The Welfare of Ramsey Optimal Policy Facing Auto-Regressive Shocks

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Abstract

With non-controllable auto-regressive shocks, the welfare of Ramsey optimal policy is the solution of a single Ricatti equation of a linear quadratic regulator. The existing theory by Hansen and Sargent (2007) refers to an additional Sylvester equation but miss another equation for computing the block matrix weighting the square of non-controllable variables in the welfare function. There is no need to simulate impulse response functions over a long period, to compute period loss functions and to sum their discounted value over this long period, as currently done so far. Welfare is computed for the case of the new-Keynesian Phillips curve with an auto-regressive cost-push shock.

JEL classification numbers: C61, C62, C73, E47, E52, E61, E63.

Keywords: Ramsey optimal policy, Stackelberg dynamic game, algorithm, forcing variables, augmented linear quadratic regulator, new-Keynesian Phillips curve.

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1 Introduction

Dynamic stochastic general equilibrium (DSGE) models include auto-regressive shocks (Smets and Wouters (2007)). For computing the welfare of Ramsey optimal policy in DSGE models, one simulates impulse response functions over a long period, one computes period loss functions and one sums their discounted value over this long period.

Since Anderson *et al.* (1996) and Hansen and Sargent (2007), the available theory uses a Riccati equation for controllable variables and a Sylvester equation for non-controllable variables in order to find the optimal policy rule and the optimal initial condition for non-predetermined variables. However, the matrix of the value function allowing to compute welfare is incomplete. Even worse, computing welfare loss using only the two matrices solutions of Riccati equation and of Sylvester equation may lead to a strictly positive value, which is impossible. A third equation is missing in order to find the matrix related to the squares of the non-controllable variable in the value function.

We include in the Lagrangian the Lagrange multiplier times the dynamic equation of the non-controllable variables. This Lagrange multiplier is omitted in Anderson *et al.* (1996), p.202. Once this Lagrangian multiplier is included, the symmetry of the Hamiltonian matrix for the full system of controllable and non-controllable variables is restored. The value function is the solution of a Riccati equation for matrices related to controllable *and* non-controllable variables.

In Anderson *et al.* (1996), the Riccati equation only on controllable variables and the Sylvester equation only on non-controllable variables corresponds to two block matrix of the solution of our Riccati equation. The missing block matrix for computing welfare related to the square of non-controllable variables is found solving this Riccati equation.

This numerical solution of this Riccati equation is coded in the linear quadratic regulator instruction `lqr` in SCILAB. We compute the welfare of Ramsey optimal policy for the new-Keynesian Phillips curve with an auto-regressive cost-push shock (Gali (2015)).

2 The Welfare of Ramsey optimal policy

To derive Ramsey optimal policy a Stackelberg leader-follower model is analyzed where the government is the leader and the private sector is the follower. Let $\mathbf{k}_t$ be an $n_k \times 1$ vector of controllable predetermined state variables with initial conditions $\mathbf{k}_0$ given, $\mathbf{x}_t$ an $n_x \times 1$ vector of non-predetermined endogenous variables free to jump at t without a given initial condition for $\mathbf{x}_0$, put together in the $(n_k + n_x) \times 1$vector $\mathbf{y}_t = (\mathbf{k}_t^T, \mathbf{x}_t^T)^T$. The $n_u \times 1$ vector $\mathbf{u}_t$ denotes government policy instruments. We include an $n_z \times 1$ vector of non-controllable autoregressive shocks $\mathbf{z}_t$. All variables are expressed as absolute or proportional deviations from a steady state.

The policy maker maximizes the following quadratic function (minimizes the quadratic loss) subject to an initial condition for $\mathbf{k}_0$ and $\mathbf{z}_0$, but not for $\mathbf{x}_0$:

$$-\frac{1}{2}\sum_{t=0}^{+\infty}\beta^t \left(\mathbf{y}_t^T\mathbf{Q}_{yy}\mathbf{y}_t + 2\mathbf{y}_t^T\mathbf{Q}_{yz}\mathbf{z}_t + \mathbf{z}_t^T\mathbf{Q}_{zz}\mathbf{z}_t + \mathbf{u}_t^T\mathbf{R}_{uu}\mathbf{u}_t\right) \tag{1}$$

