From Default Distribution to Loss Distribution: Vasicek Mertonization

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Abstract

The Vasicek-Merton (VM) loss distribution function was derived using the Vasicek and the Merton models as an alternative to the AIRB approach. A loan was modeled as a portfolio of a risk-free bond, and a weighted combination of short European vanilla and binary put options written on the assets of the firm, with the strike equal to its debt and expiration equal to maturity of the loan. An endogenous Loss Given Default (LGD) was derived on the base of the Vasicek-Merton CDF.

Introduction

Risk diversification is one of the main methods of risk management. But asset correlation reduces a diversification efficiency. The Vasicek model [12] is applied to account for the correlation. It describes distribution of defaults in a large homogeneous portfolio. Risk-managers, however, need to estimate distribution of losses which takes into account the recovery rate. To achieve this goal the AIRB approach uses an eclectic mix of the ab initio Vasicek loss distribution and the empiric LGD (both used for maturity $T = 1$), and the maturity adjustment based on some econometric estimation. Drawbacks of this mix approach are evident: the capital negativity near default probability $PD = 0$ for maturity $T > 1$ and the capital discontinuity in the neighborhood of zero. The maturity adjustment is a kind of a black box – there is no clear information about the econometric model and calibration of its parameters.

To amend the drawbacks of the AIRB approach we formulate the Vasicek-Merton model for estimation distribution of losses, which is extension of the Vasicek model on the base of the Merton approach.

Technically the passage from the Vasicek model to the Vasicek-Merton model is equivalent to the transition from the binary put European options portfolio to the portfolio with the mix of vanilla and binary put options portfolio. Indeed, the Vasicek loss distribution is equivalent to the distribution of payouts of the portfolio of binary put options, while the Vasicek-Merton loss distribution is equivalent to the distribution of payouts of the portfolio of both binary and vanilla put options.

The Vasicek approach is applied to the firms characterized by the same probability of default. In turn, the Vasicek-Merton approach requires not only the same probability of default, but additionally the same volatility of assets value. The AIRB approach does not account for the volatility of assets values, though the changes in volatility affect the loss distribution.

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We suggest to use in banking risk management the Vasicek-Merton loss distribution formula, totally based on the ab initio approach, instead of the eclectic mix of the AIRB loss distribution formula. In the present paper we study the properties of the Vasicek-Merton loss distribution and the LGD, derived on the base of the distribution.

The text of the paper is organized as follows. In Section 1 we present the revised derivation of Vasicek Loss Distribution function, followed by the more general approach to estimation of a loss, based on the technique of the European put option instead of traditional approach, based on binary put options. This more general class of the loss distribution functions includes the vanilla Vasicek distribution as a special case. The second Section is devoted to the technical results on the PDF of Vasicek-Merton loss distribution, describing the shapes of a density plot depending on parameters values with comparison to the corresponding properties of Vasicek distribution. It is shown that the general picture is more complicates than Bell-shape/U-shape dichotomy in case of Vasicek distribution. In Section 3 we derive the explicit formulas for Expected Loss, Loss Variation, Loss Given Default and Unexpected Loss (Capital Reserves) in comparison to the corresponding concepts, suggested by Vasicek and AIRB approaches. The final Section with Concluding Remarks summarizes the obtained results.

Literature Review

On the base of the Black-Scholes model Robert Merton proposed in [9] the first structural credit risk model for assessing the default probability of the firm and valuation of the debt. Merton modeled the firm's equity as an European vanilla call option on its assets. Oldrich Vasicek [12] created the model of assessing risk of loan portfolio on the base of the Merton model. The MtM credit risk model KMV Portfolio ManagerTM was constructed on the base of the Vasicek approach. This commercial model was used in the AIRB approach [3].

The positive link between PD and LGD is well-documented, see the detailed survey [2] of Altman et al. The theoretic explanation of this effect for Merton-like models was presented in [1]. This conclusion was based on the formula of conditional mean for a log-normal distributed variable, derived by Liu et al. [8]. The recovery rate valuation using this formula needs knowledge of the unobservable firm's asset value volatility. The asset value volatility was evaluated on the base of the equity volatility, see [9, p.451, (3b)].

1 Vasicek-Merton Loss Distribution

In this section we derive the Vasicek-Merton loss distribution function, which accounts for the crucial features that were dropped down by the AIRB approach. In particular, we assume that given default the terminal assets of the firm are sold at a discount $1 - w$, where $0 \le w \le 1$. In case of $w = 0$ we get exactly the Vasicek approach assuming that given firm's default, the bank gets nothing. The opposite case $w = 1$ is assumed in Merton's structural model [9]. Another feature of the proposed Vasicek-Merton loss distribution is an accurate accounting of the maturity of loans, as well as the volatility of assets.

Consider a portfolio consisting of n loans with face values 1 and maturity T, assuming that n is sufficiently large. The payout of the i-th firm is equal to the payout of a portfolio consisting of a riskless zero-coupon bond with the face value 1 and maturity T, and a weighted combination of a short binary European put option and short vanilla European put option with the underlying variable $V_i(t)/D_i$, the strike 1 and the expiration T:

$$\text{Payoff}_{VM} = \begin{cases} 1 & V_i(T) \ge D_i \\ wV_i(T)/D_i & \text{otherwise} \end{cases} = 1 - \left[(1-w)\mathbb{I}_{\{V_i(T)/D_i<1\}} + w\left(1 - V_i(T)/D_i\right)^+\right]$$

where V_i is value of assets, D_i is debt of the i-th firm. Let

$$\text{Loss}_V = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{\{V_i(T)/D_i<1\}},$$

$$\text{Loss}_M = \frac{1}{n}\sum_{i=1}^{n} (1 - V_i(T)/D_i)^+,$$

be, respectively, the Vasicek loss assuming that given firm's default, the creditors get nothing, and the Merton loss assuming that creditors get the terminal firm's assets $V_i(T)$. Then the loss of the combined portfolio

$$\text{Loss}_{VM} = (1-w)\text{Loss}_V + w\text{Loss}_M$$

is a weighted sum of these two extreme types of losses.

The way to derive the distribution function for weighted Vasicek-Merton loss is quite similar the the well-known Vasicek approach. Using the above decomposition of the portfolio loss to weighted sum, we assess its components separately (for L_V it is already done by Vasicek) and then combine them back.

Assume that the value of firm's assets $V_i(t)$ obeys a geometric Brownian motion law with the trend μ_i and assets volatility σ_i

$$dV_i(t) = \mu_i V_i(t)dt + \sigma_i V_i(t)dW_i(t)$$

where $W_i(t)$ is a standard Wiener process. Hence

$$V_i(T) = V_i(0)e^{(\mu_i-\sigma_i^2/2)T+\sigma_i\sqrt{T}X_i}, \tag{1}$$

where $X_i \sim N(0,1)$ is a standard normal distributed random variable.

This implies that the default probability of the i-th firm is equal to

$$PD_i = \mathbb{P}(V_i(T) < D_i) = \mathbb{P}(X_i < -d_-(i)) = \Phi(-d_-(i)).$$

where

$$d_-(i) = \frac{\ln\left(e^{\mu_i T}\frac{V_i(0)}{D_i}\right) - \sigma_i^2 T/2}{\sigma_i\sqrt{T}},$$

Following Vasicek [12] we assume that all firms are characterized by the same probability of default, which is equivalent to identity $d_-(i) = d_-$ for all i, which will be referred as *Vasicek homogeneity condition.*

Now assume that the variable X_i is a sum of two non-correlated standard normal shocks: Y_i is an idiosyncratic shock, Z is a systematic shock, where ρ a systematic correlation coefficient

$$X_i = \sqrt{1-\rho}Y_i + \sqrt{\rho}Z$$

Let the value z of systematic shock be a given parameter. Substituting $X_i(z) = \sqrt{1-\rho}Y_i + \sqrt{\rho}z$ into (1), we obtain in similar way, that the the *conditional* probability of default is equal to

$$PD(z) = \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right).$$

Then the conditional losses $\text{Loss}_V(z)$ and $\text{Loss}_M(z)$ are equivalent, respectively, to the future values of the binary European put option and to the vanilla European put option.

a) The value of binary option is

$$p_V(z) = PD(z) = \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right).$$

b) The value of vanilla European put option may be calculated using Black-Scholes formula, see [4]. Given systematic shock z, the equation (1) may be rewritten as

$$V_i(T) = U_i(z) \cdot e^{-(\sqrt{1-\rho}\sigma_i\sqrt{T})^2/2 + \sqrt{1-\rho}\sigma_i\sqrt{T}Y_i}$$

where

$$U_i(z) \equiv V_i(0)e^{\mu_i T - \frac{(\sqrt{\rho}\sigma_i\sqrt{T})^2}{2} + \sqrt{\rho}\sigma_i\sqrt{T}z} > 0$$

is considered as spot price of assets, while the conditional asset volatility is equal to $\sqrt{1-\rho}\sigma_i$. This implies that the future value of the put option for i-th firm is as follows

$$\begin{aligned} p_M(z,i) = e^{\mu_i T}\left[e^{-\mu_i T}\Phi(-d_-(z)) - \frac{U_i(z)}{D_i}\Phi\left(-d_+(z,i)\right)\right] =, \\ =\Phi(-d_-(z)) - L_i e^{-\frac{(\sqrt{\rho}\sigma_i\sqrt{T})^2}{2} + \sqrt{\rho}\sigma_i\sqrt{T}z}\Phi\left(-d_+(z,i)\right) \end{aligned}$$

where

$$d_-(z) = \frac{\sqrt{\rho}z - \Phi^{-1}(PD)}{\sqrt{1-\rho}}.$$

$$d_+(z,i) = d_-(z) + \sqrt{1-\rho}\sigma_i\sqrt{T},\ L_i = e^{\mu_i T}\frac{V_i(0)}{D_i}.$$

The first term $\Phi(-d_-(z))$ is common for all firms due to Vasicek homogeneity condition, however, the second term, a subtrahend, is firm-specific.

To provide an equivalence of firms with respect to value of put option written on the firm's asset, we assume the following *Vasicek-Merton homogeneity condition*: for all firms i the leverages $L_i = L$ and the volatilities $\sigma_i = \sigma$ are the same. This immediately implies that

$$p_M(z,i) = p_M(z) = \Phi(-d_-(z)) - Le^{-\frac{(\sqrt{\rho}\sigma\sqrt{T})^2}{2} + \sqrt{\rho}\sigma\sqrt{T}z}\Phi\left(-d_+(z)\right),$$

where

$$d_+(z) = d_-(z) + \sqrt{1-\rho}\sigma\sqrt{T}.$$

Moreover, Vasicek-Merton homogeneity implies the Vasicek homogeneity condition due to

$$d_-(i) = \frac{\ln L - \sigma^2 T/2}{\sigma\sqrt{T}} = d_-.$$

To save space we will use the compound parameters $\sigma_T = \sigma\sqrt{T}$ and $\widetilde{\sigma}_T = \sqrt{1-\rho}\sigma_T$. Also we define the following functions

$$\Psi(x) = \frac{\Phi(x)}{\varphi(x)},\ \mathcal{R}_\alpha(x) = \frac{\Psi(x-\alpha)}{\Psi(x)}.$$

Functions $\Psi(x)$ and $\mathcal{R}_\alpha(x)$ are obviously positive for all $x > 0$ and satisfy the following conditions.

Lemma 1. *Derivative* $\Psi'(x) = 1 + x\Psi(x) > 0$ *for all* $x \in \mathbb{R}$, $\lim_{x\to-\infty}\Psi(x) = 0$, $\lim_{x\to+\infty}\Psi(x) = +\infty$. *For any given* $\alpha > 0$ *the function* $\mathcal{R}_\alpha(x)$ *strictly decreases with respect to both* x *and* α, *moreover,* $\lim_{x\to-\infty}\mathcal{R}_\alpha(x) = 1$, $\lim_{x\to+\infty}\mathcal{R}_\alpha(x) = 0$.

See **Proof** in Appendix.

