Macroscopic analogs of quantum-mechanical phenomena and auto-transformations of functions

Harin, Alexander

Modern University for the Humanities

16 November 2020

Macroscopic analogs of quantum-mechanical phenomena and auto-transformations of functions

Alexander Harin
aaharin@gmail.com
Modern University for the Humanities

The two main goals of the present article are: 1) To prove an existence theorem for forbidden zones for the expectations of real-valued random variables. 2) To define transformations (named here as auto-transformations) of the probability density functions (PDFs) of random variables into similar PDFs having smaller sizes of their domains and to outline their basic features. Such transformations can be used also for functions beyond the scope of the probability theory. The goals are caused by the well-known problems of behavioral sciences, e.g., by the underweighting of high and the overweighting of low probabilities, risk aversion, the Allais paradox, etc.

Keywords: Expectations, Boundaries, Forbidden zones, Domains, Utility.
MSC codes: 91B06, 91B16, 91C05.

Content

1. Introduction

Multifarious bounds for moments and functions of random variables and also noise and its influence are considered in a wealth of works, see, e.g., [1], [2], [3], [4].

A man as an individual actor is a key subject of economics and some other sciences. There are a number of problems concerned with the mathematical description of the behavior of an individual. The examples are the underweighting of high and the overweighting of low probabilities, risk aversion, the Allais paradox, risk premium, etc., see, e.g., [5]. The essence of the problems consists in biases of preferences and choices of people for the uncertain and sure games in comparison with the predictions of the theory of probability.

The idea for explanation of these problems is to consider influence of noise near the boundaries of measurement intervals (see also, e.g., [6], [7]). An existence theorem for forbidden zones is proved here. These forbidden zones can be considered also as some macroscopic analogs of quantum-mechanical phenomena. There are a number of such real macroscopic forbidden zones for the expectations of the coordinates of the sides of vibrating rigid bodies near rigid boundaries.

For example, a small rigid boat or any other small rigid floating body which oscillates on the waves near a rigid moorage wall can be mentioned. For example, a washing machine (or an edgeless side of a drill) that vibrates near a rigid wall can be mentioned as well.

Auto-transformations (ATs) of probability density functions (PDFs) from, e.g., infinite to semi-infinite and finite domains are defined and outlined here. Such ATs can be useful for both general and particular goals, e.g. to put forward hypotheses and make general assumptions about such modified PDFs, to extend the above existence theorem from finite to semi-infinite intervals, etc.

2. Existence theorem for forbidden zones

2.1. Theorem

Let us consider a set $\{X_i\}$, $i = 1, \dots, n$, of random variables X_i whose values lie within an interval $[a,b]:(b-a)\in(0,\infty)$. For the sake of simplicity, X_i, μ_i, σ^2 and similar symbols will often be written without the subscript "i."

Let us consider the expectation and variance of X, and their relations.

In connection with the terms "bound" and forbidden zone," the abbreviation "r_μ" (arising from the first letter "r" of the term "restriction") will be used here, due to its convenience and consonance with the usage in previous works.

A proof is given in [8] that for the variance σ^2 of a discrete random variable with the range $[a, b]$ and expectation μ, the following inequality holds:

$$\sigma^2 \le (\mu - a)(b-\mu)\,. \tag{1}$$

An alternate proof is given in, e.g., [9] that the same inequality holds also for the variance of any real-valued random variable X. Every value $p_X(x_k)$ or $f_X(x)$ is divided into two values located at the boundaries a and b such that μ is unchanged. The variance for all the divided cases is proved to be not less than the initial one and the general inequality is obtained.

