Numerical Analysis of the Disequilibrium Monetary Growth Model: Secular Stagnation, Slow Convergence, and Cyclical Fluctuations

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Abstract

This study presents a monetary disequilibrium growth model and conducts numerical simulations to investigate how dynamic paths are affected by the initial conditions and the parameters of expectation formation. The main results are as follows. First, dynamic properties such as stable convergence and cyclical fluctuations depend on the type of expectation formation rather than on the initial regimes. Stable convergence takes an excessively long time when expectation formation is too rational and cyclical fluctuations appear when it is too adaptive. Second, when the economy converges to the steady state (i.e., the Walrasian equilibrium), persistent Keynesian unemployment is likely to appear along the dynamic path. Third, the dynamics of inflation expectation that contain the price dynamics in the feedback loop might play an important role in convergence to the steady state.

Keywords: Disequilibrium macroeconomics, Non-Walrasian analysis, Economic growth, Simulation

JEL:E12,E17,E40,O42

1 Introduction

The financial crisis of 2008 and resulting Great Depression have had huge impacts on macroeconomics as well as the real economy (Summers, 2014). In particular, persistent stagnation, that is, the downward divergence of GDP growth from its trend, and persistent unemployment are often referred to as "secular stagnation" by Summers (Summers, 2014, 2015). Hence, the study of secular stagnation has become the main subject of macroeconomics.

New Keynesian economics, the mainstream notion of macroeconomics, regards unemployment under secular stagnation as Keynesian unemployment (KU). Numerous studies have examined the mechanism of secular unemployment. The basic framework of New Keynesian economics is the dynamic stochastic general equilibrium (DSGE) model in

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which deviation from the optimal growth path arises from a stochastic shock. New Keynesian economics is thought to succeed in justifying economic policies under economic shocks by adding price rigidity into the market mechanism.[1] After the occurrence of stagnation, New Keynesian economics pays attention to the zero lower bound (ZLB) of nominal interest rates and attempts to explain secular stagnation by the decrease in the natural (real) interest rate that achieves full employment and the ZLB of nominal interest rates that prevents the adjustment of bond markets.[2]

However, as Palley (2019) points out, the lower bound of the interest rate is essentially equivalent to price rigidity; hence, the analysis of the ZLB is Pigouvian rather than Keynesian. Therefore, it is inappropriate to call the analysis of the ZLB "Keynesian." In addition, he shows the possibility of persistent unemployment due to demand shortage, even without the ZLB. For this reason, we need an alternative DSGE model to analyze depressions and reveal the mechanism of persistent depressions that cannot be explained by price rigidity.

Stiglitz (2018) criticizes the framework of the DSGE model and presents an alternative quantity-constrained model (see the appendix of his paper). In his model, supply and demand are adjusted through quantity adjustments under price rigidity; hence, the quantity of supply that the supply side expresses is not necessarily realized. Therefore, economic agents do not know realized employment and realized output in advance. For this reason, his model can directly represent involuntary unemployment such that labor demand is less than labor supply.

The quantity adjustment model of Stiglitz (2018), a kind of disequilibrium model, is based on the interpretation of Keynes (1936) by Clower (1965) and Leijonhufvud (1967). As a general disequilibrium model that presents the relationships among markets under disequilibrium, refer to Barro and Grossman (1971), Bénassy (1975), and Malinvaud (1977). In these models, the planned demand and planned supply of economic agents are not necessarily realized, and realized demand and realized supply are determined by the short-side rule. For example, when the planned goods supplied by firms under prevailing prices are higher than the goods demand expressed by households, the actual quantity is equal to the goods demand of households. In this case, goods supply is under quantity rationing; hence, firms need to change labor demand according to the quantity constraint. Then, in the labor market, labor demand can be less than the labor supplied by households. In this way, involuntary unemployment occurs, called KU in terms of disequilibrium economics. Since the determining factor of labor demand is goods demand rather than wages, disequilibrium economics can represent unemployment by a different mechanism than that of New Keynesian economics. On the contrary, when wages are too high, labor demand depends on wages, and this situation is called classical unemployment (CU).

Although disequilibrium analysis was fascinating, it was insufficiently studied and evaluated; hence, it was replaced by equilibrium analysis, including the use of the DSGE model (Backhouse and Boianovsky, 2012). However, disequilibrium analysis has recently been cited by New Keynesian economists that have investigated economic stagnation.[3]

[1] For theoretical and empirical contributions, see Smets and Wouters (2003) and Christiano et al. (2005).

[2] The idea of the ZLB is rooted in the study of Krugman (1998), who discusses the Japanese depression, with Summers (2014, 2015) spreading the idea. Eggertsson and Krugman (2012), Benigno and Fornaro (2018), and Eggertsson et al. (2019) also present New Keynesian models of the ZLB.

[3] For example, see Eggertsson et al. (2019) and Dupor et al. (2019). Schoder (2017, 2020) presents disequilibrium models that are extensions of the DSGE model.

Nevertheless, these models cannot reproduce KU in the disequilibrium analysis because the labor demand shortage represented by firms arises from nominal rigidity or the ZLB as opposed to the shortage of goods demand. In addition, the consumption behavior of households is independent of the demand–supply gap of employment; hence, those models cannot express the leakage effects between markets arising from disequilibrium. Accordingly, they analyze CU rather than KU, which may distort the analysis of disequilibrium economics.[4]

On the contrary, Ogawa (2020) presents a monetary growth model based on a disequilibrium framework and shows that persistent KU (i.e., secular stagnation) may occurs along the transitional dynamics toward the steady state. According to traditional disequilibrium analysis, his model represents the dual-decision hypothesis such that demand for and the supply of a market is determined by the quantity constraints of other markets, showing that depending on whether the goods market is demand-led or supply-led, changes in the price variables and capital–labor ratio accelerate or decelerate, which he calls the "dual-decision effect." From this, wages become rigid under the KU regime and capital accumulation becomes unstable. As a result, his study shows that goods demand is stagnant at relatively low levels and that KU continues without resorting to the ZLB or wage rigidity.[5] He shows the above results using numerical simulations. However, he checks only one initial value that produces a persistent KU path and uses relatively high parameters of expectation adjustment that correspond to adaptive expectation. Therefore, his analysis does not sufficiently show the robustness of KU paths that correspond to secular stagnation.

Based on the above discussion, this study extends Ogawa's (2020) numerical simulations to investigate in detail how the dynamic paths are affected by the initial conditions and parameters of expectation adjustment. It reveals that (1) persistent KU does not depend on the rationality of expectation adjustment and initial regimes, that is, where the economy is initially located (the KU regime, CU regime, repressed inflation (RI) regime, and equilibrium) and (2) the model shows remarkable behaviors when expectation adjustment is either extremely adaptive or extremely rational.

The remainder of this paper is organized as follows. Section 2 presents our model, which is largely based on Ogawa (2020). Section 3 presents the numerical simulations conducted to investigate how the deviation of the initial values from the steady state, regime in which an economy is located, and parameters of expectation adjustment affect the dynamic paths. In particular, we examine whether an economy converges to a steady state and shows monotonic behavior or cyclical fluctuations. Section 4 discusses the remarkable dynamic paths obtained by the numerical simulations. Section 5 summarizes the analysis and presents future research issues on secular stagnation and expectation adjustment.

2 The model

In this section, we construct the model. For the details of the model, see Ogawa (2020). In our model, identical households, the representative firm, and the government trade labor, goods, and assets (money, bonds, and equity). We suppose that the goods price

[4]Backhouse and Boianovsky (2012) state that standard textbooks of macroeconomics distortedly explain the history of Keynesian economics as including disequilibrium analysis.

[5]Along the path on which KU continues, both nominal and real interest rates are positive and wages are lower than steady-state wages.

and nominal wage are fixed in the short term, while demand–supply gaps can emerge in the goods and labor markets. Meanwhile, each asset holding demand matches its supply since we suppose that the nominal interest rate and price of capital are adjusted immediately.

