Obviously Strategy-proof Implementation of Assignment Rules: A New Characterization

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Abstract

We consider assignment problems where individuals are to be assigned at most one indivisible object and monetary transfers are not allowed. We provide a characterization of assignment rules that are Pareto efficient, non-bossy, and implementable in obviously strategy-proof (OSP) mechanisms. As corollaries of our result, we obtain a characterization of OSP-implementable fixed priority top trading cycles (FPTTC) rules, hierarchical exchange rules, and trading cycles rules. Troyan (2019) provides a characterization of OSP-implementable FPTTC rules when there are equal number of individuals and objects. Our result generalizes this for arbitrary values of those.

Keywords: Assignment problem; Obvious strategy-proofness; Pareto efficiency; Non-bossiness; Indivisible goods

JEL Classification: C78; D82

1 Introduction

We consider the problem where a set of objects are to be allocated over a set of individuals based on the individuals' preferences over the objects. Each individual can receive at most one object. An assignment rule selects an allocation (of the objects over the individuals) at every collection of preferences of the individuals.

Pareto efficiency, non-bossiness, and strategy-proofness are standard properties of an assignment rule. Pareto efficiency ensures that there is no other way to allocate the objects so that each individual is weakly better-off (and hence some individual is strictly better-off). Non-bossiness says that an individual cannot change the assignment of another one without changing her own assignment. Strategy-proofness ensures that no individual can be strictly better-off by misreporting her (true) preference. Group strategy-proofness ensures the same for every group of individuals, that is, no group of individuals can be better-off by misreporting their preferences. Here, we say a group of individuals is better-off if each member in it is weakly better-off and some member is strictly better-off.

Li (2017) introduces the notion of *obvious strategy-proofness* for an assignment rule. This notion is based on the notion of *obvious dominance* in an extensive-form game. A strategy s_i of an individual i in an extensive-form game is obviously dominant if, for any deviating strategy s'_i, starting from any earliest information set where s_i and s'_i diverge, the best possible outcome from s'_i is no better than the worst possible outcome from s_i. An assignment rule is *obviously strategy-proof (OSP)* if one can construct an extensive-form game that has an equilibrium in obviously dominant strategies. By construction, OSP depends on the extensive-form game, so two games with the same normal form may differ on this criterion.[1]

The objective of this paper is to provide the structure of OSP-implementable assignment rules. We impose two mild and desirable properties: Pareto efficiency and non-bossiness. First, we introduce the notion of *dual ownership* for *hierarchical exchange rule*. Hierarchical exchange rules are introduced in Pápai (2000) where it is shown that an assignment rule is strategy-proof, non-bossy, Pareto efficient, and reallocation-proof if and only if it is a hierarchical exchange rule. A hierarchical exchange rule works in several stages. In every stage, the objects (available in that stage) are owned by certain individuals who then trade their objects by forming top trading cycles.[2] The ownership of the objects in any stage is determined by a collection of trees, which are called *inheritance trees* in Pápai (2000). A hierarchical exchange rule satisfies *dual ownership* if for each preference profile and each stage of the hierarchical exchange rule at that preference profile, there are at most two individuals who own all the objects available in that stage. In this paper, we show that an assignment rule is OSP-implementable, Pareto efficient, and non-bossy if and only if it is a hierarchical exchange rule satisfying dual ownership (Theorem 7.1).

Although dual ownership is an intuitive and simple property (and thereby, is quite helpful for ex-

[1] This verbal description of obvious strategy-proofness is adapted from Li (2017).

[2] Top trading cycle (TTC) is due to David Gale and discussed in Shapley and Scarf (1974).

plaining it to the participating individuals), it is not so convenient for the designer to check whether a given hierarchical exchange rule satisfies this property or not. This is because, technically, one needs to check at *every* preference profile and *every* stage of the hierarchical exchange rule at that preference profile, whether at most two individuals are owning all the (available) objects in that stage or not. In view of this observation, we introduce the notion of *acyclicity* for a hierarchical exchange rule. Acyclicity is a technical property, which, as the name suggests, ensures that certain type of cycles are *not* present in the inheritance trees of a hierarchical exchange rule. The advantage of checking this property for a hierarchical exchange rule is that it only involves the collection of inheritance trees, and not anything about the state of the hierarchical exchange rule in different stages at different preference profiles. In Theorem 8.1, we show that acyclicity and dual ownership are equivalent properties of a hierarchical exchange rule. It follows as a corollary (Corollary 9.1) of Theorem 7.1 that a hierarchical exchange rule is OSP-implementable if and only if it satisfies dual ownership. In another corollary (Corollary 9.2) of Theorem 7.1, we show that a *trading cycles rule* is OSP-implementable if and only if it is a hierarchical exchange rule satisfying dual ownership. Trading cycles rules are generalization of hierarchical exchange rules. These rules are introduced in Pycia and Ünver (2017) where it is shown that an assignment rule is strategy-proof, non-bossy, and Pareto efficient if and only if it is a trading cycles rule. Note that since strategy-proofness and non-bossiness together are equivalent to group strategy-proofness (see Pápai (2000) for details), all the above-mentioned results can be reformulated in terms of group strategy-proofness.

The importance of OSP-implementability of an assignment rule is well-established in the literature. It is observed that individuals do not seem to be convinced that a strategy-proof rule is indeed not manipulable (Chen and Sönmez (2006), Hassidim et al. (2016), Hassidim et al. (2017), Rees-Jones (2018), Shorrer and Sóvágó (2018)). Obvious strategy-proofness came to the literature as a remedy by strengthening strategy-proofness in a way so that it becomes clear to the individuals that such a rule is not manipulable.

Troyan (2019) introduces the notion of *weak acyclicity* and shows that it is both necessary and sufficient condition for an FPTTC rule to be OSP-implementable (Theorem 1 in Troyan (2019)).[3] However, there is a mistake in his characterization–although weak acyclicity is a sufficient condition for OSP-implementability of an FPTTC rule, it is *not* necessary. Since FPTTC rules are special cases of hierarchical exchange rules, we obtain as a corollary (Corollary 10.2) of our result that acyclicity is a necessary and sufficient condition for OSP-implementability of an FPTTC rule. In another corollary (Corollary 10.1), we obtain that an FPTTC rule is OSP-implementable if and only if it satisfies dual ownership.[4] It is worth mentioning that Troyan (2019) assumes that the number of individuals is the same as the number of objects, whereas we derive our results for arbitrary values of those.

As we have mentioned earlier, Pápai (2000) characterizes hierarchical exchange rules as the only as-

[3]Troyan (2019) uses the term "TTC rule" to refer to an FPTTC rule in his paper.

[4]Theorem 2 in Troyan (2019) states that weak acyclicity and dual dictatorships are equivalent properties of an FPTTC rule. This result is correct on its own, but because of the mistake in Theorem 1, it is not correct that an FPTTC rule is OSP-implementable if and only if it satisfies dual dictatorships.

signment rules satisfying strategy-proofness, non-bossiness, Pareto efficiency and reallocation-proofness. Our results complement hers in two ways. Firstly, whereas strategy-proofness, non-bossiness, and Pareto efficiency are desirable, reallocation-proofness is not that desirable. So, replacing strategy-proofness and reallocation-proofness by OSP-implementability, and characterizing the relevant class of hierarchical exchange rules is a significant contribution in our opinion. Secondly, hierarchical exchange rules are somewhat complicated for participants to understand. So, finding the class of such rules that can be implemented by obviously strategy-proof mechanisms is important for their application. Nevertheless, OSP-implementability is a desirable criteria on its own, and to our understanding, providing the structure of such assignment rules is a significant contribution to the literature.

1.1 Related literature

Obvious strategy-proofness was introduced by Li (2017), who studies this property extensively for both the scenarios where monetary transfers are allowed and not allowed. When monetary transfers are not allowed, he analyses the implementability of serial dictatorship and top trading cycles rules under obvious strategy-proofness. Bade and Gonczarowski (2017) *constructively* characterize Pareto-efficient social choice rules that admit obviously strategy-proof implementations in popular domains (object assignment, single-peaked preferences, and combinatorial auctions). Pycia and Troyan (2019) characterize the full class of obviously strategy-proof mechanisms in environments without transfer. They also introduce a natural strengthening of obvious strategy-proofness called *strong obvious strategy-proofness* to characterize the well-known *random priority mechanism* as the unique mechanism that is efficient and fair. Ashlagi and Gonczarowski (2018) consider two-sided matching with one strategic side. They show that for general preferences, no mechanism that implements the men-optimal stable matching (or any other stable matching) is obviously strategy-proof for men.

1.2 Organization of the paper

The organization of this paper is as follows. In Section 2, we introduce basic notions and notations that we use throughout the paper. In Section 3, we define assignment rules and discuss their standard properties. Section 4 introduces the key notion of this paper: obvious strategy-proofness, Section 5 introduces the notions of FPTTC rules and hierarchical exchange rules, and Section 6 introduces the dual ownership property of a hierarchical exchange rule. We present our main result (characterization of all OSP-implementable, Pareto efficient, and non-bossy assignment rules) in Section 7. Section 8 introduces the acyclicity property of a hierarchical exchange rule. In Section 9, we present a characterization of OSP-implementable hierarchical exchange rules and a characterization of OSP-implementable trading cycles rules. In Section 10, we present two characterizations of OSP-implementable FPTTC rules and discuss the relation between our results regarding FPTTC rules and that of Troyan (2019). In Section 11, we discuss the relation between hierarchical exchange rules satisfying dual ownership and sequential barter with lurkers rules introduced in Bade and Gonczarowski (2017). Section 12 concludes the paper. All the proofs are collected in the Appendix.

2 Basic notions and notations

For an arbitrary finite set X, we denote by $\mathbb{L}(X)$ the set of all linear orders (i.e., complete, asymmetric and transitive binary relation) on X. An element of $\mathbb{L}(X)$ is called a preference over X. For a preference P, by R we denote the weak part of P, i.e., for all $x, y \in X$, xRy if and only if $[xPy \text{ or } x = y]$. For a preference P over X and non-empty $Y \subseteq X$, we denote by $\tau(P, Y)$ the most preferred element of Y according to P, that is, $\tau(P, Y) = y$ if and only if $y \in Y$ and yPx for all $x \in Y \setminus \{y\}$.

Let $N = \{1, \ldots, n\}$ be a (finite) set of individuals and A be a (non-empty and finite) set of objects. A function $\mu : N \to A \cup \{\emptyset\}$ is called an ***allocation***. Here, $\mu(i) = x$ for some element x of A means individual i is assigned object x in allocation μ, and $\mu(i) = \emptyset$ means individual i is not assigned any object in μ. An allocation μ is ***feasible*** if no object is assigned to more than one individual. We denote by $\mathcal{M}$ the set of all feasible allocations. For $\widehat{N} \subseteq N$, $\hat{A} \subseteq A$ such that $|\widehat{N}| = |\hat{A}| \neq 0$, we denote by $\mathcal{M}(\widehat{N}, \hat{A})$ the set of all feasible allocations of $\widehat{N}$ over $\hat{A}$, that is, $\mathcal{M}(\widehat{N}, \hat{A}) = \{\hat{\mu} \mid \hat{\mu} \text{ is a bijection from } \widehat{N} \text{ to } \hat{A}\}$. To ease our presentation, for an allocation $\hat{\mu} \in \mathcal{M}(\widehat{N}, \hat{A})$, an individual $i \in \widehat{N}$, and an object $x \in \hat{A}$, we write $\hat{\mu}(x) = i$ whenever $\hat{\mu}(i) = x$.

We assume that the set of admissible preferences of each individual is $\mathbb{L}(A)$. For ease of presentation, we denote $\tau(P, A)$ by $\tau(P)$ for a preference P. An element $P_N = (P_1, \ldots, P_n)$ of $\mathbb{L}^n(A)$ is called a ***preference profile***. Given a preference profile P_N, we denote by (P_i', P_{-i}) the preference profile obtained from P_N by changing the preference of individual i from P_i to P_i' and keeping all other preferences unchanged.

For ease of presentation we use the following convention throughout the paper: for a set $\{1, \ldots, g\}$ of integers, whenever we refer to the number $g + 1$, we mean 1. For instance, if we write $s_t \geq r_{t+1}$ for all $t = 1, \ldots, g$, we mean $s_1 \geq r_2, \ldots, s_{g-1} \geq r_g$, and $s_g \geq r_1$.

3 Assignment rules and their standard properties

A function $f : \mathbb{L}^n(A) \to \mathcal{M}$ is called an ***assignment rule*** on $\mathbb{L}^n(A)$. For an assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ and a preference profile $P_N \in \mathbb{L}^n(A)$, we denote by $f_i(P_N)$ the object that is assigned to individual i by the rule f at P_N.

An allocation μ ***Pareto dominates*** another allocation ν at a preference profile P_N if $\mu(i)R_i\nu(i)$ for all $i \in N$ and $\mu(j)P_j\nu(j)$ for some $j \in N$. An assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is called ***Pareto efficient*** at a preference profile $P_N \in \mathbb{L}^n(A)$ if there is no feasible allocation that Pareto dominates $f(P_N)$ at P_N, and it is called ***Pareto efficient*** if it is Pareto efficient at every preference profile in $\mathbb{L}^n(A)$.

Non-bossiness is a standard notion in matching theory which says that if an individual misreports her preference and her assignment does not change by the same, then the assignment of any other individual cannot change. Formally, an assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is ***non-bossy*** if for all $P_N \in \mathbb{L}^n(A)$, all $i \in N$, and all $\tilde{P}_i \in \mathbb{L}(A)$, $f_i(P_N) = f_i(\tilde{P}_i, P_{-i})$ implies $f(P_N) = f(\tilde{P}_i, P_{-i})$.[5]

An individual *i manipulates* an assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ at a preference profile $P_N \in \mathbb{L}^n(A)$ via a preference $\tilde{P}_i \in \mathbb{L}(A)$ if $f_i(\tilde{P}_i, P_{-i}) P_i f_i(P_N)$. An assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is ***strategy-proof*** if no individual can manipulate it at any preference profile.

Group strategy-proofness says that no group of individuals will have an incentive to misreport their preferences. More formally, a group of individuals $\widehat{N} \subseteq N$ *manipulates* an assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ at a preference profile $P_N \in \mathbb{L}^n(A)$ via a collection of preferences $\tilde{P}_{\widehat{N}} \in \mathbb{L}^{|\widehat{N}|}(A)$ if $f_i(\tilde{P}_{\widehat{N}}, P_{-\widehat{N}}) R_i f_i(P_N)$ for all $i \in \widehat{N}$ and $f_j(\tilde{P}_{\widehat{N}}, P_{-\widehat{N}}) P_j f_j(P_N)$ for some $j \in \widehat{N}$. An assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is ***group strategy-proof*** if no group of individuals can manipulate it at any preference profile.

4 Obviously strategy-proof assignment rules

Li (2017) introduces the notion of *obviously strategy-proof implementation*. This notion is well-known in the literature and needs no introduction. We use the following notions and notations to present this.

We denote a rooted (directed) tree by T. For a tree T, we denote its set of nodes by $V(T)$, set of all edges by $E(T)$, root by $r(T)$, and set of leaves (terminal nodes) by $L(T)$. For a node $v \in V(T)$, we denote the set of all outgoing edges from v by $E^{out}(v)$. For an edge $e \in E(T)$, we denote its source node by $s(e)$. A path in a tree is a sequence of nodes such that every two consecutive nodes form an edge.

A *leaves-to-allocations* function $\eta^{LA} : L(T) \to \mathcal{M}$ assigns a feasible allocation to each leaf of T, and a *nodes-to-individuals* function $\eta^{NI} : V(T) \setminus L(T) \to N$ assigns an individual to each internal node of T. An *edges-to-preferences* function $\eta^{EP} : E(T) \to 2^{\mathbb{L}(A)} \setminus \{\emptyset\}$ assigns each edge a subset of preferences satisfying the following criteria:

(i) for all distinct $e, e' \in E(T)$ such that $s(e) = s(e')$, we have $\eta^{EP}(e) \cap \eta^{EP}(e') = \emptyset$, and

(ii) for any $v \in V(T) \setminus L(T)$,

- (a) if there exists a path $(v^1, \ldots, v^t)$ from $r(T)$ to v and some $1 \leq r < t$ such that $\eta^{NI}(v^r) = \eta^{NI}(v)$ and $\eta^{NI}(v^s) \neq \eta^{NI}(v)$ for all $s = r+1, \ldots, t-1$, then $\bigcup_{e \in E^{out}(v)} \eta^{EP}(e) = \eta^{EP}(v^r, v^{r+1})$, and
- (b) if there is no such path, then $\bigcup_{e \in E^{out}(v)} \eta^{EP}(e) = \mathbb{L}(A)$.

An ***extensive-form assignment mechanism*** is defined as a tuple $G = \langle T, \eta^{LA}, \eta^{NI}, \eta^{EP} \rangle$, where T is a rooted tree, η^{LA} is a leaves-to-allocations function, η^{NI} is a nodes-to-individuals function, and η^{EP} is an edges-to-preferences function.

[5] The concept of non-bossiness is due to Satterthwaite and Sonnenschein (1981).

Note that for a given extensive-form assignment mechanism G, every preference profile P_N identifies a unique path from the root to some leaf in T in the following manner: for each node v, follow the outgoing edge e from v such that $\eta^{EP}(e)$ contains the preference $P_{\eta^{NI}(v)}$. If a node v lies in such a path, then we say that the preference profile P_N *passes through the node* v. Furthermore, we say two preferences P_i and P'_i of some individual i *diverge at a node* $v \in V(T) \setminus L(T)$ if $\eta^{NI}(v) = i$ and there are two distinct outgoing edges e and e' in $E^{out}(v)$ such that $P_i \in \eta^{EP}(e)$ and $P'_i \in \eta^{EP}(e')$.

For a given extensive-form assignment mechanism G, the ***extensive-form assignment rule*** f^G implemented by G is defined as follows: for all preference profiles P_N, $f^G(P_N) = \eta^{LA}(l)$, where l is the leaf that appears at the end of the unique path characterized by P_N.

In what follows, we define the notion of obvious strategy-proofness. This notion is introduced in Li (2017).

Definition 4.1. An extensive-form assignment mechanism G is ***Obviously Strategy-Proof (OSP)*** if for all $i \in N$, all nodes v such that $\eta^{NI}(v) = i$, and all $P_N, \tilde{P}_N \in \mathbb{L}^n(A)$ passing through v such that P_i and $\tilde{P}_i$ diverge at v, we have $f^G_i(P_N) R_i f^G_i(\tilde{P}_N)$.

An assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is ***OSP-implementable*** if there exists an OSP mechanism G such that $f = f^G$.[6,7]

Remark 4.1. It is worth mentioning that every OSP-implementable assignment rule is strategy-proof (see Li (2017) for details).

We present an example to illustrate the notion of OSP. This example is a modified version of Example 1 in Troyan (2019).

Example 4.1. Consider an allocation problem with three individuals $\{i, j, k\}$ and three objects $\{x, y, z\}$. In Figure 4.1, we provide an extensive-form assignment mechanism G.[8] We claim that G is OSP. To see this, consider, for instance, the preference profiles $P_N = (zxy, zxy, zxy)$ and $\tilde{P}_N = (zyx, xyz, zxy)$.[9] Note that both of them pass through the node D at which P_j and $\tilde{P}_j$ diverge. Further note that $f^G_j(P_N) = z$ and $f^G_j(\tilde{P}_N) = x$, which means G satisfies the OSP property for this instance. Similarly, one can check that G satisfies the OSP property for other instances.

[6]Definition 4.1 is taken from Troyan (2019). However, his definition has a typo as it does not mention that P_N and $\tilde{P}_N$ must pass through v. We have corrected it here.

[7]An extensive-form assignment mechanism is called an ***OSP mechanism*** if it is OSP.

[8]We use the following notations in Figure 4.1: by $(A : i)$ we mean that the node A is assigned to individual i, by xy we denote the set of preferences where x is preferred to y, by xyz we denote the set of preferences where x is preferred to y and y is preferred to z, and we denote an allocation $[(i, x), (j, y), (k, z)]$ by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

[9]We denote by (xyz, yxz, zxy) a preference profile where individuals i, j and k have preferences xyz, yxz, and zxy, respectively.

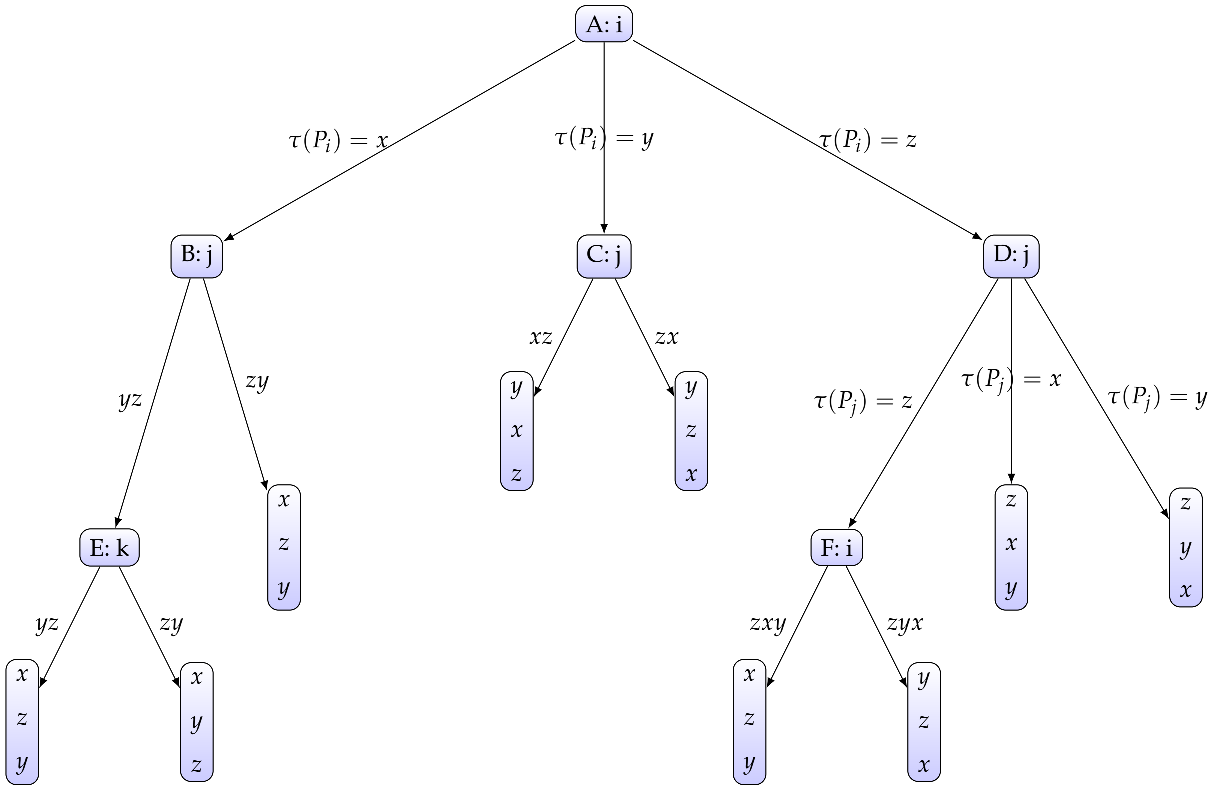


Figure 4.1

5 Hierarchical exchange rules

The notion of *hierarchical exchange rules* is introduced in Pápai (2000). These rules are generalization of *fixed priority top trading cycles (FPTTC) rules*, which we present in the next subsection.

5.1 FPTTC rules

FPTTC rules are well-known in the literature; we present a brief description for the sake of completeness. First we explain the notion of a *TTC procedure* with respect to a given endowments of the objects over the individuals. Suppose that each object is owned by exactly one individual. Note that an individual may own more than one objects. A directed graph is constructed in the following manner. The set of nodes is the same as the set of individuals. There is a directed edge from individual i to individual j if and only if individual j owns individual i's most preferred object. Note that such a graph will have exactly one outgoing edge from every node (though possibly many incoming edges to a node). Further, there may be an edge from a node to itself. It is clear that such a graph will always have a cycle. This cycle is called a *top trading cycle (TTC)*. After forming a TTC, the individuals in the TTC are assigned their most preferred objects.

We are now ready to explain FPTTC rules. For each object $x \in A$, we define the ***priority*** of x as a preference $\succ_x$ over the individuals in N. We denote the weak part of $\succ_x$ by $\succeq_x$, that is, for all $i, j \in N$,

$i \succeq_x j$ means either $i \succ_x j$ or $i = j$. We call a collection $\succ_A := (\succ_x)_{x \in A}$ a ***priority structure***.

For a given priority structure $\succ_A$, the ***FPTTC rule*** $T^{\succ_A}$ ***associated with*** $\succ_A$ is defined by an iterative procedure as follows. Consider an arbitrary preference profile $P_N \in \mathbb{L}^n(A)$.

Step 1. Each object x is owned by the individual who has the highest priority according to $\succ_x$, that is, the most preferred individual of $\succ_x$. TTC procedure is performed with respect to these endowments. Individuals who are assigned some object leave the market with their assigned objects.

$\vdots$

Step t. Consider the reduced market with the remaining individuals and objects. Each remaining object x is owned by the individual who has the highest priority among the remaining individuals according to $\succ_x$, that is, the individual who is remained in the reduced market at this step and is preferred to every other remaining individual according to $\succ_x$. TTC procedure is performed on the reduced market with respect to these endowments, and individuals who are assigned some object at this step leave the market.[10]

$\vdots$

This procedure is repeated iteratively until either all individuals are assigned or all objects are assigned. The final outcome is obtained by combining all the assignments at all steps.

5.2 Description of hierarchical exchange rules

The key step that differentiates hierarchical exchange rules from FPTTC rules is the way how objects, that were owned by an individual who is removed at a step, are passed down for the next step. In FPTTC rule, each object has a fixed linear ordering (which we have called a *priority*) of the individuals, while in hierarchical exchange rule, this is generalized to an inheritance tree, where the individual to whom objects are passed can depend endogenously on the objects the previous owner is assigned.

