Ramsey Optimal Policy in the New-Keynesian Model with Public Debt

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Abstract

In the discrete-time new-Keynesian model with public debt, Ramsey optimal policy eliminates the indeterminacy of simple-rules multiple equilibria between the fiscal theory of the price level versus new-Keynesian versus an unpleasant equilibrium. If public debt volatility is taken into account into the loss function, the interest rate responds to public debt besides inflation and output gap. Else, the Taylor rule is identical to Ramsey optimal policy when there is zero public debt. The optimal fiscal-rule parameter implies the local stability of public-debt dynamics ("passive" fiscal policy).

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1 INTRODUCTION

In a continuous-time New-Keynesian model with public debt, Barnett et al. (2020, p.3) found that:

> "*an active monetary policy, when combined with a Ricardian passive fiscal policy, à la Leeper-Woodford, may induce the onset of a Shilnikov chaotic attractor in the region of the parameter space where uniqueness of the equilibrium prevails... Paradoxically, an active interest rate feedback policy can cause nominal interest rates, inflation rates, and real interest rates unintentionally to drift downward within a Shilnikov attractor set.*"

The combination of active monetary policy and passive fiscal policy may lead to long-term global indeterminacy and unpredictability according to the definition of Shilnikov

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chaos (1965). In another example, Shilnikov chaotic dynamics and global indeterminacy are also characteristics of the Lucas (1988) endogenous growth model in its local determinacy region of the parameter space (Bella, Mattana and Venturi (2017)).

Barnett et al. (2020, proposition 3) develop policy options to control the chaotic dynamics in the sense of Ott, Grebogi and Yorke (1990) so that "*the economy supersedes irregular and cyclical behavior and approaches the intended steady state*". They interpret the interest rate response to inflation as a bifurcation parameter. For a sufficiently large response of interest to inflation in the policy rule, the controllable dynamic system shifts from the regime where "*one eigenvalue is negative and two eigenvalues have positive real parts*" to "*the desired form [which] implies three eigenvalues with negative real parts*".

This paper shows that extending Barnett's *et al.* (2020) policy recommendation valid for continuous time model to discrete time model, with the number (three) of eigenvalues of the discrete dynamic system inside the unit circle equal to the order of the dynamics of policy maker's state variables, is compatible with discrete time Blanchard and Kahn's determinacy condition when using Ramsey optimal policy *instead* of simple rules. Ramsey optimal policy determines optimal initial conditions for non-predetermined inflation and output gap, as soon as the policy maker has a non-zero probability of not reneging his policy commitment.

We compute Ramsey optimal policy in a discrete time new-Keynesian model with public debt. We compare it with Woodford's (1996) and Cochrane's (2019) approaches with simple rules and with Chatelain and Ralf's (2020) Ramsey optimal policy without debt. We get the following policy implications:

(1) For simple interest-rate and fiscal rules with zero public debt, there is a unique new-Keynesian equilibrium. But with non-zero public debt, *there is indeterminacy between three equilibria*: the new-Keynesian one, the fiscal-theory-of-the-price-level one and an unpleasant one. These three equilibria have a number of eigenvalues inside the unit circle which is strictly lower than the order of the dynamics of the policy maker's target variables. They are not robust to misspecification due to policy maker's imperfect knowledge of structural parameters and initial values of inflation, output gap, and public debt.

(2) With non-zero public debt, Ramsey optimal policy has the great virtue to eliminate the three simple-rules equilibria and the indeterminacy across them. After the policy advice of Barnett et al. (2020) leaning against chaotic dynamics, this is a second argument for negative feedback policies leading to a number of eigenvalues inside the unit circle equal to the order of the dynamics of the policy maker's target variables. A third argument is that these negative feedback policies are also robust to misspecification due to policy maker's imperfect knowledge.

(3) In the new-Keynesian model, inflation and output-gap dynamics do not depend on public-debt dynamics. For this reason, *if the volatility of public debt has a zero weight in the welfare and policy maker's loss function*, the policy implications of Ramsey optimal policy for the Taylor rule are exactly the same as in a model without public debt (Chatelain and Ralf (2020)). In addition, a surplus rule stabilizes public-debt dynamics.

(4) By contrast, *if the volatility of public debt has a non-zero weight in the welfare and policy maker's loss function*, the policy implication of Ramsey optimal policy is that the interest rate should optimally respond to deviation of public debt from its long-run equilibrium value, and not only to inflation and the output gap. In the new-Keynesian model, the theoretical sensitivity of public debt to the interest rate is much larger than to surplus or deficits. This is the origin of the welfare gain of the interest rate rule

responding to public debt.

These discrete-time results differ in several respects to the continuous-time results of Barnett *et al.* (2020). Firstly, the proposed methodology in Barnett *et al.* (2020) to control chaos, involves announcing a higher steady-state nominal interest rate and not just the nominal interest rate.

Secondly, Barnett *et al.* (2020) also recommended another alternative to avoid chaos, that is, to abandon the Taylor rule altogether and use an alternative monetary policy rule like targeting of Divisia monetary aggregates. The possibility of implementation of this second alternative weakens the case for Ramsey optimal policy. An extension may consider Divisia monetary aggregates as a policy instrument of Ramsey optimal policy instead of the funds rate.

Thirdly, the dynamics of a continuous-time version may not match the ones of a discrete-time version. For example, Barnett and Duzhak's (2008) Hopf bifurcation may arise in continuous time and in discrete time dynamical systems. By contrast, Barnett and Duzhak's (2010) flip bifurcation may arise only in the discrete time versions of the new-Keynesian model (see also Barnett and Chen (2015)).

Section two describes the policy transmission mechanism. Section three solves the Ramsey optimal policy model. Section four compares the equilibrium of our model with simple-rule multiple equilibria. Section five concludes.

