. Sections 7 and 9 present the applications of our results on betweenness and multi-dimensional domains.

We present our result for weak preferences in Section 10. Finally, in Section 11 we provide a discussion on DBRs and (fully) correlated priors.

2. Preferences and domains

We denote a finite set of alternatives by A and a finite set of n agents by N. A (strict) preference over A is defined as a linear order on A.[3] We deal with strict preferences throughout the paper, except in Section 10 where we provide the definition of weak preferences. The set of all preferences over A is denoted by $\mathscr{P}(A)$. A subset $\mathscr{D}$ of $\mathscr{P}(A)$ is called a domain. Whenever it is clear from the context, we do not use brackets to denote singleton sets.

The weak part of a preference P is denoted by R. Since P is strict, for any two alternatives x and y, xRy implies either xPy or $x = y$. The kth ranked alternative in a preference P is denoted by $P(k)$. The topset $\tau(\mathscr{D})$ of a domain $\mathscr{D}$ is defined as the set of alternatives $\cup_{P\in\mathscr{D}}P(1)$. A domain $\mathscr{D}$ is regular if $\tau(\mathscr{D}) = A$. The upper contour set $U(x,P)$ of an alternative x at a preference P is defined as the set of alternatives that are strictly preferred to x in P, that is, $U(x,P) = \{a \in A \mid aPx\}$. A set U is called an upper contour set at P if it is an upper contour set of some alternative at P. The restriction of a preference P to a subset B of alternatives is denoted by $P|_B$, more formally, $P|_B \in \mathscr{P}(B)$ such that for all $a,b \in B$, $aP|_Bb$ if and only if aPb.

Each agent $i \in N$ has a domain $\mathscr{D}_i$ (of admissible preferences). We assume that each domain $\mathscr{D}_i$ is endowed with some graph structure $G_i = \langle \mathscr{D}_i, E_i \rangle$. The graph G_i represents the proximity relation between the preferences: an edge between two preferences implies that they are close in some sense. For instance, suppose $A = \{a,b,c\}$ and $\mathscr{D}_i$ is the set of all preferences over A. Suppose that two preferences are "close" if and only if they differ by a swap of two alternatives. The graph G_i that represents this proximity relation is given in Figure 1. The alternatives that swap between two preferences are mentioned on the edge between the two.

We denote by G_N a collection of graphs $(G_i)_{i\in N}$. Whenever we use some term involving the word "graph", we mean it with respect to a collection G_N. Two preferences P_i and P_i' of an agent i are graph-local if they form an edge in G_i, and a sequence of

[3] A linear order is a complete, transitive, and antisymmetric binary relation.

preferences $(P_i^1, \ldots, P_i^t)$ is a graph-local path if every two consecutive preferences in the sequence are graph-local. A domain $\mathscr{D}_i$ is graph-connected if there is a graph-local path between any two preferences in it. We denote by $\mathscr{D}_N$ the product set $\mathscr{D}_1 \times \cdots \times \mathscr{D}_n$ of individual domains. An element of $\mathscr{D}_N$ is called a preference profile. All the domains we consider in this paper are assumed to be graph-connected.

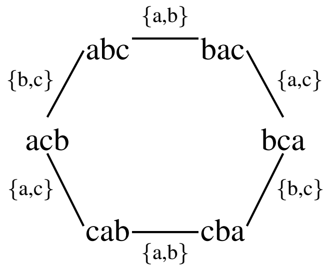

Figure 1

We use the following terminologies to ease the presentation: $P \equiv xy\cdots$ means $P(1) = x$ and $P(2) = y$; $P \equiv \cdots xy \cdots$ means x and y are consecutively ranked in P with xPy; $P \equiv \cdots x \cdots y \cdots$ means x is ranked above y. When the set of alternatives is precisely stated, say $A = \{a,b,c,d\}$, we write, for instance, $P = abcd$ to mean $P(1) = a$, $P(2) = b$, $P(3) = c$, and $P(4) = d$. We use similar notations without further explanations.

3. Random social choice functions and their properties

Let ΔA be the set of all probability distributions on A. A random social choice function (RSCF) is a mapping $\varphi : \mathscr{D}_N \to \Delta A$. We denote the probability of an alternative x at $\varphi(P_N)$ by $\varphi_x(P_N)$.

An RSCF $\varphi : \mathscr{D}_N \to \Delta A$ is **unanimous** if for all $P_N \in \mathscr{D}_N$ such that for all $i \in N$, $P_i(1) = x$ for some $x \in A$, we have $\varphi_x(P_N) = 1$. An RSCF $\varphi : \mathscr{D}_N \to \Delta A$ is **Pareto optimal** if for all $P_N \in \mathscr{D}_N$ and all $x \in A$ such that there exists $y \in A$ with yP_ix for all $i \in N$, we have $\varphi_x(P_N) = 0$. Clearly, Pareto optimality implies unanimity. An RSCF $\varphi : \mathscr{D}_N \to \Delta A$ is **tops-only** if for all $P_N, P'_N \in \mathscr{D}_N$ such that $P_i(1) = P'_i(1)$ for all $i \in N$, we have $\varphi(P_N) = \varphi(P'_N)$. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is **only-topset** if for all $P_N \in \mathscr{D}^n$, we have $\varphi_x(P_N) = 0$ for all $x \notin \tau(\mathscr{D})$.

A probability distribution ν stochastically dominates another probability distribution $\hat{\nu}$ at a preference P, denoted by $\nu P^{\text{sd}} \hat{\nu}$, if $\nu_{U(x,P_i)} \geq \hat{\nu}_{U(x,P_i)}$ for all $x \in A$ and $\nu_{U(y,P_i)} > \hat{\nu}_{U(y,P_i)}$ for some $y \in A$. We write $\nu R^{sd} \hat{\nu}$ to mean either $\nu P^{sd} \hat{\nu}$ or $\nu = \hat{\nu}$. An RSCF

$\varphi : \mathscr{D}_N \to \Delta A$ is *dominant strategy incentive compatible (DSIC) on a pair of preference* (P_i, P_i') of an agent $i \in N$, if $\varphi(P_i, P_{-i}) R_i^{\text{sd}} \varphi(P_i', P_{-i})$ for all $P_{-i} \in \mathscr{D}_{-i}$. An RSCF is **graph-local dominant strategy incentive compatible (graph-LDSIC)** if it is DSIC on every pair of graph-local preferences of each agent, and it is called **dominant strategy incentive compatible (DSIC)** if it is DSIC on *every* pair of preferences of each agent.

A set of alternatives B is a block in a pair of preferences (P, P') if it is a minimal non-empty set satisfying the following property: for all $x \in B$ and $y \notin B$, $P|_{\{x,y\}} = P'|_{\{x,y\}}$. For instance, the blocks in the pair of preferences $(abcdefg, bcadegf)$ are $\{a,b,c\}, \{d\}, \{e\}$, and $\{f,g\}$. The lower contour set $L(x,P)$ of an alternative x at a preference P is $L(x,P) = \{a \in A \mid xPa\}$. A set L is a lower contour set at a preference P if it is a lower contour set of some alternative at P. Lower contour monotonicity says that whenever an agent i unilaterally deviates from P_i to a graph-local preference P_i', the probability of each lower contour set at P_i restricted to any non-singleton block in (P_i, P_i') will weakly increase. For instance, consider our earlier example $P_i = abcdefg$ and $P_i' = bcadegf$ with non-singleton blocks $\{a,b,c\}$ and $\{f,g\}$. The lower contour sets at P_i restricted to $\{a,b,c\}$ are $\{c\}$ and $\{b,c\}$, and that restricted to $\{f,g\}$ is $\{g\}$. Lower contour monotonicity says that the probability of each of the sets $\{c\}$, $\{b,c\}$, and $\{g\}$ will weakly increase if agent i unilaterally deviates from P_i to P_i'.

Definition 3.1. An RSCF $\varphi : \mathscr{D}_N \to \Delta A$ is called **lower contour monotonic** if for all $i \in N$, all graph-local preferences $P_i, P_i' \in \mathscr{D}_i$, all non-singleton blocks B in (P_i, P_i'), and all $P_{-i} \in \mathscr{D}_{-i}$, we have $\varphi_L(P_i, P_{-i}) \geq \varphi_L(P_i', P_{-i})$ for each lower contour set L of $P_i|_B$.

4. Random Bayesian rules and their properties

A prior μ_i of an agent i is a probability distribution over $\mathscr{D}_{-i}$ which represents her belief about the preferences of the others, and a prior profile $\mu_N := (\mu_i)_{i \in N}$ is a collection of priors, one for each agent. A pair (φ, μ_N) consisting of an RSCF $\varphi : \mathscr{D}_N \to \Delta A$ and a prior profile μ_N is called a random Bayesian rule (RBR) on $\mathscr{D}_N$. When the RSCF φ is a DSCF, then it is called a deterministic Bayesian rule (DBR).

The expected outcome with respect to the belief of an agent is called her interim expected outcome. More formally, the *interim expected outcome* $\varphi(P_i, \mu_i)$ for an agent $i \in N$ at a preference $P_i \in \mathscr{D}_i$ from an RBR (φ, μ_N) on $\mathscr{D}_N$ is defined as the following probability distribution on A: for all $x \in A$,

$$\varphi_x(P_i, \mu_i) = \sum_{P_{-i} \in \mathscr{D}_{-i}} \mu_i(P_{-i}) \varphi_x(P_i, P_{-i}).$$

Example 4.1. Let $N = \{1,2\}$ and $A = \{a,b,c\}$. Consider the RBR (φ, μ_N) given in Table 1. Agent 1's belief μ_1 about agent 2's preferences is given in the top row and agent 2's belief μ_2 about agent 1's preferences in the leftmost column of the table. The rest of the table is self-explanatory. Consider the preference $P_1 = abc$ of agent 1. Her interim expected outcome at this preference is calculated as follows: $\varphi_a(P_1, \mu_1) = 0.2 \times 1 + 0.1 \times 1 + 0.05 \times 1 + 0.3 \times 0.5 + 0.15 \times 1 + 0.2 \times 1 = 0.85$, $\varphi_b(P_1, \mu_1) = 0.2 \times 0 + 0.1 \times 0 + 0.05 \times 0 + 0.3 \times 0.5 + 0.15 \times 0 + 0.2 \times 0 = 0.15$, and $\varphi_c(P_1, \mu_1) = 0.2 \times 0 + 0.1 \times 0 + 0.05 \times 0 + 0.3 \times 0 + 0.15 \times 0 + 0.2 \times 0 = 0$. Similarly, for agent 2's preference $P_2 = bca$, we have $\varphi_b(P_2, \mu_2) = 0.575$, $\varphi_c(P_2, \mu_2) = 0.06$, and $\varphi_a(P_2, \mu_2) = 0.365$.

	μ_1	0.2	0.1	0.05	0.3	0.15	0.2
μ_2	1 \ 2	abc	acb	bac	bca	cba	cab
0.25	abc	(1,0,0)	(1,0,0)	(1,0,0)	(0.5,0.5,0)	(1,0,0)	(1,0,0)
0.2	acb	(1,0,0)	(1,0,0)	(1,0,0)	(0.7,0,0.3)	(1,0,0)	(1,0,0)
0.15	bac	(1,0,0)	(1,0,0)	(0,1,0)	(0,1,0)	(0,1,0)	(1,0,0)
0.1	bca	(0,1,0)	(1,0,0)	(0,1,0)	(0,1,0)	(0,1,0)	(0,1,0)
0.2	cba	(1,0,0)	(0,0,1)	(0,0.4,0.6)	(0,1,0)	(0,0,1)	(0,0,1)
0.1	cab	(1,0,0)	(0,0.4,0.6)	(1,0,0)	(1,0,0)	(0,0,1)	(0,0,1)

Table 1

The notion of ordinal Bayesian incentive compatibility (OBIC) captures the idea of DSIC for an RBR by ensuring that no agent can improve her interim expected outcome by misreporting her preference.

Definition 4.1. An RBR (φ, μ_N) on $\mathscr{D}_N$ is *ordinal Bayesian incentive compatible (OBIC) on a pair of preferences* (P_i, P_i') of an agent $i \in N$ if $\varphi_{\mu_i}(P_i) R_i^{\text{sd}} \varphi_{\mu_i}(P_i')$. An RBR (φ, μ_N) is **graph-local ordinal Bayesian incentive compatible (graph-LOBIC)** if it is OBIC on every pair of graph-local preferences in the domain of each agent, and it is **ordinal Bayesian incentive compatible (OBIC)** if it is OBIC on *every* pair of preferences in the domain of each agent.

Note that OBIC is a weaker requirement than DSIC since if an RSCF φ is DSIC, then (φ, μ_N) is OBIC for all profiles of priors μ_N.

For ease of presentation, we use the following two terminologies in our paper. Given a property defined for an RSCF, we say an RBR (φ, μ_N) satisfies it, if φ satisfies the property. For instance, we say an RBR (φ, μ_N) is unanimous, if the RSCF φ is unanimous. We say some property holds for *almost all RBRs with RSCF* φ if there is a set $\mathscr{M}$ of profiles of priors with (Lebesgue) measure 1 such that the said property holds for each RBR (φ, μ_N) where μ_N is in $\mathscr{M}$. In other words, if a prior profile μ_N is chosen randomly, then the RBR (φ, μ_N) will satisfy the property with probability 1.

5. Results on graph-connected domains

In this section, we explore the structure of graph-LOBIC Bayesian rules on graph-connected domains. Since OBIC implies graph-LOBIC (by definition), all these results hold for OBIC RBRs as well.

Recall the definition of a block given in Page 7. The block preservation property says that if an agent unilaterally changes her preference to a graph-local preference, the total probability of any block in the two preferences will remain unchanged.

Definition 5.1. An RSCF $\varphi : \mathscr{D}_N \to \Delta A$ satisfies the **block preservation property** if for all $i \in N$, all graph-local preferences $P_i, P_i' \in \mathscr{D}_i$ of agent i, all blocks B in (P_i, P_i'), and all $P_{-i} \in \mathscr{D}_{-i}$, we have $\varphi_B(P_i, P_{-i}) = \varphi_B(P_i', P_{-i})$.

For two preferences P and P', $P \triangle P' = \{x \in A \mid U(x,P) \neq U(x,P')\}$ denotes the set of alternatives that change their relative ordering with some other alternative from P to P'. Note that the block preservation property implies $\varphi_x(P_i, P_{-i}) = \varphi_x(P_i', P_{-i})$ for all $x \notin P_i \triangle P_i'$ as such an alternative forms a singleton block in (P_i, P_i').

Proposition 5.1. *Almost all graph-LOBIC RBRs satisfy the block preservation property.*

The proof of this proposition is relegated to Appendix B.

5.1 Equivalence of graph-LOBIC and graph-LDSIC under lower contour monotonicity

The following theorem says that under lower contour monotonicity, almost all graph-LOBIC RBRs are graph-LDSIC (on any graph-connected domain).

Theorem 5.1. *If an RSCF* $\varphi : \mathscr{D}_N \to \Delta A$ *satisfies lower contour monotonicity, then almost all graph-LOBIC RBRs with RSCF* φ *are graph-LDSIC.*

The proof of this theorem is relegated to Appendix C.1.

5.2 Sufficient condition for the equivalence of unanimity and Pareto optimality

Pareto optimality is much stronger than unanimity. However, under DSIC, these two notions turn out to be equivalent for RSCFs on many domains such as the unrestricted, single-peaked, single-dipped, single-crossing, etc. In this section, we show that similar results hold with probability 1 if we replace DSIC by its weaker version OBIC. We introduce the notion of upper contour preservation property for our result.