Given
$$\frac{\ln L - \sigma_T^2/2}{\sigma_T} = d_- = -\Phi^{-1}(PD)$$
we obtain
$$L = e^{-\sigma_T \Phi^{-1}(PD) + \sigma_T^2/2}, \tag{2}$$
which implies
$$p_M(z) = \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right)\left[1 - \mathcal{R}_{\widetilde{\sigma}_T}\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right)\right],$$
while the value of weighted derivative is equal to
$$\begin{aligned} p_{VM}(z) &= (1-w)p_V(z) + wp_{VM}(z) = \\ &= \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right)\left[1 - w\mathcal{R}_{\widetilde{\sigma}_T}\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right)\right] \end{aligned} \tag{3}$$

Given $p_{VM}(z) = ELoss(z)$ – the expected portfolio loss conditional on systematic shock z, and
$$PD(z) = \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right),$$
formula (3) may be rewritten as follows
$$p_{VM}(z) = PD(z) \cdot LGD(PD(z)),$$
where
$$LGD(y) \equiv 1 - w\mathcal{R}_{\widetilde{\sigma}_T}\left(\Phi^{-1}(y)\right). \tag{4}$$
is the Loss Given Default, considered as a function of a default probability. Consequently, the conditional Recovery Rate
$$RR(PD(z)) = w\mathcal{R}_{\widetilde{\sigma}_T}\left(\Phi^{-1}(PD(z))\right) \tag{5}$$

Consider the following function
$$M_{w,\alpha}(y) = \Phi(y) - we^{-\alpha y + \alpha^2/2}\Phi(y-\alpha) = \Phi(y)\left[1 - w\mathcal{R}_\alpha(y)\right]$$
parametrized by $\alpha > 0$ and $0 \le w \le 1$. Lemma 1 immediately implies that $M_{w,\alpha}(y)$ satisfies the following conditions:
$$\lim_{y\to-\infty} M_{w,\alpha}(y) = 0, \ \lim_{y\to+\infty} M_{w,\alpha}(y) = 1,$$
moreover,
$$m_{w,\alpha}(y) \equiv M'_{w,\alpha} = \varphi(y)\left[w\alpha\Psi(y-\alpha) + (1-w)\right] > 0.$$
In other words, function $M_{w,\alpha}(y)$ satisfies the same condition as an ordinary CDF. This implies that there exists inverse function $M^{-1}_{w,\alpha}(x)$ well-defined for all $x \in (0,1)$, while $\lim_{x\to0} M^{-1}_{w,\alpha}(x) = -\infty$, $\lim_{x\to1} M^{-1}_{w,\alpha}(x) = +\infty$. Given that, we can represent the formula (3) as follows
$$p_{VM}(z) = M_{w,\widetilde{\sigma}_T}\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right).$$

Let $x = p_{VM}(z)$, then

$$z = f(x) \equiv \frac{\Phi^{-1}(PD)}{\sqrt{\rho}} - \sqrt{\frac{1-\rho}{\rho}} M^{-1}_{w,\widetilde{\sigma}_T}(x).$$

Thus

$$F_{VM}(x; PD, \rho, w, \sigma_T) = \mathbb{P}(L_{VM} < x) = \mathbb{P}(z > f(x)) = \Phi(-f(x)) =$$
$$= \Phi\left(\frac{\sqrt{1-\rho} M^{-1}_{w,\widetilde{\sigma}_T}(x) - \Phi^{-1}(PD)}{\sqrt{\rho}}\right)$$

is the Vasicek-Merton Loss distribution function.

Comparing the obtained formula with CDF of Vasicek loss distribution

$$F_V(x; PD, \rho) = \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(PD)}{\sqrt{\rho}}\right),$$

one can see that the only difference is a replacing of inverse normal CDF $\Phi^{-1}(x)$ by inverse function $M^{-1}_{w,\widetilde{\sigma}_T}$, which depends not only on probability of default PD and correlation ρ, but also takes into account volatility σ, maturity T and the parameter w.

Remark 1. AIRB approach uses the one-parameter[1] rating of firms based only on their probability of default PD. Our considerations suggest that there is one more rating parameter that have to be taken into account – the volatility of firm's assets σ. Clearly, the one-parametric rating seems more convenient due to linear ordering, it is not a big problem to "linearize" two-parametric rating (PD, σ) using, for example, a lexicographic order or a weighted sum of criteria. The attendant drawback of the Vasicek-Merton distribution function in comparison with the Vasicek one is a necessity to account for the non-observed parameters σ and w in addition to estimation of the traditional parameters – probability of default PD and correlation ρ. The usual way is to calculate the unobservable parameters as implied values on the base of observable parameters, which are combined with unobservable ones to some identities determined by the model linkage of parameters. There are different approaches in the literature exploiting this idea. The first one was first suggested in [6] and then developed in [11] and [10] proposes to consider an equity $E(t) = V(t) - D(t)$. Its value and volatility σ_E may be derived from the market statistics, which allows to calculate the unobservable market value of the firm's assets and its volatility as numerical solution of the system of two non-linear equations, see, for example, [10, Equations 1.3 and 2.1]. Another way to estimate the model parameters based on observable spread of Credit Default Swap (CDS) is developed in [5] and [7].

2 Vasicek-Merton Density Function

In this section the Vasicek-Merton PDF $f_{VM}(x) = F'_{VM}(x)$ will be studied in comparison with the Vasicek density function, which is actually the special case corresponding to the value $w = 0$. We show that the general case $w > 0$ inherits a bell-shape (or, unimodality) in case of $\rho < 1/2$. This not the case, when $\rho > 1/2$. It will be shown that unlike the U-shaped Vasicek functions with infinity values at $x = 0$ and $x = 1$ (see. e.g. [13]), the general Vasicek-Merton density function for $w > 0$ and $\rho > 1/2$ never can be U-shaped, being either strictly decreasing, or "springboard-shaped" with infinite value at $x = 0$ and second local maximum in $(0, 1)$, see Figure 1 further in this section. What of these cases will be realized depends on other parameters.

[1] To be more precise, AIRB uses one more exogenous parameter – a Loss Given Default (LGD), which is not treated as rating an can be inversely correlated to our parameter w, see more detailed discussion in Section 3.

In the vanilla Vasicek case $w = 0$ the value $\rho = 1/2$ is bifurcation point, delimiting unimodal and bimodal shapes. In general case $w > 0$ there is no closed-form description of "bifurcation surface" in parameter space, however, we present simple *sufficient* conditions describing some areas, which guarantee unimodal or bimodal (springboard-like) shape of the graph of density function.

In what follows we assume that $w > 0$ and let w, PD, ρ, $\widetilde{\sigma}_T$ be the given parameters. Then $F_{VM}(x) = \Phi(g(x))$ for

$$g(x) = \frac{\sqrt{1-\rho}M^{-1}_{w,\widetilde{\sigma}_T}(x) - \Phi^{-1}(PD)}{\sqrt{\rho}},$$

thus

$$f_{VM}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\Phi(g(x)) = \sqrt{\frac{1-\rho}{\rho}}\varphi(g(x))\frac{\mathrm{d}}{\mathrm{d}x}M^{-1}_{w,\widetilde{\sigma}_T}(x) = \sqrt{\frac{1-\rho}{\rho}}\varphi(g(x))\frac{1}{m_{w,\widetilde{\sigma}_T}(M^{-1}_{w,\widetilde{\sigma}_T}(x))},$$

where

$$m_{w,\widetilde{\sigma}_T}(y) = M'_{w,\widetilde{\sigma}_T}(y) = \varphi(y)\left[w\alpha\Psi(y-\alpha) + (1-w)\right].$$

Therefore,

$$f_{VM}(x) = \sqrt{\frac{1-\rho}{\rho}}\frac{\varphi\left(\sqrt{\frac{1-\rho}{\rho}}M^{-1}_{w,\alpha}(x) - \frac{\Phi^{-1}(PD)}{\sqrt{\rho}}\right)}{\varphi(M^{-1}_{w,\widetilde{\sigma}_T}(x))\left[w\widetilde{\sigma}_T\Psi(M^{-1}_{w,\widetilde{\sigma}_T}(x) - \widetilde{\sigma}_T) + (1-w)\right]}.$$

Clearly, substituting $w = 0$ we obtain the coincidence of functions $M^{-1}_{0,\widetilde{\sigma}_T}(x) = \Phi^{-1}(x)$ and $f_{VM}(x) = f_V(x)$.

Let $z = M^{-1}_{w,\widetilde{\sigma}_T}(x)$, $a = \sqrt{\frac{1-\rho}{\rho}} > 0$, $b = \frac{\Phi^{-1}(PD)}{\sqrt{\rho}}$, then $x = 0 \Rightarrow z = -\infty$, $x = 1 \Rightarrow z = +\infty$ and

$$\hat{f}(z) = \sqrt{\frac{1-\rho}{\rho}}\frac{\varphi(az-b)}{\varphi(z)\left[w\widetilde{\sigma}_T\Psi(z-\widetilde{\sigma}_T) + (1-w)\right]} = \sqrt{\frac{1-\rho}{\rho}}\frac{e^{-\frac{1}{2}(az-b)^2+\frac{1}{2}z^2}}{w\widetilde{\sigma}_T\Psi(z-\widetilde{\sigma}_T) + (1-w)}. \tag{6}$$

Due to bijectivity of $M^{-1} : (0,1) \to (-\infty, +\infty)$ a behavior of functions $f_{VM}(x)$ and $\hat{f}(z)$ is isomorphic, i.e., intervals of increasing/decreasing for both function are linked by bijection M^{-1}.

Lemma 2. *Let $w > 0$, then probability density function $f_{VM}(x)$ satisfies the following conditions:*

1. $$f_{VM}(0) = \begin{cases} 0 & \rho < 1/2 \\ +\infty & \rho > 1/2 \\ +\infty & \rho = 1/2 \& PD < 1/2 \\ 0 & \rho = 1/2 \& PD > 1/2 \\ +\infty & \rho = 1/2 \& PD = 1/2,\ w = 1 \\ \frac{1}{1-w} & \rho = 1/2 \& PD = 1/2,\ w < 1 \end{cases}$$
2. $f_{VM}(1) = 0$

See **Proof** in Appendix.

This result significantly differs from the Vasicek case $w = 0$ with

$$f_V(1) = \sqrt{\frac{1-\rho}{\rho}}\lim_{z\to+\infty} e^{-\frac{1}{2}(az-b)^2+\frac{1}{2}z^2} = e^{-\frac{b^2}{2}}\sqrt{\frac{1-\rho}{\rho}}\lim_{z\to+\infty} e^{-\frac{1}{2}(a^2-1)z^2+abz}.$$

Given

$$a^2 - 1 = \frac{1-2\rho}{1-\rho}, \; ab = \frac{\sqrt{1-\rho}}{\rho}\Phi^{-1}(PD),$$

we obtain

$$f_V(1) = \sqrt{\frac{1-\rho}{\rho}} \lim_{z\to+\infty} e^{-\frac{1}{2}((a^2-1)z^2-2abz+b^2)},$$

which implies that $f_V(0) = f_V(1) = 0$ for $\rho < 1/2$ and $f_V(x)$ is bell-shaped (unimodal) in interval $(0,1)$), while for $\rho > 1/2$ function $f_V(x)$ is U-shaped (bimodal) with $f_V(0) = f_V(1) = +\infty$.

For the rest of the Section our aim is to reveal the shapes of the function $f_{VM}(x)$ depending on parameters. In the Vasicek framework the threshold value $\rho = 1/2$ is delimiting for the unimodal (bell-shaped) and bimodal (U-shaped) cases of density function $f_V(x)$. In cases of the strict inequalities, $\rho < 1/2$ and $\rho > 1/2$, the probability of default PD does nor matter, though on the "bifurcation fence" $\rho = 1/2$ a value of PD determines specific behavior of $f_V(x)$. Apart from the mutual parameters ρ and PD, the Vasicek-Merton density $f_{VM}(x)$ depends also on additional parameters w and σ_T, thus is will no be surprising that "bifurcation fence", delimiting the unimodal and bimodal cases is more complicated, though the correlation ρ still plays the main role.

Differentiating $\hat{f}(z)$ with respect to z we obtain

$$\hat{f}'(z) = \sqrt{\frac{1-\rho}{\rho}} \frac{e^{-\frac{1}{2}(az-b)^2+\frac{1}{2}z^2}}{\left(w\widetilde{\sigma}_T\Psi(z-\widetilde{\sigma}_T)+(1-w)\right)^2} H(z),$$

where

$$H(z) = w\widetilde{\sigma}_T(\widetilde{\sigma}_T + ab - a^2 z)\Psi(z-\widetilde{\sigma}_T) + (1-w)(ab-(a^2-1)z) - w\widetilde{\sigma}_T,$$

$a = \sqrt{\frac{1-\rho}{\rho}} > 0$, $b = \frac{\Phi^{-1}(PD)}{\sqrt{\rho}}$. Obviously, $\hat{f}'(z) = 0 \iff H(z) = 0$.

Remark 2. In Vasicek case $w = 0$ we obtain

$$H(z) = 0 \iff z = \frac{ab}{a^2-1} = \frac{\sqrt{1-\rho}}{1-2\rho}\Phi^{-1}(PD).$$

On the other hand, $z = M^{-1}_{0,\widetilde{\sigma}_T}(x) = \Phi^{-1}(x)$ in case of $w = 0$, thus we obtain the well-known formula

$$x^* = \Phi\left(\frac{\sqrt{1-\rho}}{1-2\rho}\Phi^{-1}(PD)\right),$$

which provides the maximum of function

$$f_V(x; PD, \rho) = \sqrt{\frac{1-\rho}{\rho}} \exp\left(\frac{1}{2}\left[(\Phi^{-1}(x))^2 - \left(\frac{\sqrt{1-\rho}\Phi^{-1}(x) - \Phi^{-1}(PD)}{\sqrt{\rho}}\right)^2\right]\right).$$

in case of $\rho < 1/2$ (Bell-shaped case), and minimizes $f_V(x)$ in case of $\rho > 1/2$ (U-shaped case).

Direct calculations show that in case $w > 0$ the equation $H(z) = 0$ is equivalent to

$$(y - y_0)\Psi(y) = (y - y_0) \cdot h_0 - \gamma, \tag{7}$$

where $y = z - \widetilde{\sigma}_T$ and

$$
\begin{aligned}
y_0 &= \widetilde{\sigma}_T \frac{1-a^2}{a^2} + \frac{b}{a} = \frac{(2\rho-1)\sigma_T + \Phi^{-1}(PD)}{\sqrt{1-\rho}},\\
h_0 &= \frac{1-w}{w\widetilde{\sigma}_T}\frac{1-a^2}{a^2} = \frac{1-w}{w\sigma_T}\frac{2\rho-1}{(1-\rho)^{3/2}},\\
\gamma &= \frac{1}{a^2} - \frac{1-w}{w\widetilde{\sigma}_T}\frac{1}{a^2}\left[\widetilde{\sigma}_T\frac{1-a^2}{a^2} + \frac{b}{a}\right] = \frac{\rho}{1-\rho}\left[1 - \frac{1-w}{w\sigma_T}\frac{y_0}{\sqrt{1-\rho}}\right] =\\
&= \frac{\rho}{1-\rho}\left[1 - \frac{1-w}{w\sigma_T}\cdot\frac{(2\rho-1)\sigma_T + \Phi^{-1}(PD)}{1-\rho}\right].
\end{aligned}
$$

Depending on values of the basic parameters ρ, PD, σ_T and w the equation parameters y_0, h_0 and γ may take arbitrary values. The following statement determines the number of solutions of equation 7, depending on equation parameters. These conditions, in turn, determine the classifying relation between basic parameters.