2 NEW-KEYNESIAN MODEL WITH PUBLIC DEBT

The fiscal theory of the price level applied on the new-Keynesian model with public debt can be found in Woodford (1996, 1998), Cochrane (2019)). All variables are defined as log-deviations of their equilibrium value. In the representative household's intertemporal substitution consumption Euler equation, the expected future output gap $E_t x_{t+1}$ is *positively* correlated with the real rate of interest, equal to the nominal rate i_t minus *expected* inflation $E_t \pi_{t+1}$, where E_t is the expectation operator. The intertemporal elasticity of substitution (IES) $\gamma = 1/\sigma$ is a measure of the responsiveness of the growth rate of consumption to the interest rate, usually considered to be smaller than one. It is the inverse of σ, the relative fluctuation aversion or the relative degree of resistance to intertemporal substitution of consumption, which measures the strength of the preference for smoothing consumption over time. An independently and identically distributed additive shock $\varepsilon_{x,t}$ is taken into account:

$$x_t = E_t x_{t+1} - \gamma \left(i_t - E_t \pi_{t+1} \right) + \varepsilon_{x,t} \text{ with } \gamma > 0. \tag{1}$$

In the new-Keynesian Phillips curve, expected inflation $E_t \pi_{t+1}$ is negatively correlated with the current output gap x_t with a sensitivity $-\kappa < 0$. An independently and identically distributed additive cost-push shock $\varepsilon_{\pi,t}$ is taken into account:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + \varepsilon_{\pi,t} \text{ with } 0 < \beta < 1 \text{ and } \kappa > 0.$$

The intertemporal budget constraint of the state for public debt B_t is:

$$B_{t+1} = \left(1 + i_t\right) B_t - p_t s_t \Rightarrow b_{t+1} = \frac{B_{t+1}}{p_{t+1}} = \left(1 + i_t\right) \frac{B_t}{p_t} \frac{p_t}{p_{t+1}} - s_t.$$

The primary surplus is lump-sum tax income minus non-interest expenditures: $s_t =$

$\tau_t - g$. Define a steady state with inflation equal to zero $\pi^* = 0$, with the real rate of interest equal to the discount rate so that $i^* - \pi^* = i^* = \beta^{-1} - 1$, with steady state real public debt $\frac{B^*}{p} \geq 0$. In that steady state, the surplus pays the real interest cost of the debt: $s^* = \tau^* - g = \left(\beta^{-1} - 1\right) \frac{B^*}{p} \geq 0$. Hence, steady state surplus is a small proportion of the stock of public debt, of an order of magnitude of 1% per quarter. Woodford (1996, equation 2.9) log-linearizes the real public debt b_t equation in deviation from its steady state:

$$b_{t+1} = \beta^{-1} \left(b_t - \pi_t\right) + i_t - \left(\beta^{-1} - 1\right) s_t.$$

The marginal effect of the funds rate on future public debt (equal to one) is around 100 times larger than the marginal effect of the surplus (equal to the opposite of the discount rate) for quarterly periods. The dynamic system includes three policy targets (output gap, inflation, public debt) and two policy instrument (funds rate and primary surplus):

$$\begin{pmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ b_{t+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1+\frac{\gamma\kappa}{\beta} & -\frac{\gamma}{\beta} & 0 \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\ 0 & -\frac{1}{\beta} & \frac{1}{\beta} \end{pmatrix}}_{=\mathbf{A}} \begin{pmatrix} x_t \\ \pi_t \\ b_t \end{pmatrix} + \underbrace{\begin{pmatrix} \gamma & 0 \\ 0 & 0 \\ 1 & 1-\frac{1}{\beta} \end{pmatrix}}_{=\mathbf{B}} \begin{pmatrix} i_t & s_t \end{pmatrix} + \begin{pmatrix} \varepsilon_{x,t} \\ \varepsilon_{\pi,t} \\ 0 \end{pmatrix}. \tag{2}$$

Shocks $\varepsilon_{x,t}$, $\varepsilon_{\pi,t}$ are assumed to be independently and identically distributed with mean zero and a non-zero variance-covariance matrix. The variance covariance matrix of disturbances does not matter for seeking the optimal solution (Simon's (1956) certainty-equivalence result for the linear quadratic regulator).

The three policy targets, output gap x_t, inflation π_t and real public debt b_t, are two-time-step Kalman (1960) controllable by the two policy instruments (the interest rate i_t and surplus s_t) or only by the policy rate (using the first column $\mathbf{B}_{i.}$ of matrix $\mathbf{B}$) if $\gamma \neq 0$, $\kappa \neq 0$.

$$rank\left(\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}\right) = rank\left(\mathbf{B}_{i.}, \mathbf{AB}_{i.}, \mathbf{A}^2\mathbf{B}_{i.}\right) = 3 \text{ if } \gamma \neq 0 \text{ and } \kappa \neq 0.$$

The surplus instrument alone is only able to control public debt if $\beta \neq 1$, using the second column $\mathbf{B}_{.s}$ of matrix $\mathbf{B}$)

$$rank\left(\mathbf{B}_{.s}, \mathbf{AB}_{.\mathbf{s}}, \mathbf{A}^2\mathbf{B}_{.s}\right) = 1 \text{ if } \beta \neq 1.$$

Controllability is also checked by Barnett et al. (2020) for the continuous-time version of the new-Keynesian model with public debt. We use $\beta = 0.99$ for quarterly periods and $\kappa = 0.1$, $\gamma = 0.5$ numerical values instead of Woodford (1996) numerical values $\beta = 0.95$ for yearly periods and $\kappa = 0.3$, $\gamma = 1$ which are oversized with respect to posterior estimations U.S.A. since the 1960s (Havranek (2015), Mavroeidis *et alii* (2014)).

The transmission parameter of funds rate on future debt is equal to 1 which is nearly 100 times larger in absolute value than the transmission parameter of the surplus (0.01). This implies that parameters of the surplus rule would have to be 100 times larger than the parameters of the Taylor rule to have the same size in the parameters of the closed loop system.

3 RAMSEY OPTIMAL POLICY

3.1 Optimal Program

In a monetary policy regime indexed by j, a policy maker may re-optimize on each future period with exogenous probability $1-q$ strictly below one under "quasi commitment" (Schaumburg and Tambalotti, 2007)). Following Schaumburg and Tambalotti (2007), we assume that the mandate to minimize the loss function is delegated to a sequence of policy makers with a commitment of random duration. The degree of credibility is modelled as if it is a change of policy-maker with a given probability of reneging commitment and re-optimizing optimal plans. The length of their tenure or "regime" depends on a sequence of exogenous i.i.d. Bernoulli signals $\{\eta_t\}_{t\geq 0}$ with $E_t[\eta_t]_{t\geq 0} = 1-q$, with $0 < q \leq 1$. If $\eta_t = 1$, a new policy maker takes office at the beginning of time t. Otherwise, the incumbent stays on. A higher probability q can be interpreted as a higher credibility. A policy maker with little credibility does not give a large weight on future welfare losses. The policy maker j solves the following problem for regime j, omitting subscript j, before policy maker k starts:

$$V_0^j = -E_0 \sum_{t=0}^{t=+\infty} (\beta q)^t \left[\frac{1}{2} \left(Q_\pi \pi_t^2 + Q_x x_t^2 + Q_b b_t^2 + \mu_i i_t^2 + \mu_s s_t^2 \right) + \beta (1-q) V_t^k \right]$$

Preferences of the policy maker are given by positive weights for the three policy targets $Q_x \geq 0$, $Q_\pi \geq 0$, $Q_b \geq 0$ ($\mathbf{Q} = diag(Q_x, Q_\pi, Q_b)$ in the SCILAB algorithm). In order to insure the concavity of the LQR program, there are at least non-zero policy maker's preferences for *interest rate smoothing* and *primary surplus smoothing*, with *strictly* positive weights for these two policy instruments in the loss function: $\mu_i > 0$, $\mu_s > 0$. In the simulation grid on preferences of this paper, these strictly positive weights are also set down to 10^{-7} with respect to at least one other weight for policy targets set to 1. They are stacked in matrix $\mathbf{R} = diag(\mu_i, \mu_s)$ in the SCILAB algorithm.

