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Abstract This paper studies the valuation of fiat money in an endowment economy with specialization, costly barter, and imperfect enforcement of promises using a generalized asset-pricing framework of Lucas (1978). The environment features symmetric, competitive, and indefinitely-lived households and allows for a finite number of Markovian state variables. Money is part of a payment system allowing transfers of intrinsically useless net worth between non-negative accounts subject to a fixed time lag. Monetary policy of the issuing authority is designed in terms of sequences of nominal interest rates and transfers of new net worth. Households do not have to use money but optimally demand it from the market for goods at uniquely determined positive value if monetary policy satisfies certain conditions defining a *responsible* policy. The paper offers a general way to construct equilibria under such policies without relying on cash-in-advance constraints or other modeling shortcuts.

Keywords: Fiat money, electronic money, central bank digital currency, payment system, monetary policy, price level, inflation, multiple equilibria, sunspots, helicopter drop, universal basic income.

JEL Classification Numbers: E10, E31, E41, E51, E52, E58, G12, G21.

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1 Introduction

This paper studies the valuation of fiat money in an endowment economy with specialization, costly barter, and imperfect enforcement of private promises. The environment features symmetric, competitive, and indefinitely-lived households and allows for a finite number of Markovian state variables. Money is part of a payment system allowing transfers of intrinsically useless net worth between non-negative accounts subject to a fixed time lag. Monetary policy of the issuing authority is designed in terms of sequences of nominal interest rates and helicopter drops of new net worth in relation to nominal income.[1] Implementation of monetary policy involves paying the transfers and setting the aggregate supply of non-transferable risk-free bonds. The households are not forced to use money, but optimally decide to demand it from the market for goods if only monetary policy satisfies certain conditions defining a *responsible* policy, under which the interest rate remains low and existing net worth is not diluted too quickly. The analysis is carried under the assumption that the households are rational and have full information about the environment and monetary policy.[2]

Methodologically, the paper generalizes the asset-pricing framework of Lucas (1978) to make it applicable to fiat money without relying on modeling shortcuts such as money-in-utility-function or cash-in-advance constraints. This way, the model is able to credibly explain both the function of money in actual economies, and the reasons for which money is valued, since the incentives of its users are not distorted. The main technical contribution is an equilibrium selection mechanism able to rule out the hypothetical possibility that money might become worthless

[1]This interpretation depends on equilibrium. In the paper, monetary policy is formulated in abstract terms before it is established that money is valued.

[2]The authority might be *called* government, but the author prefers to think of a centrally coordinated banking system issuing money used in electronic payments (an idealization), or a monopoly central bank issuing a digital currency under the 100%-reserve requirement. Accordingly, the bonds are not issued by a fiscal authority, which is not part of the model. The latter dimension of monetary policy can be interpreted either as a flow of dividends from the authority, or (with symmetric households) money-financed universal basic income.

due to self-fulfilling expectations, independently of monetary policy. The mechanism is motivated economically by the postulate that rational households should always invest in more money at the margin if monetary policy is responsible, and yet the market value of money is zero, treating such situation as an arbitrage opportunity. The generalized definition of equilibrium rules out this form of arbitrage and reduces to the standard definition in models without frictions, or when money could not be valuable for other reasons, for example, due to a poorly designed monetary policy. The new definition selects a unique equilibrium whenever monetary policy is responsible, and the paper offers a general way to construct such equilibria. Since money is valued in the unique equilibrium although it is dominated as a store of value, the paper solves the long-standing theoretical puzzle known as the Hahn problem (Hahn, 1965, 1983; Hellwig, 1993), and successfully integrates the theory of money with asset pricing.[3]

The model allows for stochastic endowment growth and random shocks to monetary policy. Under a responsible policy, all trade is intermediated by money while barter and Arrow-Debreu forward markets are inactive. Competitive equilibria in the original sense of Lucas (1978), characterized by stable self-confirming expectations, are called pre-equilibria. Under a responsible policy, there is exactly one pre-equilibrium with a strictly positive value of money, which coincides with the unique equilibrium. The equilibrium value of money is determined by nominal spending since at low interest rates all endowment is sold in the market. Optimal spending is determined by expectations of lifetime income, including transfers of new net worth from the authority. If the nominal interest rate is set low (high) relative to the rate of transfers, households optimally borrow from the authority

[3]The generalized definition of equilibrium is consistent with the principle of individual rationality of competitive households and *removes* a form of irrational behavior towards money implicitly imposed by the standard definition, which is sufficient (only) in the context of frictionless models. The equilibrium selection mechanism proposed here is quite different from either that assumed in the fiscal theory of the price level (FTPL) or new-Keynesian models (see the literature overview, section 5.4), with potentially important consequences for choosing between these theories by applying the Occam razor.

(save) and equilibrium inflation is relatively low (high) as future transfers are used to repay debts (future interest payments on bonds are used to increase spending). The uniqueness of equilibrium price level is the consequence of saddle-path dynamics characterizing the accumulation of net worth. If spending is too high, this must ultimately violate the non-negativity restriction imposed on the net worth by the authority and hence is unsustainable. If spending is too low, over-accumulation of net worth must occur, which is not optimal.[4]

The rest of the paper is organized as follows. Section 2 sets up the environment and formulates the model. Section 3 defines equilibrium and motivates the proposed equilibrium selection mechanism. Section 4 constructs the pre-equilibrium with positive value of money and proves that it coincides with the unique equilibrium. Section 5 further discusses the equilibrium selection mechanism, some properties of the equilibrium, and the existing literature. Section 6 concludes.

2 Model

2.1 Timing and Information

Time is divided into periods represented by half-open intervals $[t, t+1)$. Decisions are made at t, after observing information up to and including t. No new information arrives within $(t, t+1)$.

Assumption 1 *The only state variables are* $(Z_t, e_t, u_t, s_t) \in \mathcal{R}_+ \times \mathcal{R}_+ \times U \times S$, *where* $U = \mathcal{R}^{m-2}$, $m \geq 2$, *and* $S \subset \mathcal{R}^n$ *is compact.*[5] *There is a function* $F\colon S \times S \to [0,1]$ *such that* s_t *is a stationary ergodic Markov process with cumulative transition density* $F(s, s') = \Pr(s_{t+1} \leq s' | s_t = s)$, *and stationary distribution* Φ.

[4]Chaotic dynamics are ruled out as well. This logic is analogous to the mechanics of capital accumulation in the Ramsey model (for example, see Blanchard and Fischer, 1989, ch. 2). The saddle-path stability used to be imposed ad-hoc (Brock, 1974), but this practice was challenged (Obstfeld and Rogoff, 1983) and now is often considered invalid in the context of monetary models. The present paper argues that this conclusion is incorrect.

[5]$\mathcal{R}_+$ will denote the set of strictly positive real numbers.

There is a jointly continuous function $G\colon \mathcal{R}_+ \times \mathcal{R}_+ \times U \times S \times S \to \mathcal{R}_+ \times \mathcal{R}_+ \times U$ *such that*

$$(Z_{t+1}, e_{t+1}, u_{t+1}) = G(Z_t, e_t, u_t, s_t, s_{t+1}). \qquad (1)$$

This assumption guarantees that the environment is Markovian, and all information is represented by a finite number of state variables. The economic interpretation of the state variables is postponed to later.[6]

The next assumption, preceded by a definition, guarantees that expectations can be represented by continuous functions.[7]

Definition 1 *Consider the space* $\mathcal{F}$ *of functions* $f\colon \mathcal{D} \to \mathcal{R}$, *with* $\mathcal{D}$ *a Cartesian product of* k *subsets of the real line, including one copy of* $\mathcal{R}_+$, *with corresponding argument denoted by* e. *Fix* $\varphi(x_1, \ldots, e, \ldots, x_k) \in \mathcal{F}$ *as* $\varphi(e) = e^{1-\gamma}/(1-\gamma)$, $\gamma \in (0,1) \cup (1,\infty)$, *or* $\varphi(e) = \log e$. *For any function* $g \in \mathcal{F}$, *define the norm* $\|g\|_\varphi = \sup_{x\in\mathcal{D}} |g(x_1, \ldots, e, \ldots, x_k)/\varphi(e)|$. *Any function* $g \in \mathcal{Z}$ *with the property* $\|g\|_\varphi < \infty$ *will be referred to as* φ*-bounded.*[8]

Assumption 2 *For every* φ*-bounded and jointly continuous function* $f(x, e, s, s')$ *(with* x *a vector of state variables defined on a Cartesian product of subsets of the real line), the function* $(Tf)(x, e, s) \equiv \int_S f(x, e, s, s')dF(s, s')$ *is jointly continuous.*

2.2 Consumers and Preferences

There is a large finite number of symmetric, indefinitely-lived households receiving exogenous perishable endowment which can either be consumed or given to a member of another household. All variables are in per-household terms. Households

[6] This assumption can be seen as a restriction on the environment imposed by Nature. It is consistent with the assumption of a finite number of agents (see below), none of which being capable of generating information exceeding a fixed finite number of Markovian dimensions.

[7] Again, this can be seen as a property of the environment.

[8] Intuitively, a φ-bounded function does not grow (or fall) faster than $\varphi(e)$ in any direction of its domain, asymptotically. The definition and naming are standard (Altug and Labadie, 2008, ch. 8).

occupy distinct physical locations, and the distance between any two locations is normalized to one. A household is composed of a producer who stays at home, and a consumer who can travel, carry goods, and consume at any location.