2.1 Static analysis

We suppose that identical households exist and that their population L^s grows at a constant rate $n > 0$. They hold aggregate real assets A:

$$A = (M + B + V)/P = (M + B)/P + qK, \tag{2.1}$$

where M is money holdings, B is the government-issued bond, V is the nominal value of the firm, and P is the price. q is the real market value of the capital that the firm holds K.

The households supply all labor L^s and allocate their disposable income to consumption and saving. The perceived real disposable income Y_{di} is as follows:[6]

$$Y_{di} = (1 - \tau)(Y - \delta K) - \pi(M + B + V)/P, \tag{2.2}$$

where $\tau > 0$ is the constant income tax ratio, $\delta > 0$ is the constant capital depreciation rate, and π is the expected inflation rate. The households express consumption demand C^d and saving $\dot{A}^d$ satisfying $C^d + \dot{A}^d = Y_{di}$. We suppose that the average propensity-to-consume function f^c is as follows:

$$C^d = f^c(A, r - \pi, Y_{di})Y_{di},\ 0 < f^c < 1,\ f_1^c > 0,\ f_2^c < 0,\ -f^c/Y_{di} < f_3^c \leq 0, \tag{2.3}$$

where r is the nominal interest rate. f_1^c, f_2^c, and f_3^c show the real balance (Pigou) effect on consumption, intertemporal substitution effect, and substitution effect on current income, respectively. Consumption demand depends on *realized* income. In other words, the household can revise its consumption based on realized employment and dividends. Hence, their decision is a dual decision.

The households determine their portfolio of A. Each asset demand is supposed to satisfy the following equations:

$$M^d/P = f^m(r, Y, A), \tag{2.4}$$
$$(B^d + V^d)/P = f^b(r, Y, A), \tag{2.5}$$
$$(M^d + B^d + V^d)/P = A. \tag{2.6}$$

The representative firm uses holding capital K and employs labor E to produce Y. We omit the firm's debt and inventories. The firm pays real wage w to employees and dividend ρK to asset holders, where ρ is the real net profit rate.[7] Therefore, the following equation holds:

$$Y = F(K, E) = wE + \rho K + \delta K, \tag{2.7}$$

where F is a standard neoclassical production function that is linear homogeneous and the Inada condition holds.

[6] This formulation is the same as that in Sargent (1987).

[7] As the households are identical, they are employees as well as asset holders.

The firm maximizes profit so that it solves the following problem:

$$\max_E F(K,E) - wE \text{ subject to } F(K,E) \le Y^d \text{ and } w, K \text{ are given.} \tag{2.8}$$

The firm controls production under goods demand Y^d. The solution for E is labor demand:

$$L^d = \min\{L^{d*}, \tilde{L}^d\}, \text{ where } L^{d*} = (F')^{-1}(w;K) \text{ and } \tilde{L}^d = F^{-1}(Y^d;K). \tag{2.9}$$

L^{d*} is an ordinal first-order condition without a quantity constraint. We call this *nominal* labor demand. $\tilde{L}^d$ is called *effective* labor demand, which is determined by the quantity of goods demand. For convenience, we define goods supply as follows:

$$Y^s = \min\{F(K, L^{d*}),\ F(K, L^s)\}, \tag{2.10}$$

where $F(K, L^s)$ is the physical production capacity.

The firm also purchases goods for investment I by issuing new equities. The firm decides I depending on the real market value of holding capital q. When q is larger (smaller) than 1, capital accumulation is faster (slower) than the balanced growth path:

$$I = \dot{K} + \delta K = \psi(q-1)K + (n+\delta)K, \quad \psi(0) = 0, \quad \psi > -(n+\delta), \quad \psi' > 0. \tag{2.11}$$

We suppose that q is determined as follows:[8]

$$q = q(\rho, Y^d/Y^s, r-\pi), \ q_1 > 0, \ q_2 > 0, \ q_3 < 0. \tag{2.12}$$

The market value of capital q depends on the real return rate of capital relative to bonds as well as on excess demand in the goods market. The former accords with standard neoclassical theory. The latter shows that capital accumulation is promoted when goods demand is ample, which is often assumed in disequilibrium models such as those of Böhm (1978) and Malinvaud (1980).

q is assumed to be equal to the normal q function in such equilibrium theories as those presented by Yoshikawa (1980) and Hayashi (1982) when all the factors of goods demand and supply match.

Assumption 1. q in equation (2.12) satisfies the following condition:

$$q(\rho, 1, r-\pi) = 1 \Leftrightarrow \rho = r - \pi + \xi, \tag{2.13}$$

where $\xi > 0$ is the constant risk premium.

The government purchases goods G and pays net real interest rB/P by collecting real tax T and issuing bonds and money. We assume that the size of G is proportional to the size of the economy measured by K: $G = gK$, where $g > 0$ is constant. The budget constraint is

$$G + rB/P = T + \dot{B}/P + \dot{M}/P. \ \ \dot{M}/M = \mu > 0, \quad \mu = \text{const.} \tag{2.14}$$

Taxation is assumed to be the household's net real income cash flow plus net real interest:

$$T = \tau(Y - \delta K) + rB/P. \tag{2.15}$$

[8]Furthermore, q satisfies the no-arbitrage condition in the asset market. That is, q is also determined to match demand to supply.

Goods demand consists of consumption demand and public demand:

$$Y^d = C^d + I + G. \tag{2.16}$$

For simplicity, the rationing of goods demand only affects consumption. That is, investment and government purchases are not rationed.

We next describe the transaction in the short term. The quantity adjustment dominates in real markets and the price adjustment dominates in asset markets. As capital K is given in the short term, we use the intensive form description by dividing the variables by K. Note that $y = Y/K$, $c^d = C^d/K$, $i = I/K$, $l^j = L^j/K$, $e = E/K$, $f(e) = F(1, e)$, $m = M/(PK)$, and $b = B/(PK)$.

The temporary equilibrium is the set (y, e, q, r) that satisfies the following conditions under fixed $(l^s, m, b, w, \pi) \in \mathbb{R}^4_{++} \times \mathbb{R}$:

$$y = \min\{y^d, f(l^{d*}), f(l^s)\}, \tag{2.17}$$

$$e = \min\{l^{d*}, \tilde{l}^d, l^s\}, \tag{2.18}$$

$$q = q(y - we - \delta, y^d/y^s, r - \pi), \tag{2.19}$$

$$m = m^d(r, y). \tag{2.20}$$

Proposition 1 in Ogawa (2020) shows the unique existence of a temporary equilibrium. Following his interpretation, we use the IS-LM model to describe the temporary equilibrium. The variables in the conditions above can be reduced to (y, r):

$$y = \min\{y^d(e(y), q(y, r)), y^s\}, \tag{2.21}$$

$$m = m^d(r, y). \tag{2.22}$$

The former can be interpreted as the IS condition, while the latter is the LM condition:

$$r_{IS}(y; m, b, w, \pi) = r_{LM}(y; m). \tag{2.23}$$

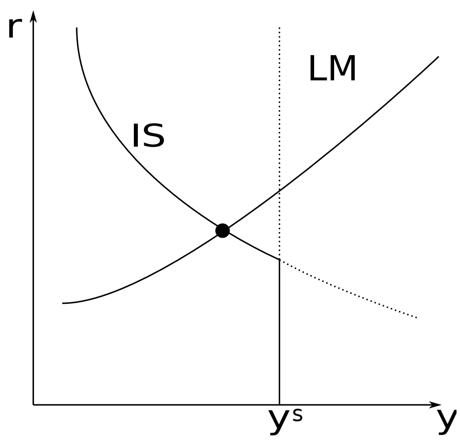

Figure 1: IS-LM interpretation of the temporary equilibrium (cited from Ogawa (2020))

Figure 1 shows the two curves. The IS curve is downward sloping when $y = y^d$ and vertical when $y = y^s$. The LM curve is upward sloping. The change in (l^s, m, b, w, π) shifts and transforms the curves.

