A Unified Characterization of Randomized Strategy-proof Rules

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A UNIFIED CHARACTERIZATION OF THE RANDOMIZED STRATEGY-PROOF RULES*

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Abstract

We show that a large class of restricted domains such as single-peaked, single-crossing, single-dipped, tree-single-peaked with top-set along a path, Euclidean, multi-peaked, intermediate (Grandmont (1978)), etc., can be characterized by using betweenness property, and we present a unified characterization of unanimous and strategy-proof random rules on these domains. As corollaries of our result, we show that all the domains we consider in this paper satisfy tops-onlyness and deterministic extreme point property. Finally, we consider weak preferences and provide a class of unanimous and strategy-proof random rules on those domains.

JEL Classification: D71, D82.
Keywords: Betweenness property, Generalized Intermediate Domains, Random Social Choice Functions, Strategy-proofness, Tops-restricted Random Min-max Rules

1. INTRODUCTION

1.1 BACKGROUND OF THE PROBLEM

We analyze the classical social choice problem of choosing an alternative from a set of feasible alternatives based on preferences of individuals in a society. Such a procedure is known as a *deterministic social choice function* (DSCF). Some desirable properties of a DSCF are *unanimity* and *strategy-proofness*. The classic Gibbard (1973)-Satterthwaite (1975) impossibility theorem states that if there are at least three alternatives and the preferences of the individuals are *unrestricted*, then every unanimous and strategy-proof DSCF is *dictatorial*.

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Although unanimity and strategy-proofness are desirable properties of a DSCF, the assumption of an unrestricted domain made in Gibbard-Satterthwaite Theorem is quite strong. Not only do there exist many political and economic scenarios where preferences of individuals satisfy natural restrictions such as *single-peakedness*, *single-dippedness*, *single-crossingness*, *Euclidean*, etc., but also the conclusion of Gibbard-Satterthwaite Theorem does not apply to such restricted domains.

The study of single-peaked domains can be traced back to Black (1948) where he shows that a *Condorcet winner* exists on such domains. Later, Moulin (1980) shows that a DSCF on a single-peaked domain is unanimous and strategy-proof if and only if it is a *min-max* rule. Peremans and Storcken (1999) show that a DSCF on a single-dipped domain is unanimous and strategy-proof if and only if it is a *monotone rule* between the left-most and the right-most alternatives. Saporiti (2014) shows that a DSCF on a single-crossing domain is unanimous and strategy-proof if and only if it is an *augmented representative voter scheme*. A domain is Euclidean if its alternatives are elements of Euclidean space and its preferences are based on Euclidean distances. Lahiri et al. (2017) and Öztürk et al. (2014) characterize the unanimous and strategy-proof DSCFs on Euclidean domains.

The horizon of social choice theory has been expanded by the concept of *random social choice functions* (RSCF). An RSCF assigns a probability distribution over the alternatives at every preference profile. The importance of RSCFs over DSCFs is well-established in the literature (see, for example, Ehlers et al. (2002), Peters et al. (2014)).

The study of RSCFs dates back to Gibbard (1977) where he shows that an RSCF on the unrestricted domain is unanimous and strategy-proof if and only if it is a *random dictatorial* rule. For the case of continuous alternatives, Ehlers et al. (2002) characterise unanimous and strategy-proof RSCFs on maximal single-peaked domains, and Border and Jordan (1983) and Dutta et al. (2002) characterise unanimous and strategy-proof DSCFs and RSCFs, respectively, on multi-dimensional single-peaked domains. Barberà and Jackson (1994) characterise efficient and strategy-proof DSCFs on multi-dimensional single-peaked domains with cardinal preferences when the range is one-dimensional. Later, Peters et al. (2014) show that every unanimous and strategy-proof RSCF on maximal single-peaked domain is a convex combination of min-max rules. Pycia and Ünver (2015) establish a similar result by using the theory of totally unimodular matrices from combinatorial integer programming. Recently, Peters et al. (2017) and Roy and Sadhukhan (2019) characterize unanimous and strategy-proof RSCFs on single-dipped domains and Euclidean domains, respectively. However, to the best of our knowledge, unanimous and strategy-proof RSCFs on domains such as single-crossing, multi-peaked, intermediate (Grandmont (1978)), and single-peaked on trees with top-set along a path have not yet been characterized in the literature.

1.2 Our motivation and contribution

Our main motivation of this paper is to present one unified characterization of unanimous and strategy-proof RSCFs that summarizes all existing results for *both DSCFs and RSCFs* and allows for new ones. We intend to do this under minimal assumption on the domains.

We show that a large class of restricted domains can be modelled by using the concept of *betweenness* (Nehring and Puppe (2007a), Nehring and Puppe (2007b)). Given a prior order over the alternatives, a preference satisfies the betweenness property with respect to an alternative a if, whenever a lies in-between (with respect to the prior order) the top-ranked alternative of the preference and some other alternative b, a is preferred to b. A domain satisfies the betweenness property with respect to an alternative if each preference in it satisfies the property with respect to that alternative. Consider the set of alternatives that appear as top-ranked for some preference in the domain. Assume the betweenness property is satisfied for each such alternatives. Then, the domain is called *generalized intermediate*.

We show that in case of finitely many alternatives, an RSCF is unanimous and strategy-proof on a *minimally rich* generalized intermediate domain if and only if it is a convex combination of the tops-restricted min-max rules. A min-max rule is tops-restricted if all its parameters belong to the top-set of the domain. We establish that all restricted domains that we have discussed so far, namely single-peaked, single-crossing, single-dipped, tree-single-peaked with top-set along a path, Euclidean, multi-peaked, and intermediate are special cases of generalized intermediate domains. Finally, we consider domains consisting of weak preferences where indifference can occur only at the top position. Single-plateaued domain is an important example of such domain. We provide a class of unanimous and strategy-proof RSCFs on these domains. Berga (1998) provides a characterization of plateau-only and strategy-proof DSCFs on single-plateaued domains; we show that a similar characterization for plateau-only and strategy-proof RSCFs does not hold.

Our result strengthens existing results for DSCFs by dropping the maximality assumption to minimal richness. Note that in a social choice problem with m alternatives, the number of preferences in the maximal single-peaked or single-dipped domain is 2^{m-1} and in a maximal single-crossing domain is $(m(m-1)/2)+1$, whereas that number can range from $2m-2$ to 2^{m-1} in a minimally rich single-peaked domain, from 2 to 2^{m-1} in a minimally rich single-dipped domain, and from $2m^*-2$ to $(m(m-1)/2)+1$ in a minimally rich single-crossing domain, where m^* is the cardinality of the top-set of the domain.

It follows from our results that minimally-rich generalized intermediate domains satisfy both tops-only property and deterministic extreme point property. Chatterji and Zeng (2018) provide a sufficient condition on a domain that guarantees tops-onlyness for the unanimous and strategy-proof RSCFs on it, however minimally-rich generalized intermediate domains do not satisfy their condition. A domain is said to satisfy the *deterministic extreme point* (DEP) property if every unanimous and strategy-proof RSCF on the domain is a convex combination of unanimous and strategy-proof DSCFs on it. This property can be utilized in finding the *optimal* RSCFs for a society. Gershkov et al. (2013) characterize the optimal DSCFs on single-crossing domains. Therefore, by means of the DEP property of single-crossing domains, one can extend their result to the case of RSCFs.

1.3 ORGANIZATION OF THE PAPER

The rest of the paper is organized as follows: Section 2 introduces the model and basic definitions. Section 3 presents our main result characterizing unanimous and strategy-proof RSCFs on minimally rich generalized intermediate domains. Section 4 contains some applications of our results. Section 5 analyzes unanimous and strategy-proof RSCFs on domains with weak preferences. Finally, Section 6 concludes the paper. The Appendix gathers all omitted proofs.

2. PRELIMINARIES

Let $N = \{1,\ldots,n\}$ be a finite set of agents. Except where otherwise mentioned, $n \geq 2$. Let $A = \{a_1,\ldots,a_m\}$ be a finite set of alternatives with a prior ordering $\prec$ given by $a_1 \prec \cdots \prec a_m$. Whenever we write minimum or maximum of a subset of A, we mean it with respect to the ordering $\prec$. By $a \preceq b$, we mean $a = b$ or $a \prec b$. For $a,b \in A$, we define $[a,b] = \{c \mid \text{ either } a \preceq c \preceq b \text{ or } b \preceq c \preceq a\}$ as the set of alternatives that lie in-between a and b, and for $B \subseteq A$, we define $[a,b]_B = [a,b] \cap B$ as the alternatives in B that lie in the interval $[a,b]$. For notational convenience, whenever it is clear from the context, we do not use braces for singleton sets, for instance we denote a set $\{i\}$ by i.

2.1 DOMAIN OF PREFERENCES

A complete, reflexive, antisymmetric, and transitive binary relation (also called a linear order) over A is called a *preference*. We denote by $\mathbb{L}(A)$ the set of all preferences over A. For $P \in \mathbb{L}(A)$ and distinct $a,b \in A$, aPb is interpreted as "a is strictly preferred to b according to P". For $P \in \mathbb{L}(A)$ and $1 \leq k \leq m$, by $r_k(P)$ we denote the k-th ranked alternative in P, i.e., $r_k(P) = a$ if and only if $|\{b \in A \mid bPa\}| = k$. Since we refer to the top-ranked alternative of a preference P very frequently, we use a simpler notation, $\tau(P)$, for that. For $P \in \mathscr{D}$ and $a \in A$, the *upper contour set* of a at P, denoted by $U(a,P)$, is defined as the set of alternatives that are as good as a in P, i.e., $U(a,P) = \{b \in A \mid bPa\}$.[1] By P^a, we denote a preference with a as the top-ranked alternative, that is, P^a is such that $\tau(P^a) = a$. Similarly, by $P^{a,b}$, we denote a preference

[1]Observe that $a \in U(a,P)$ by reflexivity.

with a as the top-ranked and b as the second-top-ranked alternatives, that is, $P^{a,b}$ is such that $\tau(P^{a,b}) = a$ and $r_2(P^{a,b}) = b$. For ease of presentation, sometimes we write $P \equiv P^{a,b}$ to mean $\tau(P) = a$ and $r_2(P) = b$.

We denote by $\mathscr{D} \subseteq \mathbb{L}(A)$ a set of admissible preferences (henceforth, will be called a domain). For $a \in A$, let $\mathscr{D}^a = \{P \in \mathscr{D} \mid \tau(P) = a\}$ denote the preferences in $\mathscr{D}$ that have a as the top-ranked alternative. For a domain $\mathscr{D}$, the top-set of $\mathscr{D}$, denoted by $\tau(\mathscr{D})$, is the set of alternatives that appear as a top-ranked alternative in some preference in $\mathscr{D}$, that is, $\tau(\mathscr{D}) = \cup_{P \in \mathscr{D}} \tau(P)$. Whenever we write $\tau(\mathscr{D}) = \{b_1, \ldots, b_k\}$, we assume without loss of generality that the indexation is such that $b_1 \prec \cdots \prec b_k$. A domain $\mathscr{D}$ is *regular* if $\tau(\mathscr{D}) = A$.

A preference profile, denoted by $P_N = (P_1, \ldots, P_n)$, is an element of $\mathscr{D}^n = \mathscr{D} \times \cdots \times \mathscr{D}$ that represents a collection of preferences one for each agent.

For $P \in \mathbb{L}(A)$ and $B \subseteq A$, the restriction of P to B, $P|_B \in \mathbb{L}(B)$ is defined as follows: for all $a, b \in B$, $aP|_B b$ if and only if aPb. For $\mathscr{D} \subseteq \mathbb{L}(A)$, $P_N \in \mathscr{D}^n$, and $B \subseteq A$, we define the restriction of the domain $\mathscr{D}$ to B as $\mathscr{D}|_B = \{P|_B \mid P \in \mathscr{D}\}$, and the restriction of the profile P_N to B as $P_N|_B = (P_1|_B, \ldots, P_n|_B)$.

2.1.1 Properties of a domain

In this section, we introduce a few properties of a domain. First, we introduce the concept of a single-peaked domain. A preference is single-peaked if it decreases as one goes far away (with respect to the ordering $\prec$) in any particular direction from its peak (top-ranked alternative). More formally, a preference P is *single-peaked* if for all $a, b \in A$, $[\tau(P) \preceq a \prec b$ or $b \prec a \preceq \tau(P)]$ implies aPb. A domain is *single-peaked* if each preference in it is single-peaked, and is *maximal single-peaked* if it contains all single-peaked preferences. For $B \subseteq A$, a domain $\mathscr{D}$ of preferences is a single-peaked domain restricted to B if $\mathscr{D}|_B$ is a single-peaked domain.

A preference P satisfies the **betweenness property** with respect to an alternative a if for all $b \in A \setminus a$, $a \in [\tau(P), b]$ implies aPb. A domain $\mathscr{D}$ satisfies the **betweenness property** with respect to an alternative a if each preference $P \in \mathscr{D}$ satisfies the property with respect to a.

Note that the betweenness property of a preference with respect to an alternative a does *not* put any restriction on the relative ordering of two alternatives if (i) both of them are different from a, or (ii) one of them lies in-between the top-ranked alternative of that preference and a, and the other one is a itself, or (iii) one of them is a and the other one lies in the other side of the top-ranked alternative. A domain $\mathscr{D}$ is **generalized intermediate** if it satisfies the betweenness property with respect to each alternative in $\tau(\mathscr{D})$.

REMARK 2.1. Note that the generalized intermediate property does not impose any restriction on the relative ordering of the alternatives outside the top-set of a domain. Furthermore, if a domain $\mathscr{D}$ satisfies this property, then $\mathscr{D}|_{\tau(\mathscr{D})}$ is single-peaked, which in particular implies that a regular domain is single-peaked if

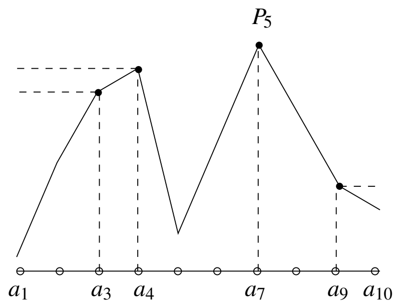

Figure 1: A graphic illustration of the preference P_5 given in Table 1

and only if it is generalized intermediate.

Note that a maximal generalized intermediate domain requires quite a few preferences to be present in the domain. In view of this, we require a *minimal* set of preferences to be present in a generalized intermediate domain. Our minimal requirement ensures that for two alternatives that are consecutive in the top-set of a domain,[2] there are two different preferences which (i) rank those two alternatives in the top-two positions, and (ii) agree on the ranking of the other alternatives.[3]

To ease our presentation, for two preferences P and P' in $\mathscr{D}$, we write $P \sim P'$ if $\tau(P) = r_2(P')$, $r_2(P) = \tau(P')$, and $r_l(P) = r_l(P')$ for all $l \geq 3$, that is, P and P' differ only on the ranking of the top two alternatives. Recall that throughout this paper, whenever we write $\tau(\mathscr{D}) = \{b_1, \ldots, b_k\}$ for a domain $\mathscr{D}$, we assume $b_1 \prec \cdots \prec b_k$.

A domain $\mathscr{D}$ with $\tau(\mathscr{D}) = \{b_1, \ldots, b_k\}$ satisfies the **minimal richness** property if for all $b_j, b_{j+1} \in \tau(\mathscr{D})$, there are $P \in \mathscr{D}^{b_j}$ and $P' \in \mathscr{D}^{b_{j+1}}$ such that $P \sim P'$. Below, we provide an example of a generalized intermediate domain satisfying the minimal richness property.

Example 1. Let the set of alternatives be $A = \{a_1, \ldots, a_{10}\}$ with prior order $a_1 \prec \cdots \prec a_{10}$. Consider the domain $\mathscr{D} = \{P_1, \ldots, P_8\}$ given in Table 1.

Note that $\tau(\mathscr{D}) = \{a_3, a_4, a_7, a_9\}$. To see that $\mathscr{D}$ is a generalized intermediate domain, consider, for instance, the preference P_3. We show that P_3 satisfies the betweenness property with respect to each alternative in $\{a_3, a_4, a_7, a_9\}$. Consider a_7. Observe that $\tau(P_3) = a_4$ and $a_7 P_3 a_j$ for all $j \in \{8, 9, 10\}$. So, P_3 satisfies the betweenness property with respect to a_7. Similarly, it can be checked that P_3 satisfies the betweenness property with respect to a_3 and a_9. It is left to the reader to verify that the other preferences in $\mathscr{D}$ satisfy the betweenness property with respect to $\{a_3, a_4, a_7, a_9\}$ and that it is minimally rich. In Figure 1, we present a pictorial description of the preference $P_5 \in \mathscr{D}$. $\square$

[2]We say two alternatives are "consecutive in the top-set" if (i) they are in the top-set of the domain, and (ii) there is no alternative in the top-set of the domain that lies strictly in-between (with respect to the prior order $\prec$) those two alternatives.

[3]This property is known as top-connectedness in the literature (Monjardet (2009), Sato (2013), Cho (2016)).

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
a_3	a_3	a_4	a_4	a_7	a_7	a_9	a_9
a_1	a_4	a_3	a_7	a_4	a_9	a_7	a_{10}
a_4	a_1	a_1	a_3	a_3	a_{10}	a_{10}	a_7
a_2	a_6	a_6	a_8	a_8	a_4	a_4	a_8
a_6	a_7	a_7	a_6	a_6	a_3	a_3	a_6
a_7	a_5	a_5	a_2	a_2	a_1	a_1	a_4
a_5	a_9	a_9	a_9	a_9	a_2	a_2	a_3
a_8	a_2	a_2	a_{10}	a_{10}	a_5	a_5	a_5
a_9	a_{10}	a_{10}	a_5	a_5	a_6	a_6	a_1
a_{10}	a_8	a_8	a_1	a_1	a_8	a_8	a_2

Table 1: The domain in Example 1

2.2 Social choice functions and their properties

In this section, we define social choice functions and discuss a few properties of those. By ΔA, we denote the set of probability distributions over A. A **random social choice function (RSCF)** is a function $\varphi : \mathscr{D}^n \to \Delta A$ that assigns a probability distribution over A at every preference profile. For $a \in A$ and $P_N \in \mathscr{D}^n$, we denote by $\varphi_a(P_N)$ the probability of a at the outcome $\varphi(P_N)$, and for $B \subseteq A$, we define $\varphi_B(P_N) = \sum_{a \in B} \varphi_a(P_N)$ as the total probability of the alternatives in B at $\varphi(P_N)$.

An RSCF is a **deterministic social choice function (DSCF)** if it selects a degenerate probability distribution at every preference profile. More formally, an RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is a DSCF if $\varphi_a(P_N) \in \{0,1\}$ for all $a \in A$ and all $P_N \in \mathscr{D}^n$.

For later reference we include the following (trivial) observation.

Remark 2.2. For all $L, L' \in \Delta A$ and all $P \in \mathbb{L}(A)$, if $L_{U(x,P)} \geq L'_{U(x,P)}$ and $L'_{U(x,P)} \geq L_{U(x,P)}$ for all $x \in A$, then $L = L'$.