Preferences at t are represented by

$$V_t = E_t \left\{ \sum_{s=0}^{\infty} \beta^s u(c_{t+s}) \right\}, \tag{2}$$

where $0 < \beta < 1$, $u(c) = c^{1-\gamma}/(1-\gamma)$ if $\gamma \in (0,1) \cup (1,\infty)$, or $u(c) = \log(c)$.

Endowment evolves according to

$$e_{t+1} = e_t \, \lambda(s_t, s_{t+1}), \tag{3}$$

where $\lambda\colon S \times S \to \mathcal{R}_+$ is a continuous function valued in a compact set of positive numbers containing 1 in the interior.

Assumption 3 *The function $\lambda(s,s')$ satisfies $w(s) \equiv \beta \int_S \lambda(s,s')^{1-\gamma} dF(s,s') < 1$, all $s \in S$.*

This assumption guarantees that the consumption-based value of the whole economy is finite. For future reference, note that it is possible to pick $\bar{w} < 1$ such that $\max_s w(s) < \bar{w}$.

2.3 Frictions

The following three assumptions summarize the properties of the environment which can be interpreted as representing specialization, costs of barter, and imperfect enforcement of private contracts, respectively.

Assumption 4 *Consuming own endowment is inefficient, and subject to a proportional waste of $\kappa_s \in (0,1]$. Consumption of endowment obtained from any other household is efficient.*

Assumption 5 *Carrying endowment involves a proportional waste of $\kappa_p \in (0,1]$*

of the carried good.

Assumption 6 *Households can offer private promises to members of other households, but a household accepting such promise loses $\kappa_e \in (0,1]$ of the discounted present value of the promised flow, under the household's consumption-based stochastic discount factor.*

The next assumption is only for convenience, since it allows to ignore barter and private promises in the analysis by making them too costly to ever be attempted.[9]

Assumption 7 *The cost parameters of assumptions (4)-(6) satisfy*

$$\kappa \equiv \min\{\kappa_s, \kappa_p, \kappa_e\} = \kappa_s.$$

2.4 Net Worth

There is a single generic institutional authority that does not consume but has well-defined incentives that can be represented by time-invariant rules. Actions of the authority are independent of any subset of households. The authority possesses sufficient power to set up all components of the environment introduced below and to secure its monopoly position.

Initially, the authority assigns a non-negative account to each household with net worth $W_0 > 0$, measured in units called dollars. At t, W_t is interpreted as end-of-period net worth carried from $t-1$, and is augmented by $G_t \geq 0$ to yield beginning-of-period net worth

$$H_t = W_t + G_t. \tag{4}$$

Accounts can be accessed from any location and used to make transfers to other households, using a pre-existing payment system controlled by the authority. A

[9] This does not affect any of the main conclusions of the paper.

transfer can immediately be verified by the recipient but requires a fixed time lag to be completed. For convenience, the frequency of the model is chosen to match that lag. Irrespective of whether households use their accounts, one can define

$$M_t \geq 0, \quad Y_t \geq 0 \tag{5}$$

as transfers to other households made at t, and transfers on the way from other households expected to arrive at the beginning of $t+1$, respectively.

A household can lend nominal funds to the authority, and the realized demand for lending is represented by B_t bonds of unit dollar face value (negative B_t represents borrowing), which can neither be transferred nor carried. The bonds are sold at discount Q_t, set by the authority, and are repaid at the end of the period.

The payment system only processes transfers satisfying

$$H_t \geq M_t + B_t Q_t, \tag{6}$$

which is referred to as the budget constraint.

The end-of-period net worth evolves according to

$$W_{t+1} = H_t - M_t + Y_t + B_t(1 - Q_t), \tag{7}$$

which implies

$$H_{t+1} = H_t - M_t + Y_t + B_t(1 - Q_t) + G_{t+1}. \tag{8}$$

If a household borrows from the authority, this is subject to

$$W_{t+1} \geq 0, \tag{9}$$

which defines $H_t - M_t - B_t Q_t + Y_t$ as maximal collateral.[10]

[10]Inequality (9) can be seen as implementing the classic collateral constraint of

In effect, it must be true in every period that

$$W_t \geq 0. \tag{10}$$

2.5 Markets and Prices

Consumers can freely travel across locations with portable accounts. Since it is technologically possible to exchange money transfers for endowment in a quid-pro-quo fashion, one can assume that there exists a competitive market price $(1/P)_t \geq 0$ of a dollar of transferred funds in terms of goods.[11]

Given this price Y_t is a choice variable restricted by

$$(1/P)_t Y_t \leq e_t. \tag{11}$$

The choices of M_t and Y_t imply that a household enjoys consumption of

$$c_t = (1/P)_t M_t + (1 - \kappa) \left[e_t - (1/P)_t Y_t \right]. \tag{12}$$

Since M_t and Y_t are the same in the aggregate, symmetry between the households requires

$$M_t = Y_t, \tag{13}$$

which will be called the Keynes law, and imposed as an equilibrium condition.[12]

Bagehot (1873), and justifies calling W_t net worth.

[11] If $(1/P)_t > 0$, one can define the price level $P_t = 1/(1/P)_t$.

[12] While this condition must hold identically (spending generates income), individual households believe that they can choose M_t and Y_t independently.

2.6 Monetary Policy

Let $Z_t > 0$ denote aggregate end-of-period net worth, defined recursively via

$$Z_{t+1} = Z_t + G_t + B_t (1 - Q_t), \tag{14}$$

Symmetry between the households requires

$$W_t = Z_t, \tag{15}$$

although the households consider them distinct.

Observation 1 *By (14), the value of Z_{t+1} is pre-determined at t.*

Monetary policy is designed in terms of two processes q_t, g_t. Since monetary policy must be defined independently of equilibrium, the processes q_t, g_t are abstract at this point, but in equilibrium will correspond to the nominal risk-free discount factor, and the rate of new net worth creation relative to nominal income, respectively. The authority chooses two continuous and bounded functions $q \colon \mathcal{S} \to (0, 1]$ and $g \colon \mathcal{S} \to [0, \infty)$, referred to as the design functions, and sets $q_t = q(s_t), g_t = g(s_t)$.

Assumption 8 *The design functions satisfy $1 - q(s) - g(s) \geq 0$, all $s \in \mathcal{S}$. There is $\mathcal{S}' \subset \mathcal{S}$ with $\phi(\mathcal{S}') > 1$, and $1 - q(s) - g(s) > 0$, all $s \in \mathcal{S}'$.*

Assumption 9 *The function $q(s)$ satisfies $q(s) > 1 - \kappa$, all $s \in \mathcal{S}$.*

A monetary policy obeying these two assumptions is called *responsible*. In equilibrium, these assumptions will guarantee that the nominal interest rate will remain sufficiently high relative to rate of new net worth creation, and sufficiently low compared to the inefficiency of consuming own endowment.[13]

Monetary policy is implemented as follows. First, given the design functions $q(s)$ and $g(s)$ satisfying assumptions 8 and 9, one constructs a new function $\tau \colon \mathcal{S} \to \mathcal{R}$.

[13]The nominal interest rate is related to the discount factor via $q = i/(1 + i)$. A responsible monetary policy allows $i = 0$ for finite (with probability one) times.

Let $\mathcal{B}$ be the set of bounded continuous functions on $\mathcal{S}$. For $f \in \mathcal{B}$, define operator $(\mathcal{A}f)(s) = 1 - q(s) - g(s) + \beta \int_S \lambda(s,s')^{1-\gamma} f(s') dF(s,s')$. Under assumption 3, $\mathcal{A}$ is a contraction mapping, so there is unique $\tau \in \mathcal{B}$ such that $\mathcal{A}\tau = \tau$ (by the Banach fixed point theorem). Under assumption 8, the fixed point satisfies $\tau(s) > 0$, all $s \in \mathcal{S}$. Also, define $h(s) \equiv \tau(s) + g(s)$, which is a strictly positive function satisfying[14]

$$h(s) > 1 - q(s). \tag{16}$$

Given these functions, the authority transfers

$$G_t = \chi(s_t) Z_t, \tag{17}$$

where $\chi(s) = g(s)/\tau(s)$, and offers net supply B_t^s of risk-free bonds, possibly negative, computed as

$$b^s(s) \equiv \frac{1}{q(s)} \frac{h(s) - 1}{\tau(s)}, \tag{18}$$

$$B_t^s = b^s(s_t) Z_t. \tag{19}$$

The bonds are sold at the discount $Q_t = q_t$.[15]

It makes little sense to impose $B_t = B_t^s$ as the market-clearing condition before establishing that money is actually valued.[16] For this reason a weaker condition will be used. Define

$$[\{x, 0\}] \equiv \begin{cases} [x, 0] & \text{if } x \le 0, \\ [0, x] & \text{if } x \ge 0, \end{cases}$$

[14] The strict positivity of $h(s)$ follows from the definition of $\mathcal{A}$, strict positivity of $\tau(s)$, and the definition of $q(s)$.

[15] The implementation of monetary policy requires predicting the households' optimal response to that policy, which is encoded in functions τ and h. The postulated implementation rules guarantee that the processes q_t and g_t will indeed possess their desired interpretation in equilibrium, while at the same time the households will find it optimal to purchase the whole supply of bonds.