What happens when the LM curve crosses the downward-sloping segment of the IS curve? In this case, production is determined by goods demand, as in the ordinal IS-LM model. Therefore, effective labor demand determines employment since $y = f(e)$ always holds, and the dominating factor of the goods and labor markets must match. That is, $y = y^d$ and $e = \tilde{y}^d$ hold. By definition, $e \leq l^s$, and this inequality strictly holds as long as the crossing point is different than the kink of the IS curve. This means that involuntary unemployment occurs because of the goods demand disequilibrium.

Next, consider the case in which the crossing point is on the vertical segment of the IS curve. In the vertical segment, $y = y^s = \min\{l^{d*}, l^s\}$ holds. Here, let us suppose that $l^{d*} < l^s$ holds because of the high real wages. Then, employment is $e = l^{d*} < \tilde{l}^d, l^s$ and unemployment occurs. Note that the unemployment mechanism is different than the former one.

To distinguish the two unemployment mechanisms, we use the following terms:[9]

- **Keynesian unemployment (KU)**
  The shortage of goods demand induces unemployment. The firm intends to raise employment under the prevailing wage since $e < l^{d*}$ holds, but it cannot because goods demand is too low.
$$y^d \leq y^s = \min\{f(l^{d*}), f(l^s)\}. \tag{2.24}$$
- **classical unemployment (CU)**
  The rigidity of nominal wages directly induces unemployment.
$$y = f(l^{d*}) \leq y^d, f(l^s). \tag{2.25}$$
- **repressed inflation (RI)**
  Excess demand occurs in the goods and labor markets.
$$y = f(l^s) < y^d, f(l^{d*}). \tag{2.26}$$
- **Equilibrium**
  Demand and supply match in both markets.
$$y = y^d = f(l^s). \tag{2.27}$$
  In particular, we call the situation in which all the demand factors and supply factors match the **Walrasian equilibrium (WE)**. The WE occurs when $f(l^s) = f(l^{d*})$ holds. In the WE, notional demand and effective demand match.

The variables (l^s, m, b, w, π) determine the regime of the temporary equilibrium. Figure 2 shows the simplified regime divisions. The equilibrium lies between KU and RI, and the WE is at the center of the disequilibrium regions.

2.2 Dynamic analysis

The variables (l^s, m, b, w, π) change over time. Their dynamic systems are as follows:
$$\dot{l}^s = l^s(n - \psi(q-1) - n) = -l^s\psi(q-1), \tag{2.28}$$
$$\dot{m} = m\{\mu - \pi - \nu_P(y^d - y^s) - \psi - n\}, \tag{2.29}$$
$$\dot{b} = \{g - \tau(y - \delta) - \mu m\} - b\{\pi + \nu_P(y^d - y^s) + \psi + n\}, \tag{2.30}$$
$$\dot{w} = w\{\nu_W(l^{d*} - l^s) - \nu_P(y^d - y^s)\}, \tag{2.31}$$
$$\dot{\pi} = \beta[\alpha\nu_P(y^d - y^s) + (1-\alpha)(\mu - n - \pi)], \tag{2.32}$$

[9] These names follow Malinvaud (1977).

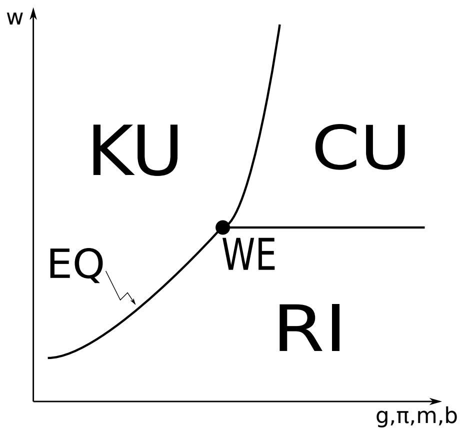

Figure 2: Regime divisions on the g, π, m, b - w plane (cited from Ogawa (2020))

where $\nu_j > 0$, $\alpha \in [0,1]$, and $\beta > 0$ are constants. The adjustment of w follows the Phillips curves of nominal wages and prices. The adjustment of π is composed of adaptive and rational adjustment. That is, the current price dynamics $\nu_P(y^d - y^s)$ and steady-state value of the inflation rate $\mu - n$ affect the dynamics. If $\alpha = 0$, the adjustment is completely rational in that the change from the current price dynamics can be ignored. If $\alpha = 1$, the adjustment process depends only on the gap between the current inflation rate and π, so that it is completely adaptive.

Our dynamic system contains the variable y or y^s, which means that the equations vary when the regime changes. Because it is discontinuous on the boundaries of regimes, the dynamic system is called "discontinuous on the right-hand side" or "piecewise continuous."

The steady state of our dynamic system is defined as the point at which the right-hand sides of all the equations above become zero:[10]

$$y^d = y^s, \tag{2.33}$$
$$l^{d*} = l^s, \tag{2.34}$$
$$q = 1, \tag{2.35}$$
$$\pi = \mu - n, \tag{2.36}$$
$$0 = g - \tau(y - \delta) - \mu(m + b). \tag{2.37}$$

Proposition 2 in Ogawa (2020) shows that the set of variables that satisfies these equations $(l_0^s, m_0, b_0, w_0, \pi_0)$ exists uniquely. In the steady states, the economy is in the WE and the Wicksellian equilibrium $\rho_0 = r_0 - \pi_0 + \xi$. Every stock variable grows at the population growth rate, and thus a balanced growth path also emerges in the steady state.

One main issue of dynamic analysis is how the economy grows when an initial state is given. We could interpret the initial state value as the value just after a certain economic shock. Then, we are interested in what disequilibrium regime appears after the shock, whether the initial disequilibrium regime switches to another regime, and how long it takes to converge to the steady state after the shock.

[10]As the system is discontinuous, the dynamics might stop on the boundaries of the regimes. This is called the pseudo equilibrium (Filippov, 1988). In the numerical experiments, the pseudo equilibrium was not detected.

One problem is that because our system has a high order, we cannot use graphical deductions. Furthermore, the system is discontinuous: linear approximation does not work properly in the neighborhood of the boundaries. Therefore, we must use numerical experiments to simulate the paths.

Before presenting the numerical experiments, we roughly deduce how the variables would work in the dynamics. Here, we check the dynamic feedback loops, which are treated in Chiarella et al. (2000), Chiarella et al. (2005), and Asada et al. (2006). First, *the Pigou and Keynes effects* stabilize the price dynamics: the increase in the price decreases goods demand through the LM market and the increase in real asset holdings, and then the price declines due to excess goods supply. Second, *the Mundell* effect destabilizes the inflation dynamics: the increase in π increases goods demand since current purchases of goods become attractive, and then the inflation pressure increases π as long as expectation adjustment is not completely rational, or $\alpha \neq 0$. Third, *the real wage effect* is ambiguous in the wage dynamics: since both goods demand and goods supply decrease in the real wage, the price inflation pressure is ambiguous.

These effects are the same as in ordinal Keynesian monetary growth models such as Chiarella and Flaschel (2000b). However, we have another effect on dynamics: *the dual-decision effect*. This is intrinsic to non-Walrasian regime-switching economic dynamics since the economy would be both demand-led and supply-led.

The dual-decision effect emphasizes the "effective" goods demand principle. In the dual-decision hypothesis, economic agents decide their expressions of demand and supply depending on *realized* production:

$$Y^d = Y^d(Y), \quad Y = \min\{Y^d, Y^s\}. \tag{2.38}$$

When production is determined by demand ($Y = Y^d$), goods demand is sensitive to changes in the economic variables in the multiplier process. When $Y = Y^s$, on the contrary, goods demand does not work in the dynamics as much, like in neoclassical models. This difference causes a regime-switching effect on the l^s (labor supply per capital stock) and w (real wage) dynamics.

Figure 3 shows the dual-decision effect. Goods supply usually lowers goods demand since investment decreases when excess goods demand is low. However, realized production is also raised by the increase in goods supply when production is determined on the supply side. This means that the goods demand–supply gap is reduced when production is on the supply side. When goods production is demand-led, the goods demand–supply gap works more strongly than when it is supply-led. Therefore, the real wage feedback is stabilized in a demand-led economy. The second loop in the figure shows that the capital accumulation feedback is destabilized in a demand-led economy since aggregate goods demand and investment have positive feedback systems (multiplier process).