The following verbal description of hierarchical exchange rules is taken from Pápai (2000). The allocation obtained by a hierarchical exchange rule can be described by the following iterative procedure. Individuals have an initial individual "endowment" of objects such that each object is exactly one individual's endowment. It is important to note that some individuals may not be endowed with any objects. Now apply the TTC procedure to this market with individual endowments. Notice that individuals who don't have endowments cannot be part of a top trading cycle, since nobody points to them, and therefore they need not point. Given that multiple endowments are allowed, after the individuals in top trading

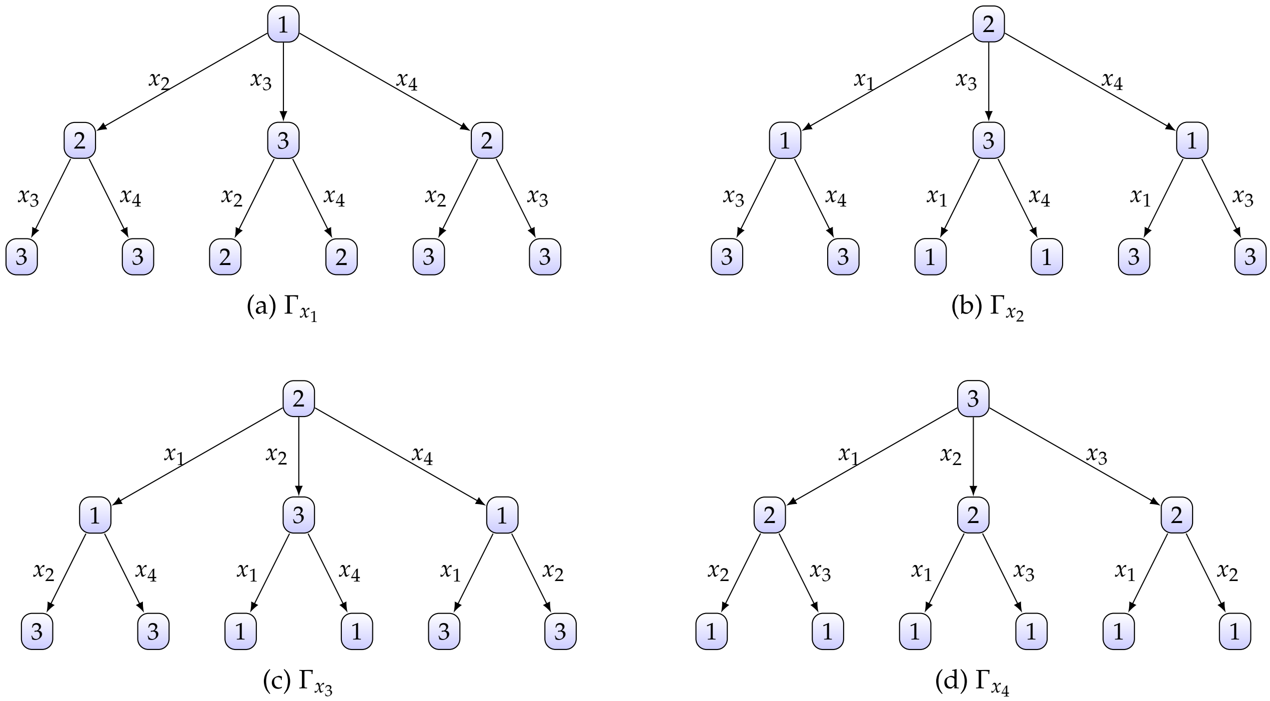
[10] In this TTC procedure, an individual i point to an individual j if j owns i's most preferred object among the remaining objects.

cycles leave the market with their most preferred objects, unassigned objects in the initial endowment sets of individuals who received their assignment may be left behind. These objects are reassigned as endowments to individuals who are still in the market, that is, they are "inherited" by individuals who have not yet received their assignments. Furthermore, the objects in the initial endowment sets of individuals who are still in the market remain the individual endowments of these individuals. Thus, notice that each unassigned object is the endowment of exactly one individual who is still in the market. Now apply the TTC procedure to this reduced market with the new endowments. Repeat this procedure until every individual has her assignment or all the objects are assigned. Since there exists at least one top trading cycle in every stage, this procedure leads to an allocation of the objects in a finite number of stages. In particular, there are at most as many stages as there are individuals or objects, whichever number is smaller, since in each stage at least one person receives her assignment. Furthermore, for any strict preferences of the individuals, the resulting allocation is unique.

A hierarchical exchange rule is determined by the initial endowments and the hierarchical endowment inheritance in later stages. While the initial endowment sets are given a priori, the hierarchical endowment inheritance may be endogenous. In particular, the inheritance of endowments may depend on the assignments made in earlier stages.

We explain how a hierarchical exchange rule works by means of the following example.

Example 5.1. Suppose $N = \{1, 2, 3\}$ and $A = \{x_1, x_2, x_3, x_4\}$. A hierarchical exchange rule is based on a collection of *inheritance trees*, one tree for each object. We will define this notion formally; for the time being we explain it through the current example. Figure 5.1 presents a collection of inheritance trees Γ_{x_1}, $\ldots, \Gamma_{x_4}$. To understand their structure, let us look at one of them, say Γ_{x_1}. Each maximal path of this tree has $\min\{|N|, |A|\} - 1 = 2$ edges. In any maximal path, each individual appears *at most* once at the nodes. For instance, individuals $1, 2$ and 3 appear at the nodes (in that order) in the left most path of Γ_{x_1}. Each object other than x_1 appears *exactly* once at the outgoing edges from the root (thus there are three edges from the root). For every subsequent node, each object other than x_1, that has *not* already appeared in the path from the root to that node, appears *exactly* once at the outgoing edges from that node. For instance, consider the node marked with 2 in the left most path of Γ_{x_1}. Since object x_2 has already appeared at the edge from the root to this node, objects x_3 and x_4 appear exactly once at the outgoing edges from this node. Thus, each object other than x_1 appears *at most* once at the edges in any maximal path of Γ_{x_1}. For instance, objects x_2 and x_3 appear at the edges (in that order) in the left most path of Γ_{x_1}. It can be verified that other inheritance trees have the same structure.



(a) Γ_{x_1} (b) Γ_{x_2} (c) Γ_{x_3} (d) Γ_{x_4}

Figure 5.1: Inheritance trees for Example 5.1

Consider the hierarchical exchange rule based on the collection of inheritance trees given in Figure 5.1. We explain how to compute the outcome of the rule at a given preference profile. Consider the preference profile P_N as given below:

P_1	P_2	P_3
x_2	x_1	x_1
x_1	x_2	x_2
x_3	x_3	x_3
x_4	x_4	x_4

Table 5.1: Preference profile for Example 5.1

The outcome is computed through a number of stages. In each stage, endowments of the individuals are determined by means of the inheritance trees and TTC procedure is performed with respect to the endowments.

Stage 1.

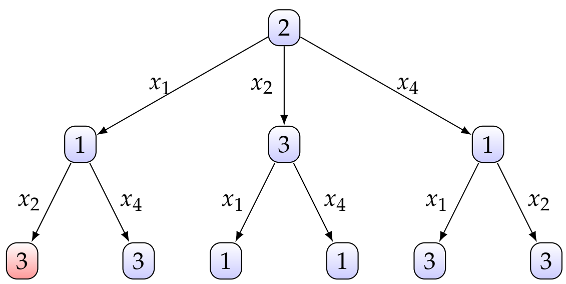
In Stage 1, the “owner“ of an object x is the individual who is assigned to the root-node of the inheritance tree Γ_x. Thus, object x_1 is owned by individual 1, objects x_2 and x_3 are owned by individual 2, and object x_4 is owned by individual 3.

Once the endowments of the individuals are decided, TTC procedure is performed with respect to the endowments to decide the outcome of Stage 1. Individuals who are assigned some object in Stage 1 leave the market with the corresponding objects. It can be verified that for the preference profile P_N given in Table 5.1, individual 1 gets object x_2 and individual 2 gets object x_1 at the outcome of TTC procedure in this stage. So, individuals 1 and 2 leave the market with objects x_2 and x_1, respectively.

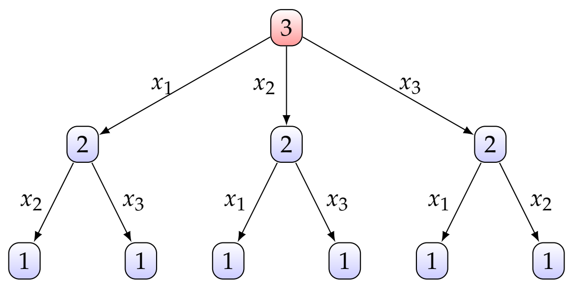
Stage 2.

As in Stage 1, the endowments of the individuals are decided first and then TTC procedure is performed with respect to the endowments. To decide the owner of a (remaining) object x, look at the root of the inheritance tree Γ_x. If the individual who appears there, say individual i, is remained in the market, then i becomes the owner of x. Otherwise, that is, if i is assigned an object in Stage 1, say y, then follow the edge from the root that is marked with y. If the individual appearing at the node following this edge, say j, is remained in the market, then j becomes the owner of x. Otherwise, that is, if j is assigned an object in Stage 1, say z, then follow the edge that is marked with z from the current node. As before, check whether the individual appearing at the end of this edge is remained in the market or not. Continue in this manner until an individual is found in the particular path who is not already assigned an object and decide that individual as the owner of x.

For the example at hand, the remaining market in Stage 2 consists of objects x_3 and x_4, and individual 3. Consider object x_3. Individual 2 appears at the root of Γ_{x_3}. Since 2 is assigned object x_1 in Stage 1, we follow the edge from the root that is marked with x_1 and come to individual 1. Since 1 is assigned object x_2, we follow the edge marked with x_2 from this node and come to individual 3. Since individual 3 is remained in the market, 3 becomes the owner of x_3. For object x_4, individual 3 appears at the root of Γ_{x_4} and she is remained in the market. So, 3 becomes the owner of x_4 in Stage 2. To emphasize the process of deciding the owner of an object, we have highlighted the node in red in the corresponding inheritance tree in Figure 5.2.



(a) Γ_{x_3}



(b) Γ_{x_4}

Figure 5.2: Stage 2

Once the endowments are decided for Stage 2, TTC procedure is performed with respect to the endowments to decide the outcome of this stage. As in Stage 1, individuals who are assigned some object in Stage 2 leave the market with the corresponding objects. It can be verified that for the current example, individual 3 gets object x_3 in this stage. So, individual 3 leave the market with objects x_3.

Stage 3 is followed on the remaining market in a similar way as Stage 2. For the current example, everybody is assigned some object by the end of Stage 2 and hence the algorithm stops in this stage. Thus, individuals 1, 2, and 3 get objects x_2, x_1, and x_3, respectively, at the outcome of the hierarchical exchange rule.

In what follows, we present a formal description of hierarchical exchange rules.

5.2.1 Inheritance trees

For a rooted tree T, the *level* of a node $v \in V(T)$ is defined as the number of edges appearing in the (unique) path from $r(T)$ to v.

Definition 5.1. For an object $x \in A$, an ***inheritance tree for*** $x \in A$ is defined as a tuple $\Gamma_x = \langle T_x, \zeta_x^{NI}, \zeta_x^{EO} \rangle$, where

(i) T_x is a rooted tree with $\max_{v \in V(T_x)} level(v) = \min\{|N|, |A|\} - 1$ and $|E^{out}(v)| = |A| - level(v) - 1$ for all $v \in V(T_x)$,

(ii) $\zeta_x^{NI} : V(T_x) \to N$ is a nodes-to-individuals function with $\zeta_x^{NI}(v) \neq \zeta_x^{NI}(\tilde{v})$ for all distinct $v, \tilde{v} \in V(T_x)$ that appear in same path, and

(iii) $\zeta_x^{EO} : E(T_x) \to A \setminus \{x\}$ is an edges-to-objects function with $\zeta_x^{EO}(e) \neq \zeta_x^{EO}(\tilde{e})$ for all distinct e, $\tilde{e} \in E(T_x)$ that appear in same path or have same source node (that is, $s(e) = s(\tilde{e})$).

In what follows, we provide two examples (for two different scenarios) of inheritance trees.

Example 5.2. Suppose $N = \{1, 2, 3\}$ and $A = \{x_1, x_2, x_3, x_4\}$. Figure 5.3 presents an example of Γ_{x_1}.

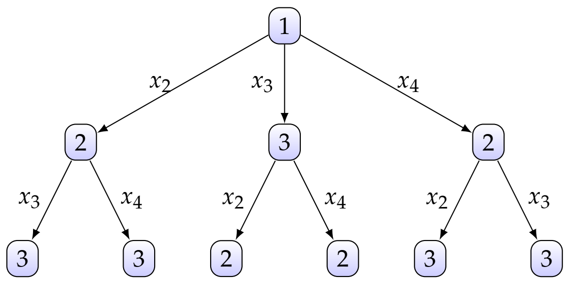


Figure 5.3: Example of Γ_{x_1}

Example 5.3. Suppose $N = \{1, 2, 3, 4\}$ and $A = \{x_1, x_2, x_3\}$. Figure 5.4 presents another example of Γ_{x_1}.

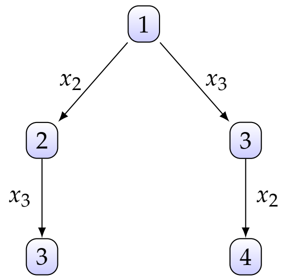


Figure 5.4: Example of Γ_{x_1}

5.2.2 Endowments

A hierarchical exchange rule works in several stages and in each stage, endowments of individuals are determined by using a (fixed) collection of inheritance trees.

Given a collection of inheritance trees $\Gamma = (\Gamma_x)_{x\in A}$, one for each object $x \in A$, we define a class of endowments $\mathcal{E}^\Gamma$ as follows:

(i) The ***initial endowment*** $\mathcal{E}_i^\Gamma(\emptyset)$ of individual i is given by

$$\mathcal{E}_i^\Gamma(\emptyset) = \{x \in A \mid \zeta_x^{NI}(r(T_x)) = i\}.$$

(ii) For all $\widehat{N} \subseteq N \setminus \{i\}$ and $\hat{A} \subseteq A$ with $|\widehat{N}| = |\hat{A}| \neq 0$, and all $\hat{\mu} \in \mathcal{M}(\widehat{N}, \hat{A})$, the ***endowment*** $\mathcal{E}_i^\Gamma(\hat{\mu})$ of individual i is given by

$$\begin{aligned}\mathcal{E}_i^\Gamma(\hat{\mu}) =\{&x \in A \setminus \hat{A} \mid \zeta_x^{NI}(r(T_x)) = i, \text{ or}\\ &\text{there exists a path } (v_x^1, \dots, v_x^{r_x}) \text{ from } r(T_x) \text{ to } v_x^{r_x} \text{ in } \Gamma_x \text{ such that } \zeta_x^{NI}(v_x^{r_x}) = i\\ &\text{and for all } s = 1, \dots, r_x - 1, \text{ we have } \zeta_x^{NI}(v_x^s) \in \widehat{N} \text{ and } \hat{\mu}(\zeta_x^{NI}(v_x^s)) = \zeta_x^{EO}(v_x^s, v_x^{s+1})\}.\end{aligned}$$

5.2.3 Iterative procedure to compute the outcome of a hierarchical exchange rule

For a given collection of inheritance trees $\Gamma = (\Gamma_x)_{x\in A}$, the ***hierarchical exchange rule*** f^Γ ***associated with*** Γ is defined by an iterative procedure with at most $\min\{|N|, |A|\}$ number of stages. Consider a preference profile $P_N \in \mathbb{L}^n(A)$.

Stage 1.

Hierarchical Endowments (Initial Endowments): For all $i \in N$, $E_1(i, P_N) = \mathcal{E}_i^\Gamma(\emptyset)$.

Top Choices: For all $i \in N$, $T_1(i, P_N) = \tau(P_i)$.

Trading Cycles: For all $i \in N$,

$$S_1(i, P_N) = \begin{cases} \{j_1, \ldots, j_g\} & \text{if there exist } j_1, \ldots, j_g \in N \text{ such that} \\ & \text{for all } s = 1, \ldots, g,\ T_1(j_s, P_N) \in E_1(j_{s+1}, P_N), \text{ and} \\ & \text{for some } \hat{s} = 1, \ldots, g,\ j_{\hat{s}} = i; \\ \emptyset & \text{otherwise.} \end{cases}$$

Since each individual can be in at most one trading cycle, $S_1(i, P_N)$ is well-defined for all $i \in N$. Furthermore, since both the number of individuals and the number of objects are finite, there is always at least one trading cycle. Note that $S_1(i, P_N) = \{i\}$ if $T_1(i, P_N) \in E_1(i, P_N)$.

Assigned Individuals: $W_1(P_N) = \{i \mid S_1(i, P_N) \neq \emptyset\}$.

Assignments: For all $i \in W_1(P_N)$, $f_i^\Gamma(P_N) = T_1(i, P_N)$.

Assigned Objects: $F_1(P_N) = \{T_1(i, P_N) \mid i \in W_1(P_N)\}$.

This procedure is repeated iteratively in the remaining reduced market. For each stage t, define $W^t(P_N) = \bigcup_{u=1}^{t} W_u(P_N)$ and $F^t(P_N) = \bigcup_{u=1}^{t} F_u(P_N)$. In what follows, we present Stage $t+1$ of f^Γ.

⋮

Stage $t+1$**.**

Hierarchical Endowments (Non-initial Endowments): Let $\mu^t \in \mathcal{M}(W^t(P_N), F^t(P_N))$ such that for all $i \in W^t(P_N)$,

$$\mu^t(i) = f_i^\Gamma(P_N).$$

For all $i \in N \setminus W^t(P_N)$, $E_{t+1}(i, P_N) = \mathcal{E}_i^\Gamma(\mu^t)$.

Top Choices: For all $i \in N \setminus W^t(P_N)$, $T_{t+1}(i, P_N) = \tau(P_i, A \setminus F^t(P_N))$.

Trading Cycles: For all $i \in N \setminus W^t(P_N)$,

$$S_{t+1}(i, P_N) = \begin{cases} \{j_1, \ldots, j_g\} & \text{if there exist } j_1, \ldots, j_g \in N \setminus W^t(P_N) \text{ such that} \\ & \text{for all } s = 1, \ldots, g,\ T_{t+1}(j_s, P_N) \in E_{t+1}(j_{s+1}, P_N), \text{ and} \\ & \text{for some } \hat{s} = 1, \ldots, g,\ j_{\hat{s}} = i; \\ \emptyset & \text{otherwise.} \end{cases}$$

Assigned Individuals: $W_{t+1}(P_N) = \{i \mid S_{t+1}(i, P_N) \neq \emptyset\}$.

Assignments: For all $i \in W_{t+1}(P_N)$, $f_i^\Gamma(P_N) = T_{t+1}(i, P_N)$.

Assigned Objects: $F_{t+1}(P_N) = \{T_{t+1}(i, P_N) \mid i \in W_{t+1}(P_N)\}$.

$$\vdots$$

This procedure is repeated iteratively until either all individuals are assigned or all objects are assigned. The hierarchical exchange rule f^Γ associated with Γ is defined as follows. For all $i \in N$,

$$f_i^\Gamma(P_N) = \begin{cases} T_t(i, P_N) & \text{if } i \in W_t(P_N) \text{ for some stage } t; \\ \emptyset & \text{otherwise.} \end{cases}$$

Since for every preference profile P_N and every individual i, there exists at most one stage t such that $i \in W_t(P_N)$, f^Γ is well-defined.

Remark 5.1. Note that a collection of inheritance trees do not uniquely identify a hierarchical exchange rule. More formally, two different collections of inheritance trees Γ and $\overline{\Gamma}$ may give rise to the same hierarchical exchange rule, that is, $f^\Gamma \equiv f^{\overline{\Gamma}}$.

6 Dual ownership

Troyan (2019) introduces the notion of *dual dictatorships* in the context of FPTTC rules.[11] We introduce a closely related notion for hierarchical exchange rules which we call ***dual ownership***. This property plays a key role in our characterization result. A hierarchical exchange rule satisfies dual ownership if for any preference profile and any stage of the hierarchical exchange rule at that preference profile, there are at most two individuals who own all the objects remained in the reduced market in that stage. Clearly, to ensure dual ownership in Stage 1 of a hierarchical exchange rule, we must have at most two individuals at the root-nodes of the inheritance trees of the hierarchical exchange rule. For instance, individuals 1, 2, and 3 own objects x_1, x_2, and x_4, respectively, in Stage 1 of the hierarchical exchange rule in Example 5.1, and consequently, the hierarchical exchange rule in this example does not satisfy dual ownership. In what follows, we present an example to illustrate the notion of dual ownership.

Example 6.1. Suppose $N = \{1, 2, 3, 4\}$ and $A = \{x_1, x_2, x_3, x_4\}$. Figure 6.1 presents a collection of inheritance trees Γ.

[11] As we have mentioned earlier, Troyan (2019) uses the term "TTC rule" to refer to an FPTTC rule.

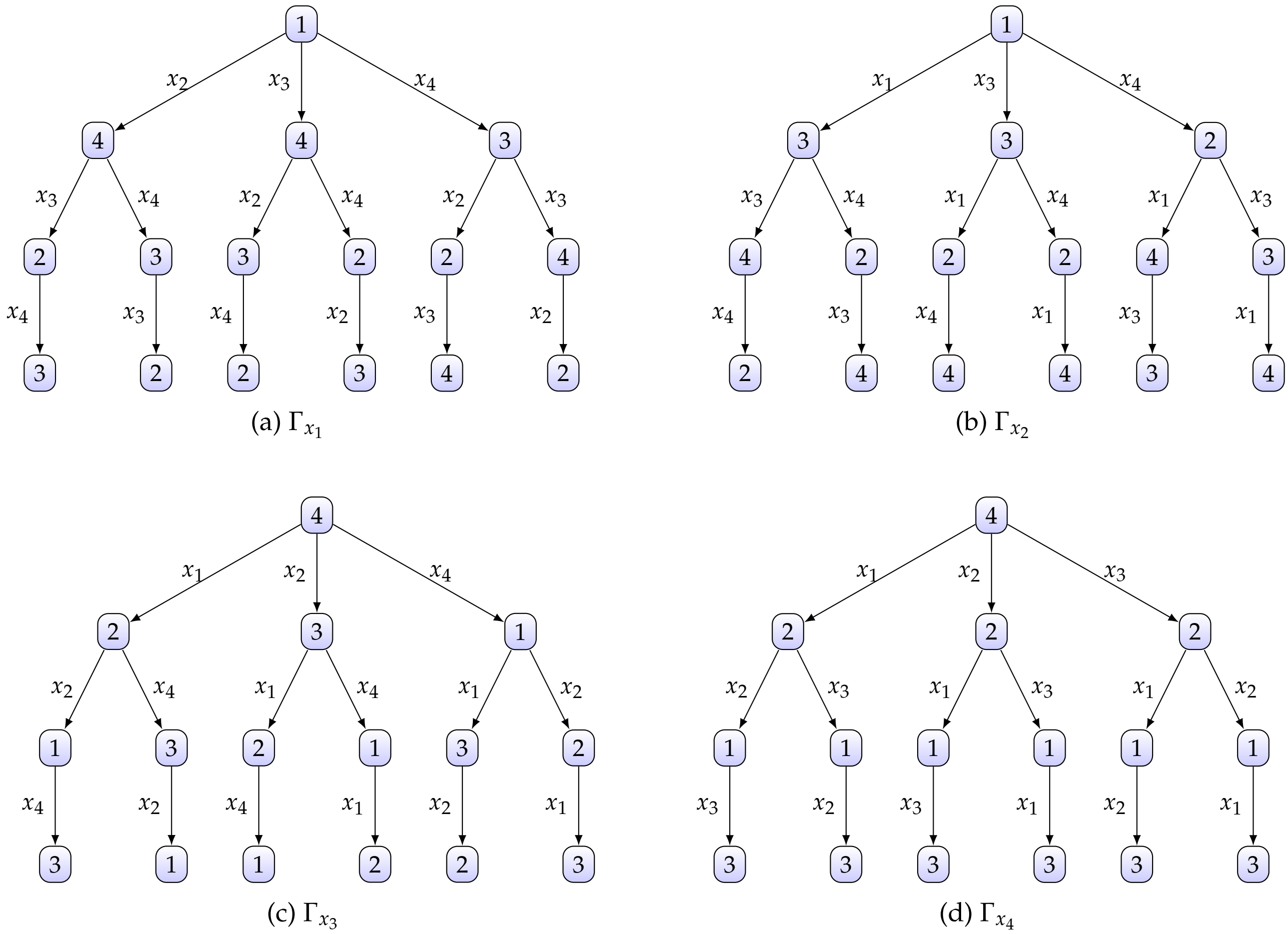


Figure 6.1: Inheritance trees for Example 6.1

Consider the hierarchical exchange rule based on these collection of inheritance trees and consider the preference profile P_N given in Table 6.1.

P_1	P_2	P_3	P_3
x_2	x_1	x_3	x_2
x_1	x_2	x_4	x_3
x_3	x_4	x_2	x_1
x_4	x_3	x_1	x_4

Table 6.1: Preference profile P_N for Example 6.1

In Stage 1, individual 1 owns the objects x_1 and x_2, and individual 4 owns the objects x_3 and x_4. At then end of this stage, individual 1 gets object x_2 and leaves the market with it.

In Stage 2, individual 4 owns the objects x_1, x_3 and x_4. At the end of this stage, individual 4 gets object x_3 and leaves the market with it.

After Stage 2, there are only 2 individuals remained in the reduced market and dual ownership cannot be violated. This concludes that the hierarchical exchange rule under consideration satisfies dual ownership at the preference profile given in Table 6.1. Note that this does not ensure the dual ownership property of the hierarchical exchange rule, as for that one needs to check the property at *every* preference profile.[12]

It should be noted from Example 6.1 that although dual ownership is an intuitive and simple property, it is somewhat time consuming to check whether a given hierarchical exchange rule satisfies this property. This is because, technically, one needs to check at every preference profile whether at most two individuals are owning all the (remaining) objects in every stage of the hierarchical exchange rule. In view of this observation, we will present an equivalent property called *acyclicity* in Section 8 which involves the inheritance trees only (and not preference profiles), and thus, is much easier to be checked.

6.1 Advantage of using hierarchical exchange rules satisfying dual ownership property

As mentioned in Troyan (2019), use of FPTTC rules in practice is rare in school choice environments as participating individuals find it difficult to understand these rules (particularly the fact that these rules are indeed strategy-proof, even though they are so in theoretical level). Hierarchical exchange rules are generalizations of FPTTC rules, and consequently, suffer from the same problem. However, as we explain in the following, when the dual ownership property is imposed on a hierarchical exchange rule, this problem reduces considerably.

In what follows, we present how a hierarchical exchange rule satisfying the dual ownership property can be explained to the participating individuals and how the explanation helps in convincing individuals that such rules are indeed strategy-proof.[13]

In Stage 1:

(1) We call at most two individuals who will be the owners in this stage.

(2) We tell them their endowed sets.

(3) We tell them that each of them can “take“ something from her endowed set (and leave the market), or “wait“ to see if she gets something better. We additionally mention that if someone chooses to “wait“, she can leave the market anytime in the future with an object from her current endowment set.