Remark 3. First consider the special case $\gamma = 0$, then equation (7) takes on the form $(y - y_0)\cdot(\Psi(y) - h_0) = 0$. Given

$$\lim_{y\to-\infty}\Psi(y) = 0,\ \lim_{y\to\infty}\Psi(y) = +\infty,\ \Psi'(y) > 0,$$

we obtain that $y = y_0$ is solution of (7) for all h_0, while the second solution $y = \Psi^{-1}(h_0)$ may exist if and only if $h_0 < 0$. In what follows we assume that $\gamma \neq 0$, thus equation (7) may be rewritten as follows

$$\Psi(y) = G(y) \equiv h_0 - \frac{\gamma}{y - y_0}.$$

Lemma 3. *Let*

(1) $h_0 = 0$ and $\gamma < 1$ then equation (7) has unique solution

(2) $h_0 = 0$, $y_0 \leq 0$, $\gamma \geq 1$ then equation (7) has no solutions

(3) $h_0 = 0$, $y_0 > 0$, $\gamma = 1$, then equation (7) has unique solution

(4) $h_0 = 0$, $y_0 > 0$ then there exists $\gamma^*(y_0) > 1$, such that the equation (7) has two different solutions for all $1 < \gamma < \gamma^*$ and has no solutions for $\gamma > \gamma^*$

(5) $h_0 < 0$, $y_0 \leq 0$ then equation (7) has unique solution

(6) $h_0 < 0$, $y_0 > 0$, and $\gamma \in (0, 1)$ then equation (7) has unique solution

(7) $h_0 > 0$, $\gamma < 0$, then equation (7) has two different solutions

(8) $h_0 > 0$, $\gamma > 0$, then then there exist $y^*(h_0, \gamma) < y^{**}(h_0, \gamma)$ such that, for all $y_0 \in (y^*, y^{**})$ equation (7) has no solutions, while for $y_0 < y^*$ or $y_0 > y^{**}$ equation (7) has two different solutions.

Proof of Lemma 3 see in Appendix.

Remark 4. It is easy to see that Lemma 4 lacks the case $h_0 < 0$, $y_0 > 0$, $\gamma > 1$. Actually, this case is impossible in terms of original parameters. Indeed,

$$h_0 = \frac{1-w}{w\sigma_T}\frac{2\rho-1}{(1-\rho)^{3/2}} < 0,$$

implies that $w < 1$ and $\rho < 1/2$, while $y_0 > 0$ implies that

$$\gamma = \frac{\rho}{1-\rho}\left[1 - \frac{1-w}{w\sigma_T}\frac{y_0}{\sqrt{1-\rho}}\right] < \frac{\rho}{1-\rho} < 1.$$

Conclusions from the Lemma 3

Lemma 3 is purely algebraic result describing a structure of the solution set of some equation without any connections to original problem. In this subsection we discuss these conclusions in terms of the basic parameters. First consider the special case $w = 1$, corresponding to a zero bankruptcy cost. This implies $h_0 = 0$, $\gamma = \frac{\rho}{1-\rho} > 0$, while the sign of

$$y_0 = \frac{(2\rho - 1)\sigma_T + \Phi^{-1}(PD)}{\sqrt{1-\rho}}$$

depends on ρ, PD and σ_T, to be more precise,

$$y_0 \leq 0 \iff PD \leq \Phi((1-2\rho)\sigma_T).$$

Let

$$\gamma^* = \sup_{y<y_0}(y_0 - y)\Psi(y),$$

then $\gamma^* = 1$ in case of $y_0 \leq 0$, otherwise, $\gamma^* > 1$ and depends on y_0, i.e., on ρ, PD and σ_T, see proof of Lemma 3 (4).

Proposition 1. *Let $w = 1$, then $\rho < 1/2$ implies that $f_{VM}(x)$ is unimodal, in case of $\rho > 1/2$ function $f_{VM}(x)$ is strictly decreasing if and only if $\frac{\rho}{1-\rho} \geq \gamma^*$, otherwise, it has additional local maximum in interval $(0, 1)$. In case of $\rho = 1/2$ function $f_{VM}(x)$ is unimodal if and only if $PD > 1/2$, otherwise, it is strictly decreasing.*

Proof. Let $\rho < 1/2$, then $f_{VM}(0) = f_{VM}(1) = 0$ due to Lemma 2, moreover, $\rho < 1/2 \Rightarrow \gamma = \frac{\rho}{1-\rho} < 1$, therefore, equation (7) has unique solution due to Lemma 3(1), which is equivalent to existence of unique $x^* \in (0, 1)$, such that $f'_{VM}(x^*) = 0$. Given $f_{VM}(x) > 0$ in $(0, 1)$, this means that x^* is unique maximum of $f_{VM}(x)$.

Let $\rho > 1/2$, then $f_{VM}(0) = +\infty$, $f_{VM}(1) = 0$ due to Lemma 2 and in case of

$$\gamma = \frac{\rho}{1-\rho} > \gamma^*$$

Lemma 3(4) implies that $f'_{VM}(x) \neq 0$ in $(0, 1)$, or, equivalently, $f_{VM}(x)$ decreases in $(0, 1)$. In the opposite case $\frac{\rho}{1-\rho} < \gamma^*$ Lemma 3 (4) implies that function $f_{VM}(x) = 0$ twice in $(0, 1)$, which corresponds to one local minimum and one local maximum, i.e., $f_{VM}(x)$ is bimodal function. Finally, $\rho = 1/2$ implies $\gamma = 1$ and $y_0 > 0 \iff PD > 1/2$. The rest statements of Proposition follow from Lemma 2 and Lemma 3 (2) and (3). □

In what follows we assume $w < 1$. Then $h_0 < 0 \iff \rho < 1/2$, $h_0 = 0 \iff \rho = 1/2$, $h_0 > 0 \iff \rho > 1/2$, while the signs and values of y_0 and γ may vary.

Proposition 2. *Let $w < 1$, $\rho < 1/2$, then function $f_{VM}(x)$ has unique maximum in $(0, 1)$.*

Proof. Assumption $\rho < 1/2$ implies $h_0 < 0$. Consider two possible cases.
a) Let $PD \leq \Phi((1-2\rho)\sigma_T) \iff y_0 \leq 0$, then Lemma 3(5) implies that equation (7) has unique solution.
b) Let $PD > \Phi((1-2\rho)\sigma_T) \iff y_0 > 0$, then

$$\gamma = \frac{\rho}{1-\rho}\left[1 - \frac{1-w}{w\sigma_T}\frac{y_0}{\sqrt{1-\rho}}\right] < \frac{\rho}{1-\rho} < 1.$$

and Lemma 3(6) implies that equation (7) has unique solution. By definition, this means that there exists unique $x^* \in (0, 1)$ satisfying $f'_{VM}(x^*) = 0$. Given $f_{VM}(0) = f_{VM}(1) = 0$ for $\rho < 1/2$ and $f_{VM}(x) > 0$ in $(0, 1)$, we obtain that x^* is unique maximum of $f_{VM}(x)$, in other words, function f_{VM} is bell-shaped. □

Proposition 3. *Let $w < 1$, $\rho = 1/2$, then the function $f_{VM}(x)$ has unique maximum in $(0,1)$ is and only if $PD > 1/2$, otherwise, $f_{VM}(x)$ strictly decreases on interval $(0,1)$.*

Proof. Assumptions $\rho = 1/2$ and $PD \le 1/2$ imply $h_0 = 0$, $y_0 \le 0$, $\gamma \ge 1$, thus equation 7 has no solutions due to Lemma 3 (2). Now assume that $\rho = 1/2$, and $PD > 1/2$, then $h_0 = 0$, $y_0 > 0$ and equation 7 has unique solution due to Lemma 3(1) in case of $\gamma < 1$ and Lemma 3(3) for $\gamma = 1$. Given $f_{VM}(0) > f_{VM}(1)$, and $f'_{VM}(x) \neq 0$ for $\rho = 1/2$, $PD \le 1/2$ we obtain that $f_{VM}(x)$ strictly decreases on $(0,1)$. In turn, $f_{VM}(0) = f_{VM}(1) = 0$ for $\rho = 1/2$, $PD > 1/2$ and the uniqueness of $x^* \in (0,1)$ satisfying $f'_{VM}(x^*) = 0$ implies the bell-shape of function $f_{VM}(x)$. □

Proposition 4. *Let $w < 1$, $\rho > 1/2$ and $PD \ge \Phi\left(\sigma_T \frac{1-2\rho+w\rho}{1-w}\right)$ then function $f_{VM}(x)$ is bi-modal.*

Proof. The statement of Proposition follows from Lemma 2 and Lemma 3(7-8) because

$$PD \ge \Phi\left(\sigma_T \frac{1-2\rho+w\rho}{1-w}\right) \iff \gamma \le 0$$

□

In case of $PD < \Phi\left(\sigma_T \frac{1-2\rho+w\rho}{1-w}\right)$, which is equivalent to $y_0 > 0$, function $f_{VM}(x)$ may be either bi-modal, or decreasing. Unlike the Vasicek case $w = 0$, in general Vasicek-Merton framework there is no a closed-form description of "bifurcation fence" delimiting unimodal and bimodal combination of basic parameters. Propositions 1-4 imply that area $\rho < 1/2$ is still unimodal, though for $\rho > 1/2$ results may vary. The bimodal shape of $f_{VM}(x)$ is guaranteed for relatively large values of PD, otherwise, there are no closed-form conditions for precise delimiting. Even in simplified case $w = 1$ the threshold value γ^* can not be found in closed form.

The Figure 1 demonstrates a bi-modal (or, rather "springboard") shape of function $f'_{VM}(x)$ may in case of $\rho > 1/2$. Given $\rho = 0.9$, $PD = 0.01$, $w = 0.5$, $\sigma_T = 4$, we obtain

$$\Phi\left(\sigma_T \frac{1-2\rho+w\rho}{1-w}\right) \approx 0.002 < PD = 0.01.$$

Computer simulations shows that in case of $\rho < 1/2$ the shapes of both functions, Vasicek $f_V(x)$ and Vasicek-Merton $f_{VM}(x)$, look similar, though, the density f_{VM} is more "concentrated", as it is shown at Figure 2.

It was mentioned above that the significant difference between f_V and f_{VM} is that the latter function has no pike at $x = 1$ regardless of the parameter values. The Figure 3 shows the behavior of $f_V(x)$ and $f_{VM}(x)$ for $\rho = 0.65$ in neighborhood of $x = 1$. To make the difference more visible we use the logarithmic scale for y-axis. The second mode at $x = 1$ is obvious for the Vasicek density, as well as for its logarithm, while the Vasicek-Merton distribution the only mode is at $x = 0$.

The following Table 1 summarizes the comparison of PDFs for vanilla Vasicek and non-trivial Vasicek-Merton distributions. The ambivalent case $\rho > 1/2$ is generated by impossibility to delimit these two cases in closed form using the function parameters.

3 Expected Loss and Loss Given Default

In this section we derive the explicit formulas for Expected Loss, Loss Variation, Loss Given Default and Unexpected Loss (Capital Reserves) in comparison to the corresponding concepts,

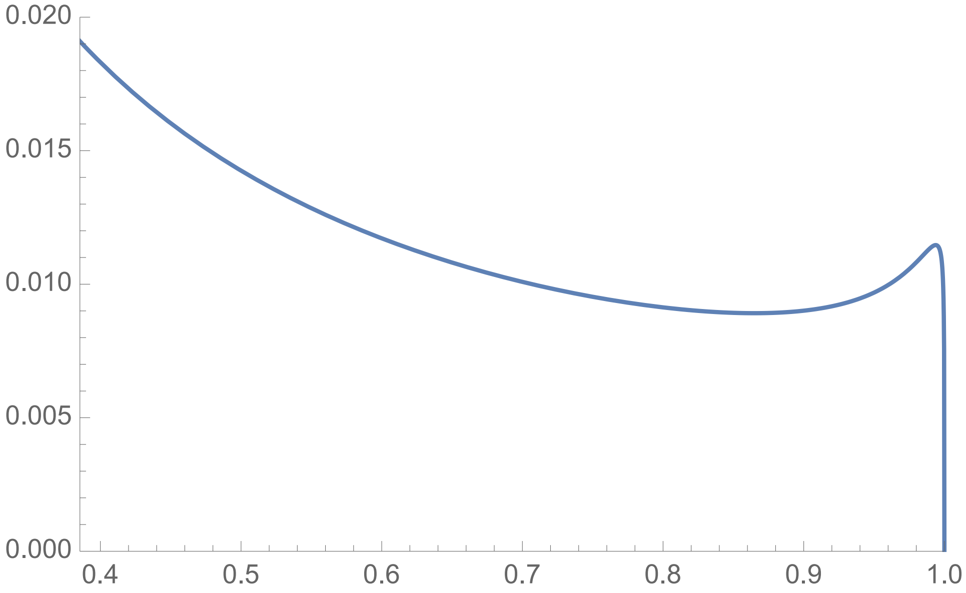

Figure 1: Springboard-shaped function f_{VM} for $\rho = 0.9$, $PD = 0.01$, $w = 0.5$, $\sigma_T = 4$.