Theorem 1. Consider a set $\{X_i\}$, $i = 1, \dots, n$, of random variables X_i whose values lie within an interval $[a, b]$. If $0 < (b\text{-}a) < \infty$ and there exists a forbidden zone of the non-zero width $\sigma^2_{\min}$ for the variances σ_i^2 of X_i, such that for all i

$$\sigma^2{}_i \ge \sigma^2{}_{\min} > 0\,, \tag{2}$$

then certain forbidden zones (or boundary bounds, or restrictions) of a non-zero width r_μ exist for the expectations μ_i of the X_i

$$a < (a + r_\mu) \le \mu_i \le (b - r_\mu) < b\,.$$

Proof. Inequalities (1) and (2) lead at the boundary, e.g., a to

$$\mu_i \ge a + \frac{\sigma^2{}_i}{b-a} \ge a + \frac{\sigma^2{}_{\min}}{b-a} \equiv a + r_\mu\,.$$

Since $0 < (b\text{-}a) < \infty$ and $\sigma^2_{\min} > 0$, the bounds r_μ are non-zero and this leads to the required inequalities those can also serve as estimations of the widths of the forbidden zones

$$a < \left(a + \frac{\sigma^2{}_{\min}}{b-a}\right) \le \mu_i \le \left(b - \frac{\sigma^2{}_{\min}}{b-a}\right) < b\,. \qquad \square \tag{3}$$

2.2. Comments to the theorem

This simple theorem supports the uniform solution (see, e.g., [10]) for the well-known fundamental problems of behavioral sciences, e.g., for the underweighting of high and the overweighting of low probabilities, risk aversion, the Allais paradox, risk premium, etc.

Indeed, the (central) allowed zone is evidently compressed by the (boundaries') forbidden zones (in comparison with the entire interval). Therefore the expectations are biased from the boundaries to the center of the probability scale (in comparison with the case of zero forbidden zones). Therefore the expectations are underweighted at high and overweighted at low probabilities, that uniformly solves the above and some other problems, at least qualitatively or partially.

Moreover, these forbidden zones can be considered as some macroscopic analogs of quantum-mechanical phenomena. The well-known forbidden zones in the energy spectrum of an electron in semiconductors can be considered as an example of such phenomena.

The theorem is proved for finite intervals. The auto-transformations proposed below will help to extend it for semi-infinite intervals.

3. Auto-transformations. Main definitions and some basic features

3.1. Auto-transformations as a tool for modifications and hypotheses

The domains of many probability density functions are infinite. Questions can arise about how such or similar PDFs could be modified if their domains were half-infinite or finite.

Generally, questions can arise about how probability density functions can be modified when their domains are modified from larger to smaller sizes of the domains. These questions may be relevant in particular in connection with possible expansions and generalizations of the results of, e.g., [10] and the above existence theorem those are obtained for finite intervals.

Such questions can be too hard to be solved immediately and exactly. So a tool is proposed here to modify probability density functions and also put forward hypotheses and make assumptions about such modified functions. It will modify mainframe probability density functions into transformed ones those will be, depending on parameters of the transformations, similar to the mainframe PDFs to a greater or lesser degree.

This tool can be named as auto-transformations of probability density functions.

3.2. Main definitions and assumptions. Some basic features

Consider the domain of the probability density function of a random variable. Suppose this domain is the infinite or a half-infinite (or a finite) interval. Further this interval is referred to as a mainframe interval (**MF-interval** or **MFI** and their boundaries – in any case as a_{MF} and b_{MF}). This PDF is referred to as a mainframe PDF (**MF-PDF** or **MFF** or ***f***).

Auto-transformations are abbreviated to **AT**s.

A half-infinite or finite part of the infinite or a half-infinite or finite MF-interval is defined as an **interval of auto-transformation** or an **auto-transformation interval** (**AT-interval** or **ATI**) under the following three determining conditions:

1. The AT-interval contains (at least at its boundary) at least one of the key points of the mainframe probability density function such as the expectation, median, or mode.
2. The part of the MF-PDF that is situated in the AT-interval is unchanged.
3. The part or parts of the MF-PDF that lie outside the ATI are mapped into the ATI.

Usually this (these) part(s) is (are) denoted as an **out-ATI part(s)**.