Inflation, output gap and public debt next period are an average between two terms. The first term, with weight q is the inflation, output gap and public debt that would prevail under the current regime upon which there is commitment. The second term with weight $1-q$ is the inflation and public debt that would be implemented under the alternative regime by policy maker k:

$$q \begin{pmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ b_{t+1} \end{pmatrix} + (1-q) \begin{pmatrix} E_t x_{t+1}^k \\ E_t \pi_{t+1}^k \\ b_{t+1}^k \end{pmatrix} = \underbrace{\begin{pmatrix} 1+\frac{\gamma\kappa}{\beta} & -\frac{\gamma}{\beta} & 0 \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\ 0 & -\frac{1}{\beta} & \frac{1}{\beta} \end{pmatrix}}_{=\mathbf{A}} \begin{pmatrix} x_t \\ \pi_t \\ b_t \end{pmatrix} + \underbrace{\begin{pmatrix} \gamma & 0 \\ 0 & 0 \\ 1 & 1-\frac{1}{\beta} \end{pmatrix}}_{=\mathbf{B}} \begin{pmatrix} i_t & s_t \end{pmatrix}$$

The optimal program for policy maker j is a discounted linear quadratic regulator (LQR) with a "credibility adjusted" discount factor βq. We apply Chatelain and Ralf (2019) algorithm using $\sqrt{\beta q}\mathbf{A}/q$ and $\sqrt{\beta q}\mathbf{B}/q$ (SCILAB code in the appendix). We seek stabilizing solutions that satisfy:

$$\sum_{t=0}^{t=+\infty} (\beta q)^t \left(\pi_t^2 + x_t^2 + b_t^2 + i_t^2 + s_t^2 \right) < +\infty$$

The policy maker seeks optimal linear feedback rules parameters stacked in matrix $\mathbf{F}$:

$$\begin{pmatrix} i_t \\ s_t \end{pmatrix} = \underbrace{\begin{pmatrix} F_x & F_\pi & F_b \\ G_x & G_\pi & G_b \end{pmatrix}}_{=\mathbf{F}} \begin{pmatrix} x_t \\ \pi_t \\ b_t \end{pmatrix}$$

Replacing feedback rules in the transmission mechanism leads to a closed loop system with transition matrix $\frac{\sqrt{\beta q}}{q}(\mathbf{A}+\mathbf{BF})$ for the algorithm:

$$\begin{pmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \\ b_t \end{pmatrix} = \frac{\sqrt{\beta q}}{q} \begin{pmatrix} 1+\frac{\gamma\kappa}{\beta}+\gamma F_x & -\frac{\gamma}{\beta}+\gamma F_\pi & \gamma F_b \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} & 0 \\ F_x-\left(\frac{1}{\beta}-1\right)G_x & -\frac{1}{\beta}+F_\pi-\left(\frac{1}{\beta}-1\right)G_\pi & \frac{1}{\beta}+F_b-\left(\frac{1}{\beta}-1\right)G_b \end{pmatrix} \begin{pmatrix} x_t \\ \pi_t \\ b_t \end{pmatrix}$$

When the Taylor rule does not include a reaction of funds rate to public debt ($F_b = 0$), the closed-loop matrix remains block triangular. In this case, we use these notations for the private sector closed-loop dynamics:

$$\mathbf{A}_{x\pi} + \mathbf{B}_{x\pi}\mathbf{F}_{x\pi} = \frac{\sqrt{\beta q}}{q} \begin{pmatrix} 1+\frac{\gamma\kappa}{\beta} & -\frac{\gamma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{pmatrix} + \frac{\sqrt{\beta q}}{q} \begin{pmatrix} -\gamma \\ 0 \end{pmatrix} \begin{pmatrix} F_x & F_\pi \end{pmatrix}$$

It is the transition matrix of the closed loop new-Keynesian model excluding public debt when shocks are not auto-regressive.

The case of the peg of policy instruments at their long-run value $\mathbf{F} = \mathbf{0}$ corresponds to an open loop system with transition matrix $\mathbf{A}$.

We vary the policy maker's preferences in all their range of possible values using a simulation grid. Specific values of preferences for households welfare would be somewhere in this locus for policy rule. Tables 1 to 4 present solutions of the four polar cases of the policy maker's preferences. For comparisons with Chatelain and Ralf (2020a), we use the numerical value $q = 1$. Results for any other value $0 < q < 1$ are available from the authors upon request or can be found copying SCILAB code in the appendix after downloading this open source software substitute to MATLAB. They do not qualitatively differ from the case $q = 1$.

3.2 Solution for Welfare Preferences

Gali (2015) uses the approximation of households welfare for an efficient steady state with this relative weight for the output gap $Q_x = \frac{\kappa}{\varepsilon}$ after normalizing $Q_\pi = 1$, with no cost on the volatility of wealth: $Q_b = 0$. This is the result of lengthy local second order Taylor development conditional to the specific new-Keynesian model of the transmission mechanism. The welfare preferences for the smoothing policy instruments have the same theoretical status as the assumption of quadratic adjustment costs for smoothing investment or consumption smoothing assumption. Here, we set $\mu_i = 1 > 0$, $\mu_s = 1 > 0$.

We set $\beta = 0.99$, $\kappa = 0.1$, $\gamma = 0.5$. Gali (2015) found $\kappa = 0.1275$. We use a slightly larger elasticity of substitution $\varepsilon = 8.2$ than $\varepsilon = 8.2$ chosen by Gali (2015) to get a round figure for $\kappa = 0.1$, other parameters being unchanged:

$$\kappa = \left(\sigma + \frac{\varphi + \alpha_L}{1 - \alpha_L}\right) \frac{(1-\theta)(1-\beta\theta)}{\theta} \frac{(1-\alpha_L)}{(1-\alpha_L + \alpha_L \varepsilon)} \tag{3}$$

with $\varepsilon > 1$, $0 < \beta = 0.99, \alpha_L = 1/3, \theta = 2/3 < 1$, $\sigma = 1 > 0$, $\varphi = 1 > 0$.