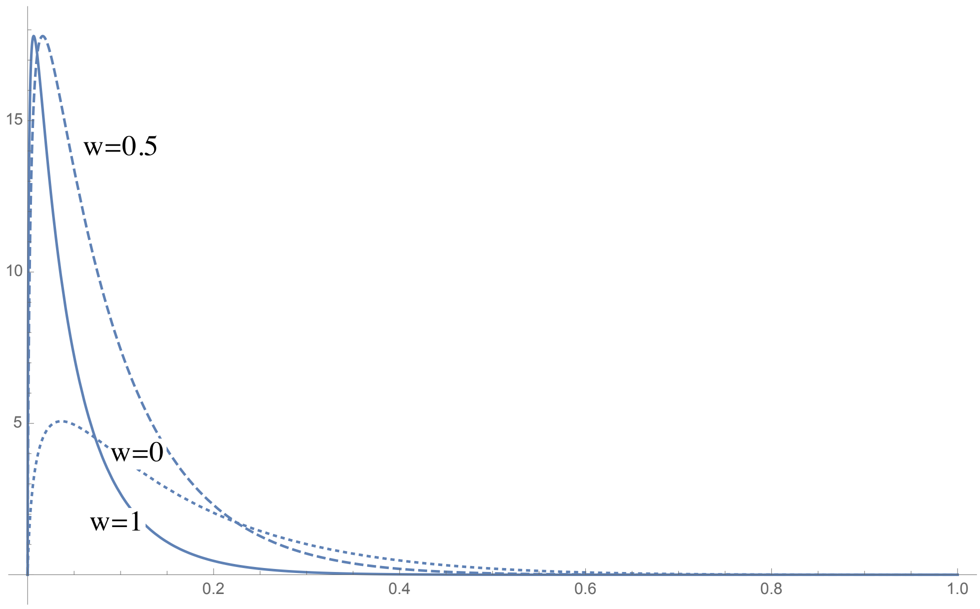

Figure 2: PDFs of Vasicek-Merton loss distributions for $\rho = 0.25$ and $w = 1$ (solid curve), $w = 0.5$ (dashed curve), $w = 0$ (dotted curve).

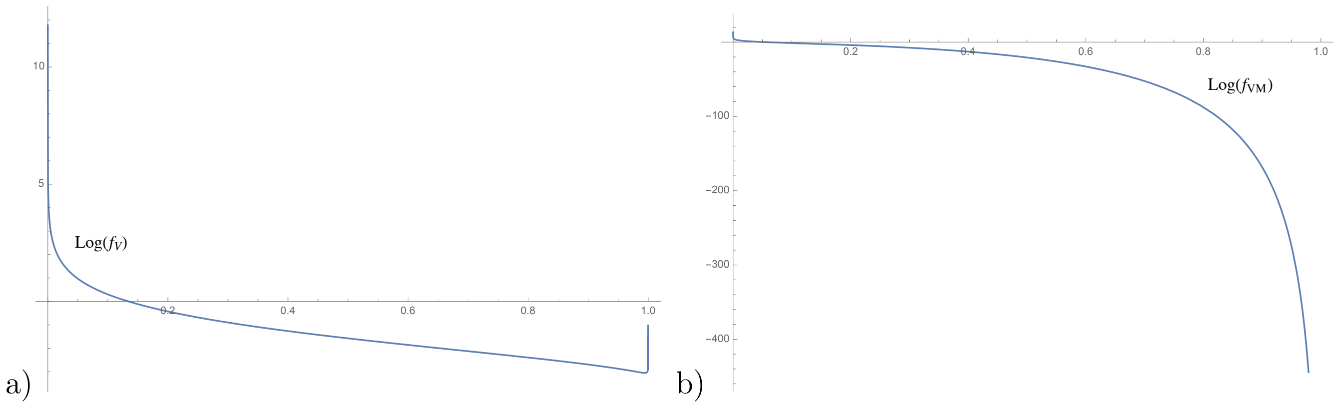

Figure 3: Density plots in logarithm scale of Vasicek (a) and Vasicek-Merton (b) loss distribution for $\rho = 0.65$.

ρ	PD	$f_V(0)$	$f_V(1)$	Shape of f_V	$f_{VM}(0)$	$f_{VM}(1)$	Shape of f_{VM}
$(0, 0.5)$	any	0	0	Bell-shaped	0	0	Bell-shaped
$(0.5, 1)$	any	$+\infty$	$+\infty$	U-shaped	$+\infty$	0	Decreasing or SBoard-shaped
0.5	$(0, 0.5)$	$+\infty$	0	Decreasing	$+\infty$	0	Decreasing
0.5	$(0.5, 1)$	0	$+\infty$	Increasing	0	0	Bell-shaped
0.5	0.5	1	1	Flat	$\begin{cases} \frac{1}{1-w}, & w<1 \\ +\infty, & w=1 \end{cases}$	0	Decreasing

Table 1: Shapes of f_V and f_{VM}.

suggested by Vasicek and AIRB approaches. Both difference and similarity of our approach to those ones are highlighted. We suggest that our approach accounts a credit maturity and an asset volatility in more proper way.

Let

$$\Phi_2(s, t; \rho) = \mathbb{P}[X_1 < s, X_2 < t],$$

be the bivariate normal CDF, where X_1, X_2 are standard normal variables with correlation ρ. Then hold following identities

$$\int_{-\infty}^{+\infty} \Phi\left(\frac{s+\sqrt{\rho}y}{\sqrt{1-\rho}}\right) \mathrm{d}\Phi(y) = \Phi(s), \tag{8}$$

$$\int_{-\infty}^{+\infty} \Phi\left(\frac{s+\sqrt{\rho}y}{\sqrt{1-\rho}}\right) \Phi\left(\frac{t+\sqrt{\rho}y}{\sqrt{1-\rho}}\right) \mathrm{d}\Phi(y) = \Phi_2(s, t; \rho). \tag{9}$$

In particular

$$\int_{-\infty}^{+\infty} \Phi\left(\frac{s+\sqrt{\rho}y}{\sqrt{1-\rho}}\right)^2 \mathrm{d}\Phi(y) = \Phi_2(s, s; \rho),$$

in case of $t = s$.

Proposition 5. *The expected loss*

$$ELoss = PD\left[1 - w \cdot \mathcal{R}_{\sigma_T}(\Phi^{-1}(PD))\right],$$

while the loss variation

$$\begin{aligned} VarLoss = {} & \Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) - PD^2 + \\ & + w^2 PD^2 \cdot \mathcal{R}_{\sigma_T}(\Phi^{-1}(PD))^2 \left[e^{\rho\sigma_T^2} \frac{\Phi_2(\Phi^{-1}(PD) - (1+\rho)\sigma_T, \Phi^{-1}(PD) - (1+\rho)\sigma_T; \rho)}{\Phi(\Phi^{-1}(PD) - \sigma_T)^2} - 1 \right] - \\ & - 2wPD^2 \cdot \mathcal{R}_{\sigma_T}(\Phi^{-1}(PD)) \left[\frac{\Phi_2(\Phi^{-1}(PD) - \sigma_T, \Phi^{-1}(PD) - \rho\sigma_T; \rho)}{\Phi(\Phi^{-1}(PD) - \sigma_T)\Phi(\Phi^{-1}(PD))} - 1 \right]. \end{aligned} \tag{10}$$

See **Proof** in Appendix.

Note that $w = 0$ implies the well-known formulas for the vanilla Vasicek default distribution

$$ELoss_V = PD, \ \mathrm{Var}Loss_V = \Phi_2(\Phi^{-1}(PD), \Phi^{-1}(PD); \rho) - PD^2,$$

see, e.g., [13]. Clearly, $ELoss < ELoss_V = PD$ in case of $w > 0$.

Note that the fraction

$$LGD(PD, w, \sigma_T) \equiv \frac{ELoss}{PD} = 1 - w \cdot \mathcal{R}_{\sigma_T}(\Phi^{-1}(PD)) \tag{11}$$

is expected Loss Given Default (LGD). In turn, an expected Recovery Rate

$$RR(PD, w, \sigma_T) = w \cdot \mathcal{R}_{\sigma_T}(\Phi^{-1}(PD))$$

is in line with conditional RR (5).

Note that the Loss Given Default does not depend on correlation ρ, though the underlying loss distribution function $F_{VM}(x)$ substantially depends on it. This amazing fact of neutrality to ρ is based on identity (8) (see proof of Proposition 5 for details), which is a specific feature of a standard normal distribution function. Comparing (11) with formula (4) of $LGD(PD(z))$ conditional on systematic shock z we may note that the functional form in both cases is the same. Given the normally distributed system shock z, we obtain that the expected probability of default

$$\int_{-\infty}^{+\infty} PD(z)\mathrm{d}\Phi(z) = \int_{-\infty}^{+\infty} \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho}z}{\sqrt{1-\rho}}\right) \mathrm{d}\Phi(z) = \Phi(\Phi^{-1}(PD)) = PD$$

due to (8), because $y = -z$ is also normally distributed. The similar considerations transform $LGD(PD(z)))$ into $LGD(PD)$.

This unity of form is very important from the following point of view. The conditional values $PD(z)$ and $LGD(z)$ may be naturally interpreted as historical data, while expected values PD and $LGD(PD)$ are rather "theoretical" ones. The common functional form implies that the aggregation of data to calibrate the expected values of the probability of default and the loss given default does not contain any cavities.

Remark 5. Assuming that $V(T)$ is log-normally distributed, in paper [8] there was obtained an explicit formula for RR in case of zero bankruptcy costs, i.e., for $w = 1$:

$$RR = L\frac{\Phi(-d_+)}{\Phi(-d_-)},$$

where

$$L = e^{\mu T}\frac{V(0)}{D},\ d_- = \frac{-\ln L - \sigma_T^2/2}{\sigma_T},\ d_+ = \frac{-\ln L + \sigma_T^2/2}{\sigma_T}.$$

Given

$$PD = \Phi(-d_-) \iff d_+ = d_- + \sigma_T = \sigma_T - \Phi^{-1}(PD)$$

and applying identity (2) we obtain

$$L = e^{rT}\frac{V(0)}{D} = e^{-\sigma_T\Phi^{-1}(PD)+\sigma_T^2/2} = \frac{\varphi(\Phi^{-1}(PD))}{\varphi\left(\Phi^{-1}(PD) - \sigma_T\right)},$$

which implies

$$RR = \frac{\varphi(\Phi^{-1}(PD))}{\varphi\left(\Phi^{-1}(PD) - \sigma_T\right)}\frac{\Phi\left(\Phi^{-1}(PD) - \sigma_T\right)}{\Phi\left(\Phi^{-1}(PD)\right)} = \mathcal{R}_{\sigma_T}(\Phi^{-1}(PD)),$$

or, equivalently,

$$LGD = 1 - RR = 1 - \mathcal{R}_{\sigma_T}(\Phi^{-1}(PD)).$$

This means that formula from paper [8] is a particular case of our result in case of $w = 1$, in other words, when the bankruptcy cost are equal to zero.

In case of $w = 0$ formula 11 implies the Vasicek case $LGD_V \equiv 1$. Now assume that $w > 0$, then the following statement holds.

Proposition 6. *$LGD(0, w, \sigma_T) = 1 - w < 1$, $LGD(1, w, \sigma_T) = 1$ and LGD increases with respect to PD and σ_T.*

Proof. Proof follows immediately from Lemma 1. □

Figure 4 illustrates the results of Proposition 6 showing three plots of the function $LGD(PD)$ for $T = 1$ and three values of $\sigma = 0.2,\ 0.5,\ 0.75$.

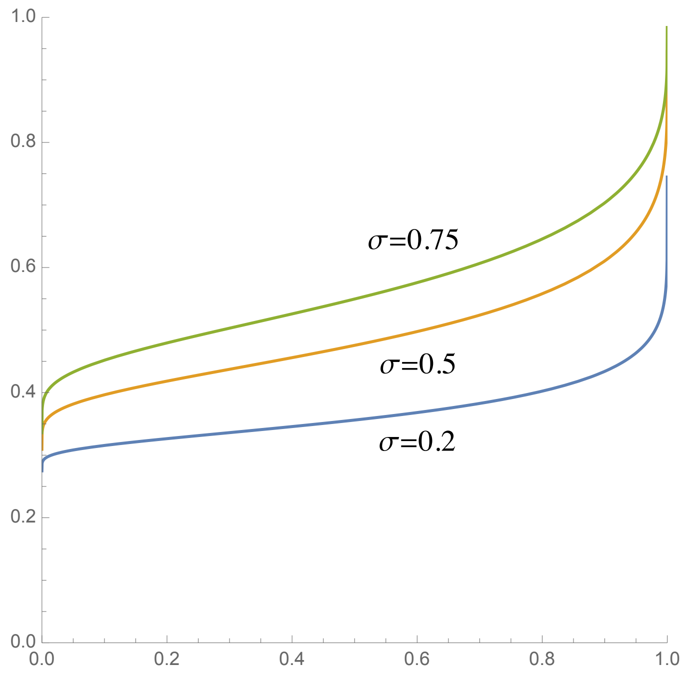

Figure 4: How LGD depends on Probability of Default

3.1 Vasicek-Merton LGD and AIRB Maturity Adjustment

Finally, we compare the Vasicek-Merton loss distribution with AIRB ones, which is the combination of 3 components: (1) Vasicek CDF of defaults with estimated probability of default for maturity $T = 1$ year; (2) LGD also for $T = 1$ chosen by a bank at its discretion; (3) Maturity adjustment constructed on the base of some econometric calculations. It is a kind of black box: there is no clear information how this adjustment was constructed except the remark "The actual form of the Basel maturity adjustments has been derived by applying a specific MtM credit risk model, similar to the KMV Portfolio Manager$^{\text{TM}}$, in a Basel consistent way. This model has been fed with the same bank target solvency (confidence level) and the same asset correlations as used in the Basel ASRF model", see [3, Note 4.6].

