On the Purchasing Power of Money in an Exchange Economy

Radwanski, Juliusz

Humboldt University of Berlin

19 November 2020

On the Purchasing Power of Money in an Exchange Economy*

Juliusz F. Radwański

juliusz.radwanski@hu-berlin.de

19th November, 2020

Preliminary draft. Comments welcome.

Abstract

A model is constructed in which completely unbacked fiat money, issued by generic supplier implementing realistically specified monetary policy designed to obey certain sufficient conditions, is endogenously accepted by rational individuals at uniquely determined price level. The model generalizes Lucas (1978) to an economy with frictions and specialization in production, without imposing the cash-in-advance constraint. The uniqueness of equilibrium is the consequence of complete characterization of both the environment, and the equilibrium concept. The results challenge the doctrine that equilibria of monetary economies are inherently indeterminate, and that money can become worthless only due to self-fulfilling expectations. The paper shows that monetary policy canonically features two dimensions, one of which corresponds to nominal interest rate, and the other to continuous helicopter drop of net worth, which in the model takes the form of universal basic income.

Keywords: fiat money, monetary policy, Hahn problem, price level, inflation, sunspots, helicopter drop, universal basic income.

JEL Classification Numbers: E10, E31, E41, E51, E52, E58, G12, G21.

*Wirtschaftswissenschaftliche Fakultät, Humboldt-Universität zu Berlin, Dorotheenstr. 1, 10099 Berlin, Germany.

1 Introduction

The main question addressed in this paper is whether there exist valuation principles applicable to intrinsically useless tokens of completely unbacked fiat money, if one does not want to rely on modeling shortcuts such as cash-in-advance constraints, or utility over real balances. The main result is that money is endogenously accepted by fully rational agents, at uniquely determined positive value, if the supplying institution obeys certain restrictions on the design of monetary policy. The latter is specified in a realistic way, making the paper interesting to readers wishing to understand policies of actual monetary authorities. The paper highlights the role of nominal net worth, a variable under direct control of the supplier of money, besides the nominal interest rate. By successfully integrating the theory of money with the theory of value, the paper challenges alternative explanations of the observed price level, and inflation.[1]

It appears useful to start with a list of properties of a successful theory integrating money with asset pricing. As a theory of money, it should (1) clarify the economic

[1]The problem of integrating the theory of money with the theory of value (asset pricing) has been recognized by neoclassical economists (Walras, 1900; Hicks, 1935; Patinkin, 1965), dissatisfied with the practice of using the ad-hoc equation of exchange (Fisher, 1911). However, a fully successful theory explaining how fiat money is valued by rational individuals has apparently not been offered. For example, Hahn (1965) observed that the theory of Patinkin (1965) did not rule out equilibrium in which money is permanently worthless, and models with money-in-utility, or equivalent, are generally known to allow for multiplicities of equilibria under standard formulations of monetary policy (Obstfeld and Rogoff, 1983; Matsuyama, 1991), a property shared with overlapping-generations models (Samuelson, 1958), and other (so-called) micro-founded models. In a detailed review of monetary literature, Hellwig (1993) concluded that the fundamental problem of why fiat money is valuable *at all*, especially in the presence of securities that dominate it as store of value (the Hahn problem) has never been solved. The well-known recent controversies around the validity of the fiscal theory of the price level (Kocherlakota and Phelan, 1999; Christiano and Fitzgerald, 2000; Buiter, 2002; Niepelt, 2004), and the new-Keynesian model of inflation (Cochrane, 2011, 2018), are interpreted by the present author as consequences of this unfortunate state of affairs. Section 5.3 offers a more complete discussion.

function of money in actual economies, offer a way to distinguish money from other securities, and explain how the supply of money is determined. As a theory of value, it should (2) explain why intrinsically useless tokens are accepted by rational individuals, which factors determine their value, and how that value is affected by monetary policy. Finally, it should (3) not contradict the evidence, and not depend on implausible assumptions. The theory offered here satisfies these postulates. It extends the asset-pricing framework of Lucas (1978) in apparently novel direction, and does not rely on the cash-in-advance constraint. The main source of progress lies in a more complete, relative to related studies, specification of *both* (a) the environment, and (b) the equilibrium concept.

Regarding (a), the paper is explicit about all components of the environment, including technological properties of the (pre-existing) payments infrastructure. In addition, it is observed that there must exist physical limits on the dimensionality and kind of available information. Specifically, all information is contained in a finite-dimensional (possibly large) vector of recursively defined Markovian state variables, which by itself does not rule out a broad class of sunspot processes that might affect the value of money for non-fundamental reasons.[2]

Regarding (b), the definition of Lucas (1978) is generalized by adding a requirement interpreted as no-arbitrage condition, trivially satisfied in economies with complete markets, and no frictions. Intuitively, a rational consumer should always strictly prefer to accept a free, non-negative lottery known to possess a winning state, even if the probability of winning is unknown, or cannot be defined ex ante. If an equilibrium with valued money (in the original sense of Lucas, 1978) *exists as economically justified possibility*, and yet money can be acquired for free, a rational consumer should be expected to strictly prefer to invest in *more* money at

[2]Specific assumptions imposing the informational limits are merely of technical nature, and thus cannot affect the generality of qualitative conclusions.

the margin. By allowing consumers to recognize all arbitrage opportunities, the new definition of equilibrium is very well motivated economically. At the same time, it naturally fits into the notion of competitive equilibrium, since consumers are allowed to individually decide whether they should give up consumption in exchange for intrinsically useless tokens.[3]

Competitive equilibria in the sense of Lucas (1978) are called pre-equilibria. Under a *reasonable* monetary policy, defined precisely in the paper, there exists exactly one pre-equilibrium with positive value of money, and a continuum of pre-equilibria in which money is worthless. By the generalized definition of equilibrium, only the former is economically interesting, and has the property that consumers strictly prefer to start every period with more, rather than less, nominal net worth. For any given level of net worth, individually formed expectations uniquely determine the supply of money to the market for goods, which in turn determines the price level that clears that market. The uniqueness is the consequence of saddle-path dynamics characterizing accumulation of net worth. A too-high consumption must ultimately violate the lower bound on net worth imposed by the supplier of money, while a too-low consumption must lead to over-accumulation of net worth, which is not optimal.[4]

The author does not think that a useful theory of money must be explicit about deviations from the idealized complete-market Arrow-Debreu setup, which can be a daunting task given the complexity of actual world. This work starts instead

[3]This idea requires a more general notion of arbitrage than formalized by Harrison and Kreps (1979), where probabilities must be assigned to outcomes. Subsection 5.1 offers additional discussion.

[4]This is analogous to the mechanics of capital accumulation in the Ramsey model (for example, see Blanchard and Fischer, 1989, ch. 2). The saddle-path stability used to be imposed ad-hoc (Brock, 1974), but this practice has been challenged (Obstfeld and Rogoff, 1983), and is now considered generally invalid in the context of monetary models. The present paper argues that this conclusion is in fact incorrect.

from the observation that frictions *exist*, whether due to asymmetric information, and/or opportunistic behavior. This is sufficient to highlight the economic function of money as transferable, easily recognizable object existing in limited supply, and study issues related to its adoption, and valuation.

The rest of the paper is organized as follows. Section 2 formulates the model. Section 3 defines equilibrium, and generalizes several results of Lucas (1978). Section 4 constructs equilibrium with positive value of money, and proves that there is no other equilibrium. Section 5 discusses the equilibrium selection mechanism, and relevant aspects of the literature. The conclusion is omitted.

2 Model

2.1 Preliminaries

The environment is simplified to avoid analytical traps associated with distributional issues. It features symmetric, infinitely-lived consumers, receiving exogenous streams of perishable, stochastically growing endowment. Consumption of goods produced by others is efficient, while consumption of own endowment results in a proportional waste of $\kappa \in (0, 1]$, which represents the degree of specialization in production.[5] Similarly, spot barter exchange, and privately negotiated forward exchange, are subject to real inefficiencies.[6] For convenience, it is assumed that

[5]The model abstracts from physical differences across goods while preserving the motive for trade.

[6]Costs of barter can be interpreted as resulting from difficulties of coordinating and executing simultaneous delivery of goods, especially if more than two parties are involved, as must be the case under the absence of double coincidence of wants (Jevons, 1875; Menger, 1892). The necessity of simultaneous delivery can in turn be explained by the inability of consumers to enforce private promises, which in the absence of *quid pro quo* delivery must expose the party accepting a promise to ex-ante cost representable as expected discounted loss given default of the counterparty. By the same logic, there must exist ex-ante costs to private

all these forms of exchange require proportional costs of at least κ to all involved parties.

Consumers interact in a spot market where any measure of money can be easily exchanged for goods.[7] They are simultaneously buyers and sellers in this market, and consider these activities independent. The market for goods is competitive in the usual sense that agents are restricted in their ability to influence the aggregate value of money (price level), of which they form expectations before trade, but otherwise take as given. Money exists in the form of transferable tokens, where transferability is defined as compatibility with a pre-existing payments infrastructure. For simplicity, the model ignores frictions in the flow of goods, which can be consumed immediately after buying, but is explicit about frictions in the flow of money, summarized by minimal technological time lag, referred to as Robertsonian lag (Robertson, 1933), separating receipts from expenditures. This lag is fixed for simplicity, and by convention only affects the timing of receipts from the market for goods, while money raised by issuing securities is available immediately. For convenience, the basic frequency at which the model operates is chosen to match the Robertsonian lag.[8]

There is a generic monetary authority called supplier of money, with monopoly on costless production of tokens that can be owned by consumers, and transferred within the payments infrastructure. The supplier of money most naturally corresponds to a consolidated, centrally coordinated banking system, issuing distinct

(also monetary) credit, which for this reason is not explicitly considered.

[7] This is consistent with individual incentives. Given the results of the paper, agents must find it optimal to maintain a market in which money can be used at minimal friction. Related points about the existence and organization of the market for money have been made by Brunner and Meltzer (1971); Alchian (1977), and more recently Howitt (2005).