Gali's (2015, chapter 3) calibration of other structural parameters is as follows. The representative household discount factor $\beta = 0.99$ for a logarithmic utility of consumption $\sigma = 1$ and a unitary Frisch elasticity of labor supply $\varphi = 1$. The production function is $Y = A_t L^{1-\alpha_L}$ where Y is output, L is labor, A_t represents the level of technology. The measure of decreasing returns to scale of labor is $0 < \alpha_L = 1/3 < 1$. The proportion of firms who do not reset their price each period $0 < \theta = 2/3 < 1$ which corresponds to an average price duration of three quarters.

Our weight is $Q_x = \kappa/\varepsilon = 0.1/8.2 = 1.2195\%$ is lower than Gali (2015): $\frac{\kappa}{\varepsilon} = 2.125\%$. The welfare results are presented in Table 3, in the group of results where the policy makers only targets inflation volatility in his loss function, so that he sets a zero weight on output gap and on public debt volatility. There are negligible changes of the optimal policy rule parameters, of the eigenvalues and of the initial anchors of inflation and output gap between the welfare case $Q_x = \frac{\kappa}{\varepsilon} = 1/82$ (Table 2, row 1) and the case $Q_x = 0$ (Table 2, row 2) for $Q_\pi = 1$, $Q_b = 0$, $\mu_i = 1$, $\mu_s = 1$ unchanged. Hence, their impulse response functions are approximately identical.

A near-zero relative weight $Q_x = \kappa/\varepsilon$ on output gap fluctuations with respect to inflation fluctuations is inconsistent with the Fed's dual mandate. This is the reason why we do a simulation grid on the policy maker's preferences in the loss function in the following sections, in order to check the sensitivity of optimal policy rule parameters to changes of policy maker's preferences.

3.3 Policy Maker's Maximal Inertia

Maximal inertia is such that the weight on the volatility of the policy targets is zero and the weight of each of the two policy instrument is any non-zero value, for example: $diag(\mathbf{Q}, \mathbf{R}) = (0, 0, 0, 1, 1)$. Changing the relative weight between the two instrument (10^7 to 10^{-7}) does not change the results.

The Hamiltonian system of the LQR with 3 state variables for the policy maker and 3 co-state variables has a transition matrix $\mathbf{H}_6$ of dimension six which is symplectic: its transpose is similar to its inverse. This implies that the list of eigenvalues of $\mathbf{H}_6$ (its spectrum $\Lambda_{\mathbf{H}}$) is such that all inverse of each eigenvalues belong the spectrum. Hence, if there is no eigenvalues exactly on the unit circle, there is an equal number (three) of eigenvalues inside the unit circle and outside the unit circle. Because of the requirement of local stability of the optimal solution, the optimal solution selects three stable eigenvalues available in the spectrum $\Lambda_{\mathbf{H}}$. This method has been extended to the solution of rational expectations models not necessarily derived from optimal behavior by Blanchard and Kahn (1980).

With maximal inertia, the spectrum includes the three eigenvalues of the open-loop transition matrix $\Lambda_{\mathbf{A}}$ where $\mathbf{F} = \mathbf{0}$ and their three inverse of these eigenvalues for for

$diag(\mathbf{Q},\mathbf{R}) = (0,0,0,1,1)$.

$$\begin{aligned}
\Lambda_{\mathbf{A}} &= \begin{pmatrix} \lambda_{x\pi} = 0.7995 & \lambda_{\pi x} = 1.251 & \lambda_b = 1.005 \end{pmatrix}, \\
\Lambda_{\mathbf{H}} &= \begin{pmatrix} 0.7995 & 1.251 & 1.005 & 1/0.7995 & 1/1.251 & 1/1.005 \end{pmatrix} \\
\Lambda_{\mathbf{A+BF}} &= \begin{pmatrix} \lambda_{x\pi} = 0.7995 & \lambda_{\pi x} = 0.7995 & \lambda_b = 0.995 \end{pmatrix}.
\end{aligned}$$

The open loop transition matrix $\mathbf{A}$ includes two unstable eigenvalues. In particular, debt dynamics is exploding without negative-feedback. Then, the maximal inertia eigenvalues $\Lambda_{\mathbf{A+BF}}$ includes the two inverse of these eigenvalues and keeps the eigenvalue inside the unit circle. For this reason, maximal inertia implies nonetheless non-zero optimal policy rule parameters. These policy parameters insures the local stability in dimension three of the closed-loop system.

$$\begin{pmatrix} i_t \\ s_t \end{pmatrix} = \begin{pmatrix} -0.907 & 1.855 & 0 \\ -2 & -0.8 & 1 \end{pmatrix} \begin{pmatrix} x_t & \pi_t & b_t \end{pmatrix}^T$$

It is optimal for the Taylor rule not to respond to public debt: there is no need to set the restriction $F_b = 0$. The optimal surplus rule parameter $G_b = 1$ is such that the public debt eigenvalue shifts from its diverging open loop value number $1/\sqrt{\beta}$ to its inverse $\sqrt{\beta}$. These eigenvalues include the factor $\sqrt{\beta}$ because the loss function is discounted.

$$\sqrt{\beta}\frac{1}{\beta} + \sqrt{\beta}\left(F_b - \left(\frac{1}{\beta} - 1\right)G_b\right) = \sqrt{\beta} = \sqrt{0.99} = 0.995 < 1$$

This optimal surplus rule and the resulting public debt eigenvalue with least effort stabilization of public debt remains unchanged when the policy maker does not care about the volatility of public debt ($\mu_b = 0$) and for any non zero weights on inflation and output gap.

The Taylor rule parameter on inflation satisfies the Taylor principle ($1.855 > 1$). The output gap parameter is negative for Ramsey optimal policy with the new-Keynesian model (Chatelain and Ralf (2020)). This is because a rise of funds rate increase future consumption and future consumption growth because of the intertemporal substitution effect for consumers. There is no income effect of funds rate nor cost of capital effect decreasing investment demand as in the investment saving equation of the IS-LM model. Hence, negative feedback aiming at decreasing future output gap implies a negative response of funds rate when current output gap is positive. Then, the intertemporal substitution mechanism implies a relative decrease of the growth rate of future consumption.

If one reverts the signs to Keynesian mechanism: κ accelerationist Phillips curve and γ (delayed cost of capital effect or cost of working capital), then the sign of the response of funds rate to output gap turns to be positive, with exactly the same numerical values ($+0.907$), as it was in the time of Keynesian economics.

The optimal surplus rule also responds to inflation and output gap. Because the marginal parameter in $\mathbf{B}$ of the surplus is nearly one hundred times smaller than the one of the funds, this leads to marginal changes of parameters relating future public debt with current inflation and current output gap than in the case of only an effect due to the Taylor rule.