where β is the policy maker's discount factor. The policymaker's preferences are the relative weights included in the matrices $\mathbf{Q}$ and $\mathbf{R}$. $\mathbf{Q}_{yy} \geq \mathbf{0}$ is a $(n_k + n_x) \times (n_k + n_x)$ positive symmetric semi-definite matrix, $\mathbf{R}_{uu} > \mathbf{0}$ is a $p \times p$ *strictly* positive symmetric definite matrix, so that the policy maker has at least a very small concern for the volatility of policy instruments. The policy transmission mechanism of the private sector's behavior is summarized by this system of equations:

$$\begin{pmatrix} E_t\mathbf{y}_{t+1} \\ \mathbf{z}_{t+1} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{yy} & \mathbf{A}_{yz} \\ \mathbf{0} & \mathbf{A}_{zz} \end{pmatrix} \begin{pmatrix} \mathbf{y}_t \\ \mathbf{z}_t \end{pmatrix} + \begin{pmatrix} \mathbf{B}_{yu} \\ \mathbf{0} \end{pmatrix} \mathbf{u}_t, \tag{2}$$

where $\mathbf{A}$ is an $(n_k + n_x + n_z) \times (n_k + n_x + n_z)$ matrix and $\mathbf{B}$ is the $(n_k + n_x + n_z) \times p$ matrix of marginal effects of policy instruments $\mathbf{u}_t$ on next period policy targets $\mathbf{y}_{t+1}$. The certainty equivalence principle of the linear quadratic regulator allows us to work with a non-stochastic model (Anderson *et al.* (1996)). Anderson *et al.* (1996) is word by word Hansen and Sargent (2007) chapter 5, so we refer only to Anderson *et al.* (1996) in what follows.

The government chooses sequences $\{\mathbf{u}_t, \mathbf{x}_t, \mathbf{k}_{t+1}\}_{t=0}^{+\infty}$ taking into account the policy transmission mechanism (2) and boundary conditions detailed below.

Essential boundary conditions are the initial conditions of predetermined variables $\mathbf{k}_0$ and $\mathbf{z}_0$ which are given. Natural boundary conditions are chosen by the policy maker to anchor the unique optimal initial values of the private sector's forward-looking variables. The policy maker's Lagrange multipliers of the private sector's forward (Lagrange multipliers) variables are *predetermined at the value zero:* $\frac{\partial L}{\partial \mathbf{x}_0} = \mu_{\mathbf{x},t=0} = 0$ in order to determine the unique optimal initial value $\mathbf{x}_0 = \mathbf{x}_0^*$ of the private sector's forward variables.

Anderson *et al.* (1996) assume a bounded discounted quadratic loss function:

$$E\left(\sum_{t=0}^{+\infty} \beta^t \left(\mathbf{y}_t^T \mathbf{y}_t + \mathbf{z}_t^T \mathbf{z}_t + \mathbf{u}_t^T \mathbf{u}_t\right)\right) < +\infty \tag{3}$$

which implies

$$\lim_{t\to+\infty} \beta^t \mathbf{z}_t = \mathbf{z}^* = \mathbf{0},\ \mathbf{z}_t \text{ bounded},$$

$$\lim_{t\to+\infty} \beta^t \mathbf{y}_t = \mathbf{y}^* = \mathbf{0} \Leftrightarrow \lim_{t\to+\infty} \frac{\partial L}{\partial \mathbf{y}_t} = \mathbf{0} = \lim_{t\to+\infty} \beta^t \mu_t,\ \mu_t \text{ bounded}.$$

The bounded discounted quadratic loss function implies to select eigenvalues of the dynamic system such that $\left|(\beta\lambda_i^2)^t\right| < |\beta\lambda_i^2| < 1$ or equivalently such that: $|\lambda_i| < 1/\sqrt{\beta}$. A preliminary step is to multiply matrices by $\sqrt{\beta}$ as follows: $\sqrt{\beta}\mathbf{A}_{yy}$ $\sqrt{\beta}\mathbf{B}_y$ in order to apply the formulas of Riccati equations for the *non*-discounted linear quadratic regulator augmented by auto-regressive shocks.

