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21 August 2020

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August, 2020

Abstract

We consider a weaker notion of strategy-proofness called upper contour strategy-proofness (UCSP) and investigate its relation with strategy-proofness (SP) for random social choice functions (RSCFs). Apart from providing a simpler way to check whether a given RSCF is SP or not, UCSP is useful in modeling the incentive structures for certain behavioral agents. We show that SP is equivalent to UCSP and elementary monotonicity on any domain satisfying the upper contour no restoration (UCNR) property. To analyze UCSP on multi-dimensional domains, we consider some block structure over the preferences. We show that SP is equivalent to UCSP and block monotonicity on domains satisfying the block restricted upper contour preservation property. Next, we analyze the relation between SP and UCSP under unanimity and show that SP becomes equivalent to UCSP and multi-swap monotonicity on any domain satisfying the multi-swap UCNR property. Finally, we show that if there are two agents, then under unanimity, UCSP alone becomes equivalent to SP on any domain satisfying the swap UCNR property. We provide applications of our results on the unrestricted, single-peaked, single-crossing, single-dipped, hybrid, and multi-dimensional domains such as lexicographically separable domains with one component ordering and domains under committee formation.

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JEL Classification: D71, D82.

Keywords: strategy-proofness; upper contour strategy-proofness; unanimity; elementary monotonicity; block monotonicity; multi-swap monotonicity;

1. Introduction

We consider a society with n agents and m alternatives. Each agent has a preference over the alternatives and a *random social choice function* (RSCF) selects a probability distribution over the alternatives at every collection of preferences of the agents. An RSCF is *strategy-proof* (SP) if no agent can increase the probability of any upper contour set of his preference by misreporting his preference. It is *upper contour strategy-proof* (UCSP) if no agent can increase the probability of an upper contour set by misreporting his preference *maintaining* the same upper contour set. Clearly, UCSP is much weaker than SP.

The main objective of this paper is to find conditions on a domain so that UCSP and SP become equivalent (with or without unanimity). The question arises: why is it important to explore such an equivalence? The most important reason is to provide a simpler way to check if a given RSCF is strategy-proof. The fact that UCSP is a significant weakening of SP is established in the literature; in fact, the proportion of the number of constraints under UCSP to that under SP goes to zero as the number agents n goes to infinity (see Chun and Yun (2019) for a detailed account of this). Another reason we study UCSP is that it can be used to model behavioral agents. Even though agents have complete preferences, they might have alternatives classified as acceptable, unacceptable, etc. Due to behavioral reasons such as ethics, stigma, or self-guilt, they might be uncomfortable to vouch for some candidate who they very much dislike. Consequently, they maintain some upper contour sets of their sincere preference while manipulating and look for an increase of the probability of these sets only.

Local strategy-proofness (LSP) is another well-studied weakening of SP (see Sato (2013), Carroll (2012), Kumar et al. (2020) for details). LSP requires SP only for profiles that are "neighbors" (or, close in some sense). The reason we work with UCSP is that there is not much progress with the LSP approach for RSCFs, particularly for multi-dimensional (separable) domains (in comparison with that for deterministic social choice functions where necessary and sufficient conditions are known). Cho (2016) shows that LSP is equivalent to SP (without unanimity) on any domain satisfying the "*swap no-restoration*" property. Multi-dimensional separable domains are not "*swap* *connected*" and hence this result does not apply to these domains (among others). To our understanding, it is "hard" to keep track of probabilities by LSP on a domain that is not swap-connected, which is why the literature has not progressed much in this direction. This motivates us to take the new approach of UCSP, which has its own importance as well.

We provide a condition on a domain so that SP becomes equivalent to UCSP and a property called *elementary monotonicity* (EM) (see Majumdar and Sen (2004), Mishra (2016)). EM is an immediate consequence of SP and cannot be avoided in characterizing SP. Our result applies to a large class of one-dimensional domains of practical importance such as the unrestricted domain, single-peaked domain, single-crossing domain, single-dipped domain, hybrid domain, etc. Next, we explore the equivalence of SP and UCSP when there is a "block structure" over the preferences. We introduce a generalization of EM called *block monotonicity* (BM) and show that if a domain satisfies the *block restricted upper contour preservation* (BRUCP) property then SP becomes equivalent with UCSP and BM. This result applies to many well-known multi-dimensional domains like Lexicographically separable domains with one component ordering and domains under committee formation. Finally, we investigate the equivalence of SP and UCSP under unanimity. We introduce a restricted version of EM called *multi-swap monotonicity* and show that if a domain satisfies the multi-swap upper contour no-restoration (multi-swap UCNR) property, then a unanimous RSCF on it is SP if and only if it is UCSP and satisfies multi-swap monotonicity. We further show that if there are two agents and the domain satisfies the swap upper contour no-restoration (swap UCNR property), then a unanimous RSCF on it is SP if and only if it is UCSP. In other words, when there are two agents, UCSP *alone* becomes equivalent to SP on the mentioned class of domains. These results apply to a large class of well-known domains such as the unrestricted domain, single-peaked domain, single-crossing domain, single-dipped domain, hybrid domain.

It is worth mentioning that our results (for both with and without unanimity) apply to many more domains relative to the domains on which the equivalence of LSP and SP is known to hold for RSCFs. Moreover, we provide our analysis for both unanimous and non-unanimous RSCFs on a common platform; to the best of our knowledge, these two cases are treated separately in the context of LSP.

1.1 RELATED LITERATURE

The literature on UCSP is quite limited: Chun and Yun (2019) introduces this notion in the context of random assignment problems and ours is the first paper to explore it for RSCFs. However, the literature on SP RSCFs is quite extensive. It dates back to Gibbard (1977) where he shows that an RSCF on the unrestricted domain is unanimous and strategy-proof if and only if it is a *random dictatorial* rule. For the case of continuous alternatives, Ehlers et al. (2002) characterise unanimous and strategy-proof RSCFs on maximal single-peaked domains, and Dutta et al. (2002) characterise unanimous and strategy-proof RSCFs on multi-dimensional single-peaked domains. Later, Peters et al. (2014) show that every unanimous and strategy-proof RSCF on maximal single-peaked domain is a convex combination of min-max rules. Pycia and Ünver (2015) establish a similar result by using the theory of totally unimodular matrices from combinatorial integer programming. Peters et al. (2017), Roy and Sadhukhan (2019), Roy and Sadhukhan (2020), and Peters et al. (2020) characterize unanimous and strategy-proof RSCFs on single-dipped domains, Euclidean domains, generalized intermediate domains, and single-peaked domains on graphs, respectively.

The rest of the paper is organized as follows. Section 2 introduces the model and basic definitions regarding domains and random social choice functions. Sections 3 and 4 present our results on the equivalence of SP and UCSP without unanimity and with unanimity, respectively. Section 5 analyses the equivalence of SP and UCSP under unanimity when there are two agents. Finally, Section 6 presents the applications of our results.

2. PRELIMINARIES

Let $N = \{1, \ldots, n\}$ be a finite set of agents. Except where otherwise mentioned, $n \geq 2$. Let A be a finite set of alternatives. For notational convenience, whenever it is clear from the context, we do not use braces for singleton sets, for instance, we denote the set $\{i\}$ by i.

2.1 PREFERENCES, DOMAINS, AND THEIR PROPERTIES

A complete, reflexive, antisymmetric, and transitive binary relation over A (also called a linear order) is called a *preference*. We denote by $\mathbb{L}(A)$ the set of all preferences over A. Let P be a preference. For distinct $a,b \in A$, aPb is interpreted as "a is strictly preferred to b according to P". For $k \in \{1,\dots,m\}$, by $r_k(P)$ we denote the k-th ranked alternative in P, that is, $r_k(P) = a$ if and only if $|\{b \in A \mid bPa\}| = k$. For $a \in A$, the *upper contour set* of a in P, denoted by $U(a,P)$, is defined as the set of alternatives that are as good as a in P, that is, $U(a,P) = \{b \in A \mid bPa\}$.[1] We say that a set of alternatives U is an upper contour set at P if $U = U(a,P)$ for some $a \in A$.

A subset $\mathscr{D}$ of $\mathbb{L}(A)$ is called a *domain* (of admissible preferences). A domain $\mathscr{D}$ is called *unrestricted* if it contains all preferences over A, that is, $\mathscr{D} = \mathbb{L}(A)$. A preference profile, denoted by P_N is an element $(P_1,\dots,P_n)$ of $\mathscr{D}^n = \mathscr{D} \times \cdots \times \mathscr{D}$.

A path is a sequence of preferences. A path $(P(1),\dots,P(k))$ is said to have an (a,b)*-restoration* for two distinct alternatives a and b if their relative ordering changes more than once along the sequence, that is, there are $1 \leq r < s < t \leq k$ such that the relative ordering of a and b in $P(r)$ and $P(s)$ are different, and that in $P(s)$ and $P(t)$ are different. A path $(P(1),\dots,P(k))$ is said to have *no-restoration* if it does not have (a,b)-restoration for all distinct $a,b \in A$.

2.2 RANDOM SOCIAL CHOICE FUNCTIONS AND THEIR PROPERTIES

A *Random Social Choice Function* (RSCF) is a function $\varphi : \mathscr{D}^n \to \Delta A$, where ΔA denotes the set of probability distributions on A.

For $B \subseteq A$ and $P_N \in \mathscr{D}^n$, we define $\varphi_B(P_N) = \sum_{a \in B} \varphi_a(P_N)$, where $\varphi_a(P_N)$ is the probability of a at $\varphi(P_N)$.

Unanimity is a well-known property of an RSCF. Unanimity ensures that whenever all the agents in a society agree on their top-ranked alternatives, that alternative is chosen (with probability 1).

Definition 2.1. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is called **unanimous** if for all $a \in A$ and all

[1] Observe that $a \in U(a,P)$ by reflexivity.

$P_N \in \mathscr{D}^n$, we have

$$[r_1(P_i) = a \text{ for all } i \in N] \Rightarrow [\varphi_a(P_N) = 1].$$

An RSCF is strategy-proof if no agent can increase the probability of any upper contour set (in his/her sincere preference) by misreporting his/her (sincere) preferences.