These two feedback loops show that the real wage dynamics become sticky and that the over- and underaccumulation of capital would occur in a Keynesian demand-led economy. In particular, the first result shows that the dual-decision effect reverses the causal relationship between price stickiness and the "Keynesian" case.

3 Numerical experiments

In this section, we simulate our dynamic system to investigate the stability and initial value dependency of the dynamic path.

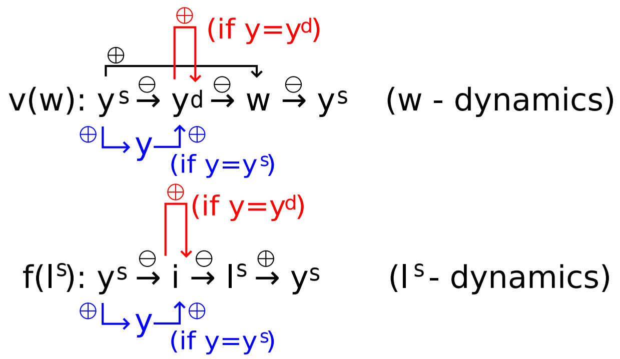

Figure 3: The y^s feedback loops with the dual-decision effect (cited from Ogawa (2020))

3.1 Settings and definitions

As stated in the previous section, our dynamic system of differential equations is discontinuous; hence, at the boundary of different regimes, the knowledge of the standard continuous dynamic system is useless (Figure 4). For this reason, we employ a method that precisely analyzes the behavior of our dynamic system at the boundary of different regimes to numerically simulate the dynamic paths. We use the MATLAB algorithm

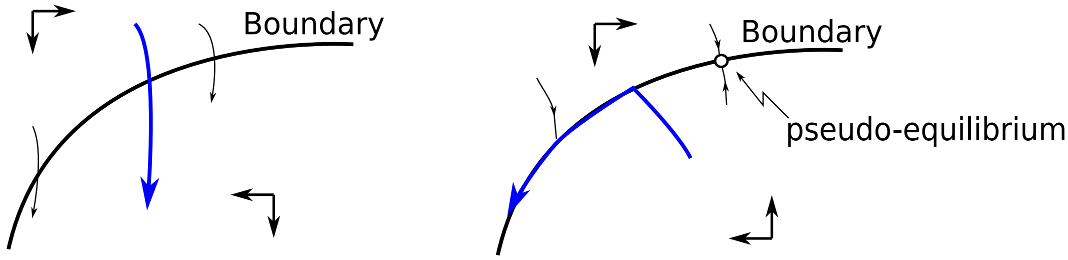

Figure 4: Crossing path (left) and sliding path (right) in a discontinuous system

`DISODE45` provided by Calvo et al. (2016). This algorithm extends the fourth- and fifth-order Runge–Kutta method and can precisely compute the behavior of the solution at the boundary, called the Filippov solution.[11] `DISODE45` can detect not only dynamic paths but also discontinuous points if we determine the initial values, period of analysis, and functions that characterize the dynamic system and discontinuous regions.

To conduct the numerical simulations, following Ogawa (2020), we specify the functions and parameters. For the parameters, we use the values of Flaschel et al. (2001):

$$F(K,E) = K^a E^{1-a} \tag{3.1}$$
$$q = (y^d/y^s)^\gamma(\rho/(r-\pi+\xi)) \tag{3.2}$$
$$c^d = 0.6483\exp(0.9044(r-\pi))((m+b+q)/y_{di})^{0.1866}y_{di} \tag{3.3}$$
$$m^d = h_1 y + h_2(r_0 - r) \tag{3.4}$$
$$\psi = i_1(\rho - r + \pi - \xi) + i_2((y^d/y^s) - 1), \tag{3.5}$$

[11]Filippov (1988) specifies the solution to piecewise continuous differential equations, which is used in not only in chemistry and electromagnetism but also in economics (Henry, 1972; Ito, 1979). In recent years, bifurcation analysis has also progressed in the field of mathematics (Kuznetsov et al., 2003; Guardia et al., 2011). The system used in `DISODE45` is an improvement of the event-driven method proposed by Piiroinen and Kuznetsov (2008).

$a = 0.34$, $\xi = 0.1500$, $\gamma = 1.4976$, $h_1 = 0.1769$, $h_2 = 2.1400$, $i_1 = 0.1363$, $i_2 = 0.0340$, $n = 0.0081$, $\mu = 0.0154$, $\delta = 0.0468$, $\tau = 0.15$, $g = 0.1250$, $\nu_P = 0.010$, $\nu_W = 0.0958$.

In this setting, we obtain the following steady-state values:

$$y_0 = 0.6276,\ r_0 = 0.0239,\ l_0^s = 0.4937,\ m_0 = 0.1110,\ b_0 = 2.3491,\ w_0 = 0.8390,\ \pi_0 = 0.0073.$$

Here, the parameters α (type of expectation formation) and β (speed of expectation adjustment) are undetermined. Since Ogawa (2020) suggests that these two parameters affect the stability of dynamic paths, we conduct our numerical analysis by changing the values of (α, β) and initial conditions.

Before analyzing the dynamic path, we define the sub-regime, which is related to the size of the relationship among three variables, y^d, $f(l^{d*})$, and $f(l^s)$; hence, six sub-regimes exist, with two sub-regimes in each regime (e.g., in the KU regime, there are $y^s = f(l^{d*})$ and $y^s = f(l^s)$). In a strict sense, the discontinuity region of the dynamic system appears at the boundary of the sub-regimes.

To characterize the dynamics, we classify the dynamic paths as follows. First, we define the "cyclicity" of a dynamic path as the case in which the change in a sub-regime detected by the numerical simulations is cyclical. When a dynamic path is non-cyclical, we define the dynamic path as "convergence" if it converges to the neighborhood of the steady state within the upper time limit T. On the contrary, we define the dynamic path as "not convergence" when it does not converge to the neighborhood of the steady state within the upper time limit T. Here, a variable is in the neighborhood of the steady state when the Euclidean norm of a vector that represents the degree of divergence from the steady-state value is less than 0.01, that is,

$$\left\| \left(\frac{l_T^s - l_0^s}{l_0^s}, \cdots, \frac{\pi_T - \pi_0}{\pi_0} \right) \right\| < 0.01. \tag{3.6}$$

For example, the Euclidean norm of a fifth-dimensional vector whose elements are all 0.005 is calculated as 0.0112. If the dynamic path shows convergence, we record the first time the path attains the above convergence condition as the convergence time.

When the dynamic path is cyclical, we check whether it tends toward convergence or divergence. We select the last two detected cycles and examine how each cycle diverges from the steady state. If the divergence of the last cycle from the steady state is less (more) than the divergence of the penultimate cycle, we define the situation as stable (unstable). We measure the divergence from the steady state at the turning point of the same sub-regime in each cycle.

Finally, we explain how to simulate the dynamic system. The time span ranges from $t = 0$ to $t = 2000$,[12] and we conduct the simulations by changing α from 0 to 1 in increments of 0.1 and β from 0.2 to 0.9 in increments of 0.1.[13] Considering that the dynamic path may depend on the initial condition, we use the initial values obtained by adding 50 random shocks to the steady-state values for each pair of (α, β). To examine how the degree of divergence from the steady state affects the dynamic paths, we conduct

[12]The parameters used in Ogawa (2020) are estimated from quarterly data; hence, $t = 2000$ corresponds to 500 years.

[13]We do not have to use a value of β less than unity. Flaschel et al. (2001) use $\beta = 0.6$, while Chiarella and Flaschel (2000a) use $\beta = 1$. Moreover, the value of Chiarella and Flaschel (1996) corresponds to $\beta = 1.1$. The size of β does not significantly affect the dynamics.

50 simulations for a 5% shock and 50 simulations for a 10% shock. Here, a 5% (10%) shock means that the degree of divergence between the initial value and steady-state value is less than 5% (10%).