Assumption 1: The matrix pair $(\sqrt{\beta}\mathbf{A}_{yy}\ \sqrt{\beta}\mathbf{B}_{yu})$ is Kalman controllable if the controllability matrix has full rank:

$$\text{rank}\ \left(\sqrt{\beta}\mathbf{B}_{yu}\ \ \beta\mathbf{A}_{yy}\mathbf{B}_{yu}\ \ \beta^{\frac{3}{2}}\mathbf{A}_{yy}^2\mathbf{B}_{yu}\ \ \ldots\ \ \beta^{\frac{n_k+n_x}{2}}\mathbf{A}_{yy}^{n_k+n_x-1}\mathbf{B}_{yu}\right) = n_k + n_x. \tag{4}$$

Assumption 2: The system is can be stabilized when the transition matrix $\mathbf{A}_{zz}$ for the non-controllable auto-regressive variables has eigenvalues such that $|\lambda_i| < 1/\sqrt{\beta}$.

The policy maker's choice can be solved with Lagrange multipliers. The Lagrangian includes not only the constraints of the private sector's policy transmission mechanisms multiplied by their respective Lagrange multipliers $2\beta^{t+1}\mu_{t+1}$, **BUT ALSO** *the constraints of the non-controllable variables dynamics with their respective Lagrange multi-*

plier $2\beta^{t+1}\nu_{t+1}$, which were omitted in Anderson et al. (1996), p.202.

$$-\frac{1}{2}\sum_{t=0}^{+\infty} \begin{array}{c} \beta^t \left[\mathbf{y}_t^T \mathbf{Q}_{yy}\mathbf{y}_t + 2\mathbf{y}_t^T \mathbf{Q}_{yz}\mathbf{z}_t + \mathbf{z}_t^T \mathbf{Q}_{zz}\mathbf{z}_t + \mathbf{u}_t^T \mathbf{R}_{uu}\mathbf{u}_t\right] + \\ 2\beta^{t+1}\mu_{t+1}\left[\mathbf{A}_{yy}\mathbf{y}_t + \mathbf{A}_{yz}\mathbf{z}_t + \mathbf{B}_{yu}u_t - \mathbf{y}_{t+1}\right] + \\ 2\beta^{t+1}\nu_{t+1}\left[\mathbf{A}_{zz}\mathbf{z}_t + \mathbf{0}_z u_t - \mathbf{z}_{t+1}\right]. \end{array} \tag{5}$$

The first order conditions are:

$$\begin{aligned} \frac{\partial L}{\partial x_t} &= \mathbf{R}\mathbf{x}_t + \beta\mathbf{B}\gamma_{t+1} = 0 \Rightarrow \mathbf{x}_t = -\beta\mathbf{R}^{-1}\mathbf{B}\gamma_{t+1} \\ \frac{\partial L}{\partial \pi_t} &= \mathbf{Q}\pi_t + \beta\mathbf{A}\gamma_{t+1} - \gamma_t = 0 \\ \frac{\partial L}{\partial z_t} &= \beta\gamma_{t+1}\mathbf{A}_{yz} + \beta\delta_{t+1}\mathbf{A}_{zz} - \delta_t = 0 \end{aligned}$$

The policy instrument are substituted by $\mathbf{x}_t = -\beta\mathbf{R}^{-1}\mathbf{B}\gamma_{t+1}$ in the transmission mechanism equation. The Hamiltonian of the linear quadratic regulator has the usual block matrices on left hand side and right hand side:

$$L = \begin{pmatrix} \mathbf{I} & -\beta\mathbf{B}_{(y,z)u}\mathbf{R}_{uu}^{-1}\mathbf{B}_{(y,z)u}^T \\ \mathbf{0} & \beta\mathbf{A}^T \end{pmatrix} \text{ and } N = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{Q} & \mathbf{I} \end{pmatrix}$$

with this particular block decomposition between controllable variables $\mathbf{y}_t$ and non-controllable variables $\mathbf{z}_t$:

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} & -\beta\mathbf{B}_{yu}\mathbf{R}_{uu}^{-1}\mathbf{B}_{yu}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \beta\mathbf{A}_{yy} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \beta\mathbf{A}_{yz} & \beta\mathbf{A}_{zz} \end{pmatrix} \begin{pmatrix} y_{t+1} \\ z_{t+1} \\ \mu_{t+1} \\ \nu_{t+1} \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{yy} & \mathbf{A}_{yz} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{zz} & \mathbf{0} & \mathbf{0} \\ -\mathbf{Q}_{yy} & -\mathbf{Q}_{yz} & \mathbf{I} & \mathbf{0} \\ -\mathbf{Q}_{yz} & -\mathbf{Q}_{zz} & \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} y_t \\ z_t \\ \mu_{t+1} \\ \nu_{t+1} \end{pmatrix}$$