[8] The lag could be made endogenous via a functional relation to real effort (shoe-leather cost) exerted by the recipient of funds, and possibly other technological inputs. The lag should however be bounded away from zero, also in a continuous-time version of the model.

but functionally equivalent monies within a single payments infrastructure, and maintaining fixed exchange rates between them, or a central bank issuing single type of money under the 100%-reserve requirement.[9] The supplier of money is canonically responsible for the design and implementation of monetary policy, but its role extends to the maintenance of other components of monetary infrastructure. In particular, it assigns nominal net worth to every consumer, provides costless accounting services, validates transfers processed through the payments infrastructure, and intermediates a centralized market for risk-free loanable funds, in which it sets the net supply. Consumers are not forced to use money, and can ignore their accounts. There is only one supplier of money, and only one type of token is technologically feasible.[10]

The design of monetary policy requires specifying two exogenous dimensions, in equilibrium corresponding to nominal discount factor, and rate of new net worth creation relative to nominal income. The corresponding dimensions of policy implementation are algorithmic, and correspond to executing the transfers, and setting the net supply of bonds in the market for loanable funds.[11] The properties of a *reasonable* monetary policy are imposed as two assumptions, requiring that (1) the nominal interest rate is set sufficiently high in relation to the rate of new net worth creation, and (2) the nominal rate remains sufficiently low relative to κ, the inefficiency of consuming own endowment.

The model abstracts from the possibility of composite securities combining trans-

[9]The supplier of money is defined by its functions, and hence may differ from an actual central bank, or government. After some modifications, it could be interpreted as the Nature, computer, or cryptocurrency protocol.

[10]This can be interpreted as technological restriction, although actual suppliers of money are usually able to legally restrict the use of tokens other than issued by themselves. As a historical example, consider the *Executive Order 6102* of 1933, restricting the possession of gold.

[11]The bonds can also be interpreted as reverse-repo lending to the supplier of money, or interest-bearing reserves unavailable for spending.

ferability with a stream of dividends, so all contingent claims backed by dividends are treated as non-transferable. This is methodologically correct, since it reflects the separating line between two fundamentally distinct types of securities that can exist in a monetary economy, and the two types could be used as spanning basis for composite securities.[12]

Consumers have full knowledge of the environment, including the design of monetary policy. They are experienced enough to formulate correct statistical models relating variables potentially relevant to their well being to exogenous, from their perspective, sources of uncertainty. This has been known as the assumption of rational expectations, and is nothing more than deliberate focus on idealized market outcomes in which expectations need not be (further) revised, rather than behavioral postulate.

2.2 Timing and Information

Time is divided into periods represented by half-open intervals $[t, t+1)$. Consumers, potential users of money, make decisions at t, after observing information up to, and including that point. No new information arrives during $(t, t+1)$. Aggregate variables are expressed per capita.

Assumption 1 *All information is contained in state variables* $(Z_t, e_t, u_t, s_t) \in \mathcal{R}_+ \times \mathcal{R}_+ \times U \times S$, *where* $U = \mathcal{R}^{m-2}$, $m \geq 2$, *and* $S \subset \mathcal{R}^n$ *is compact.*[13] *There is a function* $F\colon S \times S \to [0,1]$ *such that* s_t *is a stationary, ergodic Markov process with cumulative transition density* $F(s, s') = \Pr(s_{t+1} \leq s' | s_t = s)$, *and stationary density* Φ. *There is also a jointly continuous (vector) function* $G\colon \mathcal{R}_+ \times \mathcal{R}_+ \times U \times$

[12]Extending the model in this direction, one could obtain a version of the Modigliani and Miller (1958) theorem, which is equivalent to linearity of equilibrium valuation functional, also known as the law of one price.

[13]In what follows, $\mathcal{R}_+$ denotes the set of strictly positive real numbers.

$S \to \mathcal{R}_+ \times \mathcal{R}_+ \times U$ *such that*

$$(Z_{t+1}, e_{t+1}, u_{t+1}) = G(Z_t, e_t, u_t, s_t, s_{t+1}). \tag{1}$$

This assumption restricts the dimensionality and kind of available aggregate information. The identity of the state variables will be explained later.[14]

The next technical assumption, preceded by definition, regulates the process s_t to guarantee that expectations are continuous in the state variables.

Definition 1 *Consider the space $\mathcal{F}$ of functions $f\colon \mathcal{D} \to \mathcal{R}$, with $\mathcal{D}$ a Cartesian product of k subsets of the real line, including one copy of $\mathcal{R}_+$, with corresponding argument denoted by e. Fix $\varphi(x_1, \ldots, e, \ldots, x_k) \in \mathcal{F}$ as $\varphi(e) = e^{1-\gamma}/(1-\gamma)$, $\gamma \in (0,1) \cup (1,\infty)$, or $\varphi(e) = \log e$. For any function $g \in \mathcal{F}$, define the norm $\|g\|_\varphi = \sup_{x \in \mathcal{D}} |g(x_1, \ldots, e, \ldots, x_k)/\varphi(e)|$. Any function $g \in \mathcal{Z}$ with the property $\|g\|_\varphi < \infty$ will be referred to as φ-bounded.*[15]

Assumption 2 *For every φ-bounded and jointly continuous function $f(x, e, s, s')$ (with x a vector of state variables defined on a Cartesian product of subsets of the real line), the function $(Tf)(x, e, s) \equiv \int_S f(x, e, s, s') dF(s, s')$ is jointly continuous.*

[14] Since consumers are symmetric, individual state variables are redundant, and need not be included in the state vector. Still, competitive consumers will be allowed to consider themselves distinct from the average consumer.

[15] Intuitively, a φ-bounded function does not grow (or fall) faster than $\varphi(e)$, in any direction of its domain. The definition and naming are standard (Altug and Labadie, 2008, ch. 8).

2.3 Consumers and Preferences

Consumers are symmetric, and live indefinitely. A representative consumer receives exogenous flow of perishable endowment e_t, following

$$e_{t+1} = e_t \, \lambda(s_t, s_{t+1}), \tag{2}$$

with λ: $S \times S \to \mathcal{R}_+$ continuous, and valued in a compact set containing 1.

According to (2), u_t does not matter for the dynamics, which allows to interpret it as vector state variable unrelated to fundamentals. Similarly, the dynamics do not depend on Z_t, later given the interpretation of nominal net worth. This restriction reflects the fact that the unit of nominal measure cannot, by itself, affect the dynamics of real variables.

Preferences at t are represented by

$$V_t = E_t \left\{ \sum_{s=0}^{\infty} \beta^s u(c_{t+s}) \right\} \tag{3}$$

where $0 < \beta < 1$, $u(c) = c^{1-\gamma}/(1-\gamma)$ if $\gamma \in (0,1) \cup (1, \infty)$, or $u(c) = \log(c)$.

The following assumption guarantees that the consumption-based value of the economy is finite.

Assumption 3 *The function $\lambda(s, s')$ satisfies $w(s) \equiv \beta \int_S \lambda(s, s')^{1-\gamma} dF(s, s') < 1$, all $s \in S$.*

2.4 Markets and Prices

The value of a unit measure of money is $(1/P)_t$. Consumers take it as given, and assume that they can trade freely at this price.[16]

Let M_t be the measure of money brought to the market by a consumer willing to buy goods, which can be interpreted as endogenous supply of money. Similarly, let Y_t be the measure of money raised from the market by a consumer willing to sell goods, which corresponds to demand for money. Both satisfy

$$M_t \geq 0, \quad Y_t \geq 0. \tag{4}$$

The measure of money raised via Y_t is limited by the endowment,

$$(1/P)_t Y_t \leq e_t. \tag{5}$$

The clearing condition in the market for goods is

$$M_t = Y_t, \tag{6}$$

and will be referred to as Keynes law. While this condition must hold identically (spending generates income), it must be imposed as equilibrium condition, since consumers believe that they can choose M_t and Y_t independently.

Consumption of own endowment yields $c_t = (1-\kappa)e_t$. If M_t is spent in the market for goods, consumption increases by $(1/P)_t M_t$. If a measure of money Y_t is raised by selling endowment, inefficient consumption falls by $(1/P)_t Y_t$. Overall,

$$c_t = (1/P)_t M_t + (1-\kappa)[e_t - (1/P)_t Y_t]. \tag{7}$$

[16] This definition allows $(1/P)_t = 0$.

The price of a unit risk-free bond is Q_t, and the number of unit bonds held by a consumer leaving the market for loanable funds is B_t. The clearing condition in that market is discussed later (subsection 2.6).

2.5 Net Worth Accounting

At the end of $t-1$, a representative consumer holds W_t units of money in the form of end-of-period net worth, satisfying

$$W_t \geq 0, \tag{8}$$

which is enforced by the supplier of money (see below).

At the beginning of t, a representative consumer receives non-negative transfer of new net worth G_t, available for use immediately.[17] Define beginning-of-period net worth

$$H_t \equiv W_t + G_t, \tag{9}$$

which under the assumptions so far satisfies

$$H_t \geq 0. \tag{10}$$

A consumer is allowed to choose M_t, B_t subject to

$$H_t \geq M_t + B_t Q_t, \tag{11}$$

referred to as budget constraint.

Due to the Robertsonian lag, the receipt of income Y_t occurs at the end of period.

[17] The case of $G_t < 0$, corresponding to negative transfers of net worth, is not considered empirically interesting.

Given Y_t, the law of motion for end-of-period net worth is

$$W_{t+1} = (H_t - M_t - B_t Q_t) + Y_t + B_t. \tag{12}$$

Equivalently,

$$H_{t+1} = H_t - M_t + Y_t + B_t(1 - Q_t) + G_{t+1}. \tag{13}$$

The non-negativity restriction (8) applied to (12) yields

$$W_{t+1} = H_t - M_t + Y_t + B_t(1 - Q_t) \geq 0, \tag{14}$$

which is perceived by consumers (who have decided to use money) as restriction on choice.[18]

2.6 Design and Implementation of Monetary Policy

Let $Z_t > 0$ denote aggregate end-of-period net worth per capita, defined recursively via

$$Z_{t+1} = Z_t + G_t + B_t\left(1 - q_t\right), \tag{15}$$

where q_t is gross discount on risk-free bonds offered by the supplier of money. With symmetric consumers, it must be true that

$$W_t = Z_t, \tag{16}$$

although these variables are considered distinct by individual consumers, who also take the evolution of Z_t as given, while controlling W_t.

[18]Inequality (14) implements the classic collateral constraint of Bagehot (1873). To see this, write it as $-B_t \leq (H_t - M_t - B_t Q_t) + Y_t$. On the left, $-B_t$ represents debt to the the supplier of money, end of the period. The terms on the right represent good collateral, defined by the supplier of money as acceptable.

Observation 1 *By (15), the value of Z_{t+1} is pre-determined at t.*

The design of monetary policy is characterized by two stationary processes q_t, g_t, adapted to the filtration generated by s_t. The supplier of money sets these processes to $q_t = q(s_t), g_t = g(s_t)$, for two continuous, bounded functions $q\colon \mathcal{S} \to (0, 1]$ and $g\colon \mathcal{S} \to [0, \infty)$, referred to as design functions. In equilibrium, the processes q_t and g_t will correspond to nominal risk-free discount factor, and rate of new net worth creation, respectively.