3.2 Results

First, as Ogawa (2020) suggests, we examine the effect of the expectation adjustment parameter (α, β) on the dynamic paths. Figure 5 shows the extent to which the value of α induces the properties of dynamics, convergence, not convergence, stable cycle, and unstable cycle. The horizontal axis corresponds to α and the vertical axis corresponds to the ratio of each dynamic property to all the dynamic properties. Each line graph is colored according to its dynamic property and there are multiple line graphs of the same color according to the size of β. There are eight line graphs for each dynamic property since β takes eight values from 0.2 to 0.9.

The horizontal subplots show that the relationship between the value of α and the occurrence of each dynamic property is not related to the initial regime. In addition, each subplot shows that the value of β does not affect the results obtained. That is, whether the inflation expectation adjustment is rational or adaptive affects the property of the dynamics. When α is low, that is, π converges to the steady-state value without being affected by price changes, each dynamic property occur rarely and not convergence is somewhat likely to occur. In particular, when the initial conditions are in the RI regime, up to $\alpha = 0.5$, not convergence is most likely to be observed irrespective of the degree of divergence from the steady state. As expectation formation becomes more adaptive, not convergence is not observed. When expectation formation is intermediate (i.e., around $\alpha = 0.5$), the dynamic path is likely to converge to the steady state. If expectation formation is extremely adaptive ($\alpha = 1$), an unstable cycle almost certainly occurs. A stable cycle is observed around $\alpha = 0.3$, but it is not likely to be observed.

Next, the vertical subplots show that the effect of the degree of divergence from the initial value on the property of the dynamics. When the initial condition is in the CU or KU regime, a 10% shock from the steady state is likely to produce a 5% shock, which is consistent with our intuition. On the contrary, when the initial condition is in the RI regime, there is no difference between the 5% shock and 10% shock, which suggests that not convergence contributes to the sustainability of the RI regime.

Figures 6 and 7 show which dynamic property is observed according to the combination of (α, β) without considering the initial regime. These figures show that as the degree of divergence from the initial value becomes large, the economy is not likely to converge to the steady state within the given time.

Then, is not convergence a divergent path? Is there a kind of corridor stability that Leijonhufvud (1973) suggests and does the economy diverge from this corridor if the initial value deviates significantly from the steady state? Figure 8 shows the extent to which the convergence time changes according to the size of the shock, suggesting that it takes a longer time for the economy to converge to the steady state if the shock is large. Therefore, not convergence is not a divergent path and may be slow convergence such that the convergence time exceeds the upper limit ($t = 2000$). In the next section, we consider the mechanism of this dynamic path.

As stated above, the number of studies that emphasize KU has increased in recent years. The steady state of our model is the WE regime and the KU regime cannot be