Definition 2.2. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is **strategy-proof (SP)** if for all $i \in N$, all $P_N \in \mathscr{D}^n$, all $P'_i \in \mathscr{D}$, and all upper contour sets U of P, we have

$$\varphi_U(P_i, P_{-i}) \geq \varphi_U(P'_i, P_{-i}).$$

Next, we introduce the key notion of this paper. It is a weaker version of strategy-proofness. It says that whenever an agent misreports his/her (sincere) preference P as some preference P', the probability of only those upper contour sets, that remain the same in both these preferences, will not increase. For an example, suppose that the alternatives are $\{a,b,c,d,e,f\}$, and $P = abcdef$ and $P' = cabedf$ are two preferences in $\mathscr{D}$. Here, by $P = abcdef$, we mean $r_1(P) = a$, $r_2(P) = b$, $r_3(P) = c$, and so on. Note that $\{a,b,c\}$ and $\{a,b,c,d,e\}$ (and, trivially, $\{a,b,c,d,e,f\}$) are the sets that are upper contour sets in both P and P'. Upper contour strategy-proofness says that when an agent unilaterally misreports his/her sincere preference P as P', probabilities of the sets $\{a,b,c\}$ and $\{a,b,c,d,e\}$ will not increase. Note that this is much weaker than strategy-proofness which requires that the probabilities of each upper contour set in P, that is, each of the sets $\{a\}$, $\{a,b\}$, $\{a,b,c\}$, $\{a,b,c,d\}$, and $\{a,b,c,d,e\}$, will not increase by the mentioned misreport.

Definition 2.3. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is **upper contour strategy-proof (UCSP)** if for all $i \in N$, all $P_i, P'_i \in \mathscr{D}$, all $P_{-i} \in \mathscr{D}^{n-1}$, and all $U \subseteq A$ such that U is an upper contour set in both P_i and P'_i, we have

$$\varphi_U(P_i, P_{-i}) \geq \varphi_U(P'_i, P_{-i}).$$

3. Equivalence of upper contour strategy-proofness and strategy-proofness in the absence of unanimity

We have separate results for domains satisfying the "UCNR" property and domains satisfying the "BRUCP" property. The UCNR property majorly applies to one-dimensional domains, whereas the BRUCP property applies to multi-dimensional domains. It is worth mentioning that, as it seems, our results on UCNR domains do not follow from those on BRUCP ones.

3.1 Results on domains satisfying the UCNR property

We begin with providing an example to show that UCSP alone cannot ensure SP on the unrestricted domain. This example clarifies that the same happens on restricted domains such as single-peaked, single-crossing, single-dipped, etc. (in fact, the same RSCF will work).

Example 3.1. Suppose $N = \{1,2\}$ and $A = \{a,b\}$. Consider the RSCF $f : \mathbb{L}^2(A) \to A$ given in Table 1.

	ab	ba
ab	b	a
ba	b	a

Table 1

It is easy to check that this RSCF is upper contour strategy-proof but not strategy-proof. This is because at (ab,ab) the outcome is b but at (ab,ba) the outcome is a, which means agent 2 manipulates at (ab,ab) via ba.

In view of Example 3.1, we impose some additional restrictions on an RSCF to derive an equivalence between UCSP and SP. To ease the presentation, we denote by $\mathscr{U}(P)$ the set of all upper contour sets of a preference P, and by $P \equiv \cdots ab \cdots$ a preference in which a is ranked just above b.

Definition 3.1. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is **elementary monotonic (EM)** if for all $i \in N$, all $a,b \in A$, all $P_i \equiv \cdots ab \cdots, P_i' \equiv \cdots ba \cdots \in \mathscr{D}$ with $U(b,P_i) = U(a,P_i')$, and all

$P_{-i} \in \mathscr{D}^{n-1}$, we have $\varphi_a(P_i, P_{-i}) \geq \varphi_a(P'_i, P_{-i})$.

Next, we define the notion of *upper contour no-restoration*. A path $(P(1), \ldots, P(r))$ satisfies the *upper contour no-restoration (UCNR)* property with respect to an upper contour set U of $P(1)$ if it has no (a,b)-restoration for every $a \in U$ and every $b \notin U$. A domain satisfies *upper contour no-restoration (UCNR)* if for every two preferences P and P' and every upper contour set U of P, there is a UCNR path from P to P' such that an alternative outside U can overtake an alternative in U along the path only through a swap between the two maintaining the respective upper contour sets.

Definition 3.2. A domain $\mathscr{D}$ satisfies the **upper contour no-restoration (UCNR)** property if for all $P, P' \in \mathscr{D}$ and all $U \in \mathscr{U}(P)$, there exists a UCNR path $(P = P(1), \ldots, P(k) = P')$ in $\mathscr{D}$ with respect to U such that for all $l < k$, $aP(l)b$ and $bP(l+1)a$ for some $a \in U$ and $b \notin U$ implies $P(l) \equiv \cdots ab \cdots$ and $P(l+1) = \cdots ba \cdots$ with $U(b, P(l)) = U(a, P(l+1))$.

Remark 3.1. Let $(P(1), \ldots, P(k))$ be a UCNR path with respect to an upper contour set U of $P(1)$. The UCNR property implies that if $aP(l)b$ and $bP(l+1)a$ for some $l < k$, $a \in U$, and $b \notin U$, then

$$cP(l)d \iff cP(l+1)d \text{ for all } c \in \{a,b\} \text{ and } d \notin \{a,b\}.$$

Note that in addition to the unrestricted domain, most restricted domains of practical importance, such as single-peaked, single-crossing, single-dipped, etc., satisfy the UCNR property.

Our next theorem says that SP is equivalent to the combination of UCSP and EM for RSCFs on domains satisfying the UCNR property.

Theorem 3.1. *An RSCF on a domain satisfying the UCNR property is SP if and only if it is UCSP and EM.*

The proof of the theorem is relegated to Appendix A.

3.2 RESULTS ON DOMAINS SATISFYING THE BRUCP PROPERTY

A **block** $X \subseteq A$ in a preference P is a set of contiguous (consecutively ranked) alternatives, that is, a set of alternatives X is a block in P if there are $1 \leq s \leq t \leq m$ such that $X = \{r_s(P), \ldots, r_t(P)\}$. A pair of disjoint blocks (X,Y) in a preference P is called **adjacent** if the alternatives in Y appear just below those in X in the preference P, that is, if there are $1 \leq s \leq t < u \leq m$ such that $X = \{r_s(P), \ldots, r_t(P)\}$ and $Y = \{r_{t+1}(P), \ldots, r_u(P)\}$. We call two preferences P and P' *block adjacent* if multiple pairs of adjacent blocks flip from P to P' without changing the relative ordering of the alternatives within a block.

Definition 3.3. Two preferences P and P' are **block adjacent** if

(i) there exists a collection of adjacent blocks $(X_1,Y_1), \ldots, (X_k,Y_k)$ in P such that $(Y_1,X_1), \ldots, (Y_k,X_k)$ are adjacent blocks in P', and

(ii) aPb and $bP'a$ if and only if $a \in X_l$ and $b \in Y_l$ for some $l = \{1, \ldots, k\}$.

In such situation, we write $P' = P[(X_1,Y_1), \ldots, (X_k,Y_k)]$ and say P' is $(X_1,Y_1), \ldots, (X_k,Y_k)$ flip of P.

Block monotonicity imposes strategy-proofness restricted to swapping blocks: if an agent i unilaterally swaps a pair of blocks (X,Y), then the probability of any upper contour set of $X \cup Y$ according to his sincere preference cannot increase.

Definition 3.4. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ satisfies **block monotonicity** (BM) if for all $i \in N$, all block adjacent preferences P_i and P'_i of i with $P'_i = P_i[(X_1,Y_1), \ldots, (X_k,Y_k)]$, all $P_{-i} \in \mathscr{D}^{n-1}$, and all $l \in \{1, \ldots, k\}$, we have $\varphi_U(P_i,P_{-i}) \geq \varphi_U(P'_i,P_{-i})$ for all upper contour sets U of $P_i|_{X_l \cup Y_l}$.

The *block restricted upper contour preservation* property says that for any two preferences P and P', and any upper contour set U of P, there is a path from P to P' such that (i) any two consecutive preferences in the path differ by the swaps of multiple adjacent blocks, and (ii) whenever a pair of blocks (X,Y) swaps from a preference $\hat{P}$ to the next one in the path, it must be that the alternatives in U form an upper contour set in $\hat{P}|_{\{X \cup Y\}}$.

Definition 3.5. A domain $\mathscr{D}$ satisfies the **block restricted upper contour preservation (BRUCP)** property if for all $P, P' \in \mathscr{D}$ and all $U \in \mathscr{U}(P)$, there exists a path $(P(1) = P, \ldots, P(v) = P')$ in $\mathscr{D}$ such that for all $u \leq v$, there exist $(X_1^u, Y_1^u), \ldots, (X_k^u, Y_k^u)$ such that

(i) $P(u+1) = P(u)[(X_1^u, Y_1^u), \ldots, (X_k^u, Y_k^u)]$, and

(ii) $U \cap (X_l^u \cup Y_l^u)$ is an upper contour set of $P(u)|_{X_l^u \cup Y_l^u}$ for all $l \leq k$.

Theorem 3.2. *An RSCF on a domain satisfying the BRUCP is SP if and only if it is UCSP and BM.*

The proof of this theorem is relegated to Appendix B.

4. Equivalence of upper contour strategy-proofness and strategy-proofness in the presence of unanimity

Two preferences P and P' are called *swap-local*, denoted by $P \sim P'$, if they differ by a swap of two consecutively ranked alternatives. For instance, the preferences $abcdef$ and $abdcef$ are swap-local. Similarly, two preferences P and P' are called *multiple-swap-local*, denoted by $P \approx P'$, if the differ by the swaps of multiple pairs of consecutively ranked alternatives. For instance the preferences $abcdef$ and $abdcfe$ are multiple-swap-local as they differ by the swaps of the pairs (c,d) and (e,f). A path $(P(1) = P, \ldots, P(r) = P')$ from a preference P to a preference P' is a *multiple-swap* path if for all $s < r$, $\tau(P(s)) = \tau(P(s+1))$ implies $P(s) \sim P(s+1)$, and $\tau(P(s)) \neq \tau(P(s+1))$ implies $P(s) \approx P(s+1)$. It is a swap path if $P(s) \sim P(s+1)$ for all $s \in \{1, \ldots, r\}$. Clearly, every swap path is a multi-swap path.