[16] Households may refuse to purchase the whole supply of bonds, especially when $(1/P)_t = 0$, without violating any constraint of the environment.

and restrict the demand for bonds realized by a representative household to

$$B_t \in [\{B_t^s, 0\}]. \tag{20}$$

A household can never end up holding more bonds than is actually supplied, in absolute-value terms.

Proposition 1 *Under assumption 8, monetary policy implemented by (17) and (19) has the property that* $Z_t > 0$ *implies* $Z_{t+1} > 0$.

Proof. To prove the claim for $B_t = B_t^s$, substitute (17) and (19) into (14), and use the definition of the function $\tau(s)$. The claim is true for $B_t = 0$, since $g(s) \geq 0$. Since (14) is linear in B_t, the claim is valid for all $B_t \in [\{B_t^s, 0\}]$. ■

Letting $Z_0 > 0$ be the initial aggregate net worth, proposition 1 provides a recursive justification for $Z_t > 0$, which would otherwise need to be imposed as sequence of assumptions.

As corollary of proposition 1 and (15), $Z_t > 0$ is sufficient to guarantee that the end-of-period net worth of a representative household is strictly positive,

$$W_{t+1} > 0. \tag{21}$$

Intuitively, the net supply of risk-free bonds is engineered by the authority in a way that repayment of debt (if households are borrowers) never exhausts the end-of-period net worth. Since $G_{t+1} \geq 0$, this also implies

$$H_{t+1} > 0. \tag{22}$$

By (4) and (17), the beginning-of-period net worth of a representative household is $H_t = Z_t(1 + \chi(s_t))$. Since g is bounded and τ is bounded away from zero, one can choose

$$\bar{h} > \max_{s \in \mathcal{S}} (1 + \chi(s)), \tag{23}$$

and restrict H_t for technical reasons to the compact set

$$H_t \in [0, \bar{h} Z_t] \equiv \mathcal{H}(Z_t). \tag{24}$$

2.7 Expectations

Households form expectations of $(1/P)_t$ before trade given the available information. In the assumed Markovian environment, attention can be restricted to expectations formed in a time-invariant way. Since households are competitive, expectations cannot depend on individual state variables. Since Z_t and e_t are the only aggregate state variables, the households can be assumed to expect

$$(1/P)_t = \frac{e_t}{Z_t} \eta(Z_t, e_t, u_t, s_t), \tag{25}$$

where $\eta \colon \mathcal{R}_+ \times \mathcal{R}_+ \times \mathcal{U} \times \mathcal{S} \to [0, \infty)$ is a jointly continuous function, and e_t/Z_t is added without loss of generality. One can further restrict attention to functions η that are bounded, since otherwise a household could expect to purchase consumption exceeding the value of the whole economy.

Under expectations formed in this way, consumption (12) becomes

$$c_t = e_t \left\{ (1-\kappa) + \eta(Z_t, e_t, u_t, s_t) \left[\frac{M_t}{Z_t} - (1-\kappa) \frac{Y_t}{Z_t} \right] \right\}. \tag{26}$$

The expectations of the evolution of Z_t are represented by the model

$$Z_{t+1} = \theta(Z_t, e_t, u_t, s_t), \tag{27}$$

where $\theta \colon \mathcal{R}_+ \times \mathcal{R}_+ \times \mathcal{U} \times \mathcal{S} \to \mathcal{R}_+$ is a jointly continuous function. According to observation 1, this function is independent of s_{t+1}.

2.8 Technical Constraints

Additional constraints must be imposed for technical reasons to ensure that the choice of households is well defined. These constraints must be compatible with the idea of competitive equilibrium, where households never directly experience aggregate supply conditions and believe to be able to trade freely at the margin in all markets that are active.

The demand for bonds will be restricted to

$$B_t \in \mathcal{B}Z_t \equiv \left[-\underline{b}Z_t, \bar{b}Z_t\right], \tag{28}$$

where $\mathcal{B} \equiv [-\underline{b}, \bar{b}]$, and $\underline{b} > 0$, $\bar{b} > 0$ are chosen such that

$$\underline{b} > -\min_{s\in\mathcal{S}} b^s(s), \quad \bar{b} > \max_{s\in\mathcal{S}} b^s(s). \tag{29}$$

By construction, the interval (20), defining the feasible range for the realized bond demand is always strictly inside the interval (28).[17]

Another constraint in addition to (11) is needed to prevent households from rising infeasible funds Y_t at low values of $(1/P)_t$.[18] To find the appropriate bound, one can think in terms of M_t, since $M = Y$. From the budget constraint (6),

$$M_t \leq H_t - B_t Q_t,$$

and the right-hand side is maximized by setting $B_t = -\underline{b}Z_t$ and $Q_t = 1$. Since the beginning-of-period net worth of a representative household is $H_t = Z_t(1+\chi(s_t))$, one can choose $\bar{\bar{y}} > 0$ such that

$$\bar{\bar{y}} > \max_{s\in\mathcal{S}} \left(1 + \chi(s) + \underline{b}\right), \tag{30}$$

[17]It is always possible to find constants $\underline{b}, \bar{b}$, since $b^s(s)$ is bounded.

[18]As an example, a household might want to raise an infinite amount of money at $(1/P)_t = 0$.

and require

$$Y_t \leq \bar{\bar{y}} Z_t. \tag{31}$$

For future reference, (29) can be used to show that

$$\bar{\bar{y}} > 1/\tau(s). \tag{32}$$

Constraints (11) and (31) can be combined into a single constraint under (25). Since both bind at the same time when $\eta(Z_t, e_t, u_t, s_t) = 1/\bar{\bar{y}}$,

$$Y_t \in [0, \bar{y}(Z_t, e_t, u_t, s_t)], \quad \bar{y}(z, e, u, s) \equiv \begin{cases} \frac{1}{\eta(z,e,u,s)} z & \text{if } \eta(z, e, u, s) \geq 1/\bar{\bar{y}}, \\ \bar{\bar{y}} z & \text{if } \eta(z, e, u, s) < 1/\bar{\bar{y}}. \end{cases} \tag{33}$$

The function $\bar{y}$ is positive and jointly continuous.

2.9 Household's Problem

The state variables that a household must take into account in the decision process are H_t, Z_t, e_t, u_t, s_t, and take values in $\mathcal{X} \equiv \{(h, z, e, u, s) \colon z \in \mathcal{R}_+, h \in \mathcal{H}(z), e \in \mathcal{R}_+, u \in \mathcal{U}, s \in \mathcal{S}\}$. A household evaluates its well-being using a jointly continuous value function $v \colon \mathcal{X} \to \mathcal{R}$, which in equilibrium must be co-determined with the expectations.

One can restrict attention to value functions that represent maximized utility (2).

Proposition 2 *A value function consistent with (2) is φ-bounded.*

Proof. Let $\gamma \neq 1$ (the log case is similar). Define v_t, the homothetic version of the utility functional, via $(\sum_{s \geq 0} \beta^s) v_t^{1-\gamma}/(1-\gamma) \equiv V_t$. For fixed e_t, let $\bar{v}_t$ be attained in a hypothetical economy without frictions, and $\underline{v}_t = (1-\kappa)\bar{v}_t$ in an economy where the fraction κ of endowment is lost. Since v_t in a monetary economy is bounded between $\underline{v}_t$ and $\bar{v}_t$, V_t is φ-bounded. ■

At the beginning of t, knowing H_t, Z_t, e_t, u_t, s_t, a household forms expectations according to the functions η, θ, and chooses M_t, Y_t, B_t subject to (5), (11), (6), (9), (28), (31). The objective is to maximize

$$u(c_t) + \beta E_t[v(H_{t+1}, Z_{t+1}, e_{t+1}, u_{t+1}, s_{t+1})],$$

subject to the laws (3) and (8), and given (26).

3 Definition of Equilibrium

Informally, an equilibrium is a value function and a set of functions representing expectations such that the value function represents maximized utility functional, agents do not miss any opportunity to improve their well-being, markets clear, and expectations cannot be improved.

Define the set $\mathcal{C}(h, z, e, u, s) \subset \mathcal{R}^3$ as the set of triples m, y, b satisfying

$$0 \leq m,\ 0 \leq y, \tag{34}$$

$$b \in [-\underline{b}z, \bar{b}z], \tag{35}$$

$$h \geq m + bq(s), \tag{36}$$

$$0 \leq h - m + y + b[1 - q(s)], \tag{37}$$

$$y \leq \bar{y}(z, e, u, s), \tag{38}$$

For each $H_t, Z_t, e_t, u_t, s_t \in \mathcal{X}$, $\mathcal{C}(H_t, Z_t, e_t, u_t, s_t)$ is the set of feasible choices of M_t, Y_t, B_t. This set is non-empty, since it contains $M_t = Y_t = B_t = 0$, and compact. The mapping $\mathcal{X} \to \mathcal{C}$ is a continuous correspondence.