    To see that the owners will act truthfully in (3), first note that the owners are asked to choose between “take“ or “wait“, in particular, they are not asked to reveal their top choices. Therefore,

    (a) if any of the owners has her favorite object in her endowment, then she will “take” that object and leave the market, and

[12] The current hierarchical exchange rule indeed satisfies dual ownership at every preference profile.

[13] This explanation does not highlight many of the key features of hierarchical exchange rules satisfying the dual ownership property.

(b) if any of the owners does not have her favorite object in her endowment, then she will "wait" as she can leave the market anytime in the future with an object from her current endowment set.

(4) (i) If any of the owners chooses to "take" in (3). We get a submarket.

(ii) On the other hand, if both of them choose to "wait", we tell each of them to "take" something from other's endowment and leave the market, and again we get a submarket. Clearly, there is no question of manipulation for an individual at this step as she will simply take her favorite object from other's endowment.

In Stage 2:

(1) We call at most two individuals who will be the owners in this stage. If one of the owners in Stage 1 remains in the reduced market in Stage 2, we make her one of the owners in Stage 2.[14]

(2) We tell them their endowed sets. If one of the owners in Stage 2 was also an owner in Stage 1, all the objects in her endowment in Stage 1 must be included in her endowment in Stage 2.

(3) Same as Stage 1. For the same reason as we have discussed in (3) of Stage 1, individuals will act truthfully at this step of Stage 2.

(4) Same as Stage 1.

We continue this procedure until everyone is assigned or all objects are assigned.

The main reason why a hierarchical exchange rule satisfying dual ownership is simpler than an arbitrary hierarchical exchange rule is as follows. The dual ownership property ensures that at most two individuals will get to act in each stage. This makes it very simple for them to trade: they only interchange their favorite objects. In contrast, for an arbitrary hierarchical exchange rule, there might be arbitrary number of individuals trading their favorite objects in a stage, which makes it harder to asses what would happen if they do not do this truthfully.

7 A characterization of OSP-implementable assignment rules

In this Section, we provide a characterization of OSP-implementable assignment rules under two mild and desirable properties, namely Pareto efficiency and non-bossiness.

Theorem 7.1. *An assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is OSP-implementable, Pareto efficient and non-bossy if and only if f is a hierarchical exchange rule satisfying dual ownership.*

[14]Note that both owners in Stage 1 can not remain in the reduced market in Stage 2.

The proof of this theorem is relegated to Appendix B; here we provide an outline of it. We use the following two results of Pápai (2000) in the proof of Theorem 7.1: (i) strategy-proofness and non-bossiness together are equivalent to group strategy-proofness (Lemma 1 in Pápai (2000)), and (ii) an assignment rule is group strategy-proof, Pareto efficient, and reallocation-proof if and only if it is a hierarchical exchange rule (main theorem in Pápai (2000)). The proof of Theorem 7.1 is structured as follows.

We show that a hierarchical exchange rule is OSP-implementable if and only if it satisfies dual ownership (Lemma B.1). Note that this result features as Corollary 9.1 in our paper. Lemma B.1, together with the mentioned results of Pápai (2000), completes the proof of the "if" part of Theorem 7.1.

To prove the "only if" part of Theorem 7.1, we show that every OSP-implementable, non-bossy and Pareto efficient assignment rule is reallocation-proof (Lemma B.2). Since OSP is stronger than strategy-proofness (Remark 4.1), Lemma B.2, together with the mentioned results of Pápai (2000), implies that every OSP-implementable, non-bossy and Pareto efficient assignment rule is an OSP-implementable hierarchical exchange rule. Together with Lemma B.1, this completes the proof of the "only-if" part of Theorem 7.1.

Since OSP-implementability implies strategy-proofness (see Remark 4.1) and group strategy-proofness is equivalent to strategy-proofness and non-bossiness (see Pápai (2000) for details), we obtain the following corollary from Theorem 7.1.

Corollary 7.1. *A group strategy-proof and Pareto efficient assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is OSP-implementable if and only if f is a hierarchical exchange rule satisfying dual ownership.*

It is worth mentioning that OSP-implementability and non-bossiness together do not imply Pareto efficiency. For instance, any constant assignment rule satisfies the former two properties, but does not satisfy the latter. Furthermore, it follows from Pápai (2000) that non-bossiness and Pareto efficiency together do not imply strategy-proofness. Since OSP-implementability is stronger than strategy-proofness (by Remark 4.1), non-bossiness and Pareto efficiency cannot imply it either. Example 7.1 shows that OSP-implementability and Pareto efficiency together do not imply non-bossiness.

Example 7.1. Consider an allocation problem with three individuals $N = \{1, 2, 3\}$ and three objects $A = \{x_1, x_2, x_3\}$. Consider the assignment rule f such that

$$f = \begin{cases} \text{Serial dictatorship with priority } (1 \succ 2 \succ 3) & \text{if } x_2 P_1 x_3 \\ \text{Serial dictatorship with priority } (1 \succ 3 \succ 2) & \text{if } x_3 P_1 x_2 \end{cases}$$

Consider the preference profiles $P_N = (x_1x_2x_3, x_1x_2x_3, x_1x_2x_3)$ and $\tilde{P}_N = (x_1x_3x_2, x_1x_2x_3, x_1x_2x_3)$.[15] Note that only individual 1 changes her preference from P_N to $\tilde{P}_N$. This, together with the facts $f(P_N) = [(1, x_1), (2, x_2), (3, x_3)]$ and $f(\tilde{P}_N) = [(1, x_1), (2, x_3), (3, x_2)]$, implies f is not non-bossy. However, the OSP

[15] Here, we denote by $(x_1x_2x_3, x_2x_3x_1, x_3x_2x_1)$ a preference profile where individuals 1, 2 and 3 have preferences $x_1x_2x_3$, $x_2x_3x_1$, and $x_3x_2x_1$, respectively.

mechanism in Figure 7.1 implements f.[16] This, in particular, means that f is OSP-implementable. Also, it can be easily verified that f is Pareto efficient.

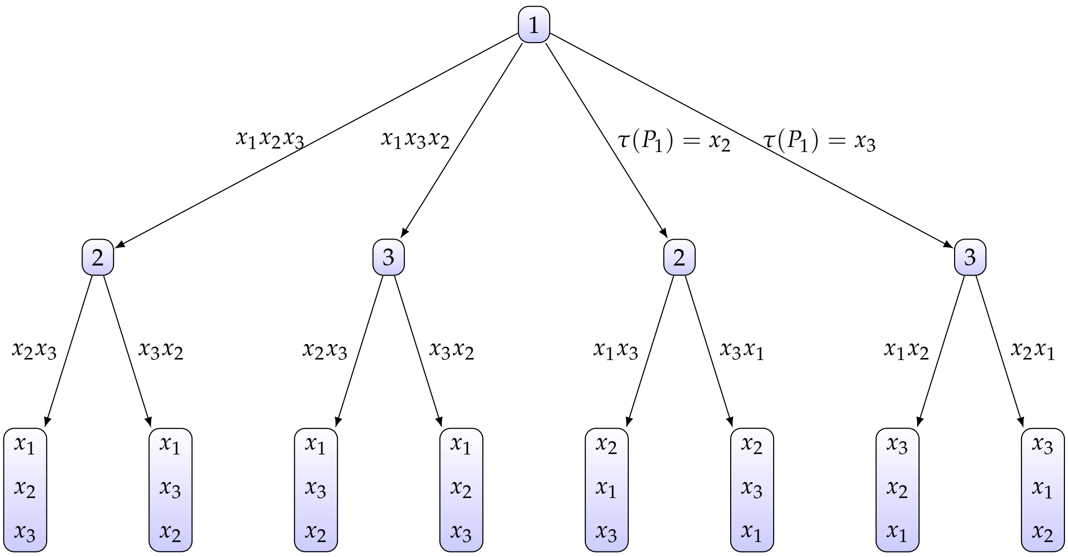


Figure 7.1: Tree Representation for Example 7.1

8 Acyclicity: equivalent to dual ownership

In this section, we introduce a property, called *acyclicity*, of a collection of inheritance trees and show that it is equivalent to the dual ownership property of the corresponding hierarchical exchange rule. A collection of inheritance tress is *acyclic* if they do not have any *inheritance cycle*. The notion of inheritance cycle is somewhat involved, so we begin with a verbal description of the same.[17] We have extended that idea for an inheritance cycle in a collection of inheritance trees. We use the following terminologies to facilitate the description (and the subsequent definitions). A path in an inheritance tree is called a ***root-path*** if it starts from the root of the inheritance tree. We use the notation π_x to denote a root-path in an inheritance tree Γ_x. For ease of presentation, sometimes we represent a root-path by the sequence of node-assignments and edge-assignments in it. For instance, we denote by $i_4 \;\; x_5 \;\; i_5 \;\; x_4 \;\; i_1$ a root-path that starts at the root-node to which individual i_4 is assigned, follows the edge to which object x_5 is assigned and goes to the node assigned to individual i_5, then follows the edge assigned to object x_4 and goes to the node assigned to individual i_1. For an individual i in a root-path π_x, we say *individual i lies in the interior of π_x*, if individual i is *not* assigned to the *last* node of π_x. For instance, individuals i_4 and i_5 lie in the interior of the aforementioned path.

[16] Apart from the notations that we have already introduced in this paper, we use the following notation in Figure 7.1: the allocation $[(1, x_1), (2, x_2), (3, x_3)]$ is denoted by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

[17] The notion of an inheritance cycle becomes simpler if we consider a priority structure in place of a collection of inheritance trees. In Section 10.2, we provide an intuitive explanation of the same.

A tuple $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$ where $i_1, i_2, i_3 \in N$ and $x_1, x_2, x_3 \in A$ are all distinct, can constitute an inheritance cycle in two ways. In the first way, individual i_h is assigned to the root-node of Γ_{x_h} for all $h = 1, 2, 3$. To explain the second way, let us present a specific instance where individuals i_1, i_2, i_3 and objects x_1, x_2, x_3 form an inheritance cycle. Suppose there exist distinct individuals $i_4, i_5, i_6 \in N \setminus \{i_1, i_2, i_3\}$, distinct objects $x_4, x_5, x_6 \in A \setminus \{x_1, x_2, x_3\}$, and a feasible allocation $\hat{\mu}$ of $\{i_4, i_5, i_6\}$ over $\{x_4, x_5, x_6\}$, say

$$\hat{\mu}(i_4) = x_5,\ \hat{\mu}(i_5) = x_4, \text{ and } \hat{\mu}(i_6) = x_6.$$

For $h = 1, \ldots, 6$, let π_{x_h} be a root-path in Γ_{x_h} as given below.

$$\pi_{x_1} : i_4\ \ x_5\ \ i_5\ \ x_4\ \ i_1$$

$$\pi_{x_2} : i_6\ \ x_6\ \ i_5\ \ x_4\ \ i_2$$

$$\pi_{x_3} : i_4\ \ x_5\ \ i_3$$

$$\pi_{x_4} : i_6\ \ x_6\ \ i_4$$

$$\pi_{x_5} : i_6\ \ x_6\ \ i_5$$

$$\pi_{x_6} : i_6$$

Note that this collection of root-paths satisfies the following properties.

(i) (a) For all $h = 1, \ldots, 6$, the last element of the root-path π_{x_h} is individual i_h and the other individuals in π_{x_h} are from the set $\{i_4, i_5, i_6\}$. For instance, consider the root-path π_{x_1}. The last element is individual i_1 and the other individuals are i_4 and i_5.

(b) For all $h = 4, 5, 6$, if there is an (outgoing) edge from individual i_h in any of the root-paths in $(\pi_{x_h})_{h \in \{1,\ldots,6\}}$, then object $\hat{\mu}(i_h)$ is assigned to that edge. For instance, there is an edge from individual i_4 in the paths π_{x_1} and π_{x_3}, and the object x_5 (which is $\hat{\mu}(i_4)$) is assigned to all these edges.

(ii) For all distinct objects $x_i, x_j \in \{x_4, x_5, x_6\}$, if individual $\hat{\mu}(x_i)$ lies in the interior of the root-path π_{x_j}, then individual $\hat{\mu}(x_j)$ does *not* lie in the root-path π_{x_i}. For instance, $\hat{\mu}(x_6) = i_6$ lies in the interior of the root-path π_{x_4}, and hence, $\hat{\mu}(x_4) = i_5$ does not lie in the root-path π_{x_6}.

In this case, the tuple $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$ is called an inheritance cycle. In general, a tuple $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$ is an inheritance cycle if one can get hold of individuals $\{i_4, \ldots, i_t\}$, objects $\{x_4, \ldots, x_t\}$, a feasible allocation $\hat{\mu}$, and a collection of paths $(\pi_{x_h})_{h \in \{1,\ldots,t\}}$ satisfying properties as stated above. In what follows, we present a formal definition.

Definition 8.1. A tuple $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$, where $i_1, i_2, i_3 \in N$ and $x_1, x_2, x_3 \in A$ are all distinct, is called an ***inheritance cycle*** in a collection of inheritance trees Γ if either individuals i_1, i_2, and i_3 are assigned to the root-nodes of Γ_{x_1}, Γ_{x_2}, and Γ_{x_3}, respectively, or there exist distinct individuals $i_4, \ldots,$ $i_t \in N \setminus \{i_1, i_2, i_3\}$, distinct objects $x_4, \ldots, x_t \in A \setminus \{x_1, x_2, x_3\}$, an allocation $\hat{\mu} \in \mathcal{M}(\{i_4, \ldots, i_t\}, \{x_4,$ $\ldots, x_t\})$, and a collection of root-paths $(\pi_{x_h})_{h \in \{1,\ldots,t\}}$ with the properties that

(i) (a) for all $h = 1, \ldots, t$, the last element of the root-path π_{x_h} is individual i_h and the other individuals in π_{x_h} are from the set $\{i_4, \ldots, i_t\}$,

(b) for all $h = 4, \ldots, t$, if there is an (outgoing) edge from individual i_h in any of the root-paths in $(\pi_{x_h})_{h \in \{1,\ldots,t\}}$, then object $\hat{\mu}(i_h)$ is assigned to that edge, and

(ii) for all distinct objects $x_i, x_j \in \{x_4, \ldots, x_t\}$, if individual $\hat{\mu}(x_i)$ lies in the interior of the root-path π_{x_j}, then individual $\hat{\mu}(x_j)$ does *not* lie in the root-path π_{x_i}.

We call a collection of inheritance trees ***acyclic*** if it contains no inheritance cycles, and call a hierarchical exchange rule ***acyclic*** if it is based on an acyclic collection of inheritance trees.

Our next theorem says that acyclicity and dual ownership are equivalent properties of a hierarchical exchange rule.

Theorem 8.1. *A hierarchical exchange rule satisfies dual ownership if and only if it is acyclic.*

The proof of this theorem is relegated to Appendix A.

9 OSP-implementability of hierarchical exchange rules and trading cycles rules

In this section, we provide a necessary and sufficient condition for a hierarchical exchange rule and a trading cycles rule to be OSP-implementable.

Corollary 9.1. *A hierarchical exchange rule is OSP-implementable if and only if it satisfies dual ownership.*

We prove this corollary as a part of the proof of Theorem 7.1 (see Lemma B.1).

Pycia and Ünver (2017) introduce a general version of hierarchical exchange rules which they call *trading cycles rules*. They show that an assignment rule is group strategy-proof and Pareto efficient if and only if it is a trading cycles rule. Combining this result with Corollary 7.1, we obtain the following corollary.

Corollary 9.2. *A trading cycles rule is OSP-implementable if and only if it is a hierarchical exchange rule satisfying dual ownership.*

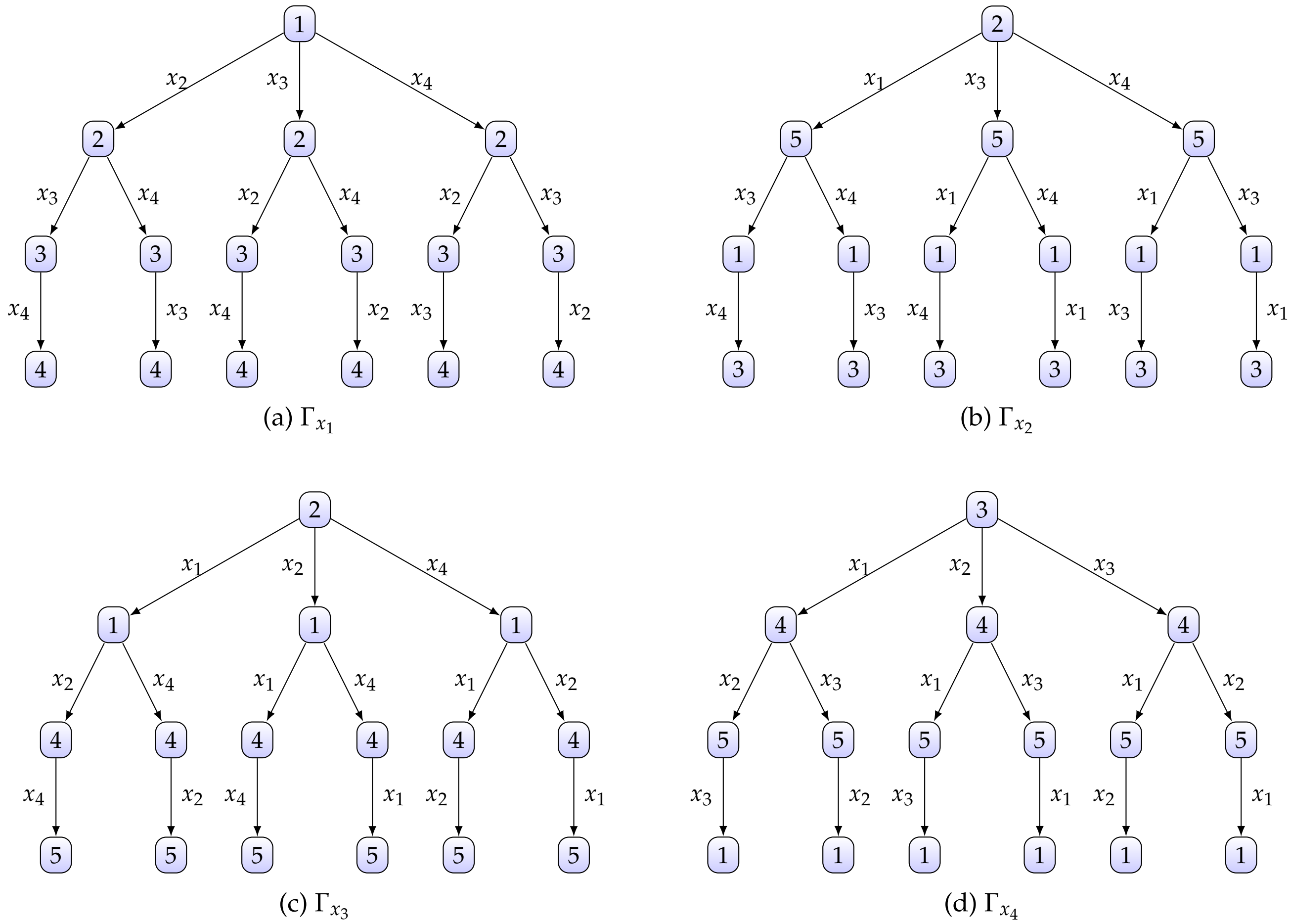
10 OSP-implementability of FPTTC rules

In this section, we consider FPTTC rules and investigate when such assignment rules are OSP-implementable. FPTTC rules are special cases of hierarchical exchange rules when the collection of inheritance trees satisfies a property, which we call the ***priority property*** (see Pápai (2000) for details).[18] In what follows, we present the definition of the priority property and explain how this property induces a priority structure for an FPTTC rule.

A collection of inheritance trees $\Gamma = (\Gamma_x)_{x \in A}$ satisfies the ***priority property*** if for all $x \in A$ and all $v, \tilde{v} \in V(T_x)$ with $level(v) = level(\tilde{v})$, we have $\zeta_x^{NI}(v) = \zeta_x^{NI}(\tilde{v})$. We provide an example of a collection of inheritance trees Γ that satisfies the priority property.

Example 10.1. Suppose $N = \{1, 2, 3, 4, 5, 6\}$ and $A = \{x_1, x_2, x_3, x_4\}$. Consider the collection of inheritance trees given in Figure 10.1. To see that the collection satisfies the priority property, consider one inheritance tree from the collection, say Γ_{x_1}. Note that individual 1 is assigned to the root-node of Γ_{x_1}, individual 2 is assigned to all nodes of Γ_{x_1} with level 1, individual 3 is assigned to all nodes of Γ_{x_1} with level 2, and individual 4 is assigned to all nodes of Γ_{x_1} with level 3. Similarly, it can be verified for any other inheritance tree that the same individual is assigned to all the nodes with a given level.

[18] Pápai (2000) uses the term "fixed endowment hierarchical exchange rule" to refer to the hierarchical exchange rule associated with a Γ satisfying the priority property.



(a) Γ_{x_1} (b) Γ_{x_2} (c) Γ_{x_3} (d) Γ_{x_4}

Figure 10.1: Γ with priority property

We now explain how for each $x \in A$, a preference $\succ_x$ over N can be constructed based on a collection of inheritance trees Γ that satisfies the priority property. For ease of presentation, by the *level of an individual i in* Γ_x, we mean the level of a node in Γ_x where i is assigned. Note that this is well-defined since Γ satisfies the priority property. Define the level of an individual in Γ_x who does not appear in Γ_x as $|A|$.

For each $x \in A$, let $\widehat{\succ}_x$ be a preference over N such that $i \widehat{\succ}_x j$ if the level of i is less than that of j. Note that such a preference $\widehat{\succ}_x$ is not unique since it does not specify the relative ranking of the individuals who do not appear in Γ_x. We say that the priority structure $\widehat{\succ}_A := (\widehat{\succ}_x)_{x \in A}$ is induced by Γ. Clearly, the induced priority structure need not be unique. In Table 10.1 we present one induced priority structure of the collection of inheritance trees presented in Example 10.1 (Figure 10.1).

P_{x_1}	P_{x_2}	P_{x_3}	P_{x_4}
1	2	2	3
2	5	1	4
3	1	4	5
4	3	5	1
5	6	3	6
6	4	6	2

Table 10.1: One induced priority structure of the collection of inheritance trees in Example 10.1 (Figure 10.1)

10.1 Counter example of Theorem 1 in Troyan (2019)

Troyan (2019) deals with the case where there are equal number of individuals and objects, i.e., $|N| = |A|$. Theorem 1 of his paper says that an FPTTC rule is OSP-implementable if and only if the priority structure satisfies a property called *weak acyclicity*. In this subsection, we provide a counter example to this theorem.

For ease of presentation, for a preference P over a set X and two subsets Y and Z of X, we write YPZ to mean that each element of Y is preferred to each element of Z, that is, yPz for all $y \in Y$ and all $z \in Z$.

Definition 10.1. (Troyan, 2019) A priority structure $\succ_A$ is said to have a ***strong cycle*** if there are three objects $x, y, z \in A$ and three individuals $i, j, k \in N$ such that $i \succ_x \{j,k\}$, $j \succ_y \{i,k\}$, and $k \succ_z \{i,j\}$. If $\succ_A$ contains no strong cycles, then we say $\succ_A$ is ***weakly acyclic***.

In what follows, we present an example to show that an FPTTC rule can be OSP-implementable even if the priority structure has a strong cycle.[19]

Example 10.2. Consider an allocation problem with four individuals $N = \{i, j, k, l\}$ and four objects $A = \{w, x, y, z\}$. Let $\succ_A$ be as follows:

$\succ_w$	$\succ_x$	$\succ_y$	$\succ_z$
i	i	l	l
j	j	j	k
k	k	k	j
l	l	i	i

Table 10.2: Priority structure for Example 10.2

[19]Note that this contradicts the "only-if" part Theorem 1 in Troyan (2019). In order to prove this part, Troyan (2019) considers an arbitrary priority structure $\succ_A$ with a strong cycle with $x, y, z \in A$ and $i, j, k \in N$ such that $i \succ_x \{j,k\}$, $j \succ_y \{i,k\}$, and $k \succ_z \{i,j\}$. To show that the FPTTC rule $T^{\succ_A}$ is not OSP-implementable, he reduces the whole problem to a restricted domain and uses a result from Li (2017). However, for the purpose of Troyan (2019), this reduction step is not correct.

Note that $i \succ_x \{j, k\}$, $j \succ_y \{i, k\}$, and $k \succ_z \{i, j\}$, which means there is a strong cycle in $\succ_A$. However, the OSP mechanism in Figure 10.2 shows that $T^{\succ_A}$ is OSP-implementable.[20]

[20] Apart from the notations we have already introduced, we use the following notations in Figure 10.2: by $x\{y,z\}$ we denote the set of preferences P such that xPy and xPz, and we denote an allocation $[(i,w),(j,x),(k,y),(l,z)]$ by

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}.$$

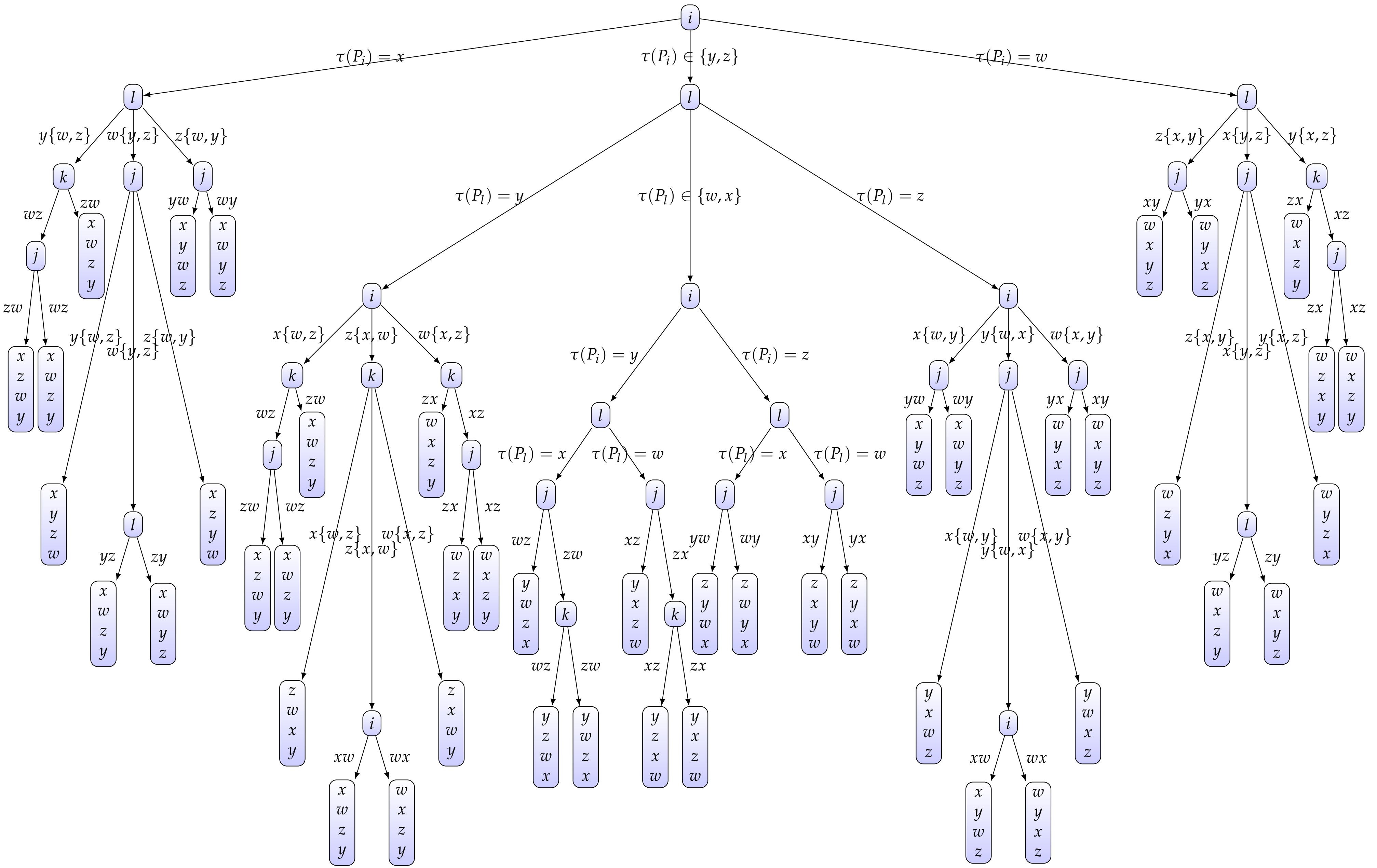


Figure 10.2: Tree representation of $T^{\succ_A}$ in Example 10.2

10.2 Characterizations of OSP-implementable FPTTC rules

As we have mentioned, Troyan (2019) deals with the case where $|N| = |A|$. In this subsection, we consider arbitrary values of $|N|$ and $|A|$ and provide two characterizations of OSP-implementable FPTTC rules for, one using the dual ownership property and the other using the acyclicity property.