Two assumptions that follow characterize a *reasonable* monetary policy.

Assumption 4 *The design functions satisfy $1 - q(s) - g(s) \geq 0$, all $s \in \mathcal{S}$. There is $\mathcal{S}' \subset \mathcal{S}$ with $\phi(\mathcal{S}') > 1$, and $1 - q(s) - g(s) > 0$, all $s \in \mathcal{S}'$.*

Assumption 5 *The function $q(s)$ satisfies $q(s) > 1 - \kappa$, all $s \in \mathcal{S}$.*

The former guarantees that the nominal discount is sufficiently low relative to the rate at which new net worth is created, which allows for zero interest rate for stochastically finite periods. The latter guarantees that the nominal discount is sufficiently high relative to the degree of inefficiency associated with consuming own endowment.[19]

Monetary policy is implemented as follows. First, given the policy design functions $q(s)$ and $g(s)$ satisfying assumptions 4 and 5, one constructs a new function $\tau\colon \mathcal{S} \to \mathcal{R}$. Let $\mathcal{B}$ be the set of bounded, continuous functions on $\mathcal{S}$. For any $f \in \mathcal{B}$, define operator $(\mathcal{A}f)(s) = 1 - q(s) - g(s) + \beta \int_S \lambda(s, s')^{1-\gamma} f(s') dF(s, s')$. Under assumption 3, the operator $\mathcal{A}$ is a contraction mapping, so there is a unique function $\tau \in \mathcal{B}$ such that $\mathcal{A}\tau = \tau$ (Banach fixed point theorem). Under assumption 4, the fixed point satisfies $\tau(s) > 0$, all $s \in \mathcal{S}$. Also, define the (strictly positive)

[19]The discount factor q is related to nominal interest rate via $q = i/(1 + i)$. The interpretation of these assumptions only makes sense in equilibrium.

function $h(s) \equiv \tau(s) + g(s)$, which satisfies[20]

$$h(s) > 1 - q(s). \tag{17}$$

Second, given these functions, the supplier of money makes per-capita transfers

$$G_t = \frac{g(s_t)}{\tau(s_t)} Z_t, \tag{18}$$

and offers net supply B_t^s of risk-free bonds, sold at discount q_t, according to

$$b^s(s) \equiv \frac{1}{q(s)} \frac{h(s) - 1}{\tau(s)}, \tag{19}$$

$$B_t^s = b^s(s_t) Z_t. \tag{20}$$

Before it can be established that money is actually valuable, it makes little sense to require $B_t = B_t^s$ as clearing condition in the market for loanable funds. Instead, a weaker condition will be imposed. Define

$$[\{x, 0\}] \equiv \begin{cases} [x, 0] & \text{if } x \leq 0, \\ [0, x] & \text{if } x \geq 0, \end{cases}$$

and restrict the demand for bonds realized by a representative consumer to

$$B_t \in [\{B_t^s, 0\}]. \tag{21}$$

A consumer may refuse to buy the pre-capita net supply of bonds (e.g., $B_t = 0$ is allowed), but can never buy more than is supplied, in terms of absolute value.

[20] The strict positivity of $h(s)$ follows from the definition of $\mathcal{A}$, strict positivity of $\tau(s)$, and the definition of $q(s)$.

Proposition 1 *Under assumption 4, monetary policy implemented by (18) and (20) has the property that* $Z_t > 0$ *implies* $Z_{t+1} > 0$.

Proof. To prove the claim for $B_t = B_t^s$, substitute (18) and (20) into (15), and use the defining properties of function $\tau(s)$. The claim is true for $B_t = 0$, since $g(s) \geq 0$. Since (15) is linear in B_t, the claim is valid for all $B_t \in [\{B_t^s, 0\}]$. ■

Letting $Z_0 > 0$ be an initial aggregate net worth, proposition 1 provides a recursive justification for $Z_t > 0$, which would otherwise need to be imposed as sequence of assumptions.

As a corollary of proposition 1, and (16), $Z_t > 0$ is sufficient to guarantee that end-of-period net worth of a representative consumer is strictly positive,

$$W_{t+1} > 0. \tag{22}$$

Intuitively, the net supply of risk-free bonds has been engineered such that repayment of debt (if consumers are net borrowers) never exhausts the end-of-period net worth. Since $G_{t+1} \geq 0$, this also implies

$$H_{t+1} > 0. \tag{23}$$

Still, consumers will be allowed to consider $H_t = 0$ as hypothetical possibility.

By (9), and (18), beginning-of-period net worth of a representative consumer satisfies $H_t = Z_t[1 + g(s_t)/\tau(s_t)]$. Since g is bounded, and τ is bounded away from zero, one can choose

$$\bar{h} > \max_{s \in \mathcal{S}} \left(1 + \frac{g(s)}{\tau(s)}\right), \tag{24}$$

and restrict H_t, for technical reasons, to the compact set

$$H_t \in [0, \bar{h} Z_t] \equiv \mathcal{H}(Z_t). \tag{25}$$

2.7 Expectations

Consumers form expectations of quantities relevant to their well-being, such as prices, and aggregate state variables. Under assumption 1, these expectations can be represented as functions of aggregate information.

Before entering the market for goods, but after observing information, a representative consumer forms expectations about the value of money using

$$(1/P)_t = \frac{e_t}{Z_t}\eta(Z_t, e_t, u_t, s_t), \tag{26}$$

where $\eta\colon \mathcal{R}_+ \times \mathcal{R}_+ \times \mathcal{U} \times \mathcal{S} \to [0,\infty)$ is a jointly continuous function. This is sufficiently general under assumption 1, and the ratio e_t/Z_t has been added without loss of generality.

Under this model, consumption (7) is

$$c_t = e_t \left\{ (1-\kappa) + \eta(Z_t, e_t, u_t, s_t) \left[\frac{M_t}{Z_t} - (1-\kappa)\frac{Y_t}{Z_t} \right] \right\}. \tag{27}$$

Similarly, the expectation of market value of risk-free bonds is formed as $Q_t = Q(s_t)$, for a continuous function $Q\colon \mathcal{S} \to (0,1]$. The functions η, Q are referred to as price functions.

Transfers of net worth are known to arrive according to $G_t = \chi(s_t)Z_t$, where

$$\chi(s) = \frac{g(s)}{\tau(s)}. \tag{28}$$

Finally, consumers believe that the aggregate state variable Z_t follows

$$Z_{t+1} = \theta(Z_t, e_t, u_t, s_t), \tag{29}$$

where $\theta\colon \mathcal{R}_+ \times \mathcal{R}_+ \times \mathcal{U} \times \mathcal{S} \to \mathcal{R}_+$ is a jointly continuous function. According to observation 1, this function is independent of s_{t+1}.

2.8 Technical Restrictions

Additional restrictions are needed to insure that the choice set is always well defined. These restrictions are by design compatible with the notion of competitive equilibrium, by which consumers never directly experience aggregate supply conditions, and believe to be able to freely trade in all active markets.

The demand for bonds is restricted to

$$B_t \in \mathcal{B}Z_t \equiv \left[-\underline{b}Z_t, \bar{b}Z_t\right], \tag{30}$$

where $\mathcal{B} \equiv [-\underline{b}, \bar{b}]$, and $\underline{b} > 0$, $\bar{b} > 0$ are chosen in a way that[21]

$$\underline{b} > -\min_{s\in\mathcal{S}} b^s(s), \quad \bar{b} > \max_{s\in\mathcal{S}} b^s(s). \tag{31}$$

By construction, the interval (21), defining the feasible range for the realized bond demand is always strictly inside the interval (30).

Another constraint in addition to (5) is needed to restrict the demand for money Y_t at low values of $(1/P)_t$, in order to prevent consumers from rising funds that are not physically available.[22] This problem can be stated in terms of M_t, since if a given $\bar{M}$ is not feasible as spending, then a representative consumer cannot raise it from the market. By the budget constraint (11),

$$M_t \leq H_t - B_tQ_t,$$

[21] It is always possible to find constants $\underline{b}, \bar{b}$, since $b^s(s)$ is bounded.

[22] As an extreme example, a consumer might want to raise *any* measure of money at $(1/P)_t = 0$.

and the right-hand side is maximized by setting $B_t = -\underline{b}Z_t$, and $Q_t = 1$. Since the beginning-of-period net worth of a representative consumer is $H_t = Z_t[1 + g(s_t)/\tau(s_t)]$, one can choose $\bar{\bar{y}} > 0$ such that

$$\bar{\bar{y}} > \max_{s\in\mathcal{S}} \left[1 + \frac{g(s)}{\tau(s)} + \underline{b}\right], \tag{32}$$

and require that

$$Y_t \leq \bar{\bar{y}} Z_t. \tag{33}$$

For future reference, (31) can be used to show that

$$\bar{\bar{y}} > 1/\tau(s). \tag{34}$$

The pair of constraints (5), (33) can be written as a single constraint. Under (26), both bind at the same time when $\eta(Z_t, e_t, u_t, s_t) = 1/\bar{\bar{y}}$, so one can write

$$Y_t \in [0, \bar{Y}(Z_t, e_t, u_t, s_t)], \quad \bar{Y}(Z_t, e_t, u_t, s_t) \equiv \begin{cases} \frac{1}{\eta(Z_t,e_t,u_t,s_t)} Z_t & \text{if } \eta(Z_t, e_t, u_t, s_t) \geq 1/\bar{\bar{y}}, \\ \bar{\bar{y}} Z_t & \text{if } \eta(Z_t, e_t, u_t, s_t) < 1/\bar{\bar{y}}. \end{cases}$$

The upper bound $\bar{Y}(Z_t, e_t, u_t, s_t)$ is strictly positive, and varies continuously with the state variables, by the assumed joint continuity of η.

2.9 Consumer's Problem

The variables relevant to a consumer are H_t, Z_t, e_t, u_t, s_t, and take values in $\mathcal{X} \equiv \{(h, z, e, u, s) \colon z \in \mathcal{R}_+, h \in \mathcal{H}(z), e \in \mathcal{R}_+, u \in \mathcal{U}, s \in \mathcal{S}\}$. The consumer evaluates her well-being using a jointly continuous value function $v \colon \mathcal{X} \to \mathcal{R}$.