Definition 2 *An equilibrium is:*

(a) A jointly continuous and bounded function $\eta\colon \mathcal{R}_+ \times \mathcal{R}_+ \times \mathcal{U} \times \mathcal{S} \to [0, \infty)$, *and a jointly continuous function* $\theta\colon \mathcal{R}_+ \times \mathcal{R}_+ \times \mathcal{U} \times \mathcal{S} \to (0, \infty)$,

(b) A jointly continuous and φ*-bounded function* $v\colon \mathcal{X} \to \mathcal{R}$,

such that:

(i) Given the functions η, θ, *the function* v *solves*

$$v(h,z,e,u,s) = \max_{(m,y,b)} \left\{ u(c) + \beta \int_S v(h',z',e',u',s')dF(s,s') \right\}, \tag{39}$$

$$\text{subject to:} \quad c = e\left[(1-\kappa) + \eta(z,e,u,s)\left(\frac{m}{z} - (1-\kappa)\frac{y}{z}\right)\right], \tag{40}$$

$$(m,y,b) \in \mathcal{C}(h,z,e,u,s), \tag{41}$$

$$h' = h - m + y + b(1-q(s)) + \chi(s')z', \tag{42}$$

$$z' = \theta(z,e,u,s), \tag{43}$$

$$e' = e\lambda(s,s'), \tag{44}$$

(ii) For each z, e, u, s, *the value* $v(z(1+\chi(s)), z, e, u, s)$ *is attained by* m, y, b *that satisfy* $m = y$, $b \in [\{b^s(s)z, 0\}]$, $w \equiv z(1+\chi(s)) - m + y + b(1-q(s)) > 0$, $y < \bar{\bar{y}}z$.

(iii) For each z, e, u, s, *if there are functions* η^p, θ^p *specified as in (a), for which a function* v^p *specified as in (b) satisfies (i)-(ii), and if* $\eta^p(z,e,u,s) > 0$, *then* $\eta(z,e,u,s) > 0$.

Condition (i) restricts the set of equilibrium value functions to those that are consistent with maximized utility functional (2). The first two sub-conditions of (ii) guarantee that individually optimal choices are consistent with market clearing. The third sub-condition of (ii) guarantees that the choices are consistent with strict positivity of end-of-period net worth, as required by condition (21), and the fourth sub-condition prevents the optimal demand for money from attaining the technical upper bound of inequality (31). Condition (iii) postulates that if money *could* be valuable without violating individual optimality and market clearing, then it must be valuable in equilibrium.[19]

[19] Conditions analogous to (i)-(ii) are sufficient to define equilibrium in the frictionless economy of Lucas (1978), where a condition analogous to (iii) would hold trivially, since money would never be considered valuable.

Any set of functions η, θ, v satisfying (a)-(b) and (i)-(ii) are referred to as pre-equilibrium. Any equilibrium is a pre-equilibrium. If all pre-equilibria feature $\eta(z, e, u, s) = 0$, then all of them are equilibria. If there is at least one pre-equilibrium with $\eta^p(z, e, u, s) > 0$, then the set of equilibria must be restricted to those pre-equilibria for which $\eta(z, e, u, s) > 0$.

By definition, expectations in a pre-equilibrium are always confirmed, which means that households do not exercise their freedom to change their models η, θ. If the awareness of other pre-equilibria can affect how the expectations are formed then the concept of pre-equilibrium is too vague, and one needs an additional equilibrium selection mechanism.

Such a mechanism is provided by condition (iii). To motivate it, consider a pre-equilibrium with responsible monetary policy, but $\eta = 0$. After observing information at t, households know that there exists a pre-equilibrium with $\eta^p > 0$ for all possible configurations of state variables, which can be interpreted as the possibility that money will become valuable in the future. Given this knowledge, rational households cannot be assumed to ignore the opportunity to acquire arbitrarily large quantities of money for free, *in addition* to their current holdings, and it can be assumed that they enter the market for goods with such commitment. By the same logic, no household should be willing to give away net worth for free, so optimal decisions of sellers and buyers would disagree. One must conclude that the existence of a pre-equilibrium with a strictly positive value of money, guaranteed by a responsible design of monetary policy, is sufficient to rule out the pre-equilibrium with worthless money.

Observation 2 *Irrespective of monetary policy, there is a pre-equilibrium with* $\eta(z, e, u, s) = 0$, *all* z, e, u, s.

The existence of this pre-equilibrium is allowed by the possibility of self-fulfilling expectations that money will always remain worthless.

The rest of this section is concerned with properties of equilibria that can be deduced from conditions (i)-(ii). The proofs generalize those in Lucas (1978), with necessary modifications to take into account the more complex environment and endowment growth.

Proposition 3 *For any functions η, θ specified as in (a), there is exactly one non-negative, jointly continuous, and φ-bounded function v satisfying (i)-(ii).*

Proof. Let $\mathcal{V}$ be the Banach space of jointly continuous, φ-bounded functions $g\colon \mathcal{X} \to \mathcal{R}$. Let $\mathcal{T}$ be an operator on $\mathcal{V}$, defined such that condition (i) of definition 2 is equivalent to $\mathcal{T}v = v$.

Applying $\mathcal{T}$ involves maximization of a jointly continuous function on a compact set, by assumption 2, and by the definition of $\mathcal{C}$. Hence, the maximum exists. Since the set $\mathcal{C}$ is given by a continuous correspondence in the state variables, the maximum is jointly continuous in h, z, e, u, s (Berge, 1963).

The function $(\mathcal{T}v)(h, z, e, u, s)$ is φ-bounded, since the maximand in (39) is a sum of two φ-bounded functions. Indeed, this is true of $u(c)$ under the assumed CRRA utility, since

$$\begin{aligned} c \leq (1-\kappa)e + (1/P)m = \; & e\left[(1-\kappa) + \eta(z,e,u,s)m/z\right] \\ \leq \; e\left[(1-\kappa) + \eta(z,e,u,s)\left(h/z - b/zq(s)\right)\right] \leq \; & e\left[(1-\kappa) + \bar{\eta}\left(\bar{h} + \underline{b}\right)\right], \end{aligned}$$

while the other part of the maximand satisfies

$$\begin{aligned} \left|\frac{\beta \int_S v(h',z',e',u',s')dF(s,s')}{\varphi(e)}\right| &= \left|\beta \int_S \frac{\varphi(e')}{\varphi(e)} \frac{v(h',z',e',u',s')}{\varphi(e')} dF(s,s')\right| \\ \leq \beta \int_S \lambda(s,s')^{1-\gamma} \left|\frac{v(h',z',e',u',s')}{\varphi(e')}\right| dF(s,s') &\leq \bar{v}\beta \int_S \lambda(s,s')^{1-\gamma} dF(s,s') < \bar{v}, \end{aligned}$$

where $\bar{v} \equiv \sup_{\mathcal{X}} \left|\frac{v(h',z',e',u',s')}{\varphi(e')}\right| < \infty$, and the last inequality follows from assumption 3. Hence, the operator $\mathcal{T}$ maps $\mathcal{V}$ into itself.

A similar argument can be used to show that for any $a > 0$ and $f \in \mathcal{V}$, there exists $\delta \in (0,1)$ such that $\mathcal{T}(f + a\varphi) \leq \mathcal{T}f + \delta a\varphi$. (Set $\delta = \bar{w}$, defined under assumption 3.) In addition, (i) $f \geq g$ implies $\mathcal{T}f \geq \mathcal{T}g$ for any $f, g \in \mathcal{V}$, and (ii) $\mathcal{T}0 \in \mathcal{V}$. Under these conditions, $\mathcal{T}$ has a unique fixed point $v = \mathcal{T}v$ in $\mathcal{V}$, and $\lim_{n\to\infty} \mathcal{T}^n f = v$ for every $f \in \mathcal{V}$, by the weighted contraction mapping theorem (Boud, 1990). Since $v \geq 0$ implies $\mathcal{T}v \geq 0$, the fixed point is non-negative. ■

Proposition 4 *In a pre-equilibrium, $v(h, z, e, u, s)$ is non-decreasing in h, and concave in h.*

Proof. To show that v is non-decreasing, consider $(\mathcal{T}f)(h, z, e, u, s)$, for any $f \in \mathcal{V}$. The maximum is attained by some $m, y, b \in \mathcal{C}$. An increase in h expands the set $\mathcal{C}$, so Tf is non-decreasing in h. This is true in particular for $\mathcal{T}v$, and then for v, since $v = \mathcal{T}v$.