Definition 5.2. A domain $\mathscr{D}$ satisfies the **upper contour preservation property** if for all $x, y \in A$ and all $P \in \mathscr{D}$ with xPy, there exists a graph-local path from P to a preference $\hat{P} \in \mathscr{D}$ with $\hat{P}(1) = x$ such that $U(P,y) = U(\hat{P},y)$.

Our next theorem says that if a domain satisfies the upper contour preservation property then almost all unanimous and graph-LOBIC RBRs on it will be Pareto optimal.

Theorem 5.2. *Suppose $\mathscr{D}_i$ satisfies the upper contour preservation property for all $i \in N$. If an RSCF $\varphi : \mathscr{D}_N \to \Delta A$ satisfies unanimity, then almost all graph-LOBIC RBRs with RSCF φ are Pareto optimal.*

The proof of this theorem is relegated to Appendix C.2.

5.3 Relation between unanimity and tops-onlyness

We use the notion of path-richness in our result. A domain satisfies the path-richness property if for every two preferences P and P' having the same top-ranked alternative, say x, the following happens: (i) if P and P' are not graph-local then there is graph-local path from P to P' such that x appears as the top-ranked alternative in each preference in the path, and (ii) if P and P' are graph-local, then from any preference $\hat{P}$ there is a path to some preference $\bar{P}$ with x as the top-ranked alternative such that for any two alternatives a, b that change their relative ranking from P to P' and for any two consecutive preferences in the path, there is a common upper contour set of the preferences such that exactly one of a and b belongs to it. For an illustration of Part (ii) of the path-richness property, suppose $A = \{a,b,c,d\}$, $P = abcd$ and $P' = adcb$, and assume that P and P' are graph-local. Consider a preference $\hat{P} = dbca$. Path-richness requires that a path of the following type must be present in the domain: $(dbca, dbac, dabc, adbc)$. To see that this path satisfies (ii), consider two alternatives that change their relative ordering from P to P', say b and c. Note that the upper contour set $\{d,b\}$ in P^1 and P^2 contains b but not c, the upper contour set $\{d,b,a\}$ in P^2 and P^3 contains b but not c, and so on. Path-richness requires that such a path must exist for every preference $\hat{P}$ in the domain.

Definition 5.3. A domain $\mathscr{D}$ satisfies the **path-richness property** if for all preferences $P,P' \in \mathscr{D}$ such that $P(1) = P'(1)$,

(i) if P and P' are not graph-local, then there is a graph-local path $(P^1 = P,\dots,P^t = P')$ such that $P^l(1) = P(1)$ for all $l = 1,\dots,t$, and

(ii) if P and P' are graph-local, then for each preference $\hat{P} \in \mathscr{D}$, there exists a graph-local path $(P^1 = \hat{P},\dots,P^t)$ with $P^t(1) = P(1)$ such that for all $l < t$ and all distinct $y,z \in P \triangle P'$, there is a common upper contour set U of P^l and $\bar{P}^{l+1}$ such that exactly one of y and z is contained in U.

Our next theorem says that the path-richness property of a domain ensures that almost all unanimous and graph-LOBIC RBRs on it are tops-only.

Theorem 5.3. *Suppose $\mathscr{D}$ satisfies the path-richness property. If an RSCF $\varphi : \mathscr{D}^n \to \Delta A$ satisfies unanimity, then almost all graph-LOBIC RBRs with RSCF φ are tops-only.*

The proof of this theorem is relegated to Appendix C.3.

Remark 5.1. Lower contour monotonicity can be weakened in a straightforward way under tops-onlyness. Let us say that an RSCF satisfies top lower contour monotonicity if it satisfies lower contour monotonicity only over (unilateral) deviations to graph-local preferences where the top-ranked alternative is changed. Thus, top lower contour monotonicity does not impose any restriction for graph-local preferences P and P' with $\tau(P) = \tau(P')$. Clearly, under tops-onlyness, lower contour monotonicity will be automatically guaranteed in all other cases, and hence, top lower contour monotonicity will be equivalent to lower contour monotonicity. Since under graph-LOBIC, unanimity implies tops-onlyness on a large class of domains, this simple observation is of great help for practical applications.

5.4 Relation between tops-onlyness and only-topsetness

We use the notion of top-connectedness in our result. Two alternatives a and b are top-connected in a domain $\mathscr{D}$ if there exist graph-local preferences $P,P' \in \mathscr{D}$ with $P \equiv xy\cdots$

and $P' \equiv yx\cdots$ such that $P|_{A\setminus\{x,y\}} = P'|_{A\setminus\{x,y\}}$.

Definition 5.4. A domain $\mathscr{D}$ is called **top-connected** if for all $x, \bar{x} \in \tau(\mathscr{D})$ there exists a sequence $(x^1 = x, x^2, \ldots, x^t = \bar{x})$ of alternatives such that x^l and x^{l+1} are top-connected for all $l < t$.

Our next result says that almost all unanimous and tops-only graph-LOBIC RBRs on a top-connected domain are only-topset.

Theorem 5.4. *Suppose $\mathscr{D}$ is top-connected. If an RSCF $\varphi : \mathscr{D}^n \to \Delta A$ satisfies unanimity and tops-onlyness, then almost all graph-LOBIC RBRs with RSCF φ are only-topset.*

The proof of this theorem is relegated to Appendix C.4.

6. THE CASE OF SWAP-CONNECTED DOMAINS

In this section, we consider graphs where two preferences are local if and only if they differ by a swap of two consecutively ranked alternatives. Formally, two preferences P and P' are swap-local if $P \triangle P' = \{x, y\}$ for some $x, y \in A$. For two swap-local preferences P and P', we say x overtakes y from P to P' if yPx and $xP'y$. A domain $\mathscr{D}_i$ is swap-connected if there is a swap-local path between any two preferences in it. We use terms like swap-LOBIC, swap-LDSIC, etc. (instead of graph-LOBIC, graph-LDSIC, etc.) to emphasize the fact that the graph is based on the swap-local structure.

When graphs are swap-connected, lower contour monotonicity boils down to the following condition called elementary monotonicity. An RSCF $\varphi : \mathscr{D}_N \to \Delta A$ is called **elementary monotonic** if for every $i \in N$, all swap-local preferences $P_i, P'_i \in \mathscr{D}_i$ of agent i, and all $P_{-i} \in \mathscr{D}_{-i}$, x overtakes some alternative from P_i to P'_i implies $\varphi_x(P_i, P_{-i}) \leq \varphi_x(P'_i, P_{-i})$.

Under swap-connectedness, Condition (ii) of the path-richness property (Definition 5.3) simplifies to the following condition: if there are two swap-local preferences having the same top-ranked alternative, say x, where two alternatives, say y and z, are swapped, then from every preference in the domain there must be a swap-local path to some preference with x as the top-ranked alternative such that the relative ranking of y and z remains the same along the path.

6.1 Equivalence of swap-LDSIC and weak elementary monotonicity under tops-onlyness

Weak elementary monotonicity (Mishra (2016)) is a restricted version of elementary monotonicity where the latter is required to be satisfied only for a particular type of profiles where all the agents agree on the ranking of alternatives from rank three onward.

Definition 6.1. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ satisfies **weak elementary monotonicity** if for all $i \in N$, and all (P_i, P_{-i}) and (P'_i, P_{-i}) such that $P_i(k) = P'_i(k) = P_j(k)$ for all $j \in N \setminus i$ and all $k > 2$, we have $\varphi_{P_i(1)}(P_i, P_{-i}) \geq \varphi_{P_i(1)}(P'_i, P_{-i})$.

Our next result says that under tops-onlyness, almost all weak elementary monotonic and swap-LOBIC RBRs are swap-LDSIC.

Theorem 6.1. *If an RSCF $\varphi : \mathscr{D}^n \to \Delta A$ satisfies tops-onlyness, then almost all weak elementary monotonic swap-LOBIC RBRs with RSCF φ are swap-LDSIC.*

The proof of this theorem is relegated to Appendix C.5.

The following corollary follows from Theorem 5.3 and Theorem 6.1.

Corollary 6.1. *Suppose $\mathscr{D}$ satisfies the path-richness property. If an RSCF $\varphi : \mathscr{D}^n \to \Delta A$ satisfies unanimity, then almost all weak elementary monotonic swap-LOBIC RBRs with RSCF φ are swap-LDSIC.*

6.2 Equivalence of swap-LOBIC and swap-LDSIC under Pareto optimality and tops-onlyness

A domain satisfies the **top-swap richness property** if for all distinct $x, y, z \in A$, whenever there are two swap-local preferences $P \equiv xyz\cdots$ and $P' \equiv yxz\cdots$ in $\mathscr{D}$, the swap-local path $(P, xzy\cdots, zxy\cdots, zyx\cdots, yzx\cdots, P')$ is in $\mathscr{D}$. Our next theorem provides a sufficient condition on a domain for the equivalence of swap-LOBIC and swap-LDSIC under Pareto optimality and tops-onlyness.

Theorem 6.2. *Suppose $|A| \geq 3$ and $\mathscr{D}$ satisfies the top-swap richness property. If an RSCF $\varphi : \mathscr{D}^n \to \Delta A$ satisfies Pareto optimality and tops-onlyness, then almost all swap-LOBIC RBRs with RSCF φ are swap-LDSIC.*

The proof of this theorem is relegated to Appendix C.6.

In Section 5.2 and Section 5.3, we provide conditions on a domain so that unanimity implies Pareto optimality, and unanimity implies tops-onlyness for almost all swap-LOBIC RBRs. We will improve Theorem 6.2 *under unanimity* as corollaries of those results.

It is worth mentioning that many restricted domains of practical importance, such as the single-peaked, hybrid, multiple single-peaked, single-crossing, single-dipped, etc., do not satisfy the top-swap richness property.[4]

7. Applications on domains satisfying the betweenness property

A **betweenness relation** β maps every pair of distinct alternatives (x,y) to a subset of alternatives $\beta(x,y)$ including x and y. We only consider betweenness relations β that are rational: for every $x \in A$, there is a preference P with $P(1) = x$ such that for all $y,z \in A$, $y \in \beta(x,z)$ implies yRz. Such a preference P is said to respect the betweenness relation β. A domain $\mathscr{D}$ respects a betweenness relation β if it contains all preferences respecting β. We denote such a domain by $\mathscr{D}(\beta)$. For a collection of betweenness relations $\mathscr{B} = \{\beta_1,\ldots,\beta_r\}$, we denote the domain $\cup_{l=1}^{r}\mathscr{D}(\beta_l)$ by $\mathscr{D}(\mathscr{B})$.

A pair of alternatives (x,y) is adjacent in β if $\beta(x,y) = \{x,y\}$. A betweenness relation β is **weakly consistent** if for all $x,\bar{x} \in A$, there is a sequence $(x^1 = x,\ldots,x^t = \bar{x})$ of adjacent alternatives in $\beta(x,\bar{x})$ such that for all $l < k$, we have $\beta(x^{l+1},\bar{x}) \subseteq \beta(x^l,\bar{x})$. A betweenness relation β is **strongly consistent** if for all $x,\bar{x} \in A$, there is a sequence $(x^1 = x,\ldots,x^t = \bar{x})$ of adjacent alternatives in $\beta(x,\bar{x})$ such that for all $l < t$ and all $w \in \beta(x^l,\bar{x})$, we have $\beta(x^{l+1},w) \subseteq \beta(x^l,\bar{x})$. A collection $\mathscr{B} = \{\beta_1,\ldots,\beta_r\}$ or a betweenness domain $\mathscr{D}(\mathscr{B})$ is strongly/weakly consistent if β_l is strongly/weakly consistent for all $l = 1,\ldots,r$.

Two betweenness relations β and β' are swap-local if for every $x \in A$, there are $P \in \mathscr{D}(\beta)$ and $P' \in \mathscr{D}(\beta')$ such that $P(1) = P'(1)$ and P and P' are swap-local. A collection $\mathscr{B}$ of betweenness relations is called swap-connected if for all $\beta,\beta' \in \mathscr{B}$, there is a sequence $(\beta^1 = \beta,\ldots,\beta^t = \beta')$ in $\mathscr{B}$ such that β^l and β^{l+1} are swap-local for all $l < t$.

[4] In Section 7, we present formal definitions of these restricted domains.

We now define the local structure on a betweenness domain $\mathscr{D}(\mathscr{B})$ in a natural way. A preference P' is graph-local to another preference P if there is no preference $P'' \in \mathscr{D}(\mathscr{B}) \setminus \{P, P'\}$ that is "more similar" to P than P' is to P, that is, there is no P'' such that for all $x, y \in A$, $P|_{\{x,y\}} = P'|_{\{x,y\}}$ implies $P|_{\{x,y\}} = P''|_{\{x,y\}}$. Our next corollary follows from Theorem 5.3.

Corollary 7.1. *Let $\mathscr{B}$ be a collection of strongly consistent and swap-connected betweenness relations. Then, almost all unanimous and graph-LOBIC RBRs on $\mathscr{D}(\mathscr{B})$ are tops-only.*

The proof of this corollary is relegated to Appendix C.7.

A domain is called graph deterministic local-global equivalent (graph-DLGE) if every graph-LDSIC DSCF on it is DSIC.

Theorem 7.1. *Let $\mathscr{B}$ be a collection of weakly consistent and swap-connected betweenness relations. Then, $\mathscr{D}(\mathscr{B})$ is a graph-DLGE domain.*

The proof of this corollary is relegated to Appendix C.8.

In what follows, we apply our results to explore the structure of LOBIC RBRs on well-known betweenness domains.

7.1 The unrestricted domain

The domain $\mathscr{P}(A)$ containing all preferences over A is called the **unrestricted domain** (over A). Since, the unrestricted domain satisfies both the upper contour preservation property and the path-richness property, it follows from Theorems 5.2 and 5.3 that unanimity implies both Pareto optimality and tops-only for almost all swap-LOBIC RBRs. Combining all these observations with Theorem 6.2 we derive the following proposition.

Proposition 7.1. *Almost all unanimous and swap-LOBIC RBRs are swap-LDSIC.*

Gibbard (1977) shows that every unanimous and DSIC RSCF on the unrestricted domain is *random dictatorial*. Let us call a domain swap random local-global equivalent (swap-RLGE) if every swap-LDSIC RSCF on it is DSIC. It follows from Cho (2016) that the unrestricted domain is swap-RLGE. Since every OBIC RBR is swap-LOBIC by definition, it follows from Proposition 7.1 that the same result as Gibbard (1977) holds (with probability 1) if we replace DSIC with the much weaker notion OBIC.

Corollary 7.2. *Almost all unanimous and swap-LOBIC RBRs on the unrestricted domain over at least three alternatives are random dictatorial.*

7.2 SINGLE-PEAKED DOMAINS ON GRAPHS

Peters et al. (2019) introduce the notion of single-peaked domains on graphs and characterize all unanimous and DSIC RSCFs on these domains. We assume that the set of alternatives is endowed with an (undirected) graph $\mathscr{G} = \langle A, E \rangle$. For $x, \bar{x} \in A$ with $x \neq \bar{x}$, a path $(x^1 = x, \ldots, x^t = \bar{x})$ from x to $\bar{x}$ in $\mathscr{G}$ is a sequence of distinct alternatives such that $\{x^i, x^{i+1}\} \in E$ for all $i = 1, \ldots, t-1$. If it is clear which path is meant, we also denote it by $[x, \bar{x}]$. We assume that $\mathscr{G}$ is connected, that is, there is a path from x to $\bar{x}$ for all distinct $x, \bar{x} \in A$. If this path is unique for all $x, \bar{x} \in A$, then $\mathscr{G}$ is called a tree. A spanning tree of $\mathscr{G}$ is a tree $T = \langle A, E_T \rangle$ where $E_T \subseteq E$. In other words, spanning tree of $\mathscr{G}$ is a tree that can be obtained by deleting some edges of $\mathscr{G}$.