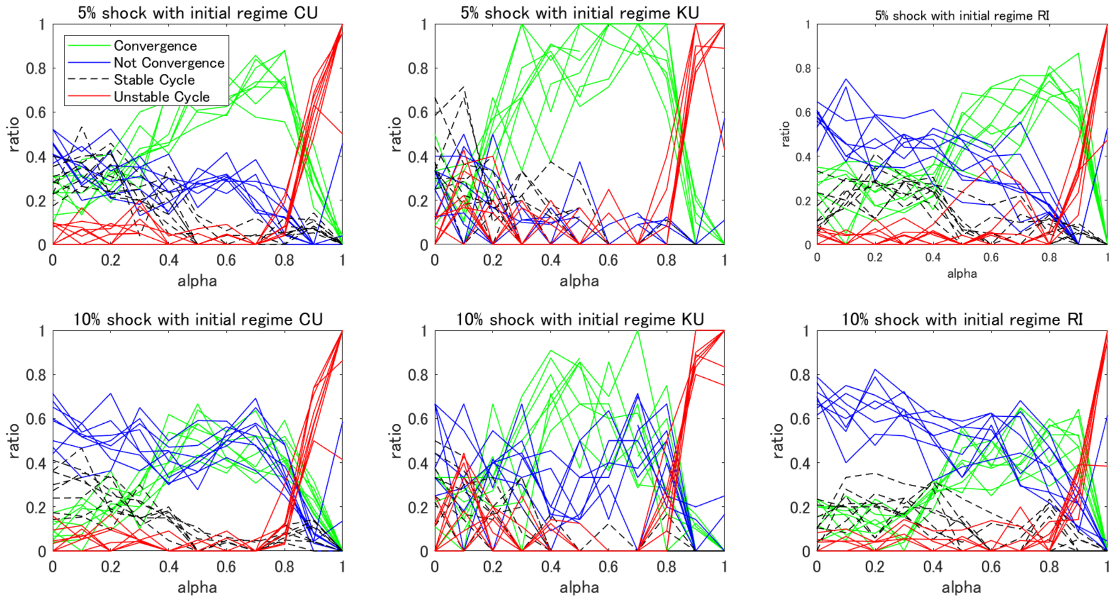

Figure 5: How α and the initial regime affect the dynamics

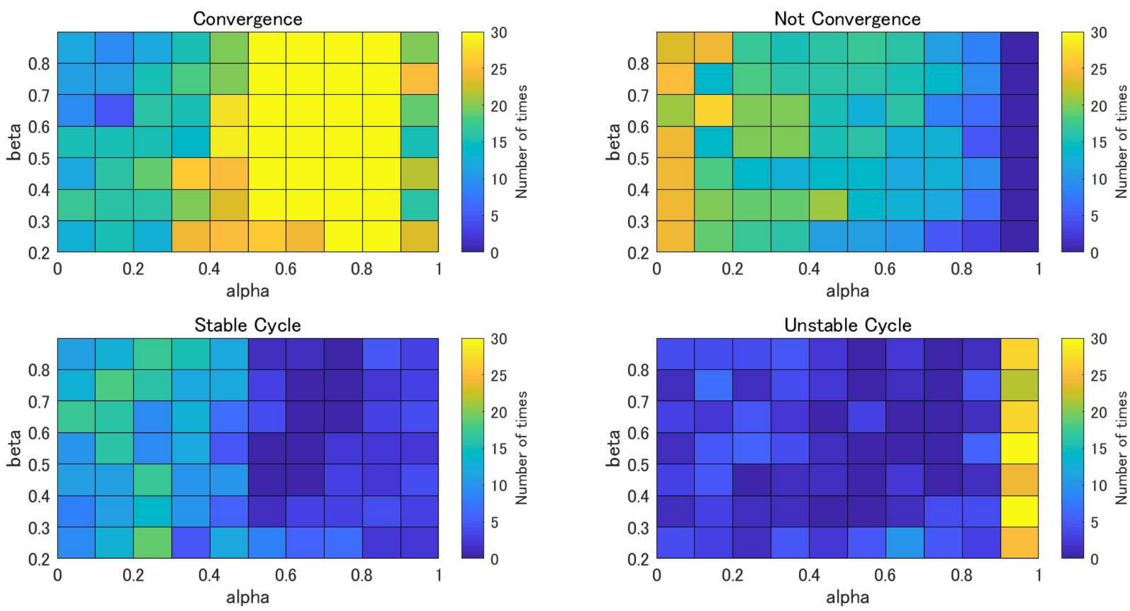

Figure 6: How α affects the dynamics: 5% shock

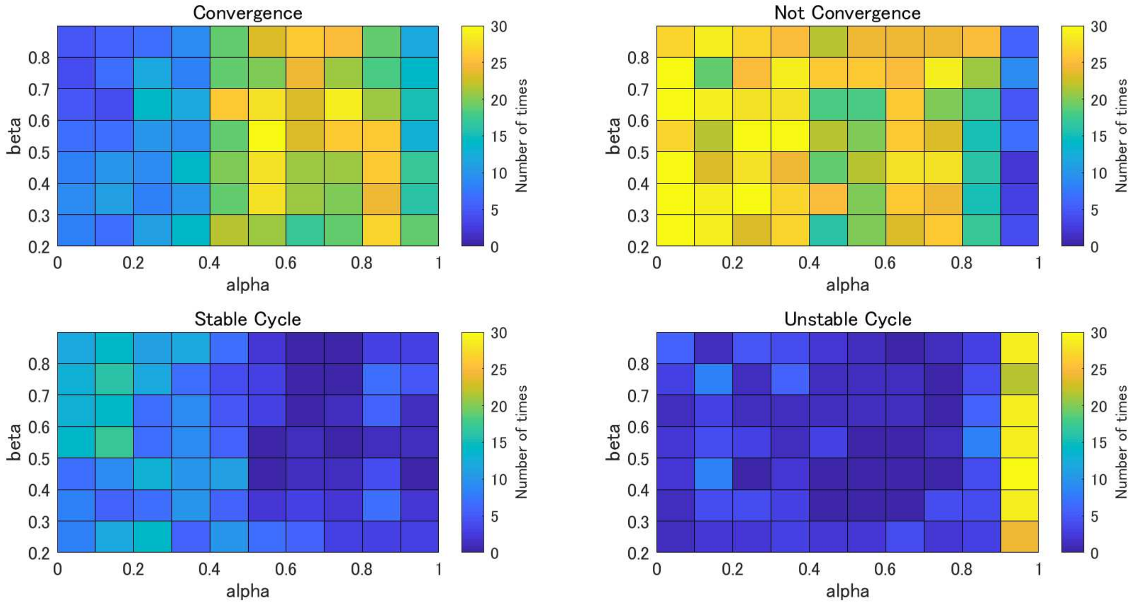

Figure 7: How α affects the dynamics: 10% shock

observed at the steady state.[14] Then, we investigate how long the economy experiences the KU regime along the transitional dynamics toward the steady state, which is shown in Figure 9. This figure suggests that at around $\alpha = 0.5$, the KU regime dominates the convergence path.[15][16] Therefore, when the economy converges to the steady state, along with the transitional dynamics, persistent KU is likely to be observed. In the next section, we consider the mechanism of persistent KU.

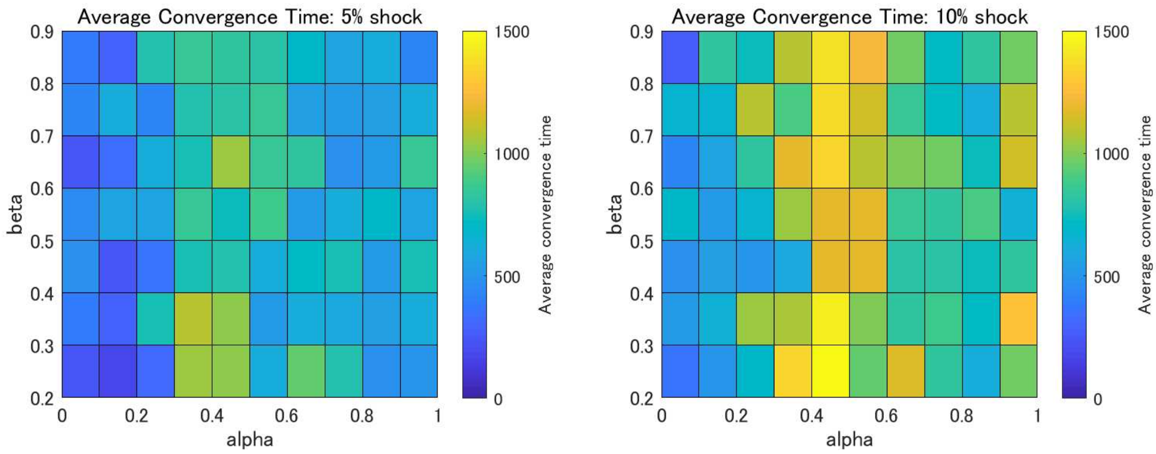

Figure 8: Convergence time

[14]Flaschel (1999) shows that the steady state can be characterized as KU if we assume that capacity utilization rate and employment rate are under unity in the steady state. By contrast, our model is based on the neoclassical model in which an economy is adjusted toward full capacity utilization and full employment in the long run.

[15]The white region shows that no convergence is observed.

[16]Nakayama and Oshima (1999) show empirical evidence that the proportion of rational expectation formation and that of adaptive expectation formation are roughly the same in Japan, which corresponds to $\alpha = 0.5$ in our model.

In summary, using numerical simulations, we find that the parameter α that determines the type of expectation formation plays an important role. If expectation formation is rational, we observe slow convergence such that it takes a very long time for the economy to converge to the steady state. On the contrary, if expectation formation is adaptive, we observe unstable cycles such that the economy diverges with cyclical fluctuations. If expectation formation is intermediate, the economy converges to the steady state and is likely to experience the KU regime along its transitional dynamics. The initial regime does not significantly affect the property of the dynamics. When the degree of divergence from the steady state is large, it takes a long time for the economy to converge to the steady state.

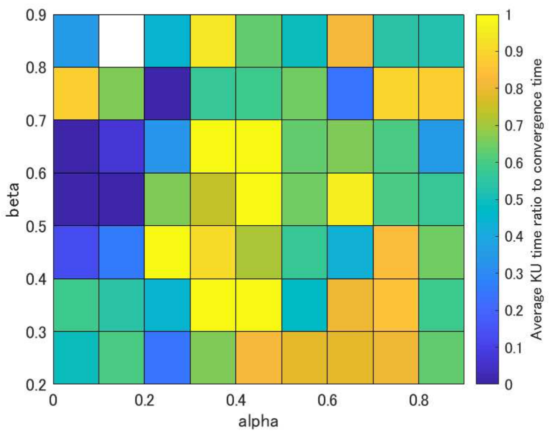

Figure 9: Average KU time ratio on the convergence path

4 Characteristic paths

4.1 Persistent KU

As shown above, KU occupies a large part of the convergence paths. This implies that the persistence of KU exists. Figures 10 and 11 show the persistent KU paths. Ogawa (2020) shows that the dual-decision effect continues KU without the ZLB constraint.

Figure 12 interprets the path using the IS-LM model. The circled numbers correspond with those in Figure 11.

At ① (moving from CU to KU), the interest rate is raised in the LM market because of the small real money balance m, and nominal goods supply $y^s = f(l^{d*})$ is small because the real wage is high. While the real wage is smoothly adjusted under CU, the lack of investment, which is induced by the high interest rate and low profit rate under CU, is enhanced under KU by the multiplier effect.

Therefore, the downward sloping segment of the IS curve shifts slightly to the right (goods demand increases with the increase in realized production $y = y^s$) and the vertical segment shifts to the right excessively. The decline in investment increases m so that the LM curve shifts downward.

The real wage is stuck at a low level, meaning that investment is stimulated by high profitability. However, low excess goods demand (below 1) also negatively affects investment. Capital accumulation slowly recovers ($f(l^s)$ adjustment) and the real wage also