Take any concave function $g(h, z, e, u, s) \in \mathcal{V}$. Fix z, e, u, s, let $h^0, h^1 \in \mathcal{H}(z)$ be chosen, and let m^i, y^i, b^i attain $(\mathcal{T}g)(h^i, z, e, u, s)$, $i \in \{0, 1\}$. Define $c^i = e[(1 - \kappa) + \eta(z, e, u, s)(m^i/z - (1-\kappa)y^i/z)]$, and $h'^i = h^i - m^i + y^i + b^i(1 - q(s)) + \chi(s')z'$, $i \in \{0, 1\}$. For $0 \leq \theta \leq 1$, define $h^\theta \equiv \theta h^0 + (1-\theta)h^1$, $(m^\theta, y^\theta, b^\theta) \equiv (\theta m^0 + (1-\theta)m^1, \ldots, \ldots)$, $c^\theta \equiv \theta c^0 + (1-\theta)c^1$, and $h'^\theta \equiv \theta h'^0 + (1-\theta)h'^1$. Note that $m^\theta, y^\theta, b^\theta$ are feasible at h^θ, $h'^\theta = h^\theta - m^\theta + y^\theta + b^\theta(1 - q(s)) + \chi(s')z'$, and that $h'^\theta \in \mathcal{H}(z)$. At h^θ, $\mathcal{T}g$ satisfies

$$\begin{aligned}(\mathcal{T}g)(h^\theta, z, e, u, s) &\geq u(c^\theta) + \beta \int_S g(h'^\theta, z', e', u', s')dF(s, s') \\ &\geq \theta(\mathcal{T}g)(h^0, z, e, u, s) + (1-\theta)(\mathcal{T}g)(h^1, z, e, u, s).\end{aligned}$$

Hence, $(\mathcal{T}g)(h, z, e, u, s)$ is concave in h for every $g \in \mathcal{V}$. Since functions that are concave in h form a Banach vector subspace of $\mathcal{V}$, the fixed point $v = \mathcal{T}v$ is concave in h. ■

The established concavity can be used to prove that:

Proposition 5 *Under (i)-(ii), if the value function v is attained by m in the interior of the feasible set at some $h, z, e, u, s \in \mathcal{X}$ for which $\eta(z, e, u, s) > 0$, then*

v is differentiable in h, and

$$\frac{\partial}{\partial h} v(h, z, e, u, s) = u'(c) \frac{e}{z} \eta(z, e, u, s) \tag{45}$$

Proof. Fix z, e, u, s, and let $f \colon \mathcal{R}_+ \to \mathcal{R}_+$ be defined by $f(A) \equiv (\mathcal{T}v)(A, z, e, u, s)$. Let $m(A)$, $y(A)$, and $b(A)$ attain $f(A)$.

Define $\tilde{u}(m) = u\left(\frac{e}{z}[\eta m + (1-\kappa)(1-\eta y]\right)$. With $\eta(z, e, u, s) > 0$, $\tilde{u}\,(m)$ is strictly concave in m. By proposition 4 and (42), $\beta \int_S v(h', z', e', u', s') dF(s, s')$ is concave in m. Therefore, the maximand in the definition of $(\mathcal{T}v)(A, z, e, u, s)$ is strictly concave in m, so $m(A)$ is unique, and varies continuously with A (Berge, 1963).

Let $h'(A) = A - m(A) + y(A) + b(A)(1 - q(s)) + \chi(s')z'$. For small ϵ, $m(A) + \epsilon$ is feasible at $A + \epsilon$, and $m(A + \epsilon) - \epsilon$ is feasible at A. Using the definition of f,

$$\begin{aligned} f(A+\epsilon) &\geq \tilde{u}(m(A+\epsilon)) + \beta \int_S g(h'(A), z', e', u', s') dF(s, s'), \\ &= \tilde{u}(m(A+\epsilon)) - \tilde{u}(m(A)) + f(A). \end{aligned} \tag{46}$$

$$\begin{aligned} f(A) &\geq \tilde{u}(m(A+\epsilon) - \epsilon) + \beta \int_S g(h'(A+\epsilon), z', e', u', s') dF(s, s'), \\ &= \tilde{u}(m(A+\epsilon) - \epsilon) - \tilde{u}(m(A+\epsilon)) + f(A+\epsilon). \end{aligned} \tag{47}$$

Combining (46) and (47),

$$\tilde{u}(m(A+\epsilon)) - \tilde{u}(m(A)) \leq f(A+\epsilon) - f(A) \leq \tilde{u}(m(A+\epsilon)) - \tilde{u}(m(A+\epsilon) - \epsilon).$$

Dividing by ϵ, taking the limit $\epsilon \to 0$, using the continuity of $m(A)$ and the definition of $\tilde{u}(m)$, one finds that $f'(A) = u'(c)\frac{e}{z}\eta(z, e, u, s)$. The partial derivative of $v(h, z, e, u, s)$ with respect to h is $f'(h)$, because $v = \mathcal{T}v$, which proves (45). ■

4 Constructing the Equilibrium

This section demonstrates by construction that under assumptions 8 and 9 there is only one equilibrium. In that equilibrium, the value of money is strictly positive.

4.1 Differentiability of the Value Function

The strict positivity of η is first imposed as a hypothesis in addition to conditions (i)-(ii) of definition 2. Under this combination of assumptions, the pre-equilibrium value function of a representative household is differentiable in h, which is established in the following three propositions.

Proposition 6 *In a pre-equilibrium with $\eta(z, e, u, s) > 0$, it is true that $m > 0$ for all z, e, u, s.*

Proof. A representative household holds $h = (1 + \chi(s))z = (h(s)/\tau(s))z$ of beginning-of-period net worth. By (42), $h - m + y + b[1 - q(s)]$ strictly improves the maximand of (39). Suppose $m = 0$, which also necessitates $y = 0$ by condition (ii) of definition 2.

If $q(s) < 1$, a household would like to optimally set the demand for bonds b to the maximal level allowed by the budget constraint (6) in order to benefit from the positive interest rate, i.e., $b = \frac{1}{q(s)} \frac{h(s)}{\tau(s)} z \equiv b^d(s)$. This can only be consistent with condition (ii) of pre-equilibrium if the supply for bonds $b^s(s)$ under the assumed monetary policy is sufficiently large. However, as seen from (18), it is true that $b^d(s) > b^s(s)$.

If $q(s) = 1$, a household does not have strict preference between saving in bonds or in money, since both result in the same value of end-of-period net worth (h). Consider a strategy of increasing m and y by a small number $\epsilon > 0$, which is feasible since $h > 0$, and because $\bar{y}(z, e, u, s) > 0$, as seen from (38). This strategy leaves the end-of-period net worth unchanged, but increases consumption by $\kappa \frac{e}{z} \eta(z, e, u, s)\epsilon > 0$, as seen from (40), and hence increases utility. It follows that

$m = 0$ cannot be optimal. ■

Proposition 7 *In a pre-equilibrium, a representative household chooses m strictly below the upper bound allowed by the set of feasible choices $\mathcal{C}(h, z, e, u, s)$.*

Proof. Define $a \equiv h - m - bq(s)$, and $w \equiv h - m + y + b[1 - q(s)]$, which are non-negative by (36), and (37). Since w is defined as in requirement (ii) of definition 2, it must satisfy $w > 0$. Combining the two definitions, $m = h + q(s)y - [1 - q(s)]a - q(s)w$. Since $q(s) > 0$, maximizing m would instead require setting $w = 0$ for any given values of y and a. ■

Proposition 8 *In a pre-equilibrium with $\eta(z, e, u, s) > 0$, the value function v is differentiable in h, at $h = (1+\chi(s))z$, for all z, e, u, s.*

Proof. By propositions (6) and (7), the value function v of a representative household is attained by m in the interior of the feasible range allowed by budget feasibility. For each z, e, u, s, since $\eta(z, e, u, s) > 0$, the conditions of proposition (5) are satisfied at $(1+\chi(s))z, z, e, u, s$, and hence the value function is differentiable in h at $h = (1+\chi(s))z$. ■

By equation (45) of proposition 5,

$$\nu \equiv \frac{\partial}{\partial h} v(h, z, e, u, s)|_{h=(1+\chi(s))z} = u'(c)\frac{e}{z}\eta(z, e, u, s) \equiv u'(c)(1/P), \tag{48}$$

where c is optimal consumption. In what follows, ν' will denote the partial derivative evaluated at state variables realized in the next period.

4.2 Pre-equilibrium with Positive Value of Money

The established differentiability of the value function can be used to explore the implications of $\eta(z, e, u, s) > 0$ in a pre-equilibrium. The problem (39) can be studied using the Lagrangian

$$\begin{aligned}\mathcal{L} \equiv\ & u\left(e\left[(1-\kappa)+\eta(z,e,u,s)\left(m/z-(1-\kappa)y/z\right)\right]\right) \\ & +\beta\int_S v\left(\left(h-m+y+b[1-q(s)]+\chi(s')z',z',e',u',s'\right)dF(s,s')\right. \\ & +\mu[h-m-bq(s)]+\phi[\bar{y}(z,e,u,s)-y],\end{aligned} \tag{49}$$

where μ and ϕ are non-negative multipliers. The first-order conditions associated with m are

$$\nu-\beta\int_S \nu' dF(s,s')-\mu=0 \tag{50}$$

$$h-m-bq(s)\geq 0,\ \mu\geq 0,\ \mu[h-m-bq(s)]=0, \tag{51}$$

The first-order conditions associated with optimal choice of y are

$$(1-\kappa)\nu-\beta\int_S \nu' dF(s,s')+\phi=0 \tag{52}$$

$$\bar{y}-y\geq 0,\ \phi\geq 0,\ \phi(\bar{y}-y)=0, \tag{53}$$

and the first-order condition for the choice of b is

$$\mu-\beta\int_S \nu' dF(s,s')\frac{1-q(s)}{q(s)}=0. \tag{54}$$

Combining (50) and (54),

$$\beta\int_S \frac{\nu'}{\nu}dF(s,s')=q(s), \tag{55}$$

according to which a household invests in bonds such that the nominal marginal rate of substitution equals the market discount factor.[20]

Using (55) in (54), one finds that $\mu/\nu=1-q(s)$. Then, the complementary

[20]While this condition is standard, it only holds as consequence of assumptions 8 and 9. The same comment applies to other equilibrium conditions.

slackness condition of (51) can be written as

$$[1 - q(s)][h - m - bq(s)] = 0, \tag{56}$$

so a household does not save 'under the bed' at positive interest rate.

