A note on gensys' minimality

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Abstract

`gensys`' non-minimality is shown analytically and necessary and sufficient conditions for vector autoregression representations of states in outputs are presented.

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1. Introduction

Sims' [7] Matlab solution algorithm to linear rational expectation models is called `gensys`. Does it deliver minimal linear time invariant state space representations? Namely, is `gensys` *sufficient* for minimal linear time invariant state space representations? The *example* produced by Komunjer and Ng [4] shows that the answer is negative: $G \not\longrightarrow MR$, since $\exists x \in U$ such that $Gx \wedge \neg MRx$, in which $G \equiv$ `gensys`, $MR \equiv$ Minimal representation, $x \equiv$ counterexample and $U \equiv$ universe (i.e. domain of discourse). This note shows such analytically, presenting necessary and sufficient conditions for vector autoregression representations of states in outputs.

2. `gensys` state space, minimality and VAR

`gensys` gives rise to the unique and stable solution $[x_{1t}\ x_{2t}]^\top = [(A_{11}\ 0)\ \ (0\ 0)]^\top [x_{1t-1}\ x_{2t-1}]^\top + [B_{11}\ B_{21}]^\top u_t,\ \forall t \in \mathbb{Z},\ x_{1t} \in \mathbb{R}^{n_{x_1}},\ x_{2t} \in \mathbb{R}^{n_{x_2}},\ u_t \in \mathbb{R}^{n_u},\ A_{11} \in \mathbb{R}^{n_{x_1} \times n_{x_1}},\ B_{11} \in \mathbb{R}^{n_{x_1} \times n_u}$ and $B_{21} \in \mathbb{R}^{n_{x_2} \times n_u}$; x_{1t} is a vector of non-expectational variables, x_{2t} is a vector of expectational variables and u_t is a vector of inputs (i.e. shocks). Such solution is the transition equation of a linear time invariant state space representation in discrete time: $[x_{1t}\ x_{2t}]^\top = [(A_{11}\ 0)\ \ (0\ 0)]^\top [x_{1t-1}\ x_{2t-1}]^\top + [B_{11}\ B_{21}]^\top u_t \longleftrightarrow x_t = Ax_{t-1} + Bu_t,\ \forall x_t \in \mathbb{R}^{n_x},\ A \in \mathbb{R}^{n_x \times n_x}$ and $B \in \mathbb{R}^{n_x \times n_u}$; x_t is a vector states such that $n_x = n_{x_1} + n_{x_2}$.

Let $M \in \mathbb{R}^{n_y \times n_x}$ give rise to $Mx_t = MAx_{t-1} + MBu_t \longleftrightarrow y_t = Cx_{t-1} + Du_t,\ \forall y_t \in \mathbb{R}^{n_y},\ C \in \mathbb{R}^{n_y \times n_x}$ and $D \in \mathbb{R}^{n_y \times n_u}$. It is the measurement equation of a linear time invariant state space representation in discrete time, in which y_t is a vector of outputs; M is called measurement matrix.

Linear time invariant state space representations are minimal *if and only if* rank $r_{\mathcal{C}} = r_{\mathcal{O}} = n_x$ for controllability matrix $\mathcal{C} = \left[\cdots\ A^{n_x-1}B\right]$ and observability matrix $\mathcal{O} = \left[\cdots\ CA^{n_x-1}\right]^\top$. Non-minimal representations can be reduced to minimal ones by the Kalman decomposition: the economic interpretation is invariant (see Franchi [2]). Assume that the representation be minimal: $x_{mt} = A_m x_{mt-1} + B_m u_t$ and $y_t = C_m x_{mt-1} + Du_t$.

Assume that D be non-singular and thus square: $n_y = n_u$. Solve the measurement equation for u_t and plug it into the transition equation: $x_{mt} = \left(A_m - B_m D^{-1} C_m\right) x_{mt-1} + B_m D^{-1} y_t = F_m x_{mt-1} + B_m D^{-1} y_t$; notice that $F_m \equiv A_m - B_m D^{-1} C_m$. Solve it backwards, satisfying causality: $x_{mt} = \sum_{j=0}^{\infty} F_m^j B_m D^{-1} y_{t-j}$ *if and only if* F_m is stable, namely, F_m's characteristic polynomial eigenvalues are less than one in

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modulus, $|\lambda_{F_m(\lambda)}| < 1$ for $F_m(\lambda) = F_m - \lambda I$ in $det\,[F_m(\lambda)] = 0$. Plug this into the measurement equation: $y_t = \sum_{j=0}^{\infty} F_m^j B_m D^{-1} y_{t-j-1} + D u_t$.