The specificity of non-controllable variables is that the following matrix includes three blocks with zeros, which is not the case for controllable variables:

$$-\beta\begin{pmatrix} \mathbf{B}_{yu} \\ \mathbf{0} \end{pmatrix}\left(\mathbf{R}_{uu}^{-1}\right)\begin{pmatrix} \mathbf{B}_{yu} \\ \mathbf{0} \end{pmatrix}^T = \begin{pmatrix} -\beta\mathbf{B}_{yu}\mathbf{R}_{uu}^{-1}\mathbf{B}_{yu}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

If $\mathbf{L}$ is non-singular, the Hamiltonian matrix $\mathbf{H} = \mathbf{L}^{-1}\mathbf{N}$ is a symplectic matrix. With the equations of the Lagrange multipliers ν_{t+1}, all the roots ρ_i of A_{zz} have their mirror roots $(1/\beta\rho_i)$ which were all missing in Anderson *et al.* (1996).

The value function for welfare involve the matrix $\mathbf{P}$ such that:

$$L_{t=0} = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}^T \begin{pmatrix} \mathbf{P}_{yy} & \mathbf{P}_{yz} \\ \mathbf{P}_{yz} & \mathbf{P}_{zz} \end{pmatrix} \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$$

A stabilizing solution of the Hamiltonian system satisfies (Anderson *et al.* (1996)):

$$\frac{\partial L}{\partial \mathbf{y}_{t=0}} = \mu_0 = \mathbf{P}_y\mathbf{y}_0 + \mathbf{P}_z\mathbf{z}_0. \tag{6}$$

The optimal rule of the augmented linear quadratic regulator is:

$$\mathbf{u}_t = \mathbf{F}_y \mathbf{y}_t + \mathbf{F}_z \mathbf{z}_t. \tag{7}$$

The matrix $\mathbf{P}$ is solution of this Riccati equation:

$$\begin{pmatrix} \mathbf{P}_{yy} & \mathbf{P}_{yz} \\ \mathbf{P}_{yz} & \mathbf{P}_{zz} \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_{yy} & \mathbf{Q}_{yz} \\ \mathbf{Q}_{yz} & \mathbf{Q}_{zz} \end{pmatrix} + \beta \begin{pmatrix} \mathbf{A}_{yy} & \mathbf{A}_{yz} \\ \mathbf{0} & \mathbf{A}_{zz} \end{pmatrix}^T \begin{pmatrix} \mathbf{P}_{yy} & \mathbf{P}_{yz} \\ \mathbf{P}_{yz} & \mathbf{P}_{zz} \end{pmatrix} \begin{pmatrix} \mathbf{A}_{yy} & \mathbf{A}_{yz} \\ \mathbf{0} & \mathbf{A}_{zz} \end{pmatrix}$$
$$- \beta \begin{pmatrix} \mathbf{A}_{yy} & \mathbf{A}_{yz} \\ \mathbf{0} & \mathbf{A}_{zz} \end{pmatrix}^T \begin{pmatrix} \mathbf{P}_{yy} & \mathbf{P}_{yz} \\ \mathbf{P}_{yz} & \mathbf{P}_{zz} \end{pmatrix} \begin{pmatrix} \mathbf{B}_{yu} \\ \mathbf{0} \end{pmatrix}$$
$$\left(\mathbf{R}_{uu} + \beta \mathbf{B}'_{yu} \mathbf{P}_{yy} \mathbf{B}_{yu}\right)^{-1} \beta \begin{pmatrix} \mathbf{B}_{yu} \\ \mathbf{0} \end{pmatrix}^T \begin{pmatrix} \mathbf{P}_{yy} & \mathbf{P}_{yz} \\ \mathbf{P}_{yz} & \mathbf{P}_{zz} \end{pmatrix} \begin{pmatrix} \mathbf{A}_{yy} & \mathbf{A}_{yz} \\ \mathbf{0} & \mathbf{A}_{zz} \end{pmatrix}$$