Dividing (52) by ν and using (55) gives $\phi/\nu = \kappa + q(s) - 1$, which is positive under assumption 9. Hence, the complementary slackness condition of (53) implies

$$y = \bar{y}(z, e, u, s). \tag{57}$$

Intuitively, under a monetary policy that guarantees sufficiently low interest rates in relation to the degree of inefficiency characterizing consumption of own endowment, households always prefer to sell all endowment in the market.

To further investigate the implication of the condition above, recall that (38) combines two constraints reflected in the definition of $\bar{y}$ (33). If $\eta < 1/\bar{\bar{y}}$, then the demand for money would attain the technical upper bound $y = \bar{\bar{y}}z$, which is impossible by the construction of that bound, and hence it must instead be true that $y = z/\eta(z, e, u, s)$, which is the same as $y(1/P) = e$ by the definition of $(1/P)$. Since $m = y$ holds in equilibrium,

$$m(1/P) = e, \tag{58}$$

which is a version of the equation of exchange (Fisher, 1911).

Using these results, one also finds that

$$\nu = u'(e)(1/P) = u'(e)\frac{e}{z}\eta(z, e, u, s). \tag{59}$$

The pre-equilibrium price function η can be found by considering the dynamics of the end-of-period net worth of a representative household,[21]

$$W_t + G_t = M_t(1 - Q_t) + Q_t W_{t+1}. \tag{60}$$

Using (55) for the discount factor in front of W_{t+1} and multiplying by $(1/P)_t$,

$$W_t(1/P)_t = M_t(1/P)_t(1 - Q_t) - G_t(1/P)_t + \beta E_t \left[\frac{\nu_{t+1}}{\nu_t} \frac{(1/P)_t}{(1/P)_{t+1}} W_{t+1}(1/P)_{t+1} \right], \tag{61}$$

where ν_t denotes the partial derivative (59) evaluated at realized state variables. Also, it is true by (15) that

$$W_t(1/P)_t = W_t \frac{e_t}{Z_t} \eta(Z_t, e_t, u_t, s_t) = e_t \eta(Z_t, e_t, u_t, s_t).$$

Substituting this in (61) and applying (59) and (58) allows to interpret the former as necessary condition for the price function η in the studied pre-equilibrium. Restated in functional form,

$$\eta(z, e, u, s) = 1 - q(s) - g(s) \frac{\eta(z, e, u, s)}{\tau(s)} + \beta \int_S \frac{u'(e')e'}{u'(e)e} \eta(z', e', u', s') dF(s, s'), \tag{62}$$

where the laws of motion for z' and e' are given by (43) and (44), respectively.

Condition (62) can be compared with the definition of the function $\tau(s)$, computed by the authority at the stage of policy implementation (re-stated here for

[21] This can be derived as follows:

$$\begin{aligned}
W_t + G_t = H_t &\equiv M_t + B_t Q_t + (H_t - M_t - B_t Q_t) \\
&= M_t + B_t Q_t + (H_t - M_t - B_t Q_t) - (H_t - M_t - B_t Q_t)(1 - Q_t) \\
&= M_t(1 - Q_t) + Q_t[M_t + B_t + (H_t - M_t - B_t Q_t)] \\
&= M_t(1 - Q_t) + Q_t[Y_t + B_t + (H_t - M_t - B_t Q_t)] \\
&= M_t(1 - Q_t) + Q_t W_{t+1}.
\end{aligned}$$

The second line subtracts a term that is zero in equilibrium by the optimality condition (56). The fourth line uses the Keynes law $M_t = Y_t$, and the last line applies (7), the definition of W_{t+1}.

convenience),

$$\tau(s) = 1 - q(s) - g(s) + \beta \int_S \frac{u'(e')e'}{u'(e)e} \tau(s') dF(s, s'). \tag{63}$$

Define a bounded and jointly continuous function $x(z, e, u, s) \equiv \eta(z, e, u, s) - \tau(s)$. Subtracting (63) from (62) and solving for $x(s)$ yields

$$x(z, e, u, s) = \frac{\tau(s)}{\tau(s) + g(s)} \beta \int_S \frac{u'(e')e'}{u'(e)e} x(z', e', u', s') dF(s, s'). \tag{64}$$

Define operator $\mathcal{Y}$ such that (64) is equivalent to $\mathcal{Y}x = x$. This operator maps the space of bounded, jointly continuous functions in variables z, e, u, s onto itself. Under assumptions 8 and 9 on monetary policy, it is true that $g(s) \geq 0$ and $\tau(s) > 0$ (as shown in subsection 2.6), so the ratio in front of β is in $(0, 1]$. Under the assumed CRRA utility, and assumption 3, this is enough to establish that $\mathcal{Y}$ is a contraction mapping, and hence there is exactly one solution to (64). Since $x(z, e, u, s) = 0$ is a solution, it must be the only solution. Hence, the unique price function consistent with pre-equilibrium with $\eta > 0$ is

$$\eta(z, e, u, s) = \tau(s), \text{ for all } z, e, u, s. \tag{65}$$

To complete the construction of the pre-equilibrium with $\eta > 0$, one must compute the law of motion for the aggregate state variable Z_t. Consider (58), written as $M_t(1/P)_t = e_t$. Using (25), and (65), this is equivalent to

$$\frac{H_t}{M_t} = h(s_t), \tag{66}$$

where $h(s) \equiv \tau(s) + g(s)$, as defined in subsection 2.6. Write the necessary condition (56) in the form

$$B_t(1 - Q_t) = \frac{1 - Q_t}{Q_t}(H_t - M_t),$$

and substitute into the law of motion (14). Using (66), and the definition of $\tau(s)$, one finds that the unique model of the form (27) consistent with the actual evolution of Z_t is

$$\frac{z'}{z} = \frac{1}{q(s)} \beta \int_S \frac{u'(e')e'}{u'(e)e} \frac{\tau(s')}{\tau(s)} dF(s, s') \equiv \tilde{\theta}(s), \tag{67}$$

and one can identify $\theta(z, e, u, s) = z\tilde{\theta}(s)$. Under the assumed monetary policy, the growth rate in Z_t does not depend on state variables other than s_t.

With the unique set of functions η, θ consistent with pre-equilibrium with $\eta > 0$, proposition 3 guarantees that there is exactly one corresponding value function v. Hence, the constructed pre-equilibrium is unique.

4.3 Uniqueness of Equilibrium

The previous subsections show that under assumptions 8 and 9 there exists a unique pre-equilibrium in which the value of money remains strictly positive. According to condition (iii) of definition 2, any equilibrium price function η must then be strictly positive. Since every equilibrium is a pre-equilibrium, there can be at most one equilibrium with $\eta > 0$. Since the pre-equilibrium of the previous section satisfies condition (iii), it is an equilibrium, and hence the only equilibrium is the one with the positive value of money.

One can also conclude that there exists exactly one equilibrium for every specification of monetary policy consistent with assumptions 8 and 9, in which the fundamental value of money is determined by condition (65). Intuitively, the role of a responsible monetary policy is to provide households with the knowledge that a monetary equilibrium *exists*, which is sufficient to make money valuable by their individual rational decisions.

5 Discussion

5.1 Behavioral Aspect of Condition (iii)

This section shows that a non-zero value of money is supported in a pre-equilibrium precisely when a representative household *decides* to assign strictly positive marginal valuation to its beginning-of-period net worth. The definition of equilibrium imposes this decision automatically under a responsible monetary policy.

By equation (48), the value function $v(h, z, e, u, s)$ in a pre-equilibrium is strictly increasing in h at $h = (1 + \chi(s))z$ whenever $\eta > 0$. To prove the converse, let η, θ, v be functions specified as in (a)-(b) of definition 2 satisfying conditions (i)-(ii). Assume that the value function used by a representative household is strictly increasing in h, i.e., that the households strictly prefers to hold more net worth, rather than less, at the opening of the market for goods.

Hypothesis 1 *The value function $v(h, z, e, u, s)$ used by a representative household is strictly increasing in h at $h = (1 + \chi(s))z$, for all z, e, u, s.*

Proposition 9 *Under Hypothesis 1, $\eta(z, e, u, s) > 0$, for all z, e, u, s, in a pre-equilibrium.*

Proof. Suppose $\eta(z, e, u, s) = 0$ for some z, e, u, s. Then, a household with value function v that is strictly increasing in h finds it optimal to set $y = \bar{y}(z, e, u, s) > 0$ and $m = 0$ to maximize the right-hand side of (39). This true in particular at $h = (1 + \chi(s))z$, which results in a violation of condition (ii) of definition 2, so it is necessary that $\eta(z, e, u, s) > 0$, for all z, e, u, s. ■

Hence, the assumption that money is valuable in a pre-equilibrium is *equivalent* to the behavioral postulate of Hypothesis 1, so condition (iii) of definition 2 could equivalently be formulated as: (iii) *For each z, e, u, s, if there are functions η^p, θ^p specified as in (a), for which a function v^p specified as in (b) satisfies (i)-(ii), and if $\eta^p(z, e, u, s) > 0$, then $v(h, z, e, u, s)$ is strictly increasing in h at $h = (1+\chi(s))z$.* This formulation would highlight the behavioral aspect of the proposed equilibrium selection mechanism, and the strict positivity of equilibrium value of money would follow from proposition 9.