Since FPTTC rules are special cases of hierarchical exchange rules, the dual ownership property of FPTTC rules implies the following: for any preference profile and any step of the FPTTC rule at that preference profile, there are at most two individuals who own all the objects remained in the reduced market at that step. Thus, we obtain the following corollary from Corollary 9.1.

Corollary 10.1. *An FPTTC rule is OSP-implementable if and only if it satisfies dual ownership.*

Next, we explain the implication of the acyclicity property for a priority structure and present a correct version of Theorem 1 in Troyan (2019) (in fact, we present a general result for arbitrary values of $|N|$ and $|A|$).

We explain the reduced form of acyclicity when a collection of inheritance trees is reduced to a priority structure. We use the following terminologies to facilitate the explanation (and the subsequent definitions). For a preference $\succ \in \mathbb{L}(N)$ and an individual $i \in N$, by $U(i, \succ)$ we denote the (strict) upper contour set $\{j \in N \mid j \succ i\}$ of i at $\succ$. Following our notational convention, we write $\tau(\succ)$ to denote the most preferred individual according to $\succ$.

When we deal with a priority structure (in place of a collection of inheritance trees), the implication of an inheritance cycle becomes simpler. We call it a *priority cycle*. The purpose of a priority cycle in a priority structure is to make the corresponding FPTTC rule *not* OSP-implementable. By Corollary 10.1, this can be done by making the corresponding FPTTC rule violate the dual ownership property. An FPTTC rule violates the dual ownership property if there exists a preference profile and a step of the FPTTC rule at that preference profile such that there are at least three owners at that step. Suppose individuals i_1, i_2, and i_3 own objects x_1, x_2, and x_3, respectively, at Step s for some s. If $s = 1$, then it has to be that i_1, i_2, and i_3 are the most preferred individuals of $\succ_{x_1}$, $\succ_{x_2}$, and $\succ_{x_3}$, respectively. If $s > 1$, then one necessary condition for making i_1, i_2, and i_3 the owners of x_1, x_2, and x_3, respectively, at Step s, is that all the individuals in the sets $U(i_1, \succ_{x_1}), U(i_2, \succ_{x_2})$, and $U(i_3, \succ_{x_3})$ are assigned before Step s. A part of the definition of priority cycle captures this necessary condition. Note that ensuring that the individuals in $U(i_1, \succ_{x_1}), U(i_2, \succ_{x_2})$, and $U(i_3, \succ_{x_3})$ are assigned before Step s is *not sufficient* to ensure that i_1, i_2, and i_3 will be the owners of x_1, x_2, and x_3, respectively, at Step s. The definition of priority cycle captures the additional requirements for this.

A tuple $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$ where $i_1, i_2, i_3 \in N$ and $x_1, x_2, x_3 \in A$ are all distinct, constitutes a priority cycle in two ways. In the first way, i_h is the most preferred individual of $\succ_{x_h}$ for all $h = 1, 2, 3$. As we have done in case of an inheritance cycle, let us explain the second way with a specific instance where individuals i_1, i_2, i_3 and objects x_1, x_2, x_3 form a priority cycle. Suppose there exist distinct individuals

$i_4, i_5 \in N \setminus \{i_1, i_2, i_3\}$ and distinct objects $x_4, x_5 \in A \setminus \{x_1, x_2, x_3\}$. For $h = 1, \ldots, 5$, let $\succ_{x_h}$ be as given below (the dots indicate that all preferences for the corresponding parts are irrelevant and can be chosen arbitrarily).

$\succ_{x_1}$	$\succ_{x_2}$	$\succ_{x_3}$	$\succ_{x_4}$	$\succ_{x_5}$
i_4	i_4	i_4	i_5	i_5
i_1	i_2	i_3	i_4	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Table 10.3: Priority structure with a priority cycle

The priority structure in Table 10.3 has the property that for all $h = 1, \ldots, 5$, the (strict) upper contour set of individual i_h at $\succ_{x_h}$ is a subset of $\{i_4, i_5\}$. For instance, the (strict) upper contour set of individual i_1 is the singleton set $\{i_4\}$. In this case, the tuple $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$ is called a priority cycle. In general, a tuple $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$ is a priority cycle if one can get hold of individuals $\{i_4, \ldots, i_t\}$, objects $\{x_4, \ldots, x_t\}$ such that their priority structure satisfies the property as stated above. In what follows, we present a formal definition.

Definition 10.2. A tuple $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$, where $i_1, i_2, i_3 \in N$ and $x_1, x_2, x_3 \in A$ are all distinct, is called a ***priority cycle*** at a priority structure $\succ_A$ if either $\tau(\succ_{x_h}) = i_h$ for all $h = 1, 2, 3$, or there exist distinct individuals $i_4, \ldots, i_t \in N \setminus \{i_1, i_2, i_3\}$ and distinct objects $x_4, \ldots, x_t \in A \setminus \{x_1, x_2, x_3\}$ such that for all $h = 1, \ldots, t$, we have $U(i_h, \succ_{x_h}) \subseteq \{i_4, \ldots, i_t\}$.

If a priority structure $\succ_A$ does not contain any priority cycle, then we say $\succ_A$ is ***acyclic***.

Remark 10.1. Note that if $\succ_A$ contains a priority cycle $[(i, j, k), (x, y, z)]$, then $i \succ_x \{j, k\}$, $j \succ_y \{i, k\}$, and $k \succ_z \{i, j\}$. Therefore every priority cycle is a strong cycle, and hence weak acyclicity implies acyclicity. However, the converse is not true (see the priority structure in Example 10.2).

The following lemma establishes a connection between the acyclicity property of a collection of inheritance trees and that of a priority structure.

Proposition 10.1. *Let Γ be a collection of inheritance trees which satisfies the priority property. Suppose a priority structure $\widehat{\succ}_A$ is induced by Γ. Then, $\widehat{\succ}_A$ is acyclic if and only if Γ is acyclic.*

The proof of this proposition is relegated to Appendix C.

We obtain the following corollary by applying Proposition 10.1 and Theorem 8.1 to Corollary 9.1. It provides a characterization of OSP-implementable FPTTC rules.

Corollary 10.2. *Let $\succ_A$ be a priority structure. The FPTTC rule $T^{\succ_A}$ is OSP-implementable if and only if $\succ_A$ is acyclic.*

10.3 Relation between dual dictatorships (Troyan, 2019) and dual ownership of FPTTC rules

As we have mentioned earlier, Troyan (2019) introduces the notion of *dual dictatorships*. It follows from Theorem 1 and Theorem 2 of his paper that an FPTTC rule is OSP-implementable if and only if it satisfies dual dictatorships, whereas Corollary 10.1 of our paper says that an FPTTC rule is OSP-implementable if and only if it satisfies dual ownership. In what follows, we clarify the difference between these two (conflicting) results and conclude that while dual dictatorships is a sufficient condition for an FPTTC rule to be OSP-implementable, it is *not* necessary.

Dual dictatorships property of an FPTTC rule requires that in any submarket, at most two individuals will own all the objects in the submarket. In contrast, dual ownership property of an FPTTC rule requires that for every preference profile and every step of that FPTTC rule at that preference profile, at most two individuals will own all the objects remained in reduced market at that step. The difference between these two properties arises from the fact that *not every* submarket occurs at some step at some preference profile of an FPTTC rule. In other words, dual dictatorships is stronger than dual ownership. We clarify this by means of the following example. We consider the same number of individuals and objects in this example as the results in Troyan (2019) are derived under that assumption.

Example 10.3. Consider the FPTTC rule given in Example 10.2. First, we argue that it satisfies dual ownership. Since either individual i or individual l appears at the top position in each priority (see Table 10.2 in Example 10.2), it follows that for any preference profile, individuals i and l will own all the objects at Step 1 of the FPTTC rule. Moreover, since there are only four individuals in the original market, for any preference profile, at any step from Step 3 onward of the FPTTC rule, there will remain at most two individuals in the corresponding submarket and hence dual ownership will be vacuously satisfied. In what follows, we show that dual ownership will also be satisfied at Step 2 for any preference profile. We distinguish three cases based on the possible assignments at Step 1.

(i) Suppose only individual i is assigned some object at Step 1. No matter whether individual i is assigned object w or object x, individuals j and l will own all the objects at Step 2.

(ii) Suppose only individual l is assigned some object at Step 1.

   (a) If l is assigned object y, then individuals i and k will own all the objects at Step 2.

   (b) If l is assigned object z, then individuals i and j will own all the objects at Step 2.

(iii) Suppose both i and l are assigned some objects at Step 1. Since there are only four individuals in the original market, only two individuals will remain in the reduced market at Step 2.

Since Cases (i), (ii), and (iii) are exhaustive, it follows that the FPTTC rule in Example 10.2 satisfies dual ownership. We now proceed to show that it does not satisfy dual dictatorships. Consider the submarket consisting of individuals i, j, and k and objects x, y, and z. Here, individuals i, j, and k will own objects x, y, and z, respectively, and hence the FPTTC rule under consideration violates dual dictatorships.

11 Relation between sequential barter with lurkers (Bade and Gonczarowski, 2017) and dual ownership

Bade and Gonczarowski (2017) introduce the notion of sequential barter with lurkers for an assignment rule and show that an assignment rule is OSP-implementable and Pareto efficient if and only if it is a sequential barter with lurkers rule (Theorem 7.2 in Bade and Gonczarowski (2017)). Since hierarchical exchange rules satisfying acyclicity (or dual ownership) characterize all OSP-implementable, Pareto efficient, as well as non-bossy assignment rules, technically they must be special cases of sequential barter with lurkers rules, obtained under the imposition of non-bossiness. The sequential barter with lurkers rules are defined in the approach of the dual ownership property.[21] However, we do not see a way to relate these two types of assignment rules as the formal definition of sequential barter with lurkers rules (as given in Bade and Gonczarowski (2017)) is constructive (computer algorithmic) while hierarchical exchange rules are defined with a functional form. In what follows, we explain how hierarchical exchange rules can be generalized to relax the non-bossiness requirement and sequential barter with lurkers rules can be seen as such a generalization.

Recall that for a hierarchical exchange rule, the ownership of the objects (that is, which object will be owned by which individual) in a given stage depends on the assignments of the owners in the previous stages. For a sequential barter with lurkers rule, this ownership depends not only on the assignments of the previous owners but also on their preferences. Thus, it is possible for a sequential barter with lurkers rules that some individuals (who are the owners in a given stage) can change the assignments of some other individuals (the individuals who remain in the market in the next stage) without changing their own assignments (by changing their preferences only). Clearly, this makes the sequential barter with lurkers rules bossy, while the hierarchical exchange rules (be it acyclic or not) remain non-bossy.

[21] The following verbal description of sequential barter with lurkers rules is taken from Bade and Gonczarowski (2017). "Sequential barter is a trading mechanism with many rounds. At each such round, there are at most two owners. Each not-yet-matched house sequentially becomes owned by one of them. Each of the owners may decide to leave with a house that she owns, or they may both agree to swap. If an owner does not get matched in the current round, she owns at least the same houses in the next round. When a lurker appears, she may ultimately get matched to any one house in some set S. A lurker is similar to a dictator in the sense that she may immediately appropriate *all but one special house* in the set S. If she favors this special house the most, she may "lurk" it, in which case she is no longer considered an owner (so there are at most two owners, and additionally any number of lurkers, each for a different house). If no agent who is entitled to get matched with this special house chooses to do so, then the lurker obtains it as a residual claimant. Otherwise, the lurker gets the second-best house in this set S."

12 Conclusion

In this paper, we have provided a characterization of OSP-implementable, Pareto efficient and non-bossy assignment rules in the context of one-sided matching problem. We have shown that such assignment rules are hierarchical exchange rules satisfying dual ownership. As corollaries of our result, we have characterized all OSP-implementable hierarchical exchange rules, all OSP-implementable trading cycles rules, and all OSP-implementable FPTTC rules.

The structure of OSP-implementable assignment rules in the context of two-sided matching problem is not much explored. Ashlagi and Gonczarowski (2018) provide a sufficient condition for a deferred acceptance rule to be OSP-implementable. We think providing a characterization of OSP-implementable rules in the context of two-sided matching problem is an important problem and we are working on it.

Appendix A Proof of Theorem 8.1

Let us first recall some of the notations used in the context of hierarchical exchange rules: for a preference profile $P_N \in \mathbb{L}^n(A)$ and a hierarchical exchange rule, $F_s(P_N)$ is the set of assigned objects in Stage s, $W_s(P_N)$ is the set of assigned individuals in Stage s, $W^s(P_N)$ is the set of assigned individuals up to Stage s (including Stage s), and $E_s(i, P_N)$ is the endowment set for individual i in Stage s.

Proof of Theorem 8.1. (If part) Suppose f^Γ does not satisfy dual ownership. We show that Γ contains an inheritance cycle.

Since f^Γ does not satisfy dual ownership, there exists a preference profile $\tilde{P}_N$ and a stage s^* of f^Γ at $\tilde{P}_N$ such that there are three individuals i_1, i_2, i_3 and three objects x_1, x_2, x_3 in the reduced market in Stage s^* with the property that individual i_h owns the object x_h for all $h = 1, 2, 3$. We proceed to show that $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$ is an inheritance cycle in Γ. We distinguish the following two cases.

Case 1: Suppose $s^* = 1$.

Since for all $h = 1, 2, 3$, individual i_h owns the object x_h in Stage 1, by the definition of f^Γ, it follows that individuals i_1, i_2, and i_3 are assigned to the root-nodes of Γ_{x_1}, Γ_{x_2}, and Γ_{x_3}, respectively. This means $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$ is an inheritance cycle in Γ.

Case 2: Suppose $s^* > 1$.

Let $\{i_4, \ldots, i_t\} \subseteq N \setminus \{i_1, i_2, i_3\}$, $\{x_4, \ldots, x_t\} \subseteq A \setminus \{x_1, x_2, x_3\}$, an allocation $\hat{\mu} \in \mathcal{M}(\{i_4, \ldots, i_t\}, \{x_4, \ldots, x_t\})$, and a collection of root-paths $(\pi_{x_h})_{h \in \{1,\ldots,t\}}$ be as follows.

(i) $\{i_4, \ldots, i_t\} = W^{s^*-1}(\tilde{P}_N)$.

(ii) For all $h = 4, \ldots, t$, $\{x_h\} = \big(F_s(\tilde{P}_N) \cap E_s(i_h, \tilde{P}_N)\big)$ where $i_h \in W_s(\tilde{P}_N)$ for some $s < s^*$. To see that this is well-defined note that by the definition of f^Γ, (a) for every $i_h \in W^{s^*-1}(\tilde{P}_N)$, there exists exactly one stage s with $s < s^*$ such that $i_h \in W_s(\tilde{P}_N)$, and (b) $E_s(i_h, \tilde{P}_N) \cap F_s(\tilde{P}_N)$ is a singleton set for all $i_h \in W_s(\tilde{P}_N)$ with $s < s^*$.

(iii) For all $h = 4, \ldots, t$, $\hat{\mu}(i_h) = f^{\Gamma}_{i_h}(\tilde{P}_N)$. By the definition of f^{Γ}, it follows that $\hat{\mu} \in \mathcal{M}(\{i_4, \ldots, i_t\}, \{x_4, \ldots, x_t\})$.

(iv) (a) For all $h = 1, 2, 3$, π_{x_h} is the root-path in Γ_{x_h} such that the "property rights" of the object x_h travels along the root-path π_{x_h} up to Stage s^* (including Stage s^*) of f^{Γ} at preference profile $\tilde{P}_N$.[22]

(b) For all $h = 4, \ldots, t$, π_{x_h} is the root-path in Γ_{x_h} such that the "property rights" of the object x_h travels along the root-path π_{x_h} until it is assigned in the iterative procedure of f^{Γ} at the preference profile $\tilde{P}_N$.[23]

It follows from the definition of f^{Γ} and the construction of $\{i_4, \ldots, i_t\}$, $\{x_4, \ldots, x_t\}$, $\hat{\mu}$, and $(\pi_{x_h})_{h \in \{1,\ldots,t\}}$ that

(i) (a) for all $h = 1, \ldots, t$, the last element of the root-path π_{x_h} is individual i_h and the other individuals in π_{x_h} are from the set $\{i_4, \ldots, i_t\}$,

(b) for all $h = 4, \ldots, t$, if there is an (outgoing) edge from individual i_h in any of the root-paths in $(\pi_{x_h})_{h \in \{1,\ldots,t\}}$, then object $\hat{\mu}(i_h)$ is assigned to that edge, and

(ii) for all distinct objects $x_i, x_j \in \{x_4, \ldots, x_t\}$, if individual $\hat{\mu}(x_i)$ lies in the interior of the root-path π_{x_j}, then individual $\hat{\mu}(x_j)$ does not lie in the root-path π_{x_i}.

This implies that $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$ is an inheritance cycle in Γ, which completes the proof of the "if" part of Theorem 8.1.

(Only-if part) Suppose Γ contains an inheritance cycle $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$. We show that f^{Γ} does not satisfy dual ownership. By the definition of an inheritance cycle, one of the following two statements must hold.

(1) Individuals i_1, i_2, and i_3 are assigned to the root-nodes of Γ_{x_1}, Γ_{x_2}, and Γ_{x_3}, respectively.

(2) There exist distinct individuals $i_4, \ldots, i_t \in N \setminus \{i_1, i_2, i_3\}$, distinct objects $x_4, \ldots, x_t \in A \setminus \{x_1, x_2, x_3\}$, an allocation $\hat{\mu} \in \mathcal{M}(\{i_4, \ldots, i_t\}, \{x_4, \ldots, x_t\})$, and a collection of root-paths $(\pi_{x_h})_{h \in \{1,\ldots,t\}}$ with the properties that

[22] More precisely, if $\pi_{x_h} = j_1\ y_1\ j_2\ y_2\ j_3\ y_3\ j_4$, then (a) there is a stage $\hat{s}$ with $\hat{s} \le s^*$ such that in all stages between $\hat{s}$ and s^* (including Stages $\hat{s}$ and s^*), x_h is owned by individual j_4 (which is essentially individual i_h), and (b) j_1, j_2 and j_3 are assigned objects y_1, y_2, and y_3, respectively, before Stage $\hat{s}$. It should be noted that the assignments of j_1, j_2, and j_3 can happen in any order, for instance j_1 might not be assigned in Stage 1 or j_2 might be assigned before j_1, etc.

[23] More precisely, if $\pi_{x_h} = j_1\ y_1\ j_2\ y_2\ j_3$ for some $x_h \in F_{s'}(\tilde{P}_N)$ with $s' < s^*$, then (a) there is a stage $\hat{s}$ with $\hat{s} \le s'$ such that in all stages between $\hat{s}$ and s' (including Stages $\hat{s}$ and s'), x_h is owned by individual j_3, and (b) j_1 and j_2 are assigned objects y_1 and y_2, respectively, before Stage $\hat{s}$. As in the previous case, it should be noted that the assignments of j_1 and j_2 can happen in any order, for instance j_1 might not be assigned in Stage 1 or j_2 might be assigned before j_1, etc.

(i) (a) for all $h = 1, \ldots, t$, the last element of the root-path π_{x_h} is individual i_h and the other individuals in π_{x_h} are from the set $\{i_4, \ldots, i_t\}$,

(b) for all $h = 4, \ldots, t$, if there is an (outgoing) edge from individual i_h in any of the root-paths in $(\pi_{x_h})_{h \in \{1,\ldots,t\}}$, then object $\hat{\mu}(i_h)$ is assigned to that edge, and

(ii) for all distinct objects $x_i, x_j \in \{x_4, \ldots, x_t\}$, if individual $\hat{\mu}(x_i)$ lies in the interior of the root-path π_{x_j}, then individual $\hat{\mu}(x_j)$ does not lie in the root-path π_{x_i}.

We distinguish the following two cases.

CASE 1: Suppose (1) holds.

Since individuals i_1, i_2, and i_3 are assigned to the root-nodes of Γ_{x_1}, Γ_{x_2}, and Γ_{x_3}, respectively, it must be that for any preference profile, individuals i_1, i_2, and i_3 own objects x_1, x_2, and x_3, respectively, in Stage 1 of f^Γ at that preference profile. Therefore f^Γ will not satisfy dual ownership.

CASE 2: Suppose (2) holds.

Assume for contradiction that f^Γ satisfies dual ownership. Consider the preference profile $\tilde{P}_N$ defined as follows. Each $i_h \in \{i_4, \ldots, i_t\}$ has a preference $\tilde{P}_{i_h}$ such that $\tau(\tilde{P}_{i_h}) = \hat{\mu}(i_h)$ and each $j \in N \setminus \{i_4, \ldots, i_t\}$ has a preference $\tilde{P}_j$ such that $\{x_4, \ldots, x_t\}\tilde{P}_j(A \setminus \{x_4, \ldots, x_t\})$. The next claim establishes some properties of the outcome of f^Γ at $\tilde{P}_N$ in Stage 1.

Claim A.1. *(a) $W_1(\tilde{P}_N) \subseteq \{i_4, \ldots, i_t\}$, (b) $f^\Gamma_i(\tilde{P}_N) = \hat{\mu}(i)$ for all $i \in W_1(\tilde{P}_N)$, and (c) $x_h \in F_1(\tilde{P}_N)$ for all $h = 4, \ldots, t$ with $i_h \in W_1(\tilde{P}_N)$.*

Proof of Claim A.1. By the properties of $(\pi_{x_h})_{h \in \{1,\ldots,t\}}$ (see (2)), it follows that $\{x_4, \ldots, x_t\} \subseteq \bigcup_{h=4}^{t} E_1(i_h, \tilde{P}_N)$. Moreover, by the construction of $\tilde{P}_N$ and the assumption on $\hat{\mu}$, we have $\tau(\tilde{P}_i) \in \{x_4, \ldots, x_t\}$ for all $i \in N$. Since $\{x_4, \ldots, x_t\} \subseteq \bigcup_{h=4}^{t} E_1(i_h, \tilde{P}_N)$ and $\tau(\tilde{P}_i) \in \{x_4, \ldots, x_t\}$ for all $i \in N$, it follows from the definition of f^Γ that $W_1(\tilde{P}_N) \subseteq \{i_4, \ldots, i_t\}$ and $f^\Gamma_i(\tilde{P}_N) = \tau(\tilde{P}_i)$ for all $i \in W_1(\tilde{P}_N)$. These two facts, along with the construction of $\tilde{P}_N$, imply

$$W_1(\tilde{P}_N) \subseteq \{i_4, \ldots, i_t\} \text{ and } f^\Gamma_i(\tilde{P}_N) = \hat{\mu}(i) \text{ for all } i \in W_1(\tilde{P}_N). \tag{A.1}$$

Since f^Γ satisfies dual ownership, it follows from the definition of f^Γ that $|W_1(\tilde{P}_N)| \leq 2$. We distinguish two cases.

(A) Suppose $|W_1(\tilde{P}_N)| = 1$.

Since $W_1(\tilde{P}_N) \subseteq \{i_4, \ldots, i_t\}$, without loss of generality, assume $W_1(\tilde{P}_N) = \{i_4\}$. By (A.1) we have $f^\Gamma_{i_4}(\tilde{P}_N) = \hat{\mu}(i_4)$. This, along with the definition of f^Γ, implies that individual i_4 is assigned to the root-node of $\Gamma_{\hat{\mu}(i_4)}$. It follows from the definition of an inheritance tree and property (2).(i) of the root-paths that $\pi_{\hat{\mu}(i_4)}$ is a single-node root-path and $\hat{\mu}(i_4) = x_4$.[24] By the assumption that

[24] A root-path in an inheritance tree is called ***single-node*** if it ends at the root of the inheritance tree itself, that is, it contains a single node.

$W_1(\tilde{P}_N) = \{i_4\}$ and the facts that $f^\Gamma_{i_4}(\tilde{P}_N) = \hat{\mu}(i_4)$, and $\hat{\mu}(i_4) = x_4$, we have $F_1(\tilde{P}_N) = \{x_4\}$.

(B) Suppose $|W_1(\tilde{P}_N)| = 2$.

Since $W_1(\tilde{P}_N) \subseteq \{i_4, \ldots, i_t\}$, without loss of generality, assume $W_1(\tilde{P}_N) = \{i_4, i_5\}$. By (A.1) we have $f^\Gamma_{i_4}(\tilde{P}_N) = \hat{\mu}(i_4)$ and $f^\Gamma_{i_5}(\tilde{P}_N) = \hat{\mu}(i_5)$. This, along with the definition of f^Γ, implies that individuals i_4 and i_5 are either assigned to the root-nodes of $\Gamma_{\hat{\mu}(i_4)}$ and $\Gamma_{\hat{\mu}(i_5)}$, respectively, or to the root-nodes of $\Gamma_{\hat{\mu}(i_5)}$ and $\Gamma_{\hat{\mu}(i_4)}$, respectively.