The matrix to be inverted in the Riccati equation is modified due to non-controllable variables:

$$\begin{pmatrix} \mathbf{B}_{yu} \\ \mathbf{0} \end{pmatrix}^T \begin{pmatrix} \mathbf{P}_{yy} & \mathbf{P}_{yz} \\ \mathbf{P}_{yz} & \mathbf{P}_{zz} \end{pmatrix} \begin{pmatrix} \mathbf{B}_{yu} \\ \mathbf{0} \end{pmatrix} = \mathbf{B}_{yu}^T \mathbf{P}_{yy} \mathbf{B}_{yu}$$

This Riccati equation is written as:

$$\begin{pmatrix} \mathbf{P}_{yy} & \mathbf{P}_{yz} \\ \mathbf{P}_{yz} & \mathbf{P}_{zz} \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_{yy} & \mathbf{Q}_{yz} \\ \mathbf{Q}_{yz} & \mathbf{Q}_{zz} \end{pmatrix}$$
$$+ \beta \begin{pmatrix} \mathbf{A}_{yy}^T \mathbf{P}_{yy} \mathbf{A}_{yy} & \mathbf{A}_{yy}^T \left(\mathbf{P}_{yy}\mathbf{A}_{yz} + \mathbf{P}_{yz}\mathbf{A}_{zz}\right) \\ \left(\mathbf{A}_{yy}^T \mathbf{P}_{yy} \mathbf{A}_{yz} + \mathbf{A}_{yy}^T \mathbf{P}_{yz} \mathbf{A}_{zz}\right)^T & \mathbf{A}_{yz}^T \left(\mathbf{P}_{yy}\mathbf{A}_{yz} + \mathbf{P}_{yz}\mathbf{A}_{zz}\right) + \mathbf{A}_{zz}^T \left(\mathbf{P}_{yz}\mathbf{A}_{yz} + \mathbf{P}_{zz}\mathbf{A}_{zz}\right) \end{pmatrix}$$
$$- \beta^2 \begin{pmatrix} \mathbf{A}_{yy}^T \mathbf{P}_{yy} \mathbf{B}_{yu} \\ \mathbf{A}_{yz}^T \mathbf{P}_{yy} \mathbf{B}_{yu} + \mathbf{A}_{zz}^T \mathbf{P}_{yz} \mathbf{B}_{yu} \end{pmatrix} \left(\mathbf{R}_{uu} + \beta \mathbf{B}'_{yu} \mathbf{P}_{yy} \mathbf{B}_{yu}\right)^{-1}$$
$$\begin{pmatrix} \mathbf{B}_{yu}^T \mathbf{P}_{yy} \mathbf{A}_{yy} & \mathbf{B}_{yu}^T \left(\mathbf{P}_{yy}\mathbf{A}_{yz} + \mathbf{P}_{yz}\mathbf{A}_{zz}\right) \end{pmatrix}$$

where $\mathbf{P}_{yy}$ solves the matrix Riccati equation (Anderson at al. (1996)):

$$\mathbf{P}_{yy} = \mathbf{Q}_{yy} + \beta \mathbf{A}_{yy}^T \mathbf{P}_y \mathbf{A}_{yy} - \beta \mathbf{A}_{yy}^T \mathbf{P}_y \mathbf{B}_y \left(\mathbf{R}_{uu} + \beta \mathbf{B}_{yu}^T \mathbf{P}_{yy} \mathbf{B}_{yu}\right)^{-1} \beta \mathbf{B}_y^T \mathbf{P}_y \mathbf{A}_{yy},$$

where $\mathbf{F}_y$ is computed knowing $\mathbf{P}_y$:

$$\mathbf{F}_y = -\left(\mathbf{R}_{uu} + \beta \mathbf{B}_{yu}^T \mathbf{P}_{yy} \mathbf{B}_{yu}\right)^{-1} \beta \mathbf{B}_y^T \mathbf{P}_y \mathbf{A}_{yy}, \tag{8}$$

where $\mathbf{P}_{yz}$ solves the matrix Sylvester equation knowing $\mathbf{P}_y$ and $\mathbf{F}_y$ (Anderson et al. (1996)):

$$\mathbf{P}_{yz} = \mathbf{Q}_{yz} + \beta \left(\mathbf{A}_{yy} + \mathbf{B}_y \mathbf{F}_y\right)^T \mathbf{P}_y \mathbf{A}_{yz} + \beta \left(\mathbf{A}_{yy} + \mathbf{B}_y \mathbf{F}_y\right)^T \mathbf{P}_z \mathbf{A}_{zz}$$