Definition 7.1. A preference P is single-peaked on $\mathscr{G}$ if there is a spanning tree T of $\mathscr{G}$ such that for all distinct $x, y \in A$ with $P(1) \neq y$, $x \in [P(1), y] \implies xPy$, where $[P(1), y]$ is the path from $P(1)$ to y in T. A domain is called **single-peaked on** $\mathscr{G}$ if it contains all single-peaked preferences on $\mathscr{G}$.

It follows from the definition that a single-peaked domain $\mathscr{D}_T$ on a tree T can be represented as a betweenness domain $\mathscr{D}(\beta^T)$ where β^T is defined as follows: $\beta^T(x, y) = [x, y]$. Single-peaked domains on graphs are well-known for the cases when the graph $\mathscr{G}$ is a line or a tree.[5] When the graph $\mathscr{G}$ is a line, then the corresponding domain is known in the literature as the **single-peaked domain**.[6]

In what follows, we argue that a single-peaked domain on a graph satisfies the upper contour preservation property. Since a single-peaked domain on a graph is a union of single-peaked domains on trees, it is enough to show that a single-peaked domain on a tree satisfies the upper contour preservation property. Consider a single-peaked domain $\mathscr{D}_T$ on a tree T. Let P be a preference with xPy for some $x, y \in A$. Suppose $P(1) = a$. Consider the path $[a, x]$ in T. Since xPy, it must be that $y \notin [a, x]$. Suppose $[a, x] = (x^1 = a, \ldots, x^k = x)$. By the definition of single-peaked domain on a tree, one

[5] A tree is called a line if it has exactly two nodes with degree one (such nodes are called leafs).

[6] A line graph can be represented by a linear order $\prec$ over the alternatives in an obvious manner: if the edges in a line graph are $\{(a_1, a_2), \ldots, (a_{m-1}, a_m)\}$, then one can take the linear order $\prec$ as $a_1 \prec \cdots \prec a_m$.

can go from P to a preference with x^2 at the top through a swap-local path maintaining the upper contour set of y. Continuing in this manner, one can go to a preference with x at the top maintaining the upper contour set of y. This concludes that $\mathscr{D}_T$ satisfies the upper contour preservation property, and hence, we obtain the following corollary from Theorem 5.2.

Corollary 7.3. *Almost all unanimous and swap-LOBIC RBRs on the single-peaked domain on a graph are Pareto optimal.*

We now argue that the betweenness relation β^T is strongly consistent. To see that β^T is strongly consistent consider two alternatives x and $\bar{x}$, and consider the unique path $[x,\bar{x}]$ between them in T. Let $[x,\bar{x}] = (x^1 = x, \dots, x^t = \bar{x})$. By the definition of β^T, the path $[x,\bar{x}]$ lies in (in fact, is equal to) $\beta^T(x,\bar{x})$. Consider $x^l \in \beta^T(x,\bar{x})$ and $w \in \beta^T(x^l,\bar{x})$. Since both w and x^{l+1} lie on the path $[x^l,\bar{x}]$, it follows that $[x^{l+1},w] \subseteq [x^l,\bar{x}]$, and hence $\beta^T(x^{l+1},w) \subseteq \beta^T(x^l,\bar{x})$. This proves that β^T is the strongly consistent (and hence is also weakly consistent). Since a betweenness relation that generates a single-peaked domain on a tree is strongly consistent, it follows from the definition of a single-peaked domain on a graph that the betweenness relation that generates such a domain also satisfies the property. It is shown in Peters et al. (2019) (see Lemma A.1 for details) that for all $x \in A$, the (sub)domain of $\mathscr{D}_{\mathscr{G}}$ containing all preferences with x as the top-ranked alternative is swap-connected, which implies that the betweenness relations generated by the spanning trees of a graph are swap-connected. Therefore, it follows from Corollary 7.1 that almost all unanimous and swap-LOBIC RBRs on the single-peaked domain on a graph are tops-only. Consequently, we obtain the following corollary from Corollary 6.1.

Corollary 7.4. *Almost all unanimous and weak elementary monotonic swap-LOBIC RBRs on the single-peaked domain on a graph are swap-LDSIC.*

It follows from Theorem 7.1 that the single-peaked domain on a graph is swap-DLGE. It is shown in Peters et al. (2019) that a DSCF on the single-peaked domain on a graph is unanimous and DSIC if and only if it is a *monotonic collection of parameters based rule* (see Theorem 5.5 in Peters et al. (2019) for details). Although Peters et al. (2019) provide the result for RSCFs, we cannot apply it as it is not known whether the single-peaked domain on a graph is RLGE or not. Therefore, we obtain the following corollary from Theorem 6.1 and Theorem 7.1.

Corollary 7.5. *Almost all unanimous, weak elementary monotonic, and swap-LOBIC DBRs on the single-peaked domain on a graph are monotonic collection of parameters based rules.*

Cho (2016) shows that the single-peaked domain is swap-RLGE. Moreover, Peters et al. (2014) show that every unanimous and DSIC RSCF on the single-peaked domain is a *probabilistic fixed ballot rule (PFBR)*. We obtain the following corollary by combining these results with Corollary 7.4.

Corollary 7.6. *Almost all unanimous, weak elementary monotonic, and swap-LOBIC RBRs on the single-peaked domain are PFBRs.*

In what follows, we provide a discussion on the structure of unanimous and swap-LOBIC RBRs on the single-peaked domain that do not satisfy weak elementary monotonicity. The structure of such RBRs depends on the specific prior profile. In the following example, we present an RSCF for three agents that is unanimous and OBIC with respect to any independent prior profile (μ_1,μ_2,μ_3) where $\mu_2(abc) \geq \frac{1}{6}$.[7] By Corollary 7.1, we know that such an RSCF will be tops-only. In Table 2, the preferences in rows and columns belong to agents 1 and 2, respectively, and the preferences written at the top-left corner of any table belong to agent 3. Note that agent 3 is the dictator for this RSCF except when she has the preference abc. When she has the preference abc, the rule violates weak elementary monotonicity over the profiles (abc,bac,abc) and (bac,bac,abc). Note that except from such violations, the rule behaves like a PFBR.

abc	abc	bac	bca	cba
abc	(1,0,0)	(0.4,0.6,0)	(0.4,0.6,0)	(0.4,0.6,0)
bac	(0.5,0.5,0)	(0.5,0.5,0)	(0.5,0.5,0)	(0.5,0.5,0)
bca	(0.5,0.5,0)	(0.5,0.5,0)	(0.5,0.5,0)	(0.5,0.5,0)
cba	(0.5,0.5,0)	(0.5,0.5,0)	(0.5,0.5,0)	(0.5,0.5,0)

bac	abc	bac	bca	cba
abc	(0,1,0)	(0,1,0)	(0,1,0)	(0,1,0)
bac	(0,1,0)	(0,1,0)	(0,1,0)	(0,1,0)
bca	(0,1,0)	(0,1,0)	(0,1,0)	(0,1,0)
cba	(0,1,0)	(0,1,0)	(0,1,0)	(0,1,0)

bca	abc	bac	bca	cba
abc	(0,1,0)	(0,1,0)	(0,1,0)	(0,1,0)
bac	(0,1,0)	(0,1,0)	(0,1,0)	(0,1,0)
bca	(0,1,0)	(0,1,0)	(0,1,0)	(0,1,0)
cba	(0,1,0)	(0,1,0)	(0,1,0)	(0,1,0)

cba	abc	bac	bca	cba
abc	(0,0,1)	(0,0,1)	(0,0,1)	(0,0,1)
bac	(0,0,1)	(0,0,1)	(0,0,1)	(0,0,1)
bca	(0,0,1)	(0,0,1)	(0,0,1)	(0,0,1)
cba	(0,0,1)	(0,0,1)	(0,0,1)	(0,0,1)

Table 2

[7] The rule is OBIC for dependent priors if: $5\mu_1(abc,abc) \geq \mu_1(bac,abc) + \mu_1(bca,abc) + \mu_1(cba,abc)$, where the first and the second preference in μ_1 denote the preferences of agents 2 and 3, respectively.

7.3 Hybrid domains

Chatterji et al. (2020) introduce the notion of hybrid domains and discuss its importance. These domains satisfy single-peaked property only over a subset of alternatives. Let us assume that $A = \{1, \dots, m\}$. Throughout this subsection, we assume that two alternatives $\underline{k}$ and $\overline{k}$ with $\underline{k} < \overline{k}$ are arbitrary but fixed.

Definition 7.2. A preference P is called $(\underline{k}, \overline{k})$-hybrid if the following two conditions are satisfied:

(i) For all $r, s \in A$ such that either $r, s \in [1, \underline{k}]$ or $r, s \in [\overline{k}, m]$, $\big[r < s < P(1)$ or $P(1) < s < r\big] \Rightarrow [sPr]$.

(ii) $\big[P(1) \in [1, \underline{k}]\big] \Rightarrow \big[\underline{k}Pr$ for all $r \in (\underline{k}, \overline{k}]\big]$ and
$\big[P(1) \in [\overline{k}, m]\big] \Rightarrow \big[\overline{k}Ps$ for all $s \in [\underline{k}, \overline{k})\big]$.[8]

A domain is $(\underline{k}, \overline{k})$-**hybrid** if it contains all $(\underline{k}, \overline{k})$-hybrid preferences. The betweenness relation β that generates a $(\underline{k}, \overline{k})$-hybrid domain is as follows: if $x < y$ then $\beta(x, y) = \{x, y\} \cup \big((x, y) \setminus (\underline{k}, \overline{k})\big)$ and if $y < x$ then $\beta(x, y) = \{x, y\} \cup \big((y, x) \setminus (\underline{k}, \overline{k})\big)$. In other words, an alternative other than x and y lies between x and y if and only if it lies in the interval $[x, y]$ or $[y, x]$ but not in the interval $(\underline{k}, \overline{k})$.

In what follows, we argue that a hybrid domain satisfies the upper contour preservation property. Consider a preference P in a $(\underline{k}, \overline{k})$-hybrid domain. Suppose xPy for some $x, y \in A$. Assume without loss of generality that $x < a$. Let $P(1) = a$ and let $U(x, P) \cap [x, a] = \{x^1 = a, \dots, x^k = x\}$ where $x^1 P x^2 P \cdots P x^k$. Note that by the definition of the $(\underline{k}, \overline{k})$-hybrid domain, from P one can go to a preference with x^2 at the top though a swap-local path by maintaining the upper contour set of y. Therefore, by repeated application of this fact, one can go to a preference with x at the top by maintaining the upper contour set of y. This shows that a hybrid domain satisfies the upper contour preservation property. Therefore, we obtain the following corollary from Theorem 5.2.

Corollary 7.7. *Almost all unanimous and swap-LOBIC RBRs on the $(\underline{k}, \overline{k})$-hybrid domain are Pareto optimal.*

[8] For two alternatives x and y, by $(x, y]$ we denote the alternatives z such that $x < z \leq y$. The interpretation of the notation $[x, y)$ is similar.

Using similar logic as we have used in the case of a single-peaked domain on a tree, it follows that the betweenness relation that generates a hybrid domain is strongly consistent. Therefore, Corollary 7.1 implies that almost all unanimous and swap-LOBIC RBRs on the $(\underline{k},\overline{k})$-hybrid domain are tops-only. Therefore, by Corollary 6.1, we obtain the following corollary.

Corollary 7.8. *Almost all unanimous and weak elementary monotonic swap-LOBIC RBRs on the $(\underline{k},\overline{k})$-hybrid domain are swap-LDSIC.*

Chatterji et al. (2020) show that every unanimous and DSIC RSCF on the hybrid domain is a $(\underline{k},\overline{k})$-*restricted probabilistic fixed ballot rule* ($(\underline{k},\overline{k})$-*RPFBR*). Since the hybrid domain is swap-RLGE (see Chatterji et al. (2020) for details), Corollary 7.8 implies the following result.

Corollary 7.9. *Almost all unanimous, weak elementary monotonic, and swap-LOBIC RBRs on the $(\underline{k},\overline{k})$-hybrid domain are $(\underline{k},\overline{k})$-RPFBRs.*

7.4 Multiple single-peaked domains

The notion of multiple single-peaked domains is introduced in Reffgen (2015). As the name suggests, these domains are union of several single-peaked domains. It is worth mentioning that these domains are different from hybrid domains–neither of them contains the other. For ease of presentation, we denote a single-peaked domain with respect to a prior ordering $\prec$ over A by $\mathscr{D}_{\prec}$.

Definition 7.3. Let $\Omega \subseteq \mathscr{P}(A)$ be a swap-connected collection of prior orderings over A. A domain $\mathscr{D}$ is called **multiple single-peaked** with respect to Ω if $\mathscr{D} = \cup_{\prec \in \Omega} \mathscr{D}_{\prec}$.

Since the prior orders in a multiple single-peaked domain are assumed to be swap-connected, it follows that preferences with the same top-ranked alternative are swap-connected. This implies that the collection $\mathscr{B}$ of betweenness relations that generate a multiple single-peaked domain is swap-connected. Using similar logic as we have used in the case of a single-peaked domain on a tree, it follows that multiple single-peaked domains are both weakly and strongly consistent betweenness domains. Therefore, Corollary 7.1 implies that almost all unanimous and swap-LOBIC RBRs on the multiple single-peaked domain are tops-only. Using similar argument as we have used in the case of a single-peaked domain on a tree, it follows that multiple single-peaked domains satisfy the upper contour preservation property. In view of these observations, we obtain the following corollaries from Theorem 5.2 and Corollary 6.1.

Corollary 7.10. *Almost all unanimous and swap-LOBIC RBRs on the multiple single-peaked domain are Pareto optimal.*

Corollary 7.11. *Almost all unanimous and weak elementary monotonic swap-LOBIC RBRs on the multiple single-peaked domain are swap-LDSIC.*

Let us assume without loss of generality that Ω contains the integer ordering $<$ over $A = \{1,\ldots,m\}$. For a class of prior ordering Ω over A, the left cut-off $\underline{k}$ is defined as the maximum (with respect to $<$) alternative with the property that $1 \prec 2 \prec \cdots \prec \underline{k} \prec x$ for all $x \notin \{1,\ldots,\underline{k}\}$ and all $\prec \in \Omega$. Similarly, define the right cut-off as the minimum alternative $\overline{k}$ such that $x \prec \overline{k} \prec \cdots \prec m-1 \prec m$ for all $x \notin \{\overline{k},\ldots,m\}$ and all $\prec \in \Omega$. Reffgen (2015) shows that a DSCF is unanimous and DSIC on a multiple single-peaked domain with left cut-off $\underline{k}$ and right cut-off $\overline{k}$ if and only if it is a $(\underline{k},\overline{k})$*-partly dictatorial generalized median voter scheme* (*$(\underline{k},\overline{k})$-PDGMVS*). Moreover, by Theorem 7.1, a multiple single-peaked domain is a swap-DLGE domain. Combining all these results with Corollary 7.11, we obtain the following corollary.