The initial optimal jumps of the inflation and output gap on public debt are obtained from optimal marginal conditions. The optimal value of the loss function is:

$$L^* = \begin{pmatrix} x_0 & \pi_0 & b_0 \end{pmatrix} \mathbf{P}_3 \begin{pmatrix} x_0 & \pi_0 & b_0 \end{pmatrix}^T$$
$$\text{with } \mathbf{P}_3 = \begin{pmatrix} \mathbf{P}_{x\pi} & \mathbf{P}_{x\pi,b} \\ \mathbf{P}_{b,x\pi} & \mathbf{P}_b \end{pmatrix}$$

Where $\mathbf{P}_3$ is a square matrix of dimension three which is the solution of a discrete algebraic Riccati equation (DARE) given by the `lqr` instruction in SCILAB. The marginal values of the loss function with respect to each state variables are equal to the co-state variable or Lagrange multiplier of each state variable. If state variables for the policy maker are jump variables of the private sector (here, inflation and output gap), there initial values is found optimizing the loss function at the initial date for each of these variables. These marginal conditions are:

$$\begin{pmatrix} (\partial L^*/\partial x)_{t=0} \\ (\partial L^*/\partial \pi)_{t=0} \end{pmatrix} = \begin{pmatrix} \gamma_{x_0} \\ \gamma_{\pi_0} \end{pmatrix} = \mathbf{P}_{x\pi} \begin{pmatrix} x_0 \\ \pi_0 \end{pmatrix} + \mathbf{P}_{x\pi,b} b_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} x_0 \\ \pi_0 \end{pmatrix} = -\mathbf{P}_{x\pi}\mathbf{P}_{x\pi,b} b_0 = \begin{pmatrix} 0.418 \\ 0.205 \end{pmatrix} b_0$$

An increase of initial public debt implies a proportional *increase* of initial output gap and inflation, as in Woodford (1996) model, a result which is obtained for $F_\pi < 1$. But, for the following periods, future values output gap and inflation do not depend on current values of public debt b_t for $t > 0$ because the parameters of marginal effects equal to zero in the closed loop matrix $\mathbf{A} + \mathbf{BF}$. In particular, because the funds rate do not react to public debt in the Taylor rule ($F_b = 0$). As seen below, this result changes when the cost of the volatility of public debt is taken into account by the policy maker ($\mu_b > 0$).

For Ramsey optimal policy with maximal inertia, the optimal initial jumps of forward-looking policy targets are chosen in order to minimize the volatility of policy instrument. They imply that the initial jumps of the policy instruments are equal to zero:

$$\begin{pmatrix} i_0/b_0 \\ s_0/b_0 \end{pmatrix} = \begin{pmatrix} -0.907 & 1.855 & 0 \\ -2 & -0.8 & 1 \end{pmatrix} \begin{pmatrix} 0.418 \\ 0.205 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Even though the policy instrument are equal to zero, the dynamic system is no longer the open loop system with matrix $\mathbf{A}$ because this matrix includes eigenvalues outside the unit circle. It is the closed loop system $\mathbf{A} + \mathbf{BF}$ with $\mathbf{F} \neq \mathbf{0}$ so that there is local stability of the dynamic system in dimension three. The dimension three corresponds to the number of policy targets for the policy maker.

With Ramsey optimal policy, there is no indeterminacy with three eigenvalues inside the unit circle despite two forward-looking policy targets and one predetermined policy target. Optimal initial conditions implies that the co-states of the forward-looking policy targets are predetermined at zero: $\gamma_{x_0} = \gamma_{\pi_0} = 0$. Because these co-state variables have a linear relation with the funds rate and its lag, these conditions can also be interpreted as if the funds rate and its lag are predetermined variables.

3.4 General Results

In all the cases where the volatility of the public debt does has zero weight in the loss function ($Q_b = 0$), the surplus rule parameters do not change. The surplus rule achieves the maximal inertia (minimal effort) for stabilizing public debt eigenvalue. The percentage deviation of surplus is equal to the percentage deviation of public debt ($G_b = 1$). It is optimal that the funds rate do not respond to public debt in the Taylor rule ($F_b = 0$): *this is not a constraint that we impose.* The Taylor rule only controls the eigenvalues of output gap and inflation. The Taylor rule parameters (F_x,F_π) are identical to those found when assuming public debt is equal to zero at all periods ($b_t = 0$). However, an increase of the weight of the surplus $\mu_s > 0$ *increases* the parameter $\mathbf{P}_{x\pi,b}$ for the optimal loss function. Hence, it increases the absolute value of the optimal initial anchor of inflation and output gap, even though the Taylor rule parameters are unchanged when varying the surplus weight $\mu_s > 0$. For a negligible weight of the surplus $\mu_s = 10^{-7}$, inflation and output gap anchors are instantaneously set to zero.

3.5 Minimizing Output Gap Volatility

Output gap can be stabilized in the first period following a change of interest rate using the Euler consumption equation. The stabilization of inflation occurs in period two. The correlation of the policy instrument with output gap of period one is then transmitted to period two by the change of output gap correlated with expected inflation of period two. Hence, a policy maker is only concerned with a single eigenvalue related to the output gap. He may leave the second eigenvalue mostly related to inflation close to one, because the cost of inflation volatility is zero in this polar case.

When the cost of changing funds rate decreases from 1 to near-zero 10^{-7}, the output gap parameter in the Taylor rule increases in absolute value from -1.15 to -2.12 (whereas the inflation rule parameter decreases from 1.79 to 1.22). The output gap eigenvalue shifts from 0.56 to 0 (which means back to equilibrium in one period).

Table 1: Minimize output gap volatility.

μ_i μ_s	Q_x Q_π Q_b $\lvert\lambda_{x\pi}\rvert$ $\lvert\lambda_{\pi x}\rvert$ $\lvert\lambda_b\rvert$	F_x F_π F_b G_x G_π G_b	$\frac{x_0}{b_0}$ $\frac{\pi_0}{b_0}$	$\frac{i_0}{b_0}$ $\frac{s_0}{b_0}$
1 1	1 0 0 0.562 0.918 0.995	-1.148 1.786 0 -2 -0.8 1	0.430 0.170	-0.191 0.004
1 10^{-7}	1 0 0 0.562 0.918 0.995	-1.148 1.786 0 -2 -0.8 1	0 0	0 1
10^{-7} 1	1 0 0 0 0.995 0.995	-2.121 1.121 0 -2 -0.8 1	0.419 0.201	-0.644 0.002
10^{-7} 10^{-7}	1 0 0 0 0.995 0.995	-2.121 1.121 0 -2 -0.8 1	0 0	0 1

3.6 Minimizing Inflation Volatility

In the new-Keynesian model, a change of the interest rate at period zero is correlated with period one output gap, which is correlated with period two inflation. Decreasing the persistence of inflation implies to decrease the persistence of output gap which is an intermediate variable in the transmission mechanism of the changes of the interest rate to inflation. Hence, both eigenvalues of the block matrix of output gap and inflation are to be modified by the interest rate rule.