Definition 4.1. A domain $\mathscr{D}$ satisfies the **multi-swap (or, swap) UCNR** property if for all $P, P' \in \mathscr{D}$ and for all $U \in \mathscr{U}(P)$, there exists a multi-swap (or, swap) path from P to P' satisfying the UCNR property with respect to U.

A domain satisfies **swap no-restoration (swap NR)** if, between any two preferences in it, there exists a swap path having no-restoration. It is easy to verify that if a domain satisfies the swap NR property, then it also satisfies the multi-swap UCNR property. We formally present this observation in the following remark for future reference.

Remark 4.1. The swap NR property implies the swap UCNR property on any domain.

Let P_i and P_i' are multiple-swap-local preferences having different top-ranked alternatives and suppose an agent unilaterally changes his preference from P_i to P_i'. Multi-swap monotonicity says that an RSCF will assign (weakly) higher probabilities to the alternatives whose ranks are strictly improved (that is, the alternatives that swap up from P_i to P_i'.).

Definition 4.2. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ satisfies **multi-swap monotonicity** if for all $i \in N$, all multiple-swap-local $P_i \equiv \cdots ab \cdots$, $P_i' \equiv \cdots ba \cdots \in \mathscr{D}$ with $\tau(P_i) \neq \tau(P_i')$, and all $P_{-i} \in \mathscr{D}^{n-1}$, we have $\varphi_b(P_i', P_{-i}) \geq \varphi_b(P_i, P_{-i})$.

Theorem 4.1. *A unanimous RSCF on a domain satisfying the multi-swap UCNR property is SP if and only if it is UCSP and multi-swap monotonicity.*

The proof is relegated to Appendix C.

Remark 4.2. Let us recall the definition of the UCNR property. Roughly speaking, it says that for all preferences P and P' in the domain and all upper contour sets U of P, there exists a UCNR path such that whenever an alternative outside U overtakes an alternative in U, it does this (i) through a swap between the two alternatives, and (ii) without changing the upper contour set. Note that if the mentioned UCNR path is multi-swap, then conditions (i) and (ii) will be automatically satisfied. This implies that if a domain satisfies the multi-swap UCNR property, it also satisfies the UCNR property.

We obtain the following corollary from Remarks 4.1 and 4.2.

Corollary 4.1. *If a domain satisfies the swap NR property, then it satisfies the UCNR property.*

It follows from Remark 4.1 and Corollary 4.1 that if a domain satisfies the swap NR property, it satisfies both the multi-swap UCNR property and the UCNR property. Therefore, we obtain the following corollary from Theorems 3.1 and 4.1.

Corollary 4.2. *If a domain satisfies the swap NR property, then every RSCF satisfying UCSP and EM on it is SP, and every unanimous RSCF satisfying UCSP and multi-swap monotonicity on it is SP.*

5. THE CASE OF TWO AGENTS

We show in this section that if we strengthen the multi-swap UCNR property by requiring the swap UCNR property, then under unanimity, UCSP alone becomes equivalent to SP for the case of two agents.

Theorem 5.1. *Let $N = \{1,2\}$ and let $\mathscr{D}$ satisfy the swap UCNR property. Then, a unanimous RSCF on $\mathscr{D}$ is SP if and only if it is UCSP.*

The proof is relegated to Appendix D.

In view of Remark 4.1 and Theorem 5.1, we have the following corollary.

Corollary 5.1. *Let $N = \{1,2\}$. If a domain satisfies the swap NR property, then every unanimous RSCF is SP if and only if it is UCSP.*

6. APPLICATIONS OF OUR RESULTS

In this section, we provide some applications of our results to well-known domains by exploring the equivalence of UCSP and SP.

6.1 THE UNRESTRICTED DOMAIN

It is easy to verify that the unrestricted domain satisfies the swap NR property. Therefore, by Corollary 4.2, we obtain Corollary 6.1 and Corollary 6.2; and Corollary 5.1, we obtain Corollary 6.3.

Corollary 6.1. *An RSCF on the unrestricted domain is SP if and only if it is UCSP and EM.*

Corollary 6.2. *A unanimous RSCF on the unrestricted domain is SP if and only if it is UCSP and multi-swap monotonicity.*

Corollary 6.3. *Let $n = 2$. A unanimous RSCF on the unrestricted domain is SP if and only if it is UCSP.*

6.2 RESTRICTED DOMAINS

In this section, we apply our results to well-known restricted domains in the literature. Let the set of alternatives be $A = \{a_1, \dots, a_m\}$ with a prior ordering $\prec$ given by $a_1 \prec \cdots \prec a_m$.

6.2.1 SINGLE-PEAKED DOMAINS

A preference is single-peaked if it decreases as one goes far from its peak (top-ranked alternative) in any particular direction. More formally, A preference P is called *single-peaked* if for all $a, b \in A$, $[r_1(P) \preceq a \prec b$ or $b \prec a \preceq r_1(P)]$ implies aPb. A domain is called *single-peaked* if each preference in the domain is single-peaked.

Corollary 6.4 and Corollary 6.5 follow from Corollary 4.2; and Corollary 6.6 follows from Corollary 5.1.

Corollary 6.4. *An RSCF on a single-peaked domain satisfying swap NR is SP if and only if it is UCSP and EM.*

Corollary 6.5. *A unanimous RSCF on a single-peaked domain satisfying swap NR is SP if and only if it is UCSP and multi-swap monotonicity.*

Corollary 6.6. *Let $n = 2$. A unanimous RSCF on a single-peaked domain satisfying swap NR is SP if and only if it is UCSP.*

6.2.2 SINGLE-DIPPED DOMAINS

A preference is single-dipped if it declines as one goes far away from its worst (bottom-ranked) alternative in any particular direction. Recall that for a preference P and some $k \in \{1, \dots, m\}$, we denote by $r_k(P)$ the k-th ranked alternative in P. A preference P is called *single-dipped* if for all $a, b \in A$, $[r_m(P) \preceq a \prec b$ or $b \prec a \preceq r_m(P)]$ implies bPa. A domain is called *single-dipped* if each preference in the domain is single-dipped, and is called *maximal single-dipped* if it contains all single-dipped preferences.

Corollary 6.7 and Corollary 6.8 follow from Corollary 4.2; Corollary 6.9 follows from Corollary 5.1.

Corollary 6.7. *An RSCF on a single-dipped domain satisfying swap NR is SP if and only if it is UCSP and EM.*

Corollary 6.8. *A unanimous RSCF on a single-dipped domain satisfying swap NR is SP if and only if it is UCSP and multi-swap monotonicity.*

Corollary 6.9. *Let $n = 2$. A unanimous RSCF on a single-peaked domain satisfying swap NR is SP if and only if it is UCSP.*

6.2.3 SINGLE-CROSSING DOMAINS

Single-crossing domains are studied in Saporiti (2009) in the context of strategic social choice. A domain $\mathscr{D}$ is *single-crossing* if there is an ordering $\lhd$ over $\mathscr{D}$ such that for all $a, b \in A$ and all $P, P' \in \mathscr{D}$, $[a \prec b, P \lhd P', \text{ and } bPa] \implies bP'a$. In words, a single-crossing domain is one for which the preferences can be ordered in a way such that every pair of alternatives switches their relative ranking at most once along that ordering. A single-crossing domain $\bar{\mathscr{D}}$ is *maximal* if there does not exist another single-crossing domain that is a strict superset of $\bar{\mathscr{D}}$. Note that a maximal single-crossing domain with m alternatives contains $m(m-1)/2+1$ preferences.[2] A domain $\mathscr{D}$ is *successive single-crossing* if there is a maximal single-crossing domain $\bar{\mathscr{D}}$ with respect to some ordering $\lhd$ and two preferences $P', P'' \in \bar{\mathscr{D}}$ with $P' \unlhd P''$ such that $\mathscr{D} = \{P \in \bar{\mathscr{D}} \mid P' \unlhd P \unlhd P''\}$.[3]

In the following example, we present a maximal single-crossing domain and a successive single-crossing domain with 5 alternatives.

Example 6.1. Let the set of alternatives be $A = \{a_1, a_2, a_3, a_4, a_5\}$ with the prior order $a_1 \prec \cdots \prec a_5$. The domain $\bar{\mathscr{D}} = \{a_1a_2a_3a_4a_5,\ a_2a_1a_3a_4a_5,\ a_2a_3a_1a_4a_5,\ a_2a_3a_4a_1a_5,\ a_2a_4a_3a_1a_5,\ a_4a_2a_3a_1a_5,\ a_4a_2a_3a_5a_1, a_4a_3a_2a_5a_1, a_4a_3a_5a_2a_1,\ a_4a_5a_3a_2a_1,\ a_5a_4a_3a_2a_1\}$ is a maximal single-crossing domain with respect to the ordering $\lhd$ given by $a_1a_2a_3a_4a_5 \lhd a_2a_1a_3a_4a_5 \lhd a_2a_3a_1a_4a_5 \lhd a_2a_3a_4a_1a_5 \lhd a_2a_4a_3a_1a_5 \lhd a_4a_2a_3a_1a_5 \lhd a_4a_2a_3a_5a_1 \lhd a_4a_3a_2a_5a_1 \lhd a_4a_3a_5a_2a_1 \lhd a_4a_5a_3a_2a_1 \lhd a_5a_4a_3a_2a_1$ since every pair of alternatives change their relative ordering at most once along this ordering. Note that the cardinality of A is 5 and that

[2]For details see Saporiti (2009).