5.2 Ruling Out Speculative Price Dynamics

The economic intuition behind condition (64) can be developed as follows. Comparing the equilibrium value of money implied by (65) with equation (58), one obtains $\tau(s_t) = Z_t/M_t$. Using this in (17) gives $g(s_t) = G_t/M_t$, and then $h(s_t) \equiv \tau(s_t) + g(s_t) = H_t/M_t$. Hence, the function $h(s)$ reflects the inverse marginal propensity to consume out of H_t, and the function $x(z, e, u, s)$ can be identified with deviation of H_t/M_t from $h(s_t)$,

$$x(Z_t, e_t, u_t, s_t) = \left(\frac{Z_t}{M_t} + \frac{G_t}{M_t}\right) - \left(\tau(s_t) + \frac{G_t}{M_t}\right) = \frac{H_t}{M_t} - h(s_t).$$

The content of condition (64) is that this difference is dynamically unstable. At given H_t, if a household decides to choose M_t according to a time-invariant rule with $H_t/M_t \geq h(s_t)$, and $H_t/M_t > h(s_t)$ with positive probability, then the ratio H_t/M_t must eventually exceed any positive bound. By $M_t(1/P)_t = e_t$, the real value of H_t must then exceed any bound relative to e_t, violating individual rationality. Similarly, if a representative household decides to choose M_t according to a time-invariant rule with $H_t/M_t \leq h(s_t)$, and $H_t/M_t < h(s_t)$ with a positive probability, then the ratio H_t/M_t must eventually turn negative, since $x(Z_t, e_t, s_t)$ must exceed any negative bound. But this cannot happen without violating the non-negativity of H_t, imposed on nominal net worth by the authority.

By assumption 9, equilibrium must be characterized by low interest rates to induce households to sell all endowment in the market for goods such that $Y_t(1/P)_t = e_t$. By the Keynes law, this is equivalent to $M_t(1/P)_t = e_t$, which can be written as

$$(1/P)_t = \frac{e_t}{H_t}\frac{H_t}{M_t}.$$

At the same time, the fundamental value of money can be defined as

$$(1/P)^*_t \equiv \frac{e_t}{Z_t}\tau(s_t) = \frac{e_t}{H_t}h(s_t).$$

Subtracting this from the previous condition,

$$(1/P)_t - (1/P)^*_t = \frac{e_t}{H_t}\left(\frac{H_t}{M_t} - h(s_t)\right),$$

so the market value of money differs from the fundamental value precisely when the aggregate inverse marginal propensity to consume H_t/M_t differs from $h(s_t)$. Since this is never optimal, as discussed below equation (65), equilibrium value of money never deviates from $(1/P)^*_t$.

5.3 Some Equilibrium Arithmetic

Substituting (13) into the law of motion for beginning-of-period net worth (8),

$$H_{t+1} - H_t = B_t(1 - Q_t) + G_{t+1}.$$

The net supply of risk-free bonds (19) is set in a way that households never find it optimal to save 'under the bed' even when $Q_t = 1$ (which is allowed for stochastically finite periods by assumption 8). Hence, the budget constraint (6) is always satisfied with equality,

$$H_t = M_t + B_tQ_t.$$

Substituting this into the previous equation and shifting the time index,

$$B_{t-1} = M_t - M_{t-1} - G_t + B_tq_t. \tag{68}$$

Condition (68) can be used to study the properties of equilibrium in much the same way in which one usually studies intertemporal budget constraints, and its economic content can be explained using discounted present values.[22] For notational convenience, let

$$\mathrm{PV}^{e}_{t,r}\left[x_{t+s}\right] \equiv E_t\left\{\sum_{s=r}^{\infty}\beta^s \frac{u'(e_{t+s})}{u'(e_t)}(e_{t+s}x_{t+s})\right\} \tag{69}$$

be the consumption-based present value of the fraction x_{t+s} of aggregate endowment starting from $t+r$.[23] Define $\pi_t + \lambda_t \equiv (M_t - M_{t-1})/M_t$ as realized growth in the volume of transactions. Dividing (68) by equilibrium price level $P_t \equiv 1/(1/P)_t$ and solving forward for B_{t-1}/P_t in the usual way gives

$$\frac{B_{t-1}}{P_t} = \mathrm{PV}^{e}_{t,0}\left[(\pi_{t+s} + \lambda_{t+s}) - g_{t+s}\right]. \tag{70}$$

According to this condition, the aggregate real demand for risk-free bonds offered by the authority is positive (negative) precisely when the growth in the nominal volume of transactions is expected to exceed (fall short of) the rate of new net worth creation. This is intuitive, since holding nominal bonds forever cannot be optimal, and households only hold them if they are planning to increase nominal spending in the future faster than allowed by the rate of new transfers. Conversely, if households are indebted, they must plan to use part of future transfers to repay their debts.[24]

Another implication of equilibrium can be obtained starting from (68) in the form

$$B_{t-1} + M_{t-1} = (B_t + M_t)q_t + M_t(1 - Q_t) - G_t.$$

[22]It is easy to confuse (68) with a budget constraint for the authority since one cannot distinguish these concepts in the data. The authority is not restricted by a budget constraint, but only by the design of monetary policy.

[23]For example, the process x_{t+s} can be interpreted as time-varying tax rate, in which case the functional returns the present value of real tax revenues.

[24]Condition (70) is formally similar to the central equation of the fiscal theory of the price level (Sargent and Wallace, 1981; Woodford, 1995). This similarity is superficial and only reflects the mathematics of present values. The same comment applies to other present-value relations developed in this section.

Dividing by the price level, solving forward, and rearranging the terms,

$$\frac{B_{t-1} + M_{t-1}}{P_t} + \mathrm{PV}^e_{t,0}\left[g_{t+s}\right] = \mathrm{PV}^e_{t,0}\left[1 - q_{t+s}\right]. \tag{71}$$

Since $H_t = M_{t-1} + B_{t-1} + G_t$, this is equivalent to

$$\frac{H_t}{P_t} + \mathrm{PV}^e_{t,1}\left[g_{t+s}\right] = \mathrm{PV}^e_{t,0}\left[1 - q_{t+s}\right], \tag{72}$$

where now g_t is included in H_t/P_t. According to this condition, the real value of the beginning-of-period nominal net worth held by the households, plus the real present value of expected transfers, together add up to the real present value of privately perceived losses associated with participation in the market for goods, where selling endowment is effectively subject to a tax of $1-q = \frac{i}{1+i}$ per unit of endowment in present-value terms. The left-hand side can be interpreted as monetary assets of households seen as buyers of endowment, which in equilibrium must be balanced by monetary liabilities of the same households seen as sellers of endowment.[25]

Condition (72) can also be written as

$$\frac{H_t}{P_t} + \mathrm{PV}^e_{t,1}\left[g_{t+s}\right] + \mathrm{PV}^e_{t,0}\left[q_{t+s}\right] = \mathrm{PV}^e_{t,0}\left[1\right], \tag{73}$$

according to which the real value of the whole economy consists of three components: (1) the real value of nominal net worth held by the households, (2) the real present value of rationally expected transfers of new net worth, (3) the real present value of expected receipts from the market for goods, postponed one period ahead due to the lag in the payment system, and hence discounted by q. Only the first component of wealth is liquid, but households never need to borrow against future

[25]This confirms that the usual accounting convention (used for historical reasons) that money is a *liability* of the issuing authority makes little economic sense. Money is not backed in any way by the authority but is endogenously accepted by its users.

nominal income and never experience borrowing constraints.[26]

One more implication of equilibrium can be obtained starting from the identity

$$M_{t-1} = M_t(1 - Q_t) - (M_t - M_{t-1}) + Q_t M_t.$$

Dividing by the price level and solving forward,

$$\frac{M_{t-1}}{P_t} = \mathrm{PV}^e_{t,0}\left[(1 - q_{t+s}) - (\pi_{t+s} + \lambda_{t+s})\right]. \tag{74}$$

Since the left-hand side is positive, the nominal interest rate must in equilibrium on average exceed the growth in the nominal volume of transactions.[27]

5.4 Literature Overview

The need to integrate the theory of money with the theory of value has been recognized by neoclassical economists (Walras, 1900; Hicks, 1935; Patinkin, 1965), dissatisfied with the practice of using the ad-hoc equation of exchange (Fisher, 1911). However, a fully successful theory explaining how fiat money is valued has never been offered. For example, Hahn (1965) observed that the classic study by Patinkin (1965) could not not rule out the solution in which money was permanently worthless and models with money-in-utility (or equivalent formulations) are known to allow multiplicities of equilibria (Obstfeld and Rogoff, 1983; Matsuyama, 1991), a property shared with overlapping-generations models (Samuelson, 1958) and many other models starting from explicit frictions. In a detailed review of

[26]This condition reflects the property of the equilibrium that barter and other forms of non-monetary exchange are never used.