We now introduce some properties of an RSCF that are standard in the literature. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is **unanimous** if for all $a \in A$ and all $P_N \in \mathscr{D}^n$, $[\tau(P_i) = a$ for all $i \in N] \Rightarrow [\varphi_a(P_N) = 1]$. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is **strategy-proof** if for all $i \in N$, all $P_N \in \mathscr{D}^n$, all $P'_i \in \mathscr{D}$, and all $x \in A$, $\varphi_{U(x,P_i)}(P_i, P_{-i}) \geq \varphi_{U(x,P_i)}(P'_i, P_{-i})$.[4] The concepts of unanimity and strategy-proofness for DSCFs are special cases of the corresponding ones for RSCFs. Two profiles $P_N, P'_N \in \mathscr{D}^n$ are *tops-equivalent* if each agent has the same top-ranked alternative in those two profiles, that is, $\tau(P_i) = \tau(P'_i)$ for all $i \in N$. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is **tops-only** if $\varphi(P_N) = \varphi(P'_N)$ for all tops-equivalent $P_N, P'_N \in \mathscr{D}^n$. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is **uncompromising** if $\varphi_B(P_N) = \varphi_B(P'_i, P_{-i})$ for all $i \in N$, all $P_N \in \mathscr{D}^n$, all $P'_i \in \mathscr{D}$, and all $B \subseteq A$ such that $B \cap [\tau(P_i), \tau(P'_i)] = \emptyset$. In words, uncompromisingness says that if an agent moves his peak (top-ranked alternative) from an

[4] Our notion of strategy-proofness (which is introduced in Gibbard (1977)) is based on first order stochastic dominance. Informally speaking, strategy-proofness ensures that the outcome an(y) agent can obtain by misreporting his/her preference will be first order stochastically dominated by the original/sincere outcome.

alternative a to another alternative b, then the probability assigned by an RSCF to each alternative outside the interval $[a,b]$ will remain unchanged. Note that an uncompromising RSCF is tops-only by definition.

2.2.1 A CLASS OF SOCIAL CHOICE FUNCTIONS

Moulin (1980) introduces the concept of min-max rules with respect to a collection of parameters. Tops-restricted min-max rules are special cases of these rules where the parameters must come from the top-set of the domain.

A DSCF $f : \mathscr{D}^n \to A$ is a **tops-restricted min-max (TM)** rule if for all $S \subseteq N$, there exists $\beta_S \in \tau(\mathscr{D})$ satisfying the conditions that $\beta_\emptyset = \max(\tau(\mathscr{D})), \beta_N = \min(\tau(\mathscr{D}))$, and $\beta_T \preceq \beta_S$ for all $S \subseteq T$ such that

$$f(P_N) = \min_{S \subseteq N} \left[\max_{i \in S} \{\tau(P_i), \beta_S\} \right].$$

If $\tau(\mathscr{D}) = A$, then a TM rule is called a **min-max** rule. In what follows, we present an example of a TM rule.

Example 2. Let $A = \{a_1, \ldots, a_{10}\}$ and $N = \{1,2,3\}$. Consider a domain $\mathscr{D}$ with $\tau(\mathscr{D}) = \{a_2, a_3, a_4, a_5, a_7, a_8, a_9\}$. Consider the TM rule, say f, with respect to the parameters given in Table 2.

β	$\beta_\emptyset$	β_1	β_2	β_3	$\beta_{\{1,2\}}$	$\beta_{\{1,3\}}$	$\beta_{\{2,3\}}$	$\beta_{\{1,2,3\}}$
	a_9	a_8	a_9	a_7	a_4	a_5	a_2	a_2

Table 2: Parameters of the TM rule in Example 2

Let (a_5, a_3, a_8) denote a profile where a_5, a_3 and a_8 are the top-ranked alternatives of agents 1, 2 and 3, respectively. The outcome of f at this profile is determined as follows.

$$\begin{aligned}
f(P_N) &= \min_{S \subseteq \{1,2,3\}} \left[\max_{i \in S} \{\tau(P_i), \beta_S\} \right] \\
&= \min \left[\max\{\beta_\emptyset\}, \max\{\tau(P_1), \beta_1\}, \max\{\tau(P_2), \beta_2\}, \max\{\tau(P_3), \beta_3\}, \right. \\
&\qquad \max\{\tau(P_1), \tau(P_2), \beta_{\{1,2\}}\}, \max\{\tau(P_1), \tau(P_3), \beta_{\{1,3\}}\}, \max\{\tau(P_2), \tau(P_3), \beta_{\{2,3\}}\}, \\
&\qquad \left. \max\{\tau(P_1), \tau(P_2), \tau(P_3), \beta_{\{1,2,3\}}\} \right] \\
&= \min \left[a_9, a_8, a_9, a_8, a_5, a_8, a_8, a_8 \right] \\
&= a_5.
\end{aligned}$$

□

Note that the outcome of a TM rule f always lies in the top-set of the corresponding domain, i.e.,

$f(P_N) \in \tau(\mathscr{D})$ for all $P_N \in \mathscr{D}^n$. Our next remark says that a TM rule on a domain can be seen as a min-max rule on the domain obtained by restricting it to its top-set. It further says that the former is strategy-proof if and only if latter is.

REMARK 2.3. Let $f : \mathscr{D}^n \to A$ be a TM rule. Define $\hat{f} : (\mathscr{D}|_{\tau(\mathscr{D})})^n \to \tau(\mathscr{D})$ such that $\hat{f}(P_N|_{\tau(\mathscr{D})}) = f(P_N)$.[5] Then, f is strategy-proof if and only if $\hat{f}$ is strategy-proof.

For DSCFs f^j, $j = 1,\dots,k$ and nonnegative numbers λ^j, $j = 1,\dots,k$, summing to 1, we define the RSCF $\varphi = \sum_{j=1}^{k} \lambda^j f^j$ as $\varphi_a(P_N) = \sum_{j=1}^{k} \lambda^j f^j_a(P_N)$ for all $P_N \in \mathscr{D}^n$ and all $a \in A$. We call φ a *convex combination* of the DSCFs f^j. So, at every profile, φ assigns probability λ^j to the outcome of f^j for all $j = 1,\dots,k$.

An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is a **tops-restricted random min-max (TRM)** rule if φ can be written as a convex combination of some TM rules on $\mathscr{D}^n$.[6] If $\tau(\mathscr{D}) = A$, then a TRM rule $\varphi : \mathscr{D}^n \to \Delta A$ is a **random min-max** rule.

3. RESULTS

3.1 UNANIMOUS AND STRATEGY-PROOF RSCFS ON GENERALIZED INTERMEDIATE DOMAINS

In this subsection, we present our main result characterizing the unanimous and strategy-proof RSCFs on the minimally rich generalized intermediate domains.

Theorem 1. *Let $\mathscr{D}$ be a minimally rich generalized intermediate domain. Then, an RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is unanimous and strategy-proof if and only if it is a TRM rule.*

The proof of this theorem is relegated to Appendix A. We provide a brief sketch of it here. The if part of the theorem follows from Moulin (1980). To see this, first note the following two facts: (i) every minimally

[5] This is well-defined since by the definition of a TM rule, f is tops-only and $f(P_N) \in \tau(\mathscr{D})$ for all $P_N \in \mathscr{D}^n$.

[6] A TRM rule can be directly defined as follows. Let $\tau(\mathscr{D}) = \{b_1,\dots,b_k\}$. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is a TRM rule if if there exists a lottery $\beta_S \in \Delta\tau(\mathscr{D})$ for each $S \subseteq N$, referred to as a **probabilistic ballot**, such that the following three conditions are satisfied:

(i) *(Regularity condition)* $\beta_N = e_{b_1}$ and $\beta_\emptyset = e_{b_k}$.

(ii) *(Monotonicity condition)* For all $S,T \subseteq N$ with $S \subset T$, we have

$$\beta_T[b_1,b_l] \geq \beta_S[b_1,b_l] \text{ for all } l = 1,2,\dots,k.$$

(iii) For all $P_N \in \mathscr{D}^n$ and $a_k \in A$, we have

$$\varphi_{a_k}(P_N) = \beta_{\mathscr{S}(a_k,P_N)}[a_1,a_k] - \beta_{\mathscr{S}(a_{k-1},P_N)}[a_1,a_{k-1}], \text{ where } \mathscr{S}(a_k,P_N) = \{i \in N \mid \tau(P_i) \preceq a_k\} \text{ and } \beta_{\mathscr{S}(a_0,P_N)}[a_1,a_0] \equiv 0.$$

rich generalized intermediate domain $\mathscr{D}$ restricted to its top-set $\tau(\mathscr{D})$ is a subset of the maximal single-peaked domain over $\tau(\mathscr{D})$, and (ii) every TRM rule on $\mathscr{D}^n$ is a random min-max rule on $\mathscr{D}^n|_{\tau(\mathscr{D})}$. In view of these observations, it is enough to show that every random min-max rule is unanimous and strategy-proof on $\mathscr{D}|_{\tau(\mathscr{D})}$. From Moulin (1980), every min-max rule on $\mathscr{D}|_{\tau(\mathscr{D})}$ is unanimous and strategy-proof, and since every random min-max rule is a convex combination of min-max rules, such rules are also unanimous and strategy-proof on $\mathscr{D}|_{\tau(\mathscr{D})}$.

We prove the only-if part of the theorem in the following two steps. In the first step, we prove a proposition that states that every unanimous and strategy-proof RSCF on a minimally rich generalized intermediate domain is uncompromising and assigns probability 1 to the top-set of the domain. We prove this proposition by using the method of induction on the number of agents. We start with the base case $n = 1$. The proposition follows trivially for this case. Assuming that the proposition holds for all cases where the number of agents is less than n, we proceed to prove it for n agents. First, we consider the set of profiles where agents 1 and 2 have the same preferences. We show that the restriction of φ to this set induces a unanimous and strategy-proof RSCF on $\mathscr{D}^{n-1}$, and claim by means of the induction hypothesis that the proposition holds (in a suitable sense) on this set of profiles. Next, we show that the same holds for the profiles where agents 1 and 2 have the same top-ranked alternatives (instead of having the same preferences). Finally, in order to prove the proposition for profiles where agents 1 and 2 have arbitrary top-ranked alternatives, we use another level of induction on the "distance" between the top-ranked alternatives of agents 1 and 2. The distance between two alternatives $b_j, b_{j+l} \in \tau(\mathscr{D})$ is defined as l. Assuming that the proposition holds for the profiles where the said distance is less than some $\hat{l}$, we prove the proposition for the profiles where it is $\hat{l}$. By induction, this completes the proof of the proposition.

For a clearer picture, we explain the first step of the proof by means of an example. Suppose that $N = \{1,2,3\}$ and $A = \{a_1, \ldots, a_{10}\}$. Let $\mathscr{D}$ be a minimally rich generalized intermediate domain with $\tau(\mathscr{D}) = \{a_1, a_4, a_5, a_8, a_9\}$. Note that if we had one agent, then trivially every unanimous and strategy-proof RSCF on $\mathscr{D}$ would be uncompromising and would assign probability 1 to the alternatives in $\{a_1, a_4, a_5, a_8, a_{10}\}$ at every profile. Suppose (as the induction hypothesis) that the same holds if we had two agents. Consider all the preference profiles P_N, where agents 1 and 2 have the same preferences. We look at the restriction of a unanimous and strategy-proof RSCF φ on these profiles. Since agents 1 and 2 have the same preferences for all these profiles, they can be treated as one agent and φ can be seen as an RSCF for two agents. By some elementary arguments, one can show that φ, when seen as a two-agent RSCF, is unanimous and strategy-proof. So, by the induction hypothesis, φ satisfies uncompromisingness and assigns probability 1 to the set $\{a_1, a_4, a_5, a_8, a_9\}$ for all these profiles. Next, we let the preferences of agents 1 and 2 differ beyond their top-ranked alternatives and extend our proposition to those profiles. We use Remark 2.2 to complete this step. Finally, we proceed to prove the proposition when agents 1 and 2 have arbitrary preferences. Here, we use another level of induction. Suppose (as the induction hypothesis) that the proposition holds over the profiles for which the top-ranked alternatives of agents 1 and 2 are at distance 1, that is, their top-ranked alternatives are either $\{a_1,a_4\}$ or $\{a_4,a_5\}$ or $\{a_5,a_8\}$ or $\{a_8,a_9\}$. We show as the induction step that the same holds over the profiles for which their top-ranked alternatives are at distance 2, that is, they are either $\{a_1,a_5\}$ or $\{a_4,a_8\}$ or $\{a_5,a_9\}$. We prove this as a general step of the induction, and thereby cover all profiles in $\mathscr{D}^3$. The details of the arguments needed to show this step is quite technical, so we do not discuss it here.

In the second step, we show that every uncompromising RSCF on $\mathscr{D}^n$ is a random min-max rule. We use results from Ehlers et al. (2002) and Peters et al. (2014) to prove this. Finally, we argue that if a random min-max rule assigns positive probability only to the alternatives in the top-set of the domain, then it is a TRM rule. This completes the proof of the only-if part of the theorem.

REMARK 3.1. Since every TRM rule is tops-only, it follows from our result that unanimity and strategy-proofness together guarantee tops-onlyness for the RSCFs on minimally rich generalized intermediate domains. Chatterji and Zeng (2018) provide a sufficient condition for a domain to be tops-only for RSCFs.[7] However, minimally rich generalized intermediate domains do not satisfy their condition.

REMARK 3.2. A domain $\mathscr{D}$ satisfies the *deterministic extreme point* (DEP) property if every unanimous and strategy-proof RSCF on $\mathscr{D}^n$ can be written as a convex combination of unanimous and strategy-proof DSCFs on $\mathscr{D}^n$. It follows from Theorem 1 that minimally rich generalized intermediate domains satisfy deterministic extreme point property.

REMARK 3.3. Barberà and Moreno (2011) introduce the notion of top-monotonicity. It can be verified that if every preference in a domain satisfies the betweenness property, then the corresponding preference profile will satisfy the top-monotonicity property. Therefore, it follows from Barberà and Moreno (2011) that generalized intermediateness guarantees the existence of voting equilibria, not only under the majority rule but also for the wide class of voting rules analyzed by Austen-Smith and Banks (2000). Moreover, these equilibria are closely connected to an extended notion of the median voter.

REMARK 3.4. It can be verified that minimally rich generalized intermediate domains are semilattice single-peaked, and hence by Proposition 3 of Chatterji and Massó (2018), it follows that they admit unanimous, anonymous, tops-only, and strategy-proof DSCFs.

4. APPLICATIONS

In this section, we demonstrate the applicability of our results by showing that a class of domains of practical importance are generalized intermediate.

[7] A domain is tops-only if every unanimous and strategy-proof RSCF on it is tops-only.

4.1 Single-peaked domains

Ehlers et al. (2002) characterize the unanimous and strategy-proof RSCFs on the maximal single-peaked domain as fixed-probabilistic-ballots rules, and Peters et al. (2014) show that such an RSCF is a convex combination of the min-max rules. Theorem 1 improves these results by relaxing the maximality assumption. Note that the number of preferences in the maximal single-peaked domain is 2^{m-1}, whereas that in a minimally rich single-peaked domain can range from $2m-2$ to 2^{m-1}.

4.2 Single-crossing domains

In this subsection, we introduce the concept of single-crossing domains and show that every single-crossing domain is generalized intermediate. Saporiti (2014) characterizes all unanimous and strategy-proof DSCFs on maximal single-crossing domains. Carroll (2012) considers a slightly more general class of single-crossing domains called successive single-crossing domains in the context of local strategy-proofness with transfers. We show that all these domains are special cases of minimally rich generalized intermediate domains.

A domain $\mathscr{D}$ is *single-crossing* if there is an ordering $\lhd$ over $\mathscr{D}$ such that for all $a,b \in A$ and all $P,P' \in \mathscr{D}$, $[a \prec b, P \lhd P', \text{ and } bPa] \implies bP'a$. In words, a single-crossing domain is one for which the preferences can be ordered in a way such that every pair of alternatives switches their relative ranking at most once along that ordering. A single-crossing domain $\bar{\mathscr{D}}$ is *maximal* if there does not exist another single-crossing domain that is a strict superset of $\bar{\mathscr{D}}$. Note that a maximal single-crossing domain with m alternatives contains $m(m-1)/2+1$ preferences.[8] A domain $\mathscr{D}$ is *successive single-crossing* if there is a maximal single-crossing domain $\bar{\mathscr{D}}$ with respect to some ordering $\lhd$ and two preferences $P', P'' \in \bar{\mathscr{D}}$ with $P' \unlhd P''$ such that $\mathscr{D} = \{P \in \bar{\mathscr{D}} \mid P' \unlhd P \unlhd P''\}$.[9]

In the following example, we present a maximal single-crossing domain and a successive single-crossing domain with 5 alternatives.

Example 3. Let the set of alternatives be $A = \{a_1, a_2, a_3, a_4, a_5\}$ with the prior order $a_1 \prec \cdots \prec a_5$. The domain $\bar{\mathscr{D}} = \{a_1a_2a_3a_4a_5,\ a_2a_1a_3a_4a_5,\ a_2a_3a_1a_4a_5,\ a_2a_3a_4a_1a_5,\ a_2a_4a_3a_1a_5,\ a_4a_2a_3a_1a_5,\ a_4a_2a_3a_5a_1,\ a_4a_3a_2a_5a_1, a_4a_3a_5a_2a_1,\ a_4a_5a_3a_2a_1,\ a_5a_4a_3a_2a_1\}$ is a maximal single-crossing domain with respect to the ordering $\lhd$ given by $a_1a_2a_3a_4a_5 \lhd a_2a_1a_3a_4a_5 \lhd a_2a_3a_1a_4a_5 \lhd a_2a_3a_4a_1a_5 \lhd a_2a_4a_3a_1a_5 \lhd a_4a_2a_3a_1a_5 \lhd a_4a_2a_3a_5a_1 \lhd a_4a_3a_2a_5a_1 \lhd a_4a_3a_5a_2a_1 \lhd a_4a_5a_3a_2a_1 \lhd a_5a_4a_3a_2a_1$ since every pair of alternatives change their relative ordering at most once along this ordering. Note that the cardinality of A is 5 and that

[8] For details see Saporiti (2009).

[9] By $P \unlhd P'$, we mean either $P = P'$ or $P \lhd P'$.

of $\bar{\mathscr{D}}$ is $5(5-1)/2+1=11$. The domain $\mathscr{D}=\{a_1a_2a_3a_4a_5,\ a_2a_1a_3a_4a_5,\ a_2a_3a_1a_4a_5,\ a_2a_3a_4a_1a_5,\ a_2a_4a_3a_1a_5,\ a_4a_2a_3a_1a_5\}$ is a successive single-crossing domain since it contains all the preferences in-between $a_1a_2a_3a_4a_5$ and $a_4a_2a_3a_1a_5$ in the maximal single-crossing domain $\bar{\mathscr{D}}$. $\square$

In the following lemmas, we show that every single-crossing domain is a generalized intermediate domain, and every successive single-crossing domain is a minimally rich general intermediate domain.