Thus: there exists a vector autoregression of infinite order $VAR(\infty)$ *if and only if* F_m is stable; there exists a vector autoregression of finite order $VAR(k)$ for $k < \infty$ *if and only if* F_m is nilpotent, namely, F_m's characteristic polynomial eigenvalues are zero, $\lambda_{F_m(\lambda)} = 0$. See Franchi [2], Franchi and Paruolo [3], Fernández-Villaverde *et al.* [1], Ravenna [6] and Franchi and Vidotto [4] for further detail.

3. Symmetric case

Let x_{1t} be symmetrically semi-measurable, namely, let half of its rows be measurable: $x_t = [x_{M1t}\; x_{N1t}\; x_{2t}]^\top$ such that $n_{x_{M1}} = n_{x_{N1}}$, $A = [(A_{11_{11}}\; A_{11_{12}}\; 0)\; (A_{11_{21}}\; A_{11_{22}}\; 0)\; (0\;0\;0)]^\top$, $B = [B_{11_{11}}\; B_{11_{21}}\; B_{21}]^\top$, $M = [1\;0\;0]$, $y_t = x_{m1t}$, $C = [A_{11_{11}}\; A_{11_{12}}\; 0]$ and $D = B_{11_{11}}$. Record $r_{\mathcal{C}}$ for $\mathcal{C}$ and $r_{\mathcal{O}}$ for $\mathcal{O}:\; n_x = r_{\mathcal{C}} = 3 > r_{\mathcal{O}} = 2$, thus, the representation is controllable, non-observable and therefrom non-minimal.

Reduce the representation to minimality by the Kalman decomposition: construct similarity transformation matrix $\mathcal{T} = [\mathcal{O}_{r_{\mathcal{O}}}\; v_{n_x - r_{\mathcal{O}}}]^\top$ such that $\bar{x}_{co\bar{o}t} = \mathcal{T}^{-1} x_t$, $\bar{A}_{co\bar{o}} = \mathcal{T}^{-1} A \mathcal{T}$, $\bar{B}_{co\bar{o}} = \mathcal{T}^{-1} B$, $\bar{C}_{co\bar{o}} = C\mathcal{T}$, $\bar{\mathcal{C}}_{co\bar{o}} = \mathcal{T}^{-1}\mathcal{C}$ and $\bar{\mathcal{O}}_{co\bar{o}} = \mathcal{O}\mathcal{T}$; select the first $r_{\mathcal{O}} = 2$ states such that $\bar{x}_{cot} = \bar{x}_{mt}$, $\bar{A}_{co} = \bar{A}_m$, $\bar{B}_{co} = \bar{B}_m$, $\bar{C}_{co} = \bar{C}_m$, $\bar{\mathcal{C}}_{co} = \bar{\mathcal{C}}_m$ and $\bar{\mathcal{O}}_{co} = \bar{\mathcal{O}}_m$.

Computing F_m, $F_m(\lambda)$ and $|\lambda_{F_m(\lambda)}|$, F_m first eigenvalue matrix $\Lambda_1 \equiv \lambda_{1Fm(\lambda)} = -[A_{11_{12}} B_{11_{21}} - A_{11_{22}} B_{11_{11}}] B_{11_{11}}^{-1}$ and F_m second eigenvalue matrix $\Lambda_2 \equiv \lambda_{2Fm(\lambda)} = \mathbf{0}$; notice that $A_{11_{12}} \in \mathbb{R}^{n_{x_{M1}} \times n_{x_{N1}}}$, $B_{11_{21}} \in \mathbb{R}^{n_{x_{N1}} \times n_u}$, $A_{11_{22}} \in \mathbb{R}^{n_{x_{N1}} \times n_{x_{N1}}}$, $B_{11_{11}} \in \mathbb{R}^{n_{x_{M1}} \times n_u}$. Thus, there exists a $VAR(k)$, $\forall k \leq \infty$, of x_t in y_t *if and only if* $|\lambda_{\Lambda_1(\lambda)}| \in [0,\, 1)$ for $\Lambda_1(\lambda) = \Lambda_1 - \lambda I$ in $det\,[\Lambda_1(\lambda)] = 0$.

Such `gensys` condition is necessary and sufficient for a vector autoregression representation of the states in the outputs in the symmetric case, furthering $|\lambda_{F_m(\lambda)}| \in [0,\, 1)$ and acting as the analytical *counterexample* to the syntactic implication 'Minimal linear time invariant state space representations *if* `gensys`'.