Proposition 2 *Any value function consistent with (3) is φ-bounded.*

Proof. Let $\gamma \neq 1$ (the log case is similar). Define v_t, the homothetic version of the utility functional, via $(\sum_{s\geq 0}\beta^s)v_t^{1-\gamma}/(1-\gamma) \equiv V_t$. For fixed e_t, let $\bar{v}_t$ be attained in a hypothetical economy without frictions, and $\underline{v}_t = (1-\kappa)\bar{v}_t$ in an economy where the fraction κ of endowment is lost. Since v_t in a monetary economy is bounded between $\underline{v}_t$ and $\bar{v}_t$, V_t is φ-bounded. ■

At the beginning of t, after observing H_t, Z_t, e_t, u_t, s_t, a representative consumer forms expectations according to functions η, Q, θ, chooses M_t, Y_t, B_t subject to (4), (5), (11), (14), (30), (33), in order to maximize

$$u(c_t) + \beta E_t[v(H_{t+1}, Z_{t+1}, e_{t+1}, u_{t+1}, s_{t+1})],$$

subject to laws (2), (13), and consumption function (27).

2.10 Transversality Condition

With rational consumers, the real value of H_t, measured at the market value of money $(1/P)_t$, cannot exceed the value of the whole economy. This observation can be used to derive an upper bound on the price function η.

Choose $\bar{w}$ such that $\max_s w(s) < \bar{w} < 1$, and consider a consumer with CRRA utility, and consumption stream $\{e_{t+s}(1-\kappa_{t+s})\}_{s\in\{0,1,2,\dots\}}$, with $\kappa_{t+s} \in [0,\kappa]$.[23]

[23]Choosing $\bar{w}$ with the given property is always possible under assumption 3. The considered consumption process is sufficiently general to cover all possible equilibrium consumption paths in the studied economy.

The utility-based value of this stream satisfies

$$
\begin{aligned}
W_t \equiv\ & e_t(1-\kappa_t) + \beta E_t\left\{\frac{u'[e_{t+1}(1-\kappa_{t+1})]}{u'[e_t(1-\kappa_t)]}e_{t+1}(1-\kappa_{t+1})\right\} \\
& + \beta^2 E_t\left\{\frac{u'[e_{t+2}(1-\kappa_{t+2})]}{u'[e_t(1-\kappa_t)]}e_{t+2}(1-\kappa_{t+2})\right\} + \dots \\
& \leq e_t + \beta E_t\left\{\frac{u'[e_{t+1}(1-\kappa)]}{u'[e_t]}e_{t+1}\right\} + \beta^2 E_t\left\{\frac{u'[e_{t+2}(1-\kappa)]}{u'[e_t]}e_{t+2}\right\} + \dots \\
& = e_t + \beta E_t\left\{\frac{u'[e_{t+1}]}{u'[e_t]}e_{t+1}(1-\kappa)^{-\gamma}\right\} + \beta^2 E_t\left\{\frac{u'[e_{t+2}]}{u'[e_t]}e_{t+2}(1-\kappa)^{-\gamma}\right\} + \dots \\
& < e_t(1-\kappa)^{-\gamma}\left[1 + \beta E_t\left\{\frac{u'[e_{t+1}]}{u'[e_t]}\frac{e_{t+1}}{e_t}\right\} + \beta^2 E_t\left\{\frac{u'[e_{t+2}]}{u'[e_t]}\frac{e_{t+1}}{e_t}\right\} + \dots\right] \\
& < e_t(1-\kappa)^{-\gamma}\left[1 + \bar{w} + \bar{w}^2 + \dots\right] \\
& = e_t[(1-\kappa)^{\gamma}(1-\bar{w})]^{-1} \equiv e_t\bar{\eta},
\end{aligned}
$$

where the second line uses the monotonicity of marginal utility, and the fourth line follows from assumption 3, and the definition of $\bar{w}$. The sequence of inequalities shows that consumption-based wealth of a representative consumer can never attain the computed bound.

A representative consumer holds $H_t = Z_t[1 + g(s_t)/\tau(s_t)] \geq Z_t$ of beginning-of-period net worth, so if $\eta(Z_t, e_t, u_t, s_t) > \bar{\eta}$, then under the model (26),

$$H_t(1/P)_t \geq Z_t(1/P)_t = e_t\eta(Z_t, e_t, u_t, s_t) > e_t\bar{\eta}.$$

Since rational consumers attempt to increase consumption at the margin at this (or larger) value of H_t, equilibrium price functions η must be bounded.

3 Definition of Equilibrium

Informally, an equilibrium is (i) a function representing subjective valuation of the environment by a representative consumer, and (ii) a set of functions representing expectations of market prices, and evolution of relevant state variables. These functions are requited to jointly satisfy the requirements that consumers exploit all opportunities to maximize utility, markets clear, and expectations are consistent with actual evolution of uncertainty.

Before defining equilibrium, let $\mathcal{C}(h, z, e, u, s) \subset \mathcal{R}^3$ be the set of triples m, y, b:

$$0 \leq m,\ 0 \leq y, \tag{35}$$

$$b \in [-\underline{b}z, \bar{b}z], \tag{36}$$

$$h \geq m + bQ(s), \tag{37}$$

$$0 \leq h - m + y + b[1 - Q(s)], \tag{38}$$

$$y \leq \frac{1}{\eta(z, e, u, s)} z, \tag{39}$$

$$y \leq \bar{\bar{y}} z. \tag{40}$$

where the last two inequalities can be combined into

$$y \leq \bar{y}(z, e, u, s) \equiv \begin{cases} \frac{z}{\eta(z,e,u,s)} & \text{if } \eta(z, e, u, s) \geq 1/\bar{\bar{y}}, \\ \bar{\bar{y}} z & \text{if } \eta(z, e, u, s) < 1/\bar{\bar{y}}. \end{cases} \tag{41}$$

For each $H_t, Z_t, e_t, u_t, s_t \in \mathcal{X}$, the set of feasible choices $M_t, Y_t, B_t \in \mathcal{C}(H_t, Z_t, e_t, u_t, s_t)$ is non-empty, since it contains the point $M_t = Y_t = B_t = 0$, and compact. The correspondence $\mathcal{X} \to \mathcal{C}$ is continuous.

Definition 2 *An equilibrium is:*

(a) A jointly continuous, bounded function $\eta \colon \mathcal{R}_+ \times \mathcal{R}_+ \times \mathcal{U} \times \mathcal{S} \to [0, \infty)$, *a continuous function* $Q \colon \mathcal{S} \to [0, 1]$, *and a jointly continuous function* $\theta \colon \mathcal{R}_+ \times \mathcal{R}_+ \times \mathcal{U} \times \mathcal{S} \to (0, \infty)$,

(b) A jointly continuous, φ*-bounded function* $v \colon \mathcal{X} \to \mathcal{R}$,

such that:

(i) Given the functions η, Q, θ, the function v solves

$$v(h,z,e,u,s) = \max_{(m,y,b)} \left\{ u(c) + \beta \int_S v(h',z',e',u',s') dF(s,s') \right\}, \tag{42}$$

$$\textit{subject to:} \quad c = e\left[(1-\kappa) + \eta(z,e,u,s)\left(\frac{m}{z} - (1-\kappa)\frac{y}{z}\right)\right], \tag{43}$$

$$(m,y,b) \in \mathcal{C}(h,z,e,u,s), \tag{44}$$

$$h' = h - m + y + b(1-Q(s)) + \chi(s')z', \tag{45}$$

$$z' = \theta(z,e,u,s), \tag{46}$$

$$e' = e\lambda(s,s'), \tag{47}$$

(ii) For each z,e,u,s, the value $v\left(z(1+\chi(s)),z,e,u,s\right)$ is attained by m,y,b that satisfy $m = y$, $b \in [\{b^s(s)z, 0\}]$, $w \equiv z(1+\chi(s)) - m + y + b(1-Q(s)) > 0$, $y < \bar{\bar{y}}z$.

(iii) For each z,e,u,s, if there are functions η^p, Q^p, θ^p specified as in (a), for which a function v^p specified as in (b) satisfies (i)-(ii), and if $\eta^p(z,e,u,s) > 0$, then $\eta(z,e,u,s) > 0$.

Condition (i) restricts the set of equilibrium value functions to those that are consistent with maximized utility functional (3). The first two sub-conditions of (ii) guarantee that individually optimal choices are consistent with market clearing in the markets for goods, and nominal risk-free loanable funds, respectively. The third sub-condition of (ii) guarantees that optimal choices are consistent with the strict positivity of end-of-period net worth, as required by condition (22), and the fourth sub-condition prevents the optimal demand for money from attaining the technical upper bound of inequality (33).

Condition (iii) postulates that if money *could* be valuable without violating individual optimality and market clearing, then the only market outcome consistent with equilibrium is when money *is* actually valuable. This equilibrium selection mechanism can be motivated economically, as discussed in the introduction, and further in subsection 5.1. Conditions (i)-(ii) are sufficient to define equilibrium in the economy of Lucas (1978), where money must be worthless, in which case a condition analogous to (iii) would be satisfied trivially.

Sets of functions η, Q, θ, v satisfying (a)-(b), and (i)-(ii), will be referred to as pre-equilibria. Any equilibrium is a pre-equilibrium. If all pre-equilibria feature $\eta(z, e, u, s) = 0$, then all of them are equilibria, since (iii) is satisfied. If there is at least one pre-equilibrium with $\eta^p(z, e, u, s) > 0$, then the set of equilibria is restricted to pre-equilibria with $\eta(z, e, u, s) > 0$.

Observation 2 *Independently of monetary policy, there is a continuum of pre-equilibria with* $\eta(z, e, u, s) = 0$ *for all* z, e, u, s.

The existence of these pre-equilibria is allowed by the possibility of self-fulfilling (rational) expectations that money is worthless. In such case, any value of $Q(s_t)$ technically clears the market for loanable funds in *real* terms, although the realized demand for bonds may fall short of the supplied quantity.

On the other hand, the law of one price must hold in a pre-equilibrium with strictly positive value of money, which guarantees that:

Proposition 3 *If* $\eta(z, e, u, s) > 0$ *for all* z, e, u, s, *then* $Q(s) = q(s)$ *in a pre-equilibrium.*

The rest of this section is concerned with other properties of pre-equilibria, i.e., the properties of equilibria that can be deduced from (i)-(ii) alone. The proofs generalize those in Lucas (1978), with necessary modifications to take into account more complex environment, and endowment growth.