Summarizing the previous considerations in Table 2, we can compare loss distributions generated by Vasicek, Vasicek-Merton and AIRB approaches. To make the comparison more accurate, the following remark is in order. The first column CDF^{-1} of Table shows "inverse CDF", which is the value of loss corresponding to the VaR quantile y. These values are formally derived for Vasicek and Vasicek-Merton cases, while AIRB approach uses the maturity adjustment coefficient

$$\lambda_B(LGD_1, PD_1, T) = LGD_1 \cdot \frac{1 + (T - 2.5) \cdot b(PD_1)}{1 - 1.5 \cdot b(PD_1)}$$
$$b(PD_1) = (0.11852 - 0.05478 \ln(PD_1))^2,$$

to the Vasicek loss $\Phi\left(\frac{\sqrt{\rho}\Phi^{-1}(y)+\Phi^{-1}(PD)}{\sqrt{1-\rho}}\right)$, which allows to account for the maturity effect as well as the fact that Loss Given Default is not necessary be equal to 1. Here PD_1 is an estimated *one-year* probability of default, also an annual Loss Given Default LGD_1 is considered as exogenous parameter chosen by bank on its descretion. The second column consists of the CDFs, the well-known Vasicek distribution $F_V(x)$, the Vasicek-Merton distribution of loss $F_{VM}(x)$ derived in present paper, and the non-common "AIRB CDF" $F_B(x)$, which is just "inverse to inverse" function.

The Vasicek model uses only two exogenous parameters — default probability PD and correlation ρ — and does not account for the credit maturity or the asset volatility. To correct

Distribution	CDF^{-1}	CDF
V	$\Phi\left(\frac{\sqrt{\rho}\Phi^{-1}(y)+\Phi^{-1}(PD)}{\sqrt{1-\rho}}\right)$	$F_V(x) = \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}(x)-\Phi^{-1}(PD)}{\sqrt{\rho}}\right)$
VM	$M_{w,\tilde{\sigma}_T}\left(\frac{\sqrt{\rho}\Phi^{-1}(y)+\Phi^{-1}(PD)}{\sqrt{1-\rho}}\right)$	$F_{VM}(x) = \Phi\left(\frac{\sqrt{1-\rho}M^{-1}_{w,\tilde{\sigma}_T}(x)-\Phi^{-1}(PD)}{\sqrt{\rho}}\right)$
AIRB	$\lambda_B \cdot \Phi\left(\frac{\sqrt{\rho}\Phi^{-1}(y)+\Phi^{-1}(PD_1)}{\sqrt{1-\rho}}\right)$	$F_B(x) = \begin{cases} \Phi\left(\frac{\sqrt{1-\rho}\Phi^{-1}\left(\frac{x}{\lambda_B}\right)-\Phi^{-1}(PD_1)}{\sqrt{\rho}}\right), & x < \lambda_B \\ 1, & \text{o/w} \end{cases}$

Table 2: Comparison of distributions

these obvious shortcomings the AIRB approach uses a correction factor $\lambda_B(LGD_1, PD_1, T)$, which accounts for the maturity T and exogenously defined one-year Loss Given Default lgd, though, does not accounts, at least, in explicit form, the asset volatility. Moreover, the correlation ρ is considered as a function of PD_1, not as independent parameter. Formula of the coefficient λ_B is not derived theoretically, being rather the empirically calibrated, thus it may be outdated in a changing circumstances. The drawbacks of this *ad hoc* approach are obvious. The maturity adjustment coefficient $\lambda_B(LGD_1, PD_1, T)$ has discontinuity when $b(PD_1) = 2/3 \iff PD_1 \approx 2.927 \cdot 10^{-6}$ and it is negative for $0 < PD_1 < 2.927 \cdot 10^{-6}$ for all $T > 1$. Moreover, an assumption on the linear dependence of the Loss Given Default on maturity term T may be very inaccurate.

The Vasicek-Merton approach developed in this paper, has wider range of discretion, being based on more detailed set of parameters. Similarly to the AIRB, the Vasicek-Merton approach suggests the amount of the capital reservation, considered as "unexpected loss", which is equal to difference between an admissible portfolio loss and an expected loss. In turn, the admissibility of loss is determined by Value art Risk at level 0.001, recommended by Basel Committee. Thus the capital reserves are determined as follows

$$K_B = \Phi\left(\frac{\sqrt{\rho}\Phi^{-1}(0.999) + \Phi^{-1}(PD_1)}{\sqrt{1-\rho}}\right) \cdot \lambda_B - PD_1 \cdot \lambda_B, \tag{12}$$

with applying the maturity adjustment coefficient λ_B to both terms of difference.

The Vasicek-Merton approach determines the capital reserves, with the same confidence level 0.999, as

$$K_{VM} = M_{w,\tilde{\sigma}_T}\left(\frac{\sqrt{\rho}\Phi^{-1}(0.999) + \Phi^{-1}(PD)}{\sqrt{1-\rho}}\right) - \text{ELoss},$$

where the expected loss

$$\text{ELoss} = PD\left[1 - w \cdot \mathcal{R}_{\sigma_T}(\Phi^{-1}(PD))\right] = M_{w,\sigma_T}(\Phi^{-1}(PD)),$$

due to obvious identity $PD = \Phi(\Phi^{-1}(PD))$, therefore,

$$K_{VM} = M_{w,\tilde{\sigma}_T}\left(\frac{\sqrt{\rho}\Phi^{-1}(0.999) + \Phi^{-1}(PD)}{\sqrt{1-\rho}}\right) - M_{w,\sigma_T}(\Phi^{-1}(PD)). \tag{13}$$

Comparing (13) and (12), the latter may be rewritten as

$$K_B = \Phi\left(\frac{\sqrt{\rho}\Phi^{-1}(0.999) + \Phi^{-1}(PD_1)}{\sqrt{1-\rho}}\right) \cdot \lambda_B - \Phi(\Phi^{-1}(PD_1)) \cdot \lambda_B.$$

Given

$$M_{w,\alpha}(y) = \Phi(y) \cdot [1 - w \cdot \mathcal{R}_\alpha(y)]\,,$$

we may interpret the AIRB coefficient λ_B as semi-empirical estimation of the theoretical LGD-multiplier $1 - w \cdot \mathcal{R}_\alpha(y)$.

Remark 6. The formula (13) implies that the capital reserves $K_{VM} \to 0$ when $\rho \to 0$, i.e., when correlation is negligible, there is no need to reserve capital additionally to expected value. On the other hand, increasing in ρ, which is typically in case of systematic crisis, leads to necessity to reserve more capital.

4 Concluding Remarks

We derived the loss distribution function of a big portfolio of loans using the Vasicek approach and the Merton model of the firm, as an alternative to the AIRB approach. We modeled a loan as a portfolio of a risk-free bond, and a weighted combination of short European vanilla and binary put options written on the assets of the firm, with the strike equal to its debt and expiration equal to maturity of the loan. The expected loss of the portfolio of loans is equal to the expected payouts of the options, hence, to the price of the options – taking into account the asset correlation. To derive the default distribution function it is sufficient to use the sample of firms with the same default probability, while in case of the loss distribution function the firms from sample should be characterized also by the same assets volatility. The Vasicek default distribution function is a particular case of our function, corresponding to the 100% bankruptcy costs.

It is shown that the unimodal (bell) shape of Vasicek distribution for $0 < \rho < 0.5$ is inherited by the general case, while the U-shape is no longer valid for $0.5 < \rho < 1$ in case of partial cost of firm's default. This shape can be bimodal, having the second internal local maximum, so the density function is rather springboard-shaped, but not U-shaped. The boundary in the space of parameters limiting unimodal and bimodal areas has no closed form description in a general non-Vasicek case. On the base of the Vasicek-Merton loss distribution we derived the endogenously defined Loss Given Default as a function of a probability of default PD and an assets volatility σ. This demonstrates that LGD, as well as expected and unexpected loss, hence, consequently, the capital reserves requirements, are not neutral to the volatility of assets values. Thus, the traditional one-parametric rating of firms, which is used by the AIRB approach, is not quite consistent, if we are interested not only in evaluation of default probability, but also in evaluation of loss given default. The two-parametric rating approach, based on both probability of default PD and an assets volatility σ turned out to be more reliable.

Moreover, we derived formulas for two variants of Loss Given Default as a function of the probability of default – the expected value and value conditional on the given systematic shock. The amazing feature of this result is the unity of their functional forms. Taking into account that the conditional values reflect the historical data, this means that aggregating of historical data on default of probability brings to the consistent estimation of expected LGD.

Appendix

List of Notions

D	Debt
$V(t)$	Asset value
T	Maturity of loan
μ	Asset value trend
σ	Asset value volatility
σ_T	$\sigma\sqrt{T}$
$\widetilde{\sigma}_T$	$\sqrt{1-\rho}\sigma\sqrt{T}$
PD	Probability of default
ρ	Asset correlation
L	Leverage
LGD	Loss Given Default
RR	Recovery Rate
λ_B	AIRB maturity adjustment
$F_{VM}(x)$	Vasicek-Merton CDF
$f_{VM}(x)$	Vasicek-Merton PDF

Proof of Lemma 1

Given $\varphi'(x) = -x\varphi(x)$, we obtain

$$\Psi'(x) = \left(\frac{\Phi(x)}{\varphi(x)}\right)' = \frac{\varphi^2(x) + x\varphi(x)\Phi(x)}{\varphi^2(x)} = \frac{\varphi(x) + x\Phi(x)}{\varphi(x)} = 1 + x\Psi(x).$$

It is obvious that $\varphi(x) + x\Phi(x) > 0$ for all $x \geq 0$. Let $x < 0$ then $y = -x > 0$ and $\varphi(x) + x\Phi(x) = \varphi(y) - y\Phi(-y) > 0$ due to the well-known upper-tail property for standard normal distribution. Moreover,

$$\lim_{x\to-\infty} \frac{\Phi(x)}{\varphi(x)} = \lim_{x\to-\infty} \frac{\varphi(x)}{-x\varphi(x)} = 0, \quad \lim_{x\to+\infty} \frac{\Phi(x)}{\varphi(x)} = \lim_{x\to+\infty} \frac{1}{\varphi(x)} = +\infty.$$

Furthermore,

$$\begin{aligned}
\frac{\partial \mathcal{R}_\alpha(x)}{\partial x} &= \frac{\Psi'(x-\alpha)\,\Psi(x) - \Psi(x-\alpha)\,\Psi'(x)}{\Psi(x)^2} = \frac{\Psi(x) - \alpha\Psi(x-\alpha)\,\Psi(x) - \Psi(x-\alpha)}{\Psi(x)^2} = \\
&= \varphi(x)^2 \frac{\frac{\Phi(x)}{\varphi(x)} - \alpha\frac{\Phi(x-\alpha)}{\varphi(x-\alpha)}\frac{\Phi(x)}{\varphi(x)} - \frac{\Phi(x-\alpha)}{\varphi(x-\alpha)}}{\Phi(x)} \\
&= \frac{\varphi(x)}{\varphi(x-\alpha)} \frac{\Phi(x)\varphi(x-\alpha) - \alpha\Phi(x-\alpha)\Phi(x) - \varphi(x)\Phi(x-\alpha)}{\Phi(x)} < 0
\end{aligned}$$

if and only if

$$G(x) = -\alpha\Phi(x-\alpha)\,\Phi(x) + \varphi(x-\alpha)\Phi(x) - \Phi(x-\alpha)\,\varphi(x) < 0.$$

Note that $\lim\limits_{x\to-\infty} G(x) = 0$ and

$$G'(x) = (x-\alpha)\Phi(x-\alpha)\,\varphi(x) - x\varphi(x-\alpha)\Phi(x).$$

It is obvious, that $G'(x) < 0$ for all $x \leq \alpha$. Assume $x > \alpha > 0$, then $\varphi(x) < \varphi(x-\alpha)$ and $(x\Phi(x))' > 0$, hence

$$G'(x) < \varphi(x-\alpha)\left[(x-\alpha)\Phi(x-\alpha) - x\Phi(x)\right] < 0,$$

which implies $G(x) < 0$ for all x. In turn, this means that $\frac{\partial \mathcal{R}_\alpha(x)}{\partial x} < 0$. Inequality $\frac{\partial \mathcal{R}_\alpha(x)}{\partial \alpha} < 0$ immediately follows from $\Psi'(x) > 0$.