Types or modes of this mapping can be chosen in accordance with the conditions of the mapping. For example, this mapping can be uniform, stepwise, triangle, convex, concave, reflecting, adhering, etc. That is, e.g., the integral of the out-ATI part of the MF-PDF can be uniformly distributed in the ATI, transformed into a step, triangle, etc.

So the summarized result of the transformation of the MF-PDF is fully enclosed in the AT-interval and consists of the following two parts:

1) The part that is identically mapped (or, in other words, is unchanged).

2) The part(s) or addition(s) that is (are) mapped from the outside into inside of the ATI.

This resultant transformed MF-PDF is referred to as an auto-transformed probability density function (**AT-PDF** or **ATF**).

A **reflection** auto-transformation is defined as an AT such that the MF-PDF f is modified to the AT-PDF such that the out-ATI part of f is reflected (as an addition) with respect to some point (dot) of the reflection (that is to one of the boundaries of the AT-interval).

This AT is, in a sense, similar to the reflection of a wave of light from a mirror.

If an auto-transformation interval is finite, especially if the mainframe interval is infinite or at least semi-infinite, then the reflection can be multiple. This can occur also in similar cases but in the absence of reflection. In all these cases the AT can be referred to as a **repeated auto-transformation** or **multiple AT** or **many-fold AT** or **two-mirror AT**.

Otherwise (and as a rule here), the ATs can be referred to as **one-fold ATs**.

An **adhesion** AT can be modified from the reflection one. The reflected part of the MF-PDF is "adhered" (as an addition) to the boundary of the ATI. In this case the MF-PDF is transformed to the probability distribution function of the mixed type, such that its discrete part is equal to the integral of the reflected and adhered part of the MF-PDF.

The **mean** of the mainframe PDF within the AT-intervals can be denoted as

$$f_{mean} \equiv \frac{1}{b_{AT} - a_{AT}} \int_{a_{AT}}^{b_{AT}} f_{MF}(x)dx \,,$$

An auto-transformation that transforms an out-ATI part of a MF-PDF into an addition that do not increase in the direction from this out-ATI part boundary to the opposite one is referred to as a **non-increasing** AT (this addition is added with the unchanged part of the MF-PDF in the AT-interval to constitute the resulting AT-PDF).

Non-increasing ATs correspond to an intuitive assumption that an auto-transformed out-ATI part should contribute near the boundary that is the closest to this out-ATI part, at least, not less than near the opposite boundary.

ATs that transform the out-ATI parts of the MF-PDFs into the total AT-intervals are referred to as **full-ATI ATs** or **filling ATs**.

Minimal distances. One of the main particular goals of the auto-transformations is to estimate possible distances from the expectations of the PDFs to the boundaries of the intervals. Let us consider reflection auto-transformations and determine conditions for the minimal distances from the expectations of AT-PDFs to the boundaries of the AT-intervals.

Suppose a reflection auto-transformation is performed with respect to an arbitrary reflection point (dot) $d \equiv d_{refl}$ and the expectation of the AT-PDF is expressed as a function of d. At that, the median m arises as a parameter of this expression.

The reflection point is shifted by some increment ε and the shift of the expectation is calculated. Further the shifts for $d > m$ and for $d+\varepsilon < m$ are compared.

This comparison reveals that the abovementioned distance is minimal when the reflection point coincides with the median of the MF-PDF (in more detail see, e.g., [11]).

The adhesion ATs provide evidently the minimal distances among all the ATs.

An AT of a PDF f is referred to as a **necessary AT** or **Norm-necessary AT** if

$$\int_{a}^{b} f(x)dx \equiv \int_{a_{AT}}^{b_{AT}} f(x)dx \geq \frac{1}{2}\int_{a_{MF}}^{b_{MF}} f(x)dx = \frac{1}{2},$$

That is the norm of the unchanged part of the MF-PDF is not less than 1/2. That is the difference between the norms calculated for the MF-PDF and its unchanged part is not more than 1/2.