Lemma 1. *Every single-crossing domain is a generalized intermediate domain.*

Proof. Let $\mathscr{D}$ be a single-crossing domain with an ordering $\triangleleft$ over the preferences. We show that $\mathscr{D}$ is a generalized intermediate domain. Suppose not and assume without loss of generality that there exist $a\in A$, $b_r,b_s\in\tau(\mathscr{D})$ and $P^{b_r}\in\mathscr{D}$ such that $b_r\prec b_s\prec a$ and $aP^{b_r}b_s$. Consider $P^{b_s}\in\mathscr{D}$. Since $b_rP^{b_r}b_s$, $b_sP^{b_s}b_r$, and $b_r\prec b_s$, it follows from the definition of a single-crossing domain that $P^{b_r}\triangleleft P^{b_s}$. By means of our assumption that $b_s\prec a$ and $aP^{b_r}b_s$, $P^{b_r}\triangleleft P^{b_s}$ implies $aP^{b_s}b_s$. However, this is a contradiction since $\tau(P^{b_s})=b_s$. This completes the proof. $\blacksquare$

Lemma 2. *Every successive single-crossing domain is a minimally rich single-crossing domain.*

Proof. It is enough to show that every successive single-crossing domain is minimally rich. Let $\mathscr{D}$ be a successive single-crossing domain. Then, by the definition of a successive single-crossing domain, there is a maximal single-crossing domain $\bar{\mathscr{D}}$ with respect to some ordering $\triangleleft$ such that $\mathscr{D}=\{P\in\bar{\mathscr{D}}\mid\tilde{P}\trianglelefteq P\trianglelefteq\tilde{\tilde{P}}\}$ for some $\tilde{P},\tilde{\tilde{P}}\in\bar{\mathscr{D}}$ with $\tilde{P}\trianglelefteq\tilde{\tilde{P}}$. Suppose $\tau(\mathscr{D})=\{b_1,\ldots,b_k\}$. We show that for all $j=1,2,\ldots,k-1$, there are $P\in\mathscr{D}^{b_j}$ and $P'\in\mathscr{D}^{b_{j+1}}$ such that $P\sim P'$. Consider $b_j,b_{j+1}\in\tau(\mathscr{D})$ and consider $\bar{P}\in\mathscr{D}^{b_j}$ and $\hat{P}\in\mathscr{D}^{b_{j+1}}$. Since $b_j\bar{P}b_{j+1}$, $b_{j+1}\hat{P}b_j$, and $b_j\prec b_{j+1}$, it follows from the definition of a single-crossing domain that $\bar{P}\triangleleft\hat{P}$. Using a similar argument, we obtain $P^{b_l}\triangleleft\bar{P}$ for all $l<j$, and $P^{b_l}>\hat{P}$ for all $l>j+1$. Therefore, there must be $P\in\mathscr{D}^{b_j}$ and $P'\in\mathscr{D}^{b_{j+1}}$ that are consecutive in the ordering $\triangleleft$, that is, $P\in\mathscr{D}^{b_j}$ and $P'\in\mathscr{D}^{b_{j+1}}$ are such that there is no $P''\in\mathscr{D}$ with $P\triangleleft P''\triangleleft P'$. We show $P\sim P'$. Suppose not. Let a be the alternative which is ranked just above b_{j+1} in P, that is, aPb_{j+1} and there is no $x\in A$ with $aPxPb_{j+1}$. Consider the preference P'' that is obtained by switching the alternatives a and b_{j+1} in P. We show $P''\notin\bar{\mathscr{D}}$. In particular, we show that both $P''\triangleleft P$ and $P'\triangleleft P''$ are impossible. This is sufficient since P and P' are consecutive in the ordering $\triangleleft$. Suppose $P''\triangleleft P$. Since aPb_{j+1}, $P\triangleleft P'$, and $b_{j+1}P'a$, by the single-crossing property of $\bar{\mathscr{D}}$, it must be that $a\prec b_{j+1}$. However, because $b_{j+1}P''a$ and aPb_{j+1}, this contradicts $P''\triangleleft P$. Now, suppose $P'\triangleleft P''$. Since $P\triangleleft P'$, there must be a pair of alternatives c,d with $c\prec d$ such that cPd and $dP'c$. Moreover, because P' and P'' are different, it must be that $\{c,d\}\neq\{a,b_{j+l}\}$. Since $c\prec d$, $dP'c$, and $P'\triangleleft P''$, by the single-crossing property of $\bar{\mathscr{D}}$, we have $dP''c$. However, by the construction of P'', we have $cP''d$, which is a contradiction. Thus, we have $P''\notin\bar{\mathscr{D}}$. This implies $\bar{\mathscr{D}}\cup P''$ is a single-crossing domain with respect to the ordering $\triangleleft'$ over $\bar{\mathscr{D}}\cup P''$, where $\triangleleft'$ is obtained by placing P'' in-between P and

P' in the ordering $\lhd$, i.e., $\lhd'$ coincides with $\lhd$ over $\bar{\mathscr{D}}$ and $P \lhd' P'' \lhd' P'$. This contradicts the fact that $\bar{\mathscr{D}}$ is a maximal single-crossing domain. Therefore, $P \sim P'$ and $\mathscr{D}$ is minimally rich. This completes the proof of the lemma. ■

4.3 Single-dipped domains

In this subsection, we introduce the concept of single-dipped domains and show that they are generalized intermediate. A preference P is *single-dipped* if it has a unique minimal element $d(P)$, the *dip* of P, such that for all $a,b \in A$, $[d(P) \preceq a \prec b$ or $b \prec a \preceq d(P)] \Rightarrow bPa$. A domain is *single-dipped* if each preference in it is single-dipped.

It is straightforward that a minimally rich single-dipped domain is a minimally rich generalized intermediate domain. Note that the number of preferences in the maximal single-dipped domain is 2^{m-1}, while that in a minimally rich single-dipped domain can range from 2 to 2^{m-1}.

It is worth mentioning that any unanimous and strategy-proof RSCF on a minimally rich single-dipped domain can give positive probability to two particular (the boundary ones) alternatives.

4.4 Single-peaked domains on trees with top-set along a path

A domain is tree-single-peaked if the alternatives are located on a tree and agents' preferences fall as one moves away from his/her top-ranked alternative along any path. Schummer and Vohra (2002) characterize the tops-only, unanimous and strategy-proof DSCFs on tree-single-peaked domains. Under the additional restriction that the top-set of the domain lie along a path, our result improves their one in two ways: first, by allowing for random rules, and second, by relaxing tops-onlyness.

We introduce a graph structure over the set of alternatives. A collection $G \subseteq \{\{a,b\} \mid a,b \in A,\ a \neq b\}$ is an *undirected graph over* A. The elements of G are *edges*. A *path* in G from a node a_1 to another a_k is a sequence of distinct nodes $\langle a_1,\ldots,a_k\rangle$ such that $\{a_i,a_{i+1}\} \in G$ for all $i = 1,\ldots,k-1$. Note that a path cannot have a cycle by definition.

A graph over A is a tree, denoted by T, if for all $a,b \in A$, there exists a unique path from a to b. Since such a path is unique in a tree, for ease of presentation we denote it by $[a,b]$. A preference P is *single-peaked on* T if for all distinct $x,y \in A$ with $y \neq \tau(P)$, $x \in [\tau(P),y] \implies xPy$. A domain is *single-peaked on* T if each preference in it is single-peaked on T.

Let T be a tree over A and let $\mathscr{D}$ be a single-peaked domain on T. Suppose $\tau(\mathscr{D}) = \{b_1,\ldots,b_k\}$. We call $\mathscr{D}$ a *single-peaked domain with top-set along a path* if $\langle b_1,\ldots,b_k\rangle$ is a path in T. In Figure 2, we present a tree in which a path is marked with red. A single-peaked domain with respect to this tree with top-set along the red path can be constructed by taking those single-peaked preferences that have top-ranked alternatives

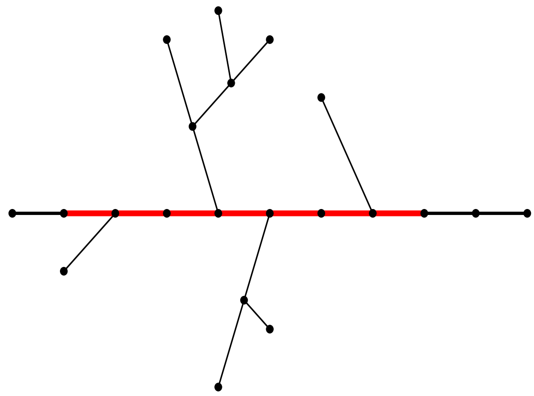

Figure 2: An example of a tree

in that path.

The following lemma says that a single-peaked domain on a tree with top-set along a path is a minimally rich generalized intermediate domain.

Lemma 3. *Let $\mathscr{D}$ be a single-peaked domain on a tree T with top-set along a path in T. Then, $\mathscr{D}$ is a minimally rich generalized intermediate domain.*

Proof. Let T be a tree and let $\pi = \langle b_1, \ldots, b_k \rangle$ be a path in it. Let $\mathscr{D}$ be a single-peaked domain on T with $\tau(\mathscr{D}) = \{b_1, \ldots, b_k\}$. Consider a linear order $\prec$ on A such that

- $b_1 \prec \cdots \prec b_k$, and
- for all $a \in A \setminus \{b_1, \ldots, b_k\}$, $a \prec b_l$ if and only if the projection of a on π is b_j for some $j \leq l$.[10]

Note that the linear order $\prec$ defined above is not unique since it does not specify the relative ordering of two alternatives that are outside the path π but have the same projection. We show that $\mathscr{D}$ is a minimally rich generalized intermediate domain with respect to $\prec$. Since $\mathscr{D}$ is single-peaked on T and $\{b_l, b_{l+1}\}$ is an edge in T for all $l \in \{1, \ldots, k-1\}$, we can always find two preferences P and P' such that $\tau(P) = r_2(P') = b_l$, $r_2(P) = \tau(P') = b_{l+1}$, and $r_l(P) = r_l(P')$ for all $l \geq 3$. Therefore, $\mathscr{D}$ is minimally rich.

Now, we show that $\mathscr{D}$ is generalized intermediate. Consider b_r and b_s with $b_r \prec b_s$. To show $\mathscr{D}$ is generalized intermediate, it is enough to show that for all P with $\tau(P) = b_r$, we have $b_s P a$ for all a with $b_s \prec a$. Assume for contradiction that there exist $P \in \mathscr{D}$ and $a \in A$ with $\tau(P) = b_r$ and $b_s \prec a$ such that aPb_s. If $a \in \{b_{s+1}, \ldots, b_k\}$, then by means of the fact that T is a tree, we have $b_s \in [b_r, a]$. However, by single-peakedness of P, this implies $b_s P a$, which is a contradiction to aPb_s. Now, suppose $a \in A \setminus \{b_{s+1}, \ldots, b_k\}$. Since $b_s \prec a$, by the definition of $\prec$, there exists $b_l \in \{b_{s+1}, \ldots, b_k\}$ such that the projection of a on π is b_l.

[10] By the projection of an alternative $a \in A$ on a path π in a tree T, we mean the alternative $b \in \pi$ that is closest (with respect to graph distance) to a, i.e., $b \in \pi$ is such that $|[a,b]| \leq |[a,c]|$ for all $c \in \pi$.

By the definition of projection and by single-peakedness of P, we have $b_l P a$. Moreover, since $b_s \in [b_r, b_l]$ it follows that $b_s P b_l$, which in turn implies $b_s P a$. However, this is a contradiction to $a P b_s$. Thus, for all P with $\tau(P) = b_r$, we have $b_s P a$ for all a with $b_s \prec a$. This proves $\mathscr{D}$ is a generalized intermediate domain. ■

4.5 Multi-peaked domains

In many practical scenarios in Economics and Political Science, preferences of individuals often exhibit *multi-peakedness* as opposed to single-peakedness. As the name suggests, multi-peaked preferences admit multiple (local) ideal points in a unidimensional policy space. We discuss a few settings where it is plausible to assume that individuals have multi-peaked preferences.

- *Preference for "Do Something" in Politics*: Davis et al. (1970) and Egan (2014) consider policy (decision) problems such as choosing alternate tax regimes, lowering health care costs, responding to foreign competition, reducing national debt, etc. They show that such a problem is perceived to be poorly addressed by the status-quo policy, and consequently some individuals prefer both liberal and conservative policies to the moderate status quo one. Clearly, such a preference will have two peaks, one on the left of the status quo and another one on the right of it.

- *Multi-stage Voting System*: Shepsle (1979), Denzau and Mackay (1981), Enelow and Hinich (1983) deal with multi-stage voting system where individuals vote on a set of issues where each issue can be thought of as a unidimensional spectrum and voting is distributed over several stages considering one issue at a time. In such a model, preference of an individual over the present issue can be affected by his/her prediction of the outcome of future issues. In other words, such a preference is not separable across issues. They show that preferences of individuals in such scenarios exhibit multi-peaked property.

- *Provision of Public Goods with Outside Options*: Barzel (1973), Stiglitz (1974), and Bearse et al. (2001) consider the problem of setting the level of tax rates to provide public funding in the education sector, and Ireland (1990) and Epple and Romano (1996) consider the same problem in the health insurance market. They show that preferences of individuals exhibit multi-peaked property due to the presence of outside options (i.e., the public good is also available in a competitive market as a private good).

- *Provision of Excludable Public Goods*: Fernandez and Rogerson (1995) and Anderberg (1999) consider public good provision models such as health insurance, educational subsidies, pensions, etc., where a government provides the public good to a particular section of individuals and show that individuals' preferences in such scenarios exhibit multi-peaked property.

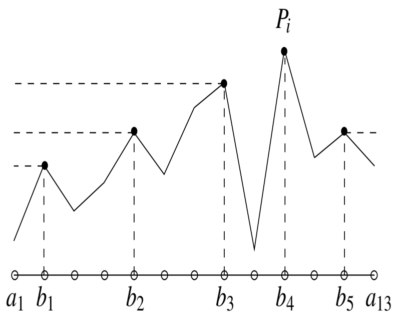

Figure 3: A multi-peaked preference

We now present a formal definition of multi-peaked domains and show that they are special cases of generalized intermediate domains. To ease our presentation, for two alternatives a and b, we denote by (a,b) the set $[a,b] \setminus \{a,b\}$.

Let $b_1 \prec \cdots \prec b_k$ be such that $(b_l, b_{l+1}) \neq \emptyset$ for all $1 \leq l < k$. Then, a preference P is *multi-peaked* with peak-set $\{b_1, \ldots, b_k\}$ if (i) $P|_{[a_1,b_1]}$ and $P|_{[b_k,a_m]}$ are single-dipped with dips at a_1 and a_m, respectively, (ii) for all $1 \leq l < k$, $P|_{[b_l,b_{l+1}]}$ is single-dipped with a dip in (b_l, b_{l+1}), and (iii) $P|_{\{b_1,\ldots,b_k\}}$ is single-peaked. A domain $\mathscr{D}$ is multi-peaked if it contains all multi-peaked preferences with peak-set $\tau(\mathscr{D})$.

In words, for a multi-peaked preference there are several (local) peaks such that the preference behaves like a single-dipped one between every two consecutive peaks and like a single-peaked one over the peaks. In Figure 3, we present a pictorial description of a multi-peaked preference.

Lemma 4. *Every multi-peaked domain is a minimally rich generalized intermediate domain.*

Proof. Let $\mathscr{D}$ be a multi-peaked domain. Suppose $\tau(\mathscr{D}) = \{b_1, \ldots, b_k\}$ with $b_1 \prec \ldots \prec b_k$. By the definition of $\mathscr{D}$, for all $b_l, b_{l+1} \in \tau(\mathscr{D})$, there are preferences $P, P' \in \mathscr{D}$ such that $\tau(P) = b_l$, $\tau(P') = b_{l+1}$ and $P \sim P'$. This shows $\mathscr{D}$ is minimally rich. Now, we prove $\mathscr{D}$ is a generalized intermediate domain. Consider b_r and b_s where $b_r \prec b_s$. We show that for all P with $\tau(P) = b_r$, we have $b_s P a$ for all $a \in A$ with $b_s \prec a$. Consider $P \in \mathscr{D}$ with $\tau(P) = b_r$ and consider $a \in A$ with $b_s \prec a$. Since $b_s \prec a$ there exists b_l with $b_s \preceq b_l$ such that $a \in [b_l, b_{l+1}]$. By the definition of multi-peaked domains, we have $b_s P b_l$ and $b_l P a$, which implies $b_s P a$. This proves that $\mathscr{D}$ is a generalized intermediate domain. ■

REMARK 4.1. Note that for both applications 4.4 and 4.5, the top-set of the domain is (exogenously) known to the designer. Domains with exogenously given characteristics are not new to the literature, for instance Alcalde-Unzu and Vorsatz (2018) consider domains where the top-ranked alternative of each agent is known to the designer and Pramanik and Sen (2016) consider domains where the indifference classes are known to the designer.

4.6 Euclidean domains

Roy and Sadhukhan (2019) consider Euclidean domains and show that every unanimous and strategy-proof RSCF on such domains is a random minmax rule.

For ease of presentation, we assume that the set of alternatives are (finitely many) elements of the interval $[0,1]$.[11] In particular, we assume $0 = a_1 < \cdots < a_m = 1$. Suppose that the individuals are located at arbitrary locations in $[0,1]$ and they derive their preferences using Euclidean distances of the alternatives from their own locations. We call such preferences Euclidean. A preference P is *Euclidean* if there is $x \in [0,1]$, called the location of P, such that for all alternatives $a,b \in A$, $|x-a| < |x-b|$ implies aPb. A domain is *Euclidean* if it contains *all* Euclidean preferences.

Lemma 5. *Every Euclidean domain is a minimally rich generalized intermediate domain.*

Proof. Let $\mathscr{D}$ be a Euclidean domain. Then, by definition, it is regular single-peaked, and by Remark 2.1, it is generalized intermediate. It remains to show that $\mathscr{D}$ is minimally rich. Consider a_r and a_{r+1} for some $r \in \{1,\ldots,m-1\}$. By the definition of Euclidean domain, there are two preferences P and P' in $\mathscr{D}$ with location $\frac{a_r+a_{r+1}}{2}$ such that $\tau(P) = r_2(P') = a_r$, $r_2(P) = \tau(P') = a_{r+1}$, and $r_l(P) = r_l(P')$ for $l \geq 3$. This completes the proof of the lemma. ■

4.7 Intermediate domain

Grandmont (1978) introduces the concept of intermediate domains and shows that under some conditions on the distribution of voters over preferences, majority rule is transitive on these domains. However, to the best of our knowledge, no characterization of unanimous and strategy-proof RSCFs on these domains is available in the literature. Under a mild condition on these domains (mainly to avoid non-transitive preferences), we show that these domains are special cases of generalized intermediate domains, and consequently, we provide a characterization of unanimous and strategy-proof RSCFs on those.

Throughout this section, we denote by X an open convex subset of the Euclidean space E^2, and whenever we refer to a line, we mean a line in X (that is, a collection of points in X that constitute a line).

A preference P is *between* two preferences P_1 and P_2, denoted by $P \in (P_1,P_2)$, if for all $a,b \in A$, aP_1b and aP_2b imply aPb. A domain $\{P_x\}_{x\in X}$ satisfies the *intermediate property* if for every x' and $x'' \in X$, $x \in (x',x'')$ implies $P_x \in (P_{x'},P_{x''})$.[12]

Grandmont (1978) provides a characterization of the intermediate domains where preferences are allowed to be weak (i.e., can have indifferences) and non-transitive. In the following lemma, we modify his/her result for the situation where preferences are strict and transitive (i.e., linear orders).

[11] With abuse of notation, we denote by $[0,1]$ the set of all real numbers in-between 0 and 1.

[12] With slight abuse of notation, by $x \in (x',x'')$, we mean $x = \lambda x' + (1-\lambda)x''$ for some real number $\lambda \in (0,1)$.

Lemma 6. *Let a domain $\{P_x\}_{x \in X}$ satisfy the intermediate property. Then, for every pair of alternatives (a,b), exactly one of the following statements must hold:*

(i) *aP_xb for all $x \in X$.*

(ii) *bP_xa for all $x \in X$.*

(iii) *There exist $q = (q_1,q_2) \in E^2$; $(q_1,q_2) \neq (0,0)$ and $\kappa \in \mathbb{R}$ such that for all $(x_1,x_2) \in X$, aP_xb implies $q_1x_1 + q_2x_2 \geq \kappa$ and bP_xa implies $q_1x_1 + q_2x_2 \leq \kappa$.*

Proof. Suppose that both (i) and (ii) do not hold. We show that then (iii) must hold. Consider $a,b \in A$. Let $A_1 = \{x \in X \mid aP_xb\}$ and $A_2 = \{x \in X \mid bP_xa\}$. By our assumption that both (i) and (ii) do not hold, it follows that both A_1 and A_2 are non-empty. Moreover, by definition, A_1 and A_2 are disjoint, and by the intermediate property, both A_1 and A_2 are convex. Therefore, by Hyperplane separation theorem (Rockafellar (1970), Theorem 11.3), there exist $q = (q_1,q_2) \in E^2$; $(q_1,q_2) \neq (0,0)$ and $\kappa \in \mathbb{R}$ such that for all $(x_1,x_2) \in X$, aP_xb implies $q_1x_1 + q_2x_2 \geq \kappa$ and bP_xa implies $q_1x_1 + q_2x_2 \leq \kappa$. This completes the proof of the lemma. ■

Note that for a domain satisfying the intermediate property and for a pair of alternatives (a,b) that satisfies (iii) in Lemma 6, the object $((q_1,q_2),\kappa)$ identifies the line: $q_1x_1 + q_2x_2 = \kappa$. We denote such a line by $l(a,b)$. Lemma 6 implies that a is preferred to b on one side of this line, and b is preferred to a on the other side.[13] Since such a line separates the preferences with respect to a and b, we call it the separating line for a and b. In what follows, we introduce the concept of strict intermediate property.