Proposition 4 *For any functions* η, Q, θ *specified as in (a), there is exactly one* *non-negative, jointly continuous, and φ-bounded function v satisfying (i)-(ii).*

Proof. Let $\mathcal{V}$ be the Banach space of jointly continuous, φ-bounded functions $g\colon \mathcal{X} \to \mathcal{R}$. Let $\mathcal{T}$ be an operator on $\mathcal{V}$, defined such that condition (i) of definition 2 is equivalent to $\mathcal{T}v = v$.

Applying $\mathcal{T}$ involves maximization of a jointly continuous function on a compact set, by assumption 2, and by the definition of $\mathcal{C}$. Hence, the maximum exists. Since the set $\mathcal{C}$ is given by a continuous correspondence in the state variables, the maximum is jointly continuous in h, z, e, u, s (Berge, 1963).

The function $(\mathcal{T}v)(h, z, e, u, s)$ is φ-bounded, since the maximand in (42) is a sum of two φ-bounded functions. Indeed, this is true of $u(c)$ under the assumed CRRA utility, since

$$\begin{aligned} c &\le (1-\kappa)e + (1/P)m = \ e\left[(1-\kappa) + \eta(z,e,u,s)m/z\right] \\ &\le \ e\left[(1-\kappa) + \eta(z,e,u,s)\left(h/z - b/zQ(s)\right)\right] \le \ e\left[(1-\kappa) + \bar{\eta}\left(\bar{h} + \underline{b}\right)\right], \end{aligned}$$

while the other part of the maximand satisfies

$$\begin{aligned} &\left|\frac{\beta\int_S v(h',z',e',u',s')dF(s,s')}{\varphi(e)}\right| = \left|\beta\int_S \frac{\varphi(e')}{\varphi(e)}\frac{v(h',z',e',u',s')}{\varphi(e')}dF(s,s')\right| \\ &\le \beta\int_S \lambda(s,s')^{1-\gamma}\left|\frac{v(h',z',e',u',s')}{\varphi(e')}\right| dF(s,s') \le \bar{v}\beta\int_S \lambda(s,s')^{1-\gamma}dF(s,s') < \bar{v}, \end{aligned}$$

where $\bar{v} \equiv \sup_{\mathcal{X}}\left|\frac{v(h',z',e',u',s')}{\varphi(e')}\right| < \infty$, and the last inequality follows from assumption 3. Hence, the operator $\mathcal{T}$ maps $\mathcal{V}$ into itself.

A similar argument can be used to show that for any $a > 0$ and $f \in \mathcal{V}$, there exists $\delta \in (0,1)$ such that $\mathcal{T}(f + a\varphi) \le \mathcal{T}f + \delta a\varphi$. (Set $\delta = \bar{w}$, defined in section 2.10.) In addition, (i) $f \ge g$ implies $\mathcal{T}f \ge \mathcal{T}g$ for any $f, g \in \mathcal{V}$, and (ii) $\mathcal{T}0 \in \mathcal{V}$. Under these conditions, $\mathcal{T}$ has a unique fixed point $v = \mathcal{T}v$ in $\mathcal{V}$, and $\lim_{n\to\infty}\mathcal{T}^n f = v$ for every $f \in \mathcal{V}$, by the weighted contraction mapping theorem (Boud, 1990). Since

$v \geq 0$ implies $\mathcal{T}v \geq 0$, the fixed point is non-negative. ■

Proposition 5 *In a pre-equilibrium, $v(h, z, e, u, s)$ is non-decreasing in h, and concave in h.*

Proof. To show that v is non-decreasing, consider $(\mathcal{T}f)(h, z, e, u, s)$, for any $f \in \mathcal{V}$. The maximum is attained by some $m, y, b \in \mathcal{C}$. An increase in h expands the set $\mathcal{C}$, so Tf is non-decreasing in h. This is true in particular for $\mathcal{T}v$, and then for v, since $v = \mathcal{T}v$.

Take any concave function $g(h, z, e, u, s) \in \mathcal{V}$. Fix z, e, u, s, let $h^0, h^1 \in \mathcal{H}(z)$ be chosen, and let m^i, y^i, b^i attain $(\mathcal{T}g)(h^i, z, e, u, s)$, $i \in \{0, 1\}$. Define $c^i = e[(1-\kappa)+\eta(z, e, u, s)(m^i/z-(1-\kappa)y^i/z)]$, and $h'^i = h^i - m^i + y^i + b^i(1-Q(s)) + \chi(s')z'$, $i \in \{0, 1\}$. For $0 \leq \theta \leq 1$, define $h^\theta \equiv \theta h^0 + (1-\theta)h^1$, $(m^\theta, y^\theta, b^\theta) \equiv (\theta m^0 + (1-\theta)m^1, \ldots, \ldots)$, $c^\theta \equiv \theta c^0 + (1-\theta)c^1$, and $h'^\theta \equiv \theta h'^0 + (1-\theta)h'^1$. Note that $m^\theta, y^\theta, b^\theta$ are feasible at h^θ, $h'^\theta = h^\theta - m^\theta + y^\theta + b^\theta(1-Q(s)) + \chi(s')z'$, and that $h'^\theta \in \mathcal{H}(z)$. At h^θ, $\mathcal{T}g$ satisfies

$$
\begin{aligned}
(\mathcal{T}g)(h^\theta, z, e, u, s) &\geq u(c^\theta) + \beta \int_S g(h'^\theta, z', e', u', s')dF(s, s') \\
&\geq \theta(\mathcal{T}g)(h^0, z, e, u, s) + (1-\theta)(\mathcal{T}g)(h^1, z, e, u, s).
\end{aligned}
$$

Hence, $(\mathcal{T}g)(h, z, e, u, s)$ is concave in h for every $g \in \mathcal{V}$. Since functions that are concave in h form a Banach vector subspace of $\mathcal{V}$, the fixed point $v = \mathcal{T}v$ is concave in h. ■

The established concavity can be used to prove that:

Proposition 6 *Under (i)-(ii), if the value function v is attained by m in the interior of the feasible set at some $h, z, e, u, s \in \mathcal{X}$ for which $\eta(z, e, u, s) > 0$, then*

v is differentiable in h, and

$$\frac{\partial}{\partial h}v(h,z,e,u,s) = u'(c)\frac{e}{z}\eta(z,e,u,s) \tag{48}$$

Proof. Fix z, e, u, s, and let $f\colon \mathcal{R}_+ \to \mathcal{R}_+$ be defined by $f(A) \equiv (\mathcal{T}v)(A,z,e,u,s)$. Let $m(A)$, $y(A)$, and $b(A)$ attain $f(A)$.

Define $\tilde{u}(m) = u\left(\frac{e}{z}[\eta m + (1-\kappa)(1-\eta y]\right)$. With $\eta(z,e,u,s) > 0$, $\tilde{u}\,(m)$ is strictly concave in m. By proposition 5, $\beta \int_S v(h',z',e',u',s')dF(s,s')$ is concave in h', and hence in m, by (45). Therefore, the maximand in the definition of $(\mathcal{T}v)(A,z,e,u,s)$ is strictly concave in m, so $m(A)$ is unique, and varies continuously with A (Berge, 1963).

Let $h'(A) = A - m(A) + y(A) + b(A)(1-Q(s)) + \chi(s')z'$. For sufficiently small ϵ, $m(A)+\epsilon$ is feasible at $A+\epsilon$, and $m(A+\epsilon)-\epsilon$ is feasible at A. Using the definition of f,

$$\begin{aligned} f(A+\epsilon) &\geq \tilde{u}(m(A+\epsilon)) + \beta \int_S g(h'(A),z',e',u',s')dF(s,s'), \\ &= \tilde{u}(m(A+\epsilon)) - \tilde{u}(m(A)) + f(A). \end{aligned} \tag{49}$$

$$\begin{aligned} f(A) &\geq \tilde{u}(m(A+\epsilon)-\epsilon) + \beta \int_S g(h'(A+\epsilon),z',e',u',s')dF(s,s'), \\ &= \tilde{u}(m(A+\epsilon)-\epsilon) - \tilde{u}(m(A+\epsilon)) + f(A+\epsilon). \end{aligned} \tag{50}$$

Combining (49) and (50),

$$\tilde{u}(m(A+\epsilon)) - \tilde{u}(m(A)) \leq f(A+\epsilon) - f(A) \leq \tilde{u}(m(A+\epsilon)) - \tilde{u}(m(A+\epsilon)-\epsilon).$$

Dividing by ϵ, taking the limit $\epsilon \to 0$, using the continuity of $m(A)$, and the definition of $\tilde{u}(m)$, one has that $f'(A) = u'(c)\frac{e}{z}\eta(z,e,u,s)$. The partial derivative of $v(h,z,e,u,s)$ with respect to h is given by $f'(h)$, because $v = \mathcal{T}v$, which proves (48). ■

4 Constructing the Unique Equilibrium

This section shows by construction that under assumptions 4 and 5 there is only one equilibrium of the model. In that equilibrium, the value of money remains strictly positive.

4.1 Differentiability of the Value Function

The strict positivity of η is first imposed as hypothesis, in addition to conditions (i)-(ii) of definition 2. Under this combination of assumptions, the pre-equilibrium value function of a representative consumer must be differentiable in h, which is established by the three propositions that follow.

Proposition 7 *In a pre-equilibrium with $\eta(z,e,u,s) > 0$, the supply of money represented by m satisfies $m > 0$ for all z,e,u,s.*

Proof. A representative consumer holds $h = (1+\chi(s))z = (h(s)/\tau(s))z$ in beginning-of-period net worth. By (45), $h-m+y+b[1-Q(s)]$ strictly improves the maximand of (42). Suppose $m = 0$, which also necessitates $y = 0$ in equilibrium, by condition (ii) of definition 2.

If $Q(s) < 1$, a consumer would like to optimally set the demand for bonds b to the maximal level allowed by the budget constraint (11), in order to benefit from the positive interest rate, i.e., $b = \frac{1}{q(s)}\frac{h(s)}{\tau(s)}z \equiv b^d(s)$, where $Q(s) = q(s)$ has been used. This can only be consistent with condition (ii) of pre-equilibrium if the supply for bonds $b^s(s)$, under the assumed monetary policy, is sufficiently large. However, as seen from (19), it is instead true that $b^d(s) > b^s(s)$.