[3]By $P \unlhd P'$, we mean either $P = P'$ or $P \lhd P'$.

of $\bar{\mathscr{D}}$ is $5(5-1)/2+1 = 11$. The domain $\mathscr{D} = \{a_1a_2a_3a_4a_5,\ a_2a_1a_3a_4a_5,\ a_2a_3a_1a_4a_5,\ a_2a_3a_4a_1a_5,\ a_2a_4a_3a_1a_5,\ a_4a_2a_3a_1a_5\}$ is a successive single-crossing domain since it contains all the preferences in-between $a_1a_2a_3a_4a_5$ and $a_4a_2a_3a_1a_5$ in the maximal single-crossing domain $\bar{\mathscr{D}}$. □

Lemma 6.1. *Every successive single-crossing domain satisfies swap NR property.*

Proof: Let $\mathscr{D}$ ba a successive single-crossing domain. We show that $\mathscr{D}$ satisfies swap NR property. Consider $P, P' \in \mathscr{D}$. Without loss of generality assume that $P \lhd P'$. Let $\mathscr{P} = \{\hat{P} \in \mathscr{D} | P \unlhd \hat{P} \unlhd P'\}$. Construct a path $(P(1) = P, \ldots, P(k) = P')$ from $\mathscr{P}$ such that for all $s < t$, $P(s) \lhd P(t)$. Note that this path is a swap path as for all $s < k$, we have $P(s) \sim P(s+1)$. We show that this path has no-restoration. Assume for contradiction there exist $a, b \in A$ such that the path has (a,b)-restoration. As $aP(1)b$, this means there exists $s < t$ such that $bP(s)a$ and $aP(t)b$. Since $1 < s$, by the construction, we have $P(1) \lhd P(s)$. This together with $aP(1)b$ and $bP(s)a$ imply $a \prec b$. Again, as $s < t$ we have by the construction $P(s) \lhd P(t)$. Therefore, $a \prec b$ and $bP(s)$ imply $bP(t)a$, a contradiction to our assumption that $aP(t)b$. This completes the proof of the lemma. ■

In view of Lemma 6.1, Corollary 6.10 and Corollary 6.11 follow from Corollary 4.2; Corollary 6.12 follows from Corollary 5.1.

Corollary 6.10. *An RSCF on a successive single-crossing domain is SP if and only if it is UCSP and EM.*

Corollary 6.11. *A unanimous RSCF on a successive single-crossing domain is SP if and only if it is UCSP and multi-swap monotonicity.*

Corollary 6.12. *Let $n = 2$. A unanimous RSCF on a successive single-crossing domain is SP if and only if it is UCSP.*

6.2.4 Hybrid domains

Hybrid domains are introduced in Chatterji et al. (2020). These domains are, in a sense, a mixture of the single-peaked domain and the unrestricted domain. Preferences in such a domain satisfy single-peakedness only outside an interval of the alternatives.

Definition 6.1. Let $1 \leq \underline{k} < \overline{k} \leq m$. A preference P_i is called $(\underline{k}, \overline{k})$**-hybrid** if the following two conditions are satisfied:

(i) For all $a_r, a_s \in L$ or $a_r, a_s \in R$, $[a_r \prec a_s \prec r_1(P_i)$ or $r_1(P_i) \prec a_s \prec a_r] \Rightarrow [a_s P_i a_r]$.

(ii) $[r_1(P_i) \in L] \Rightarrow [a_{\underline{k}} P_i a_r$ for all $a_r \in M$ with $a_r \neq a_{\underline{k}}]$ and
$[r_1(P_i) \in R] \Rightarrow [a_{\overline{k}} P_i a_s$ for all $a_s \in M$ with $a_s \neq a_{\overline{k}}]$.

Let $\mathscr{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ denote **the** $(\underline{k}, \overline{k})$**-hybrid domain** which contains *all* $(\underline{k}, \overline{k})$-hybrid preferences. Note that $\mathscr{D}_{\mathrm{H}}(\underline{k}', \overline{k}') \subseteq \mathscr{D}_{\mathrm{H}}(\underline{k}, \overline{k})$ for all $\underline{k} \leq \underline{k}' < \overline{k}' \leq \overline{k}$.

Chatterji et al. (2020) show that every $(\underline{k}, \overline{k})$-hybrid domain satisfies the swap NR property. Therefore, by Corollary 4.2, we obtain Corollary 6.13; and Corollary 6.14; and by Corollary 5.1, we obtain Corollary 6.15.

Corollary 6.13. *An RSCF on $\mathscr{D}_H(\underline{k}, \overline{k})$ is SP if and only if it is UCSP and EM.*

Corollary 6.14. *A unanimous RSCF on $\mathscr{D}_H(\underline{k}, \overline{k})$ is SP if and only if it is UCSP and multi-swap monotonicity.*

Corollary 6.15. *Let $n = 2$. A unanimous RSCF on $\mathscr{D}_H(\underline{k}, \overline{k})$ is SP if and only if it is UCSP.*

6.3 Lexicographically separable domains

Let $M = \{1, \ldots, m\}$ be a finite set of m components. For each component k, the component set A^k contains finitely many alternatives available in component k and $|A_k| \geq 2$. For any $K \subseteq M$, $A^K = \prod_{k \in K} A^k$, denotes the set of alternatives available in components in K. The set of (multi-dimensional) alternatives is given by A^M. For ease of presentation, we write A instead of A^M.

We start the investigation from *lexicographically separable* preferences. First, a lexicographic order, that is, a linear order over M, is fixed to characterize an agent's attitude towards all components. Second, on each component set, a linear order is independently specified, which is referred to as a *marginal preference*. Last, a lexicographically separable preference over A is established such that given two distinct alternatives, according to the most important disagreed component, the alternative owning a better element is always preferred.

Definition 6.2. A preference P is **lexicographically separable**, if there exists a (unique) lexicographic order P^0 and a (unique) marginal preference P^k for each $k \in M$ such that for all $a, b \in A$, we have $[a_l P^l b_l$ and $a_k = b_k$ for all $k P^0 l] \implies [aPb]$.

Evidently, a lexicographically separable preference P can be uniquely represented by an $m+1$-tuple of the lexicographic order P^0 and marginal preferences $P^1, \ldots, P^m$, that is, $P = (P^0, P^1, \ldots, P^m)$. Let $\mathscr{D}_{LS} = (\mathscr{D}_0, \mathscr{D}_1, \ldots, \mathscr{D}_m)$ denote the lexicographically separable domain containing all lexicographically separable preferences with $\mathscr{D}_0$ as the set of lexicographic orders and marginals $\mathscr{D}_k$, $k \in M$.

6.3.1 One component order lexicographic domain

In this section we assume that $\mathscr{D}_0 = \{P_0\}$. Without loss of generality assume that $1 P_0 \ldots P_0 m$.

Proposition 6.1. *Let $\mathscr{D}_{LS}$ be a lexicographic domain such that $\mathscr{D}_0 = \{P_0\}$ and each marginal domain $\mathscr{D}_k$ satisfies the no-restoration property. Then $\mathscr{D}_{LS}$ satisfies the BRUCP property.*

The proof is relegated to Appendix E.

6.3.2 Domain under committee formation

Consider the problem where a committee has to be formed by taking members from a given set of candidates. For each candidate, the designer has to decide whether to take him or not. The domain arising in this problem can be modeled as a multi-dimensional lexicographic domain as follows. Let $\mathscr{D}_0 = \mathbb{L}(M)$, $A_k = \{0, 1\}$, and $\mathscr{D}_k = \mathbb{L}(A_k)$ for all $k \in M$. We call $\mathscr{D}_{LS} = (\mathscr{D}_0, \mathscr{D}_1, \ldots, \mathscr{D}_m)$ the domain under committee formation.

Proposition 6.2. *The domain under committee formation satisfies the BRUCP property.*

The proof is relegated to Appendix F.

A. Proof of Theorem 3.1

Proof: Let $\mathscr{D}$ satisfy the UCNR property. We show that every unanimous RSCF on $\mathscr{D}$ is SP if and only if it is UCSP and EM. The "only if" part of the theorem follows from the definitions; we only prove the "if" part. Let φ be a unanimous RSCF satisfying UCSP and EM. Since φ does not satisfy unanimity, it is enough to assume that there is exactly one agent.[4] Consider $P, P' \in \mathscr{D}$ and an upper contour set $U \subseteq A$ of P. We show that $\varphi_U(P) \geq \varphi_U(P')$. Since $\mathscr{D}$ satisfies the UCNR property there exists a UCNR path $(P = P(1), \ldots, P(k) = P')$ in $\mathscr{D}$ with respect to U such that for all $l < k$, $\big[aP(l)b$ and $bP(l+1)a$ for some $a \in U$ and $b \notin U\ \big]$ implies $\big[P(l) \equiv \cdots ab \cdots$ and $P(l+1) = \cdots ba \cdots$ with $U(b, P(l)) = U(a, P(l+1))\big]$.

To show $\varphi_U(P) \geq \varphi_U(P')$, it is enough to show that $\varphi_U(P(l)) \geq \varphi_U(P(l+1))$ for all $l < k$. Consider $P(l)$ such that $l < k$. Let us call a subset $\hat{U}$ of U a block of U in the preference $P(l)$ if it is a maximal subset of U such that its elements are consecutively ranked in $P(l)$, that is, there is no alternative $b \notin U$ that is ranked (according to $P(l)$) between two elements of $\hat{U}$ and there is no $\bar{U}$ with $\hat{U} \subsetneq \bar{U} \subseteq U$ that satisfies the same property. Let $\mathscr{U} = \{U_1, \ldots, U_r\}$ be the collection of all blocks of U. Clearly, $\mathscr{U}$ forms a partition of U. For each $U_s \in \mathscr{U}$, let $U_s' = \{a \in U_s \mid aP(l)b$ and $bP(l+1)a$ for some $b \notin U\}$. In other words, U_s' is the elements of U_s that are overtaken by some alternative outside U from $P(l)$ to $P(l+1)$. Let $U_s'' = U_s \setminus U_s'$.

Consider $U_s \in \mathscr{U}$. Let $\underline{a} \in U_s$ be the worst alternative of U_s according to $P(l)$. We claim that the set U_s' is either empty or $\{\underline{a}\}$. Assume for contradiction that $a \in U_s'$ for some $a \neq \underline{a}$. By the definition of U_s', there is an alternative $c \notin U$ such that $aP(l)c$ and $cP(l+1)a$. Since $\mathscr{D}$ satisfies the UCNR property, by Remark 3.1 we have $xP(l)y \iff xP(l+1)y$ for all $x \in \{a, c\}$ and all $y \notin \{a, c\}$. Since $aP(l)c$, by the definition of U_s and $\underline{a}$, it must be that $aP(l)\underline{a}P(l)c$. This, together with the fact that $cP(l+1)a$, implies either $\underline{a}P(l+1)a$ or $cP(l+1)\underline{a}$, each of which is a contradiction to the UCNR property.

To show $\varphi_U(P(l)) \geq \varphi_U(P(l+1))$, it is sufficient to show $\varphi_{U_s}(P(l)) \geq \varphi_{U_s}(P(l+1))$ for each $U_s \in \mathscr{U}$, which can be ensured by showing $\varphi_{U_s'}(P(l)) \geq \varphi_{U_s'}(P(l+1))$ and $\varphi_{U_s''}(P(l)) \geq \varphi_{U_s''}(P(l+1))$. We show this in the following two claims.