where $\mathbf{P}_{zz}$, which is missing in Anderson et al. (1996), solves the matrix Sylvester equation knowing P_y, F_y and P_{yz}:

$$\mathbf{P}_{zz} = \mathbf{Q}_{zz} + \mathbf{A}_{yz}^T (\mathbf{P}_{yy}\mathbf{A}_{yz} + \mathbf{P}_{yz}\mathbf{A}_{zz}) + \mathbf{A}_{zz}^T (\mathbf{P}_{yz}\mathbf{A}_{yz} + \mathbf{P}_{zz}\mathbf{A}_{zz})$$
$$- \beta^2 \left(\mathbf{A}_{yz}^T\mathbf{P}_{yy}\mathbf{B}_{yu} + \mathbf{A}_{zz}^T\mathbf{P}_{yz}\mathbf{B}_{yu}\right) \left(\mathbf{R}_{uu} + \beta\mathbf{B}_{yu}^{\prime}\mathbf{P}_{yy}\mathbf{B}_{yu}\right)^{-1}$$
$$\mathbf{B}_{yu}^T (\mathbf{P}_{yy}\mathbf{A}_{yz} + \mathbf{P}_{yz}\mathbf{A}_{zz})$$

Now, at last, we know $\mathbf{P}_{zz}$ *so that we can compute the welfare of Ramsey optimal policy*:

Proposition 1 *The welfare of Ramsey optimal policy is:*

$$-\begin{pmatrix} \mathbf{k}_0 \\ \mathbf{z}_0 \end{pmatrix}^T \begin{pmatrix} \mathbf{P}_{kk} - \mathbf{P}_{kk}\mathbf{P}_{xx}^{-1}\mathbf{P}_{xk} & \mathbf{P}_{kz} - \mathbf{P}_{kx}\mathbf{P}_{xx}^{-1}\mathbf{P}_{xz} \\ \mathbf{P}_{zx} - \mathbf{P}_{zk}\mathbf{P}_{xx}^{-1}\mathbf{P}_{xk} & \mathbf{P}_{zz} - \mathbf{P}_{zk}\mathbf{P}_{xx}^{-1}\mathbf{P}_{xz} \end{pmatrix} \begin{pmatrix} \mathbf{k}_0 \\ \mathbf{z}_0 \end{pmatrix}$$

Proof. Welfare is a function of controllable non-predetermined variables $\mathbf{x}_0$, controllable predetermined variables $\mathbf{k}_0$ and non controllable predetermined auto-regressive shocks $\mathbf{z}_0$:

$$\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{k}_0 \\ \mathbf{z}_0 \end{pmatrix}^T \begin{pmatrix} \mathbf{P}_{xx} & \mathbf{P}_{xk} & \mathbf{P}_{xz} \\ \mathbf{P}_{kx} & \mathbf{P}_{kk} & \mathbf{P}_{kz} \\ \mathbf{P}_{zk} & \mathbf{P}_{zx} & \mathbf{P}_{zz} \end{pmatrix} \begin{pmatrix} \mathbf{x}_0 \\ \mathbf{k}_0 \\ \mathbf{z}_0 \end{pmatrix}$$

Ramsey optimal initial anchor of non-predetermined variables $\mathbf{x}_0$ is (Ljungqvist L. and Sargent T.J. (2012), chapter 19):

$$\frac{\partial L}{\partial \mathbf{x}_0} = \mathbf{P}_{xk}\mathbf{k}_0 + \mathbf{P}_{xx}\mathbf{x}_0 + \mathbf{P}_{xz}\mathbf{z}_0 = \mathbf{0} \Rightarrow \mathbf{x}_0 = \mathbf{P}_{xx}^{-1}\mathbf{P}_{xk}\mathbf{k}_0 + \mathbf{P}_{xx}^{-1}\mathbf{P}_{xz}\mathbf{z}_0$$