Corollary 7.12. *Let Ω be a class of swap-connected prior orderings over A with the left cut-off $\underline{k}$ and the right cut-off $\overline{k}$. Then, almost all unanimous and weak elementary monotonic swap-LOBIC DBRs on the multiple single-peaked domain with respect to Ω are $(\underline{k},\overline{k})$-PDGMVSs.*

7.5 Domains under partitioning

The notion of domains under partitioning is introduced in Mishra and Roy (2012). Such domains arise when a group of objects are to be partitioned based on the preferences of the agents over different partitions.

Let X be a finite set of objects and let A be the set of all partitions of X.[9] For instance, if $X = \{x,y,z\}$, then elements of A are $\{\{x\},\{y\},\{z\}\}$, $\{\{x\},\{y,z\}\}$, $\{\{y\},\{x,z\}\}$, $\{\{z\},\{x,y\}\}$, and $\{\{x,y,z\}\}$. We say that two objects are together in a partition if they

[9] A partition of a set is a set of subsets of that set that are mutually exclusive and exhaustive.

are contained in a common element (subset of X) of the partition. For instance, objects x and y are together in the partition $\{\{z\},\{x,y\}\}$. If two objects are not together in a partition, we say they are separated. For three distinct partitions $X_1,X_2,X_3 \in A$, we say X_2 lies between X_1 and X_3 if for every two objects x and y, x and y are together in both X_1 and X_3 implies they are also together in X_2, and x and y are separate in both X_1 and X_3 implies they are also separate in X_2. For instance, any of the partitions $\{\{x\},\{y,z\}\}$ or $\{\{y\},\{x,z\}\}$ or $\{\{z\},\{x,y\}\}$ lies between $\{\{x\},\{y\},\{z\}\}$ and $\{\{x,y,z\}\}$. This follows from the fact that no two objects are together (or separated) in both $\{\{x\},\{y\},\{z\}\}$ and $\{\{x,y,z\}\}$, so the betweenness condition is vacuously satisfied. For another instance, consider the partitions $\{\{x,y\},\{z\}\}$ and $\{\{x,z\},\{y\}\}$. The only partition that lies between these two partitions is $\{\{x\},\{y\},\{z\}\}$. To see this, note that y are z are separate in both the partitions (and no two objects are together in both), and $\{\{x\},\{y\},\{z\}\}$ is the only partition (other than the two) in which y and z are separated.

Definition 7.4. A domain $\mathscr{D}$ is **intermediate** if for all $P \in \mathscr{D}$ and every two partitions $X_1,X_2 \in A$, X_1 lies between $P(1)$ and X_2 implies $X_1 P X_2$.

By definition, intermediate domains are betweenness domains. In Table 3, we present three preferences in an intermediate domain with three objects that have different structure of the top-ranked partition. Note that the betweenness relation does not specify the ordering of $\{\{a,b\},\{c\}\}$, $\{\{\{a,c\},\{b\}\}\}$, and $\{\{a\},\{b,c\}\}$ when $\{\{a\},\{b\},\{c\}\}$ is the top-ranked partition. Therefore, there are six preferences with $\{\{a\},\{b\},\{c\}\}$ as the top-ranked partition, P^1 is one of them. It is worth noting that an intermediate domain is not swap-connected. For instance, the preferences P_2 and P_3 are graph-local but not swap-local.

P_1	P_2	P_3
$\{\{a\},\{b\},\{c\}\}$	$\{\{a,b\},\{c\}\}$	$\{\{a,b,c\}\}$
$\{\{a,b\},\{c\}\}$	$\{\{a,b,c\}\}$	$\{\{a,b\},\{c\}\}$
$\{\{a,c\},\{b\}\}$	$\{\{a\},\{b\},\{c\}\}$	$\{\{a,c\},\{b\}\}$
$\{\{a\},\{b,c\}\}$	$\{\{a,c\},\{b\}\}$	$\{\{a\},\{b,c\}\}$
$\{\{a,b,c\}\}$	$\{\{a\},\{b,c\}\}$	$\{\{a\},\{b\},\{c\}\}$

Table 3

Proposition 7.2. *The intermediate domain is strongly consistent.*

The proof of this proposition is relegated to Appendix C.9.

By Corollary 7.1 and Proposition 7.2, it follows that almost all unanimous and DSIC RBRs on the intermediate domain are tops-only. This is a major step towards characterizing almost all unanimous and DSIC RBRs on the intermediate domain. It is worth mentioning that the structure of unanimous and DSIC RSCFs are yet not explored on the intermediate domain and it follows from Corollary 7.1 that every such rule is tops-only.

It is shown in Mishra and Roy (2012) that a DSCF is unanimous and DSIC on the intermediate domain if and only if it is a *meet aggregator*. Moreover, by Theorem 7.1 and Proposition 7.2, every intermediate domain is graph-DLGE. Combining these results with Remark 5.1, we obtain the following corollary.

Corollary 7.13. *Almost all top lower contour monotonic graph-LOBIC DBRs on the intermediate domain are meet aggregators.*

8. Non-regular domains

In this section, we consider two important non-regular domains, namely single-dipped and single-crossing domains. Let the alternatives be $A = \{1, \dots, m\}$.

8.1 Single-dipped domains

A preference is single-dipped if there is a "dip" (the worst alternative) of it so that as one moves farther away from it, preference increases. These domains arise in the context of locating a "public bad" (such as garbage dump, nuclear plant, wind mill, etc.).

Definition 8.1. A preference P is **single-dipped** if it has a unique minimal element $d(P)$, the *dip* of P, such that for all $x, y \in A$, $[d(P) \leq x < y$ or $y < x \leq d(P)] \Rightarrow yPx$. A domain is single-dipped if it contains all single-dipped preferences.

Since the single-dipped domain is swap-connected, it satisfies top-connectedness. Therefore, we obtain the following corollary from Theorem 5.4.

Corollary 8.1. *Almost all unanimous and swap-LOBIC RBRs on the single-dipped domain are only-topset.*

Note that the topset of a single-dipped domain consists of the alternatives 1 and m. Therefore, by Corollary 8.1, to analyze the structure of unanimous and swap-LOBIC RBRs on a single-dipped domain, we can assume that there are only two alternatives. Thus, a single-dipped domain becomes the unrestricted domain over two alternatives, and hence we obtain the following corollaries from Theorem 5.3 and Corollary 6.1.

Corollary 8.2. *Almost all unanimous and weak elementary monotonic swap-LOBIC RBRs on the single-dipped domain are tops-only.*

Corollary 8.3. *Almost all unanimous and weak elementary monotonic swap-LOBIC RBRs on the single-dipped domain are swap-LDSIC.*

It is shown in Peters et al. (2017) that an RSCF on the single-dipped domain is unanimous and DSIC if and only if it is a *random committee rule*. We obtain the following corollary by combining this result with Corollary 8.3 and the fact that every swap-LDSIC RSCF on the single-dipped domain is DSIC (see Cho (2016) for details).

Corollary 8.4. *Almost all unanimous, weak elementary monotonic, and swap-LOBIC RBRs on the single-dipped domain are random committee rules.*

8.2 Single-crossing domains

A domain is single-crossing if its preferences can be ordered in a way so that no two alternatives change their relative ranking more than once along that ordering. Such domains are used in models of income taxation and redistribution, local public goods and stratification, and coalition formation (see Saporiti (2009) for details).

Definition 8.2. A domain $\mathscr{D}$ is **single-crossing** if there is an ordering $\lhd$ over $\mathscr{D}$ such that for all $x, y \in A$ and all $P, P' \in \mathscr{D}$, $[x < y, P \lhd P', \text{ and } yPx] \implies yP'x$.

Since the single-crossing domain is swap-connected, it satisfies top-connectedness. Therefore, we obtain the following corollary from Theorem 5.4.

Corollary 8.5. *Almost all unanimous and swap-LOBIC RBRs on the single-crossing domain are only-topset.*

By Corollary 8.5, to analyze the structure of unanimous and swap-LOBIC RBRs on a single-crossing domain, we can restrict a single-crossing domain to its topset. To see that a single-crossing domain satisfies the path-richness property, consider an alternative a and suppose that there are two swap-local preferences $P \equiv a \cdots xy \cdots$ and $P' \equiv a \cdots yx \cdots$. Since P and P' are swap-local, they must be consecutive in the ordering $\lhd$. Assume without loss of generality that $P \lhd P'$. This means $x\hat{P}y$ for all $\hat{P}$ with $\hat{P} \lhd P$ and $y\bar{P}x$ for all $\bar{P}$ with $P' \lhd \hat{P}$. Consider any preference $\tilde{P}$. If $x\tilde{P}y$, then $\tilde{P} \lhd P$, and hence from $\tilde{P}$ one can go to the preference P following the path given by $\lhd$ maintaining the relative ordering between x and y. On the other hand, if $y\tilde{P}x$, then one can go from $\tilde{P}$ to the preference P' following the path given by $\lhd$. This shows that a single-crossing domain satisfies the path-richness property, and hence we obtain the following corollaries from Theorem 5.3 and Corollary 6.1.

Corollary 8.6. *Almost all unanimous and weak elementary monotonic swap-LOBIC RBRs on the single-crossing domain are tops-only.*

Corollary 8.7. *Almost all unanimous and weak elementary monotonic swap-LOBIC RBRs on the single-crossing domain are swap-LDSIC.*

Roy and Sadhukhan (2019) show that an RSCF on the single-crossing domain is unanimous and DSIC if and only if it is a *tops-restricted probabilistic fixed ballot rules (TPFBRs)*. Moreover, Cho (2016) shows that every swap-LDSIC RSCF on the single-crossing domain is DSIC. Combining these results with Corollary 8.7, we obtain the following corollary.

Corollary 8.8. *Almost all unanimous, weak elementary monotonic, and swap-LOBIC RBRs on any single-crossing domain are TPFBRs.*

9. Applications on multi-dimensional separable domains

Multi-dimensional separable domains comprise the main application of our general model. We assume that the alternative set can be decomposed as a Cartesian product, i.e., $A = A^1 \times \cdots \times A^k$, where $1,\ldots,k$ are the components/dimensions with $k \geq 2$, and for each component $l \in K$, the component set A^l contains at least two elements. Thus, an alternative x is a vector of k elements, and hence we denote it $(x^1,\ldots,x^k)$. For $l \in K$, we denote by A^{-l} the set $A^1 \times \cdots \times A^{l-1} \times A^{l+1} \times \cdots \times A^k$ and by x^{-l} an element of A^{-l}.

A preference $P \in \mathscr{P}(A)$ is *separable* if there exists a (unique) marginal preference P^l for each $l \in K$ such that for all $x, y \in A$, we have $[x^l P^l y^l$ for some $l \in K$ and $x^{-l} = y^{-l}] \Rightarrow [xPy]$. A domain is called separable if each preference in it is separable.

9.1 Lexicographically separable domains

A preference P is **lexicographically separable** if there exists a (unique) component order $P^0 \in \mathscr{P}(K)$ and a (unique) marginal preference $P^j \in \mathscr{P}(A^j)$ for each $j \in K$ such that for all $x, y \in A$, we have $\left[x^l P^l y^l \text{ for some } l \in K \text{ and } x^j = y^j \text{ for all } jP^0 l\right] \Rightarrow [xPy]$. A lexicographically separable preference P can be uniquely represented by a $(k+1)$-tuple consisting of a lexicographic order P^0 over the components and marginal preferences $P^1, \ldots, P^k$. We write $P = (P^0, P^1, \ldots, P^k)$ to denote such a preference.

Let $\mathscr{D}_i^0$ be a collection of swap-connected component orderings, and for each component l, let $\mathscr{D}_i^l$ be a collection of swap-connected marginal preferences over the elements A^l. We denote by $\mathscr{L}_i = (\mathscr{D}_i^0, \mathscr{D}_i^1, \ldots, \mathscr{D}_i^k)$ the lexicographically separable domain containing all lexicographically separable preferences with component orders in $\mathscr{D}_i^0$ and marginal preferences in $\mathscr{D}_i^1 \times \cdots \times \mathscr{D}_i^k$, that is, $\mathscr{L}_i = \{(P_i^0, P_i^1, \ldots, P_i^k) \mid (P_i^0, P_i^1, \ldots, P_i^k) \in \mathscr{D}_i^0 \times \mathscr{D}_i^1 \times \cdots \times \mathscr{D}_i^k\}$.

Two preferences $P = (P^0, P^1, \ldots, P^k)$ and $\bar{P} = (\bar{P}^0, \bar{P}^1, \ldots, \bar{P}^k)$ are lex-local if there exists $l \in K \cup \{0\}$ such that P^l and $\bar{P}^l$ are swap-local and $P^j = \bar{P}^j$ for all $j \neq l$.

We introduce a simpler (and stronger) version of lower contour monotonicity for lexicographic domains, which we call lex-monotonicity.

Definition 9.1. An RSCF $\varphi : \mathscr{L}_N \to A$ is **lex-monotonic** if for all $i \in N$, all $P_i, \bar{P}_i \in \mathscr{L}_i$ such that P_i and $\bar{P}_i$ are lex-local, and all $P_{-i} \in \mathscr{L}_{-i}$, we have

(i) if $\bar{P}_i^0$ is an (l, j)-swap of P_i^0 for some $l, j \in K$, then φ is lower contour monotonic on the pair $((P_i, P_{-i}), (\bar{P}_i, P_{-i}))$, and

(ii) if $\bar{P}_i^l$ is an (x^l, y^l)-swap of P_i^l for some $x^l, y^l \in A^l$, then $\varphi_{(a^{-l}, y^l)}(P_i, P_{-i}) \leq \varphi_{(a^{-l}, y^l)}(\bar{P}_i, P_{-i})$ for all $a^{-l} \in A^{-l}$.

The following corollary is obtained from Theorem 5.1.

Corollary 9.1. *If an RSCF $\varphi : \mathscr{L}_N \to A$ satisfies lex-monotonicity, then almost all lex-LOBIC RBRs with RSCF φ are lex-LDSIC.*

Since lex-LDSIC implies DSIC for DSCFs on lexicographic domains in which component orderings are swap-connected and marginal domains are regular and swap-DLGE (see Kumar et al. (2020) for details), it follows that for almost all lex-monotonic DSCFs, OBIC and DSIC are equivalent.

Corollary 9.2. *Let $\mathscr{D}^0$ be swap-connected, and let $\mathscr{D}^l$ be regular and swap-DLGE for all $l = 1,\dots,k$. Suppose $\mathscr{L}_i = (\mathscr{D}^0, \mathscr{D}^1, \dots, \mathscr{D}^k)$ for all $i \in N$. If a DSCF $f : \mathscr{L}_N \to A$ satisfies lex-monotonicity, then almost all lex-LOBIC DBRs with DSCF f are DSIC.*

9.2 Full separable domains

For a collection of marginal preferences $(P^1,\dots,P^k)$, the collection of all separable preferences with marginals as $(P^1,\dots,P^k)$ is denoted by $\mathscr{S}(P^1,\dots,P^k)$. Similarly, for a collection of marginal domains $(\mathscr{D}^1,\dots,\mathscr{D}^k)$, the set of all separable preferences with marginals in $(\mathscr{D}^1,\dots,\mathscr{D}^k)$ is denoted by $\mathscr{S}(\mathscr{D}^1,\dots,\mathscr{D}^k)$, that is, $\mathscr{S}(\mathscr{D}^1,\dots,\mathscr{D}^k) = \cup_{(P^1,\dots,P^k)\in(\mathscr{D}^1,\dots,\mathscr{D}^k)} \mathscr{S}(P^1,\dots,P^k)$. A separable domain of the form $\mathscr{S}(\mathscr{D}^1,\dots,\mathscr{D}^k)$ is called a full separable domain. Throughout this subsection, we assume that the marginal domains are betweenness domains satisfying swap-connectedness and consistency, for instance, they can be any domain we have discussed so far except the intermediate domain. For $P_N \in \mathscr{S}(\mathscr{D}^1,\dots,\mathscr{D}^k)$, we denote its restriction to a component $l \in K$ by P_N^l, that is, $P_N^l = (P_1^l,\dots,P_n^l)$. We introduce the local structure in a full separable domain in a natural way.