Definition 4.1. A domain $\{P_x\}_{x \in X}$ satisfies the *strict intermediate* property if

(i) there are no three distinct separating lines of the domain that pass through a common point, that is, for all three distinct (unordered) pairs (x_1,y_1), (x_2,y_2), and (x_3,y_3), we have $l(x_1,y_1) \cap l(x_2,y_2) \cap l(x_3, y_3) = \emptyset$,[14] and

(ii) there exists a line l that intersects with all the separating lines of the domain, that is, for all pairs $x,y \in A$ satisfying (iii) in Lemma 6, we have $l \cap l(x,y) \neq \emptyset$.

We provide an example of a domain that satisfies the strict intermediate property. It is worth noting from this example that (i) strictness is indeed a mild condition, and (ii) the strict intermediate property does *not* imply the single-crossing property.

[13]There is no restriction on the relative preference over a and b for the preferences P_x when x lies on this line.

[14]By distinct (unordered pairs), we mean that $\{x_i,y_i\} \neq \{x_j,y_j\}$ for all $i,j \in \{1,2,3\}$ with $i \neq j$.

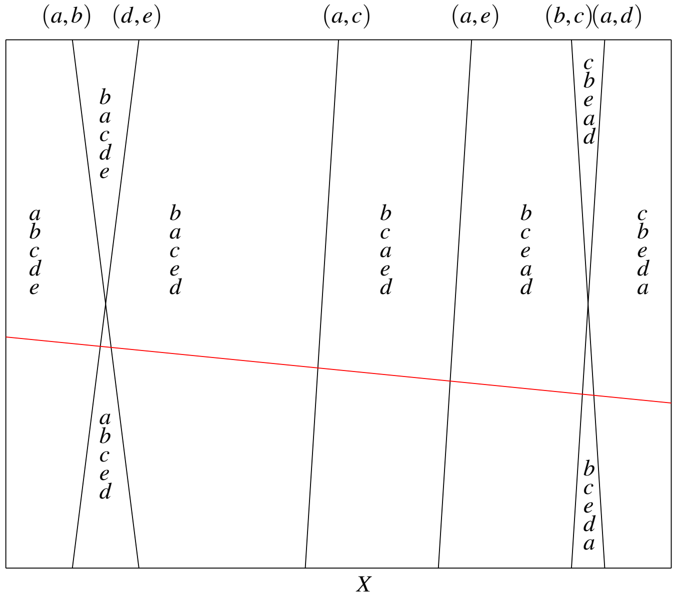

Figure 4: The set of separating lines of the domain in Example 4

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
a	a	b	b	b	b	b	c	c
b	b	a	a	c	c	c	b	b
c	c	c	c	a	e	e	e	e
d	e	d	e	e	a	d	a	d
e	d	e	d	d	d	a	d	a

Table 3: The domain in Example 4

Example 4. Let X be the open set in Figure 4 and let $\{P_x\}_{x\in X} = \{abcde, abced, bacde, baced, bcaed, bcead, bceda, cbead, cbeda\}$ be a domain satisfying intermediate property. For each pair of alternatives, the separating line is indicated in the figure. Note that for the pairs (b,d), (b,c), etc., there are no separating lines. Further note that P_x is constant over all points x that are enclosed by some separating lines of the domain (this follows from Lemma 6). Such P_xs are mentioned in the respective region in Figure 4.

Clearly, the domain $\{P_x\}_{x\in X}$ satisfies strict intermediate property since no three separating lines pass through a common point and the line l (marked with red) intersects with all these lines. It is left to the reader to verify that the domain $\{P_x\}_{x\in X}$ is not a single-crossing domain. □

It is worth noting that the domain in Example 4 is a minimally rich generalized intermediate domain. Our next lemma shows that this fact is true in general.

Lemma 7. *Every domain $\{P_x\}_{x\in X}$ satisfying strict intermediate property is a generalized intermediate domain.*

The proof of this lemma is relegated to Appendix B.

5. THE CASE OF INDIFFERENCE AT THE TOP

In this section, we investigate the structure of the unanimous and strategy-proof RSCFs when indifference can occur at the top position. A particular class of these domains are known as *single-plateaued* domains. Importance of such domains is well-established in the literature (see Berga (1998) for more details).

A weak preference is a transitive and complete binary relation. For a weak preference R, we denote its indifference part by I and strict part by P. We denote the set of top-ranked alternatives in R by $\tau(R)$ and call it the *plateau* of R. In this section, we consider weak preferences R such that the size of the plateau can be at most two and the rest part of the preference is strict, that is, $|\tau(R)| \leq 2$ and aIb if and only if $a,b \in \tau(R)$. We denote a collection of such preferences by $\bar{\mathscr{D}}$.

For a domain $\bar{\mathscr{D}}$, we define $\tau(\bar{\mathscr{D}}) = \{x \mid x \in \tau(R) \text{ for some } R \in \bar{\mathscr{D}}\}$ as the set of alternatives that appear in the plateau of some preference in $\bar{\mathscr{D}}$. Following our terminology for the case of strict preferences, we write the elements of $\tau(\bar{\mathscr{D}})$ as $\{b_1,\dots,b_k\}$, where $b_1 \prec \cdots \prec b_k$.

An RSCF $\varphi : \bar{\mathscr{D}}^n \to \Delta A$ is *unanimous* if for all $R_N \in \bar{\mathscr{D}}^n$, $\cap_{i\in N}\tau(R_i) \neq \emptyset$ implies $\varphi_{\cap_{i\in N}\tau(R_i)}(R_N) = 1$. An RSCF $\varphi : \bar{\mathscr{D}}^n \to \Delta A$ is *strategy-proof* if for all $i \in N$, all $R_N \in \bar{\mathscr{D}}^n$, all $R'_i \in \bar{\mathscr{D}}$, and all $x \in A$, $\varphi_{U(x,R_i)}(R_N) \geq \varphi_{U(x,R_i)}(R'_i,R_{-i})$.[15]

A *tie-breaking* of a preference $R \in \bar{\mathscr{D}}$ is defined as a strict preference $\hat{P}$ such that for all $a,b \in A$, aPb implies $a\hat{P}b$. In other words, if R is strict, then its tie-breaking is R itself, and if $|\tau(R)| = 2$, then in a tie-breaking the top two alternatives appear as the first and the second ranked alternatives, and rest of the preference remains the same. A domain $\bar{\mathscr{D}}$ satisfies the *tie-braking property* if each preference R in it has a tie-breaking present in it.

We define the suitable version of the betweenness property for weak preferences. A preference R satisfies the **betweenness property** with respect to an alternative a if for all $b \in A \setminus a$ with $b \notin \tau(R)$, $a \in [x,b]$ for some $x \in \tau(R)$ implies aPb. A domain $\bar{\mathscr{D}}$ satisfies the **betweenness property** with respect to an alternative a if each preference $R \in \bar{\mathscr{D}}$ satisfies the property with respect to a. A domain $\bar{\mathscr{D}}$ is **generalized intermediate** if it satisfies the betweenness property with respect each alternative in $\tau(\bar{\mathscr{D}})$. Note that this means that for each $R \in \bar{\mathscr{D}}$ with $|\tau(R)| = 2$, $\tau(R)$ can only be of the form $\{b_t, b_{t+1}\}$ for some $t \in \{1,\dots,k-1\}$ (that is, it cannot be like $\{b_s,b_t\}$ where $t-s \geq 2$). We use the same definition for minimal richness as in the case of strict preferences, that is, $\bar{\mathscr{D}}$ satisfies minimal richness if its strict preferences satisfy the same.

Our next theorem says that every unanimous and strategy-proof RSCF on a minimally rich generalized intermediate domain satisfying the tie-breaking property is tops-restricted, that is, the top-set of the domain gets probability 1 at all profiles.

[15] For a weak preference R and an alternative a, $U(a,R) = \{b \in A | bRa\}$.

Theorem 2. *Let $\bar{\mathscr{D}}$ be a minimally rich generalized intermediate domain satisfying the tie-breaking property and let $\varphi : \bar{\mathscr{D}} \to \Delta A$ be a unanimous and strategy-proof RSCF. Then $\varphi_{\tau(\bar{\mathscr{D}})}(R_N) = 1$ for all $R_N \in \bar{\mathscr{D}}^n$.*

The proof of this theorem is relegated to Appendix C.

In what follows, we analyze the structure of unanimous and strategy-proof RSCFs on $\bar{\mathscr{D}}$. With slight abuse of terminology, we define a tie-breaking mapping of an agent $i \in N$ as $\pi_i : \bar{\mathscr{D}}^n \to A$ such that $\pi_i(R_N) \in \tau(R_i)$ for all $R_N \in \bar{\mathscr{D}}^n$. For a collection of tie-breaking mappings $\boldsymbol{\pi} = (\pi_i)_{i\in N}$ and a profile R_N, we write $\boldsymbol{\pi}(R_N)$ to denote the profile $(\pi_1(R_N), \ldots, \pi_n(R_N))$. A collection of tie-breaking mappings $\boldsymbol{\pi}$ is called unanimous and strategy-proof if each π_i in it is unanimous and strategy-proof.

We now define a new class of RSCFs by composing a TRM rule with a collection of tie-breaking mappings. Let φ be any arbitrary TRM rule and $\boldsymbol{\pi}$ be an arbitrary collection of unanimous and strategy-proof tie-breaking mappings. Since φ is tops-only, it can be viewed as a function from A^n to ΔA. Define the composition of φ and $\boldsymbol{\pi}$ as the RSCF $\bar{\varphi} : \bar{\mathscr{D}}^n \to \Delta A$ defined as $\bar{\varphi}(R_N) = \varphi(\boldsymbol{\pi}(R_N))$ for all $R_N \in \bar{\mathscr{D}}^n$. Our next theorem says that every composition of a TRM rule and a unanimous and strategy-proof collection of tie-braking mappings is unanimous and strategy-proof on a minimally rich generalized intermediate domain.

Theorem 3. *Let $\bar{\mathscr{D}}$ be a minimally rich generalized intermediate domain. Then, for any TRM rule φ and any collection of unanimous and strategy-proof tie-breaking mappings $\boldsymbol{\pi}$, the composition of φ and $\boldsymbol{\pi}$ is unanimous and strategy-proof.*

The proof of this theorem is relegated to Appendix D.

The natural question arises as to what happens with the converse of Theorem 3. Berga (1998) shows that if we replace unanimity by plateau-onlyness, then converse of Theorem 3 holds for deterministic rules.[16] In what follows, we provide an example of a unanimous and strategy-proof RSCF that cannot not be written as a composition as in Theorem 3, which in particular means that the converse of Theorem 3 is not true. Furthermore, it can be verified that the RSCF in the example is also plateau-only, which says that the converse of Theorem 3 does not hold even under plateau-onlyness.

Example 5. Let $A = \{a_1, a_2, a_3\}$ and $N = \{1, 2\}$. Consider the domain $\bar{\mathscr{D}} = \{a_1a_2a_3, a_2a_1a_3, a_2a_3a_1, a_3a_2a_1, [a_3a_2]a_1\}$. Here, for instance, by $[a_3a_2]a_1$ we denote the preference R such that $\tau(R) = \{a_2, a_3\}$, and the second ranked alternative is a_1. Consider the RSCF φ given in Table 4. It can be verified that this rule is unanimous and strategy-proof. It is plateau-only too. We argue that this rule cannot be written as a composition of a TRM rule and a collection of unanimous and strategy-proof tie-breaking mappings. Assume for contradiction this rule is a composition of a TRM rule and a collection of unanimous and

[16]An RSCF $\varphi : \bar{\mathscr{D}}^n \to \Delta A$ is plateau-only if for all R_N, R'_N such that $\tau(R_i) = \tau(R'_i)$ for all $i \in N$, we have $\varphi(R_N) = \varphi(R'_N)$.

strategy-proof tie-breaking mappings. By the definition of a TRM rule, we can deduce the following facts about its parameters: $\beta_1 = \varphi(a_1a_2a_3, a_3a_2a_1)$ and $\beta_2 = \varphi(a_3a_2a_1, a_1a_2a_3)$, which implies $\beta_1 = (0.3, 0.4, 0.3)$ and $\beta_2 = (0.2, 0.4, 0.4)$. Consider the profile $([a_3a_2]a_1, [a_3a_2]a_1)$. By unanimity, $\pi_i([a_3a_2]a_1, [a_3a_2]a_1) \in \{a_2, a_3\}$ for all $i = 1, 2$. Since $\varphi_{a_2}([a_3a_2]a_1, [a_3a_2]a_1) > 0$ and $\varphi_{a_3}([a_3a_2]a_1, [a_3a_2]a_1) > 0$, it must be that $\pi_1([a_3a_2]a_1, [a_3a_2]a_1) \neq \pi_2([a_3a_2]a_1, [a_3a_2]a_1)$. Suppose $\pi_1([a_3a_2]a_1, [a_3a_2]a_1) = a_2$ and $\pi_2([a_3a_2]a_1, [a_3a_2]a_1) = a_3$. But this means $\varphi([a_3a_2]a_1, [a_3a_2]a_1) = (0, 0.3, 0.7)$, which is a contradiction as $\varphi([a_3a_2]a_1, [a_3a_2]a_1) = (0, 0.5, 0.5)$. A similar contradiction as before emerges by considering the opposite case: $\pi_1([a_3a_2]a_1, [a_3a_2]a_1) = a_3$ and $\pi_2([a_3a_2]a_1, [a_3a_2]a_1) = a_2$. This shows that φ can not be written as a composition of a TRM rule and a collection of unanimous and strategy-proof tie-breaking mappings.

$1\backslash 2$	$a_1a_2a_3$	$a_2a_1a_3$	$a_2a_3a_1$	$a_3a_2a_1$	$[a_3a_2]a_1$
$a_1a_2a_3$	$(1,0,0)$	$(0.3,0.7,0)$	$(0.3,0.7,0)$	$(0.3,0.4,0.3)$	$(0.3,0.7,0)$
$a_2a_1a_3$	$(0.2,0.8,0)$	$(0,1,0)$	$(0,1,0)$	$(0,0.7,0.3)$	$(0,1,0)$
$a_2a_3a_1$	$(0.2,0.8,0)$	$(0,1,0)$	$(0,1,0)$	$(0,0.7,0.3)$	$(0,1,0)$
$a_3a_2a_1$	$(0.2,0.4,0.4)$	$(0,0.6,0.4)$	$(0,0.6,0.4)$	$(0,0,1)$	$(0,0,1)$
$[a_2a_3]a_1$	$(0.2,0.8,0)$	$(0,1,0)$	$(0,1,0)$	$(0,0,1)$	$(0,0.5,0.5)$

Table 4: The RSCF in Example 5

$\square$

It is worth noting from Example 5 that the structure of unanimous (plateau-only) and strategy-proof RSCFs on single-plateaued domains is fairly complicated. This is particularly because unanimity (plateau-onlyness) and strategy-proofness do not uniquely determine the outcomes at profiles that involve plateaus. For instance, in Example 5, any vector of the form $(0, \delta, 1-\delta)$, where $\delta \in [0,1]$, can be the outcome at the profile $([a_2a_3]a_1, [a_3a_2]a_1)$ (maintaining unanimity (plateau-onlyness) and strategy-proofness). We leave the problem of characterizing all unanimous (plateau-only) and strategy-proof rules on single-plateaued domains for future research.

6. Conclusion

In this paper, we have shown that in case of finitely many alternatives, an RSCF on a minimally rich generalized intermediate domain is unanimous and strategy-proof if and only if it can be written as a convex combination of the tops-restricted min-max rules. As applications of our result, we have obtained a characterization of the unanimous and strategy-proof RSCFs on restricted domains such as single-peaked, single-crossing, single-dipped, single-peaked on a tree with top-set along a path, Euclidean, multi-peaked, and intermediate domain (Grandmont (1978)). We have also analyzed the structure of unanimous and strategy-proof RSCFs on domains containing weak preferences for which indifference can occur only at the top two positions.

To our understanding, our results apply to all well-known restricted domains in one dimension. An interesting problem would be to see to what extent one can enlarge a generalized intermediate domain ensuring the existence of a non-random-dictatorial, unanimous, and strategy-proof (not necessarily tops-restricted random min-max) random rule. This will give some idea of the robustness of the generalized intermediate domains as possibility domains. Another interesting problem would be to explore the generalized intermediate domains for multiple dimensions. We leave all these problems for future research.

A. PROOF OF THEOREM 1

First, we prove a proposition that constitutes a major step in this proof.

Proposition 1. *Let $\mathscr{D}$ be a minimally rich generalized intermediate domain and let $\varphi : \mathscr{D}^n \to \Delta A$ be a unanimous and strategy-proof RSCF. Then,*

(i) $\varphi_{\tau(\mathscr{D})}(P_N) = 1$ *for all* $P_N \in \mathscr{D}^n$*, and*

(ii) φ *is uncompromising.*

We prove a sequence of lemmas which we will use in the proof of Proposition 1. The following lemma establishes that a generalized intermediate domain restricted to its top-set is single-peaked.

Lemma 8. *Let $\mathscr{D}$ be a generalized intermediate domain. Then, $\mathscr{D}|_{\tau(\mathscr{D})}$ is single-peaked.*

Proof. Let $\mathscr{D}$ be a generalized intermediate domain with $\tau(\mathscr{D}) = \{b_1, \ldots, b_k\}$. We show that $\mathscr{D}|_{\tau(\mathscr{D})}$ is single-peaked. Without loss of generality, assume by contradiction that there exists $P \in \mathscr{D}$ such that $\tau(P) = b_j$ and $b_{l'} P b_l$ for some l, l' with $l' < l < j$. This means P violates the betweenness property with respect to b_l, which is a contradiction since $\mathscr{D}$ is a generalized intermediate domain and $b_l \in \tau(\mathscr{D})$. This completes the proof of the lemma. ■

In what follows, we prove a technical lemma that we use repeatedly in the proof of Proposition 1. We use the following notation in this lemma: for $X, Y \subseteq A$ and a preference P, XPY means xPy for all $x \in X$ and $y \in Y$.

Lemma 9. *Let $\mathscr{D}$ be a domain and let $\varphi : \mathscr{D}^n \to \Delta A$ be a strategy-proof RSCF. Let $P_N \in \mathscr{D}^n$, $P'_i \in \mathscr{D}$, and B, $C \subseteq A$ be such that BP_iC, BP'_iC, and $P_i|_C = P'_i|_C$. Suppose $\varphi_C(P_N) = \varphi_C(P'_i, P_{-i})$ and $\varphi_a(P_N) = \varphi_a(P'_i, P_{-i})$ for all $a \notin B \cup C$. Then, $\varphi_a(P_N) = \varphi_a(P'_i, P_{-i})$ for all $a \in C$.*

Proof. First note that since $\varphi_C(P_N) = \varphi_C(P'_i, P_{-i})$ and $\varphi_a(P_N) = \varphi_a(P'_i, P_{-i})$ for all $a \notin B \cup C$, $\varphi_B(P_N) = \varphi_B(P'_i, P_{-i})$. Suppose $b \in C$ is such that $\varphi_b(P_N) \neq \varphi_b(P'_i, P_{-i})$ and $\varphi_a(P_N) = \varphi_a(P'_i, P_{-i})$ for all $a \in C$ with aP_ib. In other words, b is the maximal element of C according to P_i that violates the assertion of the lemma. Without loss of generality, assume that $\varphi_b(P_N) < \varphi_b(P'_i, P_{-i})$. Since BP_iC, $\varphi_B(P_N) = \varphi_B(P'_i, P_{-i})$, and $\varphi_a(P_N) = \varphi_a(P'_i, P_{-i})$ for all $a \notin B$ with aP_ib, it follows that $\varphi_{U(b,P_i)}(P_N) < \varphi_{U(b,P_i)}(P'_i, P_{-i})$. This implies agent i manipulates at P_N via P'_i, which is a contradiction. This completes the proof of the lemma. ■

Proof of Proposition 1

Now, we are ready to complete the proof of Proposition 1.