If $Q(s) = 1$, the consumer does not have a strict preference between saving in bonds or in money, since both result in the same value h of the resulting end-of-period net worth. Consider a strategy of increasing m and y by a small number $\epsilon > 0$, which is feasible since $h > 0$, and because $\bar{y}(z, e, u, s) > 0$, as seen from (41). This strategy leaves the end-of-period net worth unchanged, but increases consumption by $\kappa\frac{e}{z}\eta(z, e, u, s)\epsilon > 0$, as seen from (43), and hence increases utility. It follows that $m = 0$ cannot be optimal. ■

Proposition 8 *In a pre-equilibrium, a representative consumer chooses m strictly below the upper bound allowed by the set of feasible choices $\mathcal{C}(h, z, e, u, s)$, for all z, e, u, s.*

Proof. Define $a \equiv h - m - bQ(s)$, and $w \equiv h - m + y + b[1 - Q(s)]$, which are non-negative by (37), and (38). Since w is defined as in requirement (ii) of definition 2, it must satisfy $w > 0$ in a pre-equilibrium. Combining the two definitions, $m = h + Q(s)y - [1 - Q(s)]a - Q(s)w$. Since $Q(s) = q(s) > 0$, maximizing m requires setting $w = 0$, for any given values of y and a, which contradicts $w > 0$. Hence, the choice of m by a representative consumer can never attain the upper bound permitted by budget feasibility. ■

Proposition 9 *In a pre-equilibrium with $\eta(z, e, u, s) > 0$, the value function v is differentiable in h, at $h = (1+\chi(s))z$, for all z, e, u, s.*

Proof. By propositions (7) and (8), the value function v of a representative consumer (for whom $h = [1+\chi(s)]z$, by definition) is attained by m in the interior of the feasible set. For each z, e, u, s, since $\eta(z, e, u, s) > 0$, the conditions of proposition (6) are satisfied at $[1+\chi(s)]z, z, e, u, s$, and hence the value function is differentiable in h at $h = [1+\chi(s)]z$. ■

By equation (48) of proposition 6, the partial derivative is

$$\nu \equiv \frac{\partial}{\partial h} v(h, z, e, u, s)|_{h=[1+\chi(s)]z} = u'(c)\frac{e}{z}\eta(z, e, u, s) \equiv u'(c)(1/P), \tag{51}$$

where c is optimal consumption of a representative consumer. In what follows, ν' will denote the partial derivative evaluated at next period's realizations of the state variables.

4.2 Pre-equilibrium with Positive Value of Money

The established differentiability of the value function can be used to explore the implications of $\eta(z, e, u, s) > 0$ in a pre-equilibrium. The problem (42) can be studied using the Lagrangian

$$\begin{aligned}\mathcal{L} \equiv\ & u\left(e\left[(1-\kappa) + \eta(z, e, u, s)\left(m/z - (1-\kappa)y/z\right)\right]\right) \\ & + \beta \int_S v\left((h - m + y + b[1 - Q(s)] + \chi(s')z', z', e', u', s'\right) dF(s, s') \\ & + \mu[h - m - bQ(s)] + \phi(\bar{y} - y),\end{aligned} \tag{52}$$

where μ and ϕ are non-negative multipliers, and $h = [1+\chi(s)]z$. The first-order necessary conditions associated with m are

$$\nu - \beta \int_S \nu' dF(s, s') - \mu = 0 \tag{53}$$

$$h - m - bQ(s) \geq 0,\ \mu \geq 0,\ \mu[h - m - bQ(s)] = 0, \tag{54}$$

The first-order necessary conditions associated with optimal choice of y are

$$(1-\kappa)\nu - \beta \int_S \nu' dF(s, s') + \phi = 0 \tag{55}$$

$$\bar{y} - y \geq 0,\ \phi \geq 0,\ \phi(\bar{y} - y) = 0, \tag{56}$$

and the first-order necessary condition for the choice of b is

$$\mu - \beta \int_S \nu' dF(s,s') \frac{1 - Q(s)}{Q(s)} = 0. \tag{57}$$

Combining (53) and (57),

$$\beta \int_S \frac{\nu'}{\nu} dF(s,s') = Q(s), \tag{58}$$

according to which the consumer invests in bonds such that the expected discounted nominal marginal rate of intertemporal substitution equals the nominal market discount factor.[24]

Using (58) in (57), one finds that $\mu/\nu = 1 - Q(s)$. Then, the complementary slackness condition of (54) can be written as

$$[1 - Q(s)][h - m - bQ(s)] = 0. \tag{59}$$

Intuitively, consumers do not save money 'under the bed' at a positive market interest rate.

Dividing (55) by ν, and using (58), results in $\phi/\nu = \kappa + Q(s) - 1$, which is positive under assumption 5 on monetary policy. Hence, the complementary slackness condition of (56) implies

$$y = \bar{y}(z,e,u,s). \tag{60}$$

Intuitively, under a monetary policy that guarantees sufficiently low interest rates in relation to the degree of inefficiency characterizing consumption of own endowment, consumers always prefer to sell all endowment in the market, in order to spend the proceeds later.

[24]While this condition is standard, it only holds as consequence of assumptions 4 and 5. The same comment applies to other equilibrium conditions.

Condition (60) is not sufficient to determine which of the two constraints underlying the joint constraint (41) actually binds. If (39) does not bind, then it must be true that $y = \bar{\bar{y}}z$, by (40). This is inconsistent with condition (ii) of definition 2, since it would mean that the demand for money y attains the technical upper bound, which is not feasible, as described in subsection 2.8. Intuitively, the value of money in a competitive equilibrium must be sufficiently high to allow consumers to sell all endowment brought to the market for goods.

This result can be used to discard all price functions that do not satisfy $\eta(z, e, u, s) \geq 1/\bar{\bar{y}}$, and interpret (60) as $y = z/\eta(z, e, u, s)$, which is the same as $y(1/P) = e$. Since all endowment is sold at the market price, no endowment is consumed inefficiently, implying $c = e$. Hence,

$$\nu = u'(e)(1/P) = u'(e)\frac{e}{z}\eta(z, e, u, s). \tag{61}$$

Also, by the Keynes law,

$$m(1/P) = e, \tag{62}$$

which is a version of the equation of exchange (Fisher, 1911).

The pre-equilibrium price function η can be found by considering the dynamics of end-of-period net worth of a representative consumer,[25]

$$W_t + G_t = M_t(1 - Q_t) + Q_t W_{t+1}. \tag{63}$$

Using (58) to expand the discount factor Q_t in front of W_{t+1}, and multiplying by $(1/P)_t$,

$$W_t(1/P)_t = M_t(1/P)_t(1 - Q_t) - G_t(1/P)_t + \beta E_t\left[\frac{\nu_{t+1}}{\nu_t}\frac{(1/P)_t}{(1/P)_{t+1}}W_{t+1}(1/P)_{t+1}\right], \tag{64}$$

where ν_t denotes the partial derivative (61), evaluated at state variables realized at t. Also, it is true by (16) that

$$W_t(1/P)_t = W_t\frac{e_t}{Z_t}\eta(Z_t, e_t, u_t, s_t) = e_t\eta(Z_t, e_t, u_t, s_t).$$

Substituting this in (64), and applying (61), and (62), allows to interpret (64) as a necessary condition for the price function η in a pre-equilibrium characterized by $\eta > 0$. Restating this condition in the functional form (for generic values of z, e, s) gives

$$\eta(z, e, u, s) = 1 - Q(s) - g(s)\frac{\eta(z, e, u, s)}{\tau(s)} + \beta\int_S \frac{u'(e')e'}{u'(e)e}\eta(z', e', u', s')dF(s, s'), \tag{65}$$

where the laws of motion for z' and e' are given by (46), and (47), respectively.

[25]This can be derived as follows:

$$\begin{aligned}
W_t + G_t = H_t &\equiv M_t + B_tQ_t + (H_t - M_t - B_tQ_t) \\
&= M_t + B_tQ_t + (H_t - M_t - B_tQ_t) - (H_t - M_t - B_tQ_t)(1 - Q_t) \\
&= M_t(1 - Q_t) + Q_t[M_t + B_t + (H_t - M_t - B_tQ_t)] \\
&= M_t(1 - Q_t) + Q_t[Y_t + B_t + (H_t - M_t - B_tQ_t)] \\
&= M_t(1 - Q_t) + Q_tW_{t+1}.
\end{aligned}$$

The second line subtracts a term that is zero in equilibrium, by the optimality condition (59). The fourth line uses the Keynes law $M_t = Y_t$, and the last line applies (12), the definition of W_{t+1}.

Condition (65) can be compared with the definition of function $\tau(s)$, re-stated for convenience,

$$\tau(s) = 1 - q(s) - g(s) + \beta \int_S \frac{u'(e')e'}{u'(e)e} \tau(s') dF(s, s'). \tag{66}$$

Subtracting (66) from (65), one can define a new function $x(z, e, u, s) \equiv \eta(z, e, u, s) - \tau(s)$, which must be bounded, and jointly continuous. Proceeding in this way, using $Q(s) = q(s)$, and solving for $x(s)$ yields

$$x(z, e, u, s) = \frac{\tau(s)}{\tau(s) + g(s)} \beta \int_S \frac{u'(e')e'}{u'(e)e} x(z', e', u', s') dF(s, s'). \tag{67}$$

Define operator $\mathcal{Y}$ such that (67) is equivalent to $\mathcal{Y}x = x$. This operator maps the space of bounded, jointly continuous functions in variables z, e, u, s onto itself. Under assumptions 4 and 5 on monetary policy, it is true that $g(s) \geq 0$ and $\tau(s) > 0$ (as shown in subsection 2.6), so the ratio in front of β is in $(0, 1]$. Under the assumed CRRA utility, and assumption 3, this is enough to establish that $\mathcal{Y}$ is a contraction mapping, and hence there is exactly one solution to (67). Since $x(z, e, u, s) = 0$ is a solution, it must be the only solution. Hence, the unique price function consistent with pre-equilibrium with $\eta > 0$ is

$$\eta(z, e, u, s) = \tau(s), \text{ for all } z, e, u, s. \tag{68}$$

To complete the construction of the pre-equilibrium with $\eta > 0$, one must compute the law of motion for the aggregate state variable Z_t. Consider (62), written as $M_t(1/P)_t = e_t$. Using (26), and (68), this is equivalent to

$$\frac{H_t}{M_t} = h(s_t), \tag{69}$$

where $h(s) \equiv \tau(s) + g(s)$, as defined in subsection 2.6. Write the necessary condition (59) in the form

$$B_t(1 - Q_t) = \frac{1 - Q_t}{Q_t}(H_t - M_t),$$

and substitute into the law of motion (15). Using (69), and the definition of $\tau(s)$, one finds that the unique model of the form (29) consistent with the actual evolution of Z_t is

$$\frac{z'}{z} = \frac{1}{q(s)}\beta \int_S \frac{u'(e')e'}{u'(e)e}\frac{\tau(s')}{\tau(s)}dF(s, s') \equiv \tilde{\theta}(s), \tag{70}$$

and one can identify $\theta(z, e, u, s) = z\tilde{\theta}(s)$. Under the assumed monetary policy, the growth rate in Z_t does not depend on state variables other than s_t.