4. Complete and asymmetric case

Let x_{1t} be fully measurable, namely, let all of its rows be measurable: $M = [1\;0]$, $y_t = x_{1t}$, $C = [A_{11}\; 0]$ and $D = B_{11}$. Record $r_{\mathcal{C}}$ for $\mathcal{C}$ and $r_{\mathcal{O}}$ for $\mathcal{O}:\; n_x = r_{\mathcal{C}} = 2 > r_{\mathcal{O}} = 1$, thus, the representation is controllable, non-observable and therefrom non-minimal.

Reduce the representation to minimality by the Kalman decomposition: construct $\mathcal{T} = [\mathcal{O}_{r_{\mathcal{O}}}\; v_{n_x - r_{\mathcal{O}}}]^\top = [(A_{11}\; 0)\; (0\; 1)]^\top$ and proceed as before, selecting the first $r_{\mathcal{O}} = 1$ states, so that $[x_{mt}\; y_t]^\top = [A_m\; C_m]^\top x_{mt-1} + [B_m\; D]^\top u_t \longleftrightarrow [A_{11}^{-1} x_{1t}\; x_{1t}]^\top = [A_{11}\; A_{11}^2]^\top A_{11}^{-1} x_{1t-1} + [A_{11}^{-1} B_{11}\; B_{11}]^\top u_t$.

Computing F_m, $F_m(\lambda)$ and $|\lambda_{F_m(\lambda)}|$, $\lambda_{F_m(\lambda)} = F_m = A_{11} - A_{11}^{-1} B_{11} B_{11}^{-1} A_{11}^2 = \mathbf{0}$. Thus, there exists a $VAR(k)$, $\forall k < \infty$, of x_t in y_t.

The scenario of x_{1t} asymmetric semi-measurability, namely, $n_{x_{M1}} \neq n_{x_{N1}}$, is best studied case by case.

5. Conclusion

This note's conclusion prescribes the reduction of `gensys`' representation to minimality as hereby shown.

References

[1] Fernández-Villaverde J., Rubio-Ramírez J., Sargent T. and Watson M. (2007), "ABCs (and Ds) of understanding VARs", American economic review.

[2] Franchi M. (2013), "Comment on: Ravenna F. 2007. Vector autoregressions and reduced form representations of DSGE models. Journal of Monetary Economics 54, 2048-2064.", Dipartimento di scienze statistiche empirical economics and econometrics working papers series.

[3] Franchi M. and Paruolo P. (2014), "Minimality of state space solutions of DSGE models and existence conditions for their VAR representation", Computational economics.

[4] — and Vidotto A. (2013), "A check for finite order VAR representations of DSGE models", Economics letters.

[5] KOMUNJER I. AND NG S. (2011), "Dynamic identification of dynamic stochastic general equilibrium models", Econometrica.
[6] RAVENNA F. (2007), "Vector autoregressions and reduced form representations of DSGE models", Journal of monetary economics.
[7] SIMS C. (2001), "Solving linear rational expectations models", Computational economics.

APPENDIX

MATLAB commands for symmetric case.

```
% gensys state space (symmetric case)
syms a1111 a1112 a1121 a1122 b1111 b1121 b21
A=[a1111 a1112 0; a1121 a1122 0; zeros(1,3)];
B=[b1111; b1121; b21];
M=[1 0 0]; C=M*A; D=M*B;

% Controllability and observability
Con=[B A*B A*A*B];
fprintf('Controllability matrix rank')
rc=rank(Con)
Obs=[C; C*A; C*A*A];
fprintf('Observability matrix rank')
ro=rank(Obs)

% Similarity transformation
v=[0 0 1];
T=[Obs(1:2, 1:3); v];
invT=inv(T);

% Canonical and minimal decomposition
Ad = invT*A*T;
Bd = invT*B;
Cd = C*T;
Am = [Ad(1:2, 1:2)];
Bm = [Bd(1:2, 1:1)];
Cm = Cd(1:1, 1:2);

% Minimal controllability and observability
Conm=[Bm Am*Bm];
fprintf('Minimal controllability matrix rank')
rcm=rank(Conm)
Obsm=[Cm; Cm*Am];
fprintf('Minimal observability matrix rank')
rom=rank(Obsm)

% Minimal VAR representation
Fm=Am-Bm*inv(D)*Cm;
fprintf('Minimal VAR representation condition eigenvalues')
lambdas_Fm=eig(Fm)
```