Proof. We prove this proposition by using induction on the number of agents. Let $\mathscr{D}$ be a generalized intermediate domain with $\tau(\mathscr{D}) = \{b_1, \ldots, b_k\}$.

Let $|N| = 1$ and let $\varphi : \mathscr{D} \to \Delta A$ be a unanimous and strategy-proof RSCF. Then, by unanimity, $\varphi_{\tau(\mathscr{D})}(P_N) = 1$ for all $P_N \in \mathscr{D}$, and hence φ satisfies uncompromisingness.

Assume that the proposition holds for all sets with $k < n$ agents. We prove it for n agents. Let $|N| = n$ and let $\varphi : \mathscr{D}^n \to \Delta A$ be a unanimous and strategy-proof RSCF. Suppose $N^* = N \setminus \{1\}$. Define the RSCF $g : \mathscr{D}^{n-1} \to \Delta A$ for the set of voters N^* as follows: for all $P_{N^*} = (P_2, P_3, \ldots, P_n) \in \mathscr{D}^{n-1}$,

$$g(P_2, P_3, \ldots, P_n) = \varphi(P_2, P_2, P_3, P_4, \ldots, P_n).$$

Evidently, g is a well-defined RSCF satisfying unanimity and strategy-proofness (See Lemma 3 in Sen (2011) for a detailed argument). Hence, by the induction hypothesis, $g_{\tau(\mathscr{D})}(P_{N^*}) = 1$ for all $P_{N^*} \in \mathscr{D}^{n-1}$ and g satisfies uncompromisingness. In terms of φ, this implies $\varphi_{\tau(\mathscr{D})}(P_N) = 1$ for all $P_N \in \mathscr{D}^n$ with $P_1 = P_2$.

We complete the proof of Proposition 1 by using the following lemmas. In the next lemma, we show that $\varphi_{\tau(\mathscr{D})}(P_N) = 1$ and φ is tops-only over all profiles P_N where agents 1 and 2 have the same top alternative.

Lemma 10. *Let $P_N, P'_N \in \mathscr{D}^n$ be two tops-equivalent profiles such that $P_1, P_2 \in \mathscr{D}^{b_j}$ for some $b_j \in \tau(\mathscr{D})$. Then, $\varphi_{\tau(\mathscr{D})}(P_N) = 1$ and $\varphi(P_N) = \varphi(P'_N)$.*

Proof. Note that since g is uncompromising, g satisfies tops-onlyness. Because g is tops-only and P_1, $P_2 \in \mathscr{D}^{b_j}$, we have $g(P_1, P_{-\{1,2\}}) = g(P_2, P_{-\{1,2\}})$, and hence $\varphi(P_1, P_1, P_{-\{1,2\}}) = \varphi(P_2, P_2, P_{-\{1,2\}})$. We show $\varphi(P_1, P_2, P_{-\{1,2\}}) = \varphi(P_1, P_1, P_{-\{1,2\}})$. Using strategy-proofness of φ for agent 2, we have $\varphi_{U(x,P_1)}(P_1, P_1, P_{-\{1,2\}}) \geq \varphi_{U(x,P_1)}(P_1, P_2, P_{-\{1,2\}})$ for all $x \in A$, and using that for agent 1, we have $\varphi_{U(x,P_1)}(P_1, P_2, P_{-\{1,2\}}) \geq \varphi_{U(x,P_1)}(P_2, P_2, P_{-\{1,2\}})$ for all $x \in A$. Since $\varphi(P_1, P_1, P_{-\{1,2\}}) = \varphi(P_2, P_2, P_{-\{1,2\}})$, it follows from Remark 2.2 that $\varphi(P_1, P_1, P_{-\{1,2\}}) = \varphi(P_1, P_2, P_{-\{1,2\}})$. Using a similar logic, we have $\varphi(P'_1, P'_1, P'_{-\{1,2\}}) = \varphi(P'_1, P'_2, P'_{-\{1,2\}})$. Because g is tops-only and P_N, P'_N are tops-equivalent, we have $g(P_1, P_{-\{1,2\}}) = g(P'_1, P'_{-\{1,2\}})$. This implies $\varphi(P_1, P_1, P_{-\{1,2\}}) = \varphi(P'_1, P'_1, P'_{-\{1,2\}})$, and hence $\varphi(P_1, P_2, P_{-\{1,2\}}) =$

$\varphi(P_1', P_2', P_{-\{1,2\}}')$. Moreover, as $\varphi_{\tau(\mathscr{D})}(P_1, P_1, P_{-\{1,2\}}) = 1$, it follows that $\varphi_{\tau(\mathscr{D})}(P_1, P_2, P_{-\{1,2\}}) = 1$. This completes the proof of the lemma. ■

Lemma 11. *Let $1 \leq j \leq j+l \leq k$ and let $P_N, P_N' \in \mathscr{D}^n$ be such that $P_1, P_2 \in \mathscr{D}^{b_j}$ and $P_1', P_2' \in \mathscr{D}^{b_{j+l}}$, and $\tau(P_i) = \tau(P_i')$ for all $i \neq 1,2$. Then, $\varphi_b(P_N) = \varphi_b(P_N')$ for all $b \notin [b_j, b_{j+l}]_{\tau(\mathscr{D})}$.*

Proof. By uncompromisingness of g and the fact that $g_{\tau(\mathscr{D})}(P_{N^*}) = 1$ for all $P_{N^*} \in \mathscr{D}^{n-1}$, we have $g_b(P_1, P_{-\{1,2\}}) = g_b(P_1', P_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l}]_{\tau(\mathscr{D})}$. Moreover, since g is tops-only and $\tau(P_i) = \tau(P_i')$ for all $i \in \{3, 4, \ldots, n\}$, we have $g(P_1', P_{-\{1,2\}}) = g(P_1', P_{-\{1,2\}}')$. By the definition of g, $g(P_1, P_{-\{1,2\}}) = \varphi(P_1, P_1, P_{-\{1,2\}})$ and $g(P_1', P_{-\{1,2\}}) = \varphi(P_1', P_1', P_{-\{1,2\}})$. As $\tau(P_1) = \tau(P_2)$ and $\tau(P_1') = \tau(P_2')$, Lemma 10 implies $\varphi(P_1, P_2, P_{-\{1,2\}}) = \varphi(P_1, P_1, P_{-\{1,2\}})$ and $\varphi(P_1', P_2', P_{-\{1,2\}}') = \varphi(P_1', P_1', P_{-\{1,2\}}')$. Combining all these observations, we have $\varphi_b(P_1, P_2, P_{-\{1,2\}}) = \varphi_b(P_1', P_2', P_{-\{1,2\}}')$ for all $b \notin [b_j, b_{j+l}]_{\tau(\mathscr{D})}$. This completes the proof of the lemma. ■

Lemma 12. *Let $1 \leq j \leq j+l \leq k$ and let $P_N, P_N' \in \mathscr{D}^n$ be such that $P_1, P_2, P_1' \in \mathscr{D}^{b_j}$ and $P_2' \in \mathscr{D}^{b_{j+l}}$, and $\tau(P_i) = \tau(P_i')$ for all $i \neq 1,2$. Then, $\varphi_c(P_N) = \varphi_c(P_N')$ for all $c \notin U(b_{j+l}, P_1') \cap U(b_j, P_2')$.*

Proof. By Lemma 10, $\varphi(P_1, P_2, P_{-\{1,2\}}) = \varphi(P_1', P_1', P_{-\{1,2\}}')$. Hence, it suffices to show that $\varphi_c(P_1', P_1', P_{-\{1,2\}}') = \varphi_c(P_1', P_2', P_{-\{1,2\}}')$ for $c \notin U(b_{j+l}, P_1') \cap U(b_j, P_2')$. We prove this for $c \notin U(b_{j+l}, P_1')$, the proof of the same when $c \notin U(b_j, P_2')$ follows from symmetric argument.

Consider $c \notin U(b_{j+l}, P_1')$. By strategy-proofness of φ,

$$\varphi_{U(c,P_1')}(P_1', P_1', P_{-\{1,2\}}') \geq \varphi_{U(c,P_1')}(P_1', P_2', P_{-\{1,2\}}') \geq \varphi_{U(c,P_1')}(P_2', P_2', P_{-\{1,2\}}').$$

Moreover, by Lemma 11, $\varphi_b(P_1', P_1', P_{-\{1,2\}}') = \varphi_b(P_2', P_2', P_{-\{1,2\}}')$ for all $b \notin [b_j, b_{j+l}]_{\tau(\mathscr{D})}$, and hence $\varphi_B(P_1', P_1', P_{-\{1,2\}}') = \varphi_B(P_2', P_2', P_{-\{1,2\}}')$ for all $B \subseteq A$ such that $[b_j, b_{j+l}]_{\tau(\mathscr{D})} \subseteq B$. Since $c \notin U(b_{j+l}, P_1')$ and $\tau(P_1') = b_j$, by the definition of a generalized intermediate domain, we have $[b_j, b_{j+l}]_{\tau(\mathscr{D})} \subseteq U(c, P_1')$, and hence $\varphi_{U(c,P_1')}(P_1', P_1', P_{-\{1,2\}}') = \varphi_{U(c,P_1')}(P_2', P_2', P_{-\{1,2\}}')$. Thus, we have

$$\varphi_{U(c,P_1')}(P_1', P_1', P_{-\{1,2\}}') = \varphi_{U(c,P_1')}(P_1', P_2', P_{-\{1,2\}}'). \tag{1}$$

Suppose that $d \in A$ is ranked just above c in P_1'. Then, $[b_j, b_{j+l}]_{\tau(\mathscr{D})} \subseteq U(d, P_1')$, and hence

$$\varphi_{U(d,P_1')}(P_1', P_1', P_{-\{1,2\}}') = \varphi_{U(d,P_1')}(P_1', P_2', P_{-\{1,2\}}'). \tag{2}$$

Subtracting (2) from (1), we have $\varphi_c(P_1', P_1', P_{-\{1,2\}}') = \varphi_c(P_1', P_2', P_{-\{1,2\}}')$, which completes the proof of the lemma. ■

Recall that for two preferences P and P', we write $P \sim P'$ to mean $\tau(P) = r_2(P')$, $r_2(P) = \tau(P')$, and $r_l(P) = r_l(P')$ for all $l > 2$.

Lemma 13. *Let $P^{b_j,b_{j+1}}, P^{b_{j+1},b_j} \in \mathscr{D}$ be such that $P^{b_j,b_{j+1}} \sim P^{b_{j+1},b_j}$. Then, for all $i \in N$ and all $P_{-i} \in \mathscr{D}^{n-1}$,*

$$[\varphi_{\tau(\mathscr{D})}(P^{b_j,b_{j+1}}, P_{-i}) = 1] \implies [\varphi_{\tau(\mathscr{D})}(P^{b_{j+1},b_j}, P_{-i}) = 1].$$

Proof. As $P^{b_j,b_{j+1}} \sim P^{b_{j+1},b_j}$, by strategy-proofness, $\varphi_a(P^{b_j,b_{j+1}}, P_{-i}) = \varphi_a(P^{b_{j+1},b_j}, P_{-i})$ for all $a \notin \{b_j, b_{j+1}\}$. Thus $\varphi_{\tau(\mathscr{D})}(P^{b_j,b_{j+1}}, P_{-i}) = 1$ implies $\varphi_{\tau(\mathscr{D})}(P^{b_{j+1},b_j}, P_{-i}) = 1$. This completes the proof of the lemma. ■

To simplify notations for the following lemma, for $j < l$, we define the distance from b_l to b_j, denoted by $b_l - b_j$, as $l - j$.

Lemma 14. *The RSCF φ is tops-only and $\varphi_{\tau(\mathscr{D})}(P_N) = 1$ for all $P_N \in \mathscr{D}^n$.*[17]

Proof. We prove this lemma by using induction on the distance between the top-ranked alternatives of agents 1 and 2.

Consider l such that $0 \leq l \leq k-1$. Suppose $\varphi_{\tau(\mathscr{D})}(P_N) = 1$ and $\varphi(P_N) = \varphi(\tilde{P}_N)$ for all tops-equivalent profiles $P_N, \tilde{P}_N \in \mathscr{D}^n$ with $|\tau(P_2) - \tau(P_1)| \leq l$. We show $\varphi_{\tau(\mathscr{D})}(P'_N) = 1$ and $\varphi(P'_N) = \varphi(\tilde{P}'_N)$ for all tops-equivalent profiles $P'_N, \tilde{P}'_N \in \mathscr{D}^n$ with $|\tau(P'_2) - \tau(P'_1)| = l + 1$.

Let P_N and P'_N be such that $P_1, P'_1 \in \mathscr{D}^{b_j}$, $P_2 \in \mathscr{D}^{b_{j+l}}$, $P'_2 \in \mathscr{D}^{b_{j+l+1}}$, and $\tau(P_i) = \tau(P'_i)$ for all $i \neq 1,2$. Further, let $\bar{P}_1 \equiv P^{b_j,b_{j+1}}$, $\hat{P}_1 \equiv P^{b_{j+1},b_j}$, $\hat{P}_2 \equiv P^{b_{j+l},b_{j+l+1}}$, and $\bar{P}_2 \equiv P^{b_{j+l+1},b_{j+l}}$ be such that $\bar{P}_u \sim \hat{P}_u$ for all $u = 1,2$. Note that such preferences exist by the definition of a minimally rich generalized intermediate domain. By the induction hypothesis, $\varphi(P_N) = \varphi(P'_1, \hat{P}_2, P'_{-\{1,2\}})$. We prove the following claims.

Claim 1. $\varphi_{\tau(\mathscr{D})}(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = 1$ and $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, P'_2, P'_{-\{1,2\}})$.

By the induction hypothesis, $\varphi_{\tau(\mathscr{D})}(P'_1, \hat{P}_2, P'_{-\{1,2\}}) = 1$ and $\varphi(P_N) = \varphi(\bar{P}_1, \hat{P}_2, P'_{-\{1,2\}}) = \varphi(P'_1, \hat{P}_2, P'_{-\{1,2\}})$. Let $P''_1 \in \{P'_1, \bar{P}_1\}$. By Lemma 12,

$$\varphi_c(P''_1, P''_1, P'_{-\{1,2\}}) = \varphi_c(P''_1, \hat{P}_2, P'_{-\{1,2\}}) \quad \text{for all } c \notin U(b_{j+l}, P''_1) \cap U(b_j, \hat{P}_2), \tag{3}$$

and

$$\varphi_c(P''_1, P''_1, P'_{-\{1,2\}}) = \varphi_c(P''_1, \bar{P}_2, P'_{-\{1,2\}}) \quad \text{for all } c \notin U(b_{j+l+1}, P''_1) \cap U(b_j, \bar{P}_2). \tag{4}$$

[17] Chatterji and Zeng (2018) provide a sufficient condition for a domain to be tops-only for RSCFs. However, generalized intermediate domains do not satisfy their condition.

As $\tau(\hat{P}_2) - \tau(P_1'') \leq l$, it follows from the induction hypothesis that $\varphi_{\tau(\mathscr{D})}(P_1'', P_1'', P'_{-\{1,2\}}) = \varphi_{\tau(\mathscr{D})}(P_1'', \hat{P}_2, P'_{-\{1,2\}}) = 1$. Since $U(b_{j+l}, P_1'') \cap U(b_j, \hat{P}_2) \cap \tau(\mathscr{D}) = [b_j, b_{j+l}]_{\tau(\mathscr{D})}$, (3) implies

$$\varphi_b(P_1'', P_1'', P'_{-\{1,2\}}) = \varphi_b(P_1'', \hat{P}_2, P'_{-\{1,2\}}) \quad \text{for all } b \notin [b_j, b_{j+l}]_{\tau(\mathscr{D})}. \tag{5}$$

Moreover, since $\hat{P}_2 \equiv P^{b_{j+l}, b_{j+l+1}}$, $\bar{P}_2 \equiv P^{b_{j+l+1}, b_{j+l}}$, and $\varphi_{\tau(\mathscr{D})}(P_1'', \hat{P}_2, P'_{-\{1,2\}}) = 1$, by Lemma 13, $\varphi_{\tau(\mathscr{D})}(P_1'', \bar{P}_2, P'_{-\{1,2\}}) = 1$. This, in particular, implies $\varphi_{\tau(\mathscr{D})}(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = 1$. Because $U(b_{j+l+1}, P_1'') \cap U(b_j, \bar{P}_2) \cap \tau(\mathscr{D}) = [b_j, b_{j+l+1}]_{\tau(\mathscr{D})}$, (4) implies

$$\varphi_b(P_1'', P_1'', P'_{-\{1,2\}}) = \varphi_b(P_1'', \bar{P}_2, P'_{-\{1,2\}}) \quad \text{for all } b \notin [b_j, b_{j+l+1}]_{\tau(\mathscr{D})}. \tag{6}$$

Combining (5) and (6), $\varphi_b(P_1'', \hat{P}_2, P'_{-\{1,2\}}) = \varphi_b(P_1'', \bar{P}_2, P'_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l+1}]_{\tau(\mathscr{D})}$. Since $\hat{P}_2 \equiv P^{b_{j+l}, b_{j+l+1}}$ and $\bar{P}_2 \equiv P^{b_{j+l+1}, b_{j+l}}$, we have by strategy-proofness that $\varphi_{\{b_{j+l}, b_{j+l+1}\}}(P_1'', \hat{P}_2, P'_{-\{1,2\}}) = \varphi_{\{b_{j+l}, b_{j+l+1}\}}(P_1'', \bar{P}_2, P'_{-\{1,2\}})$. Let $B' = [b_j, b_{j+l+1}]_{\tau(\mathscr{D})} \setminus \{b_{j+l}, b_{j+l+1}\}$. Then, $\varphi_{B'}(P_1'', \hat{P}_2, P'_{-\{1,2\}}) = \varphi_{B'}(P_1'', \bar{P}_2, P'_{-\{1,2\}})$. Note that by Lemma 8, $\hat{P}_2|_{B'} = \bar{P}_2|_{B'}$. Therefore, by applying Lemma 9 with $B = \{b_{j+l}, b_{j+l+1}\}$ and $C = B'$, we have

$$\varphi_b(P_1'', \hat{P}_2, P'_{-\{1,2\}}) = \varphi_b(P_1'', \bar{P}_2, P'_{-\{1,2\}}) \quad \text{for all } b \neq b_{j+l}, b_{j+l+1}. \tag{7}$$

By the induction hypothesis, $\varphi(\bar{P}_1, \hat{P}_2, P'_{-\{1,2\}}) = \varphi(P_1', \hat{P}_2, P'_{-\{1,2\}})$. Again, by Lemma 8, $b_{j+l}\bar{P}_1 b_{j+l+1}$ and $b_{j+l}P_1' b_{j+l+1}$, which implies $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(P_1', \bar{P}_2, P'_{-\{1,2\}})$. Using a similar logic, $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, P_2', P'_{-\{1,2\}})$. This completes the proof of Claim 1. $\square$

Claim 2. $\varphi_c(P_1', \bar{P}_2, P'_{-\{1,2\}}) = \varphi_c(P_N')$ for all $c \notin U(b_{j+l+1}, P_1') \cap U(b_j, P_2')$.