Clearly,

$$\lim_{x\to+\infty} \mathcal{R}_\alpha(x) = \lim_{x\to+\infty} \frac{\Phi(x-\alpha)\varphi(x)}{\Phi(x)\varphi(x-\alpha)} = \lim_{x\to+\infty} \frac{\varphi(x)}{\varphi(x-\alpha)} = \lim_{x\to+\infty} e^{-\alpha x+\alpha^2/2} = 1.$$

Note that

$$\mathcal{R}_\alpha(x) = \frac{\Phi(x-\alpha)\varphi(x)}{\Phi(x)\varphi(x-\alpha)} = e^{\alpha^2/2}\frac{e^{-\alpha x}\Phi(x-\alpha)}{\Phi(x)}$$

and

$$\lim_{x\to-\infty} e^{-\alpha x}\Phi(x-\alpha) = \lim_{x\to-\infty} \frac{\Phi(x-\alpha)}{e^{\alpha x}} = \lim_{x\to-\infty} \frac{\varphi(x-\alpha)}{\alpha e^{\alpha x}} = \frac{e^{-\frac{1}{2}\alpha^2}}{\alpha}\varphi(z) = 0,$$

therefore, applying the L'Hospital rule twice we obtain

$$\lim_{x\to-\infty} \frac{e^{-\alpha x}\Phi(x-\alpha)}{\Phi(x)} = \lim_{x\to-\infty} \frac{-\alpha e^{-\alpha x}\Phi(x-\alpha) + e^{-\alpha x}\varphi(x-\alpha)}{\varphi(x)} =$$

$$= e^{-\frac{1}{2}\alpha^2} - \alpha \lim_{x\to-\infty} \frac{e^{-\alpha x}\Phi(x-\alpha)}{\varphi(x)} = e^{-\frac{1}{2}\alpha^2} - \alpha \lim_{x\to-\infty} \frac{\Phi(x-\alpha)}{\varphi(x-\alpha)} = e^{-\frac{1}{2}\alpha^2},$$

therefore,

$$\lim_{x\to-\infty} \mathcal{R}_\alpha(x) = e^{\alpha^2/2} \lim_{x\to-\infty} \frac{e^{-\alpha x}\Phi(x-\alpha)}{\Phi(x)} = 1.$$

Proof of Lemma 2

Formula (6) implies

$$f_{VM}(0) = \lim_{z\to-\infty} \hat{f}(z),\ f_{VM}(1) = \lim_{z\to+\infty} \hat{f}(z),$$

in particular,

$$f_{VM}(1) = \sqrt{\frac{1-\rho}{\rho}} \lim_{z\to+\infty} \frac{e^{-\frac{1}{2}(az-b)^2+\frac{1}{2}z^2}}{w\widetilde{\sigma}_T\Psi(z-\widetilde{\sigma}_T) + (1-w)} = \sqrt{\frac{1-\rho}{\rho}} \lim_{z\to+\infty} \frac{\varphi(z-\widetilde{\sigma}_T)e^{-\frac{1}{2}(az-b)^2+\frac{1}{2}z^2}}{w\widetilde{\sigma}_T\Phi(z-\widetilde{\sigma}_T) + (1-w)\varphi(z-\widetilde{\sigma}_T)} =$$

$$=\sqrt{\frac{1-\rho}{2\pi\rho}} \lim_{z\to+\infty} \frac{e^{-\frac{1}{2}(az-b)^2+\frac{1}{2}z^2-\frac{1}{2}(z-\alpha)^2}}{w\alpha\Phi(z-\alpha) + (1-w)\varphi(z-\alpha)} =$$

$$=\sqrt{\frac{1-\rho}{2\pi\rho}} \lim_{z\to+\infty} \frac{e^{-\frac{1}{2}(az-b)^2+\widetilde{\sigma}_T z-\frac{1}{2}\widetilde{\sigma}_T^2}}{w\widetilde{\sigma}_T\Phi(z-\widetilde{\sigma}_T) + (1-w)\varphi(z-\widetilde{\sigma}_T)} = 0$$

provided that $w > 0$ and regardless of other parameters. Moreover, applying the L'Hospital rule we obtain

$$f_{VM}(0) = \sqrt{\frac{1-\rho}{2\pi\rho}} \lim_{z\to-\infty} \frac{e^{-\frac{1}{2}(az-b)^2+\widetilde{\sigma}_T z-\frac{1}{2}\widetilde{\sigma}_T^2}}{w\widetilde{\sigma}_T\Phi(z-\widetilde{\sigma}_T) + (1-w)\varphi(z-\widetilde{\sigma}_T)} =$$

$$=\sqrt{\frac{1-\rho}{\rho}} \lim_{z\to-\infty} \frac{(-a^2z+ab+\widetilde{\sigma}_T)e^{-\frac{1}{2}(az-b)^2+\widetilde{\sigma}_T z-\frac{1}{2}\widetilde{\sigma}_T^2}}{e^{-\frac{1}{2}(z-\widetilde{\sigma}_T)^2}(\widetilde{\sigma}_T-(1-w)z)} =$$

$$= e^{-b^2/2}\sqrt{\frac{1-\rho}{\rho}} \lim_{z\to-\infty} \frac{-a^2z+ab+\widetilde{\sigma}_T}{\widetilde{\sigma}_T-(1-w)z} e^{-\frac{1}{2}\left(a^2-1\right)z^2-abz}.$$

Note that

$$\lim_{z\to-\infty} \frac{-a^2z+ab+\widetilde{\sigma}_T}{\widetilde{\sigma}_T-(1-w)z} = \begin{cases} \frac{a^2}{1-w} & w<1 \\ +\infty & w=1 \end{cases},$$

however, in both cases the condition $a^2 - 1 > 0 \iff \rho < 1/2$ implies $f_{VM}(0) = 0$, because the term $e^{-\frac{1}{2}\left(a^2-1\right)z^2} \to 0$ when $z \to -\infty$ suppressing all other terms. Analogously, the condition $a^2 - 1 < 0 \iff \rho > 1/2$ implies $f_{VM}(0) = +\infty$ regardless of the value of w.

Now assume that $\rho = 1/2$, which implies $a = 1$, $b = \sqrt{2}\Phi^{-1}(PD)$, $\widetilde{\sigma}_T = \sigma_T/\sqrt{2}$ and

$$f_{VM}(0) = e^{-(\Phi^{-1}(PD))^2} \lim_{z\to-\infty} \frac{-z + \sqrt{2}\Phi^{-1}(PD) + \sigma_T/\sqrt{2}}{\sigma_T/\sqrt{2} - (1-w)z} e^{-\sqrt{2}\Phi^{-1}(PD)z},$$

while

$$\lim_{z\to-\infty} \frac{-z + \sqrt{2}\Phi^{-1}(PD) + \sigma_T/\sqrt{2}}{w\sigma_T/\sqrt{2} + (1-w)(-z + \sigma_T/\sqrt{2})} = \begin{cases} \frac{1}{1-w} & w < 1 \\ +\infty & w = 1 \end{cases}.$$

As before, the condition $PD < 1/2 \iff \Phi^{-1}(PD) < 0$ implies $f_{VM}(0) = +\infty$, moreover, $PD > 1/2 \Rightarrow f_{VM}(0) = 0$ regardless of the value of w. However, the case $PD = 1/2 \iff \Phi^{-1}(PD) = 0$ and $w < 1$ implies

$$f_{VM}(0) = \frac{1}{1-w},$$

otherwise, $f_{VM}(0) = +\infty$.

Proof of Lemma 3

Apart from the formal proof we present its "visual sketch" for all listed cases. Value $y = y_0$ can not be solution of (7) with exception of the very special case $\gamma = 0$, which was considered in Remark 3. Thus this equation may be rewritten as

$$\Psi(y) = h_0 - \frac{\gamma}{y - y_0},$$

thus all solutions of (7) may be identified as intersection points of two graphs in the coordinate system $(h.y)$: non-parametric curve $h = \Psi(y)$ and parametric hyperbola

$$h = h_0 - \frac{\gamma}{y - y_0}. \tag{14}$$

Then the following series of graphs bring the clear visual presentation of all possible cases, where the blue curve is graph of $\Psi(y)$, while graph of hyperbola (14) is an orange one. Clearly in case $h_0 \le 0$ one of the branches of hyperbola (14) lies beneath zero, while $\Psi(y) > 0$, thus we can consider only one branch with positive values, while in case $h_0 > 0$ both hyperbola branches may have the intersection points with graph of function $\Psi(y)$, see Figures 5 and 6.

Case (1). Let $h_0 = 0$ and $\gamma < 1$, then equation (7) may be rewritten as

$$(y_0 - y)\Psi(y) = \gamma. \tag{15}$$

Assume first that $\gamma < 0$, then equation (15) has no solutions in $y < y_0$. Moreover, the function $(y_0 - y)\Psi(y)$ strictly decreases in area $(y_0, +\infty)$ from 0 to $-\infty$, there exists unique solution of equation (15).

Now assume $\gamma \in (0, 1)$, then equation (15) has no solutions in area $(y_0, +\infty)$. Let $W(y) = (y_0 - y)\Psi(y)$, then $W(y_0) = 0$ and

$$\lim_{y\to-\infty} W(y) = \lim_{y\to-\infty} \frac{(y_0 - y)\Phi(y)}{\varphi(y)} = \lim_{y\to-\infty} \frac{-\Phi(y) + (y_0 - y)\varphi(y)}{-y\varphi(y)} = 1.$$