(I) Suppose i_4 and i_5 are assigned to the root-nodes of $\Gamma_{\hat{\mu}(i_4)}$ and $\Gamma_{\hat{\mu}(i_5)}$, respectively. Using similar logic as for (A), it follows that $F_1(\tilde{P}_N) = \{x_4, x_5\}$.

(II) Suppose i_4 and i_5 are assigned to the root-nodes of $\Gamma_{\hat{\mu}(i_5)}$ and $\Gamma_{\hat{\mu}(i_4)}$, respectively. It follows from property (2).(ii) of the root-paths that both of $\pi_{\hat{\mu}(i_4)}$ and $\pi_{\hat{\mu}(i_5)}$ are single-node root-paths, and $\hat{\mu}(i_4) = x_5$ and $\hat{\mu}(i_5) = x_4$. By the assumption that $W_1(\tilde{P}_N) = \{i_4, i_5\}$ and the facts that $f^\Gamma_{i_4}(\tilde{P}_N) = \hat{\mu}(i_4)$, $f^\Gamma_{i_5}(\tilde{P}_N) = \hat{\mu}(i_5)$, $\hat{\mu}(i_4) = x_5$, and $\hat{\mu}(i_5) = x_4$, it follows that $F_1(\tilde{P}_N) = \{x_4, x_5\}$.

Since Cases (A) and (B) are exhaustive, it follows that $x_h \in F_1(\tilde{P}_N)$ for all $h = 4, \ldots, t$ with $i_h \in W_1(\tilde{P}_N)$. This, along with (A.1), completes the proof of Claim A.1. □

By Claim A.1, $W_1(\tilde{P}_N) \subsetneq \{i_4, \ldots, i_t\}$, $f^\Gamma_i(\tilde{P}_N) = \hat{\mu}(i)$ for all $i \in W_1(\tilde{P}_N)$, and $x_h \in F_1(\tilde{P}_N)$ for all $h = 4, \ldots, t$ with $i_h \in W_1(\tilde{P}_N)$. We proceed to show that there will be a stage s^* such that $W^{s^*}(\tilde{P}_N) = \{i_4, \ldots, i_t\}$ and $f^\Gamma_i(\tilde{P}_N) = \hat{\mu}(i)$ for all $i \in W^{s^*}(\tilde{P}_N)$. If $W_1(\tilde{P}_N) = \{i_4, \ldots, i_t\}$, then $s^* = 1$ and we are done. Suppose $W_1(\tilde{P}_N) \subsetneq \{i_4, \ldots, i_t\}$, that is, $W_1(\tilde{P}_N)$ is a proper subset of $\{i_4, \ldots, i_t\}$. Since $W_1(\tilde{P}_N) \subsetneq \{i_4, \ldots, i_t\}$, $f^\Gamma_i(\tilde{P}_N) = \hat{\mu}(i)$ for all $i \in W_1(\tilde{P}_N)$, and $x_h \in F_1(\tilde{P}_N)$ for all $h = 4, \ldots, t$ with $i_h \in W_1(\tilde{P}_N)$, using similar argument as for Claim A.1, it follows from the assumption regarding $\{i_1, \ldots, i_t\}$, $\{x_1, \ldots, x_t\}$, $\hat{\mu}$, $(\pi_{x_h})_{h \in \{1,\ldots,t\}}$, and the construction of $\tilde{P}_N$ that $W_2(\tilde{P}_N) \subseteq \big(\{i_4, \ldots, i_t\} \setminus W_1(\tilde{P}_N)\big)$, $f^\Gamma_i(\tilde{P}_N) = \hat{\mu}(i)$ for all $i \in W_2(\tilde{P}_N)$, and $x_h \in F_2(\tilde{P}_N)$ for all $h = 4, \ldots, t$ with $i_h \in W_2(\tilde{P}_N)$. If $W_1(\tilde{P}_N) \cup W_2(\tilde{P}_N) = \{i_4, \ldots, i_t\}$, then $s^* = 2$ and we are done. Otherwise, continuing in this manner, we obtain a stage $s^* > 2$ of f^Γ at $\tilde{P}_N$ such that $W^{s^*}(\tilde{P}_N) = \{i_4, \ldots, i_t\}$ and $f^\Gamma_{i_h}(\tilde{P}_N) = \hat{\mu}(i_h)$ for all $h = 4, \ldots, t$.

Since $W^{s^*}(\tilde{P}_N) = \{i_4, \ldots, i_t\}$ and $f^\Gamma_{i_h}(\tilde{P}_N) = \hat{\mu}(i_h)$ for all $h = 4, \ldots, t$, by the assumptions for Case 2, we have $x_h \in E_{s^*+1}(i_h, \tilde{P}_N)$ for all $h = 1, 2, 3$. This implies that individuals i_1, i_2, and i_3 own the objects x_1, x_2, and x_3, respectively, in Stage $s^* + 1$ of f^Γ at $\tilde{P}_N$, a contradiction to our assumption that f^Γ satisfies dual ownership. This completes the proof of the "only-if" part of Theorem 8.1. ■

Appendix B Proof of Theorem 7.1

We use Theorem 8.1 (which is presented after Theorem 7.1 in the body of the paper) in the proof of Theorem 7.1. Therefore, for the sake of completeness, we have already presented the proof of Theorem 8.1 in the previous appendix (Appendix A).

We first prove two lemmas which we will combine with two results of Pápai (2000) to complete the proof of Theorem 7.1. In Lemma B.1, we show that a hierarchical exchange rule is OSP-implementable if and only if it satisfies dual ownership, and in Lemma B.2, we show that every OSP-implementable, non-bossy and Pareto efficient assignment rule is reallocation-proof.

B.1 Lemma B.1 and its proof

Lemma B.1. *A hierarchical exchange rule is OSP-implementable if and only if it satisfies dual ownership.*

Before we formally start proving Lemma B.1, to facilitate the proof we introduce the notion of a reduced tree structure and make two observations about it.

B.1.1 Reduced tree structure

For an inheritance tree $\Gamma_a = \langle T_a, \zeta_a^{NI}, \zeta_a^{EO}\rangle$ and an edge $(v,v') \in E(T_a)$, we say that an inheritance tree $\tilde{\Gamma}_a = \langle \tilde{T}_a, \tilde{\zeta}_a^{NI}, \tilde{\zeta}_a^{EO}\rangle$ ***is obtained by collapsing the edge*** (v,v') if

(i) $V(\tilde{T}_a) = V(T_a) \setminus \Big(\{v\} \cup \{v'' \mid \text{ there exists a path in } T_a \text{ from } v \text{ to } v'' \text{ which does not contain } v'\}\Big)$,

(ii) $E(\tilde{T}_a) = \Big(E(T_a) \cap \big(V(\tilde{T}_a) \times V(\tilde{T}_a)\big)\Big) \cup \{(\hat{v}, v')\}$, where $\hat{v}$ is the parent node of v in T_a. If $v = r(T_a)$, then $\hat{v}$ does not exist, and consequently, we take $\{(\hat{v}, v')\} = \emptyset$,

(iii) $\tilde{\zeta}_a^{NI}(v) = \zeta_a^{NI}(v)$ for all $v \in V(\tilde{T}_a)$, and

(iv) $\tilde{\zeta}_a^{EO}(e) = \zeta_a^{EO}(e)$ for all $e \in \Big(E(T_a) \cap \big(V(\tilde{T}_a) \times V(\tilde{T}_a)\big)\Big)$ and $\tilde{\zeta}_a^{EO}(\hat{v}, v') = \zeta_a^{EO}(\hat{v}, v)$.

For an inheritance tree $\Gamma_a = \langle T_a, \zeta_a^{NI}, \zeta_a^{EO}\rangle$ and an edge $(v,v') \in E(T_a)$, we say that an inheritance tree $\tilde{\Gamma}_a = \langle \tilde{T}_a, \tilde{\zeta}_a^{NI}, \tilde{\zeta}_a^{EO}\rangle$ ***is obtained by dropping the edge*** (v,v') if

(i) $V(\tilde{T}_a) = V(T_a) \setminus \{v'' \mid \text{ there exists a path in } T_a \text{ from } v \text{ to } v'' \text{ which contains } v'\}$,

(ii) $E(\tilde{T}_a) = E(T_a) \cap \big(V(\tilde{T}_a) \times V(\tilde{T}_a)\big)$,

(iii) $\tilde{\zeta}_a^{NI}(v) = \zeta_a^{NI}(v)$ for all $v \in V(\tilde{T}_a)$, and

(iv) $\tilde{\zeta}_a^{EO}(e) = \zeta_a^{EO}(e)$ for all $e \in E(\tilde{T}_a)$.

For an inheritance tree $\Gamma_a = \langle T_a, \zeta_a^{NI}, \zeta_a^{EO}\rangle$, we denote an edge $(v,v') \in E(T_a)$ by (i,x) if $\zeta_a^{NI}(v) = i$ and $\zeta_a^{EO}(v,v') = x$ in Γ_a. By the construction of Γ_a, $\zeta_a^{EO}(v,v') = x$ implies $a \neq x$.

For a pair $(i,x) \in N \times A$ and a collection of inheritance trees $\Gamma = (\Gamma_x)_{x \in A}$, we define the ***reduced collection*** $\Gamma \setminus (i,x)$ as follows:

(i) If $a = x$, then drop the inheritance tree Γ_a.

(ii) If $a \neq x$ and $\zeta_a^{NI}(r(T_a)) = i$, then $\Gamma_a \setminus (i,x)$ is obtained by collapsing the edge (i,x) in Γ_a.[25]

(iii) If $a \neq x$ and $\zeta_a^{NI}(r(T_a)) \neq i$, then $\Gamma_a \setminus (i,x)$ is obtained by collapsing all edges (i,x) and dropping all edges (j,x) with $j \neq i$ in Γ_a.

For $(i,x),(j,y) \in N \times A$ and a collection of inheritance trees $\Gamma = (\Gamma_x)_{x\in A}$, we denote the reduced collection $\Big(\Gamma \setminus (i,x)\Big) \setminus (j,y)$ by $\Gamma \setminus \Big((i,x),(j,y)\Big)$.

Remark B.1. For $(i,x),(j,y) \in N \times A$ and a collection of inheritance trees $\Gamma = (\Gamma_x)_{x\in A}$, we have $\Gamma \setminus \Big((i,x),(j,y)\Big) = \Gamma \setminus \Big((j,y),(i,x)\Big)$.

Example B.1. Suppose $N = \{1,2,3,4,5\}$ and $A = \{x_1,x_2,x_3,x_4\}$. Consider the collection of inheritance trees Γ given in Figure B.1.

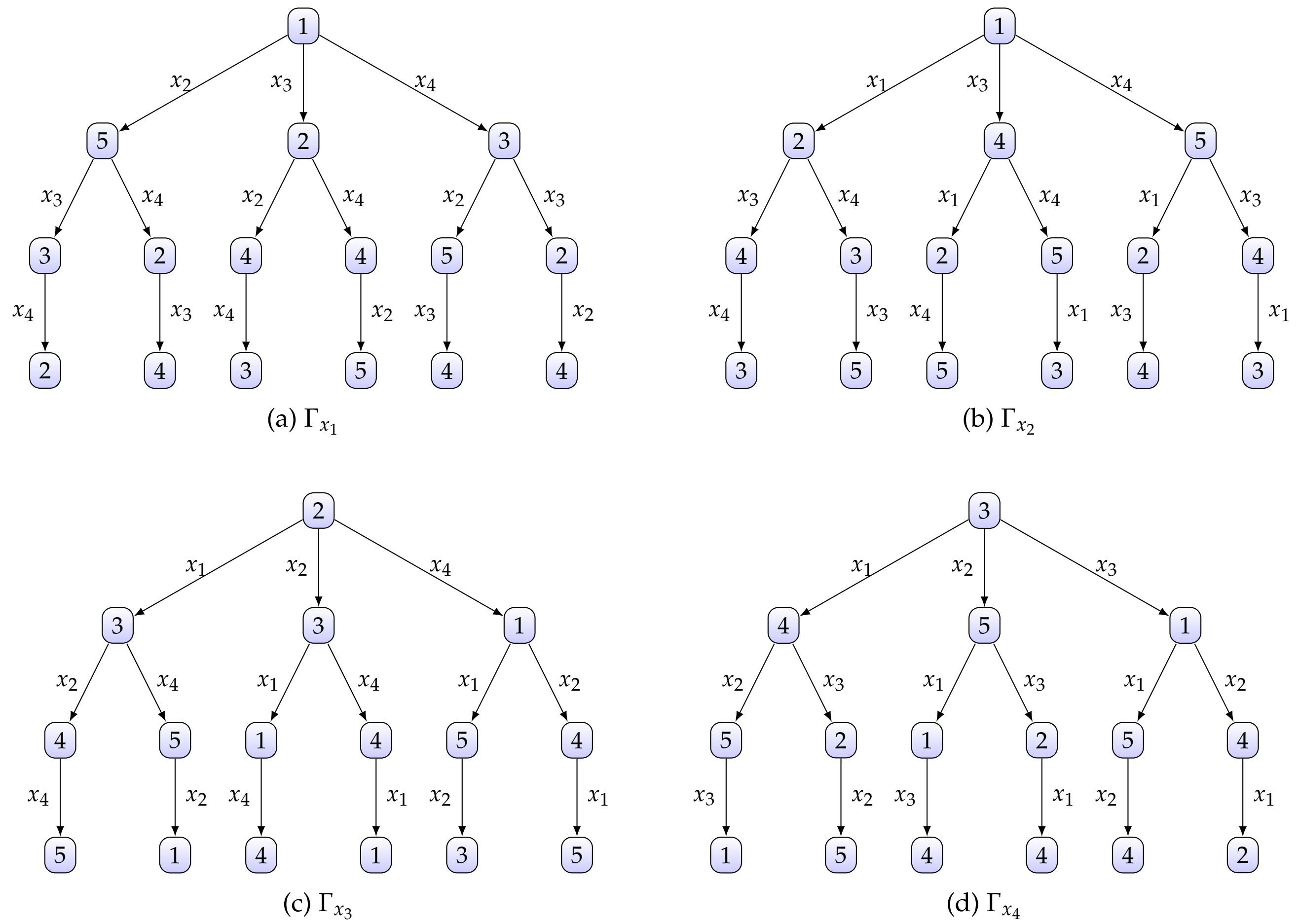
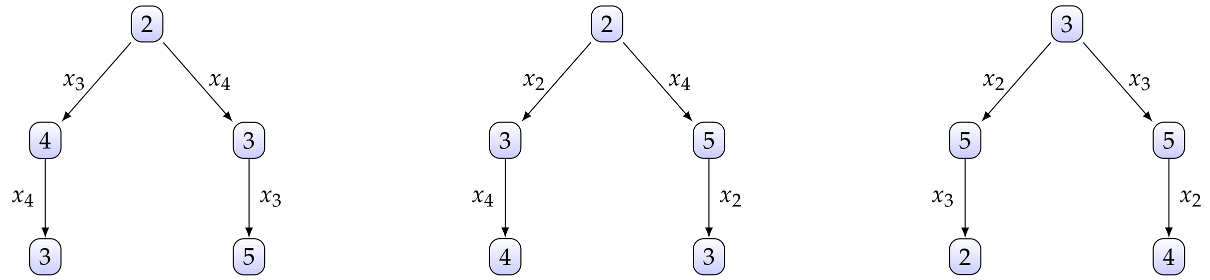


Figure B.1: Collection of inheritance trees Γ for Example B.1

Consider the pair $(1,x_1) \in N \times A$. The reduced collection $\Gamma \setminus (1,x_1)$ is given in Figure B.2.

[25] Note that in this case, there is only one such edge (i,x).



(a) $\Gamma_{x_2} \setminus (1, x_1)$ (b) $\Gamma_{x_3} \setminus (1, x_1)$ (c) $\Gamma_{x_4} \setminus (1, x_1)$

Figure B.2: Reduced collection $\Gamma \setminus (1, x_1)$

B.1.2 Two observations

Let $\mathcal{T}(\Gamma) = \{i \mid \zeta_x^{NI}(r(T_x)) = i \text{ for some } x \in A\}$ be the set of individuals who appear at the root-node of some inheritance tree in the collection of inheritance trees Γ. We now make two observations. The first observation is straightforward, and see Step 2.a in the "Necessity Proof" of Pápai (2000) for the second observation.

Observation B.1. *If Γ is acyclic, then $|\mathcal{T}(\Gamma)| \leq 2$.*

Observation B.2. *Suppose $\zeta_x^{NI}(r(T_x)) = i$ for some $x \in A$ and some $i \in N$. Then, for all $P_N \in \mathbb{L}^n(A)$, $f_i^\Gamma(P_N) R_i x$.*

B.1.3 Proof of Lemma B.1

(If part) Suppose f^Γ satisfies dual ownership. By Theorem 8.1, Γ is acyclic. We show that f^Γ is OSP-implementable by using induction on the number of individuals, which we refer to as the *size of the market*.

Base Case: Suppose $|N| = 1$.[26] The following extensive-form assignment mechanism, labeled as G^1, implements f^Γ.

Step 1. Ask the only individual which object is her top choice and assign her that object.

It is simple to check that the extensive-form assignment mechanism G^1 is OSP. Since the OSP mechanism G^1 implements f^Γ, it follows that f^Γ is OSP-implementable. Now, we proceed to prove the induction step.

Induction Hypothesis: Assume that f^Γ is OSP-implementable for $|N| \leq m$. We show f^Γ is OSP-implementable for $|N| = m + 1$. Since Γ is acyclic, by Observation B.1, we have $|\mathcal{T}(\Gamma)| \leq 2$. We distinguish the following two cases.

Case A: $|\mathcal{T}(\Gamma)| = 1$.

Let $\mathcal{T}(\Gamma) = \{i\}$. Define the extensive-form assignment mechanism G^{m+1} as follows:

[26] With only one individual, Γ is trivially acyclic.

Step 1. Ask individual i which object is her top choice and assign her that object, say x.

Step 2. Consider the reduced market $(N \setminus \{i\}, A \setminus \{x\})$ where individual i is removed from the market together with the object x she is assigned. This reduced market $(N \setminus \{i\}, A \setminus \{x\})$ is of size m.

Claim B.1. *$\Gamma \setminus (i, x)$ is acyclic.*

Proof of Claim B.1. We denote the reduced collection $\Gamma \setminus (i, x)$ by $(\bar{\Gamma}_a)_{a \in A \setminus \{x\}}$. Assume for contradiction that $(\bar{\Gamma}_a)_{a \in A \setminus \{x\}}$ contains an inheritance cycle, say $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$. By the definition of an inheritance cycle, one of the following two statements must hold.

(1) Individuals i_1, i_2, and i_3 are assigned to the root-nodes of $\bar{\Gamma}_{x_1}$, $\bar{\Gamma}_{x_2}$, and $\bar{\Gamma}_{x_3}$, respectively.

(2) There exist distinct individuals $i_4, \ldots, i_t \in N \setminus \{i, i_1, i_2, i_3\}$, distinct objects $x_4, \ldots, x_t \in A \setminus \{x, x_1, x_2, x_3\}$, an allocation $\bar{\mu} \in \mathcal{M}(\{i_4, \ldots, i_t\}, \{x_4, \ldots, x_t\})$, and a collection of root-paths $(\bar{\pi}_{x_h})_{h \in \{1, \ldots, t\}}$, where $\bar{\pi}_{x_h}$ lies in the reduced tree $\bar{\Gamma}_{x_h}$ for all $h = 1, \ldots, t$, with the properties that

  (i) (a) for all $h = 1, \ldots, t$, the last element of the root-path $\bar{\pi}_{x_h}$ is individual i_h and other individuals in $\bar{\pi}_{x_h}$ are from the set $\{i_4, \ldots, i_t\}$,

  (b) for all $h = 4, \ldots, t$, if there is an (outgoing) edge from individual i_h in any of the root-paths in $(\bar{\pi}_{x_h})_{h \in \{1, \ldots, t\}}$, then object $\bar{\mu}(i_h)$ is assigned to that edge, and

  (ii) for all distinct objects $x_i, x_j \in \{x_4, \ldots, x_t\}$, if individual $\bar{\mu}(x_i)$ lies in the interior of the root-path $\bar{\pi}_{x_j}$, then individual $\bar{\mu}(x_j)$ does not lie in the root-path $\bar{\pi}_{x_i}$.

We distinguish the following two cases.

CASE 1: Suppose (1) holds.

Let $\{j_1, j_2, j_3, j_4\} \subseteq N$, $\{y_1, y_2, y_3, y_4\} \subseteq A$, an allocation $\hat{\mu} \in \mathcal{M}(\{j_4\}, \{y_4\})$, and a collection of root-paths $(\pi_{y_h})_{h \in \{1, \ldots, 4\}}$ be as follows

(a) $j_h = i_h$ for all $h = 1, 2, 3$, and $j_4 = i$.

(b) $y_h = x_h$ for all $h = 1, 2, 3$, and $y_4 = x$.

(c) $\hat{\mu}(j_4) = y_4$.

(d) π_{y_h} is the root-path in Γ_{y_h} to the root of $\bar{\Gamma}_{y_h}$ for all $h = 1, 2, 3$, and π_{y_4} is the single-node root-path in Γ_{y_4}.

By the assumption of Case A, $\mathcal{T}(\Gamma) = \{i\}$, which implies that individual i is assigned to the root-node of Γ_x. This, together with the definition of $\{j_1, j_2, j_3, j_4\}$, $\{y_1, y_2, y_3, y_4\}$, $\hat{\mu}$, and $(\pi_{y_h})_{h \in \{1, \ldots, 4\}}$, and the assumption of Case 1, implies that

$$\left[\begin{array}{ll}\text{(a)} & \text{for all } h=1,\ldots,4, \text{ the last element of the root-path } \pi_{y_h} \text{ is individual } j_h \\ & \text{and the other individuals in } \pi_{y_h} \text{ are from the set } \{j_4\}, \text{ and} \\ \text{(b)} & \text{if there is an (outgoing) edge from individual } j_4 \text{ in any of the root-paths in } (\pi_{y_h})_{h\in\{1,\ldots,4\}}, \\ & \text{then object } \hat{\mu}(j_4) \text{ is assigned to that edge.}\end{array}\right] \tag{B.1}$$

This shows that $[(j_1, j_2, j_3), (y_1, y_2, y_3)]$ satisfies (i) of the definition of inheritance cycle (Definition 8.1).

Now, we proceed to show that (ii) of Definition 8.1 is also satisfied by $[(j_1, j_2, j_3), (y_1, y_2, y_3)]$, that is, we show that for all distinct objects $y_i, y_j \in \{y_4\}$, if individual $\hat{\mu}(y_i)$ lies in the interior of the root-path π_{y_j}, then individual $\hat{\mu}(y_j)$ does not lie in the root-path π_{y_i}. Since $\{y_4\}$ is a singleton set, this condition is vacuously satisfied. Therefore, $[(j_1, j_2, j_3), (y_1, y_2, y_3)]$ is an inheritance cycle at Γ, a contradiction to the fact that Γ is acyclic.

Case 2: Suppose (2) holds.

Let $\{j_1, \ldots, j_{t+1}\} \subseteq N$, $\{y_1, \ldots, y_{t+1}\} \subseteq A$, an allocation $\hat{\mu} \in \mathcal{M}(\{j_4, \ldots, j_{t+1}\}, \{y_4, \ldots, y_{t+1}\})$, and a collection of root-paths $(\pi_{y_h})_{h\in\{1,\ldots,t+1\}}$ be as follows.

(a) $j_h = i_h$ for all $h = 1, \ldots, t$, and $j_{t+1} = i$.

(b) $y_h = x_h$ for all $h = 1, \ldots, t$, and $y_{t+1} = x$.

(c) $\hat{\mu}(j_h) = \bar{\mu}(i_h)$ for all $h = 4, \ldots, t$, and $\hat{\mu}(j_{t+1}) = y_{t+1}$.

(d) π_{y_h} is the minimal root-path in Γ_{y_h} that contains the nodes of $\bar{\pi}_{x_h}$ in $\bar{\Gamma}_{x_h}$ for all $h = 1, \ldots, t$, and $\pi_{y_{t+1}}$ is the single-node root-path in $\Gamma_{y_{t+1}}$.[27]

By the assumption of Case A, $\mathcal{T}(\Gamma) = \{i\}$, which implies that individual i is assigned to the root-node of Γ_x. This, together with the definition of $\{j_1, \ldots, j_{t+1}\}$, $\{y_1, \ldots, y_{t+1}\}$, $\hat{\mu}$, and $(\pi_{y_h})_{h\in\{1,\ldots,t+1\}}$, and the assumption of Case 2, implies that

$$\left[\begin{array}{ll}\text{(a)} & \text{for all } h=1,\ldots,t+1, \text{ the last element of the root-path } \pi_{y_h} \text{ is individual } j_h \\ & \text{and the other individuals in } \pi_{y_h} \text{ are from the set } \{j_4,\ldots,j_{t+1}\}, \text{ and} \\ \text{(b)} & \text{for all } h=4,\ldots,t+1, \text{ if there is an (outgoing) edge from individual } j_h \text{ in any of the root-paths} \\ & \text{in } (\pi_{y_h})_{h\in\{1,\ldots,t+1\}}, \text{ then object } \hat{\mu}(j_h) \text{ is assigned to that edge.}\end{array}\right] \tag{B.2}$$

This shows that $[(j_1, j_2, j_3), (y_1, y_2, y_3)]$ satisfies (i) of Definition 8.1.