By (6), $\varphi_b(P_1', P_1', P'_{-\{1,2\}}) = \varphi_b(P_1', \bar{P}_2, P'_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l+1}]_{\tau(\mathscr{D})}$. Since $[b_j, b_{j+l+1}]_{\tau(\mathscr{D})} \subseteq U(b_{j+l+1}, P_1') \cap U(b_j, P_2')$, we have $\varphi_c(P_1', P_1', P'_{-\{1,2\}}) = \varphi_c(P_1', \bar{P}_2, P'_{-\{1,2\}})$ for all $c \notin U(b_{j+l+1}, P_1') \cap U(b_j, P_2')$. Moreover, by Lemma 12, $\varphi_c(P_1', P_1', P'_{-\{1,2\}}) = \varphi_c(P_N')$ for all $c \notin U(b_{j+l+1}, P_1') \cap U(b_j, P_2')$. Hence, $\varphi_c(P_1', \bar{P}_2, P'_{-\{1,2\}}) = \varphi_c(P_N')$ for all $c \notin U(b_{j+l+1}, P_1') \cap U(b_j, P_2')$. This completes the proof of Claim 2. $\square$

Claim 3. $\varphi_b(P_1', \bar{P}_2, P'_{-\{1,2\}}) = \varphi_b(P_N')$ for all $b \in [b_j, b_{j+l+1}]_{\tau(\mathscr{D})}$.

First, we show $\varphi_{b_j}(P_1', \bar{P}_2, P'_{-\{1,2\}}) = \varphi_{b_j}(P_N')$. By Claim 1, $\varphi(P_1', \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, P_2', P'_{-\{1,2\}})$. Moreover, as $\tau(\bar{P}_1) = \tau(P_1') = b_j$, by strategy-proofness, $\varphi_{b_j}(\bar{P}_1, P_2', P'_{-\{1,2\}}) = \varphi_{b_j}(P_N')$. Combining, we have $\varphi_{b_j}(P_1', \bar{P}_2, P'_{-\{1,2\}}) = \varphi_{b_j}(P_N')$.

Now, we complete the proof of Claim 3 by induction. Consider $s < l+1$. Suppose $\varphi_{b_{j+r}}(P_1', \bar{P}_2, P'_{-\{1,2\}}) = \varphi_{b_{j+r}}(P_N')$ for all $0 \leq r \leq s$. We show $\varphi_{b_{j+s+1}}(P_1', \bar{P}_2, P'_{-\{1,2\}}) = \varphi_{b_{j+s+1}}(P_N')$. We show this in two steps. In Step 1, we show that if an alternative outside $\tau(\mathscr{D})$ appears above b_{j+s+1} in the preference P_1', then it receives zero probability at $\varphi(P_N')$. In Step 2, we use this fact to complete the proof of the claim.

STEP 1. Consider $c \in A \setminus \tau(\mathscr{D})$ such that $cP'_1 b_{j+s+1}$. We show $\varphi_c(P'_N) = 0$. Assume for contradiction that $\varphi_c(P'_N) > 0$. Since $cP'_1 b_{j+s+1}$, by the definition of a generalized intermediate domain, we have $b_{j+s+1}P'_2 c$. Let $t \in \{2,\ldots,k-j-l\}$ be such that $U(b_{j+s+1},P'_2) \cap \tau(\mathscr{D}) = [b_{j+s+1}, b_{j+l+1}]_{\tau(\mathscr{D})} \cup [b_{j+l+2}, b_{j+l+t}]_{\tau(\mathscr{D})}$. By Claim 1, $\varphi_{\tau(\mathscr{D})}(P'_1,\bar{P}_2,P'_{-\{1,2\}}) = 1$, and hence

$$\begin{aligned}\varphi_{U(b_{j+s+1},P'_2)}(P'_1,\bar{P}_2,P'_{-\{1,2\}}) &= \varphi_{[b_{j+s+1},b_{j+l+1}]_{\tau(\mathscr{D})}}(P'_1,\bar{P}_2,P'_{-\{1,2\}}) + \varphi_{[b_{j+l+2},b_{j+l+t}]_{\tau(\mathscr{D})}}(P'_1,\bar{P}_2,P'_{-\{1,2\}})\\ &= 1 - \varphi_{[b_1,b_{j+s}]_{\tau(\mathscr{D})}}(P'_1,\bar{P}_2,P'_{-\{1,2\}}) - \varphi_{[b_{j+l+t+1},b_k]_{\tau(\mathscr{D})}}(P'_1,\bar{P}_2,P'_{-\{1,2\}}). \end{aligned} \tag{8}$$

By Claim 2, $\varphi_{b_i}(P'_1,\bar{P}_2,P'_{-\{1,2\}}) = \varphi_{b_i}(P'_N)$ for all $i \in [1,j-1] \cup [j+l+t+1,k]$, and by the assumption of Claim 3, $\varphi_{b_i}(P'_1,\bar{P}_2,P'_{-\{1,2\}}) = \varphi_{b_i}(P'_N)$ for all $i \in [j,j+s]$. Combining all these observations, we have $\varphi_{[b_1,b_{j+s}]_{\tau(\mathscr{D})}}(P'_1,\bar{P}_2,P'_{-\{1,2\}}) = \varphi_{[b_1,b_{j+s}]_{\tau(\mathscr{D})}}(P'_N)$ and $\varphi_{[b_{j+l+t+1},b_k]_{\tau(\mathscr{D})}}(P'_1,\bar{P}_2,P'_{-\{1,2\}}) = \varphi_{[b_{j+l+t+1},b_k]_{\tau(\mathscr{D})}}(P'_N)$. Note that the sets $[b_1,b_{j+s}]_{\tau(\mathscr{D})}$, $U(b_{j+s+1},P'_2)$, $[b_{j+l+t+1},b_k]_{\tau(\mathscr{D})}$, and $\{c\}$ are pairwise disjoint. Therefore, $\varphi_{[b_1,b_{j+s}]_{\tau(\mathscr{D})}}(P'_N) + \varphi_{U(b_{j+s+1},P'_2)}(P'_N) + \varphi_{[b_{j+l+t+1},b_k]_{\tau(\mathscr{D})}}(P'_N) + \varphi_c(P'_N) \le 1$, and hence

$$\begin{aligned}\varphi_{U(b_{j+s+1},P'_2)}(P'_N) &\le 1 - \varphi_{[b_1,b_{j+s}]_{\tau(\mathscr{D})}}(P'_N) - \varphi_{[b_{j+l+t+1},b_k]_{\tau(\mathscr{D})}}(P'_N) - \varphi_c(P'_N)\\ &= 1 - \varphi_{[b_1,b_{j+s}]_{\tau(\mathscr{D})}}(P'_1,\bar{P}_2,P'_{-\{1,2\}}) - \varphi_{[b_{j+l+t+1},b_k]_{\tau(\mathscr{D})}}(P'_1,\bar{P}_2,P'_{-\{1,2\}}) - \varphi_c(P'_N). \end{aligned} \tag{9}$$

As $\varphi_c(P'_N) > 0$, (8) and (9) imply $\varphi_{U(b_{j+s+1},P'_2)}(P'_1,\bar{P}_2,P'_{-\{1,2\}}) > \varphi_{U(b_{j+s+1},P'_2)}(P'_N)$, which implies agent 2 manipulates at P'_N via $\bar{P}_2$, a contradiction. This completes Step 1.

STEP 2. In this step, we complete the proof of Claim 3. By Claim 1, it is sufficient to show that $\varphi_{b_{j+s+1}}(\bar{P}_1,P'_2,P'_{-\{1,2\}}) = \varphi_{b_{j+s+1}}(P'_N)$.

Suppose $\varphi_{b_{j+s+1}}(\bar{P}_1,P'_2,P'_{-\{1,2\}}) > \varphi_{b_{j+s+1}}(P'_N)$. Consider $d \in U(b_{j+s+1},P'_1) \setminus \tau(\mathscr{D})$. By Step 1, $\varphi_d(P'_1,\bar{P}_2,P'_{-\{1,2\}}) = \varphi_d(P'_N)$, and by Claim 1, $\varphi_d(P'_1,\bar{P}_2,P'_{-\{1,2\}}) = \varphi_d(\bar{P}_1,P'_2,P'_{-\{1,2\}})$. Now, consider $d \in U(b_{j+s+1},P'_1) \cap \tau(\mathscr{D})$ such that $d \neq b_{j+s+1}$. This implies $d = b_{j'}$ for some $j' \le j+s$. By Claim 2 and the assumption of Claim 3, $\varphi_d(P'_1,\bar{P}_2,P'_{-\{1,2\}}) = \varphi_d(P'_N)$. By Claim 1, $\varphi(P'_1,\bar{P}_2,P'_{-\{1,2\}}) = \varphi(\bar{P}_1,P'_2,P'_{-\{1,2\}})$. Combining all these observations, we have $\varphi_d(\bar{P}_1,P'_2,P'_{-\{1,2\}}) = \varphi_d(P'_N)$ for all $d \in U(b_{j+s+1},P'_1) \setminus b_{j+s+1}$. Therefore, $\varphi_{b_{j+s+1}}(\bar{P}_1,P'_2,P'_{-\{1,2\}}) > \varphi_{b_{j+s+1}}(P'_N)$ implies $\varphi_{U(b_{j+s+1},P'_1)}(\bar{P}_1,P'_2,P'_{-\{1,2\}}) > \varphi_{U(b_{j+s+1},P'_1)}(P'_N)$, which implies agent 1 manipulates at P'_N via $\bar{P}_1$.

Now, suppose $\varphi_{b_{j+s+1}}(\bar{P}_1,P'_2,P'_{-\{1,2\}}) < \varphi_{b_{j+s+1}}(P'_N)$. By Claim 1, $\varphi_{\tau(\mathscr{D})}(\bar{P}_1,P'_2,P'_{-\{1,2\}}) = 1$. Let $u \le j$ be such that $U(b_{j+s+1},\bar{P}_1) \cap \tau(\mathscr{D}) = [b_u, b_{j+s+1}]_{\tau(\mathscr{D})}$. Then, by the assumption of Claim 3, $\varphi_b(\bar{P}_1,P'_2,P'_{-\{1,2\}}) = \varphi_b(P'_N)$ for all $b \in [b_j,b_{j+s}]_{\tau(\mathscr{D})}$, and by Claim 2, $\varphi_b(\bar{P}_1,P'_2,P'_{-\{1,2\}}) = \varphi_b(P'_N)$ for all $b \in [b_u,b_{j-1}]_{\tau(\mathscr{D})}$. Therefore, $\varphi_{b_{j+s+1}}(\bar{P}_1,P'_2,P'_{-\{1,2\}}) < \varphi_{b_{j+s+1}}(P'_N)$ implies $\varphi_{U(b_{j+s+1},\bar{P}_1)}(\bar{P}_1,P'_2,P'_{-\{1,2\}}) < \varphi_{U(b_{j+s+1},\bar{P}_1)}(P'_N)$, which implies agent 1 manipulates at $(\bar{P}_1,P'_2,P'_{-\{1,2\}})$ via P'_1. This completes the proof of Claim 3. $\square$

We are now ready to complete the proof of Lemma 14. First, we show $\varphi_{\tau(\mathscr{D})}(P'_N) = 1$. By Claim 3, $\varphi_b(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_b(P'_N)$ for all $b \in [b_j, b_{j+l+1}]_{\tau(\mathscr{D})}$. By Claim 2, $\varphi_b(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_b(P'_N)$ for all $b \in [b_1, b_{j-1}]_{\tau(\mathscr{D})} \cup [b_{j+l+2}, b_k]_{\tau(\mathscr{D})}$. Combining all these observations, we have $\varphi_{\tau(\mathscr{D})}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_{\tau(\mathscr{D})}(P'_N)$. Moreover, by Claim 1, $\varphi_{\tau(\mathscr{D})}(P'_1, \bar{P}_2, P'_{-\{1,2\}}) = 1$, and hence $\varphi_{\tau(\mathscr{D})}(P'_N) = 1$.

Now, we show $\varphi(P'_N) = \varphi(\tilde{P}'_N)$ for all tops-equivalent profiles $P'_N, \tilde{P}'_N \in \mathscr{D}^n$. By claims 1, 2, and 3, we have $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(P'_N)$. Moreover, as $\tilde{P}'_1 \in \mathscr{D}^{b_j}$ and $\tilde{P}'_2 \in \mathscr{D}^{b_{j+l+1}}$, applying claims 1, 2, and 3 to $\tilde{P}'_N$, we have $\varphi(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}}) = \varphi(\tilde{P}'_N)$. Hence, to show $\varphi(P'_N) = \varphi(\tilde{P}'_N)$, it is enough to show $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$. Recall that $\hat{P}_2 \equiv P^{b_{j+l}, b_{j+l+1}}$. Since $\tau(\hat{P}_2) - \tau(P'_1) = l$ and $\tau(P'_i) = \tau(\tilde{P}'_i)$ for all $i \neq 1,2$, by the assumption of Lemma 14, we have $\varphi(\bar{P}_1, \hat{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, \hat{P}_2, \tilde{P}'_{-\{1,2\}})$. Also, by (7), $\varphi_b(\bar{P}_1, \hat{P}_2, P'_{-\{1,2\}}) = \varphi_b(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}})$ for all $b \neq b_{j+l}, b_{j+l+1}$, which implies $\varphi_b(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_b(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$ for all $b \neq b_{j+l}, b_{j+l+1}$. Using similar arguments as for the proof of (7), it follows that $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\hat{P}_1, \bar{P}_2, P'_{-\{1,2\}})$ for all $b \neq b_j, b_{j+1}$, and hence $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$ for all $b \neq b_j, b_{j+1}$. Note that if $l \geq 1$, then $\varphi_b(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_b(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$ for all $b \in A$. Now suppose $l = 0$. We show $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$ for $\tau(\bar{P}_1) = b_j$ and $\tau(\bar{P}_2) = b_{j+1}$. Because $\varphi_b(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_b(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$ for all $b \neq b_j, b_{j+1}$ and all tops-equivalent $P'_{-\{1,2\}}, \tilde{P}'_{-\{1,2\}} \in \mathscr{D}^{n-2}$, we have $\varphi_b(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi_b(\bar{P}_1, \bar{P}_2, \tilde{P}'_3, P'_{-\{1,2,3\}})$ for all $b \neq b_j, b_{j+1}$. As $\tau(P'_3) = \tau(\tilde{P}'_3)$, by Lemma 8, $b_j P'_3 b_{j+1}$ if and only if $b_j \tilde{P}'_3 b_{j+1}$. Therefore, if $\varphi_{b_j}(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) \neq \varphi_{b_j}(\bar{P}_1, \bar{P}_2, \tilde{P}'_3, P'_{-\{1,2,3\}})$, then agent 3 manipulates either at $(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}})$ via $\tilde{P}'_3$ or at $(\bar{P}_1, \bar{P}_2, \tilde{P}'_3, P'_{-\{1,2,3\}})$ via P'_3. Hence, $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, \bar{P}_2, \tilde{P}'_3, P'_{-\{1,2,3\}})$. Continuing in this manner, we have $\varphi(\bar{P}_1, \bar{P}_2, P'_{-\{1,2\}}) = \varphi(\bar{P}_1, \bar{P}_2, \tilde{P}'_{-\{1,2\}})$. Therefore, $\varphi(P'_N) = \varphi(\tilde{P}'_N)$ for all tops-equivalent profiles $P'_N, \tilde{P}'_N \in \mathscr{D}^n$. This completes the proof of the lemma. $\blacksquare$

Lemma 15. *The RSCF φ satisfies uncompromisingness.*

Proof. We prove this in two steps. In Step 1, we provide a sufficient condition for uncompromisingness, and in Step 2, we use that to prove the lemma.

STEP 1. In this step, we show that φ is uncompromising if the following happens: for all $j < k$, all $P_i \equiv P^{b_j, b_{j+1}} \in \mathscr{D}$, all $P'_i \equiv P^{b_{j+1}, b_j} \in \mathscr{D}$, and all $P_{-i} \in \mathscr{D}^{n-1}$,

$$\varphi_b(P_i, P_{-i}) = \varphi_b(P'_i, P_{-i}) \;\; \forall b \notin [\tau(P_i), \tau(P'_i)]. \tag{10}$$

Suppose (10) holds. Since φ is tops-only, (10) implies that for all $P_i \in \mathscr{D}^{b_j}$, all $P'_i \in \mathscr{D}^{b_{j+1}}$, all P_{-i}, and all $b \notin [\tau(P_i), \tau(P'_i)]$,

$$\varphi_b(P_i, P_{-i}) = \varphi_b(P'_i, P_{-i}). \tag{11}$$

Similarly, for all $\bar{P}_i \in \mathscr{D}^{b_{j+1}}$, all $\bar{P}'_i \in \mathscr{D}^{b_{j+2}}$, all P_{-i}, and all $b \notin [\tau(\bar{P}_i), \tau(\bar{P}'_i)]$, we have

$$\varphi_b(\bar{P}_i, P_{-i}) = \varphi_b(\bar{P}'_i, P_{-i}). \tag{12}$$

Combining (11) and (12), we have $\varphi_b(P_i, P_{-i}) = \varphi_b(\bar{P}'_i, P_{-i})$ for all $P_i \in \mathscr{D}^{b_j}$, all $\bar{P}'_i \in \mathscr{D}^{b_{j+2}}$, all P_{-i}, and all $b \notin [\tau(P_i), \tau(\bar{P}'_i)]$. Continuing in this manner, we have $\varphi_b(P_i, P_{-i}) = \varphi_b(P'_i, P_{-i})$ for all $P_i, P'_i \in \mathscr{D}$, all P_{-i}, and all $b \notin [\tau(P_i), \tau(P'_i)]$, which implies φ is uncompromising.

STEP 2. In this step, we show that φ satisfies (10). We do this in two further steps. In Step 2.a., we show (10) for agents 1 and 2, and in Step 2.b., we show this for other agents.

STEP 2.a. It is enough to show (10) for agent 1, the proof of the same for agent 2 follows from symmetric argument. Without loss of generality, assume $\tau(P_2) = b_{j+l}$. Note that by Lemma 14, $\varphi_{\tau(\mathscr{D})}(P_N) = 1$. Therefore, by Lemma 12, $\varphi_b(P_1, P_2, P_{-\{1,2\}}) = \varphi_b(P_2, P_2, P_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l}]_{\tau(\mathscr{D})}$ and $\varphi_b(P'_1, P_2, P_{-\{1,2\}}) = \varphi_b(P_2, P_2, P_{-\{1,2\}})$ for all $b \notin [b_{j+1}, b_{j+l}]_{\tau(\mathscr{D})}$. This implies $\varphi_b(P_1, P_2, P_{-\{1,2\}}) = \varphi_b(P'_1, P_2, P_{-\{1,2\}})$ for all $b \notin [b_j, b_{j+l}]_{\tau(\mathscr{D})}$. By strategy-proofness, $\varphi_{\{b_j, b_{j+1}\}}(P_1, P_2, P_{-\{1,2\}}) = \varphi_{\{b_j, b_{j+1}\}}(P'_1, P_2, P_{-\{1,2\}})$. Let $B' = [b_j, b_{j+l}]_{\tau(\mathscr{D})} \setminus \{b_j, b_{j+1}\}$. Since $P_1|_{B'} = P'_1|_{B'}$, by applying Lemma 9 with $B = \{b_j, b_{j+1}\}$ and $C = B'$, we have $\varphi_b(P_1, P_2, P_{-\{1,2\}}) = \varphi_b(P'_1, P_2, P_{-\{1,2\}})$ for all $b \neq b_j, b_{j+l}$. This proves (10) for agent 1. Therefore, by Step 1, we have for all $i \in \{1,2\}$, all $P_i \in \mathscr{D}$, all $P'_i \in \mathscr{D}$, and all $P_{-i} \in \mathscr{D}^{n-1}$,

$$\varphi_b(P_i, P_{-i}) = \varphi_b(P'_i, P_{-i}) \ \ \forall b \notin [\tau(P_i), \tau(P'_i)]. \tag{13}$$

This completes Step 2.a.