When the cost of changing funds rate decreases from 1 to near-zero 10^{-7}, the output gap parameter in the Taylor rule increases in absolute value from -0.993 to -4.12. The inflation rule parameter begins from 2.16 corresponding to the modulus measuring output gap and inflation persistence $|\lambda_{x\pi}| = |\lambda_{\pi x}| = 0.77$. Such a low persistence, sufficiently far for the unit root, can be already considered as a highly successful stabilization policy by practitioners of monetary policy in the real world. The gap of the policy targets from their equilibrium value goes down to $0.77^6 = 0.21\%$ after six quarters.

Only for information, in the extreme case where the cost of changing the interest rate is nearly zero ($\mu_i = 10^{-7}$), the inflation rule parameter reaches its maximal and unrealistic value of $F_\pi = 21$ which implies the zero lower bound of the persistence of inflation and output ($|\lambda_{x\pi}| = |\lambda_{\pi x}| = 0.006$): Inflation and output are back to equilibrium next quarter whatever the magnitude of the shock of the current period.

Table 2: Minimize inflation volatility (2nd row: Welfare Case).

μ_i μ_s	Q_x Q_π Q_b $\lvert\lambda_{x\pi}\rvert$ $\lvert\lambda_{\pi x}\rvert$ $\lvert\lambda_b\rvert$	F_x F_π F_b G_x G_π G_b	$\frac{x_0}{b_0}$ $\frac{\pi_0}{b_0}$	$\frac{i_0}{b_0}$ $\frac{s_0}{b_0}$
1 1	0 1 0 0.782 0.782 0.995	-0.993 2.164 0 -2 -0.8 1	0.434 0.165	-0.075 0.0008
1 1	**0.1/8.2** 1 0 0.781 0.781 0.995	-0.997 2.163 0 -2 -0.8 1	0.434 0.165	-0.078 0.0008
1 10^{-7}	0 1 0 0.782 0.782 0.995	-0.993 2.164 0 -2 -0.8 1	0 0	0 1
10^{-7} 1	0 1 0 0.006 0.006 0.975	-4.121 21.209 0 -2 -0.8 1	0.490 0.025	-1.486 0.00001
10^{-7} 10^{-7}	0 1 0 0.006 0.006 0.975	-4.121 21.209 0 -2 -0.8 1	0.004 0.0002	-0.012 0.992

Locus of Taylor rule parameters when $Q_b = 0$:

When $Q_b = 0$ and $\mu_s > 0$ and our given numerical parameters of the transmission mechanism, we draw the locus of the reduced form values of Taylor rule parameters of Ramsey optimal policy. Table 3 and figure 3 provide the boundaries of the triangle of the linear quadratic regulator (LQR) reduced form Taylor rule parameters, obtained by a simulation grid, varying *the weights in the loss function in three dimensions* $\mu_x \geq 0$, $\mu_\pi \geq 0$, $\mu_i > 0$. The sides of the LQR triangle correspond to the cases where the central bank minimizes only the variance of inflation (inflation nutter) without taking into account the zero lower bound constraint on the policy interest rate ($\mu_i = 10^{-7} > 0$), or minimizes only the variance of output gap without taking into account the zero lower bound ($\mu_i = 10^{-7} > 0$), or seeks only maximal inertia of the policy rate ($\mu_i \to +\infty$). This is taken from Chatelain and Ralf (2020a).

Table 3: Taylor rule parameters for $Q_b = 0$, $\mu_s > 0$, $\kappa = 0.1$, $\gamma = 0.5$, $\beta = 0.99$.

Minimize only:	Q_x	Q_π	μ_i	$\lvert\lambda_{x\pi}\rvert$	$\lvert\lambda_{\pi x}\rvert$	F_x	F_π
Inflation	0	1	10^{-7}	0.006	0.006	**−4.12**	**21.21**
Inflation output gap	1	4	10^{-7}	0	0.82	−2.47	4.75
Inflation output gap	1	1	10^{-7}	0	0.90	−2.30	3.03
Inflation output gap	1	1/4	10^{-7}	0	0.95	−2.21	2.09
Output gap	1	0	10^{-7}	0	0.995	**−2.12**	**1.21**
Output gap interest	4	0	1	0.36	0.95	−1.46	1.67
Output gap interest	1	0	1	0.56	0.91	−1.14	1.78
Output gap interest	1/4	0	1	0.68	0.87	−0.98	1.83
Interest rate	0	0	$1(+\infty)$	0.80	0.80	**−0.90**	**1.85**
Inflation interest	0	1/4	1	0.79	0.79	−0.93	1.94
Inflation interest	0	1	1	0.78	0.78	−0.99	2.16
Inflation interest	0	4	1	0.75	0.75	−1.14	2.75

The three points for Taylor rule in bold in Table 3 corresponds to the three apexes of the Ramsey LQR triangle in Figure 3.

A similar analysis can be made for the alternative monetary policy transmission mechanism with $\gamma < 0$ and $\kappa < 0$, see figure 4. The numerical values of the Taylor rule parameters are the same in absolute value. But this time, the output gap rule parameter is *positive*. The new locus is symmetric with respect to the horizontal axis of the locus with opposite signs of the transmission mechanism.

3.7 Minimizing Public Debt Volatility

If $\mu_b > 0$ and μ_s sufficiently large with respect to μ_i, the policy maker's cares about public debt volatility and it is relatively costly for him to use the surplus instrument with respect to the funds rate instrument. In order to reduce a lot the persistence (auto-correlation and eigenvalue) of public debt, the Taylor rule is then called into action in addition to the surplus rule.

In this polar case, as there is no cost of inflation and output gap, the two first eigenvalues related to these variables remain relatively large.

When the cost of changing funds rate, surplus and public debt are all equal to 1, the funds rate increases by 0.25 in proportion to public debt over its long run target and the surplus increases by 1.52. Public debt persistence falls from minimal effort 0.995 ($F_b = 0$, $G_b = 1$) to 0.361 ($F_b = 0.25$, $G_b = 1.3$).

We provide only for information the results of unrealistic extreme preferences leading to unrealistic policy rule parameters forcing zero persistence of public debt dynamics ($|\lambda_b| = 0$). In this case, whatever the magnitude of shocks, public debt reaches its equilibrium next quarter without transitory dynamics.

In the extreme case where the cost of changing surplus rate is nearly zero ($\mu_s = 10^{-7}$) with respect to one for the cost of changing funds rate. Because of its relative costs, the interest rate does not respond to public debt: $F_b = 0$. With near-zero cost, the surplus parameter reaches its maximal and completely unrealistic value $G_b = 99.8$, nearly 100 times the deviation of debt from its long run value.

When the cost of changing surplus is one with respect to near zero for the cost of changing funds rate (Table 4, third row), public debt persistence is zero for $F_b = 0.7$ and $G_b = 1.34$.