[4]This fact is well-known in the literature.

Claim A.1. $\varphi_{U_s'}(P(l)) \geq \varphi_{U_s'}(P(l+1))$.

Proof of Claim A.1. If $U_s' = \emptyset$, then there is nothing to prove. Suppose $U_s' = \{\underline{a}\}$. Let $b \notin U$ be such that $\underline{a}P(l)b$ and $bP(l+1)\underline{a}$. Since $\mathscr{D}$ satisfies the UCNR property, by Remark 3.1 this means $xP(l)y \iff xP(l+1)y$ for all $x \in \{\underline{a}, b\}$ and all $y \notin \{\underline{a}, b\}$. This in turn implies $U(b, P(l)) = U(\underline{a}, P(l+1))$. Therefore, by EM, we have $\varphi_{\underline{a}}(P(l)) \geq \varphi_{\underline{a}}(P(l+1))$. □

Claim A.2. $\varphi_{U_s''}(P(l)) \geq \varphi_{U_s''}(P(l+1))$.

Proof of Claim A.2. By the construction of U_s, either the best alternative (according to $P(l)$) of U_s is the top-ranked alternative in $P(l)$ or there is an alternative $b \notin U$ which is ranked just above the best alternative of U_s. Similarly, either the worst alternative of U_s is the bottom-ranked alternative in $P(l)$ or there is some alternative $c \notin U$ that is ranked just below the worst of alternative of U_s. In the rest part of the prove, we prove certain facts about b and c; if any of them does not exist, then these facts are vacuously true. Note that by the UCNR property, $bP(l)a$ for all $a \in U_s$ implies $bP(t)a$ for all $a \in U_s$, all $t \geq l+1$, which in particular implies $bP(l+1)a$ for all $a \in U_s$.

Let z be the worst alternative according to $P(l+1)$ among the alternatives which b overtakes from $P(l)$ to $P(l+1)$. We claim $zP(l+1)a$ for all $a \in U_s''$. To see this, consider $a \in U_s''$. Suppose $z \in U$. Since $zP(l)b$ and $bP(l+1)z$, by Remark 3.1, we have $bP(l)x \iff zP(l+1)x$ for all $x \notin \{b, z\}$, which in turn implies $zP(l+1)a$. Suppose $z \notin U$. As $zP(l)bP(l)a$, by the UCNR property, this means $zP(l+1)a$. We now prove the following fact.

Fact A.1. $U(x, P(l)) = U(z, P(l+1))$.

Proof of Fact A.1. Assume for contradiction $U(b, P(l)) \neq U(z, P(l+1))$. This means either there exists $x \in U(b, P(l))$ such that $x \notin U(z, P(l+1))$ or there exists $y \in U(z, P(l+1))$ such that $y \notin U(b, P(l))$. First assume that there there exists $x \in U(b, P(l))$ such that $x \notin U(z, P(l+1))$. Note that $x \neq b$ as by the definition of z, $bP(l+1)z$. Thus $xP(l)b$. This, together with the fact that $bP(l+1)zP(l+1)x$, implies z is not the worst alternative according to $P(l+1)$ among the alternatives that overtake b from $P(l)$ to $P(l+1)$, a contradiction to our assumption on z.

Now assume that there exists $y \notin U(b,P(l))$ such that $y \in U(z,P(l+1))$. Note that $z \neq y$, as by our assumption on z, $zP(l)b$. Suppose $z \in U$. Since $zP(l)b$ and $bP(l+1)z$, by the UCNR property this means $yP(l+1)z \implies yP(l)z$. Moreover, as $zP(l)b$, we have $yP(l)zP(l)b$, a contradiction to the fact that $y \notin U(x,P(l))$. Suppose $z \notin U$. By the UCNR property, $zP(l)bP(l)y$ and $yP(l+1)z$ together imply $y \notin U$. Consider $a \in U_s''$. As $zP(l+1)a$ and $yP(l+1)z$, we have $yP(l+1)a$. Note that by the definition of U_s, $bP(l)y$ and $y \notin U$ together imply $aP(l)y$. But this is a contradiction to $a \in U_s''$ as $aP(l)y$ and $yP(l+1)a$. This completes the proof of Fact A.1. □

Using Fact A.1 and UCSP together, we have

$$\varphi_{U(x,P(l))}(P(l)) = \varphi_{U(x,P(l))}(P(l+1)). \tag{1}$$

Suppose $\underline{a} \in U_s'$. By the UCNR property, this means c overtakes $\underline{a}$ from $P(l)$ to $P(l+1)$ and $U(\underline{a},P(l)) \setminus \underline{a}$ is an upper-contour set of $P(l+1)$. Thus, by UCSP,

$$\varphi_{U(\underline{a},P(l))\setminus\underline{a}}(P(l)) = \varphi_{U(\underline{a},P(l))\setminus\underline{a}}(P(l+1)). \tag{2}$$

Note that by the definition of U_s'', $\underline{a} \in U_s'$ implies $U_s'' = U(\underline{a},P(l)) \setminus (U(b,P(l)) \cup \underline{a})$. Therefore, subtracting (1) from (2) we get $\varphi_{U_s''}(P(l)) = \varphi_{U_s''}(P(l+1))$. Suppose $\underline{a} \notin U_s'$. That means $U_s' = \emptyset$ and hence, $U_s = U_s''$. Let $d \notin U$ be the most preferred alternative in $P(l+1)$ among the alternatives which overtake c from $P(l)$ to $P(l+1)$. Using similar arguments as in the proof of Fact A.1, we can show that $U(\underline{a},P_l) = U(d,P(l+1)) \setminus d$. By upper contour strategy-proofness, this implies

$$\varphi_{U(\underline{a},P(l))}(P(l)) = \varphi_{U(\underline{a},P(l))}(P(l+1)). \tag{3}$$

Subtracting (1) from (3), we have $\varphi_{U_s''}(P(l)) = \varphi_{U_s''}(P(l+1))$. This completes the proof of Claim A.2. □

Claim A.1 and Claim A.2 complete the proof of the theorem. ■

B. PROOF OF THEOREM 3.2

Proof: Let $\mathscr{D}$ be a domain satisfying the BRUCP property and φ be an RSCF on $\mathscr{D}$. We show that φ is SP if and only if it is UCSP and BM. The "only if" part of the theorem follows from the definitions; we only prove the "if" part. Suppose φ is UCSP and BM. As we have argued in the proof of Theorem 3.1, because φ does not satisfy unanimity, it is enough to assume that there is exactly one agent. Consider $P, P' \in \mathscr{D}$. We show that for all $U \in \mathscr{U}(P)$, $\varphi_U(P) \geq \varphi_U(P')$. Since $\mathscr{D}$ satisfies the BRUCP property, there exists a path $(P(1) = P, \ldots, P(v) = P')$ in $\mathscr{D}$ such that $P(u+1) = P(u)[(X_1,Y_1),\ldots,(X_k,Y_k)]$ and $U|_{X_l \cup Y_l}$ is an upper contour set of $P(u)|_{X_l \cup Y_l}$ for all $l \leq k$ and all $u < v$.

Take $u < v$. In order to prove the theorem it is sufficient to show that $\varphi_U(P(u)) \geq \varphi_U(P(u+1))$. By the definition of π, $P(u+1)$ is a flip of $P(u)$, that is, $P(u+1) = P(u)[(X_1,Y_1),\ldots,(X_k,Y_k)]$ where $(X_1,Y_1),\ldots,(X_k,Y_k)$ is a collection of adjacent blocks. Without loss of generality assume that $X_1 P(u) Y_1 P(u) X_2 P(u) \cdots P(u) Y_k$. Let $l \in \{0, \ldots, k\}$ and $z \in A$ be such that $Y_l P(u) z P(u) X_{l+1}$ where $Y_0 = X_{k+1} = \emptyset$. Since $P(u)$ and $P(u+1)$ are adjacent and $z \notin \cup_{l=1}^{k}(X_l \cup Y_l)$, we have $aP(u)z$ if and only if $aP(u+1)z$ for all $a \in A \setminus z$. This implies $U(z,P(u)) = U(z,P(u+1))$ and hence by UCSP, $\varphi_z(P(u)) = \varphi_z(P(u+1))$. Since l and z are arbitrary, we have for all $l \in \{0,\ldots,k\}$,

$$\varphi_z(P(u)) = \varphi_z(P(u+1)) \text{ for all } z \text{ with } Y_l P(u) z P(u) X_{l+1}. \tag{4}$$

For $l \in \{0,\ldots,k\}$, let $U_l \subseteq U$ be such that $Y_l P(u) U_l P(u) X_{l+1}$. Therefore, by (4) we have

$$\varphi_{U_l}(P(u)) = \varphi_{U_l}(P(u+1)) \text{ for all } l \in \{0,\ldots,k\}. \tag{5}$$

For $l \in \{1,\ldots,k\}$, let $\hat{U}_l = U \cap (X_l \cup Y_l)$. We claim $\varphi_{\hat{U}_l}(P(u)) \geq \varphi_{\hat{U}_l}(P(u+1))$ for all $l \in \{1,\ldots,k\}$. Take $l \in \{1,\ldots,k\}$. Since $U|_{X_l \cup Y_l}$ is an upper contour set of $P(u)|_{X_l \cup Y_l}$, $\hat{U}_l$ is an upper contour set of $P(u)|_{X_l \cup Y_l}$, and hence by BM, $\varphi_{\hat{U}_r}(P(u)) \geq \varphi_{\hat{U}_r}(P(u+1))$.