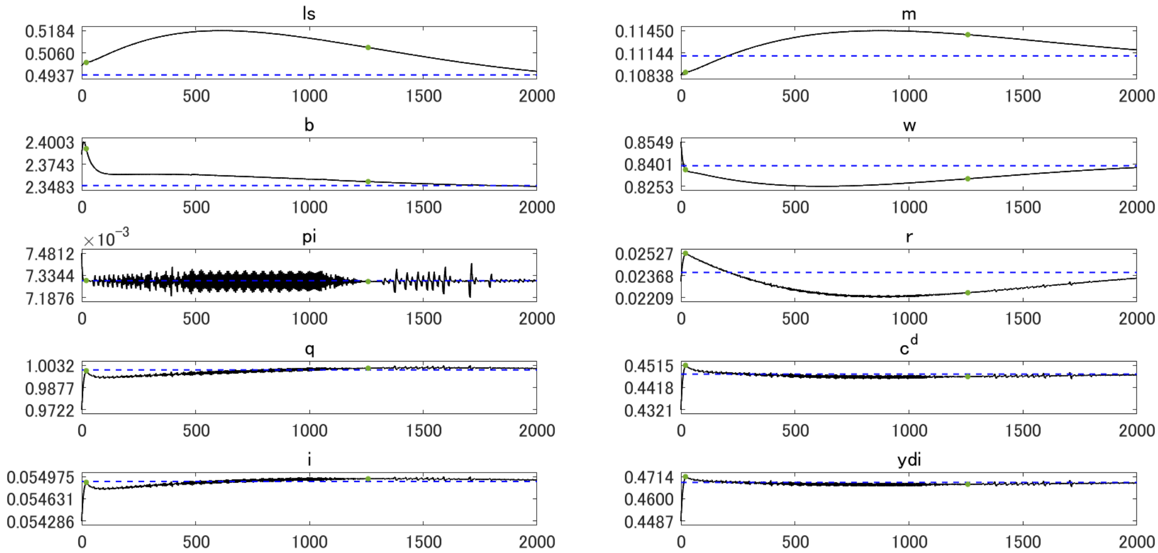

Figure 10: Dynamics of the variables under persistent KU

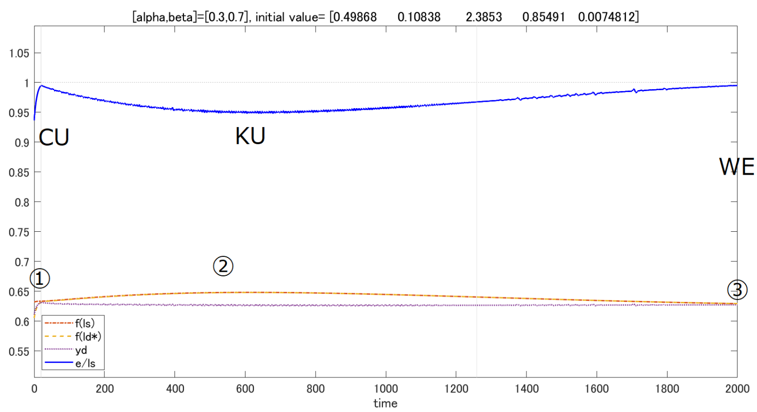

Figure 11: Dynamics of employment and production under persistent KU

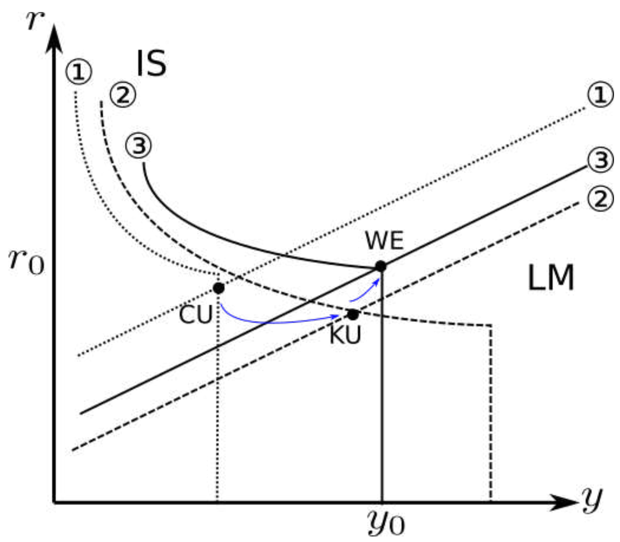

Figure 12: IS-LM interpretation of persistent KU

slowly recovers with a decline in l^s through capital accumulation. Finally, the economy converges to the steady state ③.

KU becomes persistent because of low investment demand with a multiplier process and sticky low wage, which expands the demand–supply gap in the goods market.

A fiscal stimulus is means of resolving persistent KU. The increase in g directly raises goods demand. However, the increase in g changes the steady-state values of the economic variables. y_0 also increases such that the difference between current y and y_0 is extended, which means that the change in the convergence time is ambiguous. The setting of revenue (tax, issuing bonds, or issuing money) is also a problem. Policy analysis is a future issue.

4.2 Slow convergence with rational adjustment

When the inflation expectation adjustment is close to rational (low α) and the expected inflation rate is affected little, the convergence to the steady state takes a long time. Why is not convergence detected many times when α is low?

Figures 13 and 14 show the path under $\alpha = 0$ (completely rational adjustment). The expected inflation rate immediately converges to π_0 and the goods demand–supply gap per capital stock is small. However, convergence is not detected even when $t = 5000$. This is unexpected since convergence to the steady state (and the balanced growth path) is delayed even though the destabilizing effect on π (the Mundell effect) disappears when $\alpha = 0$.

Figure 13 implies that capital stock K is overaccumulated and price P is persistently low.[17] The dynamics of π stop immediately at π_0 since the Mundell effect does not work. The value of i is always close to the steady-state value, which means that the capital accumulation rate is nearly constant. Therefore, the sticky low price slows the convergence.

This accords with New Keynesian economics in the sense that it emphasizes stickiness in convergence paths, but the expectation adjustment process generates stickiness. As

[17]Note that $m = M/(PK)$, $b = B/(PK)$, and $w = W/P$ retain high values and $l^s = L^s/K$ is lower than its steady-state value.

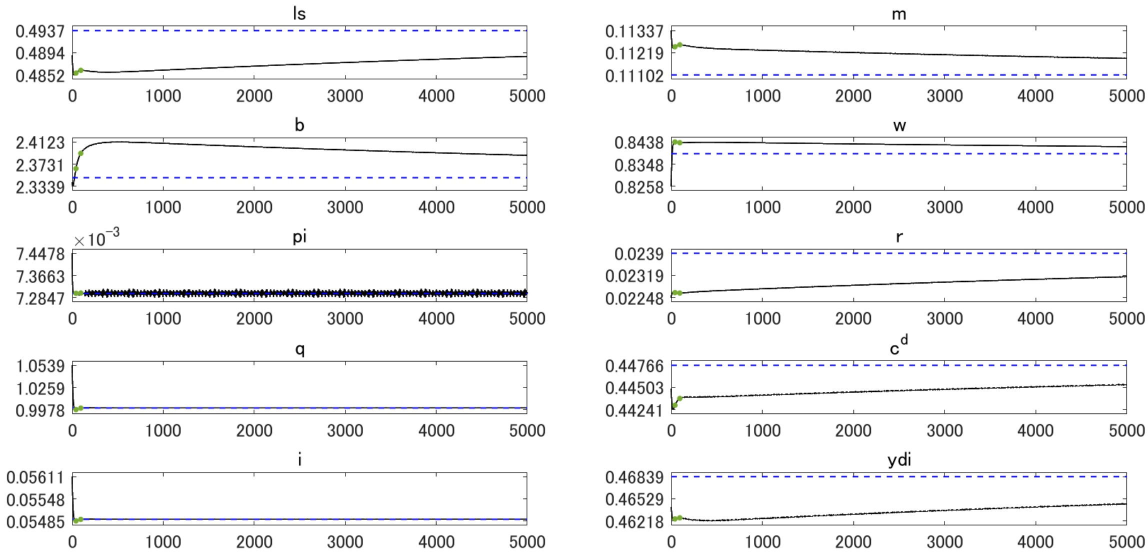

Figure 13: Dynamics of the variables in slow convergence

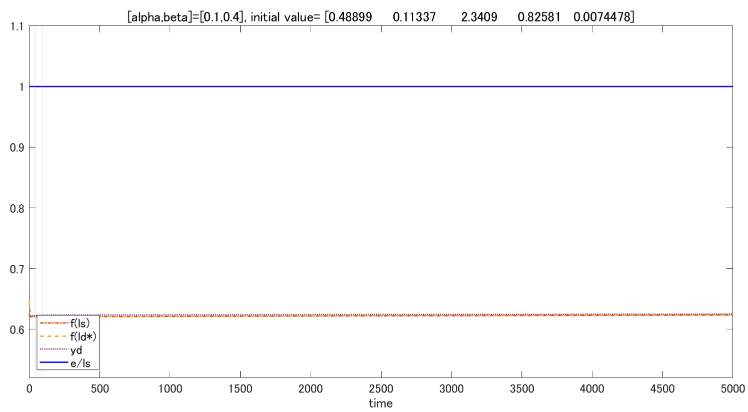

Figure 14: Dynamics of employment and production in slow convergence

the adjustment type is almost rational, the Mundell effect hardly works. When α is low, the feedback loop of π, which contains the price dynamics, weakens the price dynamics. Furthermore, the disequilibrium adjustment in price dynamics is weak because the goods demand–supply gap is smaller than K.[18] The sticky price plays no role in inducing the disequilibrium regime.