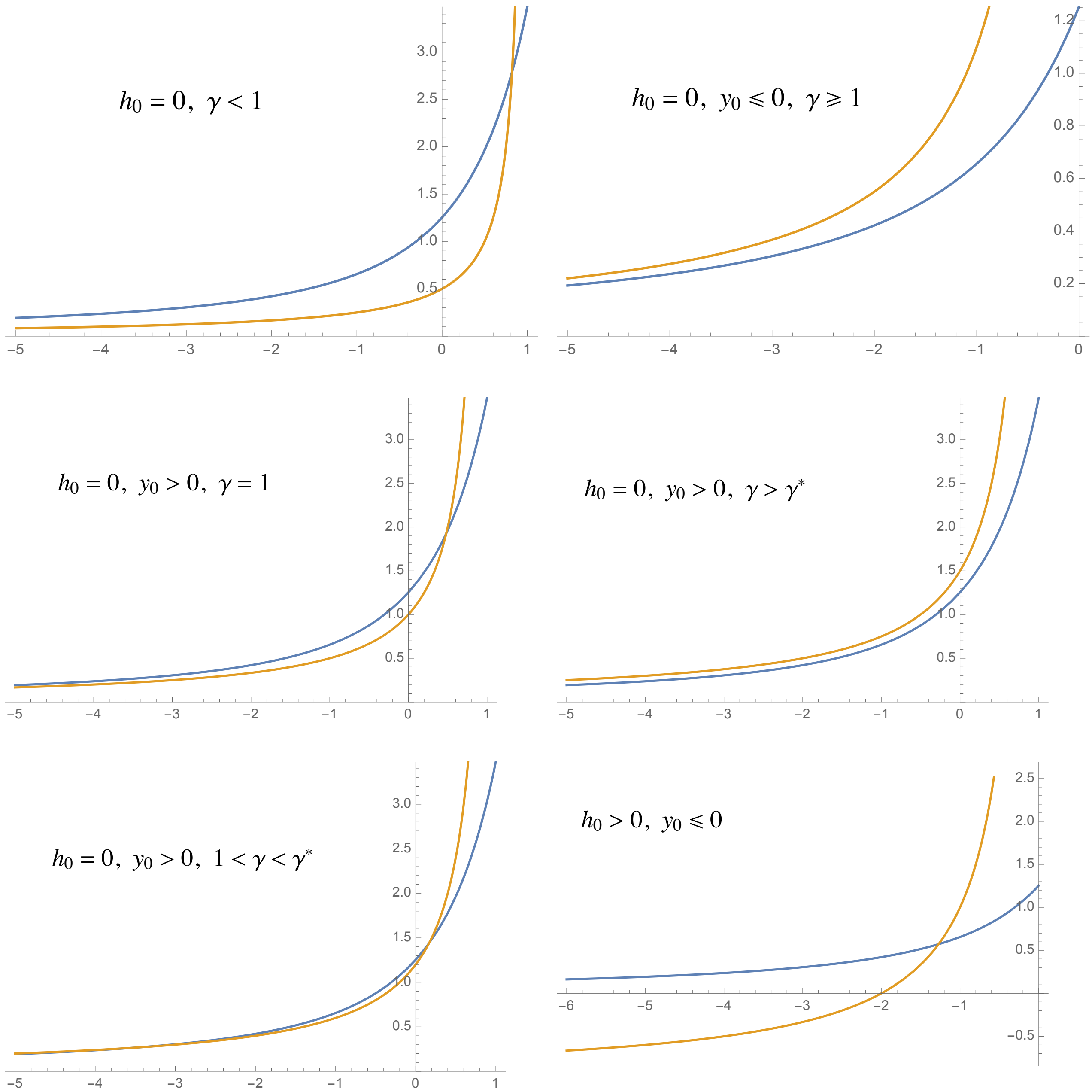

Figure 5: Lemma 3, Cases 1-5

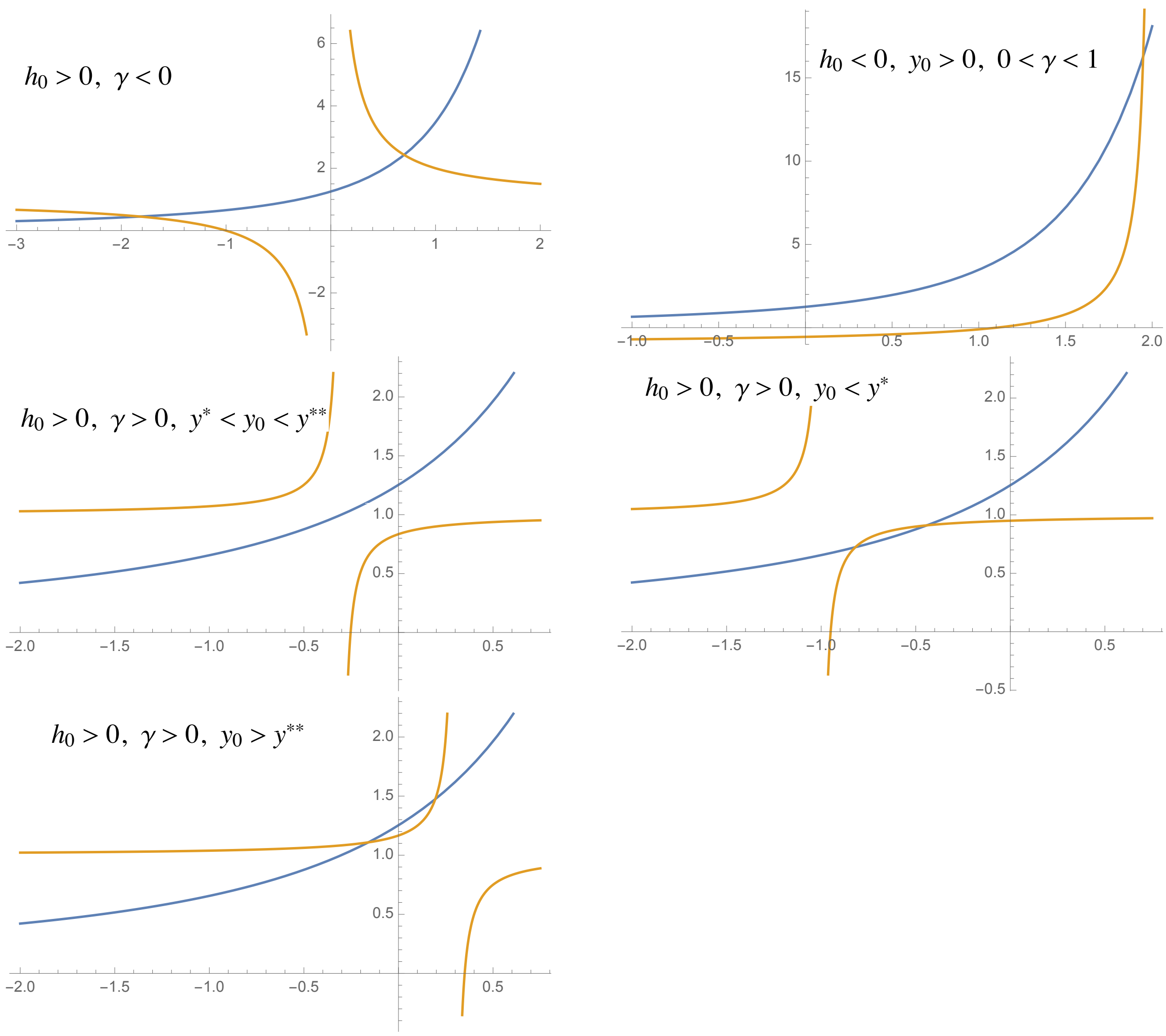

Figure 6: Lemma 3, Cases 6-8

This implies that for each $\gamma \in (0,1)$ there exists at least one solution of equation (15). Assume that there are multiple solutions and let y_L be a left-most solution, while $y_R > y_L$ is a right-most one. By definition $W'(y_L) < 0$ because $W(y) > W(y_L) = \gamma$ for all $y < y_L$. Moreover,

$$W'(y) = -\Psi(y) + (y_0 - y)(1 + y\Psi(y)) = y_0 - y - \Psi(y)(y^2 - y_0 y + 1),$$

thus $W'(y_0) = -\Psi(y_0) < 0$, which implies $W'(y_R) < 0$. Also, we obtain $W(y_L) = W(y_R) = \gamma$ and $W'(y_L) < 0$, $W'(y_R) < 0$, therefore, there exists at least one intermediate value $y_M \in (y_L, y_R)$ such that $W(y_M) = \gamma$, $W'(y_M) > 0$.

Given $y < y_0$, we obtain that $(y_0 - y_L) \cdot W'(y_L) < 0$, $(y_0 - y_L) \cdot W'(y_M) > 0$, $(y_0 - y_R) \cdot W'(y_R) < 0$, while y_L, y_M, y_R satisfy equation

$$W(y) = \gamma \iff \Psi(y) = \frac{\gamma}{y_0 - y}.$$

Let

$$\begin{aligned} U(y) = (y_0 - y)\left[y_0 - y - \frac{\gamma}{y_0 - y}\left(y^2 - y_0 y + 1\right)\right] = \\ = (y_0 - y)^2 + \gamma y(y_0 - y) - \gamma, \end{aligned}$$

then the previous consideration imply that $U(y_L) < 0$, $U(y_M) > 0$, $U(y_L) < 0$, in other words, quadratic equation

$$(y_0 - y)^2 + \gamma y(y_0 - y) - \gamma = 0$$

has two different roots in area $(-\infty, y_0)$. Substituting $t = y_0 - y$ we obtain the equivalent quadratic equation

$$(1 - \gamma)t^2 + \gamma y_0 t - \gamma = 0$$

with two *positive* roots. However, this is impossible, because $1 - \gamma > 0$, $-\gamma < 0$. This contradiction implies that $y_L = y_R$ is unique solution of equation (15).

Case (2). Let $h_0 = 0$, $y_0 \le 0$ and $\gamma \ge 1$. Then

$$\lim_{y \to -\infty} y^2 W'(y) = \lim_{y \to -\infty} \frac{(y_0 y^2 - y^3)\varphi(y) - \Phi(y)(y^4 - y_0 y^3 + y^2)}{\varphi(y)} = \tag{16}$$

$$= \lim_{y \to -\infty} \frac{(2y_0 y - 3y^2)\varphi(y) - (y_0 y^3 - y^4)\varphi(y) - \varphi(y)(y^4 - y_0 y^3 + y^2) - \Phi(y)(4y^3 - 3y_0 y^2 + 2y)}{-y\varphi(y)} =$$

$$= -2y_0 + 4 \lim_{y \to -\infty} \frac{y\varphi(y) + y^2\Phi(y)}{\varphi(y)} - 3y_0 \lim_{y \to -\infty} \frac{y\Phi(y)}{\varphi(y)} = y_0$$

which implies that in case of $y_0 < 0$ function $W(y)$ strictly decreases in neighborhood of $-\infty$. Assuming $y_0 = 0$ and applying L'Hospital's rule several times, we obtain

$$\lim_{y \to -\infty} -y^3 W'(y) = \lim_{y \to -\infty} \frac{y^4\varphi(y) + \Phi(y)(y^5 + y^3)}{\varphi(y)} = -2,$$

therefore in case of $y_0 \le 0$ there exists $y_1 \le y_0$, such that $W'(y) \le 0$ for all $y \in (-\infty, y_1)$. Show that this implies $y_1 = y_0$. Let on the contrary, $y_1 < y_0$,the by definition $\gamma_1 = W(y_1) \in (0,1)$ and $W(y_1 + \varepsilon) > \gamma_1$ for all $\varepsilon > 0$ sufficiently small. Given $W(y_0) = 0$, this implies that there exists at least one $y_2 \in (y_1, y_0)$ satisfying $W(y_2) = \gamma_1$. As result, we obtain that there exists at least two different solution of equation $W(y) = \gamma_1 \in (0,1)$, which contradicts to the statement (2) proved above. This contradiction implies $y_1 = y_0$, i.e., function $W(y)$ strictly decreases in $(-\infty, y_0)$, while

$$\lim_{y \to -\infty} W(y) = 1.$$

This means that equation $W(y) = \gamma$ has no solutions for all $\gamma \geq 1$.

Case (3). Let $h_0 = 0$, $y_0 > 0$ and $\gamma = 1$, then (16) implies that there exists $y^* \leq y_0$, such that $W'(y) \geq 0$ for all $y \in (-\infty, y^*)$. Let y^* be a maximum number with this property, then $y^* < y_0$, because of $W'(y_0) < 0$, and y^* is a local maximum of $W(y)$, thus, $W(y^*) > 1$. This implies that there exists at least one solution of equation $W(y) = 1$. Assume that there are multiple solutions and let y_L be a left-most solution, while $y_R > y_L$ is a right-most one. By definition $W'(y_L) < 0$ because $W(y) > W(y_L) = 1$ for all $y < y_L$. Repeating the similar consideration from the Case (2), we obtain that there exist two solutions of linear equation

$$y_0 t - 1 = 0,$$

because of $\gamma = 1$. This obvious contradiction implies that solution of equation $W(y) = 1$ is unique.

Case (4). Let $h_0 = 0$, $y_0 > 0$ and $\gamma > 1$. The proof of Case (3) implies that there exist $y^* < y_1 < y_0$ such that, $W(y_1) = 1$, $W(y) < 1$ for all $y \in (y_1, y_0)$, $W(y) > 1$ for all $y \in (-\infty, y_1)$ and

$$\gamma^* \equiv W(y^*) = \max_{y<y_0} W(y) > 1.$$

This immediately implies the statement (4).

Case (5). Let $h_0 < 0$, $y_0 \leq 0$. Given $\gamma \neq 0$ there are two possible sub-cases.

(5a) Assume first $\gamma < 0$, then $G(y) < 0$ for all $y < y_0$, moreover, for $y > y_0$ function $G(y)$ strictly decreases from $+\infty$ to $h_0 < 0$, while $\Psi(y)$ strictly increases from $\Psi(y_0) > 0$ to $+\infty$. Therefore, there exists unique intersection point in area $(y_0, +\infty)$.

(5b) Assume that $\gamma > 0$, then function $G(y)$ is negative for all $y > y_0$. Let

$$\hat{y} = y_0 + \frac{\gamma}{h_0} < y_0$$

then $G(y)$ increases from 0 to $+\infty$ in interval $(\hat{y}, y_0)$, while $\Psi(y)$ increases from $\Psi(\hat{y}) > 0$ to $\Psi(y_0) > 0$. This implies that in interval $(\hat{y}, y_0)$ there exists at least one one solution of equation $\Psi(y) = G(y)$, which may be rewritten as follows

$$V(y) \equiv \Psi(y) + \frac{\gamma}{y - y_0} = h_0. \tag{17}$$

Assume that there exists more than one solution of equation 17 and let y_L is the left-most solution, while $y_R > y_L$ is the right-most one. Note that, $V(\hat{y}) = \Psi(\hat{y}) > 0$, $\lim_{y \to y_0} V(y) = -\infty$, which implies that in both y_L and y_R function $V(y)$ decreases, i.e., $V(y_L) = V(y_R) = h_0$, $V'(y_L) < 0$, $V'(y_R) < 0$, where

$$V'(y) = 1 + y\Psi(y) - \frac{\gamma}{(y - y_0)^2}$$

due to identity $\Psi'(y) = 1 + y\Psi(y)$. This implies that there exists at least one intermediate point $y_L < y_M < y_R$ satisfying $V(y_M) = h_0$, $V'(y_M) > 0$.

Given $(y - y_0)^2 > 0$, the previous considerations imply $(y_L - y_0)^2 V'(y_L) < 0$, $(y_M - y_0)^2 V'(y_M) > 0$, $(y_R - y_0)^2 V'(y_R) < 0$. Moreover, y_L, y_M, y_R satisfy the equation

$$\Psi(y) = h_0 - \frac{\gamma}{y - y_0},$$

therefore, the inequalities $U(y_L) < 0$, $U(y_M) > 0$, $U(y_R) < 0$ hold, where

$$\begin{aligned} U(y) = (y - y_0)^2 \cdot \left(1 + y \cdot \left(h_0 - \frac{\gamma}{y - y_0}\right) - \frac{\gamma}{(y - y_0)^2}\right) = \\ =(y - y_0)^2 + h_0 y \cdot (y - y_0)^2 - \gamma y \cdot (y - y_0) - \gamma. \end{aligned}$$

Substituting $y = y_0 - t$, we obtain that the polynomial

$$\begin{aligned}\widetilde{U}(t) = U(y_0 - t) &= t^2 + h_0(y_0 - t) \cdot t^2 + \gamma(y_0 - t) \cdot t - \gamma = \\ &= - h_0 t^3 + (1 - \gamma + h_0 y_0)t^2 + \gamma y_0 t - \gamma\end{aligned}$$

satisfies $\widetilde{U}(t_L) < 0$, $\widetilde{U}(t_M) > 0$, $\widetilde{U}(t_R) < 0$, where $t_L = y_0 - y_L$, $t_M = y_0 - y_M$, $t_R = y_0 - y_R$. Clearly, $0 < t_R < t_M < t_L < t_0 = -\gamma/h_0$, which implies that polynomial $\widetilde{U}(t)$ has two different roots in interval $(0, t_L)$. Given $\widetilde{U}(t_L) < 0$ and $-h_0 > 0$, we obtain that there exists the third root in area $(t_L, +\infty)$. As result, our assumption about non-uniqueness of solutions of equation (17) implies that the polynomial

$$\widetilde{U}(t) = -h_0 t^3 + (1 - \gamma + h_0 y_0)t^2 + \gamma y_0 t - \gamma$$

has three different positive roots. Show that this is impossible.

Due to Descartes' rule of signs, the necessary condition of existence of three positive roots is $1 - \gamma + h_0 y_0 < 0$, $\gamma y_0 > 0$, given $h_0 < 0$, $\gamma > 0$. This contradicts to assumption $y_0 \leq 0$, thus the solution of (17) must be unique.