Since $U = (\cup_{l=0}^{k} U_l) \cup (\cup_{l=1}^{k} \hat{U}_l)$, it follows that $\varphi_U(P(u)) \geq \varphi_U(P(u+1))$. This completes the proof of the theorem. ■

C. Proof of Theorem 4.1

Proof: Let $\mathscr{D}$ satisfy the multi-swap UCNR property. We show that every unanimous RSCF on $\mathscr{D}$ is SP if and only if it is UCSP and multi-swap monotonicity. The "only if" part of the theorem follows from the definitions; we only prove the "if" part. Let φ be a unanimous RSCF satisfying UCSP and multi-swap monotonicity. We first prove a lemma.[5]

Lemma C.1. *Let $P_i, P_i' \in \mathscr{D}$ be such that $P_i \sim P_i'$ with aP_ib and $bP_i'a$ for some $a,b \in A$. Further let $P_j, P_j' \in \mathscr{D}$ be such that $P_j|_{\{a,b\}} = P_j'|_{\{a,b\}}$. Then for all $P_{-\{i,j\}} \in \mathscr{D}^{n-2}$,*

$$[\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})] \implies [\varphi(P_i, P_j', P_{-\{i,j\}}) = \varphi(P_i', P_j', P_{-\{i,j\}})].$$

Proof: By UCSP,

$$\varphi_x(P_i, P_j', P_{-\{i,j\}}) = \varphi_x(P_i', P_j', P_{-\{i,j\}}) \text{ for all } x \notin \{a,b\}. \tag{6}$$

Therefore, to show $\varphi(P_i, P_j', P_{-\{i,j\}}) = \varphi(P_i', P_j', P_{-\{i,j\}})$, it is enough to show $\varphi_a(P_i, P_j', P_{-\{i,j\}}) = \varphi_a(P_i', P_j', P_{-\{i,j\}})$. Without loss of generality assume that aP_jb and $aP_j'b$. We first prove the lemma for P_j, P_j' such that $P_j \approx P_j'$. Since aP_jb and $aP_j'b$, there exists an upper contour set $U \in \mathscr{U}(P_j)$ such that $U \in \mathscr{U}(P_j')$, $a \in U$, and $b \notin U$. By UCSP this means

$$\varphi_U(P_i, P_j, P_{-\{i,j\}}) = \varphi_U(P_i, P_j', P_{-\{i,j\}}), \text{ and} \tag{7}$$

$$\varphi_U(P_i', P_j, P_{-\{i,j\}}) = \varphi_U(P_i', P_j', P_{-\{i,j\}}). \tag{8}$$

[5] This lemma is similar to Lemma 1 of Theorem 1 in Chatterji and Zeng (2018).

Now we have

$$\begin{aligned}
\varphi_a(P_i,P'_j,P_{-\{i,j\}}) &= \varphi_U(P_i,P'_j,P_{-\{i,j\}}) - \varphi_{U\setminus a}(P_i,P'_j,P_{-\{i,j\}}) \\
&= \varphi_U(P_i,P_j,P_{-\{i,j\}}) - \varphi_{U\setminus a}(P_i,P'_j,P_{-\{i,j\}}) \text{ by (7)} \\
&= \varphi_U(P'_i,P_j,P_{-\{i,j\}}) - \varphi_{U\setminus a}(P_i,P'_j,P_{-\{i,j\}}) \text{ by the hypothesis of the lemma} \\
&= \varphi_U(P'_i,P_j,P_{-\{i,j\}}) - \varphi_{U\setminus a}(P'_i,P'_j,P_{-\{i,j\}}) \text{ by (6)} \\
&= \varphi_U(P'_i,P'_j,P_{-\{i,j\}}) - \varphi_{U\setminus a}(P'_i,P'_j,P_{-\{i,j\}}) \text{ by (8)} \\
&= \varphi_a(P'_i,P'_j,P_{-\{i,j\}}). \qquad (9)
\end{aligned}$$

This completes the proof of the lemma for P_j and P'_j with $P_j \approx P'_j$.

Now, we prove the lemma for arbitrary P_j and P'_j in $\mathscr{D}$ with aP_jb and aP'_jb. By the definition of the domain, there exists a multi-swap path $\{P(s)\}_{s=1}^t \in \mathscr{D}$ connecting P_j and P'_j such that $aP(s)b$ for all $s \in \{1,\dots,t\}$. As $P(s) \approx P(s+1)$ for all $s < t$, applying (9) repeatedly, we get

$$\begin{aligned}
&[\varphi(P_i,P(1),P_{-\{i,j\}}) = \varphi(P'_i,P(1),P_{-\{i,j\}})] \implies [\varphi(P_i,P(2),P_{-\{i,j\}}) = \varphi(P'_i,P(2),P_{-\{i,j\}})] \\
&[\varphi(P_i,P(2),P_{-\{i,j\}}) = \varphi(P'_i,P(2),P_{-\{i,j\}})] \implies [\varphi(P_i,P(3),P_{-\{i,j\}}) = \varphi(P'_i,P(3),P_{-\{i,j\}})] \\
&\qquad\qquad\qquad\qquad\cdots \\
&[\varphi(P_i,P(t-1),P_{-\{i,j\}}) = \varphi(P'_i,P(t-1),P_{-\{i,j\}})] \implies [\varphi(P_i,P(t),P_{-\{i,j\}}) = \varphi(P'_i,P(t),P_{-\{i,j\}})].
\end{aligned}$$

This completes the proof of the lemma. ■

Next, we prove a claim.

Claim C.1. *$\varphi(P_i,P_{-i}) = \varphi(P'_i,P_{-i})$ for all $P_i \sim P'_i$ with $\tau(P_i) = \tau(P'_i)$ and all $P_{-i} \in \mathscr{D}^{n-1}$.*

Proof of Claim C.1. Suppose $P_i \sim P'_i$, aP_ib, and bP'_ia, where $\tau(P_i) \neq a$. Assume for contradiction that the claim does not hold, that is, $\varphi(P_i,P_{-i}) \neq \varphi(P'_i,P_{-i})$. Consider $j \neq i$. Suppose aP_jb. Since aP_ib, by Lemma C.1 and our assumption for contradiction, we have $\varphi(P_i,\bar{P}_j,P_{-\{i,j\}}) \neq \varphi(P'_i,\bar{P}_j,P_{-\{i,j\}})$ where $\bar{P}_j = P_i$. Similarly, if bP_ja, then $\varphi(P_i,\bar{P}_j,P_{-\{i,j\}}) \neq \varphi(P'_i,\bar{P}_j,P_{-\{i,j\}})$ where $\bar{P}_j = P'_i$. Continuing in this manner we have $\varphi(P_i,\bar{P}_{-i}) \neq \varphi(P'_i,\bar{P}_{-i})$ where for all $j \neq i$, $\bar{P}_j = P_i$ if aP_jb, and $\bar{P}_j = P'_i$ if bP_ja. But this is a contradiction to unanimity as $\tau(P_i) = \tau(P_i')$. This completes the proof of the claim. □

We are now ready to complete the proof of the theorem. Suppose $P_i, P_i' \in \mathscr{D}$, $P_{-i} \in \mathscr{D}^{n-1}$ and $U \in \mathscr{U}(P)$. We show that $\varphi_U(P_i, P{-}i) \geq \varphi_U(P_i', P_{-i})$. Since $\mathscr{D}$ satisfies the multi-swap UCNR property, there exists a multi-swap path $\{P(s)\}_{s=1}^{t}$ connecting P_i to P_i' such that for all $x \in U$ and $y \notin U$, $\big[xP(r)y$ and $yP(r+1)x$ for some $r < t\big]$ implies $\big[yP(s)x$ for all $s > r+1\big]$.

Suppose $\tau(P(1)) = \tau(P(2))$. By Claim C.1, $\varphi(P(1), P_{-i}) = \varphi(P(2), P_{-i})$ and hence, $\varphi_U(P(1), P_{-i}) = \varphi_U(P(2), P_{-i})$. Suppose $\tau(P(1)) \neq \tau(P(2))$. By the definition of multi-swap path, this means $P(1) \approx P(2)$, that is, there exist $k_1, \ldots, k_p \in \{1, \ldots, m-1\}$ such that (i) $r_{k_q}(P(1)) = r_{k_q+1}(P(2))$ and $r_{k_q+1}(P(1)) = r_{k_q}(P(2))$ for all $q \in \{1, \ldots, p\}$, and (ii) $r_l(P(1)) = r_l(P(2))$ for all $l \notin \cup_{q \in \{1,\ldots,p\}} \{k_q, k_q+1\}$. By UCSP, (i) $\varphi_{\{r_{k_q}(P(1)), r_{k_q+1}(P(1))\}}(P(1), P_{-i}) = \varphi_{\{r_{k_q}(P(1)), r_{k_q+1}(P(1))\}}(P(2), P_{-i})$ for all $q \in \{1, \ldots, p\}$, and (ii) $\varphi_a(P(1), P_{-i}) = \varphi_a(P(2), P_{-i})$ for all $a \notin \cup_{q \in \{1,\ldots,p\}} \{r_{k_q}(P(1)), r_{k_q+1}(P(1))\}$, and by multi-swap monotonicity, $\varphi_{r_{k_q}(P(1))}(P(1), P_{-i}) \geq \varphi_{r_{k_q}(P(1))}(P(2), P_{-i})$ for all $q \in \{1, \ldots, p\}$.

Suppose for each $q \in \{1, \ldots, p\}$ either $\{r_{k_q}(P(1)), r_{k_q+1}(P(1))\} \subseteq U$ or $\{r_{k_q}(P(1)), r_{k_q+1}(P(1))\} \nsubseteq U$. This implies U is an upper contour set of both $P(1)$ and $P(2)$, and hence by UCSP, $\varphi_U(P(1), P_{-i}) = \varphi_U(P(2), P{-}i)$. Now suppose that there exists $q \in \{1, \ldots, p\}$ such that $r_{k_q}(P(1)) \in U$ but $r_{k_q+1}(P(1)) \notin U$. This means for all $w < q$, $\{r_{k_w}(P(1)), r_{k_w+1}(P(1))\} \subseteq U$, and for all $z > q$, $\{r_{k_z}(P(1)), r_{k_z+1}(P(1))\} \cap U = \emptyset$. Therefore, $\varphi_U(P(1), P_{-i}) \geq \varphi_U(P(2), P{-}i)$. Now we use induction to prove that $\varphi_U(P(1), P_{-i}) \geq \varphi_U(P(r), P{-}i)$ for all $r \in \{2, \ldots, t\}$.

Induction Hypothesis: Suppose $\varphi_U(P(1), P_{-i}) \geq \varphi_U(P(s), P{-}i)$ for all $2 \leq s < r$ and some $2 < r \leq t$.