Definition 9.2. Let $\mathscr{D}^l$ be swap-connected for all $l \in K$. Two preferences $P, \bar{P} \in \mathscr{S}(\mathscr{D}^1,\dots,\mathscr{D}^k)$ are **sep-local** if one of the following two holds:

(i) $P \triangle \bar{P} = \{x,y\}$ where x,y are such that $|\{l \mid x^l \neq y^l\}| \geq 2$.

(ii) $P \triangle \bar{P} = \{((a^{-l},x^l),(a^{-l},y^l)) \mid a^{-l} \in A^{-l}\}$, where $l \in K$ and $x^l, y^l \in A^l$ swap from P^l to $\bar{P}^l$.

Thus, (i) in Definition 9.2 says that exactly one pair of alternatives (x,y), that vary over at least two components, swap from P to P', and (ii) in Definition 9.2 says that multiple pairs of alternatives of the form $((a^{-l},x^l),(a^{-l},y^l))$, where $a^{-l} \in A^{-l}$, swaps from P to P'. This structure makes the lower contour monotonicity property simpler: it imposes elementary monotonicity to every pair of swapping alternatives. We call it **sep-monotonicity**.

Remark 9.1. Let $l \in K$ and let $\pi^l = (\pi^l(1), \ldots, \pi^l(t))$ be a swap-local path in $\mathscr{D}^l$ such that the relative ordering of two alternatives $x^l, y^l \in A^l$ remains the same along the path. Then, for every component ordering $P^0 \in \mathscr{P}(K)$ having l as the worst component, and for every collection of marginal preferences $(P^1, \ldots, P^{l-1}, P^{l+1}, \ldots, P^k)$ over components other than l, the relative ordering of any two alternatives in the set $\{a \in A \mid a^l \in \{x^l, y^l\}\}$ will remain the same along the sep-local path $((P^0, P^1, \ldots, P^{l-1}, \pi^l(1), P^{l+1}, \ldots, P^k), \ldots, (P^0, P^1, \ldots, P^{l-1}, \pi^l(t), P^{l+1}, \ldots, P^k))$ in the domain $\mathscr{S}(\mathscr{D}^1, \ldots, \mathscr{D}^k)$.

For notational convenience, we denote a domain $\mathscr{S}(\mathscr{D}^1, \ldots, \mathscr{D}^k)$ by $\mathscr{S}$ in the following results. The following corollary is obtained from Theorem 5.1.

Corollary 9.3. *If an RSCF $\varphi : \mathscr{S}^n \to \Delta A$ satisfies sep-monotonicity, then almost all sep-LOBIC RBRs with RSCF φ are sep-LDSIC.*

It is worth mentioning that Corollary 9.3 holds as long as the marginal domains are swap-connected.

Our next two propositions are derived by using Theorem 5.3. An RSCF $\varphi : \mathscr{S}^n \to \Delta A$ satisfies **component-unanimity** if for each component $l \in K$ and each $P_N \in \mathscr{S}^n$ such that $P_i^l(1) = x^l$ for all $i \in N$ and some $x^l \in A^l$, we have $\varphi^l_{x^l}(P_N) = 1$.

Proposition 9.1. *Almost all unanimous and sep-LOBIC RBRs on a full separable domain satisfies component-unanimity.*

The proof of this proposition is relegated to Appendix C.10.

Proposition 9.2. *Almost all unanimous and sep-LOBIC RBRs on a full separable domain are tops-only.*

The proof of this proposition is relegated to Appendix C.11.

For random rules, to the best of our knowledge, it is still not known whether sep-LDSIC implies DSIC or not. However, the same is shown for DSCFs on domains having unrestricted marginals (see Kumar et al. (2020) for details). Thus, it follows from Corollary 9.3 that for almost all sep-monotonic DSCFs, OBIC and DSIC are equivalent on such domains.

9.2.1 MARGINAL DECOMPOSABILITY OF RANDOM RULES

Breton and Sen (1999) show that every unanimous and DSIC DSCF on a multi-dimensional (full) separable domain is decomposable: its outcome in a particular dimension depends only on the (marginal) preferences of agents in that dimension. In our view, a suitable version of decomposability for random rules is marginal decomposability, which we investigate for sep-LOBIC rules in this section.

The marginal distribution of an RSCF $\varphi : \mathscr{S}^n \to \Delta A$ over component $l \in K$ at a preference profile P_N, denoted by $\varphi^l(P_N)$, is defined as $\varphi^l_{x^l}(P_N) = \sum_{x^{-l} \in A^{-l}} \varphi_{(x^l, x^{-l})}(P_N)$ for all $x^l \in A^l$.

Definition 9.3. An RSCF $\varphi : \mathscr{S}^n \to \Delta A$ is **marginally decomposable** if for all $l \in K$ and all $P_N, \bar{P}_N \in \mathscr{S}^n$ with $P^l_N = \bar{P}^l_N$, we have $\varphi^l(P_N) = \varphi^l(\bar{P}_N)$.

Remark 9.2. Note that for a DSCF $f : \mathscr{S}^n \to A$, marginal decomposability is equivalent to decomposability defined as follows: a DSCF $f : \mathscr{S}^n \to A$ is decomposable if for all $l \in K$ and all $P_N, \bar{P}_N \in \mathscr{S}^n$ with $P^l_N = \bar{P}^l_N$, we have $f^l(P_N) = f^l(\bar{P}_N)$. Here, $f^l(P_N)$ denotes the l-th component of $f(P_N)$. Thus, our notion of marginal decomposability indeed generalizes the notion of decomposability for random rules.

Theorem 9.1. *Almost all unanimous and sep-LOBIC RBRs on a full separable domain are marginally decomposable.*

The proof of this theorem is relegated to Appendix C.12.

10. Weak preferences

A weak preference is a complete and transitive binary relation. We denote a weak preference by R and the set of all weak preferences by $\mathscr{R}(A)$. For a weak preference R, we denote its strict part by P and indifference part by I. An indifference class of a preference is the maximal set of alternatives that are indifferent to each other.

As in the case of strict preferences, we assume that each domain $\mathscr{D}_i \subseteq \mathscr{R}_i$ is endowed with a graph structure with respect to which it is connected. We generalize the definition of a block for weak preferences in the following way. A set of alternatives B is a block in a pair of preferences (R, R') if it is a minimal non-empty set satisfying the following properties: (i) for all $x \in B$ and $y \notin B$, $P|_{\{x,y\}} = P'|_{\{x,y\}}$, and (ii) B is not a strict subset of an indifference class of R and an indifference class of R'.

Note that the technical definition of lower contour monotonicity and block preservation property (Proposition 5.1 and Theorem 5.1) do not involve the assumption of strict preferences, therefore we continue to use the same definitions for weak preferences. Our next two results say that Proposition 5.1 and Theorem 5.1 continue to hold in this scenario.

Proposition 10.1. *Almost all graph-LOBIC RBRs satisfy the block preservation property.*

The proof of this proposition is relegated to Appendix B.

Theorem 10.1. *If an RSCF $\varphi : \mathscr{D}_N \to \Delta A$ satisfies lower contour monotonicity, then almost all graph-LOBIC RBRs with RSCF φ are graph-LDSIC.*

The proof of this theorem is relegated to Appendix C.1.

11. Discussion

11.1 The case of DBRs

A probability distribution ν on a finite set S is generic if for all subsets U and V of S, $\nu(U) = \nu(V)$ implies $U = V$. Majumdar and Sen (2004) show that on the unrestricted domain, every unanimous DBR that is OBIC with respect to a generic prior is dictatorial, and Mishra (2016) shows that under elementary monotonicity, the notions DSIC and OBIC with respect to generic priors are equivalent. It can be verified that all our results hold for generic priors if we restrict our attention to DBRs. Additionally, our results establish further structure (such as tops-onlyness, Pareto optimality, only-topsetness, decomposability) of OBIC DBRs with respect to generic priors.

11.2 Fully correlated priors

Note that the priors we consider in this paper are partially correlated: prior of an agent is independent of her own preference, while it may be correlated over the preferences of other agents. The natural question arises here as to what will happen if the prior of an agent depends on her own preferences too. Firstly, our proof technique for Theorem 5.1 will fail, but more importantly, Theorem 5.1 will not even hold anymore. It can be verified from the proof of Proposition 5.1 that if an RSCF is graph-LOBIC but not graph-LDSIC then it must satisfy a system of equations . The proof follows from the fact that the set of priors that satisfy such a system of equations has Lebesgue measure zero. However, if an agent has two different priors for two local preferences, then we cannot obtain such a system of equations on a given prior (what we obtain are equations involving different priors), and consequently, nothing can be concluded about the Lebesgue measure of such priors. We illustrate this with the following example.

Suppose that there are two agents 1 and 2, and three alternatives a, b, and c. Consider two swap-local preferences bac and bca of agent 1. Consider the anti-plurality rule with the tie-breaking criteria as $a \succ b \succ c$. In Table 4, we present this rule when agent 1 has preferences bac and bca, and 2 has any preference. It is well-known (and also can be verified from the example) that anti-plurality rule is not swap-LDSIC. However, it is swap-LOBIC over the mentioned preferences of agent 1 if her prior satisfies the following conditions: $\mu_1(bca|cab) + \mu_1(cba|cab) - \mu_1(acb|cab) - \mu_1(cab|cab) \geq 0$ and $\mu_1(acb|cba) + \mu_1(cab|cba) - \mu_1(bca|cba) - \mu_1(cba|cba) \geq 0$. It is clear that the Lebesgue measure of such priors is not zero (as we have argued, the equality is imposed on two different priors $\mu_1(\cdot|cab)$ and $\mu_1(\cdot|cba)$). It can be verified that if one considers all possible restrictions arising from all possible swap-local preferences of each agent, the resulting priors for which the rule is LOBIC will have Lebesgue measure strictly bigger than zero.

1 \ 2	abc	acb	bac	bca	cba	cab
cab	a	a	a	c	a	c
cba	b	c	b	b	c	b

Table 4

APPENDIX

A. PRELIMINARIES FOR THE PROOFS

Consider an RSCF $\varphi : \mathscr{D}_N \to \Delta A$. A prior profile μ_N is called compatible with φ if for all $i \in N$, all $P_i, P_i' \in \mathscr{D}_i$, and all $X \subsetneq A$,

$$\sum_{R_{-i}} \mu_i(R_{-i})(\varphi_X(R_i, R_{-i}) - \varphi_X(R_i', R_{-i})) = 0 \tag{1}$$

$$\implies \varphi_X(R_i, R_{-i}) - \varphi_X(R_i', R_{-i}) = 0 \text{ for all } R_{-i}.$$

Let $\mathscr{M}(\varphi)$ denote the set of all priors that are compatible with φ.

Claim A.1. *For every RSCF φ, the Lebesgue measure of the set $\mathscr{M}(\varphi)$ is* 1.

Proof of Claim A.1. The proof of this claim follows from elementary measure theory; we provide a sketch of it for the sake of completeness. First note that for a given RSCF

φ, (1) is equivalent to an equation of the form:

$$x_1\alpha_1 + \cdots + x_k\alpha_k = 0, \tag{2}$$

where α's are some constants and x's are non-negative variables summing up to 1 (that is, probabilities). The question is if x's are drawn randomly (uniformly) from the space $\{(x_1,\ldots,x_k) \mid x_l \geq 0 \text{ for all } l \text{ and } \sum_l x_l = 1\}$, what is the Lebesgue measure of the priors for which (2) will be satisfied? Clearly, if α's are all zeros, (2) will be satisfied for all priors. We argue that it if α's are not all zeros, then (2) can be satisfied only for a set of priors with Lebesgue measure zero, which will complete the proof. However, this follows from the facts that the solutions of (2) form a hyperplane and that the Lebesgue measure of a hyperplane is zero (because of dimensional reduction, such as the Lebesgue measure of a line in a plane is zero, that of a plane in a cube is zero, etc.).[10] ■

B. Proof of Proposition 5.1 and Proposition 10.1

Proof. Let (φ, μ_N) be a graph-LOBIC RBR. Since we prove the claim for almost all graph-LOBIC RBRs, in view of Claim A.1, we assume that μ_N is compatible with φ. Consider graph-local preferences $R_i, R_i' \in \mathscr{D}_i$ and $R_{-i} \in \mathscr{D}_{-i}$. Suppose that B is a block in (R_i, R_i'). Let $U_B(R_i) = \{x \in A \mid xP_ib \text{ for all } b \in B\}$ be the set of alternatives that are strictly preferred to each element of B according to R_i. By the definition of a block in (R_i, R_i'), it follows that both $U_B(R_i)$ and $U_B(R_i) \cup B$ are upper contour sets in each of the preferences R_i and R_i'. Since R_i and R_i' are graph-local, by graph-LOBIC,

$$\sum_{R_{-i}\in\mathscr{D}_{-i}} \mu_i(R_{-i})\varphi_{U_B(R_i)}(R_i, R_{-i}) = \sum_{R_{-i}\in\mathscr{D}_{-i}} \mu_i(R_{-i})\varphi_{U_B(R_i)}(R_i', R_{-i}) \tag{3}$$

and

$$\sum_{R_{-i}\in\mathscr{D}_{-i}} \mu_i(R_{-i})\varphi_{U_B(R_i)\cup B}(R_i, R_{-i}) = \sum_{R_{-i}\in\mathscr{D}_{-i}} \mu_i(R_{-i})\varphi_{U_B(R_i)\cup B}(R_i', R_{-i}). \tag{4}$$

Subtracting (3) from (4), we have

$$\sum_{R_{-i}\in\mathscr{D}_{-i}} \mu_i(R_{-i})\left(\varphi_B(R_i, R_{-i}) - \varphi_B(R_i', R_{-i})\right) = 0.$$

[10] For a detailed argument, suppose that exactly one α, say α_1 is not zero. Note that this assumption gives maximum freedom for the values of x's and thereby maximize the Lebesgue measure of the solution space of (2). However, this means in any solution x_1 must be zero, the measure of which in the solution space is zero.

Since μ_N is compatible with φ, this means $\varphi_B(R_i, R_{-i}) = \varphi_B(R'_i, R_{-i})$ for all $R_{-i} \in \mathscr{D}_{-i}$, which completes the proof. ■

C. Other proofs

In view of Proposition 5.1, whenever we prove some statement for almost all graph-LOBIC RBRs with RSCF φ in this section, we assume that the RSCF φ satisfies the block preservation property.