When the cost of changing surplus and changing funds rate are both near zero, public debt persistence $|\lambda_b|$ is zero for $F_b = -0.9$ and $G_b = 9.9$.

Table 4: Minimizing public debt volatility.

μ_i / μ_s	Q_x / $\lvert\lambda_{x\pi}\rvert$	Q_π / $\lvert\lambda_{\pi x}\rvert$	Q_b / $\lvert\lambda_b\rvert$	F_x / G_x	F_π / G_π	F_b / G_b	$\frac{x_0}{b_0}$ / $\frac{\pi_0}{b_0}$	$\frac{i_0}{b_0}$ / $\frac{s_0}{b_0}$
1	0	0	1	−2.022	1.639	0.258	0.465	−0.530
1	0.991	0.939	0.361	−3.072	−0.911	1.518	0.093	0.005
1	0	0	1	−0.906	1.854	0	0.737	−0.0006
10^{-7}	0.799	0.799	0.0009	−89.74	83.56	99.838	0.360	63.78
10^{-7}	0	0	1	−3.575	1.357	0.692	0.481	−0.961
1	0.957	0.995	0	−2.730	−0.727	1.347	0.048	0
10^{-7}	0	0	1	−0.210	1.026	−0.909	0.481	−0.955
10^{-7}	0.958	0.987	0	−20.85	1.639	9.954	0.053	0.009

4 CONTROL OF THE INDETERMINACY OF SIMPLE RULE MULTIPLE EQUILIBRIA

4.1 Multiple Equilibria with Simple Rules

This section compares the simple rule policy maker's multiple equilibria for the same new-Keynesian transmission mechanism. In Ramsey optimal policy, optimal initial transversality conditions provide optimal initial values of policy instruments (interest rate and taxes) that anchors the initial value of forward-looking policy targets (inflation and output gap).

Non-predetermined variables depend explicitly on their expected future values in Blanchard and Kahn (1980). By contrast, in the simple rule equilibria, the policy instruments (interest rate and taxes) do not depend explicitly on their expected future value in the simple rules. Nonetheless, the policy instruments (interest rate and taxes) are assumed to be non-predetermined variables without initial condition. With simple rule equilibria, not only inflation and output gap have no given initial conditions, but also interest rate and taxes have no given initial conditions.

Definition 1 *Type I determinacy à la Blanchard and Kahn (1980), for $n \geq 1$ predetermined controllable policy targets, for $m \geq 1$ non-predetermined controllable policy targets, for $1 \leq k \leq n+m$ non-predetermined policy instruments and for* ***fixed*** *values of policy rule parameters $\mathbf{F} \in \mathbb{R}^k \times \mathbb{R}^{n+m}$. One applies Blanchard and Kahn (1980) determinacy condition on the closed loop matrix $\mathbf{A}+\mathbf{BF}$ of the dynamical system of the economy including policy maker's simple rules. The number of eigenvalues of $\mathbf{A}+\mathbf{BF}$ strictly inside the unit circle should be equal to the number n of predetermined policy targets. Conversely, the number of eigenvalues of $\mathbf{A}+\mathbf{BF}$ outside the unit circle should be equal to the number m of non-predetermined policy targets.*

Blanchard and Kahn (1980) solution amounts to solve an algebraic Riccati equation in order to restrict the dynamics of the system to a stable subspace of dimension n instead of evolving in a stable space of dimension $n+m$ for the policy targets.

In the case of the new-Keynesian model with public debt, inflation and output gap are non-predetermined variables, public debt is the only predetermined variable, if one assumes interest rate and taxes are non-predetermined variables. Blanchard and Kahn (1980) determinacy condition implies that one and only one of the three eigenvalues of the matrix $\mathbf{A}+\mathbf{BF}$ of the dynamic system is inside the unit circle, for a fixed set of policy-maker's rule parameters $\mathbf{F}$.

Definition 2 *Type I multiple equilibria indeterminacy à la Blanchard and Kahn (1980) arises for $n \geq 1$ predetermined controllable policy targets, for $m \geq 1$ non-predetermined controllable policy targets, for $1 \leq k \leq n+m$ non-predetermined policy instruments and for* ***fixed*** *values of policy rule parameters $\mathbf{F} \in \mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$. Type I indeterminacy corresponds to the case where the number of eigenvalues of $\mathbf{A}+\mathbf{BF}$ inside the unit circle is strictly larger than the number n of predetermined policy targets.*

The following definition of type II indeterminacy generalizes Leeper's (1991) multiple equilibria indeterminacy obtained for $n = 1$ predetermined controllable policy target (public debt) and for $m = 1$ non-predetermined controllable policy target (inflation), for $1 \leq k = 2 \leq n+m = 2$ non-predetermined policy instruments (interest rate and taxes) and for policy makers **varying** the values of policy rule parameters in matrix $\mathbf{F}$ in $\mathbb{R}^2 \times \mathbb{R}^2$. Blanchard and Kahn type I determinacy leads to type II indeterminacy for the policy maker when he varies $\mathbf{F}$ in $\mathbb{R}^2 \times \mathbb{R}^2$. The first equilibrium has inflation eigenvalue inside the unit circle ("passive" monetary policy) and the public debt eigenvalue outside the unit circle ("active" fiscal policy). The second equilibrium has inflation eigenvalue outside the unit circle ("active" monetary policy) and the public debt eigenvalue inside the unit circle ("passive" fiscal policy).

Proposition 3 *Policy maker's type II multiple equilibria indeterminacy à la Leeper (1991) arises for $n \geq 1$ predetermined controllable policy targets and for $m \geq 1$ non-predetermined controllable policy targets, for $1 \leq k \leq n+m$ non-predetermined policy instruments and for policy makers* ***varying*** *the values of policy rule parameters $\mathbf{F}$ in $\mathbb{R}^k \times \mathbb{R}^{n+m}$. Blanchard and Kahn's (1980) type I determinacy for the matrix $\mathbf{A}+\mathbf{BF}$ implies that there are up to $N_{\mathbf{A}+\mathbf{BF}} \leq \binom{n+m}{n}$ equilibria solutions of algebraic Riccati equations selecting a stable subspace of dimension n in a space of dimension $n+m$ when the policy maker varies the policy rule parameters $\mathbf{F}$ in $\mathbb{R}^{n+m} \times \mathbb{R}^{n+m}$.*