Hence, the welfare matrix of Ramsey optimal policy is:

$$\begin{pmatrix} \mathbf{0} & -\mathbf{P}_{xx}^{-1}\mathbf{P}_{xk} & -\mathbf{P}_{xx}^{-1}\mathbf{P}_{xz} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}^T \begin{pmatrix} \mathbf{P}_{xx} & \mathbf{P}_{xk} & \mathbf{P}_{xz} \\ \mathbf{P}_{kx} & \mathbf{P}_{kk} & \mathbf{P}_{kz} \\ \mathbf{P}_{zk} & \mathbf{P}_{zx} & \mathbf{P}_{zz} \end{pmatrix} \begin{pmatrix} \mathbf{0} & -\mathbf{P}_{xx}^{-1}\mathbf{P}_{xk} & -\mathbf{P}_{xx}^{-1}\mathbf{P}_{xz} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\left(\mathbf{P}_{xx}^{-1}\mathbf{P}_{xk}\right)^T & \mathbf{1} & \mathbf{0} \\ -\left(\mathbf{P}_{xx}^{-1}\mathbf{P}_{xz}\right)^T & \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{kk} - \mathbf{P}_{kk}\mathbf{P}_{xx}^{-1}\mathbf{P}_{xk} & \mathbf{P}_{kz} - \mathbf{P}_{kx}\mathbf{P}_{xx}^{-1}\mathbf{P}_{xz} \\ \mathbf{0} & \mathbf{P}_{zx} - \mathbf{P}_{zk}\mathbf{P}_{xx}^{-1}\mathbf{P}_{xk} & \mathbf{P}_{zz} - \mathbf{P}_{zk}\mathbf{P}_{xx}^{-1}\mathbf{P}_{xz} \end{pmatrix}$$
$$= \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_{kk} - \mathbf{P}_{kk}\mathbf{P}_{xx}^{-1}\mathbf{P}_{xk} & \mathbf{P}_{kz} - \mathbf{P}_{kx}\mathbf{P}_{xx}^{-1}\mathbf{P}_{xz} \\ \mathbf{0} & \mathbf{P}_{zx} - \mathbf{P}_{zk}\mathbf{P}_{xx}^{-1}\mathbf{P}_{xk} & \mathbf{P}_{zz} - \mathbf{P}_{zk}\mathbf{P}_{xx}^{-1}\mathbf{P}_{xz} \end{pmatrix}$$

■

3 New Keynesian Phillips Curve Example

The new-Keynesian Phillips curve constitutes the monetary policy transmission mechanism:

$$\pi_t = \beta E_t [\pi_{t+1}] + \kappa x_t + z_t \text{ where } \kappa > 0,\ 0 < \beta < 1,$$

where x_t represents the output gap, i.e. the deviation between (log) output and its efficient level. π_t denotes the rate of inflation between periods $t-1$ and t and plays the role of the vector of forward-looking variables $\mathbf{x}_t$ in the above general case. β denotes the discount factor. E_t denotes the expectation operator. The cost push shock z_t includes an exogenous auto-regressive component:

$$z_t = \rho z_{t-1} + \varepsilon_t \text{ where } 0 < \rho < 1 \text{ and } \varepsilon_t \text{ i.i.d. normal } N\left(0, \sigma_\varepsilon^2\right),$$

where ρ denotes the auto-correlation parameter and ε_t is identically and independently distributed (i.i.d.) following a normal distribution with constant variance σ_ε^2. The welfare loss function is such that the policy target is inflation and the policy instrument is the output gap (Gali (2015), chapter 5):

$$\max -\frac{1}{2} E_0 \sum_{t=0}^{t=+\infty} \beta^t \left(\pi_t^2 + \frac{\kappa}{\varepsilon} x_t^2\right)$$

$$\begin{pmatrix} E_t \pi_{t+1} \\ z_{t+1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \pi_t \\ z_t \end{pmatrix} + \mathbf{B} x_t + \begin{pmatrix} 0_y \\ 1 \end{pmatrix} \varepsilon_t$$