Now, we proceed to show that (ii) of Definition 8.1 is also satisfied by $[(j_1, j_2, j_3), (y_1, y_2, y_3)]$, that is, we show that for all distinct objects $y_i, y_j \in \{y_4, \ldots, y_{t+1}\}$, if individual $\hat{\mu}(y_i)$ lies in the interior of the

[27] Note that such root-paths exist by the construction of $\bar{\Gamma}_{x_h}$.

root-path π_{y_j}, then individual $\hat{\mu}(y_j)$ does not lie in the root-path π_{y_i}. Assume for contradiction that y_i, $y_j \in \{y_4, \ldots, y_{t+1}\}$ are two distinct objects such that individual $\hat{\mu}(y_i)$ lies in the interior of the root-path π_{y_j} and individual $\hat{\mu}(y_j)$ lies in the root-path π_{y_i}. Since $\hat{\mu}(y_i)$ lies in the interior of π_{y_j}, the fact that $\pi_{y_{t+1}}$ is a single-node root-path, implies $y_j \neq y_{t+1}$. Suppose $y_i = y_{t+1}$. Since $\pi_{y_{t+1}}$ is single-node root-path, $\hat{\mu}(y_j)$ lies in π_{y_i}, and j_{t+1} is the last element of $\pi_{y_{t+1}}$, it follows that $\hat{\mu}(y_j) = j_{t+1}$. However, by the definition of $\hat{\mu}$, $\hat{\mu}(y_j) = j_{t+1}$ implies $y_j = y_{t+1}$, which contradicts the fact $y_j \neq y_{t+1}$. So, it must be that $y_i \neq y_{t+1}$. Combining the facts that $y_i, y_j \in \{y_4, \ldots, y_{t+1}\}$, $y_j \neq y_{t+1}$, and $y_i \neq y_{t+1}$, we obtain $y_i, y_j \in \{y_4, \ldots, y_t\}$. However, since $y_i, y_j \in \{y_4, \ldots, y_t\}$, it follows from the definition of $\{j_1, \ldots, j_{t+1}\}$, $\{y_1, \ldots, y_{t+1}\}$, $\hat{\mu}$, and $(\pi_{y_h})_{h \in \{1,\ldots,t+1\}}$ that individual $\bar{\mu}(x_i)$ lies in the interior of the root-path $\bar{\pi}_{x_j}$ and individual $\bar{\mu}(x_j)$ lies in the root-path $\bar{\pi}_{x_i}$, which contradicts (2).(ii). This proves that $[(j_1, j_2, j_3), (y_1, y_2, y_3)]$ satisfies (ii) of Definition 8.1. Therefore, $[(j_1, j_2, j_3), (y_1, y_2, y_3)]$ is an inheritance cycle at Γ, a contradiction to the fact that Γ is acyclic. This completes the proof of Claim B.1. □

By the induction hypothesis and Claim B.1, it follows that there exists an OSP mechanism G^m that implements f^Γ restricted to the reduced market $(N \setminus \{i\}, A \setminus \{x\})$. Run the extensive-form assignment mechanism G^m on the reduced market $(N \setminus \{i\}, A \setminus \{x\})$.

By definition, the extensive-form assignment mechanism G^{m+1} implements f^Γ. This extensive-form assignment mechanism is OSP for individual i since she receives her top choice. For every other individual, her first decision node comes after i has been assigned, and hence, her strategic decision is equivalent to that under the OSP mechanism that implements f^Γ restricted to the reduced market. Thus, the above extensive-form assignment mechanism is OSP for all individuals, and hence, f^Γ is OSP-implementable.

Case B: $|\mathcal{T}(\Gamma)| = 2$.

Let $\mathcal{T}(\Gamma) = \{i, j\}$. Let $A_i = \{x \in A \mid \zeta_x^{NI}(r(T_x)) = i\}$ and $A_j = \{y \in A \mid \zeta_y^{NI}(r(T_y)) = j\}$. Define the extensive-form assignment mechanism G^{m+1} as follows:

Step 1. For each $x \in A_i$, ask i if her top choice is x. If i answers "Yes" for some x, assign her this x, and go to Step 1(a). Otherwise, jump to Step 2.

Step 1(a). We now have a reduced market $(N \setminus \{i\}, A \setminus \{x\})$ of size m.

Claim B.2. *$\Gamma \setminus (i, x)$ is acyclic.*

The proof of Claim B.2 follows by using similar logic as for the proof of Claim B.1. The only adjustment needed for the proof of Claim B.2 over the proof of Claim B.1 is that instead of $\mathcal{T}(\Gamma) = \{i\}$ (which is an assumption of Case A), which means individual i is assigned to the root-node of every inheritance tree, we need to consider $x \in A_i$ (which is an assumption of Step 1 in Case B) meaning that individual i is assigned to the root-node of the inheritance tree for x.

By the induction hypothesis and Claim B.2, it follows that there exists an OSP mechanism G^m that implements f^Γ restricted to the reduced market $(N \setminus \{i\}, A \setminus \{x\})$. Run the extensive-form assignment mechanism G^m on the reduced market $(N \setminus \{i\}, A \setminus \{x\})$.

Step 2. For each $y \in A_j$, ask j if her top choice is y. If j answers "Yes" for some y, assign her this y, and go to Step 2(a). Otherwise, jump to Step 3.

Step 2(a). We now have a reduced market $(N \setminus \{j\}, A \setminus \{y\})$ of size m. Similar to Claim B.2, we have the following claim.

Claim B.3. *$\Gamma \setminus (j, y)$ is acyclic.*

By the induction hypothesis and Claim B.3, it follows that there exists an OSP mechanism G^m that implements f^Γ restricted to the reduced market $(N \setminus \{j\}, A \setminus \{y\})$. Run the extensive-form assignment mechanism G^m on the reduced market $(N \setminus \{j\}, A \setminus \{y\})$.

Step 3. If the answers to both Step 1 and Step 2 are "No", then i's top choice belongs to A_j, and j's top choice belongs to A_i. Ask i for her top choice x, and j for her top choice y. Assign x to i and y to j, and go to Step 3(a).

Step 3(a). We now have a reduced market $(N \setminus \{i, j\}, A \setminus \{x, y\})$ of size $m - 1$.

Claim B.4. *$\Gamma \setminus \Big((i, x), (j, y)\Big)$ is acyclic.*

Proof of Claim B.4. Let us denote the reduced collection $\Gamma \setminus \Big((i, x), (j, y)\Big)$ by $(\bar{\Gamma}_a)_{a \in A \setminus \{x,y\}}$. Assume for contradiction that $(\bar{\Gamma}_a)_{a \in A \setminus \{x,y\}}$ contains an inheritance cycle, say $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$. By the definition of an inheritance cycle, one of the following two statements must hold.

(1) Individuals i_1, i_2, and i_3 are assigned to the root-nodes of $\bar{\Gamma}_{x_1}$, $\bar{\Gamma}_{x_2}$, and $\bar{\Gamma}_{x_3}$, respectively.

(2) There exist distinct individuals $i_4, \ldots, i_t \in N \setminus \{i, j, i_1, i_2, i_3\}$, distinct objects $x_4, \ldots, x_t \in A \setminus \{x, y, x_1, x_2, x_3\}$, an allocation $\bar{\mu} \in \mathcal{M}(\{i_4, \ldots, i_t\}, \{x_4, \ldots, x_t\})$, and a collection of root-paths $(\bar{\pi}_{x_h})_{h \in \{1,\ldots,t\}}$, where $\bar{\pi}_{x_h}$ lies in the reduced tree $\bar{\Gamma}_{x_h}$ for all $h = 1, \ldots, t$, with the properties that

   (i) (a) for all $h = 1, \ldots, t$, the last element of the root-path $\bar{\pi}_{x_h}$ is individual i_h and other individuals in $\bar{\pi}_{x_h}$ are from the set $\{i_4, \ldots, i_t\}$,

   (b) for all $h = 4, \ldots, t$, if there is an (outgoing) edge from individual i_h in any of the root-paths in $(\bar{\pi}_{x_h})_{h \in \{1,\ldots,t\}}$, then object $\bar{\mu}(i_h)$ is assigned to that edge, and

   (ii) for all distinct objects $x_i, x_j \in \{x_4, \ldots, x_t\}$, if individual $\bar{\mu}(x_i)$ lies in the interior of the root-path $\bar{\pi}_{x_j}$, then individual $\bar{\mu}(x_j)$ does not lie in the root-path $\bar{\pi}_{x_i}$.

We distinguish the following two cases.

CASE 1: Suppose (1) holds.

Let $\{j_1, \ldots, j_5\} \subseteq N$, $\{y_1, \ldots, y_5\} \subseteq A$, an allocation $\hat{\mu} \in \mathcal{M}(\{j_4, j_5\}, \{y_4, y_5\})$, and a collection of root-paths $(\pi_{y_h})_{h\in\{1,\ldots,5\}}$ be as follows.

(a) $j_h = i_h$ for all $h = 1, 2, 3$, and $j_4 = i$ and $j_5 = j$.

(b) $y_h = x_h$ for all $h = 1, 2, 3$, and $y_4 = y$ and $y_5 = x$.

(c) $\hat{\mu}(j_4) = y_5$ and $\hat{\mu}(j_5) = y_4$.

(d) π_{y_h} is the root-path in Γ_{y_h} to the root of $\bar{\Gamma}_{y_h}$ for all $h = 1, 2, 3$, and π_{y_4} and π_{y_5} are the single-node root-paths in Γ_{y_4} and Γ_{y_5}, respectively.

By the assumption of Step 3 in Case B, $x \in A_j$ and $y \in A_i$, which imply that individuals i and j are assigned to the root-nodes of Γ_y and Γ_x, respectively. This, together with the definition of $\{j_1, \ldots, j_5\}$, $\{y_1, \ldots, y_5\}$, $\hat{\mu}$, and $(\pi_{y_h})_{h\in\{1,\ldots,5\}}$, and the assumption of Case 1, implies that

$$\left[\begin{array}{ll} \text{(a)} & \text{for all } h = 1, \ldots, 5, \text{ the last element of the root-path } \pi_{y_h} \text{ is individual } j_h \\ & \text{and the other individuals in } \pi_{y_h} \text{ are from the set } \{j_4, j_5\}, \text{ and} \\ \text{(b)} & \text{for all } h = 4, 5, \text{ if there is an (outgoing) edge from individual } j_h \text{ in any of the root-paths} \\ & \text{in } (\pi_{y_h})_{h\in\{1,\ldots,5\}}, \text{ then object } \hat{\mu}(j_h) \text{ is assigned to that edge.} \end{array}\right] \tag{B.3}$$

This shows that $[(j_1, j_2, j_3), (y_1, y_2, y_3)]$ satisfies (i) of Definition 8.1.

Now, we proceed to show that (ii) of Definition 8.1 is also satisfied by $[(j_1, j_2, j_3), (y_1, y_2, y_3)]$, that is, we show that for all distinct objects $y_i, y_j \in \{y_4, y_5\}$, if individual $\hat{\mu}(y_i)$ lies in the interior of the root-path π_{y_j}, then individual $\hat{\mu}(y_j)$ does not lie in the root-path π_{y_i}. Since π_{y_4} and π_{y_5} are single-node root-paths, no individual can be in the interior of π_{y_4} or in the interior of π_{y_5}. This proves that $[(j_1, j_2, j_3), (y_1, y_2, y_3)]$ satisfies (ii) of Definition 8.1. Therefore, $[(j_1, j_2, j_3), (y_1, y_2, y_3)]$ is an inheritance cycle at Γ, a contradiction to the fact that Γ is acyclic.

CASE 2: Suppose (2) holds.

Let $\{j_1, \ldots, j_{t+2}\} \subseteq N$, $\{y_1, \ldots, y_{t+2}\} \subseteq A$, an allocation $\hat{\mu} \in \mathcal{M}(\{j_4, \ldots, j_{t+2}\}, \{y_4, \ldots, y_{t+2}\})$, and a collection of root-paths $(\pi_{y_h})_{h\in\{1,\ldots,t+2\}}$ be as follows.

(a) $j_h = i_h$ for all $h = 1, \ldots, t$, and $j_{t+1} = i$ and $j_{t+2} = j$.

(b) $y_h = x_h$ for all $h = 1, \ldots, t$, and $y_{t+1} = y$ and $y_{t+2} = x$.

(c) $\hat{\mu}(j_h) = \bar{\mu}(i_h)$ for all $h = 4, \ldots, t$, and $\hat{\mu}(j_{t+1}) = y_{t+2}$ and $\hat{\mu}(j_{t+2}) = y_{t+1}$.

(d) π_{y_h} is the minimal root-path in Γ_{y_h} that contains the nodes of $\bar{\pi}_{x_h}$ in the reduced tree $\bar{\Gamma}_{x_h}$ for all $h = 1, \ldots, t$, and $\pi_{y_{t+1}}$ and $\pi_{y_{t+2}}$ are the single-node root-paths in $\Gamma_{y_{t+1}}$ and $\Gamma_{y_{t+2}}$, respectively.[28]

[28]Such root-paths will exist by the construction of $\bar{\Gamma}_{x_h}$.

By the assumption of Step 3 in Case B, $x \in A_j$ and $y \in A_i$, which imply that individuals i and j are assigned to the root-nodes of Γ_y and Γ_x, respectively. This, together with the definition of $\{j_1, \ldots, j_{t+2}\}$, $\{y_1, \ldots, y_{t+2}\}$, $\hat{\mu}$, and $(\pi_{y_h})_{h \in \{1,\ldots,t+2\}}$, and the assumption of Case 2, implies that

$$\left[\begin{array}{ll} \text{(a)} & \text{for all } h = 1, \ldots, t+2, \text{ the last element of the root-path } \pi_{y_h} \text{ is individual } j_h \\ & \text{and the other individuals in } \pi_{y_h} \text{ are from the set } \{j_4, \ldots, j_{t+2}\}, \text{ and} \\ \text{(b)} & \text{for all } h = 4, \ldots, t+2, \text{ if there is an (outgoing) edge from individual } j_h \text{ in any of the root-paths} \\ & \text{in } (\pi_{y_h})_{h \in \{1,\ldots,t+2\}}, \text{ then object } \hat{\mu}(j_h) \text{ is assigned to that edge.} \end{array}\right] \tag{B.4}$$

This shows that $[(j_1, j_2, j_3), (y_1, y_2, y_3)]$ satisfies (i) of Definition 8.1.

Now, we proceed to show that (ii) of Definition 8.1 is also satisfied by $[(j_1, j_2, j_3), (y_1, y_2, y_3)]$, that is, we show that for all distinct objects $y_i, y_j \in \{y_4, \ldots, y_{t+2}\}$, if individual $\hat{\mu}(y_i)$ lies in the interior of the root-path π_{y_j}, then individual $\hat{\mu}(y_j)$ does not lie in the root-path π_{y_i}. Assume for contradiction that y_i, $y_j \in \{y_4, \ldots, y_{t+2}\}$ are two distinct objects such that individual $\hat{\mu}(y_i)$ lies in the interior of the root-path π_{y_j} and individual $\hat{\mu}(y_j)$ lies in the root-path π_{y_i}. Since $\hat{\mu}(y_i)$ lies in the interior of π_{y_j}, the fact that $\pi_{y_{t+1}}$ and $\pi_{y_{t+2}}$ are single-node root-paths, implies $y_j \notin \{y_{t+1}, y_{t+2}\}$. Suppose $y_i \in \{y_{t+1}, y_{t+2}\}$. Since $\pi_{y_{t+1}}$ and $\pi_{y_{t+2}}$ are single-node root-paths, $\hat{\mu}(y_j)$ lies in π_{y_i}, and j_h is the last element of π_{y_h} for all $h = t+1, t+2$, it follows that $\hat{\mu}(y_j) \in \{j_{t+1}, j_{t+2}\}$. However, by the definition of $\hat{\mu}$, $\hat{\mu}(y_j) \in \{j_{t+1}, j_{t+2}\}$ implies $y_j \in \{y_{t+1}, y_{t+2}\}$, which contradicts the fact $y_j \notin \{y_{t+1}, y_{t+2}\}$. So, it must be that $y_i \notin \{y_{t+1}, y_{t+2}\}$. Combining the facts that $y_i, y_j \in \{y_4, \ldots, y_{t+2}\}$, $y_j \notin \{y_{t+1}, y_{t+2}\}$, and $y_i \notin \{y_{t+1}, y_{t+2}\}$, we obtain $y_i, y_j \in \{y_4, \ldots, y_t\}$. However, since $y_i, y_j \in \{y_4, \ldots, y_t\}$, it follows from the definition of $\{j_1, \ldots, j_{t+2}\}$, $\{y_1, \ldots, y_{t+2}\}$, $\hat{\mu}$, and $(\pi_{y_h})_{h \in \{1,\ldots,t+2\}}$ that individual $\bar{\mu}(x_i)$ lies in the interior of the root-path $\bar{\pi}_{x_j}$ and individual $\bar{\mu}(x_j)$ lies in the root-path $\bar{\pi}_{x_i}$, which contradicts (2).(ii). This proves that $[(j_1, j_2, j_3), (y_1, y_2, y_3)]$ satisfies (ii) of Definition 8.1. Therefore, $[(j_1, j_2, j_3), (y_1, y_2, y_3)]$ is an inheritance cycle at Γ, a contradiction to the fact that Γ is acyclic. This completes the proof of Claim B.4. □

By the induction hypothesis and Claim B.4, it follows that there exists an OSP mechanism G^{m-1} that implements f^Γ restricted to the reduced market $(N \setminus \{i, j\}, A \setminus \{x, y\})$. Run the extensive-form assignment mechanism G^{m-1} on the reduced market $(N \setminus \{i, j\}, A \setminus \{x, y\})$.

By definition, the extensive-form assignment mechanism G^{m+1} implements f^Γ. We show that G^{m+1} is OSP for all individuals by showing it for the case where $|N| = 4$. The proof for other cases is similar.

Consider an allocation problem with four individuals $N = \{i_1, i_2, i_3, i_4\}$ and five objects $A = \{x_1, x_2, x_3, x_4, x_5\}$. Let Γ be an acyclic collection of inheritance trees such that $\mathcal{T}(\Gamma) = \{i_1, i_2\}$, $A_{i_1} = \{x_1, x_2\}$, and $A_{i_2} = \{x_3, x_4, x_5\}$. In Figure B.3, we provide the structure of the extensive-form assignment mechanism G^4 which implements the hierarchical exchange rule f^Γ.

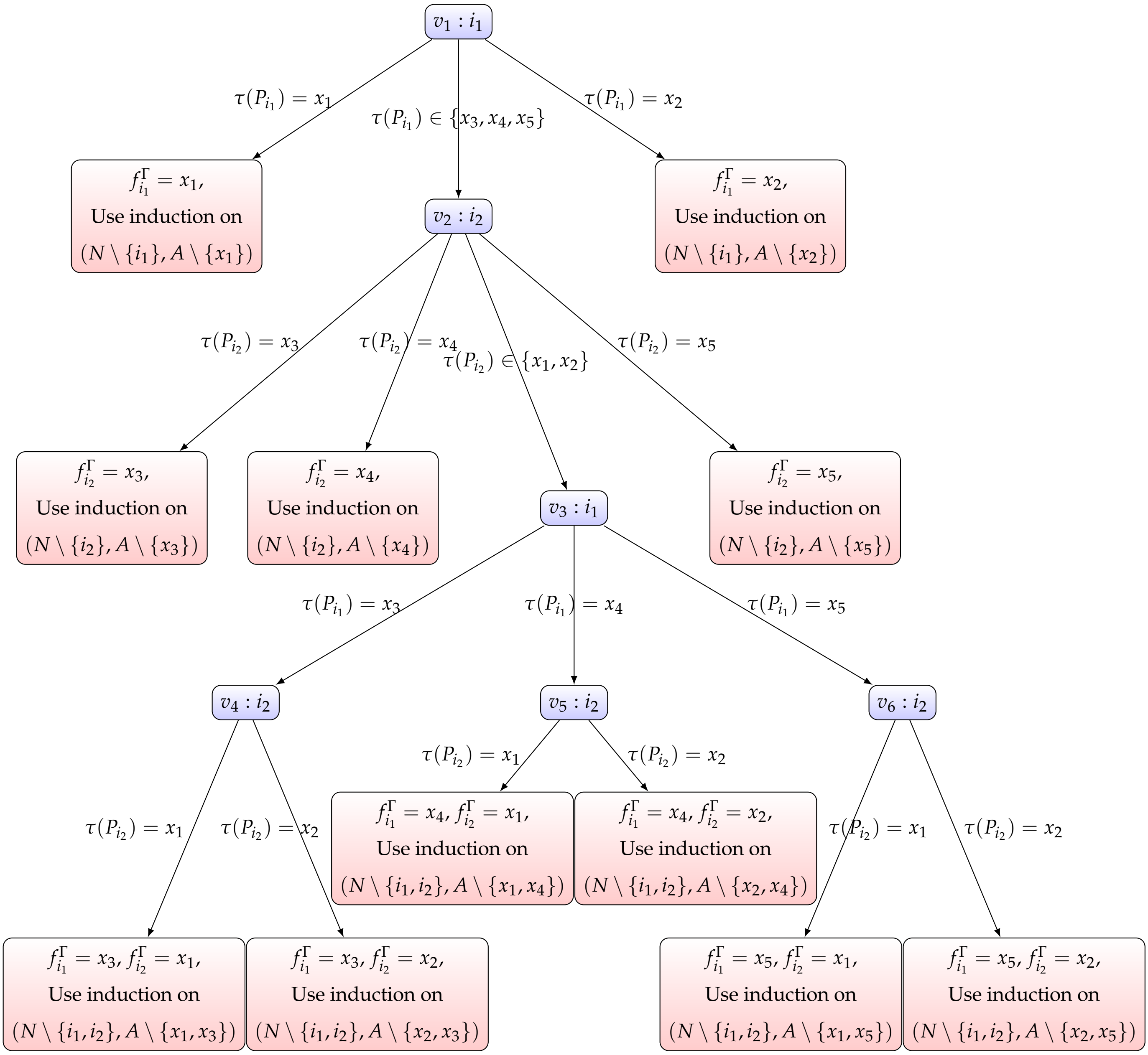


Figure B.3: Structure of G^4

In Figure B.3, node v_1 (which is the root-node of G^4) is assigned to individual i_1 and there are $|A_{i_1}| + 1$ outgoing edges from this node, node v_2 is assigned to individual i_2 and there are $|A_{i_2}| + 1$ outgoing edges from this node, and node v_3 is assigned to individual i_1 and there are $|A_{i_2}|$ outgoing edges from this node. Nodes v_4, v_5, and v_6 are assigned to individual i_2 and there are $|A_{i_1}|$ outgoing edges from each of these nodes.

It follows from the definition of G^4 and Observation B.2 that G^4 satisfies the OSP property at node v_1 (for individual i_1). We distinguish two cases.

(i) Suppose $\tau(P_{i_1}) \in \{x_1, x_2\}$.

Individual i_1 receives her top choice. The first decision node of every other individual comes after i_1 has been assigned, and hence, their strategic decisions are equivalent to that under the OSP mechanism that implements f^Γ restricted to the reduced market.

(ii) Suppose $\tau(P_{i_1}) \in \{x_3, x_4, x_5\}$.

It follows from the definition of G^4 and Observation B.2 that G^4 satisfies the OSP property at node v_2 (for individual i_2).

(a) Suppose $\tau(P_{i_2}) \in \{x_3, x_4, x_5\}$. Individual i_2 receives her top choice. For every other individual, her strategic decision is equivalent to that under the OSP mechanism that implements f^Γ restricted to the reduced market.

(b) Suppose $\tau(P_{i_2}) \in \{x_1, x_2\}$. Both i_1 and i_2 receive their top choices. The first decision node of every other individual comes after i_1 and i_2 have been assigned, and hence, their strategic decisions are equivalent to that under the OSP mechanism that implements f^Γ restricted to the reduced market.

Since Cases (i) and (ii) are exhaustive, it follows that the extensive-form assignment mechanism G^4 is OSP for all individuals, and hence, f^Γ is OSP-implementable for this particular instance.

Since Case A and Case B are exhaustive, it follows that f^Γ is OSP-implementable for $|N| = m + 1$. This completes the proof of the induction step, and thereby completes the proof of the "if" part of Lemma B.1.

(Only-if part) Suppose f^Γ does not satisfy dual ownership. We show that f^Γ is not OSP-implementable. Since f^Γ does not satisfy dual ownership, there exist a preference profile P'_N and a stage s^* of f^Γ at P'_N such that there are three individuals i_1, i_2, i_3 and three objects x_1, x_2, x_3 in the reduced market in Stage s^* with the property that for all $h = 1, 2, 3$, individual i_h owns the object x_h in Stage s^*.

Note that if an assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is not OSP-implementable on some restricted domain $\tilde{\mathcal{P}}_N \subseteq \mathbb{L}^n(A)$, then f is not OSP-implementable on the whole domain $\mathbb{L}^n(A)$ (see Li (2017) for details). We distinguish the following two cases.

CASE A: Suppose $s^* = 1$.

Consider the restricted domain $\tilde{\mathcal{P}}_N$ defined as follows. Each $l \in N \setminus \{i_1, i_2, i_3\}$ has only one (admissible) preference P'_l, and each individual in $\{i_1, i_2, i_3\}$ has two preferences, defined as follows (the dots indicate that all preferences for the corresponding parts are irrelevant and can be chosen arbitrarily).[29]

Individual i_1	Individual i_2	Individual i_3
$x_2 x_3 x_1 \ldots$	$x_3 x_1 x_2 \ldots$	$x_1 x_2 x_3 \ldots$
$x_3 x_2 x_1 \ldots$	$x_1 x_3 x_2 \ldots$	$x_2 x_1 x_3 \ldots$

Table B.1

[29]For instance, $x_1 x_2 x_3 \ldots$ indicates (any) preference that ranks x_1 first, x_2 second, and x_3 third.

In Table B.2, we present some facts regarding the outcome of f^Γ on the restricted domain $\tilde{\mathcal{P}}_N$. These fact are deduced by the construction of $\tilde{\mathcal{P}}_N$ along with the assumptions for Case A.