C.1 Proof of Theorem 5.1 and Theorem 10.1

Proof. Let $\varphi : \mathscr{D}_N \to \Delta A$ be an RSCF satisfying lower contour monotonicity and the block preservation property. We show that φ is graph-LDSIC. Consider graph-local preferences $R_i, R'_i \in \mathscr{D}_i$, $R_{-i} \in \mathscr{D}_{-i}$, and $x \in A$. We show $\varphi_{U(x,R_i)}(R_i, R_{-i}) \geq \varphi_{U(x,R_i)}(R'_i, R_{-i})$. Let $B_1, \ldots, B_t$ be the blocks in (R_i, R'_i) such that for all $l < t$ and all $b \in B_l$ and $b' \in B_{l+1}$, we have bP_ib'. Suppose that $x \in B_l$ for some $l \in \{1, \ldots, t\}$.

Let $\hat{B}_l = \{b \in B_l \mid bP_ix\}$ be the set of alternatives (possibly empty) in B_l that are (strictly) preferred to x. Note that the set $B_l \setminus \hat{B}_l$ is lower contour set of $R_i|_{B_l}$. Therefore, by lower contour monotonicity,

$$\varphi_{B_l \setminus \hat{B}_l}(R'_i, R_{-i}) \geq \varphi_{B_l \setminus \hat{B}_l}(R_i, R_{-i}). \tag{5}$$

Furthermore, by the block preservation property, we have

$$\varphi_{B_l}(R'_i, R_{-i}) = \varphi_{B_l}(R_i, R_{-i}). \tag{6}$$

Subtracting (5) from (6), we have

$$\varphi_{\hat{B}_l}(R_i, R_{-i}) \geq \varphi_{\hat{B}_l}(R'_i, R_{-i}). \tag{7}$$

Note that $U(x,R_i) = B_1 \cup \cdots \cup B_{l-1} \cup \hat{B}_l$. This means $\varphi_{U(x,R_i)}(R_i, R_{-i}) = \varphi_{B_1 \cup \cdots \cup B_{l-1}}(R_i, R_{-i}) + \varphi_{\hat{B}_l}(R_i, R_{-i})$ and $\varphi_{U(x,R_i)}(R'_i, R_{-i}) = \varphi_{B_1 \cup \cdots \cup B_{l-1}}(R'_i, R_{-i}) + \varphi_{\hat{B}_l}(R'_i, R_{-i})$. By the block preservation property, $\varphi_{B_1 \cup \cdots \cup B_{l-1}}(R_i, R_{-i}) = \varphi_{B_1 \cup \cdots \cup B_{l-1}}(R'_i, R_{-i})$, and by (7) , $\varphi_{\hat{B}_l}(R_i, R_{-i}) \geq \varphi_{\hat{B}_l}(R'_i, R_{-i})$. Combining these observations, we have $\varphi_{U(x,R_i)}(R_i, R_{-i}) \geq \varphi_{U(x,R_i)}(R'_i, R_{-i})$, which completes the proof. ■

C.2 Proof of Theorem 5.2

Proof. Let $\mathscr{D}_i$ satisfy upper contour preservation property for all $i \in N$ and suppose that $\varphi : \mathscr{D}_N \to \Delta A$ is an RSCF satisfying unanimity and the block preservation property. We show that φ is Pareto optimal. Consider $P_N \in \mathscr{D}_N$ such that xP_iy for all $i \in N$ and some $x,y \in A$. We show that $\varphi_y(P_N) = 0$. Assume for contradiction $\varphi_y(P_N) > 0$. Consider $i \in N$. By the upper contour preservation property there exists a graph-local path $(P_i^1 = P_i, \dots, P_i^t)$ such that $P_i^t(1) = x$ and $U(P_i,y) = U(P_i^l,y)$ for all $l = 1,\dots,t$. Since $U(y,P_i^1) = U(y,P_i^2)$, we have $y \notin P_i^1 \triangle P_i^2$, which implies that $\{y\}$ is a singleton block in (P_i^1,P_i^2). By the block preservation property, this implies $\varphi_y(P_i^2,P_{-i}) = \varphi_y(P_i,P_{-i})$. Continuing in this manner, we reach a preference profile (P_i^t,P_{-i}) such that $P_i^t(1) = x$ and $\varphi_y(P_i^t,P_{-i}) > 0$. By applying the same argument to the agents $j \in N \setminus \{i\}$ we can construct a preference profile P'_N such that $P'_j(1) = x$ for all $j \in N$ and $\varphi_y(P'_N) > 0$. Since $P'_j(1) = x$ for all $j \in N$, by unanimity we have $\varphi_x(P'_N) = 1$, which contradicts that $\varphi_y(P'_N) > 0$. ∎

C.3 Proof of Theorem 5.3

We use the following lemma in our proof.

Lemma C.1. *Suppose an RSCF $\varphi : \mathscr{D}^n \to \Delta A$ satisfies unanimity and the block preservation property. Let $P_i, P'_i \in \mathscr{D}$ be graph-local and let $P_{-i} \in \mathscr{D}^{n-1}$ be such that $\varphi_x(P_i,P_{-i}) \neq \varphi_x(P'_i,P_{-i})$ for some $x \in P_i \triangle P'_i$. Consider an agent $j \neq i$ and suppose that there is a graph-local path $(P_j^1 = P_j, \dots, P_j^t = \bar{P}_j)$ such that for all $l < t$ and for every two alternatives $a,b \in P_i \triangle P'_i$, there is a common upper contour set U of both P_j^l and P_j^{l+1} such that exactly one of a and b is contained in U. Then $\varphi_x(P_i,\bar{P}_j,P_{-\{i,j\}}) \neq \varphi_x(P'_i,\bar{P}_j,P_{-\{i,j\}})$.*

Proof of Lemma C.1. Suppose $\varphi_x(P_i,P_j^l,P_{-\{i,j\}}) \neq \varphi_x(P'_i,P_j^l,P_{-\{i,j\}})$ for some $l < t$ and some $x \in P_i \triangle P'_i$. It is enough to show that $\varphi_x(P_i,P_j^{l+1},P_{-\{i,j\}}) \neq \varphi_x(P'_i,P_j^{l+1},P_{-\{i,j\}})$. Let a and $\bar{a}$ be the alternatives, if exist, that are ranked just above and just below x, respectively, in $P_j^l|_{P_i \triangle P'_i}$. More formally, let $a \in P_i \triangle P'_i$ be such that $aP_j^l x$ and no alternative in $P_i \triangle P'_i$ is ranked between a and x, and let $\bar{a} \in P_i \triangle P'_i$ be such that $xP_j^l \bar{a}$ and no alternative in $P_i \triangle P'_i$ is ranked between x and a. Let U be the common upper contour set of P_j^l and P_j^{l+1} such that $U \cap \{a,x\} = a$, and $\widehat{U}$ be the common upper contour set of P_j^l and P_j^{l+1} such that $\widehat{U} \cap \{x,\bar{a}\} = x$. Here, U might be empty and

$\widehat{U}$ might be A. Consider the set of alternatives $B = U \setminus \widehat{U}$. Note that B can be expressed as a union of blocks in (P_j^l, P_j^{l+1}). Therefore, by applying the block preservation property to each block in B, we obtain $\varphi_B(P_i, P_j^l, P_{-\{i,j\}}) = \varphi_B(P_i, P_j^{l+1}, P_{-\{i,j\}})$ and $\varphi_B(P_i', P_j^l, P_{-\{i,j\}}) = \varphi_B(P_i', P_j^{l+1}, P_{-\{i,j\}})$. Moreover, since each $c \in B \setminus x$ is a block in (P_i, P_i'), we have by the block preservation property, $\varphi_c(P_i, P_j^l, P_{-\{i,j\}}) = \varphi_c(P_i', P_j^l, P_{-\{i,j\}})$ and $\varphi_c(P_i, P_j^{l+1}, P_{-\{i,j\}}) = \varphi_c(P_i', P_j^{l+1}, P_{-\{i,j\}})$ for all $c \in B \setminus x$. Combining these observations, it follows that $\varphi_x(P_i, P_j^{l+1}, P_{-\{i,j\}}) \neq \varphi_x(P_i', P_j^{l+1}, P_{-\{i,j\}})$. ■

Proof of Theorem 5.3. Let $\mathscr{D}$ satisfy the path-richness property (see Definition 5.3) and suppose that $\varphi : \mathscr{D}^n \to \Delta A$ is an RSCF satisfying unanimity and the block preservation property. We show that φ is tops-only. Assume for contradiction that $\varphi(P_i, P_{-i}) \neq \varphi(P_i', P_{-i})$ for some $P_i, P_i' \in \mathscr{D}$ with $P_i(1) = P_i'(1)$ and some $P_{-i} \in \mathscr{D}^{n-1}$. By means of Condition (i) of the path-richness property, it is enough to assume that P_i and P_i' are graph-local. Therefore, by the block preservation property, it follows that $\varphi_x(P_i, P_{-i}) \neq \varphi_x(P_i', P_{-i})$ for some $x \in P_i \triangle P_i'$.

Consider $j \in N \setminus \{i\}$. By Condition (ii) of the path-richness property, there is a path $(P_j^1 = P_j, \ldots, P_j^t = P_j')$ with $P_j'(1) = P_i(1)$ such that for all $l < t$ and for every two alternatives $a, b \in P_i \triangle P_i'$, there is a common upper contour set U of both P_j^l and P_j^{l+1} such that exactly one of a and b is contained in U. By applying Lemma C.1, it follows that $\varphi_x(P_i, P_j', P_{-i}) \neq \varphi_x(P_i', P_j', P_{-i})$. By applying this logic to all agents except i, we construct $P_{-i}' \mathscr{D}^{n-1}$ such that $P_j'(1) = P_i(1)$ for all $j \neq i$ and $\varphi_x(P_i, P_{-i}') \neq \varphi_x(P_N')$. However, since (P_i, P_{-i}') and (P_i', P_{-i}') are unanimous preference profiles with the top-ranked alternative different from x, $\varphi_x(P_i, P_{-i}') = \varphi_x(P_i', P_{-i}') = 0$, a contradiction. ■

C.4 Proof of Theorem 5.4

Proof. Let $\mathscr{D}$ satisfy top-connectedness and suppose that $\varphi : \mathscr{D}^n \to \Delta A$ is a unanimous, tops-only RSCF satisfying the block preservation property. We show that φ is only-topset. Assume for contradiction $\varphi_x(P_N) > 0$ for some $x \notin \tau(\mathscr{D})$ and some $P_N \in \mathscr{D}^n$. Let $P_2(1) = a$ and $P_1(1) = b$. Since $\mathscr{D}$ is top-connected, there exists a sequence $(x^l = a, x^2, \ldots, x^t = b)$ such that x^l, x^{l+1} are top-connected for all $l < t$. Since x^1 and x^2 are top-connected, there exist graph-local preferences $P_2', P_2'' \in \mathscr{D}$ such that $P_2' \equiv x^1 x^2 \cdots$ and $P_2'' \equiv x^2 x^1 \cdots$ with $P_2'|_{A \setminus \{x^1, x^2\}} = P_2''|_{A \setminus \{x^1, x^2\}}$. Because φ is tops-only and $P_2'(1) =$

$P_2(1)$, $\varphi_x(P_1,P'_2,P_{-\{1,2\}}) = \varphi_x(P_1,P_2,P_{-\{1,2\}})$. Again by the block preservation property $\varphi_x(P_1,P''_2,P_{-\{1,2\}}) = \varphi_x(P_1,P'_2,P_{-\{1,2\}})$. Combining these we have $\varphi_x(P_1,P''_2,P_{-\{1,2\}}) = \varphi_x(P_1,P_2,P_{-\{1,2\}})$. Continuing in this manner, we can arrive at a preference $\tilde{P}_2$ such that $\tilde{P}_2(1) = b$ and $\varphi_x(P_1,\tilde{P}_2,P_{-\{1,2\}}) = \varphi_x(P_1,P_2,P_{-\{1,2\}})$. Applying the same argument to the agents in $\{3,\dots,n\}$ we construct $\tilde{P}_N \in \mathscr{D}^n$ such that $\tilde{P}_1 = P_1$, $\tilde{P}_i(1) = b$ for all $i \in \{2,\dots,n\}$ and $\varphi_x(\tilde{P}_N) = \varphi_x(P_N) > 0$. However this contradicts unanimity. ■

C.5 Proof of Theorem 6.1

Proof. Let $\mathscr{D}$ be swap-connected and suppose that $\varphi : \mathscr{D}_N \to \Delta A$ is a tops-only RSCF satisfying weak elementary monotonicity and the block preservation property. We show that φ is swap-LDSIC.

Let P_i and P'_i be two swap-local preferences. If $\tau(P_i) = \tau(P'_i)$, then by tops-onlyness, $\varphi(P_i,P_{-i}) = \varphi(P'_i,P_{-i})$, and we are done. So, suppose $P_i \equiv ab\cdots$ and $P'_i \equiv ba\cdots$. Assume for contradiction that $\varphi_a(P_i,P_{-i}) < \varphi_a(P'_i,P_{-i})$. By the block preservation property, $\varphi_{\{a,b\}}(P_i,P_{-i}) = \varphi_{\{a,b\}}(P'_i,P_{-i})$, and hence our assumption for contradiction means $\varphi_b(P_i,P_{-i}) > \varphi_b(P'_i,P_{-i})$. Consider an agent $j \in N \setminus i$ such that $\tau(P_j) \notin \{a,b\}$. Note that since $\mathscr{D}_j$ is swap-connected one of the following two cases must hold for P_j: (i) there is a swap-local path from P_j to a preference $P'_j \equiv a\cdots$ such that b does not appear as the top-ranked alternative in any preference in the path, (ii) there is a swap-local path from P_j to a preference $P'_j \equiv b\cdots$ such that a does not appear as the top-ranked alternative in any preference in the path.

Suppose Case (i) holds. Let B be the set of alternative that appear as the top-ranked alternative in some preference in the mentioned path. Consider the outcomes of φ when agent j changes her preferences along the path, while all other agents keep their preferences unchanged. By tops-onlyness, the outcome can change only when the top-ranked alternative changes along the path. Moreover, by the definition of swap-local path, the top-ranked alternative can change along the path only through a swap between two alternatives in B. By block preservation, this implies that the probability of the two swapping alternatives can only change in any such situations, and hence, the probability of the alternatives outside B will remain unchanged at the end of the path. Since $b \notin B$, this yields $\varphi_b(P_i,P_j,P_{-\{i,j\}}) = \varphi_b(P_i,P'_j,P_{-\{i,j\}})$ and $\varphi_b(P'_i,P_j,P_{-\{i,j\}}) = \varphi_b(P'_i,P'_j,P_{-\{i,j\}})$. This, together with our assumption for contradiction that $\varphi_b(P_i,P_{-i}) > \varphi_b(P_i',P_{-i})$, implies $\varphi_b(P_i,P_j',P_{-\{i,j\}}) > \varphi_b(P_i',P_j',P_{-\{i,j\}})$. Now, since $P_i \triangle P_i' = \{a,b\}$, we have by block preservation, $\varphi_{\{a,b\}}(P_i,P_j',P_{-\{i,j\}}) = \varphi_{\{a,b\}}(P_i',P_j',P_{-\{i,j\}})$. Because $\varphi_b(P_i,P_j',P_{-\{i,j\}}) > \varphi_b(P_i',P_j',P_{-\{i,j\}})$, this yields $\varphi_a(P_i,P_j',P_{-\{i,j\}}) < \varphi_a(P_i',P_j',P_{-\{i,j\}})$. Using similar logic, we can conclude for Case (ii) that $\varphi_a(P_i,P_j',P_{-\{i,j\}}) < \varphi_a(P_i',P_j',P_{-\{i,j\}})$.