We show $\varphi_U(P_i, P_{-i}) \geq \varphi_U(P(r), P_{-i})$. Suppose $\tau(P(r-1)) = a$ and $\tau(P(r)) = b$. If $a = b$, then by Claim C.1, $\varphi(P(r-1), P_{-i}) = \varphi(P(r), P_{-i})$ and hence, $\varphi_U(P_i, P_{-i}) \geq \varphi_U(P(r), P{-}i)$. If $a \neq b$, then as $P(r-1) \approx P(r)$, there exist $k_1, \ldots, k_p \in \{1, \ldots, m-1\}$ such that (i) $r_{k_q}(P(r-1)) = r_{k_q+1}(P(r))$ and $r_{k_q+1}(P(r-1)) = r_{k_q}(P(r))$ for all $q \in \{1, \ldots, p\}$, and (ii) $r_l(P(r-1)) = r_l(P(r))$ for all $l \notin \cup_{q \in \{1,\ldots,p\}} \{k_q, k_q+1\}$. Therefore, by

UCSP,

$$\varphi_{\{r_{k_q}(P(r-1)),r_{k_q+1}(P(r-1))\}}(P(r-1),P_{-i}) = \varphi_{\{r_{k_q}(P(r-1)),r_{k_q+1}(P(r-1))\}}(P(r),P_{-i}) \text{ for all } q \in \{1,\dots,p\}, \text{ an} \tag{10}$$

$$\varphi_c(P(r-1),P_{-i}) = \varphi_c(P(r),P-i) \text{ for all } c \notin \cup_{q\in\{1,\dots,p\}}\{r_{k_q}(P(r-1)),r_{k_q+1}(P(r-1))\}, \tag{11}$$

and by multi-swap monotonicity,

$$\varphi_{r_{k_q}(P(r-1))}(P(r-1),P_{-i}) \geq \varphi_{r_{k_q}(P(r-1))}(P(r),P-i) \text{ for all } q \in \{1,\dots,p\}. \tag{12}$$

Suppose for all $q \in \{1,\dots,p\}$ either $r_{k_q}(P(r-1)),r_{k_q+1}(P(r-1)) \in U$ or $r_{k_q}(P(r-1)),r_{k_q+1}(P(r-1)) \notin U$. This, together with (10), (11), and (12), implies $\varphi_U(P(r-1), P_{-i}) = \varphi_U(P(r),P-i)$ and hence, by the induction hypothesis, $\varphi_U(P_i,P_{-i}) \geq \varphi_U(P(r), P-i)$. Suppose there exists $q \in \{1,\dots,p\}$ such that $|\{r_{k_q}(P(r-1)),r_{k_q+1}(P(r-1))\} \cap U| = 1$, that is, either $r_{k_q}(P(r-1)) \in U$ or $r_{k_q+1}(P(r-1)) \in U$. Let $r_{k_q}(P(r-1)) = x$ and $r_{k_q+1}(P(r-1)) = y$. Since $xP(r-1)y$ and $yP(r)x$, by the multi-swap UCNR property, we must have xP_iy, which in particular implies $x \in U$ and $y \notin U$. Therefore, by (10), (11), and (12), we have $\varphi_U(P(r-1),P_{-i}) \geq \varphi_U(P(r),P_{-i})$, and hence by the induction hypothesis, $\varphi_U(P_i,P_{-i}) \geq \varphi_U(P(r),P_{-i})$. This completes the proof of the theorem by induction. ■

D. Proof of Theorem 5.1

Proof: Let $n = 2$ and let $\mathscr{D}$ satisfy the swap UCNR property. We show that every unanimous RSCF on $\mathscr{D}$ is SP if and only if it is UCSP. The "only if" part of the theorem follows from the definitions; we only prove the "if" part. Let φ be a unanimous and UCSP RSCF. In view of Theorem 4.1, it is sufficient to show that φ satisfies multi-swap monotonicity. We first prove a claim.

Claim D.1. *Let $P_i,P_j \in \mathscr{D}$ be such that aP_ib and aP_jb for some $a,b \in A$. Then, $\varphi_b(P_i, P_j) = 0$.*

Proof of Claim D.1. Consider an upper contour set U of P_i such that $a \in U$ but $b \notin U$. Since $P_i, P_j \in \mathscr{D}$, there exists a swap path $(P(1) = P_i, \ldots, P(k) = P_j)$ in $\mathscr{D}$ satisfying the UCNR property with respect to U. As aP_ib and aP_jb, this means $aP(l)b$ for all $l \in \{1, \ldots, k\}$. Assume for contradiction that $\varphi_b(P_i, P_j) > 0$. Consider the profile $(P(2), P_j)$. Suppose $\tau(P_i) = \tau(P(2))$. By Claim C.1, we have $\varphi(P_i, P_j) = \varphi(P(2), P_j)$, and hence, $\varphi_b(P(2), P_j) > 0$. Suppose $\tau(P_i) \neq \tau(P(2))$. Since $P_i \sim P(2)$, we have $\tau(P_i) = r_2(P(2))$, $r_2(P_i) = \tau(P(2))$, and $r_t(P_i) = r_t(P(2))$ for all $y \geq 3$. As $aP(l)b$ for all $l \in \{1, \ldots, k\}$, we must have $b = r_s(P_i) = r_s(P(2))$ for some $s \geq 3$. By UCSP, this means $\varphi_b(P_i, P_j) = \varphi_b(P(2), P_j)$. Thus, $\varphi_b(P(2), P_j) > 0$. Continuing in this manner, we can show that $\varphi_b(P(k), P_j) > 0$ where $P(k) = P_j$.. However, this is a contradiction to unanimity as aP_jb implies $\tau(P_j) \neq b$. □

Now, we are ready to show that φ satisfies multi-swap monotonicity. Suppose not. Since $\mathscr{D}$ satisfies the swap UCNR, this means there exist $P_i, P_i', P_j \in \mathscr{D}$ with $P_i \sim P_i'$, where $P_i \equiv ab\cdots$ and $P_i' \equiv ba\cdots$, such that

$$\varphi_a(P_i, P_j) < \varphi_a(P_i', P_j). \tag{13}$$

As $a = \tau(P_i) = r_2(P_i')$ and $b = r_2(P_i) = \tau(P_i')$, by UCSP we have

$$\varphi_{\{a,b\}}(P_i, P_j) = \varphi_{\{a,b\}}(P_i', P_j). \tag{14}$$

Suppose aP_jb. Combining (13) and (14), we have $\varphi_b(P_i, P_j) > 0$. Moreover, as aP_ib and aP_jb, by Claim D.1 we have $\varphi_b(P_1, P_2) = 0$, which contradicts our earlier finding that $\varphi_b(P_1, P_2) > 0$. Suppose bP_ja. By (13), we have $\varphi_a(P_i', P_j) > 0$. Again, as $bP_i'a$ and bP_ja, by Claim D.1 we have $\varphi_a(P_1', P_2) = 0$, a contradiction to our earlier finding that $\varphi_a(P_1', P_2) > 0$. This completes the proof of Theorem 5.1. ■

E. Proof of Proposition 6.1

We first introduce a few notations to facilitate the presentation of the proof of Proposition 6.1 in this section and the proof of Proposition 6.2 in Appendix F. For $k \in \{0, \ldots, m\}$, for a path $\pi^k = (\hat{P}^k(1), \ldots, \hat{P}^k(t))$ in $\mathscr{D}_k$, and for a collection of marginal preferences

$(P^0,P^1,\ldots,P^{k-1},P^{k+1},\ldots,P^m)$, we denote by $(\pi^k,P^0,P^1,\ldots,P^{k-1},P^{k+1},\ldots,P^m)$ the path $(P(1),\ldots,P(t))$ from $(P^0,P^1,\ldots,P^{k-1},\hat{P}^k(1),P^{k+1},\ldots,P^m)$ to $(P^0,P^1,\ldots,P^{k-1},$ $\hat{P}^k(t),P^{k+1},\ldots,P^m)$ such that for all $s\leq t$, $P^l(s)=P^l$ for all $l\neq k$ and $P^k(s)=\hat{P}^k(s)$. In other words, the path $(P(1),\ldots,P(t))$ follows the path π^k over the kth component, while having fixed marginal preferences $(P^0,P^1,\ldots,P^{k-1},P^{k+1},\ldots,P^m)$ over the other components.

For $\widehat{M}\subseteq M$ and $\underset{\sim}{x}\in A^{\widehat{M}}$, we denote by $A(\underset{\sim}{x})$ the set of alternatives that coincide with $\underset{\sim}{x}$ over the components in $\widehat{M}$, that is, $A(\underset{\sim}{x})=\underset{\sim}{x}\times A^{M\setminus\widehat{M}}$.

Proof: Consider two preferences $P=(P^0,P^1,\ldots,P^m)$ and $\bar{P}=(P^0,\bar{P}^1,\ldots,\bar{P}^m)$ in $\mathscr{D}_{LS}$ and an upper contour set U of P. We show that there exists a path $\pi=(P(1)=P,$ $\ldots,P(v)=\bar{P})$ in $\mathscr{D}_{LS}$ such that for all $u\leq v$, there exist $(X_1^u,Y_1^u),\ldots,(X_k^u,Y_k^u)$ such that

(i) $P(u+1)=P(u)[(X_1^u,Y_1^u),\ldots,(X_k^u,Y_k^u)]$, and

(ii) $U\cap(X_l^u\cup Y_l^u)$ is an upper contour set of $P(u)|_{X_l^u\cup Y_l^u}$ for all $l\leq k$.

For each $k\in M$, let $\pi^k=(P^k(1),\ldots,P^k(t_k))$ be a path in $\mathscr{D}_k$ from P_k to $\bar{P}_k$ having no-restoration. Such a path exists by the assumption of Proposition 6.1. Consider the path π defined as follows:

$$\pi=\big((\pi^1,P^0,P^2,\ldots,P^m),(\pi^2,P^0,\bar{P}^1,P^3,\ldots,P^m),\ldots,(\pi^m,P^0,\bar{P}^1,\ldots,\bar{P}^{m-1})\big).$$

In words, the path π starts from the preference $(P^0,P^1,P^2,\ldots,P^m)$ and goes to the preference $(P^0,\bar{P}^1,\bar{P}^2,\ldots,\bar{P}^m)$. First it follows the path π^1 from $(P^0,P^1,P^2,\ldots,P^m)$ to $(P^0,\bar{P}^1,P^2,\ldots,P^m)$, then it follows the path π^2 from the preference $(P^0,\bar{P}^1,P^2,\ldots,P^m)$ to $(P^0,\bar{P}^1,\bar{P}_2,\ldots,P^m)$, and so on. Thus, the path π changes the marginal preferences sequentially following the corresponding paths in the marginal domains.