Case (6). Let $h_0 < 0$, $y_0 > 0$ and $\gamma \in (0, 1)$. Considerations similar to Case (5b) show that there exists at leas one solution of equation

$$V(y) \equiv \Psi(y) + \frac{\gamma}{y - y_0} = h_0.$$

in interval $(-\infty, y_0)$. Moreover, assumption on non-uniqueness of this solution leads to conclusion that there exist points $y_L < y_M < y_R < y_0$ satisfying $V(y_L) = V(y_M) = V(y_R) = h_0$, and $V'(y_L) < 0$, $V'(y_M) > 0$, $V'(y_R) < 0$. The continuity of $V'(y)$ implies that there exist $y_M < y_1 < y_R$ and $y_L < y_2 < y_M$ such that

$$V'(y_1) = V'(y_2) = 0, \; V''(y_1) < 0, \; V''(y_2) > 0.$$

Note that

$$V'(y) = 1 + y\Psi(y) - \frac{\gamma}{(y - y_0)^2}$$

and

$$V''(y) = y + (1 + y^2)\Psi(y) + \frac{2\gamma}{(y - y_0)^3}.$$

This implies that

$$\begin{aligned}\lim_{y\to-\infty} y^2 V'(y) &= \lim_{y\to-\infty} \frac{y^2\varphi(y) + y^3\Phi(y)}{\varphi(y)} - \gamma = \lim_{y\to-\infty} \frac{2y\varphi(y) + 3y^2\Phi(y)}{-y\varphi(y)} - \gamma = \\ &= -3 \lim_{y\to-\infty} \frac{y\Phi(y)}{\varphi(y)} - 2 - \gamma = -3 \lim_{y\to-\infty} \frac{\Phi(y) + y\varphi(y)}{-y\varphi(y)} - 2 - \gamma = 1 - \gamma > 0,\end{aligned}$$

therefore $V'(y) > 0$ in some neighborhood of $+\infty$. Given $V'(y_L) < 0$, this implies that there exists $y_3 < y_L$ satisfying $V'(y_3) = 0$ and $V''(y_3) < 0$. Direct calculations show that

$$\lim_{y\to-\infty} y^2 V''(y) = \lim_{y\to-\infty} \frac{y^3\varphi(y) + y^2(1 + y^2)\Phi(y)}{\varphi(y)} = 8 > 0,$$

which implies that second derivative $V''(y)$ is positive in some neighborhood of $-\infty$, thus it changes its sign in $(-\infty, y_3)$ at least once. In addition to this, $V''(y)$ changes sign at least ones in interval (y_3, y_2) and at least once in (y_2, y_1).

Moreover, $y_1,\ y_2,\ y_3$ satisfy the equation

$$V'(y) = 0 \iff \Psi(y) = \frac{1}{y}\left(\frac{\gamma}{(y_0-y)^2} - 1\right),$$

therefore, the second derivative $V''(y)$ calculated in these points is equal to

$$y + (1+y^2)\frac{1}{y}\left(\frac{\gamma}{(y_0-y)^2} - 1\right) - \frac{2\gamma}{(y_0-y)^3} =$$

$$\frac{1}{(y_0-y)^3}\left(y(y_0-y)^3 + \frac{y-y_0}{y}\left(\gamma-(y_0-y)^2\right) + y(y_0-y)\left(\gamma-(y_0-y)^2\right) - 2\gamma\right) = \frac{U(y)}{y(y_0-y)^3},$$

where

$$U(y) = \gamma(y-y_0) - (y_0-y)^3 + \gamma y^2(y_0-y) - 2\gamma.$$

Due to previous considerations cubic polynomial $U(y)$ has (at least, though, actually exactly) three roots in area $(-\infty, y_0)$. Substituting $t = y_0 - y > 0$ for $y < y_0$ and rearranging terms we obtain the polynomial

$$U(y_0-t) = (\gamma-1)t^3 - 2\gamma y_0 t^2 + \gamma(3+y_0^2)t - 2\gamma y_0,$$

which presumably has three positive roots. However, $\gamma - 1 < 0$, therefore due to Descartes' rule of signs this polynomial has only two or zero positive roots. This contradiction imply that there is only one solution of equation(17).

Case (7) Let $h_0 > 0$, $\gamma < 0$, then (7) may be rewritten as follows

$$\Psi(y) = G(y) \equiv h_0 - \frac{\gamma}{y-y_0}.$$

where the right-hand side function strictly decreases in both areas $y < y_0$ and $y > y_0$, while the left-hand side one strictly increases. Therefore, in case of $h_0 > 0$ a graph of $\Psi(y)$ intersects both branches of hyperbola $h_0 + \frac{\gamma}{y-y_0}$, once for $y < y_0$, and once for $y > y_0$, while in case of $h_0 \leq 0$ there exists only one intersection for $y > y_0$.

Case (8). Finally, let $h_0 > 0$, $\gamma > 0$. Let $\hat{y}_0 = \Psi^{-1}(h_0)$, then equation

$$\Psi(y) = h_0 - \frac{\gamma}{y-\hat{y}_0}$$

has no solutions. Indeed, the point $(\hat{y}_0, h_0)$ belongs to the graph of $\Psi(y)$ in the coordinate system (y, h), while the other points of tis graph belong to "quadrants" $\{y > \hat{y}_0,\ h > h_0\}$ and $\{y < \hat{y}_0,\ h < h_0\}$ due to $\Psi'(y) > 0$. On the other hand, the hyperbola branches belong to "quadrants" $\{y < \hat{y}_0,\ h > h_0\}$ and $\{y > \hat{y}_0,\ h < h_0\}$, therefore, there are no intersection points of these two graphs. Moreover, this also holds for all y_0 sufficiently close to $\hat{y}_0$. On the contrary, for $y_0 << \Psi^{-1}(h_0)$ there appear intersection of the graph $h = \Psi(y)$ with the right-hand side branch of hyperbola

$$h = h_0 - \frac{\gamma}{y-y_0},\ y > y_0,$$

while for $y_0 >> \Psi^{-1}(h_0)$ there will be intersection for $y < y_0$. The delimiter points y^*, y^{**} may be determined as tangential points of $h = \Psi(y)$ with, correspondingly, the left-hand side and right-hand side branches of hyperbola. Clearly, these points can not be represented in the closed form.

Proof Proposition 5

By definition

$$F_{VM}(x) = \Phi\left(\sqrt{\frac{1-\rho}{\rho}} M^{-1}_{w,\widetilde{\sigma}_T}(x) - \frac{\Phi^{-1}(PD)}{\sqrt{\rho}}\right).$$

Let

$$y(x) = \sqrt{\frac{1-\rho}{\rho}} M^{-1}_{w,\widetilde{\sigma}_T}(x) - \frac{\Phi^{-1}(PD)}{\sqrt{\rho}},$$

then $y(0) = -\infty$, $y(1) = +\infty$ and

$$x = M_{w,\widetilde{\sigma}_T}\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho}y}{\sqrt{1-\rho}}\right),$$

which implies

$$\begin{aligned} ELoss = \int_0^1 x \mathrm{d}F_{VM}(x) = \int_{-\infty}^{+\infty} M_{w,\widetilde{\sigma}_T}\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho}y}{\sqrt{1-\rho}}\right) \mathrm{d}\Phi(y) = \\ = \int_{-\infty}^{+\infty} \Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho}y}{\sqrt{1-\rho}}\right) \mathrm{d}\Phi(y) - \\ -we^{\widetilde{\sigma}_T^2/2} \int_{-\infty}^{+\infty} e^{-\widetilde{\sigma}_T \frac{\Phi^{-1}(PD)+\sqrt{\rho}y}{\sqrt{1-\rho}}} \Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho}y}{\sqrt{1-\rho}} - \widetilde{\sigma}_T\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \mathrm{d}y. \end{aligned}$$

After substitution of $\widetilde{\sigma}_T = \sqrt{1-\rho}\sigma_T$ the previous formula boils down to

$$\begin{aligned} ELoss = \int_{-\infty}^{+\infty} \Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho}y}{\sqrt{1-\rho}}\right) \mathrm{d}\Phi(y) - \\ -we^{\sigma_T^2/2 - \sigma_T\Phi^{-1}(PD)} \int_{-\infty}^{+\infty} \Phi\left(\frac{\Phi^{-1}(PD) - \sigma_T + \sqrt{\rho}z}{\sqrt{1-\rho}}\right) \mathrm{d}\Phi(z), \end{aligned}$$

where $z = y + \sqrt{\rho}\sigma_T$. Applying the first formula of (9) twice for $s = \Phi^{-1}(PD)$ and $s = \Phi^{-1}(PD) - \sigma_T$, we obtain

$$\begin{aligned} EL = PD - we^{\sigma_T^2/2 - \sigma_T\Phi^{-1}(PD)} \Phi\left(\Phi^{-1}(PD) - \sigma_T\right) = \\ = PD \cdot \left[1 - w \frac{\varphi(\Phi^{-1}(PD))}{\varphi(\Phi^{-1}(PD) - \sigma_T)} \frac{\Phi\left(\Phi^{-1}(PD) - \sigma_T\right)}{\Phi\left(\Phi^{-1}(PD)\right)}\right] = \\ = PD \cdot \left[1 - w\mathcal{R}_{\sigma_T}(\Phi^{-1}(PD))\right] \end{aligned}$$

The similar considerations show that

$$\begin{aligned} \mathrm{Var}Loss = \int_{-\infty}^{+\infty} M\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho}y}{\sqrt{1-\rho}}\right)^2 \mathrm{d}\Phi(y) - ELoss^2 = \\ \int_{-\infty}^{+\infty} \Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho}y}{\sqrt{1-\rho}}\right)^2 \mathrm{d}\Phi(y) + \\ +w^2 \int_{-\infty}^{+\infty} e^{-2\widetilde{\sigma}_T \frac{\Phi^{-1}(PD)+\sqrt{\rho}y}{\sqrt{1-\rho}} + \widetilde{\sigma}_T^2} \Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho}y}{\sqrt{1-\rho}} - \widetilde{\sigma}_T\right)^2 \mathrm{d}\Phi(y) - ELoss^2 - \qquad (18) \\ -2w \int_{-\infty}^{+\infty} e^{-\widetilde{\sigma}_T \frac{\Phi^{-1}(PD)+\sqrt{\rho}y}{\sqrt{1-\rho}} + \widetilde{\sigma}_T^2/2} \Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho}y}{\sqrt{1-\rho}}\right) \Phi\left(\frac{\Phi^{-1}(PD) + \sqrt{\rho}y}{\sqrt{1-\rho}} - \widetilde{\sigma}_T\right) \mathrm{d}\Phi(y) \end{aligned}$$

Moreover,

$$\int_{-\infty}^{+\infty} e^{-2\widetilde{\sigma}_T \frac{\Phi^{-1}(PD)+\sqrt{\rho}y}{\sqrt{1-\rho}} + \widetilde{\sigma}_T^2} \Phi\left(\frac{\Phi^{-1}(PD)+\sqrt{\rho}y}{\sqrt{1-\rho}} - \widetilde{\sigma}_T\right)^2 \mathrm{d}\Phi(y) = e^{(1+\rho)\sigma_T^2 - 2\sigma_T\Phi^{-1}(PD)} \times$$

$$\times \int_{-\infty}^{+\infty} \Phi\left(\frac{\Phi^{-1}(PD)-(1+\rho)\sigma_T+\sqrt{\rho}x}{\sqrt{1-\rho}}\right)^2 \mathrm{d}\Phi(x) =$$

$$= e^{\rho\sigma_T^2}\left(\frac{\varphi(\Phi^{-1}(PD))}{\varphi(\Phi^{-1}(PD)-\sigma_T)}\right)^2 \Phi_2(\Phi^{-1}(PD)-(1+\rho)\sigma_T, \Phi^{-1}(PD)-(1+\rho)\sigma_T; \rho)$$

where $x = y + 2\sqrt{\rho}\sigma_T$ and

$$\int_{-\infty}^{+\infty} e^{-\widetilde{\sigma}_T \frac{\Phi^{-1}(PD)+\sqrt{\rho}y}{\sqrt{1-\rho}} + \widetilde{\sigma}_T^2/2} \Phi\left(\frac{\Phi^{-1}(PD)+\sqrt{\rho}y}{\sqrt{1-\rho}}\right) \Phi\left(\frac{\Phi^{-1}(PD)+\sqrt{\rho}y}{\sqrt{1-\rho}} - \alpha\right) \mathrm{d}\Phi(y) =$$

$$= e^{\sigma_T^2/2 - \sigma_T\Phi^{-1}(PD)} \int_{-\infty}^{+\infty} \Phi\left(\frac{\Phi^{-1}(PD)-\rho\sigma_T+\sqrt{\rho}z}{\sqrt{1-\rho}}\right) \times$$

$$\times \Phi\left(\frac{\Phi^{-1}(PD)-\sigma_T+\sqrt{\rho}z}{\sqrt{1-\rho}}\right) \cdot \mathrm{d}\Phi(z) =$$

$$= \frac{\varphi(\Phi^{-1}(PD))}{\varphi(\Phi^{-1}(PD)-\sigma_T)} \Phi_2(\Phi^{-1}(PD)-\rho\sigma_T, \Phi^{-1}(PD)-\sigma_T; \rho)$$

where $z = y + \sigma_T\sqrt{\rho}$ due to (9). Given $PD = \Phi(\Phi^{-1}(PD))$ and substituting these formulas in (18) we obtain (10).

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