Note that the preceding argument holds no matter what the preferences of the agents in $N \setminus \{i,j\}$ are. Therefore, by repeated application of this argument for each agent $j \in N \setminus i$ with $\tau(P_j) \notin \{a,b\}$, we obtain $P_{-i}' \in \mathscr{D}_{-i}$ of the agents in $N \setminus i$ such that (i) $\tau(P_j') \in \{a,b\}$ for each $j \in N \setminus i$, and (ii) $\varphi_a(P_i,P_{-i}') < \varphi_a(P_i',P_{-i}')$.

We now complete the proof by means of tops-onlyness. If $P_j' \equiv a\cdots$ then let $P_j'' = P_i$, and if $P_j' \equiv b\cdots$ then let $P_j'' = P_i'$. By tops-onlyness, $\varphi(P_i,P_{-i}') = \varphi(P_i,P_{-i}'')$ and $\varphi(P_i',P_{-i}') = \varphi(P_i',P_{-i}'')$, and hence, $\varphi_a(P_i,P_{-i}'') < \varphi_a(P_i',P_{-i}'')$. However, since for each $j \in N$, either $P_j'' \equiv P_i$ or $P_j'' \equiv P_i'$, this violates weak elementary monotonicity, a contradiction. ■

C.6 Proof of Theorem 6.2

Proof. Let $\mathscr{D}$ satisfy the top-swap richness property and suppose that $\varphi : \mathscr{D}^n \to \Delta A$ is a Pareto optimal, tops-only RSCF satisfying the block preservation property. We show that φ is swap-LDSIC. To show that φ is swap-LDSIC, by Theorem 6.1, it is sufficient to show that φ is weak elementary monotonic. Consider swap-local preferences $P_i, \bar{P}_i \in \mathscr{D}$ such that $P_i \equiv ab\cdots$ and $\bar{P}_i \equiv ba\cdots$. Assume for contradiction that $\varphi_b(P_i,P_{-i}) > \varphi_b(\bar{P}_i,P_{-i})$ for some $P_{-i} \in \mathscr{D}^{n-1}$ such that $P_i(k) = \bar{P}_i(k) = P_j(k)$ for all $j \in N \setminus i$ and all $k > 2$. Let c be the alternative such that $P_i \equiv abc\cdots$. Because P_i and $\bar{P}_i$ are swap-local, this means $\bar{P}_i \equiv bac\cdots$. Consider $P_i^1 \in \mathscr{D}$ such that $P_i^1 = acb\cdots$ and P_i^1 and P_i are swap-local, that is $P_i^1 \triangle P_i = \{b,c\}$. By tops-onlyness of φ, $\varphi(P_i^1,P_{-i}) = \varphi(P_i,P_{-i})$. Next, consider $P_i^2 \in \mathscr{D}$ such that $P_i^2 \equiv cab\cdots$ and P_i^2 and P_i^1 are swap-local. By the block preservation property, $\varphi_b(P_i^2,P_{-i}) = \varphi_b(P_i^1,P_{-i})$. Now, consider $P_i^3 \in \mathscr{D}$ such that $P_i^3 \equiv cba\cdots$ and P_i^3 and P_i^2 are swap-local. By tops-onlyness of φ, $\varphi(P_i^3,P_{-i}) = \varphi(P_i^2,P_{-i})$. Finally, consider $P_i^4 \in \mathscr{D}$ such that $P_i^4 \equiv bca\cdots$ and P_i^4 and P_i^3 are swap-local. Since bP_i^4c and bP_jc for all $j \in N \setminus i$, we have by Pareto optimality, $\varphi_c(P_i^4,P_{-i}) = 0$. Moreover, by the block preservation property, we have $\varphi_b(P_i^4,P_{-i}) = \varphi_b(P_i^3,P_{-i}) + \varphi_c(P_i^3,P_{-i})$. This, together with the fact that $\varphi_b(P_i^3, P_{-i}) = \varphi_b(P_i, P_{-i})$, implies $\varphi_b(P_i^4, P_{-i}) \geq \varphi_b(P_i, P_{-i})$. By our assumption, this means that $\varphi_b(P_i^4, P_{-i}) > \varphi_b(\bar{P}_i, P_{-i})$. Since $P_i^4(1) = \bar{P}_i(1)$ which contradicts that φ is tops-only. ■

C.7 Proof of Corollary 7.1

First, we state some important observations about betweenness domains which we will use in the proof.

Observation C.1. *Consider an alternative $x \in A$ and let $\mathscr{D}^x(\beta)$ be the set of all preferences with top-ranked alternative x and satisfying the betweenness condition β. Then, the domain $\mathscr{D}^x(\beta)$ is swap-connected.*

Observation C.2. *Let $x, y \in A$ and let $P \in \mathscr{D}(\beta)$ be such that $P(1) = x$ and $U(y, P) \cup y = \beta(x, y)$. Then, for all $\hat{P} \equiv x \cdots$, there is a swap-local path from $\hat{P}$ to P such that no alternative overtakes y along the path.*

Observation C.3. *Let $\mathscr{D}(\beta)$ be strongly consistent. Let $x, \bar{x} \in A$ and let $(x^1 = x, \ldots, x^t = \bar{x})$ be a sequence of adjacent alternatives in $\beta(x, \bar{x})$ such that for all $l < t$ and all $w \in \beta(x^l, \bar{x})$, we have $\beta(x^{l+1}, w) \subseteq \beta(x^l, \bar{x})$. Then, for all $l < t$, there exist $P \equiv x^l \cdots$ and $P' \equiv x^{l+1} \cdots$ such that $\beta(x^l, x^t)$ is an upper contour set in both P and P'. To see this, consider x^l. Since $\mathscr{D}(\beta)$ is strongly consistent, there is a preference $P \in \mathscr{D}(\beta)$ such that $\beta(x^l, x^t)$ is an upper contour set of P. Name the alternatives in $\beta(x^l, x^t)$ as $w_1, \ldots, w_u$ such that $\beta(x^{l+1}, w_r) \subsetneq \beta(x^{l+1}, w_s)$ implies $r < s$. Since $\mathscr{D}(\beta)$ is strongly consistent, we have $\beta(x^{l+1}, w) \subseteq \beta(x^l, x^t)$ for all $w \in \beta(x^l, x^t)$, and hence there is a preference P', graph-local to P, satisfying the betweenness relation β such that $P' \equiv w_1 w_2 \cdots w_{u-1} w_u \cdots$. Therefore, $U(w_u, P') \cup w_u = \beta(x^l, x^t)$.*

We are now ready to start the proof. To ease the presentation, for a path π, we denote by π^{-1} the path π in the reversed direction, that is, if $\pi = (P^1, P^2, \ldots, P^t)$, then $\pi^{-1} = (P^t, P^{t-1}, \ldots, P^1)$.

Proof of Corollary 7.1. Let $\mathscr{B}$ be a collection of strongly consistent and swap-connected betweenness relations. We show that $\mathscr{D}(\mathscr{B})$ satisfies the path-richness property.

First, we show $\mathscr{D}(\mathscr{B})$ satisfies Condition (i) of the path-richness property (see Definition 5.3). Consider P and P' with $P(1) = P'(1)$ that are not graph-local. If $P, P' \in \mathscr{D}(\beta)$

for some $\beta \in \mathscr{B}$, then by Observation C.1 there is a swap-local path from P to P' such that the top-ranked alternative does not change along the path. Suppose $P \in \mathscr{D}(\beta)$ and $P' \in \mathscr{D}(\hat{\beta})$ for some $\beta, \hat{\beta} \in \mathscr{B}$. Let $P(1) = P'(1) = x$ and let $(\beta^1 = \beta, \ldots, \beta^t = \hat{\beta})$ be a swap-local path. By the swap-connectedness of $\mathscr{B}$, there are swap-local preferences $P^1 \in \mathscr{D}(\beta^1)$ and $P^2 \in \mathscr{D}(\beta^2)$ with $P^1(1) = P^2(1) = x$. By Observation C.1, there is a swap-local path π^1 from P to P^1 in $\mathscr{D}(\beta^1)$ such that x remains at the top-position in all the preferences in the path. Thus, the path (π^1, P^2) from P to P^2 satisfies Condition (i) of the path-richness property. Continuing in this manner, we can construct a path from P to P' that satisfies Condition (i) of the path-richness property.

Now, we show $\mathscr{D}(\mathscr{B})$ satisfies Condition (ii) of the path-richness property, that is, for all $P, P' \in \mathscr{D}(\mathscr{B})$ with $P(1) = P'(1)$, if P and P' are graph-local, then for each preference $\hat{P} \in \mathscr{D}(\mathscr{B})$, there exists a graph-local path $(P^1 = \hat{P}, \ldots, P^v)$ with $P^v(1) = P(1)$ such that for all $l < v$ and all distinct $a, b \in P \triangle P'$, there is a common upper contour set U of both P^l and P^{l+1} such that exactly one of a and b is contained in U. Since P and P' are graph-local with $P(1) = P'(1)$, by means of the fact that the collection $\mathscr{B}$ is swap-connected, it follows that P and P' are swap-local. So assume that $P \equiv w \cdots yz \cdots$ and $P' \equiv w \cdots zy \cdots$. Consider $\hat{P} \in \mathscr{D}(\mathscr{B})$. Suppose $\hat{P}(1) = x$ and $y\hat{P}z$. Let $\hat{P} \in \mathscr{D}(\beta)$ for some $\beta \in \mathscr{B}$. We construct a path from $\hat{P}$ to a preference with w as the top-ranked alternative maintaining Condition (ii) of the path-richness property with respect to y and z in two steps. For ease of presentation, we denote $\hat{P}$ by P^1.

Step 1: Since β is strongly consistent, there is a sequence $(x^1 = x, \ldots, x^t = y)$ of adjacent alternatives in $\beta(x^1, x^t)$ such that for all $l < t$ and all $u \in \beta(x^l, x^t)$, $\beta(x^l, x^t) \supseteq \beta(x^{l+1}, u)$. By Observation C.2, there is a path π^1 from P^1 to a preference $\bar{P}^1$ with $\bar{P}^1(1) = x^1$ such that $U(x^t, \bar{P}^1) \cup x^t = \beta(x^1, x^t)$ and no alternative overtakes x^t along the path. Consider x^2. By Observation C.3, there is a preference P^2 with $P^2(1) = x^2$ such that P^2 is graph-local to $\bar{P}^1$ and $\beta(x^1, x^t)$ is an upper contour set in P^2. Since $z \notin \beta(x^1, x^t)$ and $\beta(x^1, x^t)$ is a common upper contour set of $\bar{P}^1$ and P^2, Condition (ii) of the path-richness property is satisfied with respect to x^t and z on the path $(\bar{P}^1, P^2)$. As in the case for P^1 and $\bar{P}^1$, by Observation C.2, we can construct a swap-local path π^2 from P^2 to some preference $\bar{P}^2$ with $\bar{P}^2(1) = x^2$ such that $U(x^t, \bar{P}^2) \cup x^t = \beta(x^2, x^t)$ and no alternative overtakes x^t

along the path. As in the case for $\bar{P}^1$ and P^2, by Observation C.3, there is a preference P^3 with $P^3(1) = x^3$ such that P^3 is graph-local to $\bar{P}^2$ and $\beta(x^2, x^t)$ is an upper contour set in P^3. It follows that the path (π^1, π^2, P^3) from P^1 to the preference P^3 satisfies Condition (ii) of the path-richness property with respect to x^t and z. Continuing in this manner, we can construct a path $\hat{\pi}$ in $\mathscr{D}(\beta)$ from $\hat{P}$ to a preference $\hat{\hat{P}}$ with $\hat{\hat{P}}(1) = y$ such that Condition (ii) of the path-richness property is satisfied along the path.

Step 2: Consider the preference $P \equiv w \cdots yz \cdots$. Let $P \in \mathscr{D}(\tilde{\beta})$ for some $\tilde{\beta} \in \mathscr{B}$. Using similar argument as in Step 1, we can construct a path $\tilde{\pi}$ in $\mathscr{D}(\tilde{\beta})$ from P to some $\tilde{P}$ with $\tilde{P}(1) = y$ such that Condition (ii) of the path-richness property is satisfied with respect to y and z.

Step 3: Since $\hat{\hat{P}}(1) = \tilde{P}(1) = y$ and the collection $\mathscr{B}$ is swap-connected, there is a swap-local path $\bar{\pi}$ in $\mathscr{D}(\mathscr{B})$ from $\hat{\hat{P}}$ to $\tilde{P}$ such that y stays as the top-ranked alternative in each preference of the path. Clearly, such a path will satisfy Condition (ii) of the path-richness property with respect to y and z.

Consider the path $(\hat{\pi}, \bar{\pi}, \tilde{\pi}^{-1})$ from $\hat{P}$ to P. By construction, this path satisfies Condition (ii) of the path-richness property with respect to y and z, which completes the proof. ∎

C.8 Proof of Theorem 7.1

Proof. Kumar et al. (2020) show that a domain $\mathscr{D}$ is graph-DLGE if and only if it satisfies the following property: for all distinct $P, P' \in \mathscr{D}$ and all $a \in A$, there exists a path π from P to P' with no (a,b)-restoration for all $b \in L(a,P)$. Here, a path is said to have no (a,b)-restoration if the relative ranking of a and b is reversed at most once along π. In what follows, we show that $\mathscr{D}(\mathscr{B})$ satisfies the above-mentioned property when $\mathscr{B}$ is weakly consistent and swap-connected. Consider two preferences $P \in \mathscr{D}(\beta)$ and $P' \in \mathscr{D}(\beta')$ for some $\beta, \beta' \in \mathscr{B}$ and $a \in A$. We show that there is a path π from P to P' that has no (a,x)-restoration for all $x \in L(a,P)$. By Observation C.3, from P and P' there are graph-local paths $\hat{\pi}$ and $\bar{\pi}$, respectively, to some preferences $\hat{P}$ and $\bar{P}$ with a as the top-ranked alternatives such that no alternative overtakes a along each of the paths. Let $\tilde{\pi}$ be a swap-local path joining $\hat{P}$ and $\bar{P}$ such that a remains the top-ranked alternative throughout the path. Consider the path $(\hat{\pi}, \tilde{\pi}, \bar{\pi}^{-1})$. No alternative in $L(a,P)$ overtakes

a along the path $\hat{\pi}$. So, if there is an (a,x)-restoration for some $x \in L(a,P)$ in the path $(\hat{\pi}, \tilde{\pi}, \bar{\pi}^{-1})$, then it must be that the restoration happens in the path $\bar{\pi}^{-1}$. However, then a must overtake x in this path, which means x overtakes a in the reversed path $\bar{\pi}$, which is not possible by the construction of the path $\bar{\pi}$. This completes the proof. ■

C.9 Proof of Proposition 7.2

Proof. Consider $X, \bar{X} \in A$. We show that there is a sequence $(X^1 = X, \ldots, X^t = \bar{X})$ of adjacent alternatives in $\beta(X, \bar{X})$ such that for all $l < t$ and all $W \in \beta(X^l, X^t)$, we have $\beta(X^{l+1}, W) \subseteq \beta(X^l, X^t)$. Let $l < t$ and consider $W \in \beta(X^l, X^t)$. We show $\beta(X^{l+1}, W) \subseteq \beta(X^l, X^t)$. Take $Z \notin \beta(X^l, X^t)$. Because Z does not lie in $\beta(X^l, X^t)$, there must be a pair (a,b) of objects such that either (i) a and b are together in both X^l and X^t, but separate in Z, or (ii) a and b are separate in both X^l and X^t, but together in Z. Because both X^{l+1} and W are in $\beta(X^l, X^t)$, it must hold that in case (i) a and b are together in both X^{l+1} and W, and in case (ii) they are separate in both X^{l+1} and W. In case (i), a and b are together in both X^{l+1} and W but they are separate in Z. Therefore, Z cannot lie in $\beta(X^{l+1}, W)$. On the other hand, in case (ii) a and b are separate in both X^{l+1} and W, but they are together in Z. Therefore, Z cannot lie in $\beta(X^{l+1}, W)$. This completes the proof. ■

C.10 Proof of Proposition 9.1

We first prove some lemmas which we later use in the proof of the proposition.