Preference profile	Individual i_1	Individual i_2	Individual i_3	$f^\Gamma_{i_1}$	$f^\Gamma_{i_2}$	$f^\Gamma_{i_3}$
$\tilde{P}^1_N$	$x_2x_3x_1\dots$	$x_3x_1x_2\dots$	$x_1x_2x_3\dots$	x_2	x_3	x_1
$\tilde{P}^2_N$	$x_2x_3x_1\dots$	$x_1x_3x_2\dots$	$x_1x_2x_3\dots$	x_2	x_1	x_3
$\tilde{P}^3_N$	$x_2x_3x_1\dots$	$x_3x_1x_2\dots$	$x_2x_1x_3\dots$	x_1	x_3	x_2
$\tilde{P}^4_N$	$x_2x_3x_1\dots$	$x_1x_3x_2\dots$	$x_2x_1x_3\dots$	x_2	x_1	x_3
$\tilde{P}^5_N$	$x_3x_2x_1\dots$	$x_3x_1x_2\dots$	$x_1x_2x_3\dots$	x_3	x_2	x_1
$\tilde{P}^6_N$	$x_3x_2x_1\dots$	$x_1x_3x_2\dots$	$x_1x_2x_3\dots$	x_3	x_2	x_1
$\tilde{P}^7_N$	$x_3x_2x_1\dots$	$x_3x_1x_2\dots$	$x_2x_1x_3\dots$	x_1	x_3	x_2
$\tilde{P}^8_N$	$x_3x_2x_1\dots$	$x_1x_3x_2\dots$	$x_2x_1x_3\dots$	x_3	x_1	x_2

Table B.2: Partial outcome of f^Γ on $\tilde{\mathcal{P}}_N$

Assume for contradiction that f^Γ is OSP-implementable on $\tilde{\mathcal{P}}_N$. So, there exists an OSP mechanism $\tilde{G}$ that implements f^Γ on $\tilde{\mathcal{P}}_N$. Note that since $f^\Gamma(\tilde{P}^1_N) \neq f^\Gamma(\tilde{P}^8_N)$, there exists a node in the OSP mechanism $\tilde{G}$ that has at least two edges. Also, note that since each individual $l \in N \setminus \{i_1, i_2, i_3\}$ has exactly one preference in $\tilde{\mathcal{P}}_l$, whenever there are more than one outgoing edges from a node, the node must be assigned to some individual in $\{i_1, i_2, i_3\}$. Consider the first node (from the root) v that has two edges and, without loss of generality, assume $\eta^{NI}(v) = i_1$. Consider the preference profiles $\tilde{P}^3_N$ and $\tilde{P}^5_N$. Note that both of them pass through the node v at which $\tilde{P}^3_{i_1}$ and $\tilde{P}^5_{i_1}$ diverge. Further note that $x_3\tilde{P}^3_{i_1}x_1$, $f^\Gamma_{i_1}(\tilde{P}^3_N) = x_1$, and $f^\Gamma_{i_1}(\tilde{P}^5_N) = x_3$. However, the facts that $x_3\tilde{P}^3_{i_1}x_1$, $f^\Gamma_{i_1}(\tilde{P}^3_N) = x_1$, and $f^\Gamma_{i_1}(\tilde{P}^5_N) = x_3$ together contradict OSP-implementability of f^Γ on $\tilde{\mathcal{P}}_N$.

Case B: Suppose $s^* > 1$.

Recall that for the preference profile P'_N, $F^{s^*-1}(P'_N)$ is the set of assigned objects up to Stage $s^* - 1$ (including Stage $s^* - 1$) of f^Γ at P'_N. Fix a preference $\hat{P} \in \mathbb{L}(F^{s^*-1}(P'_N))$ over these objects.

Consider the restricted domain $\tilde{\mathcal{P}}_N$ defined as follows. Each $l \in N \setminus \{i_1, i_2, i_3\}$ has only one (admissible) preference P'_l, and each individual in $\{i_1, i_2, i_3\}$ has two preferences, defined as follows.[30]

Individual i_1	Individual i_2	Individual i_3
$\hat{P}x_2x_3x_1\dots$	$\hat{P}x_3x_1x_2\dots$	$\hat{P}x_1x_2x_3\dots$
$\hat{P}x_3x_2x_1\dots$	$\hat{P}x_1x_3x_2\dots$	$\hat{P}x_2x_1x_3\dots$

Table B.3

[30]For instance, $\hat{P}x_1x_2x_3\dots$ denotes a preference where objects in $F^{s^*-1}(P'_N)$ are ranked at the top according to the preference $\hat{P}$, objects x_1, x_2, and x_3 are ranked consecutively after that (in that order), and the ranking of the rest of the objects is arbitrarily.

In Table B.4, we present some facts regarding the outcome of f^Γ on the restricted domain $\tilde{\mathcal{P}}_N$ that can be deduced by the construction of the restricted domain $\tilde{\mathcal{P}}_N$ along with the assumptions for Case B. The verification of these facts is left to the reader.

Preference profile	Individual i_1	Individual i_2	Individual i_3	$f^\Gamma_{i_1}$	$f^\Gamma_{i_2}$	$f^\Gamma_{i_3}$
$\tilde{P}^1_N$	$\hat{P}x_2x_3x_1\dots$	$\hat{P}x_3x_1x_2\dots$	$\hat{P}x_1x_2x_3\dots$	x_2	x_3	x_1
$\tilde{P}^2_N$	$\hat{P}x_2x_3x_1\dots$	$\hat{P}x_1x_3x_2\dots$	$\hat{P}x_1x_2x_3\dots$	x_2	x_1	x_3
$\tilde{P}^3_N$	$\hat{P}x_2x_3x_1\dots$	$\hat{P}x_3x_1x_2\dots$	$\hat{P}x_2x_1x_3\dots$	x_1	x_3	x_2
$\tilde{P}^4_N$	$\hat{P}x_2x_3x_1\dots$	$\hat{P}x_1x_3x_2\dots$	$\hat{P}x_2x_1x_3\dots$	x_2	x_1	x_3
$\tilde{P}^5_N$	$\hat{P}x_3x_2x_1\dots$	$\hat{P}x_3x_1x_2\dots$	$\hat{P}x_1x_2x_3\dots$	x_3	x_2	x_1
$\tilde{P}^6_N$	$\hat{P}x_3x_2x_1\dots$	$\hat{P}x_1x_3x_2\dots$	$\hat{P}x_1x_2x_3\dots$	x_3	x_2	x_1
$\tilde{P}^7_N$	$\hat{P}x_3x_2x_1\dots$	$\hat{P}x_3x_1x_2\dots$	$\hat{P}x_2x_1x_3\dots$	x_1	x_3	x_2
$\tilde{P}^8_N$	$\hat{P}x_3x_2x_1\dots$	$\hat{P}x_1x_3x_2\dots$	$\hat{P}x_2x_1x_3\dots$	x_3	x_1	x_2

Table B.4: Partial outcome of f^Γ on $\tilde{\mathcal{P}}_N$

Using a similar argument as for Case A, it follows from Table B.4 that f^Γ is not OSP-implementable on $\tilde{\mathcal{P}}_N$. This completes the proof of the "only-if" part of Lemma B.1. ■

B.2 Lemma B.2 and its proof

Lemma B.2 involves the notion of reallocation-proof assignment rules, which we present first.

Definition B.1 (Pápai 2000)**.** An assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is *manipulable through reallocation* if there exist $P_N \in \mathbb{L}^n(A)$, distinct individuals $i, j \in N$, and $\tilde{P}_i \in \mathbb{L}(A)$, $\tilde{P}_j \in \mathbb{L}(A)$ such that

(i) $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) R_i f_i(P_N)$,

(ii) $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$, and

(iii) $f_i(P_N) = f_i(\tilde{P}_i, P_{-i}) \neq f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$ and $f_j(P_N) = f_j(\tilde{P}_j, P_{-j}) \neq f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$.

An assignment rule is ***reallocation-proof*** if it is not manipulable through reallocation.

Lemma B.2. *Suppose an assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is OSP-implementable, non-bossy and Pareto efficient. Then, f is reallocation-proof.*

Proof of Lemma B.2. Since f is OSP-implementable, by Remark 4.1, f is strategy-proof. Assume for contradiction that f is not reallocation-proof. Then, there exist $P_N \in \mathbb{L}^n(A)$, distinct individuals $i, j \in N$, and $\tilde{P}_i \in \mathbb{L}(A)$, $\tilde{P}_j \in \mathbb{L}(A)$ such that

(i) $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) R_i f_i(P_N)$,

(ii) $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$, and

(iii) $f_i(P_N) = f_i(\tilde{P}_i, P_{-i}) \neq f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$ and $f_j(P_N) = f_j(\tilde{P}_j, P_{-j}) \neq f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$.

Using non-bossiness, $f_i(P_N) = f_i(\tilde{P}_i, P_{-i})$ implies $f(P_N) = f(\tilde{P}_i, P_{-i})$, and $f_j(P_N) = f_j(\tilde{P}_j, P_{-j})$ implies $f(P_N) = f(\tilde{P}_j, P_{-j})$. Combining the facts that $f(P_N) = f(\tilde{P}_i, P_{-i})$ and $f(P_N) = f(\tilde{P}_j, P_{-j})$, we have

$$f(P_N) = f(\tilde{P}_i, P_{-i}) = f(\tilde{P}_j, P_{-j}). \tag{B.5}$$

Claim B.5. $\left\{f_i(P_N), f_j(P_N), f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}), f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})\right\} \subseteq A$.

Proof of Claim B.5. Assume for contradiction that $f_i(P_N) = \emptyset$. By (B.5), we have $f_i(P_N) = f_i(\tilde{P}_j, P_{-j})$. Because $f_i(P_N) = \emptyset$ and $f_i(P_N) = f_i(\tilde{P}_j, P_{-j})$, we have $f_i(\tilde{P}_j, P_{-j}) = \emptyset$. Since f is strategy-proof, $f_i(\tilde{P}_j, P_{-j}) = \emptyset$ implies $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = \emptyset$. However, as $f_i(P_N) = \emptyset$ and $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = \emptyset$, we have a contradiction to $f_i(P_N) \neq f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$. So, it must be that

$$f_i(P_N) \neq \emptyset. \tag{B.6}$$

Using a similar argument, we have

$$f_j(P_N) \neq \emptyset. \tag{B.7}$$

Since $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$, (B.7) implies $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) \neq \emptyset$. Also, the fact $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) R_i f_i(P_N)$, together with (B.6), implies $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) \neq \emptyset$. This completes the proof of Claim B.5. □

Claim B.6. $f_i(P_N) = f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$.

Proof of Claim B.6. Assume for contradiction that $f_i(P_N) \neq f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$. Let $f_i(P_N) = w$, $f_j(P_N) = x$, $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = y$, and $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = z$. By Claim B.5, we have $w, x, y, z \neq \emptyset$. Since $f_i(P_N) = w$ and $f_j(P_N) = x$, feasibility implies $w \neq x$. Similarly by means of feasibility, $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = y$ and $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = z$ together imply $y \neq z$. Since $f_i(P_N) \neq f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$, we have $w \neq y$. Similarly $f_j(P_N) \neq f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$ implies $x \neq z$, and $f_i(P_N) \neq f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$ implies $w \neq z$. Moreover, $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$ implies $x \neq y$. However, the facts $w, x, y, z \neq \emptyset$, $w \neq x$, $y \neq z$, $w \neq y$, $x \neq z$, $w \neq z$, and $x \neq y$ together imply w, x, y, and z are all distinct objects.

Since $f_i(P_N) \neq f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$, $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) R_i f_i(P_N)$ implies $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_i f_i(P_N)$. The facts $f_i(P_N) = w$, $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = z$, and $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_i f_i(P_N)$ together imply $z P_i w$. Since $z P_i w$ and $f_i(P_N) = w$, by strategy-proofness, we have

$$f_i(P'_i, P_{-i}) \neq z \text{ for all } P'_i \in \mathbb{L}(A). \tag{B.8}$$

By (B.5) we have $f_i(P_N) = f_i(\tilde{P}_j, P_{-j})$. This, along with the fact that $f_i(P_N) = w$, yields $f_i(\tilde{P}_j, P_{-j}) = w$. Since f is strategy-proof, the facts $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = y$ and $f_i(\tilde{P}_j, P_{-j}) = w$ together imply $y \tilde{R}_i w$, which, along with the fact that $w \neq y$, yields $y\tilde{P}_i w$. Also, combining the facts that $f_i(P_N) = w$ and $f_i(P_N) = f_i(\tilde{P}_i, P_{-i})$, we have $f_i(\tilde{P}_i, P_{-i}) = w$. Since $y\tilde{P}_i w$ and $f_i(\tilde{P}_i, P_{-i}) = w$, by strategy-proofness, we have

$$f_i(P_i', P_{-i}) \neq y \text{ for all } P_i' \in \mathbb{L}(A). \tag{B.9}$$

Moreover, since $zP_i w$ and $f_i(\tilde{P}_j, P_{-j}) = w$, by strategy-proofness, we have

$$f_i(P_i', \tilde{P}_j, P_{-i,j}) \neq z \text{ for all } P_i' \in \mathbb{L}(A). \tag{B.10}$$

Let $\hat{P}_i$ rank z first, y second, and w third. Since f is strategy-proof and non-bossy, the fact $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = y$ and (B.10) imply

$$f(\hat{P}_i, \tilde{P}_j, P_{-i,j}) = f(\tilde{P}_i, \tilde{P}_j, P_{-i,j}). \tag{B.11}$$

Similarly, by strategy-proofness and non-bossiness, the fact that $f_i(P_N) = w$ along with (B.8) and (B.9), yields

$$f(\hat{P}_i, P_{-i}) = f(P_N). \tag{B.12}$$

By (B.12) we have $f_j(\hat{P}_i, P_{-i}) = f_j(P_N)$. This, along with the fact $f_j(P_N) = x$, yields $f_j(\hat{P}_i, P_{-i}) = x$. Also, the facts $f_j(P_N) = x$, $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = y$, and $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$ together imply $yP_j x$. Since $yP_j x$ and $f_j(\hat{P}_i, P_{-i}) = x$, by strategy-proofness, we have

$$f_j(\hat{P}_i, P_j', P_{-i,j}) \neq y \text{ for all } P_j' \in \mathbb{L}(A). \tag{B.13}$$

Let $\hat{P}_j$ rank y first and z second. By (B.11) we have $f_j(\hat{P}_i, \tilde{P}_j, P_{-i,j}) = f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$. This, along with the fact $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = z$, yields $f_j(\hat{P}_i, \tilde{P}_j, P_{-i,j}) = z$. Since f is strategy-proof and non-bossy, the fact $f_j(\hat{P}_i, \tilde{P}_j, P_{-i,j}) = z$ and (B.13) imply $f(\hat{P}_i, \hat{P}_j, P_{-i,j}) = f(\hat{P}_i, \tilde{P}_j, P_{-i,j})$. This, along with (B.11), yields

$$f(\hat{P}_i, \hat{P}_j, P_{-i,j}) = f(\tilde{P}_i, \tilde{P}_j, P_{-i,j}). \tag{B.14}$$

Because $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = y$ and $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = z$, (B.14) implies $f_i(\hat{P}_i, \hat{P}_j, P_{-i,j}) = y$ and $f_j(\hat{P}_i, \hat{P}_j, P_{-i,j}) = z$. However, since $z\hat{P}_i y$ and $y\hat{P}_j z$, the facts $f_i(\hat{P}_i, \hat{P}_j, P_{-i,j}) = y$ and $f_j(\hat{P}_i, \hat{P}_j, P_{-i,j}) = z$ together contradict Pareto efficiency.

So, it must be that $f_i(P_N) = f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$. This completes the proof of Claim B.6. □

Since f is Pareto efficient, $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$ implies that there exists $k \in N \setminus \{j\}$ such that $f_k(P_N) = f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$. Also, the facts $f_k(P_N) = f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$ and $f_i(P_N) \neq f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$ together imply $k \neq i$. Let $f_i(P_N) = a$, $f_j(P_N) = b$, and $f_k(P_N) = c$. Combining the facts that $f_k(P_N) = f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$ and $f_k(P_N) = c$, we have $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = c$. Also the fact $f_i(P_N) = a$ along with Claim B.6, implies $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = a$. Let $f_k(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = d$.

Claim B.7. *a, b, and c are distinct objects, $d \in A$, and a, c, and d are distinct objects.*

Proof of Claim B.7. Since $f_i(P_N) = a$, $f_j(P_N) = b$, and $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = c$, by Claim B.5, we have $a \neq \emptyset$, $b \neq \emptyset$, and $c \neq \emptyset$. Moreover, since $f_i(P_N) = a$, $f_j(P_N) = b$, and $f_k(P_N) = c$, feasibility implies a, b, and c are all distinct objects.

Now, we show $d \in A$. Assume for contradiction that $d = \emptyset$. Consider the preference profiles presented in Table B.5. In addition to the structure provided in the table, suppose that $P_j^1 = P_j^3$, $P_j^2 = P_j^4$, and $P_k^1 = P_k^2$. Here, l denotes an individual (might be empty) other than i, j, k. Note that such an individual does not change her preference across the mentioned preference profiles.

Preference profiles	Individual i	Individual j	Individual k	...	Individual l
P_N^1	$\tilde{P}_i$	$ca\ldots$	$bc\ldots$	...	P_l
P_N^2	$\tilde{P}_i$	$cba\ldots$	$bc\ldots$	...	P_l
P_N^3	$\tilde{P}_i$	$ca\ldots$	P_k	...	P_l
P_N^4	$\tilde{P}_i$	$cba\ldots$	P_k	...	P_l

Table B.5: Preference profiles for Claim B.7

The facts $f_j(P_N) = b$, $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = c$, and $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$ together imply cP_jb. Moreover, $f_j(P_N) = b$ and (B.5) yield $f_j(\tilde{P}_i, P_{-i}) = b$. Since cP_jb and $f_j(\tilde{P}_i, P_{-i}) = b$, by strategy-proofness, we have

$$f_j(\tilde{P}_i, P'_j, P_{-i,j}) \neq c \text{ for all } P'_j \in \mathbb{L}(A). \tag{B.15}$$

By strategy-proofness and non-bossiness, the fact $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = a$ and (B.15) imply

$$f(P_N^3) = f(\tilde{P}_i, \tilde{P}_j, P_{-i,j}). \tag{B.16}$$

The facts $f_k(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = d$ and $d = \emptyset$ together imply $f_k(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = \emptyset$. Moreover, $f_k(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = \emptyset$ and (B.16) imply $f_k(P_N^3) = \emptyset$. Since f is strategy-proof and non-bossy, $f_k(P_N^3) = \emptyset$ yields $f(P_N^1) = f(P_N^3)$. This, together with (B.16), implies

$$f(P_N^1) = f(\tilde{P}_i, \tilde{P}_j, P_{-i,j}). \tag{B.17}$$

Similarly, by strategy-proofness and non-bossiness, the fact $f_j(\tilde{P}_i, P_{-i}) = b$ and (B.15) imply $f(P_N^4) = f(\tilde{P}_i, P_{-i})$. This, along with (B.5), yields

$$f(P_N^4) = f(P_N). \tag{B.18}$$

Since $f_j(P_N) = b$ and $f_k(P_N) = c$, by (B.18) we have $f_j(P_N^4) = b$ and $f_k(P_N^4) = c$. By strategy-proofness, $f_k(P_N^4) = c$ implies $f_k(P_N^2) \in \{b, c\}$. Suppose $f_k(P_N^2) = c$. Since $f_k(P_N^2) = c$ and $f_k(P_N^4) = c$, by non-bossiness and the fact that $f_j(P_N^4) = b$, we have $f_j(P_N^2) = b$. However, $f_j(P_N^2) = b$ and $f_k(P_N^2) = c$ together contradict Pareto efficiency. So, it must be that

$$f_k(P_N^2) = b. \tag{B.19}$$

Since $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = a$, by (B.17) we have $f_j(P_N^1) = a$. Also, by (B.19) we have $f_j(P_N^2) \neq b$. By strategy-proofness, the facts $f_j(P_N^1) = a$ and $f_j(P_N^2) \neq b$ imply $f_j(P_N^2) = a$. Since $f_j(P_N^1) = a$ and $f_j(P_N^2) = a$, by non-bossiness and (B.17), we have

$$f(P_N^2) = f(\tilde{P}_i, \tilde{P}_j, P_{-i,j}). \tag{B.20}$$

However, since $f_k(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = \emptyset$, by (B.20) we have $f_k(P_N^2) = \emptyset$, a contradiction to (B.19). So, it must be that

$$d \in A. \tag{B.21}$$

Since $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = c$, $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = a$, and $f_k(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = d$, feasibility implies a, c, and d are all distinct objects. This completes the proof of Claim B.7. □

Claim B.8. *cP_kd.*

Proof of Claim B.8. Assume for contradiction that dR_kc. By Claim B.7, this means dP_kc. Suppose $b = d$. Because dP_kc, this implies bP_kc. Also, the facts $f_j(P_N) = b$, $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = c$, and $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})P_jf_j(P_N)$ together imply cP_jb. However, since cP_jb and bP_kc, the facts $f_j(P_N) = b$ and $f_k(P_N) = c$ together contradict Pareto efficiency. So, it must be that $b \neq d$. This, along with Claim B.7, yields that a, b, c, and d are all distinct objects.

Consider the preference profiles presented in Table B.6. In addition to the structure provided in the table, suppose $P_j^1 = P_j^3$, $P_j^2 = P_j^4$, and $P_k^1 = P_k^2$.

Preference profiles	Individual i	Individual j	Individual k	...	Individual l
P_N^1	$\tilde{P}_i$	$ca\ldots$	$dbc\ldots$	...	P_l
P_N^2	$\tilde{P}_i$	$cba\ldots$	$dbc\ldots$	...	P_l
P_N^3	$\tilde{P}_i$	$ca\ldots$	P_k	...	P_l
P_N^4	$\tilde{P}_i$	$cba\ldots$	P_k	...	P_l

Table B.6: Preference profiles for Claim B.8

The fact $f_j(P_N) = b$ and (B.5) yield $f_j(\tilde{P}_i, P_{-i}) = b$. Moreover, the facts $f_j(P_N) = b$, $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = c$, and $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j})P_jf_j(P_N)$ together imply cP_jb. Since cP_jb and $f_j(\tilde{P}_i, P_{-i}) = b$, by strategy-proofness, we have

$$f_j(\tilde{P}_i, P_j', P_{-i,j}) \neq c \text{ for all } P_j' \in \mathbb{L}(A). \tag{B.22}$$

By strategy-proofness and non-bossiness, the fact $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = a$ and (B.22) imply

$$f(P_N^3) = f(\tilde{P}_i, \tilde{P}_j, P_{-i,j}). \tag{B.23}$$

The fact $f_k(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = d$ and (B.23) imply $f_k(P_N^3) = d$. Since f is strategy-proof and non-bossy, $f_k(P_N^3) = d$ yields $f(P_N^1) = f(P_N^3)$. This, together with (B.23), implies

$$f(P_N^1) = f(\tilde{P}_i, \tilde{P}_j, P_{-i,j}). \tag{B.24}$$

Similarly, by strategy-proofness and non-bossiness, the fact $f_j(\tilde{P}_i, P_{-i}) = b$ and (B.22) imply $f(P_N^4) = f(\tilde{P}_i, P_{-i})$. This, along with (B.5), yields

$$f(P_N^4) = f(P_N). \tag{B.25}$$

Since $f_j(P_N) = b$ and $f_k(P_N) = c$, by (B.25) we have $f_j(P_N^4) = b$ and $f_k(P_N^4) = c$. By strategy-proofness, dP_kc and $f_k(P_N^4) = c$ together imply $f_k(P_N^2) \in \{b, c\}$. Suppose $f_k(P_N^2) = c$. Since $f_k(P_N^2) = c$ and $f_k(P_N^4) = c$, by non-bossiness and the fact that $f_j(P_N^4) = b$, we have $f_j(P_N^2) = b$. However, $f_j(P_N^2) = b$ and $f_k(P_N^2) = c$ together contradict Pareto efficiency. So, it must be that

$$f_k(P_N^2) = b. \tag{B.26}$$

Since $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = a$, by (B.24) we have $f_j(P_N^1) = a$. Also, by (B.26) we have $f_j(P_N^2) \neq b$. By strategy-proofness, the facts $f_j(P_N^1) = a$ and $f_j(P_N^2) \neq b$ together imply $f_j(P_N^2) = a$. Since $f_j(P_N^1) = a$ and $f_j(P_N^2) = a$, by non-bossiness and (B.24), we have

$$f(P_N^2) = f(\tilde{P}_i, \tilde{P}_j, P_{-i,j}). \tag{B.27}$$

However, since $f_k(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = d$, by (B.27) we have $f_k(P_N^2) = d$, a contradiction to (B.26). This completes the proof of Claim B.8. □

Fix a preference $\hat{P} \in \mathbb{L}(A \setminus \{a, b, c\})$ over the objects in $A \setminus \{a, b, c\}$. Consider the preference profiles presented in Table B.7.[31] Assume that $P_k^5 = P_k^{10} = P_k^{11}$.

[31] For instance, $abc\hat{P}$ denotes the preference that ranks a first, b second, c third, and follows $\hat{P}$ for the ranking of the rest of the objects.