With the unique set of functions η, Q, θ consistent with pre-equilibrium with $\eta > 0$, proposition 4 guarantees that there is exactly one corresponding value function v. Hence, the constructed pre-equilibrium is unique.

4.3 Uniqueness of Equilibrium

The previous subsections show that under assumptions 4 and 5 there exists a unique pre-equilibrium in which the value of money remains strictly positive. According to condition (iii) of definition 2, any equilibrium price function η must then be strictly positive. Since every equilibrium is a pre-equilibrium, there can be at most one equilibrium with $\eta > 0$. Since the pre-equilibrium of the previous section satisfies condition (iii), it is an equilibrium, and hence there exists only one equilibrium with strictly positive value of money.

One must also conclude that there exists a unique equilibrium with valued money for every specification of monetary policy consistent with assumptions 4 and 5, in which case the fundamental value of money is completely characterized by condition (68). Intuitively, the role of a *reasonable* monetary policy is to provide consumers with the knowledge that a monetary equilibrium *exists*, in which case money is endogenously accepted in exchange by individual decisions of rational competitive users.

5 Discussion

5.1 Equilibrium Selection Mechanism

Condition (iii) of definition 2 is technically an equilibrium selection mechanism, extending the list of two requirements in Lucas (1978, p. 1432). However, it rests on the principle of individual rationality, and hence is very well motivated.

Since conditions (i)-(ii) are satisfied in the pre-equilibrium with valued money of section 4, rational and fully informed consumers must see no reason to reject this outcome a priori, before the market for goods opens. If monetary policy is designed to satisfy assumptions 4 and 5, it must be common knowledge that there exists a (unique) pre-equilibrium with a positive value of money, at every given realization of state variables. Then, only if investors are willing to agree that the value of money is indeed zero upon observing this outcome, the alternative equilibrium could materialize in which money would be worthless. But this extreme form of learning from the price would arguably contradict individual rationality, since it costs nothing (on the margin) to disagree with the market value of zero, especially under the common knowledge that the pre-equilibrium with valued money *could* materialize. Hence, a strict preference for acquiring *more* money at the margin can safely be assumed as credible off-equilibrium strategy.

5.1.1 Behavioral Aspect of Condition (iii)

This section shows that the non-zero market value of money is supported in a pre-equilibrium precisely when a representative consumer individually *decides* to assign strictly positive marginal valuation to her beginning-of-period net worth. Hence, the generalized definition of equilibrium can be interpreted as imposing this decision automatically, conditional on a *reasonable* monetary policy.

By equation (51), a pre-equilibrium value function $v(h, z, e, u, s)$ must be strictly increasing in h, at the level of beginning-of-period net worth $h = [1 + \chi(s)]z$, for each z, e, u, s, whenever $\eta > 0$. To prove the converse, let η, Q, θ, v be functions specified as in (a)-(b) of definition 2, and satisfying conditions (i)-(ii). Consider a behavioral postulate that the value function used by a representative consumer is strictly increasing in h, i.e., that consumers strictly prefer to hold more net worth, rather than less, at the opening of the market.

Hypothesis 1 *The value function $v(h, z, e, u, s)$, used by a representative consumer, is strictly increasing in h, at $h = [1 + \chi(s)]z$, for all z, e, u, s.*

Proposition 10 *Under Hypothesis 1, $\eta(z, e, u, s) > 0$, for all z, e, u, s, in a pre-equilibrium.*

Proof. Suppose $\eta(z, e, u, s) = 0$ for some z, e, u, s. Then, a consumer with value function v that is strictly increasing in h finds it optimal to set $y = \bar{y}(z, e, u, s) > 0$ and $m = 0$ to maximize the right-hand side of (42). This true in particular at $h = [1 + \chi(s)]z$, which results in a violation of condition (ii) of definition 2, so it is necessary that $\eta(z, e, u, s) > 0$, for all z, e, u, s. ■

Hence, the assumption that money is valuable in a pre-equilibrium is *equivalent* to the behavioral postulate of Hypothesis 1, so condition (iii) of definition 2 could be alternatively formulated as: (iii) *For each z, e, u, s, if there are functions η^p, Q^p, θ^p*

specified as in (a), for which a function v^p *specified as in (b) satisfies (i)-(ii), and if* $\eta^p(z,e,u,s) > 0$*, then* $v(h,z,e,u,s)$ *is strictly increasing in* h*, at* $h = [1+\chi(s)]z$. This would highlight the behavioral aspect of the equilibrium selection mechanism, and the strict positivity of equilibrium value of money would follow from proposition 10.

5.2 Ruling Out Speculative Price Dynamics

Economic intuition behind condition (67) can be developed as follows. Comparing the equilibrium value of money implied by (68) with equation (62), one obtains $\tau(s_t) = Z_t/M_t$. Using this in (18) gives $g(s_t) = G_t/M_t$, and then $h(s_t) \equiv \tau(s_t) + g(s_t) = H_t/M_t$. Hence, the function $h(s)$ reflects the inverse marginal propensity to consume out of H_t, and the function $x(z,e,s)$ can be identified with deviation of H_t/M_t from $h(s_t)$,

$$x(Z_t, e_t, s_t) = \left(\frac{Z_t}{M_t} + \frac{G_t}{M_t}\right) - \left(\tau(s_t) + \frac{G_t}{M_t}\right) = \frac{H_t}{M_t} - h(s_t).$$

The content of condition (67) is that this difference is dynamically unstable. At a given H_t, if a representative consumer decides to choose M_t according to a time-invariant rule with $H_t/M_t \geq h(s_t)$, and $H_t/M_t > h(s_t)$ with positive probability, then the ratio H_t/M_t must eventually exceed any positive bound. By $M_t(1/P)_t = e_t$, the real value of H_t must then exceed any bound relative to e_t, violating individual rationality. Similarly, if a representative consumer decides to choose M_t according to a time-invariant rule with $H_t/M_t \leq h(s_t)$, and $H_t/M_t < h(s_t)$ with a positive probability, then the aggregate ratio H_t/M_t must eventually turn negative, since $x(Z_t, e_t, s_t)$ must exceed any negative bound. But this cannot happen without violating the non-negativity of H_t, imposed by the supplier of money on nominal net worth.

By assumption 5, equilibrium must be characterized by low interest rates in order to induce consumers to sell all endowment in the market for goods, such that $Y_t(1/P)_t = e_t$. By the Keynes law, this is equivalent to $M_t(1/P)_t = e_t$, which can be written as

$$(1/P)_t = \frac{e_t}{H_t}\frac{H_t}{M_t}.$$

At the same time, the fundamental value of money can be defined as

$$(1/P)^*_t \equiv \frac{e_t}{Z_t}\tau(s_t) = \frac{e_t}{H_t}h(s_t).$$

Subtracting this from the previous condition,

$$(1/P)_t - (1/P)^*_t = \frac{e_t}{H_t}\left(\frac{H_t}{M_t} - h(s_t)\right),$$

so the market value of money differs from the fundamental value precisely when the aggregate 'inverse marginal propensity to consume' H_t/M_t differs from $h(s_t)$. Since this is never optimal, as discussed below equation (68), equilibrium value of money never deviates from $(1/P)^*_t$.

5.3 Relation to Existing Literature

The contribution of this paper is to argue that the concept of equilibrium of Lucas (1972, 1978) can be successfully applied to a monetary economy with explicit existence of frictions, and realistically specified monetary policy, without imposing the cash-in-advance (CIA) constraint.[26]

The conclusion that the value of money is uniquely determined should not be taken as justification for selecting the unique bounded solution for inflation in standard

[26] Hence, the paper follows a different path than the subsequent work of R. Lucas (see Sargent, 2015).

new-Keynesian models (Woodford, 2003; Galí, 2015).[27] In these models, nominal net worth is undefined, while in the present paper it plays the key role of nominal scale variable, allowing the supplier of money to engineer essentially any path of inflation with no regard to the Taylor principle. This can be accomplished by the second dimension of monetary policy, corresponding to a continuous helicopter drop of new net worth.[28]

This work complements the fiscal theory of the price level (FTPL), in which the supplier of money (often simply called government) issues fiat money and interest-bearing debt as liabilities (Sargent and Wallace, 1981; Leeper, 1991; Sims, 1994; Woodford, 1995). The FTPL interprets q_t, g_t as determinants of government surpluses, interpreted as seigniorage and taxes, respectively. Given a pre-determined measure of nominal liabilities, the price level is defined as the unique conversion factor that makes their real value equal to the present value of the surpluses. The assumption that the government is actually able to issue valuable nominal liabilities, and commit to a given path or real surpluses, is known as non-Ricardian fiscal policy (Woodford, 1995), and has been subject to much controversy (Kocherlakota and Phelan, 1999; Christiano and Fitzgerald, 2000; Buiter, 2002; Niepelt, 2004). The present study provides the implementation theory missing in the bare formulation of the FTPL, confirming that the supplier of money can indeed follow a non-Ricardian policy. This, however, raises the question of the validity of the usual interpretation of the FTPL, since the supplier of money is by construction a generic *monetary* authority, while fiscal government is not even present in the

[27]The usual argument for this relies on the so-called Taylor principle (Taylor, 1993; Clarida et al., 2000), according to which the monetary authority must commit to raising interest rate sufficiently strongly in response to inflation. This has been forcefully criticized by Cochrane (2011, 2018) as lacking economic justification, and empirical support.

[28]The idea of helicopter drop of *money*, originally due to Friedman (1969), has recently been under renewed interest (Bernanke, 2002, 2003; Buiter, 2014; Benigno and Nisticò, 2020; Galí, 2020).

model. Related, the interpretation assigned to q_t, g_t by the FTPL reverses their economic meaning. The nominal interest rate implied by the discount factor q_t is best seen as tax on the receipts from the market for goods, while g_t is the flow of seigniorage, defined as real revenue of those agents who receive the transfers first.

On the surface, the paper supports the practice of imposing the CIA constraint, interpreted as equilibrium condition in the market for goods. This is only valid under a *reasonable* design of monetary policy, so imposing the CIA constraint can seriously misrepresent individual incentives to use money. Perhaps more importantly, the present model offers a way to clearly distinguish money from other securities, which is missing in models relying on an ad-hoc specifications.