Because the production capital ratio $y = Y/K$ maintains a low value, we can interpret this path as the low capital utilization path. If this is the case, the utilization rate should be included in the price dynamics equation. However, labor and capital are fully utilized since the regime is RI ($y = f(l^s)$). Therefore, we should interpret this path as the case of a shortage of labor supply (population).[19]

Therefore, our system induces slow convergence when (1) labor supply is lacking relative to capital, (2) labor and capital are substitutable for production, and (3) the inflation expectation is not affected by current price dynamics.

4.3 Cyclical dynamics

When expectation adjustment is completely adaptive ($\alpha = 1$), the dynamics are almost an unstable cycle. This is interesting from a theoretical perspective but not a good example of a real-world approximation; see Appendix A. We attain cyclical dynamics in the area $\alpha < 1$; see Figures 6 and 7. However, the cyclical dynamics detected in this study are not interesting, as they only show that persistent KU with the continuous switching of goods supply occurs. As shown in the persistent KU example, the two factors of goods supply remain close on the path. The detected cycle is thus only an error term of these two. That is, cyclical regime switching is also detected in the area $\alpha < 1$, but the dynamics that can be treated as a kind of business cycle are limited to the case of $\alpha = 1$ in our study.

5 Concluding remarks

In this study, we investigated how the expectation adjustment parameters and initial state change the dynamics of the monetary growth model presented by Ogawa (2020). We obtained the following results:

1. The dynamic properties such as cyclical dynamics and feasibility of convergence depend on the type of expectation adjustment rather than on the initial regime. Convergence takes an excessively long time when the adjustment is rational and cyclical dynamics appear when it is adaptive.

2. When the economy converges to the steady state (which is the WE), persistent KU is likely to appear on the path.

3. The dynamics of the inflation expectation that contain the price dynamics in the feedback loop might play an important role in convergence to the steady state (balanced growth path).

[18]The price dynamics depend on $y^d - y^s$, or the demand–supply gap divided by K.

[19]As the production factors are substitutable, the quantities of labor and capital do not directly restrict production.

The second result is important for the analysis of secular stagnation. The persistent KU path is unrestricted by the ZLB constraint, type of expectation adjustment, and initial state. In particular, the persistent KU path often appears when the expectation is mixed (around $\alpha = 0.5$), which is realistic. We should thus encourage the Keynesian analysis of the quantity adjustment rather than simple "Keynesian" analysis, which blames the rigidities of the economic variables on unemployment, to find a way of removing secular stagnation.

An important future research issue is the adjustment process of expectation. Sargent and Wallace (1973) adopt the dynamic optimization to monetary growth model with perfect foresight to discuss the stability of dynamics. The DSGE model is also based on rational expectation, which corresponds to perfect foresight when the model is reduced to a deterministic version. However, Gelain et al. (2019) use a mixture of rational and adaptive expectation and find that the simulation shows a better fit than the rational expectation model.[20] Expectation theory in the disequilibrium model is incompletely established because it is complicated. For instance, Neary and Stiglitz (1983) and Weddepohl and Yildirim (1993) show that optimistic or pessimistic expectation for future demand or supply affects current disequilibrium regimes. Since expectation is affected by current transactions, a recurrence of expectation would occur. Therefore, forward expectation is still a difficult topic in disequilibrium models, making it an important future research issue.

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[20]Their model lacks theoretical consistency when they introduce hybrid expectation. Therefore, we should carefully evaluate their conclusions.

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A Cyclical dynamics

In this appendix, we see the cyclical dynamics under $\alpha = 1$. Figures A.1 and A.2 show an unstable cycle. The cyclical transitions of the regimes detected under $\alpha = 1$ are common, and every dynamic shows WE $\rightarrow$ RI $\rightarrow$ CU $\rightarrow$ KU $\rightarrow$ WE $\rightarrow \cdots$. Furthermore, the qualitative dynamics of the variables are also common. This implies that some bifurcation might occur.[21] The fatal problem of this path is that KU occurs because of the relative increase in l^s rather than the decrease in y^d. As Ogawa (2020) points out, this unexpected situation might arise from the setting of aggregate goods demand, which is unaffected by the income distribution. The dynamics affected by the income distribution are thus a future research issue.

[21] Ogawa (2020) varies the parameters (α, β) with a constant initial value and finds that the dynamics change from a monotone to a stable cycle around $\alpha = 0.94$ and a stable cycle to an unstable cycle around $\alpha = 0.98$. The mathematical analysis of the bifurcations of discontinuous dynamic systems, such as Kuznetsov et al. (2003) and Guardia et al. (2011), are restricted by low-dimensional cases.

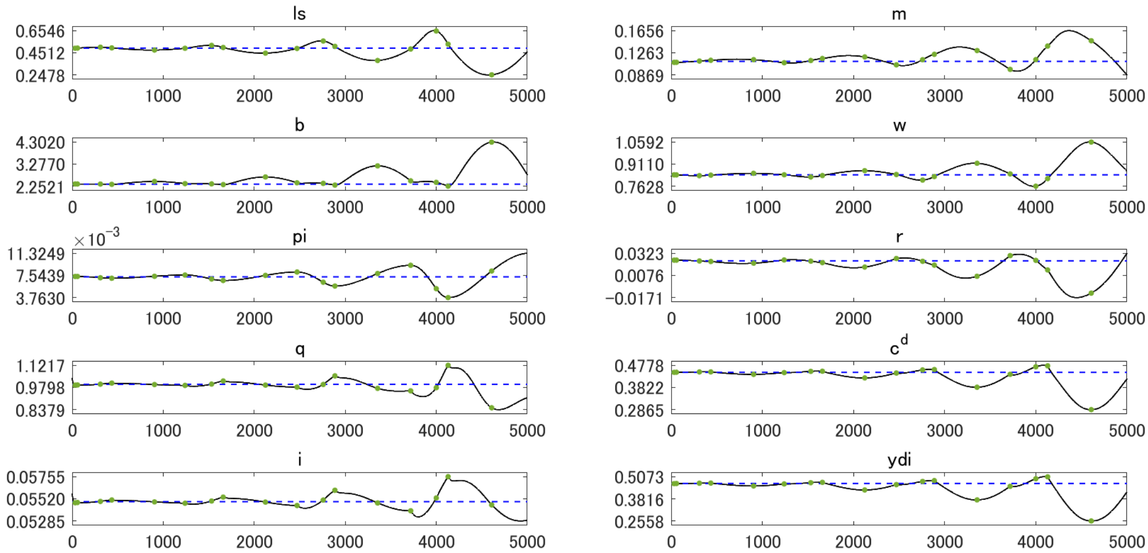

Figure A.1: Dynamics of the variables in an unstable cycle

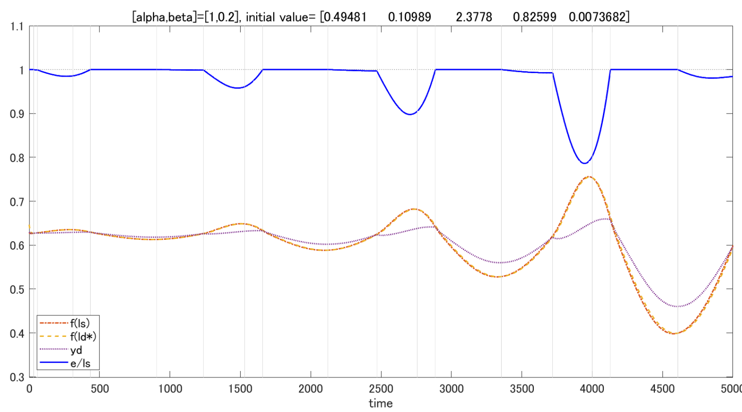

Figure A.2: Dynamics of employment and production in an unstable cycle