There is one controllable non-predetermined variable: $\mathbf{x}_t = \pi_t$. There is no controllable predetermined variable ($\mathbf{k}_t = \mathbf{0}$). Gali's (2015) calibration is:

$$\sqrt{\beta}\mathbf{A} = \sqrt{0.99} \begin{pmatrix} \mathbf{A}_{xx} = -\frac{1}{\beta} = \frac{1}{0.99} & \mathbf{A}_{xz} = -\frac{1}{\beta} = -\frac{1}{0.99} \\ 0 & \mathbf{A}_{zz} = \rho = 0.8 \end{pmatrix},$$

$$\sqrt{\beta}\mathbf{B} = \sqrt{0.99} \begin{pmatrix} \mathbf{B}_x = -\frac{\kappa}{\beta} = -\frac{0.1275}{0.99} \\ \mathbf{B}_z = 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} \mathbf{Q}_{xx} = 1 & \mathbf{Q}_{xz} = 0 \\ \mathbf{Q}_{xz} = 0 & \mathbf{Q}_{zz} = 0 \end{pmatrix}, \mathbf{R} = \frac{\kappa}{\varepsilon} = \frac{0.1275}{6}$$

One multiplies matrices by $\sqrt{\beta}$ in order to take the discount factor in the Riccati equation. The welfare matrix is computed using SCILAB lqr instruction in the appendix:

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_{xx} & \mathbf{P}_{xz} \\ \mathbf{P}_{xz} & \mathbf{P}_{zz} \end{pmatrix} = \begin{pmatrix} 1.7518055 & -1.1389181 \\ -1.1389181 & 3.4285107 \end{pmatrix}$$

Taking into account the optimal initial anchor of inflation ($\pi_0 = 0.65$ for $z_0 = 1$), the welfare matrix is:

$$\begin{pmatrix} 0 & -\mathbf{P}_{xx}^{-1}\mathbf{P}_{xz} \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} \mathbf{P}_{xx} & \mathbf{P}_{xz} \\ \mathbf{P}_{xz} & \mathbf{P}_{zz} \end{pmatrix} \begin{pmatrix} 0 & -\mathbf{P}_{xx}^{-1}\mathbf{P}_{xz} = 0.6504 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{P}_{zz} - \mathbf{P}_{xz}\mathbf{P}_{xx}^{-1}\mathbf{P}_{xz} \end{pmatrix}$$

The welfare loss of Gali's (2015) impulse response functions with Ramsey optimal initial condition: $\pi_0 = 0.65$ for $z_0 = 1$ is:

$$W = -\left(\mathbf{P}_{zz} - \mathbf{P}_{xz}\mathbf{P}_{xx}^{-1}\mathbf{P}_{xz}\right) z_0^2 = -2.688 \cdot z_0^2$$

We found the same value simulating impulse response functions over two hundred periods, computing period loss function and a discounted sum of these period loss functions over two hundred periods. Additional results on this example can be found in Chatelain and Ralf (2019).

Using only the information available in Anderson et al (1996), e.g. assuming $\mathbf{P}_{zz} = 0$ *for the missing block matrix in the value function, welfare loss would be strictly positive* $\mathbf{P}_{xz}\mathbf{P}_{xx}^{-1}\mathbf{P}_{xz} = 0.74 > -2.688$*, which is impossible.*

This paper is part of a broader project which evaluates the bifurcations of dynamic systems which occurs for Ramsey optimal policy versus discretion equilibrium (Chatelain and Ralf (2021) or versus simple rules (Chatelain and Ralf (2020c). In particular, an Hopf bifurcation occurs for the new-Keynesian model (Chatelain and Ralf (2020a)). Super-inertial interest rate rules are not solutions of Ramsey optimal monetary policy (Chatelain and Ralf (2020b). Ramsey optimal policy eliminates multiple equilibria such as the fiscal theory of the price level in the frictionless model (Chatelain and Ralf (2020d) or in the new-Keynesian model (Chatelain and Ralf (2020e)).

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4 Appendix

The numerical solution of the welfare matrix is obtained using Scilab code:

```
beta1=0.99; eps=6; kappa=0.1275; rho=0.8;
Qpi=1; Qz=0 ; Qzpi=0; R=kappa/eps;
A1=[1/beta1 -1/beta1 ; 0 rho] ;
A=sqrt(beta1)*A1;
B1=[-kappa/beta1 ; 0];
B=sqrt(beta1)*B1;
Q=[Qpi Qzpi ;Qzpi Qz ];
Big=sysdiag(Q,R);
[w,wp]=fullrf(Big);
C1=wp(:,1:2);
D12=wp(:,3:$);
M=syslin('d',A,B,C1,D12);
[Fy,Py]=lqr(M);
Py
Py(2,2)-Py(1,2)*inv(Py(1,1))*Py(1,2)
```