Lemma C.2. *Let $P \in \mathscr{S}$, $l \in K$, and $x, y \in A$ be such that $x^l P^l y^l$ and xPy. Then, for every component $j \neq l$ there is a sep-local path from P to a lexicographic preference $\bar{P} \in \mathscr{S}$ having same marginal preferences as P, and l and j as the lexicographic best and worst components, respectively, such that the x and y do not swap along the path.*

Proof. Assume without loss of generality, $l = 1$ and $j = m$. First, make the component 1 lexicographically best (without changing the marginal preferences of P) by swapping consecutively ranked alternatives multiple times in the following manner: each time swap a pair of consecutively ranked alternatives a and b where $a^1 P^1 b^1$ and bPa. Note that since $x^1 P^1 y^1$ and xPy, x and y are never swapped in this step. Having made 1 the lexicographically best component, the component 2 can be made lexicographically second-best in the following manner: each time swap a pair of consecutively ranked alternatives a and b in P where $a^1 = b^1$, $a^2 P^2 b^2$, and bPa. As we have explained for the case of component 1, alternatives x and y will not swap in this process. Continuing in this manner, we can finally obtain a preference $\bar{P}$ with lexicographic ordering over the components as $1\bar{P}^0 \cdots \bar{P}^0 k$ through a sep-local path along which the alternatives x and y are not swapped. ■

Lemma C.3. *Let $P \in \mathscr{S}$ be a preference such that xPy for some alternatives x and y that differ in at least two components. Then, there is a sep-local path $(P^1 = P, \dots, P^t = \hat{P})$ with $\hat{P}(1) = x$ such that $xP^l y$ for all $l < t$.*

Proof. Since xPy, there is a component l such that $x^l P^l y^l$. Assume without loss of generality $l = 1$. Consider component 2 . By Lemma C.2, there is a sep-local path π^1 from P to a preference $\bar{P}$ having components 1 and 2 as the lexicographic best and the worst components, respectively, such that x and y do not swap along the path. Since 2 is the lexicographically worst component of $\bar{P}$, we can construct a sep-local path from $\bar{P}$ to a preference $\bar{\bar{P}}$ such that (i) the marginal preferences in each component other than 2 and the lexicographic ordering over the components of each preference in the path remains the same as $\bar{P}$, and (ii) x^2 appears at the top-position of $\bar{\bar{P}}^2$. Since component 1 is the lexicographic best component in all these preferences and x^1 is preferred to y^1 in the marginal preference in component 1 for all these preferences, it follows that x remains ranked above y along the path. Repeating this process for all the components $3, \dots, k$, we can construct a path having no swap between x and y from P to a preference $\tilde{P}$ having (i) the same marginal preference as P in component 1, and (ii) x^t at the top-position of the marginal preference in component t for all $t > 1$.

Starting from the preference $\tilde{P}$, make component 1 lexicographically worst through a sep-local path without changing the marginal preferences. Since x^l is weakly preferred to y^l in each component l in each preference of this path, x will remain ranked above y throughout the path. Finally, move x^1 to the top-position in the marginal preference in component 1 thorough a(ny) swap-local path. Since x and y are different in at least two components, there is a component j lexicographically dominating component 1 (as it is the worst component) such that x^j is preferred to y^j in its marginal preference. Therefore, x will be ranked above y throughout the path. Note that in the final preference, for each component t, x^t appears at the top-position in the marginal preference in component t, and hence the alternative x appears at the top-position in it. ■

Lemma C.4. *Let $\varphi : \mathscr{S}^n \to \Delta A$ be a unanimous RSCF satisfying the block preservation property. Then $\varphi(P_N) = \varphi(\bar{P}_N)$ for all $P_N, \bar{P}_N$ such that $P^l_N = \bar{P}^l_N$ for all $l \in K$.*

Proof. It is enough to show that $\varphi(P_i, P_{-i}) = \varphi(\bar{P}_i, P_{-i})$ where $P^l_i = \bar{P}^l_i$ for all $l \in K$. Since preferences with the same marginals are swap-connected, we can assume without loss of generality that P_i and $\bar{P}_i$ are swap-local with the swap of alternatives x and y. Assume for contradiction $\varphi(P_i, P_{-i}) \neq \varphi(\bar{P}_i, P_{-i})$. By the block preservation property, this means $\varphi_x(P_i, P_{-i}) \neq \varphi_x(\bar{P}_i, P_{-i})$. By Lemma C.3, for all $j \in N \setminus i$, there is a sep-local path $(P^1_j = P_j, \dots, P^t_j = \bar{P}_j)$ with $\bar{P}_j(1) = P_i(1)$ satisfying the property that for all $l < t$ there is a common upper contour set U of both P^l_j and P^{l+1}_j such that exactly one of x and y is contained in U.[11] By Lemma C.1, we have $\varphi_x(P_i, \bar{P}_j, P_{-\{i,j\}}) \neq \varphi_x(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}})$. Continuing in this manner, we can construct $\bar{P}_{-i} \in \mathscr{S}^{n-1}$ such that $\bar{P}_j(1) = P_i(1)$ for all $j \neq i$ and $\varphi_x(P_i, \bar{P}_{-i}) \neq \varphi_x(\bar{P}_i, \bar{P}_{-i})$. However, since $(P_i, \bar{P}_{-i})$ and $(\bar{P}_i, \bar{P}_{-i})$ are unanimous preference profiles with the top-ranked alternative different from x, $\varphi_x(P_i, \bar{P}_{-i}) = \varphi_x(\bar{P}_i, \bar{P}_{-i}) = 0$, a contradiction. ■

Proof of Proposition 9.1. Let $\varphi : \mathscr{S}^n \to \Delta A$ be a unanimous RSCF satisfying the block preservation property. We show that φ satisfies component-unanimity. Consider $P_N \in \mathscr{S}^n$ such that $P^l_i(1) = x^l$ for all $i \in N$, some $l \in K$, and some $x^l \in A^l$. Assume for contradiction $\varphi^l_{x^l}(P_N) \neq 1$. Without loss of generality assume $l = 1$. By Lemma C.2 and Lemma C.4, we can assume that P_N is a profile of lexicographic preferences with each agent i having the component ordering $1 P^0_i \cdots P^0_i k$. Fix some alternative x^k in component k and consider some agent i. As we have argued in the proof of Lemma C.3, there is a sep-local path from P_i to a preference $\bar{P}_i$ such that each preference in the path has the same lexicographic ordering over the components as P_i, $\bar{P}^k_i(1) = x^k$, and $\bar{P}^l_i = P^l_i$ for all $l \neq k$. By construction, for all $x^{-k} \in A^{-k}$ and $y^k, z^k \in A^k$, each pair of alternatives $((x^{-k}, y^k), (x^{-k}, z^k))$ forms a block for any two consecutive (sep-local) preferences in

[11] Note that the statement of Lemma C.3 is slightly different from what we mention here. Since any two consecutive preferences in a sep-local path differ by swaps of multiple pairs of consecutively ranked alternatives, these two statements are equivalent.

the path. This in particular implies $\varphi^1_{x^1}(\bar{P}_i, P_{-i}) = \varphi^1_{x^1}(P_N)$. Continuing this way, we can construct $\bar{P}_N \in \mathscr{S}^n$ such that $\bar{P}^k_i(1) = x^k$ for all $i \in N$ and $\varphi^1_{x^1}(\bar{P}_N) = \varphi^1_{x^1}(P_N)$.

Let $\bar{\bar{P}}_N$ be the profile of lexicographic preferences that has same marginal preferences as $\bar{P}$ and has lexicographic ordering over the components as $1\bar{\bar{P}}^0_i \ldots \bar{\bar{P}}^0_i k \bar{\bar{P}}^0_i k-1$ for all $i \in N$. That is, the components $k-1$ and k are swapped from $\bar{P}^0_i$ to $\bar{\bar{P}}^0_i$. By Lemma C.4, $\varphi(\bar{\bar{P}}_N) = \varphi(\bar{P}_N)$. Now, by using similar logic as for component k, we can construct $\hat{P}_N \in \mathscr{S}^n$ such that $\hat{P}^{k-1}_i(1) = x^{k-1}$ for all $i \in N$ and $\varphi^1_{x^1}(\hat{P}_N) = \varphi^1_{x^1}(P_N)$. Continuing in this manner, we can arrive at $\tilde{P}_N \in \mathscr{S}^n$ such that $\tilde{P}^t_i(1) = x^t$ for all $t \in K$ and all $i \in N$ and $\varphi^1_{x^1}(\tilde{P}_N) = \varphi^1_{x^1}(P_N)$. However, since $\tilde{P}_N$ is unanimous with $\tilde{P}_i(1) = x$ for all $i \in N$, we have $\varphi_x(\tilde{P}_N) = 1$, which in particular implies $\varphi^1_{x^1}(\tilde{P}_N) = 1$, a contradiction. ■

C.11 PROOF OF PROPOSITION 9.2

Proof. Let $\varphi : \mathscr{S}^n \to \Delta A$ be a unanimous RSCF satisfying the block preservation property. We show that φ is tops-only. Consider $P_N, \bar{P}_N \in \mathscr{S}^n$ with $P_i(1) = \bar{P}_i(1)$ for all $i \in N$. If $P^l_N = \bar{P}^l_N$ for all $l \in K$, then we are done by Lemma C.4. It is sufficient to assume that only one agent, say i, changes her marginal preference to a swap-local preference in exactly one component, say t, and nothing else changes from P_N to $\bar{P}_N$. That is, P^t_i and $\bar{P}^t_i$ are swap-local with the swap of some y^t and z^t, $P^t_j = \bar{P}^t_j$ for all $j \in N \setminus i$, and $P^l_N = \bar{P}^l_N$ for all $l \neq t$. Assume without loss of generality, $t = k$. Furthermore, in view of Lemma C.4, let us assume that all agents have the same component ordering Q^0 in both P_N and $\bar{P}_N$ where Q^0 is given by $1Q^0 \ldots Q^0 k$. We need to show $\varphi(P_N) = \varphi(\bar{P}_N)$. Assume for contradiction $\varphi(P_N) \neq \varphi(\bar{P}_N)$. Since k is the worst component in P^0_i, by block preservation property, this implies $\varphi_{(x^{-k}, y^k)}(P_N) \neq \varphi_{(x^{-k}, y^k)}(\bar{P}_N)$ for some (x^{-k}, y^k).

Consider P^k_j for some $j \neq i$. By our assumption on the marginal domains, there is a swap-local path $\pi^k = (\pi^k(1) = P^k_j, \ldots, \pi^k(t) = \hat{P}^k_j)$ in $\mathscr{D}^k$ with $\hat{P}^k_j(1) = P^k_i(1)$ such that for any two consecutive preferences in the path there is a common upper contour set U such that exactly one of y^k and z^k is contained in U. By Remark 9.1, the path $((P^0_j, P^1_j, \ldots, P^{k-1}_j, \pi^k(1)), \ldots, (P^0_j, P^1_j, \ldots, P^{k-1}_j, \pi^k(t)))$ satisfies the property that for all $l < t$ and all $u, v \in P_i \triangle \bar{P}_i$ there is a common upper contour set U of both $(P^0_j, P^1_j, \ldots, P^{k-1}_j, \pi^k(l))$ and $(P^0_j, P^1_j, \ldots, P^{k-1}_j, \pi^k(l+1))$ such that exactly one of u and v is contained in U, and hence by Lemma C.1, we have $\varphi_{(x^{-k}, y^k)}(P_i, \hat{P}_j, P_{-\{i,j\}}) \neq$

$\varphi_{(x^{-k},y^k)}(\bar{P}_i,\hat{P}_j,P_{-\{i,j\}})$, where $\hat{P}_j = (P_j^0,P_j^1,\ldots,P_j^{k-1},\hat{P}_j^k)$. Continuing in this manner, we can construct $\hat{P}_{-i} \in \mathscr{S}^{n-1}$ such that for all $j \in N \setminus i$, $\hat{P}_j^k(1) = P_i^k(1)$ and $\hat{P}_j^l = P_j^l$ for all $l \neq k$, and $\varphi_{(x^{-k},y^k)}(P_i,\hat{P}_{-i}) \neq \varphi_{(x^{-k},y^k)}(\bar{P}_i,\hat{P}_{-i})$. Note that the preference profiles $(P_i,\hat{P}_{-i})$ and $(\bar{P}_i,\hat{P}_{-i})$ are component-unanimous for component k, and hence by Proposition 9.1, $\varphi^k_{P_i^k(1)}(P_i,\hat{P}_{-i}) = \varphi^k_{P_i^k(1)}(\bar{P}_i,\hat{P}_{-i}) = 1$. This implies $\varphi_{(x^{-k},y^k)}(P_i,\hat{P}_{-i}) = \varphi_{(x^{-k},y^k)}(\bar{P}_i,\hat{P}_{-i}) = 0$, which contradicts $\varphi_{(x^{-k},y^k)}(P_i,\hat{P}_{-i}) \neq \varphi_{(x^{-k},y^k)}(\bar{P}_i,\hat{P}_{-i})$. ■

C.12 Proof of Theorem 9.1

Proof. By means of Proposition 9.2, it is enough to prove Theorem 9.1 for every RSCF satisfying tops-onlyness and block preservation. Let $\varphi : \mathscr{S}^n \to \Delta A$ be a tops-only RSCF satisfying the block preservation property. We show that φ is marginally decomposable. Let $P_N, \bar{P}_N \in \mathscr{S}^n$ be such that $P_N^l = \bar{P}_N^l$ for some $l \in K$. Since φ is tops-only we assume without loss of generality that the l-th component is top-ranked according to the lexicographic ordering over the components in P_i and $\bar{P}_i$ for all $i \in N$. Consider an agent $j \in N$. Since component l is the lexicographic best component in both P_j and $\bar{P}_j$, for each $a^l \in A^l$, the set of alternatives $B(a^l) = \{(x^{-l},a^l) \mid x^{-l} \in A^{-l}\}$ can be expressed as a union of blocks in $(P_j,\bar{P}_j)$. Therefore, by applying the block preservation property to each block in $B(a^l)$, we obtain $\varphi_{B(a^l)}(P_j,P_{-j}) = \varphi_{B(a^l)}(\bar{P}_j,P_{-j})$ for all $a^l \in A^l$. Continuing in this manner, it follows that $\varphi_{B(a^l)}(P_N) = \varphi_{B(a^l)}(\bar{P}_N)$ for all $a^l \in A^l$. By the definition of marginal distribution, this means $\varphi^l(P_N) = \varphi^l(\bar{P}_N)$ which completes the proof. ■

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