Much of modern thinking about money is rooted in the so-called portfolio tradition (Hicks, 1935; Keynes, 1930; Tobin, 1958; Friedman, 1956). It is interesting to note that equilibrium selection mechanism proposed here would not be operational under that interpretation, which abstracts from the market for goods, and treats money as purely speculative asset. In contrast, consumers in the present model optimally plan to spend money as soon as possible, also when the nominal rate of interest is zero, which appears consistent with empirical evidence, and the anecdotal fact that rational consumers treat money as *hot potato*, rather than investment asset.[29]

By successfully integrating a theory of money based on reduced-form frictions with asset pricing, this paper finds middle ground between imposing the CIA constraint, and starting from more specific assumptions about the environment, with prominent examples of OLG models (Samuelson, 1958; Grandmont and Laroque, 1973; Wallace, 1980), turnpike models (Townsend, 1980), models of self-insurance

[29] By highlighting these differences, the paper contributes to the literature rejecting the portfolio interpretation for lacking both internal consistency, and economic intuition (for a survey, see Kohn, 1988).

against idiosyncratic risks (Bewley, 1977; Aiyagari, 1994), or models in which the technology of exchange is restricted to infrequent bilateral meetings (Kiyotaki and Wright, 1989; Trejos and Wright, 1995; Shi, 1995; Lagos and Wright, 2005). Many of these theories rely on specific assumptions which are often overly restrictive, and not always necessary (or even sufficient) to address the question at hand. While some admit enough tractability, others do not. For example, the approach based on bilateral meetings is plagued with analytical difficulties arising from the need to keep track of random changes in the distribution of money across agents, which can only be resolved by imposing additional stylized assumptions (Howitt, 2003). Moreover, following the contribution of Kocherlakota (1998), it is now clear that all explicit monetary environments must share the properties that (1) there exists a fundamental motive for bilateral exchange, and (2) there is no publicly available record (*memory*) of individual actions, which in particular implies that private promises are not credible. The present paper imposes these properties more directly, which can still be seen as minimalist's way of *looking frictions in the face* (Hicks, 1935).

References

Aiyagari, Rao S. (1994), "Uninsured idiosyncratic risk and aggregate saving." *Quarterly Journal of Economics*, 109, 659–684.

Alchian, Armen A. (1977), "Why money?" *Journal of Money, Credit and Banking*, 9, 133–140.

Altug, Sumru and Pamela Labadie (2008), *Asset pricing for dynamic economies.* Cambridge University Press.

Bagehot, Walter (1873), *Lombard Street: a Description of the Money Market.* Henry S. King & Company 65 Cornhill & 12 Paternoster Row.

Benigno, Pierpaolo and Salvatore Nisticò (2020), "The economics of helicopter money." Technical report, CEPR Discussion Paper No. 14555.

Berge, Claude (1963), *Topological spaces.* Oliver and Boyd.

Bernanke, Ben (2002), "Deflation: Making sure it does not happen here." *Remarks by Governor Ben Bernanke Before the National Economists Club Washington, DC.*

Bernanke, Ben (2003), "Some thoughts on monetary policy in Japan." *Remarks before the Japan Society of Monetary Economics, Tokyo.*

Bewley, Truman (1977), "The permanent income hypothesis: A theoretical formulation." *Journal of Economic Theory*, 16, 252–292.

Blanchard, Olivier J. and Stanley Fischer (1989), *Lectures on Macroeconomics.* MIT Press.

Boud, John H. III (1990), "Recursive utility and the ramsey problem." *Journal of Economic Theory*, 50, 326–345.

Brock, William A. (1974), "Money and growth: The case of long run perfect foresight." *International Economic Review*, 750–777.

Brunner, Karl and Allan H. Meltzer (1971), "The uses of money: Money in the theory of an exchange economy." *American Economic Review*, 61, 784–805.

Buiter, Willem H. (2002), "The fiscal theory of the price level: A critique." *Economic Journal*, 112, 459–480.

Buiter, Willem H. (2014), "The simple analytics of helicopter money: Why it works – always." *Economics*, 8.

Christiano, Lawrence J. and Terry J. Fitzgerald (2000), "Understanding the fiscal theory of the price level." *Economic Review*, 36, 2–37.

Clarida, Richard, Jordi Galí, and Mark Gertler (2000), "Monetary policy rules and macroeconomic stability: Evidence and some theory." *Quarterly Journal of Economics*, 115, 147–180.

Cochrane, John H. (2011), "Determinacy and identification with taylor rules." *Journal of Political Economy*, 119, 565–615.

Cochrane, John H. (2018), "Michelson-Morley, Fisher, and Occam: The radical implications of stable quiet inflation at the zero bound." *NBER Macroeconomics Annual*, 32, 113–226.

Fisher, Irving (1911), *The purchasing power of money*. Macmillan. Assisted by H. G. Brown.

Friedman, Milton (1956), "The quantity theory of money: A restatement." *Studies in the quantity theory of money*, 5, 3–31.

Friedman, Milton (1969), "The optimum quantity of money." In *The optimum quantity of money, and other essays*, Adline Publishing Company.

Galí, Jordi (2015), *Monetary policy, inflation, and the business cycle: An introduction to the new Keynesian framework and its applications*. Princeton University Press.

Galí, Jordi (2020), "The effects of a money-financed fiscal stimulus." *Journal of Monetary Economics (forthcoming)*.

Grandmont, Jean-Michel and Guy Laroque (1973), "Money in the pure consumption loan model." *Journal of Economic Theory*, 6, 382–395.

Hahn, Frank H. (1965), *On some problems of proving the existence of an equilibrium in a monetary economy*. Macmillan.

Harrison, J. Michael and David M. Kreps (1979), "Martingales and arbitrage in multiperiod securities markets." *Journal of Economic Theory*, 20, 381–408.

Hellwig, Martin F. (1993), "The challenge of monetary theory." *European Economic Review*, 37, 215–242.

Hicks, John R. (1935), "A suggestion for simplifying the theory of money." *Economica*, 2, 1–19.

Howitt, Peter (2003), "Comment on "Search, Money, and Capital: A Neoclassical Dichotomy" by S. Borağan Aruoba and Randall Wright." *Journal of Money, Credit and Banking*, 1107–1110.

Howitt, Peter (2005), "Beyond search: Fiat money in organized exchange." *International Economic Review*, 46, 405–429.

Jevons, William S. (1875), *Money and mechanism of exchange*. D. Appleton & Co.

Keynes, John Maynard (1930), *A Treatise on money (2 Vols)*. Macmillan.

Kiyotaki, Nobuhiro and Randall Wright (1989), "On money as a medium of exchange." *Journal of Political Economy*, 97, 927–954.

Kocherlakota, Narayana R. (1998), "Money is memory." *Journal of Economic Theory*, 81, 232–251.

Kocherlakota, Narayana R. and Christopher Phelan (1999), "Explaining the fiscal theory of the price level." *Federal Reserve Bank of Minneapolis Quarterly Review*, 23, 14–23.

Kohn, Meir (1988), "The finance constraint theory of money: A progress report." *The Jerome Levy Economics Institute Working Paper*.

Lagos, Ricardo and Randall Wright (2005), "A unified framework for monetary theory and policy analysis." *Journal of Political Economy*, 113, 463–484.

Leeper, Eric M. (1991), "Equilibria under 'active' and 'passive' monetary and fiscal policies." *Journal of Monetary Economics*, 27, 129–147.

Lucas, Robert E. (1972), "Expectations and the neutrality of money." *Journal of Economic Theory*, 4, 103–124.

Lucas, Robert E. (1978), "Asset prices in an exchange economy." *Econometrica*, 1429–1445.

Matsuyama, Kiminori (1991), "Endogenous price fluctuations in an optimizing model of a monetary economy." *Econometrica*, 1617–1631.

Menger, Karl (1892), "On the origins of money." *Economic Journal*, 2, 239–255.

Modigliani, Franco and Merton H. Miller (1958), "The cost of capital, corporation finance and the theory of investment." *American Economic Review*, 48, 261–297.

Niepelt, Dirk (2004), "The fiscal myth of the price level." *Quarterly Journal of Economics*, 119, 277–300.

Obstfeld, Maurice and Kenneth Rogoff (1983), "Speculative hyperinflations in maximizing models: Can we rule them out?" *Journal of Political Economy*, 91, 675–687.

Patinkin, Don (1965), *Money, interest, and prices; an integration of monetary and value theory*.

Robertson, Dennis H. (1933), "Saving and hoarding." *Economic Journal*, 43, 399–413.

Samuelson, Paul A. (1958), "An exact consumption-loan model of interest with or without the social contrivance of money." *Journal of Political Economy*, 66, 467–482.

Sargent, Thomas J. (2015), "Robert E. Lucas Jr.'s collected papers on monetary theory." *Journal of Economic Literature*, 53, 43–64.

Sargent, Thomas J. and Neil Wallace (1981), "Some unpleasant monetarist arithmetic." *Federal reserve bank of minneapolis quarterly review*, 5, 1–17.

Shi, Shouyong (1995), "Money and prices: A model of search and bargaining." *Journal of Economic Theory*, 67, 467–496.

Sims, Christopher A. (1994), "A simple model for study of the determination of the price level and the interaction of monetary and fiscal policy." *Economic Theory*, 4, 381–399.

Taylor, John B. (1993), "Discretion versus policy rules in practice." In *Carnegie-Rochester Conference Series on Public Policy*, volume 39, 195–214.

Tobin, James (1958), "Liquidity preference as behavior towards risk." *Review of Economic Studies*, 25, 65–86.

Townsend, Robert M. (1980), "Models of money with spatially separated agents." In *Models of Monetary Economies* (Kareken J. H. and Wallace N., eds.), 265–303, Federal Reserve Bank of Minneapolis.

Trejos, Alberto and Randall Wright (1995), "Search, bargaining, money, and prices." *Journal of Political Economy*, 103, 118–141.

Wallace, Neil (1980), "The overlapping generations model of fiat money." In *Models of Monetary Economies* (Kareken J. H. and Wallace N., eds.), 49–82, Federal Reserve Bank of Minneapolis.

Walras, Léon (1900), *Elements of pure economics*. Homewood. Translated from French by William Jaffé (1954).

Woodford, Michael (1995), "Price-level determinacy without control of a monetary aggregate." In *Carnegie-Rochester conference series on public policy*, volume 43, 1–46.

Woodford, Michael (2003), *Interest and prices*. Princeton University Press.