STEP 2.b. In this step, we show (10) for agents $i \in \{3, \ldots, n\}$. It is enough to show this for $i = 3$. If $P_1 = P_2$, then by the induction hypothesis, $\varphi_b(P_3, P_{-3}) = g_b(P_1, P_3, P_{-\{1,2,3\}}) = g_b(P_1, P'_3, P_{-\{1,2,3\}}) = \varphi_b(P'_3, P_{-3})$ for all $P_3, P'_3 \in \mathscr{D}$ and all $b \notin [\tau(P_3), \tau(P'_3)]$. Let $\tau(P_1) = b_p$ and $\tau(P_2) = b_q$. Since $\varphi_{\tau(\mathscr{D})}(P_N) = 1$ for all $P_N \in \mathscr{D}^n$, it follows from Lemma 12 that $\varphi_b(P_1, P_1, P_3, P_{-\{1,2,3\}}) = \varphi_b(P_1, P_2, P_3, P_{-\{1,2,3\}})$ for all $b \notin [b_p, b_q]_{\tau(\mathscr{D})}$ and $\varphi_b(P_1, P_1, P'_3, P_{-\{1,2,3\}}) = \varphi_b(P_1, P_2, P'_3, P_{-\{1,2,3\}})$ for all $b \notin [b_p, b_q]_{\tau(\mathscr{D})}$. Combining all these observations, we have

$$\varphi_b(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \varphi_b(P_1, P_2, P'_3, P_{-\{1,2,3\}}) \text{ for all } b \notin [b_p, b_q]_{\tau(\mathscr{D})} \cup [b_j, b_{j+1}]_{\tau(\mathscr{D})}. \tag{14}$$

Also, by strategy-proofness,

$$\varphi_{\{b_j, b_{j+1}\}}(P_1, P_2, P_3, P_{-\{1,2,3\}}) = \varphi_{\{b_j, b_{j+1}\}}(P_1, P_2, P'_3, P_{-\{1,2,3\}}). \tag{15}$$

Now, we distinguish two cases.

Case 1. Suppose $p,q \leq j+1$ or $p,q \geq j$.
Let $B' = [b_p,b_q]_{\tau(\mathscr{D})} \setminus [b_j,b_{j+1}]_{\tau(\mathscr{D})}$. Then, by (14) and (15), $\varphi_{B'}(P_1,P_2,P_3,P_{-\{1,2,3\}}) = \varphi_{B'}(P_1,P_2,P'_3,P_{-\{1,2,3\}})$. Since $P_3|_{B'} = P'_3|_{B'}$, by applying Lemma 9 with $B = \{b_j,b_{j+1}\}$ and $C = B'$, $\varphi_b(P_1,P_2,P_3,P_{-\{1,2,3\}}) = \varphi_b(P_1,P_2,P'_3,P_{-\{1,2,3\}})$ for all $b \in B'$. Therefore,

$$\varphi_b(P_1,P_2,P_3,P_{-\{1,2,3\}}) = \varphi_b(P_1,P_2,P'_3,P_{-\{1,2,3\}}) \text{ for all } b \notin \{b_j,b_{j+1}\}. \tag{16}$$

This completes Step 2.b. for Case 1.

Case 2. Suppose $p < j \leq j+1 < q$ or $q < j \leq j+1 < p$.
We prove the lemma for the case $p < j \leq j+1 < q$, the proof of the same for the case $q < j \leq j+1 < p$ follows from symmetric arguments. By (13), for all $b \notin [b_j,b_q]_{\tau(\mathscr{D})}$, we have $\varphi_b(P_1,P_2,P_3,P_{-\{1,2,3\}}) = \varphi_b(P_1,P_3,P_3,P_{-\{1,2,3\}})$ and $\varphi_b(P_1,P_2,P'_3,P_{-\{1,2,3\}}) = \varphi_b(P_1,P_3,P'_3,P_{-\{1,2,3\}})$. Moreover, since $\tau(P_1) \leq b_{j+1}$, $\tau(P_3) = b_j$ and $\tau(P'_3) = b_{j+1}$, it follows from (16) that $\varphi_b(P_1,P_3,P_3,P_{-\{1,2,3\}}) = \varphi_b(P_1,P_3,P'_3,P_{-\{1,2,3\}})$ for all $b \notin [b_j,b_{j+1}]_{\tau(\mathscr{D})}$. Combining all these observations, $\varphi_b(P_1,P_2,P_3,P_{-\{1,2,3\}}) = \varphi_b(P_1,P_2,P'_3,P_{-\{1,2,3\}})$ for all $b \notin [b_j,b_q]_{\tau(\mathscr{D})}$. By strategy-proofness, $\varphi_{\{b_j,b_{j+1}\}}(P_1,P_2,P_3,P_{-\{1,2,3\}}) = \varphi_{\{b_j,b_{j+1}\}}(P_1,P_2,P'_3,P_{-\{1,2,3\}})$. Let $B' = [b_j,b_q]_{\tau(\mathscr{D})} \setminus \{b_j,b_{j+1}\}$. Since $P_3|_{B'} = P'_3|_{B'}$, by applying Lemma 9 with $B = \{b_j,b_{j+1}\}$ and $C = B'$, we have $\varphi_b(P_1,P_2,P_3,P_{-\{1,2,3\}}) = \varphi_b(P_1,P_2,P'_3,P_{-\{1,2,3\}})$ for all $b \in B'$. Hence,

$$\varphi_b(P_1,P_2,P_3,P_{-\{1,2,3\}}) = \varphi_b(P_1,P_2,P'_3,P_{-\{1,2,3\}}) \text{ for all } b \notin \{b_j,b_{j+1}\},$$

which completes Step 2.b. for Case 2.

Since cases 1 and 2 are exhaustive, this completes Step 2, and consequently the proof of Lemma 15. ■

Proposition 1 now follows from Lemma 14 and Lemma 15. ■

Now, we come back to the proof of Theorem 1. Our proof uses the following theorem which is taken from Peters et al. (2014).

Theorem 4 (Theorem $3(a)$ in Peters et al. (2014))**.** *Let $\mathscr{D}$ be the maximal single-peaked domain. Then, every tops-only and strategy-proof RSCF $\varphi : \mathscr{D}^n \to \Delta A$ is a convex combination of some tops-only and strategy-proof DSCFs $f : \mathscr{D}^n \to A$.*

Our next lemma presents the structure of an uncompromising and strategy-proof RSCF on a regular single-peaked domain.

Lemma 16. *Let $\mathscr{D}$ be a regular single-peaked domain and let $\varphi : \mathscr{D}^n \to \Delta A$ be uncompromising and strategy-proof. Then, φ is a convex combination of the generalized min-max rules on $\mathscr{D}^n$.*[18]

Proof. Note that since φ is uncompromising, φ is tops-only. Let $\hat{\mathscr{D}}$ be the maximal single-peaked domain. Let $\hat{\varphi} : \hat{\mathscr{D}}^n \to \Delta A$ be the tops-only extension of φ on $\hat{\mathscr{D}}$. More formally, for all $\hat{P}_N \in \hat{\mathscr{D}}^n$, $\hat{\varphi}(\hat{P}_N) = \varphi(P_N)$, where $P_N \in \mathscr{D}^n$ is such that P_N and $\hat{P}_N$ are tops-equivalent. This is well-defined as φ is tops-only and $\mathscr{D}$ is regular. Since $\hat{\mathscr{D}}$ is single-peaked and φ is strategy-proof, $\hat{\varphi}$ is also strategy-proof. Hence, by Theorem 4, $\hat{\varphi}$ is a convex combination of the generalized min-max rules on $\hat{\mathscr{D}}^n$. By the definition of $\hat{\varphi}$, this implies φ is a convex combination of the generalized min-max rules on $\mathscr{D}^n$, which completes the proof. ■

Finally, we are ready to complete the proof of Theorem 1.

Proof. (If Part) Let $\mathscr{D}$ be a generalized intermediate domain with $\tau(\mathscr{D}) = \{b_1, \ldots, b_k\}$ and let $\varphi : \mathscr{D}^n \to \Delta A$ be a TRM rule. Since φ is a TRM rule, it is unanimous by definition. We show that φ is strategy-proof. Let $\varphi = \sum_{l=1}^{t} \lambda_l f_l$, where λ_ls are non-negative numbers summing to 1 and f_ls are TM rules. To show φ is strategy-proof, it is enough to show that f_ls are strategy-proof. For all $l \in \{1, \ldots, t\}$, define $\hat{f}_l : (\mathscr{D}|_{\tau(\mathscr{D})})^n \to \tau(\mathscr{D})$ as $\hat{f}_l(P_N|_{\tau(\mathscr{D})}) = f_l(P_N)$. Note that by Lemma 8, $\mathscr{D}|_{\tau(\mathscr{D})}$ is a single-peaked domain. Therefore, it follows from Moulin (1980) that $\hat{f}_l$ is strategy-proof for all $l = 1, \ldots, t$. By Remark 2.3, this implies f_l is strategy-proof for all $l = 1, \ldots, t$. This completes the proof of the if part.

(Only-if Part) Let $\mathscr{D}$ be a generalized intermediate domain with $\tau(\mathscr{D}) = \{b_1, \ldots, b_k\}$ and let $\varphi : \mathscr{D}^n \to \Delta A$ be a unanimous and strategy-proof RSCF. Define $\hat{\varphi} : (\mathscr{D}|_{\tau(\mathscr{D})})^n \to \Delta \tau(\mathscr{D})$ as $\hat{\varphi}_b(P_N|_{\tau(\mathscr{D})}) = \varphi_b(P_N)$ for all $b \in \tau(\mathscr{D})$. This is well-defined as by Proposition 1, $\varphi_{\tau(\mathscr{D})}(P_N) = 1$ for all $P_N \in \mathscr{D}^n$ and φ is tops-only. Because φ satisfies uncompromisingness, $\hat{\varphi}$ also satisfies uncompromisingness. Hence, by Lemma 16, $\hat{\varphi}$ is convex combination of generalized min-max rules on $(\mathscr{D}|_{\tau(\mathscr{D})})^n$. Moreover, since φ is unanimous, $\hat{\varphi}$ is a also unanimous. This implies $\hat{\varphi}$ is a convex combination of the min-max rules on $(\mathscr{D}|_{\tau(\mathscr{D})})^n$. By the definition of $\hat{\varphi}$, this implies φ is a TRM rule. This completes the proof of the only-if part. ■

B. Proof of Lemma 7

First we prove a lemma which we repeatedly use in the proof of Lemma 7.

Lemma 17. *Let $\{P_x\}_{x \in X}$ be a strict intermediate domain. Then for all distinct $a, b, c \in A$, the separating lines of the pairs (a, b) and (b, c) do not intersect.*

[18] If the set of alternatives is an interval of real numbers, then every uncompromising RSCF on the maximal single-peaked domain is strategy-proof (see Lemma 3.2 in Ehlers et al. (2002)). However, the same does not hold for the case of finitely many alternatives.

Proof. Let $\{P_x\}_{x\in X}$ be a strict intermediate domain. Assume for contradiction that there exist distinct $a,b,c \in A$ such that the separating lines of (a,b) and (b,c) intersect. Since $\{P_x\}_{x\in X}$ is strict, no three separating lines of $\{P_x\}_{x\in X}$ intersect at a common point. Therefore, we can choose an open (see Figure 5) ball such that no separating line other than those of the pairs (a,b) and (b,c) passes through that open ball. Consider the regions X_1 and X_2 in Figure 5. Consider $x \in X_1$. Since aP_xb and bP_xc, by transitivity, we have aP_xc. Now, consider $y \in X_2$. Again, since bPa and cPb, by transitivity, we have cPa. Since the relative preference over a and c is changing from X_1 to X_2, it must be that the separating line of (a,c) intersects at least one of these regions. However, this is a contradiction to our assumption that no separating line other than those of (a,b) and (b,c) intersects this open ball. This completes the proof of the lemma.

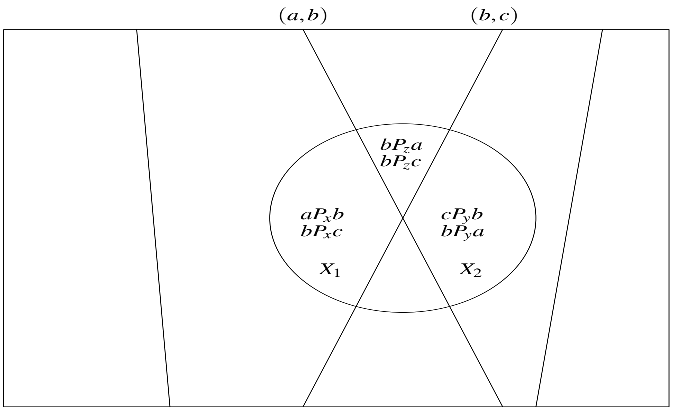

Figure 5: Diagram for the proof of Lemma 17

■

Now we prove Lemma 7.

Proof. Let $\{P_x\}_{x\in X}$ be a domain satisfying strict intermediate property. Since the number of alternatives is finite, there are finitely many preferences in the domain $\{P_x\}_{x\in X}$. Consider a preference $P \in \{P_x\}_{x\in X}$. Let $X_P = \{x \in X | P_x = P\}$. Since there are finitely many preferences in the domain $\{P_x\}_{x\in X}$, we can find a finite collection of parallel lines $\{l_1,\dots,l_k\}$ such that for each $P \in \{P_x\}_{x\in X}$, there exists $l \in \{l_1,\dots,l_k\}$ such that $X_P \cap l \neq \emptyset$. This implies that $\{P_x\}_{x\in X} = \cup_{i=1}^{k}\{P_x\}_{x\in l_i}$. Since $\{P_x\}_{x\in X}$ satisfies strict intermediate property, there exists a line $\hat{l}$ that intersects all the separating lines (as defined in Lemma 6). We assume that (i) $\hat{l} \in \{l_1,\dots,l_k\}$, and (ii) no l_i passes through the point of intersection of any two separating lines. This assumption is without of loss of generality because for (i), we can start with $\hat{l}$ and can consider a collection of parallel lines satisfying the required properties, and for (ii), since we have finitely many separating lines and hence finitely many points of intersection of those, we can always choose the lines $\{l_1,\dots,l_k\}$ by avoiding those points.

Now we show that $\cup_{i=1}^{k}\{P_x\}_{x\in l_i}$ is a generalized intermediate domain satisfying minimal richness. We show this using the following three claims.

Claim 1. For each $l \in \{l_1,\ldots,l_k\}$, the family of preferences $\{P_x\}_{x\in l}$ is a generalized intermediate domain satisfying minimal richness.

Consider $l \in \{l_1,\ldots,l_k\}$. Let $x_1,\ldots,x_s$ be the points of intersection of the line l with the separating lines of $\{P_x\}_{x\in X}$. Note that $s \leq k$ since there can be separating lines of $\{P_x\}_{x\in X}$ that do not intersect with l. Assume without loss of generality that $x_j \in (x_{j-1},x_{j+1})$ for all $j \in \{2,\ldots,s-1\}$, that is, the points $\{x_1,\ldots,x_s\}$ are ordered in a particular direction. Consider $x \in l$ such that $x_1 \in (x,x_2)$. Such a point x can always be chosen as X is open and $x_1 \in X$. Let $P_x = P_1$. By Lemma 6, $P_y = P_1$ for all $y \in [x,x_1)$. By our assumption of x_1, there exists a separating line, say for the pair of alternatives (a,b), that intersects l at x_1. This implies there exists $P_2 \in \{P_x\}_{x\in l}$ such that $P_y = P_2$ for all $y \in (x_1,x_2)$. By Lemma 6, P_1 and P_2 differ only over the ordering of the pair (a,b). Again, by Lemma 6, the preference P_{x_1} is either P_1 or P_2. Continuing in this manner, we can get hold of a sequence of preferences $\{P_j\}_{j\in\{1,\ldots,s+1\}}$ such that (i) $\{P_x\}_{x\in l} = \{P_1,\ldots,P_{s+1}\}$, and (ii) for all $j = \{2,\ldots,s\}$, P_j and P_{j+1} differ only over the ordering of a particular pair of alternatives. This implies that $\{P_x\}_{x\in l}$ is minimally rich.

Next, we show $\{P_1,\ldots,P_{s+1}\}$ is a generalized intermediate domain with respect to the ordering given by P_1. Assume for contradiction that there exist $c,d,e \in A$ with cP_1dP_1e such that $d,e \in \tau(\{P_1,\ldots,P_{s+1}\})$ and cPd for some $P \in \{P_1,\ldots,P_{s+1}\}$ with $\tau(P) = e$. Let $x_e \in X$ be such that $P_{x_e} = P$. Since $d \in \tau(\{P_1,\ldots,P_{s+1}\})$ and cP_1d, it follows that the separating line of the pair (c,d) intersects with l. Let x_t be this point of intersection. Since cPd by our assumption, $x_e \in (x_1,x_t)$. Consider $x_d \in X$ such that $\tau(P_{x_d}) = d$. Such a point x_d must exist since $d \in \tau(\{P_1,\ldots,P_{s+1}\})$ Then, it must be that $x_t \in (x_1,x_d)$. Also, dP_1e and ePd together imply $x_d \in (x_1,x_e)$. But this contradicts the fact that $x_e \in (x_1,x_t)$. This implies that $\{P_1,\ldots,P_{s+1}\}$ is a generalized intermediate domain completing the proof of Claim 1. $\square$

Recall that by our assumption, $\hat{l} \in \{l_1,\ldots,l_k\}$. Therefore, by applying Claim 1 for $l = \hat{l}$, it follows that $\{P_x\}_{x\in\hat{l}}$ is a minimally rich generalized intermediate domain with respect to some ordering, say $\prec$. Suppose $\tau(\{P_x\}_{x\in\hat{l}}) = \{b_1,\ldots,b_r\}$, where $b_1 \prec b_2 \prec \ldots \prec b_r$.

Claim 2. For all $l \in \{l_1,\ldots,l_k\}$, there exist s and t with $1 \leq s \leq t \leq r$ such that $\{P_x\}_{x\in l}$ is a generalized intermediate domain with $\tau(\{P_x\}_{x\in l}) = \{b_s,\ldots,b_t\}$.

Consider $l \in \{l_1,\ldots,l_k\} \setminus \hat{l}$. Let $y_1,\ldots,y_q$ be the points of intersection of l with the separating lines such that $y_j \in (y_{j-1},y_{j+1})$ for all $j \in \{2,\ldots,q-1\}$. Similarly, let $x_1,\ldots,x_p$ be the points of intersection of $\hat{l}$ with the separating lines such that $x_j \in (x_{j-1},x_{j+1})$ for all $j \in \{2,\ldots,p-1\}$. Assume without loss of generality that $\overrightarrow{\boldsymbol{x_p x_1}} = \overrightarrow{\boldsymbol{y_q y_1}}$, that is, the direction along which the points $x_1,\ldots,x_p$ are counted is the same as that along which the points $y_1,\ldots,y_q$ are counted (see Figure 6).