Necessary auto-transformations ensure the condition that provides a number of properties of transformed PDFs (see, e.g., [11]). These properties include questions whether ATs are one-fold, filling or many-fold. They include comparisons of the values of the additions with the mean values f_{mean} of the mainframe PDFs within the AT-intervals. They include questions whether the additions of the ATs are stepwise, convex, triangle or concave.

An AT of a PDF f is referred to as a **sufficient AT** or **Norm-sufficient AT** if

$$\int_{a_{MF}}^{a} f(x)dx + \int_{b}^{b_{MF}} f(x)dx \equiv \delta_{out} \ll \int_{a_{MF}}^{b_{MF}} f(x)dx = 1,$$

that is the difference between the norms calculated for the MF-PDF and unchanged part of the AT-PDF is negligibly small in comparison with the norm calculated for the MF-PDF.

For the normal distribution, the auto-transformation interval that corresponds to the “three-sigma rule” can be used as a sufficient AT-interval.

Generally, a sufficient auto-transformation (and its auto-transformation interval) can be referred to as sufficient with respect to a certain parameter (e.g., the dispersion), if the difference between the values of this parameter calculated for the MF-PDF for the MFI and ATI is negligibly small in comparison with the value of this parameter calculated for the MFI.

The sufficient auto-transformations are evidently the most prospective auto-transformations for hypotheses and estimations.

Conclusions

The purposes of the present report are to provide particular mathematical solutions for the well-known problems of behavioral sciences and to generalize these particular solutions.

The two main new results of the report are:

First result. The mathematical support for the uniform solution (see, e.g., [10]) for the well-known fundamental problems (the underweighting of high and the overweighting of low probabilities, etc.) of behavioral sciences is gained in the form of existence theorem 1.

Estimation (3) for the widths of the forbidden zones of the theorem is obtained as well. These zones can be also considered as macroscopic analogs of quantum-mechanical phenomena.

Second result. The concept of the auto-transformations is proposed.

The ATs can be considered as a new tool to modify and transform the PDFs of r.v.s into similar PDFs having smaller sizes of their domains. Evidently, this tool can be developed for transformations of functions even beyond the scope of the probability theory as well.

The tool can be used, e.g., for hypotheses and assumptions.

References

[1] Arnold, L., H. Crauel, and V. Wihstutz (1983). Stabilization of linear systems by noise, *SIAM J. Control Optim.* **21**(1), 451–461.

[2] Cacoullos, T. (1982). On Upper and Lower Bounds for the Variance of a Function of a Random Variable. *Ann. Probab.* **10**(3), 799–809.

[3] Chernoff, H. (1981). The identification of an element of a large population in the presence of noise. *Ann. Probab.* **9**(1), 533–536.

[4] Prekopa, A. (1990). The discrete moment problem and linear programming, *Discrete Appl. Math.* **27**(3), 235{254.

[5] Thaler, R. (2016). Behavioral Economics: Past, Present, and Future. *Am. Econ. Rev.* **106**(7), 1577–1600.

[6] Schoemaker, P., and J. Hershey (1992). Utility measurement: Signal, noise, and bias. *Organ. Behav. Hum. Dec.* **52**(1), 397–424.

[7] Aczél, J., and D. R. Luce (2007). A behavioral condition for Prelec's weighting function on the positive line without assuming W(1)=1. *J. Math. Psychol.* **51**, 126–129.

[8] Bhatia, R. and C. Davis (2000). A better bound on the variance, *Am. Math. Mon.* **107**, 353–357.

[9] Harin, A. (2018). Forbidden zones for the expectation. New mathematical results for behavioral and social sciences. MPRA Paper, No 86650.

[10] Harin, A. (2012) Data dispersion in economics (II) – Inevitability and consequences of restrictions. *Rev. Econ. Fin.* 2(4), 24-36.

[11] Harin, A. (2020). Behavioral sciences and auto-transformations of functions, MPRA No 99286.