[27]This condition should not be confused with the so-called Taylor principle postulated in new-Keynesian literature as necessary pre-condition for non-explosive inflation (Taylor, 1993; Clarida et al., 2000; Woodford, 2003; Galí, 2015). The Taylor principle is supposed to apply to *changes* in the nominal interest rate in response to changes in inflation in the activist Taylor (1993) rule, which is *not* the type of monetary policy assumed here. Consistent with empirical evidence by Cochrane (2018), the present model does not predict that inflation must ultimately become unstable if the Taylor principle is violated although the nominal rate could even be pegged at an arbitrary level.

monetary literature, Hellwig (1993) concluded that the fundamental problem of why fiat money is valuable *at all*, especially in the presence of securities that dominate it as a store of value (the Hahn problem) had not been solved, and his analysis appears valid also today. The recent controversies around the fiscal theory of the price level (Kocherlakota and Phelan, 1999; Christiano and Fitzgerald, 2000; Buiter, 2002; Niepelt, 2004), and new-Keynesian models of inflation (Cochrane, 2011, 2018) can be seen as reflecting this unfortunate state of affairs.

This paper generalizes the asset-pricing framework of Lucas (1978) to a monetary economy with explicit frictions. The decisions of competitive individuals to accept money follow from their optimizing behavior, making the approach fundamentally different from subsequent studies of Lucas (1980, 1982, 1984), relying on the cash-in-advance (CIA) constraint (Clower, 1967; Grandmont and Younes, 1972). In particular, it is shown that allowing the agents to be sufficiently sophisticated to recognize arbitrage opportunities implied by the existence of money as a traded object in the market for goods, together with a complete characterization of the environment, are enough to establish the uniqueness of equilibrium and hence overturn the doctrine that equilibria of monetary models are inherently indeterminate, or that equilibrium price level can be affected by self-fulfilling expectations.

The uniqueness of equilibrium proven in this paper does not offer justification for selecting the unique bounded solutions for inflation in standard new-Keynesian models (Woodford, 2003; Galí, 2015). The usual argument for this relies on the so-called Taylor principle (Taylor, 1993; Clarida et al., 2000), according to which the monetary authority must commit to raising the interest rate sufficiently strongly in response to inflation. This reasoning has been forcefully criticized by Cochrane (2011, 2018) for lacking economic justification and empirical support, and the present paper complements this critique by showing that the pre-conditions for price stability are quite different from those postulated in that literature.[28]

[28]Moreover, aggregate nominal net worth plays no role in new-Keynesian models, while it defines the nominal scale of the economy in the present model, allowing the authority to engineer essentially any path of inflation via a helicopter drop of

This work can be seen as complementary to the fiscal theory of the price level (FTPL), in which the authority, often called the government, issues fiat money and interest-bearing debt as nominal liabilities (Sargent and Wallace, 1981; Leeper, 1991; Sims, 1994; Woodford, 1995). The FTPL interprets q_t, g_t as defining seigniorage and taxes, respectively, which are components of real government surplus. Given a pre-determined measure of outstanding nominal liabilities, the price level is defined as the unique conversion factor that makes their real value equal to the discounted present value of the surpluses. The assumption that the government is indeed able to issue valuable nominal liabilities and commit to a given path of seigniorage and taxes has been known as *non-Ricardian* fiscal policy (Woodford, 1995) and has been subject to much controversy (Kocherlakota and Phelan, 1999; Christiano and Fitzgerald, 2000; Buiter, 2002; Niepelt, 2004). The present study can be seen as providing the implementation theory missing in the bare formulation of the FTPL, confirming that the authority can indeed follow a non-Ricardian policy. However, the equilibrium selection mechanism employed here is very different from that postulated by the FTPL. Moreover, the usual interpretation of the FTPL appears invalid since a fiscal government is not even part of the model and no taxes are raised by the authority. Incidentally, the interpretation assigned to q_t, g_t by the FTPL reverses their economic meaning since the nominal interest rate implied by the discount factor q_t is, in fact, a *tax* on the receipts from the market for goods, while g_t is the flow of *seigniorage*, defined as real revenue of those who are the first to receive transfers of new net worth.

It may appear as if the present paper justifies the practice of imposing the CIA constraint in an ad-hoc way, interpreted as an equilibrium condition in the market for goods. However, this is only valid under a responsible monetary policy, so imposing the CIA constraint with no regard to the policy may seriously misrepresent individual incentives and has often been interpreted as an artificial restriction on

new net worth. The idea of a helicopter drop of *money*, originally due to Friedman (1969), has recently been under renewed interest (Bernanke, 2002, 2003; Buiter, 2014; Benigno and Nisticò, 2020; Galí, 2020).

behavior. The model differs from Lucas and Stokey (1987), where the CIA constraint is imposed only on a subset of transactions while other transactions are exempt from it. The authors interpret the latter as intermediated by credit, although their model admits an alternative interpretation of barter. The present paper does not disallow credit from the authority, but private credit is assumed too costly to be used in equilibrium.

Much of modern thinking about money is rooted in the so-called portfolio tradition (Hicks, 1935; Keynes, 1930; Tobin, 1958; Friedman, 1956). It is interesting to note that the equilibrium selection mechanism proposed here would not be operational under this interpretation of money, which abstracts from the market for goods and treats money as a purely speculative investment asset. By highlighting this problem with the portfolio tradition, this paper contributes to the literature rejecting it for lacking both internal consistency and economic intuition.[29] Instead of holding money, households in the present model optimally plan to spend it as soon as possible, which appears consistent with the intuition (and anecdotal evidence) that rational households treat money as a hot potato, or at least would do so in the absence of transaction costs reducing the advantage of interest-bearing securities over money. This is intuitive since money in the present model is not backed by any stream of dividends, so holding it cannot be optimal.

The model starts from the list of properties characterizing actual economic environments. Prominent examples of this approach include OLG models (Samuelson, 1958; Grandmont and Laroque, 1973; Wallace, 1980), turnpike models (Townsend, 1980), models of self-insurance against idiosyncratic risks (Bewley, 1977; Aiyagari, 1994), or models in which the technology of exchange is restricted to infrequent bilateral meetings (Kiyotaki and Wright, 1989; Trejos and Wright, 1995; Shi, 1995; Lagos and Wright, 2005). Some of these theories rely on specific assumptions which are quite restrictive or stylized and sometimes make it difficult to easily interpret the results. While some admit analytical tractability, others do not. For

[29]See Kohn (1988) for a survey of the related literature.

example, the approach based on bilateral meetings is plagued with analytical difficulties arising from the need to keep track of random changes in the distribution of money across agents, which can only be resolved by imposing additional technical assumptions such as quasi-linear utility (Howitt, 2003). The role of restricting the frequency of meetings in that literature is to place a limit on the velocity of circulation, which in the present paper is achieved by introducing the payment system with a pre-determined technological time lag. This allows maintaining the assumption of a competitive market for goods in which agents can interact without restrictions and to meaningfully define the competitive equilibrium value of money. It should be noted that the model does not rely on the existence of a Walrasian auctioneer since households can compute the unique market-clearing price level already before trade based on their information.

This work connects with the older literature on the transactions demand for money, as surveyed, for example, by Ostroy and Starr (1990). While the contribution of that literature was to better understand the origins of frictions, it has not offered the answer to why money could be *expected* to circulate at a unique equilibrium value depending on the design of the monetary policy.

Since the focus is on fiat money, the paper fundamentally differs from studies where the value of money is guaranteed by a form of real backing, perhaps implicit, and usually impossible to detect in equilibrium.[30]

6 Concluding Remarks

While this paper does not offer a completely new theory of money, it re-formulates the classical theory in the language of modern asset pricing, and shows that it can be applied to a model economy closely corresponding to an actual economy relying on an electronic payment system. Although the model is based on several idealized

[30]Examples include Obstfeld and Rogoff (1983); Del Negro and Sims (2015); Hall and Reis (2016); Benigno (2020).

assumptions, the author believes that it very accurately reflects the economic function of money, and the incentives of its users. For this reason, it can serve as a workhorse for studying topics related to money and monetary policy.

Several conclusions can be drawn from the analysis already. First, money is never *held* as part of the optimal portfolio but circulates between trading parties like the anecdotal hot potato, with velocity restricted by technological properties of the payment system. Second, if the marginal cost of producing money is zero, there is no economic justification in allowing the authority to distribute seigniorage rents unequally across the society, which can only be seen as contributing to economically unjustified wealth inequality. Third, since two responsible monetary policies that differ in terms of bond supplies and transfers (and hence equilibrium price processes) generally possess the same efficiency properties, the supply of credit from the authority does not necessarily improve the efficiency of resource allocation, but may simply be aimed at extracting seigniorage rents.[31] Fourth, the model predicts that barter and Arrow-Debreu forward markets for privately negotiated contracts remain inactive under a responsible monetary policy since monetary infrastructure provides the society with a less costly alternative. Since all income in the resulting equilibrium is nominal, the model highlights the central role of monetary policy in shaping individual expectations of lifetime income and wealth. This suggests that wealth effects associated with changes in monetary policy can easily be of first-order importance as drivers of business activity, especially if monetary policy is not designed correctly, or implemented non-transparently.[32]

[31]Even in a Pareto-efficient allocation, the marginal value of a new dollar is positive (equal to the inverse price level), so the incentives to reap private seigniorage rents may be too strong to resist.

[32]The theory should not be misinterpreted as predicting that money or monetary policy are neutral. This could only be true under the idealized conditions.

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