Preference profiles	Individual i	Individual j	Individual k	$\ldots$	Individual l
P_N^1	$abc\hat{P}$	$cab\hat{P}$	$acb\hat{P}$	$\ldots$	P_l
P_N^2	$abc\hat{P}$	$cba\hat{P}$	$acb\hat{P}$	$\ldots$	P_l
P_N^3	$acb\hat{P}$	$cab\hat{P}$	$acb\hat{P}$	$\ldots$	P_l
P_N^4	$acb\hat{P}$	$cab\hat{P}$	$cab\hat{P}$	$\ldots$	P_l
P_N^5	$acb\hat{P}$	$cab\hat{P}$	$cd\ldots$	$\ldots$	P_l
P_N^6	$bca\hat{P}$	$cba\hat{P}$	$acb\hat{P}$	$\ldots$	P_l
P_N^7	$bca\hat{P}$	$cba\hat{P}$	$cab\hat{P}$	$\ldots$	P_l
P_N^8	$cab\hat{P}$	$cab\hat{P}$	$cab\hat{P}$	$\ldots$	P_l
P_N^9	$cab\hat{P}$	$cba\hat{P}$	$cab\hat{P}$	$\ldots$	P_l
P_N^{10}	$cab\hat{P}$	$cab\hat{P}$	$cd\ldots$	$\ldots$	P_l
P_N^{11}	$cab\hat{P}$	$cba\hat{P}$	$cd\ldots$	$\ldots$	P_l
P_N^{12}	$cba\hat{P}$	$cab\hat{P}$	$acb\hat{P}$	$\ldots$	P_l
P_N^{13}	$cba\hat{P}$	$cba\hat{P}$	$acb\hat{P}$	$\ldots$	P_l
P_N^{14}	$cba\hat{P}$	$cab\hat{P}$	$cab\hat{P}$	$\ldots$	P_l
P_N^{15}	$cba\hat{P}$	$cba\hat{P}$	$cab\hat{P}$	$\ldots$	P_l

Table B.7: Preference profiles for Lemma B.2

The facts $f_j(P_N) = b$, $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = c$, and $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) P_j f_j(P_N)$ together imply cP_jb. Since cP_jb and $f_j(P_N) = b$, by strategy-proofness, we have

$$f_j(P'_j, P_{-j}) \neq c \text{ for all } P'_j \in \mathbb{L}(A). \tag{B.28}$$

Combining the fact $f_j(P_N) = b$ with (B.5), we have $f_j(\tilde{P}_i, P_{-i}) = f_j(\tilde{P}_j, P_{-j}) = b$. Since f is strategy-proof, the facts $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = a$ and $f_j(\tilde{P}_i, P_{-i}) = b$ together imply $a\tilde{R}_jb$, which along with Claim B.7, yields $a\tilde{P}_jb$. Since $a\tilde{P}_jb$ and $f_j(\tilde{P}_j, P_{-j}) = b$, by strategy-proofness, we have

$$f_j(P'_j, P_{-j}) \neq a \text{ for all } P'_j \in \mathbb{L}(A). \tag{B.29}$$

However, since $f_j(\tilde{P}_j, P_{-j}) = b$, by strategy-proofness and non-bossiness along with (B.28) and (B.29), we have $f(P_j^5, P_{-j}) = f(\tilde{P}_j, P_{-j})$. By (B.5), this, in particular, means

$$f_i(P_j^5, P_{-j}) = a, f_j(P_j^5, P_{-j}) = b, \text{ and } f_k(P_j^5, P_{-j}) = c. \tag{B.30}$$

By moving the preferences of the individuals $l \in \{i, k\}$ from P_l to P_l^5 one by one, and by applying strategy-proofness and non-bossiness on (B.30) each time, we conclude

$$f_i(P^5_N) = a, f_j(P^5_N) = b, \text{ and } f_k(P^5_N) = c. \tag{B.31}$$

Using strategy-proofness and non-bossiness, we obtain from (B.31) that

$$f_i(P^4_N) = a, f_j(P^4_N) = b, \text{ and } f_k(P^4_N) = c. \tag{B.32}$$

By strategy-proofness, the facts cP_jb and $f_j(\tilde{P}_i, P_{-i}) = b$ together imply

$$f_j(\tilde{P}_i, P'_j, P_{-i,j}) \neq c \text{ for all } P'_j \in \mathbb{L}(A). \tag{B.33}$$

Since f is strategy-proof, the fact $f_j(\tilde{P}_i, P_{-i}) = b$ and (B.33) imply $f_j(\tilde{P}_i, P^{11}_j, P_{-i,j}) = b$. Moreover, since $f_j(\tilde{P}_i, P_{-i}) = b$ and $f_j(\tilde{P}_i, P^{11}_j, P_{-i,j}) = b$, by non-bossiness, we have $f(\tilde{P}_i, P^{11}_j, P_{-i,j}) = f(\tilde{P}_i, P_{-i})$. This, together with (B.5), yields

$$f(\tilde{P}_i, P^{11}_j, P_{-i,j}) = f(P_N). \tag{B.34}$$

By (B.5) we have $f_i(P_N) = f_i(\tilde{P}_j, P_{-j})$. This, along with the fact that $f_i(P_N) = a$, yields $f_i(\tilde{P}_j, P_{-j}) = a$. Since f is strategy-proof, the facts $f_i(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = c$ and $f_i(\tilde{P}_j, P_{-j}) = a$ together imply $c\tilde{R}_ia$, which along with Claim B.7, yields $c\tilde{P}_ia$. Also, the fact $f_i(P_N) = a$, together with (B.34), implies $f_i(\tilde{P}_i, P^{11}_j, P_{-i,j}) = a$. Since $c\tilde{P}_ia$ and $f_i(\tilde{P}_i, P^{11}_j, P_{-i,j}) = a$, by strategy-proofness, we have $f_i(P^{11}_i, P^{11}_j, P_{-i,j}) = a$. Moreover, since $f_i(\tilde{P}_i, P^{11}_j, P_{-i,j}) = a$ and $f_i(P^{11}_i, P^{11}_j, P_{-i,j}) = a$, by non-bossiness, we have $f(P^{11}_i, P^{11}_j, P_{-i,j}) = f_i(\tilde{P}_i, P^{11}_j, P_{-i,j})$. This, together with (B.34), implies

$$f_i(P^{11}_i, P^{11}_j, P_{-i,j}) = a, f_j(P^{11}_i, P^{11}_j, P_{-i,j}) = b, \text{ and } f_k(P^{11}_i, P^{11}_j, P_{-i,j}) = c. \tag{B.35}$$

Using strategy-proofness and non-bossiness, we obtain from (B.35) that

$$f_i(P^{11}_N) = a, f_j(P^{11}_N) = b, \text{ and } f_k(P^{11}_N) = c. \tag{B.36}$$

Again, using strategy-proofness and non-bossiness, we obtain from (B.36) that

$$f_i(P^9_N) = a, f_j(P^9_N) = b, \text{ and } f_k(P^9_N) = c. \tag{B.37}$$

Since f is strategy-proof, the fact $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = a$ and (B.33) imply $f_j(\tilde{P}_i, P^{10}_j, P_{-i,j}) = a$. Since $f_j(\tilde{P}_i, \tilde{P}_j, P_{-i,j}) = a$ and $f_j(\tilde{P}_i, P^{10}_j, P_{-i,j}) = a$, by non-bossiness, we have $f(\tilde{P}_i, P^{10}_j, P_{-i,j}) = f(\tilde{P}_i, \tilde{P}_j, P_{-i,j})$. This, in particular, means

$$f_i(\tilde{P}_i, P_j^{10}, P_{-i,j}) = c, f_j(\tilde{P}_i, P_j^{10}, P_{-i,j}) = a, \text{ and } f_k(\tilde{P}_i, P_j^{10}, P_{-i,j}) = d. \tag{B.38}$$

From Claim B.8, we have cP_kd. Since f is strategy-proof and cP_kd, (B.38) implies $f_k(\tilde{P}_i, P_j^{10}, P_k^{10}, P_{-i,j,k}) = d$. Moreover, since $f_k(\tilde{P}_i, P_j^{10}, P_{-i,j}) = d$ and $f_k(\tilde{P}_i, P_j^{10}, P_k^{10}, P_{-i,j,k}) = d$, by non-bossiness, (B.38) implies

$$f_i(\tilde{P}_i, P_j^{10}, P_k^{10}, P_{-i,j,k}) = c, f_j(\tilde{P}_i, P_j^{10}, P_k^{10}, P_{-i,j,k}) = a, \text{ and } f_k(\tilde{P}_i, P_j^{10}, P_k^{10}, P_{-i,j,k}) = d. \tag{B.39}$$

Using strategy-proofness and non-bossiness, we obtain from (B.39) that

$$f_i(P_N^{10}) = c, f_j(P_N^{10}) = a, \text{ and } f_k(P_N^{10}) = d. \tag{B.40}$$

By strategy-proofness, (B.37) implies $f_j(P_N^8) \in \{a, b\}$. Suppose $f_j(P_N^8) = b$. Since $f_j(P_N^8) = b$ and $f_j(P_N^9) = b$, by non-bossiness, (B.37) implies $f_k(P_N^8) = c$. However, since $f_k(P_N^8) = c$, (B.40) contradicts strategy-proofness. So, it must be that $f_j(P_N^8) = a$. By strategy-proofness, (B.32) implies $f_i(P_N^8) \in \{a, c\}$. This, along with the fact that $f_j(P_N^8) = a$, yields

$$f_i(P_N^8) = c \text{ and } f_j(P_N^8) = a. \tag{B.41}$$

Using strategy-proofness and non-bossiness, we obtain from (B.41) that

$$f_i(P_N^{14}) = c \text{ and } f_j(P_N^{14}) = a. \tag{B.42}$$

By strategy-proofness, (B.42) implies $f_j(P_N^{15}) \in \{a, b\}$. Suppose $f_j(P_N^{15}) = a$. Since $f_j(P_N^{14}) = a$ and $f_j(P_N^{15}) = a$, by non-bossiness and (B.42), we have $f_i(P_N^{15}) = c$. However, since $f_i(P_N^{15}) = c$, (B.37) contradicts strategy-proofness. So, it must be that $f_j(P_N^{15}) = b$. By strategy-proofness, (B.37) implies $f_i(P_N^{15}) \in \{a, b\}$. This, along with the fact that $f_j(P_N^{15}) = b$, yields $f_i(P_N^{15}) = a$. By non-bossiness, this and (B.37) imply

$$f_i(P_N^{15}) = a, f_j(P_N^{15}) = b, \text{ and } f_k(P_N^{15}) = c. \tag{B.43}$$

Using strategy-proofness and non-bossiness, we obtain from (B.43) that

$$f_i(P_N^7) = a, f_j(P_N^7) = b, \text{ and } f_k(P_N^7) = c. \tag{B.44}$$

By (B.42) we have $f_k(P_N^{14}) \notin \{a, c\}$. By strategy-proofness, the fact $f_k(P_N^{14}) \notin \{a, c\}$ implies $f_k(P_N^{12}) =$

$f_k(P_N^{14})$. This, by non-bossiness and (B.42), implies

$$f_i(P_N^{12}) = c \text{ and } f_j(P_N^{12}) = a. \tag{B.45}$$

By strategy-proofness, (B.45) implies $f_i(P_N^3) \in \{a, c\}$. Suppose $f_i(P_N^3) = c$. Since $f_i(P_N^{12}) = c$ and $f_i(P_N^3) = c$, by non-bossiness and (B.45), we have $f_j(P_N^3) = a$. However, $f_i(P_N^3) = c$ and $f_j(P_N^3) = a$ together contradict Pareto efficiency. So, it must be that $f_i(P_N^3) = a$. By strategy-proofness, (B.31) implies $f_k(P_N^3) \in \{a, c\}$. This, along with the fact that $f_i(P_N^3) = a$, yields

$$f_i(P_N^3) = a \text{ and } f_k(P_N^3) = c. \tag{B.46}$$

Using strategy-proofness and non-bossiness, we obtain from (B.46) that

$$f_i(P_N^1) = a \text{ and } f_k(P_N^1) = c. \tag{B.47}$$

By (B.47) we have $f_j(P_N^1) \notin \{a, c\}$. By strategy-proofness, $f_j(P_N^1) \notin \{a, c\}$ implies $f_j(P_N^2) = f_j(P_N^1)$. This, by non-bossiness and (B.47), implies

$$f_i(P_N^2) = a \text{ and } f_k(P_N^2) = c. \tag{B.48}$$

By (B.43) we have $f_i(P_N^{15}) = a$ and $f_k(P_N^{15}) = c$. By strategy-proofness, $f_k(P_N^{15}) = c$ implies $f_k(P_N^{13}) \in \{a, c\}$. Suppose $f_k(P_N^{13}) = c$. Since $f_k(P_N^{15}) = c$ and $f_k(P_N^{13}) = c$, by non-bossiness and the fact that $f_i(P_N^{15}) = a$, we have $f_i(P_N^{13}) = a$. However, $f_i(P_N^{13}) = a$ and $f_k(P_N^{13}) = c$ together contradict Pareto efficiency. So, it must be that $f_k(P_N^{13}) = a$. By strategy-proofness, (B.45) implies $f_j(P_N^{13}) \in \{a, b\}$. This, along with the fact that $f_k(P_N^{13}) = a$, yields $f_j(P_N^{13}) = b$. By strategy-proofness, (B.48) implies $f_i(P_N^{13}) \in \{a, b, c\}$. This, together with the facts that $f_j(P_N^{13}) = b$ and $f_k(P_N^{13}) = a$, implies

$$f_i(P_N^{13}) = c, f_j(P_N^{13}) = b, \text{ and } f_k(P_N^{13}) = a. \tag{B.49}$$

By strategy-proofness, (B.44) implies $f_k(P_N^6) \in \{a, c\}$. Suppose $f_k(P_N^6) = c$. Since $f_k(P_N^7) = c$ and $f_k(P_N^6) = c$, by non-bossiness and (B.44), we have $f_i(P_N^6) = a$. However, $f_i(P_N^6) = a$ and $f_k(P_N^6) = c$ together contradict Pareto efficiency. So, it must be that $f_k(P_N^6) = a$. Also, by (B.49) we have $f_i(P_N^{13}) = c$ and $f_j(P_N^{13}) = b$. By strategy-proofness, $f_i(P_N^{13}) = c$ implies $f_i(P_N^6) \in \{b, c\}$. Suppose $f_i(P_N^6) = c$. Since $f_i(P_N^{13}) = c$ and $f_i(P_N^6) = c$, by non-bossiness and the fact that $f_j(P_N^{13}) = b$, we have $f_j(P_N^6) = b$. However, $f_i(P_N^6) = c$ and $f_j(P_N^6) = b$ together contradict Pareto efficiency. So, it must be that $f_i(P_N^6) = b$. Combining the facts that $f_i(P_N^6) = b$ and $f_k(P_N^6) = a$, we have

$$f_i(P_N^6) = b \text{ and } f_k(P_N^6) = a. \tag{B.50}$$

Now we complete the proof of Lemma B.2. Consider the restricted domain $\tilde{\mathcal{P}}_N \subseteq \mathbb{L}^n(A)$ with only three preference profiles as follows.

Preference profiles	Individual i	Individual j	Individual k	...	Individual l
P_N^6	$bca\hat{P}$	$cba\hat{P}$	$acb\hat{P}$	...	P_l
P_N^7	$bca\hat{P}$	$cba\hat{P}$	$cab\hat{P}$	...	P_l
P_N^{14}	$cba\hat{P}$	$cab\hat{P}$	$cab\hat{P}$	...	P_l

Table B.8: Preference profiles of $\tilde{\mathcal{P}}_N$

By (B.42), (B.44), and (B.50), we have

Preference profiles	$f_i(P_N)$	$f_j(P_N)$	$f_k(P_N)$
P_N^6	b		a
P_N^7	a	b	c
P_N^{14}	c	a	

Table B.9: Partial outcome of f on $\tilde{\mathcal{P}}_N$

Since f is OSP-implementable on $\mathbb{L}^n(A)$, it must be OSP-implementable on the restricted domain $\tilde{\mathcal{P}}_N$. Let $\tilde{G}$ be an OSP mechanism that implements f on $\tilde{\mathcal{P}}_N$.

Note that since $f(P_N^6) \neq f(P_N^7)$, there exists a node in the OSP mechanism $\tilde{G}$ that has at least two edges. Also, note that since each individual $l \in N \setminus \{i,j,k\}$ has exactly one preference in $\tilde{\mathcal{P}}_l$, whenever there are at least two outgoing edges from a node, that node must be assigned to some individual in $\{i,j,k\}$. Consider the first node (from the root) v that has two edges.

Suppose $\eta^{NI}(v) = i$. Consider the preference profiles P_N^7 and P_N^{14}. Note that both of them pass through the node v at which P_i^7 and P_i^{14} diverge. Further note that cP_i^7a, $f_i(P_N^7) = a$, and $f_i(P_N^{14}) = c$. However, the facts that cP_i^7a, $f_i(P_N^7) = a$, and $f_i(P_N^{14}) = c$ together contradict OSP-implementability of f on $\tilde{\mathcal{P}}_N$. So, it must be that $\eta^{NI}(v) \neq i$.

Suppose $\eta^{NI}(v) = k$. Consider the preference profiles P_N^6 and P_N^{14}. Note that both of them pass through the node v at which P_k^6 and P_k^{14} diverge. Further note that $f_k(P_N^6) = a$, $f_k(P_N^{14}) \notin \{a,c\}$, and $aP_k^{14}x$ for all $x \in A \setminus \{a,c\}$. Since $aP_k^{14}x$ for all $x \in A \setminus \{a,c\}$, the facts that $f_k(P_N^6) = a$ and $f_k(P_N^{14}) \notin \{a,c\}$ together contradict OSP-implementability of f on $\tilde{\mathcal{P}}_N$. So, it must be that $\eta^{NI}(v) \neq k$.

Since $\eta^{NI}(v) \neq i$ and $\eta^{NI}(v) \neq k$, it must be that $\eta^{NI}(v) = j$. We distinguish the following two cases.

CASE 1: $f_j(P^6_N) = c$.

Consider the preference profiles P^6_N and P^{14}_N. Note that both of them pass through the node v at which P^6_j and P^{14}_j diverge. Further note that $cP^{14}_j a$, $f_j(P^6_N) = c$, and $f_j(P^{14}_N) = a$. However, the facts that $cP^{14}_j a$, $f_j(P^6_N) = c$, and $f_j(P^{14}_N) = a$ together contradict OSP-implementability of f on $\tilde{\mathcal{P}}_N$.

CASE 2: $f_j(P^6_N) \neq c$.

Consider the preference profiles P^6_N and P^{14}_N. Note that both of them pass through the node v at which P^6_j and P^{14}_j diverge. Further note that $f_j(P^6_N) \notin \{a,b,c\}$, $f_j(P^{14}_N) = a$, and $aP^6_j x$ for all $x \in A \setminus \{a,b,c\}$. Since $aP^6_j x$ for all $x \in A \setminus \{a,b,c\}$, the facts that $f_j(P^6_N) \notin \{a,b,c\}$ and $f_j(P^{14}_N) = a$ together contradict OSP-implementability of f on $\tilde{\mathcal{P}}_N$. This completes the proof of Lemma B.2. ■

B.3 Completion of the proof of Theorem 7.1

We present two results from Pápai (2000), which we use to complete the proof of Theorem 7.1.

Theorem B.1 (Main theorem in Pápai (2000))**.** *An assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is group strategy-proof, Pareto efficient, and reallocation-proof if and only if f is a hierarchical exchange rule.*

Lemma B.3 (Lemma 1 in Pápai (2000))**.** *An assignment rule $f : \mathbb{L}^n(A) \to \mathcal{M}$ is group strategy-proof if and only if it is strategy-proof and non-bossy.*

Proof of Theorem 7.1. (If part) Let f be a hierarchical exchange rule satisfying dual ownership. By Lemma B.1, f is OSP-implementable. Moreover, since f is a hierarchical exchange rule, by Theorem B.1, f is group strategy-proof and Pareto efficient. The fact that f is group strategy-proof along with Lemma B.3, implies f is non-bossy. This completes the proof of the "if" part of Theorem 7.1.

(Only-if part) Let f be an OSP-implementable, non-bossy, and Pareto efficient assignment rule. By Lemma B.2, f is reallocation-proof. Since f is OSP-implementable, by Remark 4.1, f is strategy-proof. This, together with Lemma B.3 and the fact that f is non-bossy, implies f is group strategy-proof. Since f is group strategy-proof, Pareto efficient, and reallocation-proof, by Theorem B.1, f is a hierarchical exchange rule. Moreover, since f is an OSP-implementable hierarchical exchange rule, by Lemma B.1, f is a hierarchical exchange rule satisfying dual ownership. This completes the proof of the "only-if" part of Theorem 7.1. ■

Appendix C Proof of Proposition 10.1

Proof of Proposition 10.1. (If part) Suppose Γ is acyclic. Assume for contradiction that $\widehat{\succ}_A$ contains a priority cycle, say $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$. By the definition of priority cycle, this means one of the following two statements must hold.

(1) $\tau(\widehat{\succ}_{x_h}) = i_h$ for all $h = 1, 2, 3$.

(2) There exist distinct individuals $i_4, \dots, i_t \in N \setminus \{i_1, i_2, i_3\}$ and distinct objects $x_4, \dots, x_t \in A \setminus \{x_1, x_2, x_3\}$ such that for all $h = 1, \dots, t$, we have $U(i_h, \succ_{x_h}) \subseteq \{i_4, \dots, i_t\}$.

We distinguish the following two cases.

CASE 1: Suppose (1) holds.

Since $\widehat{\succsim}_A$ is induced by Γ, the fact that $\tau(\widehat{\succsim}_{x_h}) = i_h$ for all $h = 1, 2, 3$ implies i_1, i_2, and i_3 are assigned to the root-nodes of Γ_{x_1}, Γ_{x_2}, and Γ_{x_3}, respectively. So, it must be that $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$ is an inheritance cycle at Γ, a contradiction to the fact Γ is acyclic.

CASE 2: Suppose (2) holds.

Consider the restriction of the priority structure $\widehat{\succsim}_A$ to the submarket consisting of $\{i_4, \dots, i_t\}$ and $\{x_4, \dots, x_t\}$. Let $\hat{\mu} \in \mathcal{M}(\{i_4, \dots, i_t\}, \{x_4, \dots, x_t\})$ be the allocation such that $\hat{\mu}(i_h) = x_h$ for all $h = 4, \dots, t$. Let $\hat{\hat{\mu}} \in \mathcal{M}(\{i_4, \dots, i_t\}, \{x_4, \dots, x_t\})$ be the outcome of TTC procedure (note that here the roles of individuals and objects are interchanged) at this restricted priority structure (in the submarket consisting of $\{i_4, \dots, i_t\}$ and $\{x_4, \dots, x_t\}$) with respect to the initial endowments corresponding to $\hat{\mu}$. Since $U(i_h, \succ_{x_h}) \subseteq \{i_4, \dots, i_t\}$ for all $h = 4, \dots, t$, we have (for details see Roth and Postlewaite (1977))

$$\left[\begin{array}{l} \text{(a) for all } h = 4, \dots, t, \text{ we have } \hat{\hat{\mu}}(x_h) \widehat{\succsim}_{x_h} i_h, \text{ and} \\ \text{(b) for all distinct objects } x_i, x_j \in \{x_4, \dots, x_t\}, i_i \mathrel{\widehat{\succ}_{x_j}} \hat{\hat{\mu}}(x_j) \text{ implies } \hat{\hat{\mu}}(x_i) \mathrel{\widehat{\succ}_{x_i}} i_j. \end{array}\right] \tag{C.1}$$

Since Γ satisfies the priority property, $\widehat{\succsim}_A$ is induced by Γ, and $U(i_h, \succ_{x_h}) \subseteq \{i_4, \dots, i_t\}$ for all $h = 1, \dots, t$, (C.1) implies that there exists a collection of root-paths $(\pi_{x_h})_{h \in \{1, \dots, t\}}$ satisfying the following properties.

(i) (a) For all $h = 1, 2, 3$, the last element of the root-path π_{x_h} is individual i_h and the other individuals in π_{x_h} are from the set $\{i_4, \dots, i_t\}$. For all $h = 4, \dots, t$, the last element of the root-path π_{x_h} is individual $\hat{\hat{\mu}}(x_h)$ and the other individuals in π_{x_h} are from the set $\{i_4, \dots, i_t\}$.

(b) For all $h = 4, \dots, t$, if there is an (outgoing) edge from individual i_h in any of the root-paths in $(\pi_{x_h})_{h \in \{1, \dots, t\}}$, then object $\hat{\mu}(i_h)$ is assigned to that edge.

(ii) For all distinct objects $x_i, x_j \in \{x_4, \dots, x_t\}$, if individual $\hat{\mu}(x_i)$ lies in the interior of the root-path π_{x_j}, then individual $\hat{\mu}(x_j)$ does not lie in the root-path π_{x_i}.

Construct $\{j_1, \dots, j_t\} \subseteq N$ as follows. For all $h = 1, 2, 3$, let $j_h = i_h$. For all $h = 4, \dots, t$, let $j_h = \hat{\hat{\mu}}(x_h)$. Since $\hat{\hat{\mu}} \in \mathcal{M}(\{i_4, \dots, i_t\}, \{x_4, \dots, x_t\})$, by the construction of $\{j_1, \dots, j_t\}$, it follows that $\{j_4, \dots, j_t\} = \{i_4, \dots, i_t\}$. This, together with the fact that $\hat{\mu} \in \mathcal{M}(\{i_4, \dots, i_t\}, \{x_4, \dots, x_t\})$, implies $\hat{\mu} \in \mathcal{M}(\{j_4, \dots, j_t\}, \{x_4, \dots, x_t\})$. Since $\{j_4, \dots, j_t\} = \{i_4, \dots, i_t\}$ and $\hat{\mu} \in \mathcal{M}(\{j_4, \dots, j_t\}, \{x_4, \dots, x_t\})$, by the construction of $\{j_1, \dots, j_t\}$, along with the properties of the collection of root-paths $(\pi_{x_h})_{h \in \{1, \dots, t\}}$, we obtain the following facts.

(i) (a) For all $h = 1, \dots, t$, the last element of the root-path π_{x_h} is individual j_h and the other individuals in π_{x_h} are from the set $\{j_4, \dots, j_t\}$.

(b) For all $h = 4, \dots, t$, if there is an (outgoing) edge from individual j_h in any of the root-paths in $(\pi_{x_h})_{h\in\{1,\dots,t\}}$, then object $\hat{\mu}(j_h)$ is assigned to that edge.

(ii) For all distinct objects $x_i, x_j \in \{x_4, \dots, x_t\}$, if individual $\hat{\mu}(x_i)$ lies in the interior of the root-path π_{x_j}, then individual $\hat{\mu}(x_j)$ does not lie in the root-path π_{x_i}.

These facts imply that $[(j_1, j_2, j_3), (x_1, x_2, x_3)]$ is an inheritance cycle at Γ, a contradiction to the fact that Γ is acyclic. This completes the proof of the "if" part of Proposition 10.1.

(Only-if part) Suppose $\widehat{\succ}_A$ is acyclic. Assume for contradiction that Γ contains an inheritance cycle, say $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$. By the definition of inheritance cycle, this means one of the following two statements must hold.

(1) Individuals i_1, i_2, and i_3 are assigned to the root-nodes of Γ_{x_1}, Γ_{x_2}, and Γ_{x_3}, respectively.

(2) There exist distinct individuals $i_4, \dots, i_t \in N \setminus \{i_1, i_2, i_3\}$, distinct objects $x_4, \dots, x_t \in A \setminus \{x_1, x_2, x_3\}$, an allocation $\hat{\mu} \in \mathcal{M}(\{i_4, \dots, i_t\}, \{x_4, \dots, x_t\})$, and a collection of root-paths $(\pi_{x_h})_{h\in\{1,\dots,t\}}$ with the properties that

(i) (a) for all $h = 1, \dots, t$, the last element of the root-path π_{x_h} is individual i_h and the other individuals in π_{x_h} are from the set $\{i_4, \dots, i_t\}$,

(b) for all $h = 4, \dots, t$, if there is an (outgoing) edge from individual i_h in any of the root-paths in $(\pi_{x_h})_{h\in\{1,\dots,t\}}$, then object $\hat{\mu}(i_h)$ is assigned to that edge, and

(ii) for all distinct objects $x_i, x_j \in \{x_4, \dots, x_t\}$, if individual $\hat{\mu}(x_i)$ lies in the interior of the root-path π_{x_j}, then individual $\hat{\mu}(x_j)$ does not lie in the root-path π_{x_i}.

We distinguish the following two cases.

CASE 1: Suppose (1) holds.

Since $\widehat{\succ}_A$ is induced by Γ, the fact that individual i_h is assigned to the root-node of Γ_{x_h} for all $h = 1, 2, 3$ implies $\tau(\widehat{\succ}_{x_h}) = i_h$ for all $h = 1, 2, 3$. This implies that $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$ is a priority cycle at $\widehat{\succ}_A$, a contradiction to the fact that $\widehat{\succ}_A$ is acyclic.

CASE 2: Suppose (2) holds.

Since Γ satisfies the priority property and $\widehat{\succ}_A$ is induced by Γ, by the construction of $\widehat{\succ}_A$ and the assumption of Case 2, we have $U(i_h, \succ_{x_h}) \subseteq \{i_4, \dots, i_t\}$ for all $h = 1, \dots, t$. This implies that $[(i_1, i_2, i_3), (x_1, x_2, x_3)]$ is a priority cycle at $\widehat{\succ}_A$, a contradiction to the fact that $\widehat{\succ}_A$ is acyclic. This completes the proof of the "only-if" part of Proposition 10.1. ■

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