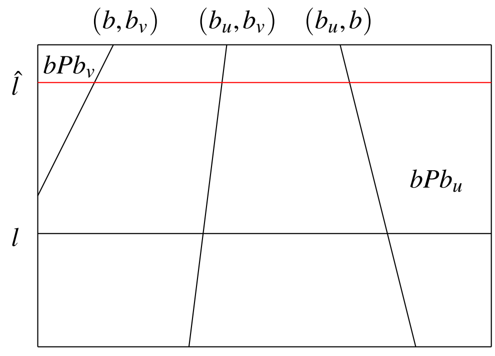

Figure 7: Diagram for the proof of Lemma 7

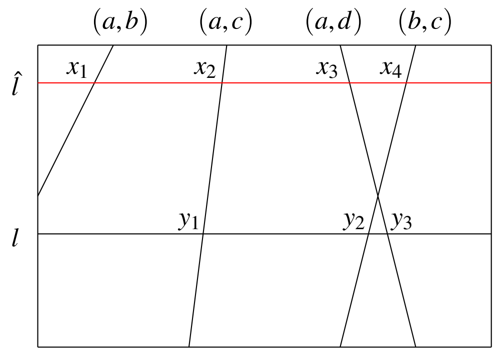

Figure 6: Diagram for the proof of Lemma 7

First, we show $\tau(\{P_x\}_{x\in l}) \subseteq \tau(\{P_x\}_{x\in\hat{l}})$. Consider $b \in \tau(\{P_x\}_{x\in l})$. Assume for contradiction that $b \notin \tau(\{P_x\}_{x\in\hat{l}})$. Since $\min_{\prec} \tau(\{P_x\}_{x\in\hat{l}}) = b_1$, this implies $b_1 \prec b$. Suppose $b_r \prec b$. Then, it must be that for all preferences in $\{P_x\}_{x\in\hat{l}}$, b_r is ranked above b, and hence the separating line of the pair (b_r,b) does not intersect with $\hat{l}$. However, since $b \in \tau(\{P_x\}_{x\in l})$, there must be a separating line of the pair (b_r,b). This is a contradiction to our assumption that $\hat{l}$ intersects with all separating lines. This shows $b \prec b_r$. Now, suppose $b_u \prec b \prec b_v$ where b_u and b_v are two consecutive alternatives (with respect to the ordering $\prec$) in the top-set $\tau(\{P_x\}_{x\in\hat{l}})$.[19] Since $b_u \prec b \prec b_v$ and $b \notin \tau(\{P_x\}_{x\in\hat{l}})$, by Lemma 6, there must be x_e, x_f and x_g with $x_f \in (x_e,x_g)$ such that the separating lines of the pairs (b,b_v), (b_u,b_v), and (b_u,b) intersect $\hat{l}$ at x_e, x_f, and x_g, respectively. By Lemma 17, no two of these separating lines intersect. Note that $b = \tau(P_z)$ for some $z \in X$ implies that z must be on the left side of the separating line of (b,b_v) and on the right side of the separating line of (b_u,b) (see Figure 7). However, as it is evident from Figure 7, there cannot be any such z. Moreover, this is true in general since the separating lines of (b,b_v) and (b_u,b) do not intersect. This shows $b \in \tau(\{P_x\}_{x\in\hat{l}})$, and hence $\tau(\{P_x\}_{x\in l}) \subseteq \tau(\{P_x\}_{x\in\hat{l}})$.

Next, we show that for all b,b_u,b_v such that $b_u,b_v \in \tau(\{P_x\}_{x\in l})$ and $b_u \preceq b \preceq b_v$, we have $b \in \tau(\{P_x\}_{x\in l})$. Suppose not. Assume without loss of generality that b_u and b_v are consecutive in $\tau(\{P_x\}_{x\in l})$, that is,

[19] By consecutive in $\tau(\{P_x\}_{x\in\hat{l}})$, we mean $(b_u,b_v) \cap \tau(\{P_x\}_{x\in\hat{l}}) = \emptyset$.

$(b_u, b_v) \cap \tau(\{P_x\}_{x \in l}) = \emptyset$. Recall that by our assumption, all the separating lines of $\{P_x\}_{x \in X}$ intersect $\hat{l}$. Suppose that the separating lines of the pairs (b_u, b), (b_u, b_v), and (b, b_v) intersect $\hat{l}$ at x_e, x_f, and x_g, respectively, where $x_f \in (x_e, x_g)$. By Lemma 17, no two of those three separating lines intersect each other. This, together with the fact that $b_u, b_v \in \tau(\{P_x\}_{x \in l})$, implies that the separating lines of the pairs (b_u, b), (b_u, b_v), and (b, b_v) intersect l at y_h, y_i, and y_j, respectively, where $y_i \in (y_h, y_j)$ (see Figure 8). By Lemma 17, $b_u \preceq \tau(P_{y_i}) \preceq b_v$. However, since $bP_{y_i}b_u$ and $bP_{y_i}b_v$, it must be that $\tau(P_{y_i}) \neq b_u, b_v$. This is a contradiction since $(b_u, b_v) \cap \tau(\{P_x\}_{x \in \hat{l}}) = \emptyset$. This completes the proof of Claim 2. $\square$

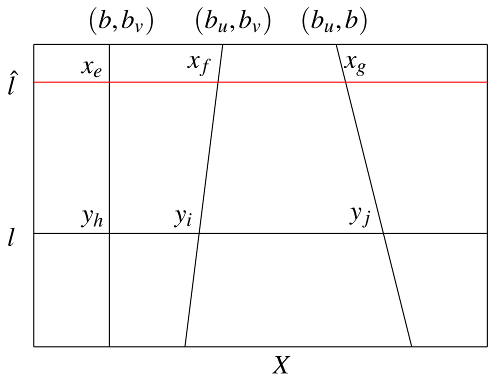

Figure 8: Diagram for the proof of Lemma 7

Claim 3. For all $l \in \{l_1, \dots, l_k\}$, all $\bar{P} \in \{P_x\}_{x \in l}$, and all $b_v \in \{b_1, \dots, b_r\}$, $\bar{P}$ satisfies the betweenness property with respect to b_v.

If $b_v \in \tau(\{P_x\}_{x \in l})$, then Claim 3 follows from Claim 2. Suppose $b_v \notin \tau(\{P_x\}_{x \in l})$. Without loss of generality, assume $b_v \prec b_s$ where $b_s = \min \tau(\{P_x\}_{x \in l})$. Let $a \prec b_v$. It is enough to show that $b_v \bar{P} a$. Since $b_v \prec b_s$ and $b_s P b_v$ for all $P \in \{P_x\}_{x \in l}$, it must be that the separating line of (b_v, b_s) does not intersect l. Let $b_t = \max_{\prec} \tau(\{P_x\}_{x \in l})$. Suppose that the points of intersection of $\hat{l}$ with the separating lines of (a, b_v), (b_v, b_s), and (b_s, b_t) are x_c, x_d, and x_e, respectively. Because $a \prec b_v \prec b_s$ and $b_v \in \tau(\{P_x\}_{x \in \hat{l}})$, we have $x_d \in (x_c, x_e)$. By Lemma 17, separating lines of (a, b_v) and (b_v, b_s) cannot intersect each other. This, together with the fact that the separating line of (b_v, b_s) does not intersect l, implies that the separating line of (a, b_v) too does not intersect l (see Figure 9). This, in particular, implies $b_v \bar{P} a$, which completes the proof of Claim 3. $\square$

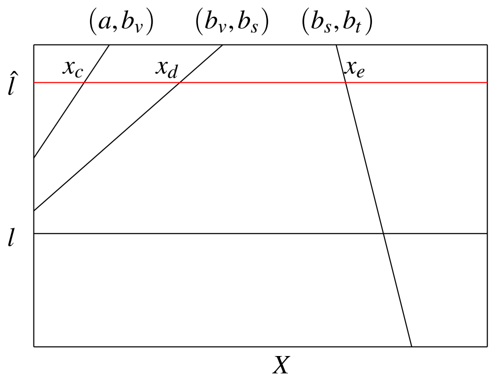

Figure 9: Diagram for the proof of Lemma 7

Now, the proof of Lemma 7 follows from Claim 2 and Claim 3. ■

C. PROOF OF THEOREM 2

Proof. Let $\widehat{\mathscr{D}} \subseteq \bar{\mathscr{D}}$ be the set of strict preferences in $\bar{\mathscr{D}}$, i.e., $\widehat{\mathscr{D}} = \{R \in \bar{\mathscr{D}} \mid |\tau(R)| = 1\}$. We first prove a claim.

Claim 1: $\widehat{\mathscr{D}}$ is a minimally rich generalized intermediate domain and $\tau(\bar{\mathscr{D}}) = \tau(\widehat{\mathscr{D}})$.

Since $\widehat{\mathscr{D}} \subseteq \bar{\mathscr{D}}$, we have $\tau(\widehat{\mathscr{D}}) \subseteq \tau(\bar{\mathscr{D}})$. As $\bar{\mathscr{D}}$ is a generalized intermediate domain, all preferences in it satisfies betweenness property with respect to each alternative in $\tau(\bar{\mathscr{D}})$. This means all preferences in $\bar{\mathscr{D}}$ satisfies betweenness property with respect to each alternatives in $\tau(\widehat{\mathscr{D}})$ and hence, $\widehat{\mathscr{D}}$ is a generalized intermediate domain.

Suppose $\tau(\bar{\mathscr{D}}) = \{b_1,\dots,b_k\}$. Since $\bar{\mathscr{D}}$ is minimally rich, for every $b_t, b_{t+1} \in \tau(\bar{\mathscr{D}})$ there exists R, $R' \in \bar{\mathscr{D}}$ such that $r_1(R) = r_2(R') = b_t$ and $r_1(R') = r_2(R) = b_{t+1}$ and $r_l(R) = r_l(R')$ for all $l \geq 2$. Note that both R and R' are strict preferences and hence, $R, R' \in \widehat{\mathscr{D}}$. This means $b_t, b_{t+1} \in \tau(\widehat{\mathscr{D}})$. Similarly, we can show this for other alternatives in $\bar{\mathscr{D}}$. This shows $\tau(\bar{\mathscr{D}}) = \tau(\widehat{\mathscr{D}})$ and $\widehat{\mathscr{D}}$ is minimally rich. □

Let $\hat{\varphi} : \widehat{\mathscr{D}}^n \to \Delta A$ be an RSCF defined as $\hat{\varphi}(R_N) = \varphi(R_N)$ for all $R_N \in \widehat{\mathscr{D}}^n$. Note that $\hat{\varphi}$ is well defined. Moreover, since φ is unanimous and strategy-proof, $\hat{\varphi}$ is also unanimous and strategy-proof. Since $\widehat{\mathscr{D}}$ is a minimally-rich generalized intermediate domain (by Claim 1), by Theorem 1, $\hat{\varphi}$ is TRM rule. Hence, $\hat{\varphi}_c(R_N) = 0$ for all $R_N \in \widehat{\mathscr{D}}^n$ and all $c \notin \tau(\widehat{\mathscr{D}})$.

Assume for contradiction there exists a profile $\tilde{R}_N \in \bar{\mathscr{D}}^n$ such that $\varphi_c(\tilde{R}_N) > 0$ for some $c \notin \tau(\bar{\mathscr{D}})$. Let $i \in N$ be such that $|\tau(\tilde{R}_i)| = 2$. By the tie-breaking property, there exists $\hat{R}_i \in \bar{\mathscr{D}}$ with $\{\tau(\hat{R}_i), r_2(\hat{R}_i)\} = \tau(\tilde{R}_i)$ and $r_l(\tilde{R}_i) = r_{l+1}(\hat{R}_i)$ for all $l \geq 2$. By strategy-proofness this means $\varphi_a(\tilde{R}_N) = \varphi_a(\hat{R}_i), \tilde{R}_{-i})$ for all $a \notin \{b_t, b_{t+1}\}$. This implies $\varphi_c(\hat{R}_i), \tilde{R}_{-i}) > 0$. Continuing in this manner we can reach a profile $\hat{R}_N$ where for all $j \in N$, $\hat{R}_j = \tilde{R}_j$ if $|\tau(\tilde{R}_j)| = 1$ and $\{\tau(\hat{R}_j), r_2(\hat{R}_j)\} = \tau(\tilde{R}_j)$ and $r_l(\tilde{R}_j) = r_{l+1}(\hat{R}_j)$ for all $l \geq 2$ if $|\tau(\tilde{R}_j)| = 2$. Furthermore, $\varphi_c(\hat{R}_N) > 0$. But this is a contradiction as $\hat{R}_N \in \widehat{\mathscr{D}}^n$ and by Claim 1

$\tau(\bar{\mathscr{D}}) = \tau(\widehat{\mathscr{D}})$. This completes the proof of the theorem. ■

D. Proof of Theorem 3

Proof. Let $\bar{\mathscr{D}}$ a minimally rich generalized intermediate domain such that $|\tau(R)| \leq 2$ for all $R \in \bar{\mathscr{D}}$. Further, let φ be a TRM rule and $\boldsymbol{\pi}$ be a collection of unanimous and strategy-proof tie-breaking rules. Consider the RSCF $\bar{\varphi}(R_N) : \bar{\mathscr{D}}^n \to \Delta A$ defined as $\bar{\varphi}(R_N) = \varphi(\boldsymbol{\pi}(R_N))$. We show that $\bar{\varphi}$ is unanimous and strategy-proof. Since $\varphi_{\tau(\bar{\mathscr{D}})}(\boldsymbol{\pi}(R_N)) = 1$ for all $R_N \in \bar{\mathscr{D}}^n$, for the rest of the proof for some $a \preceq b$, by $[a,b]$ we denote $[a,b]_{\tau(\bar{\mathscr{D}})}$.

Unanimity: Consider a profile $\tilde{R}_N \in \bar{\mathscr{D}}^n$ such that $\cap_{i\in N}\tau(\tilde{R}_i) \neq \emptyset$. Since all $\tilde{R}_i$s satisfy betweenness property with respect to each alternative in $\tau(\bar{\mathscr{D}})$, we have $\cap_{i\in N}\tau(\tilde{R}_i) = [a,b]$ for some $a \preceq b$. Note that unanimity of π_is imply $\pi_i(\tilde{R}_N) \in [a,b]$ for all $i \in N$. Hence, $[\min_{i\in N}\pi_i(R_N), \max_{i\in N}\pi_i(R_N)] \subseteq [a,b]$. By the definition of φ, $\varphi_{[\min_{i\in N}\pi_i(R_N),\max_{i\in N}\pi_i(R_N)]}(\boldsymbol{\pi}(R_N)) = 1$ for all $R_N \in \bar{\mathscr{D}}^n$. Combining all these observations we get $\varphi_{[a,b]}(\boldsymbol{\pi}(\tilde{R}_N)) = 1$ which implies $\bar{\varphi}_{\cap_{i\in N}\tau(R_i)}(\tilde{R}_N) = 1$. This completes the proof that $\bar{\varphi}$ is unanimous.

Strategy-proofness: Consider a profile $\tilde{R}_N \in \bar{\mathscr{D}}^n$ and an agent $i \in N$. We show that by changing his/her preference to any $\tilde{R}'_i \in \bar{\mathscr{D}}$ agent i cannot manipulate. Consider an upper contour set $U(a,\tilde{R}_i)$ for some $a \in A$. We show that $\bar{\varphi}_{U(a,\tilde{R}_i)}(\tilde{R}_N) \geq \bar{\varphi}_{U(a,\tilde{R}_i)}(\tilde{R}'_i,\tilde{R}_{-i})$. Note that $\bar{\varphi}(\tilde{R}_N) = \varphi(\pi_1(\tilde{R}_N),\ldots,\pi_n(\tilde{R}_N))$ and $\bar{\varphi}(\tilde{R}'_i,\tilde{R}_{-i}) = \varphi(\pi_1(\tilde{R}'_i,\tilde{R}_{-i}),\ldots,\pi_n(\tilde{R}'_i,\tilde{R}_{-i}))$. Combining all these observations, to complete the proof it is enough to show that $\varphi_{U(a,\tilde{R}_i)}(\pi_1(\tilde{R}_N),\ldots,\pi_n(\tilde{R}_N)) \geq \varphi_{U(a,\tilde{R}_i)}(\pi_1(\tilde{R}'_i,\tilde{R}_{-i}),\ldots,\pi_n(\tilde{R}_N))$. Note that by the definition of TRM rule,

$$\varphi_b(\pi_1(\tilde{R}_N),\ldots,\pi_n(\tilde{R}_N)) \neq \varphi_b(\pi_1(\tilde{R}'_i,\tilde{R}_{-i}),\ldots,\pi_n(\tilde{R}_N)) \text{ for all } b \neq [\pi_1(\tilde{R}_N),\pi_1(\tilde{R}'_i,\tilde{R}_{-i})]. \tag{17}$$

Let $i \neq 1$. Note that by the definition of π_1, $\pi_1(\tilde{R}_N),\pi_1(\tilde{R}'_i,\tilde{R}_{-i}) \in \tau(\tilde{R}_1)$. Since $|\tau(\tilde{R}_1)| \leq 2$ and π_1 is strategy-proof, we have either $\pi_1(\tilde{R}'_i,\tilde{R}_{-i}) \preceq \pi_1(\tilde{R}_N) \preceq \min\tau(\tilde{R}_i)$ or $\max\tau(\tilde{R}_i) \preceq \pi_1(\tilde{R}_N) \preceq \pi_1(\tilde{R}'_i,\tilde{R}_{-i})$. Without loss of generality assume that $\pi_1(\tilde{R}'_i,\tilde{R}_{-i}) \preceq \pi_1(\tilde{R}_N) \preceq \min\tau(\tilde{R}_i)$. Let $U(a,\tilde{R}_i) = [b_r,b_s]$. This means $b_r \preceq \min\tau(\tilde{R}_i) \preceq b_s$. Note that if $b_r \preceq \pi_1(\tilde{R}'_i,\tilde{R}_{-i})$ or $\pi_1(\tilde{R}_N) \prec b_r$, then $[\pi_1(\tilde{R}_N),\pi_1(\tilde{R}'_i,\tilde{R}_{-i})] \subseteq [b_r, b_s]$ or $[\pi_1(\tilde{R}_N),\pi_1(\tilde{R}'_i,\tilde{R}_{-i})] \cap [b_r,b_s] = \emptyset$ and hence, by (17), $\varphi_{[b_r,b_s]}(\pi_1(\tilde{R}_N),\ldots,\pi_n(\tilde{R}_N)) = \varphi_{[b_r,b_s]}(\pi_1(\tilde{R}'_i, \tilde{R}_{-i}),\ldots,\pi_n(\tilde{R}_N))$. So, assume $\pi_1(\tilde{R}'_i,\tilde{R}_{-i}) \prec b_r \preceq \pi_1(\tilde{R}_N)$. Assume for contradiction $\varphi_{[b_r,b_s]}(\pi_1(\tilde{R}_N),\ldots, \pi_n(\tilde{R}_N)) < \varphi_{[b_r,b_s]}(\pi_1(\tilde{R}'_i,\tilde{R}_{-i}),\ldots,\pi_n(\tilde{R}_N))$. Together with (17), this implies

$$\varphi_{[b_r,\pi_1(\tilde{R}_N)]}(\pi_1(\tilde{R}_N),\ldots,\pi_n(\tilde{R}_N)) < \varphi_{[b_r,\pi_1(\tilde{R}_N)]}(\pi_1(\tilde{R}'_i,\tilde{R}_{-i}),\ldots,\pi_n(\tilde{R}_N)). \tag{18}$$

Consider a single-peaked preference profile R_N such that $\tau(R_j) = \pi_j(\tilde{R}_N)$ and $U(b_r,R_1) = [b_r,\pi_1(\tilde{R}_N)]$.

Further consider R_1', a single-peaked preference with $\tau(R_1') = \pi_1(\tilde{R}_i', \tilde{R}_{-i})$. In view of (18), these profiles imply $\varphi_{U(b_r,R_1)}(R_N) < \varphi_{U(b_r,R_1)}(R_1', R_{-1})$ which means agent 1 manipulates at R_N via R_1'. But this is a contradiction since by Theorem 1, φ is strategy-proof on the set of single-peaked preference profiles.

Let $i = 1$. By the definition of π_1, $\pi_1(\tilde{R}_N) \in \tau(\tilde{R}_1)$ and $\pi_1(\tilde{R}_1', \tilde{R}_{-1}) \in \tau(\tilde{R}_1')$. Consider a single-peaked preference profile R_N where $\tau(R_j) = \pi_j(R_N)$ and $U(a, \tilde{R}_1) = U(a, R_1)$. Further consider R_1', a single-peaked preference with $\tau(R_1') = \pi_1(\tilde{R}_1', \tilde{R}_{-1})$. By strategy-proofness of φ, $\varphi_{U(a,R_1)}(R_N) \geq \varphi_{U(a,R_1)}(R_1', R_{-1})$, which is exactly what we want to show. This completes the proof of the theorem. ■

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