Let $P(u),P(u+1)$ be two consecutive preferences in π. Suppose $P(u)=(P^0,\bar{P}^1,$ $\ldots,\bar{P}^{k-1},\tilde{P}^k,P^{k+1},\ldots,P^m)$ and $P(u+1)=(P^0,\bar{P}^1,\ldots,\bar{P}^{k-1},\hat{P}^k,P^{k+1},\ldots,P^m)$ where $\hat{P}_k=\tilde{P}_k[(a_k,b_k)]$ for some $a_k,b_k\in A_k$. This implies $P(u+1)=P(u)[(X_1^u,Y_1^u),\ldots,(X_t^u,$ $Y_t^u)]$, where $t=|A_1\times\cdots\times A_{k-1}|$, and for all $s\in\{1,\ldots,t\}$, $(X_s^u,Y_s^u)=(A((\underset{\sim}{x},a_k)),$ $A((\underset{\sim}{x},b_k)))$ where $\underset{\sim}{x}$ is the s-th ranked alternative of the lexicographic preference $(P^0,\bar{P}^1,$

$\dots,\bar{P}^{k-1})$ over the components 1 to $k-1$, that is, $\underset{\sim}{x} = r_s(P^0,\bar{P}^1,\dots,\bar{P}^{k-1})$.

We proceed to show $U\cap(X_s^u\cup Y_s^u)$ is an upper contour set of $P(u)|_{X_s^u\cup Y_s^u}$. Note that as π^k does not have (a_k,b_k) restoration and $\hat{P}_k=\tilde{P}_k[(a_k,b_k)]$, it must be that $a_kP_kb_k$. This, together with the fact that $P(u)$ has the same marginal preferences as P on the components $k+1,\dots,m$, implies that for all $s\in\{1,\dots,t\}$ and for all $x,y\in X_s^u\cup Y_s^u$, $xP(u)y$ if and only if xPy. Therefore, for all $x,y\in X_s^u\cup Y_s^u$, the facts xPy, $x\in U\cap(X_s^u\cup Y_s^u)$, and $y\notin U\cap(X_s^u\cup Y_s^u)$ together imply that $xP(u)y$. This implies $U\cap(X_s^u\cup Y_s^u)$ is an upper contour set of $P(u)|_{X_s^u\cup Y_s^u}$. ■

F. Proof of Proposition 6.1

Proof: Let $P=(P^0,P^1,\dots,P^m),\bar{P}=(\bar{P}^0,\bar{P}^1,\dots,\bar{P}^m)\in\mathscr{D}_{LS}$ and let U be an upper contour set of P. We show that there exists a path $\pi=(P(1)=P,\dots,P(v)=\bar{P})$ in $\mathscr{D}_{LS}$ such that for all $u\le v$, there exists $(X_1^u,Y_1^u),\dots,(X_k^u,Y_k^u)$ such that

(i) $P(u+1)=P(u)[(X_1^u,Y_1^u),\dots,(X_k^u,Y_k^u)]$, and

(ii) $U\cap(X_l^u\cup Y_l^u)$ is an upper contour set of $P(u)|_{X_l^u\cup Y_l^u}$ for all $l\le k$.

Without loss of generality assume that $1\bar{P}^0 2\bar{P}^0\cdots\bar{P}^0 m$. Let $P_0^0=P^0$, and for all $k\in\{1,\dots,m-1\}$, let P_k^0 be the preference in which the components $1,\dots,k$ appear sequentially at the top k positions and the relative ordering of the remaining components match with P^0, that is, $r_l(P_k^0)=l$ for all $l\le k$, and for all $p,q\in\{k+1,\dots,m\}$, pP_k^0q if and only if pP^0q. By our construction, $P_{m-1}^0=\bar{P}^0$. Let $\pi_k^0=(P_{k-1}^0(1)=P_{k-1}^0,\dots,P_{k-1}^0(t)=P_k^0)$ be the path in $\mathscr{D}_0$ from P_{k-1}^0 to P_k^0 that moves component k to the k-th rank through a sequence of swaps, that is, for all $s\le t-1$, the component k swaps up from the preference $P^0(s)$ to the preference $P^0(s+1)$ without any other change in the preferences. Consider the path π defined as follows

$$\pi=\big((P_0^0,P^1,P^2,\dots,P^m),(P_0^0,\bar{P}^1,P^2,\dots,P^m),(\pi_1^0,\bar{P}^1,P^2,\dots,P^m),(P_1^0,\bar{P}^1,\bar{P}^2,\dots,P^m)\\ \dots,(\pi_{m-1}^0,\bar{P}^1,\dots,\bar{P}^{m-1},P^m),(P_{m-1}^0,\bar{P}^1,\dots,\bar{P}^{m-1},\bar{P}^m)\big).$$

In words, the path π (i) starts from the preference $(P_0^0,P^1,P^2,\dots,P^m)$ and changes the marginal preference in component 1 from P^1 to $\bar{P}^1$, (ii) changes the component ordering from P_0^0 to P_1^0 along the path π_1^0, (iii) changes the marginal preference in component 2 from P^2 to $\bar{P}^2$, (iv) changes the component ordering from P_1^0 to P_2^0 along the path π_2^0, and so on till the path reaches the preference $(P_{m-1}^0, \bar{P}^1, \dots, \bar{P}^{m-1}, \bar{P}^m)$. Let $P(u), P(u+1)$ be two consecutive preferences in π. We distinguish two cases based on the structure of the preferences $P(u)$ and $P(u+1)$.

Case 1: Suppose $P(u) = (P_{k-1}^0, \bar{P}^1, \dots, \bar{P}^{k-1}, P^k, P^{k+1} \dots, P^m)$ and $P(u+1) = (P_{k-1}^0, \bar{P}^1, \dots, \bar{P}^{k-1}, \bar{P}^k, P^{k+1}, \dots, P^m)$ where $\bar{P}^k = P^k[(0,1)]$ for some $k \in \{2, \dots, m\}$.

The proof for this case is similar to the proof of Proposition 6.2; we provide it for the sake of completeness. By the assumption of the case, we have $P(u+1) = P(u)[(X_1^u, Y_1^u), \dots, (X_t^u, Y_t^u)]$ where $t = |A_1 \times \cdots \times A_{k-1}|$ and for all $s \in \{1, \dots, t\}$, $(X_s^u, Y_s^u) = (A((\underset{\sim}{x}, 0)), A((\underset{\sim}{x}, 1)))$ where $\underset{\sim}{x} = r_s(P_{k-1}^0, \bar{P}^1, \dots, \bar{P}^{k-1})$. We show $U \cap (X_s^u \cup Y_s^u)$ is an upper contour set of $P(u)|_{X_s^u \cup Y_s^u}$. Note that for all $x, y \in (X_s^u \cup Y_s^u)$, $x_l = y_l$ for all $l \in \{1, \dots, k-1\}$. Moreover, as both $P(u)$ and P have the same marginal preferences over the components $k+1, \dots, m$, we have for all $x, y \in (X_s^u \cup Y_s^u)$, $xP(u)y$ if and only if xPy. Therefore, for all $x, y \in X_s^u \cup Y_s^u$, the facts that xPy, $x \in U \cap (X_s^u \cup Y_s^u)$, and $y \notin U \cap (X_s^u \cup Y_s^u)$, imply $xP(u)y$. This means $U \cap (X_s^u \cup Y_s^u)$ is an upper contour set of $P(u)|_{X_s^u \cup Y_s^u}$. This completes the proof of the theorem for Case 1.

Case 2: Suppose $P(u) = (\hat{P}^0, \bar{P}^1, \dots, \bar{P}^k, P^{k+1}, \dots, P^m)$ and $P(u+1) = (\tilde{P}^0, \bar{P}^1, \dots, \bar{P}^k, P^{k+1}, \dots, P^m)$ where $\tilde{P}^0 = \hat{P}^0[(r_v(\hat{P}^0), r_{v+1}(\hat{P}^0))]$ with $r_l(\hat{P}^0) = r_l(\tilde{P}^0) = l$ for all $l \leq k-1$ and $r_{v+1}(\hat{P}^0) = k$ for some $v \in \{k, \dots, m\}$.

Let $r_v(\hat{P}^0) = j$. By the construction of the path π_k^0, it follows that $j > k$. This implies $P(u+1) = P(u)[(X_1^u, Y_1^u), \dots, (X_t^u, Y_t^u)]$ where $t = |A_1 \times \cdots \times A_{k-1}|$, and for all $s \in \{1, \dots, t\}$, $(X_s^u, Y_s^u) = (A((\underset{\sim}{x}, r_1(P^j), r_2(\bar{P}^k))), A((\underset{\sim}{x}, r_2(P^j), r_1(\bar{P}^k))))$ where $\underset{\sim}{x} = r_s(\hat{P}^0, \bar{P}^1, \dots, \bar{P}^{k-1})$. We show that $U \cap (X_s^u \cup Y_s^u)$ is an upper contour set of $P(u)|_{X_s^u \cup Y_s^u}$. Note that along the path π_k^0, k overtakes j for the first time at $P(u+1)$. Therefore, it must be that jP^0k. Take $x, y \in (X_s^u \cup Y_s^u)$. For all $l \in \{k+1, \dots, m\}$, since $P^l(u) = P^l$, we have $x^l P^l y^l$ if and only if $x^l P^l(u) y^l$. Moreover, as $r_p(\hat{P}^0) = r_p(\tilde{P}^0) = p$ for all $p \in \{1, \dots, k-1\}$, $x^l \neq y^l$ implies $l \in \{k, \dots, m\}$. Combining all these observations and the fact that jP^0k and $j\hat{P}^0k$, we get $xP(u)y$ if and only if xPy. Therefore, the facts that xPy,

$x \in U \cap (X_s^u \cup Y_s^u)$, and $y \notin U \cap (X_s^u \cup Y_s^u)$ imply $xP(u)y$, and hence $U \cap (X_s^u \cup Y_s^u)$ is an upper contour set of $P(u)|_{X_s^u \cup Y_s^u}$. This completes the proof of the theorem for Case 2.

Since Case 1 and Case 2 are exhaustive, the proof of the theorem is complete. ■

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