Proof. Blanchard and Kahn (1980) type I determinacy condition implies that the number of eigenvalues of $\mathbf{A}+\mathbf{BF}$ strictly inside the unit circle should be equal to the number n of predetermined and controllable policy targets. Blanchard and Kahn (1980) solution amounts to solve an algebraic Riccati equation in order to restrict the dynamics of the system to a stable subspace of dimension n instead of evolving in a stable space of dimension $n+m$ for the policy targets. Because $(\mathbf{A}, \mathbf{B})$ is a controllable pair including real numbers as elements, any real or complex conjugate values for each eigenvalues of $\mathbf{A}+\mathbf{BF}$ can be reached when **varying** the values of policy rule parameters $\mathbf{F}$ in $\mathbb{R}^k \times \mathbb{R}^{n+m}$ (Wonham (1967) theorem). Therefore, the policy maker has the choice to select and set n eigenvalues inside the unit circle among $n+m$ eigenvalues. In this case, Freiling ((2002), Theorem 3.3, Remark (b)) finds $N_{\mathbf{M}}$ solutions for Riccati equations. The correspondence of Freiling's (2002) notations with our case is as follows: $\mathbf{M} = \mathbf{A}+\mathbf{BF}$, $n = 1$ is the dimension of $\mathbf{M}$-invariant subspaces sought for Blanchard and Kahn (1980) stable dynamics and $m = 2$ is the dimension of their $\mathbf{M}$-invariant complementary subspaces.

"If $\mathbf{M}$ is semi-simple, i.e., if the geometric multiplicity of each eigenvalue of $\mathbf{M}$ is one, then the number $N_{\mathbf{M}}$ of all n-dimensional $\mathbf{M}$-invariant subspaces, which is an upper bound for the number of all solutions of (ARE) [algebraic Riccati equation] is $\binom{n+m}{n}$. Moreover $N_{\mathbf{M}} = \binom{n+m}{n}$ if and only if all eigenvalues of $\mathbf{M}$ are simple" [distinct]. (Freiling (2002), Theorem 3.3, Remark (b), p. 253). ■

For the new-Keynesian model with public debt, because $(\mathbf{A}, \mathbf{B})$ is a controllable pair, the policy maker can control the three eigenvalues of the economic system $\lambda_{\pi x}(\mathbf{F})$, $\lambda_{x\pi}(\mathbf{F})$, $\lambda_b(\mathbf{F})$ varying the rule parameters $\mathbf{F} \in \mathbb{R}^{2x3}$. In particular, the policy maker can avoid the highly specific cases where all eigenvalues of $\mathbf{A} + \mathbf{BF}$ are not simple (distinct). The policy maker has the choice between $N_{\mathbf{A}+\mathbf{BF}} = \binom{3}{1} = 3$ equilibria, where $\lambda_{\pi x}(\mathbf{F})$ or $\lambda_{x\pi}(\mathbf{F})$ or $\lambda_b(\mathbf{F})$ is the only eigenvalue inside the unit circle, with the two others outside the unit circle. Each of these equilibria corresponds to a distinct 1-dimensional $(\mathbf{A} + \mathbf{BF})$-invariant subspace, that is, to distinct eigenvectors related to each eigenvalue.

Proposition 4 *Policy maker's type II unique equilibrium determinacy "à la Leeper (1991)" arises in two cases such that* $\binom{n+m}{n} = 1$*, which is obtained either for* $n = 0$ *and* $m > 0$ *or for the case where* $n > 0$ *and* $m = 0$.

Proof. In the case $n = 0$ and $m > 0$, the controllable predetermined variables are replaced by non-controllable autoregressive forcing variables. For example, Gali (2015) assumes zero net supply of public debt and includes a cost-push auto-regressive shock. In the case $n > 0$ and $m = 0$, this corresponds to old-Keynesian models without non-predetermined variables. ■

4.2 Mapping Multiple Equilibria in the Taylor Rule Parameters Plane

4.2.1 Mapping Eigenvalues in the Plane of Taylor Rule Parameters

To simplify the comparisons, we maintain the usual assumption in the literature of fiscal and monetary interactions ($F_b = 0$), so that $\mathbf{A} + \mathbf{BF}$ is block triangular. This comparison refers to the Figues in the plane of Taylor rule parameters (F_π, F_x) mapping eigenvalues $(\lambda_{\pi x}, \lambda_{x\pi})$ inside or outside the unit circle for given parameters of the monetary policy transmission mechanism, computed in Chatelain and Ralf (2020) (see Appendix B).

These eigenvalues $(\lambda_{\pi x}, \lambda_{x\pi})$ are the roots of the characteristic polynomial $p(\lambda)$ of the closed loop matrix $\mathbf{A}_{x\pi} + \mathbf{B}_{x\pi}\mathbf{F}_{x\pi}$ of new-Keynesian model with output gap and inflation:

$$\mathbf{A}_{x\pi} + \mathbf{B}_{x\pi}\mathbf{F}_{x\pi} = \begin{pmatrix} 1 + \frac{\gamma\kappa}{\beta} & -\frac{\gamma}{\beta} \\ -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{pmatrix} + \begin{pmatrix} -\gamma \\ 0 \end{pmatrix} \begin{pmatrix} F_x & F_\pi \end{pmatrix}$$

$$p(\lambda) = \det(\mathbf{A}_{x\pi} + \mathbf{B}_{x\pi}\mathbf{F}_{x\pi} - \lambda \mathbf{I}_2) = \lambda^2 - T\lambda + D = (\lambda - \lambda_{\pi x})(\lambda - \lambda_{x\pi}) = 0$$

$$p(1) = -T + D = (1 - \lambda_{\pi x})(1 - \lambda_{x\pi}), \; p(-1) = T + D = (-1 - \lambda_{\pi x})(-1 - \lambda_{x\pi})$$

The Taylor rule parameters (F_x, F_π) are affine functions of the trace T and determinant D of the closed loop matrix $\mathbf{A}_{x\pi} + \mathbf{B}_{x\pi}\mathbf{F}_{x\pi}$. There exists a stability triangle where both eigenvalues of of $\mathbf{A}_{x\pi} + \mathbf{B}_{x\pi}\mathbf{F}_{x\pi}$ are inside the unit circle. This stability triangle (ABC in Figure 1) include real eigenvalues such that $-1 < \lambda_{\pi x} \leq \lambda_{x\pi} < 1$ (region 4.1 in Figure 1) and complex conjugate eigenvalues such that $|\lambda_{\pi x}| = |\lambda_{x\pi}| < 1$ (region 4.2 in Figure 1). Its center Ω corresponds to the fully stabilized dynamic system with zero eigenvalues $\lambda_{\pi x} = \lambda_{x\pi} = 0$.

This stability triangle implies that the Taylor rule parameter of inflation satisfies the Taylor principle. But it also implies that the Taylor rule parameter of output gap is strictly negative, because of the intertemporal substitution effect of the Euler equation. A rise of interest rate is positively correlated with future consumption and output gap. Hence, if output gap is currently positive, it is necessary to decrease interest rate so that future consumption will decrease.

As a thought experiment, if the sign of the transmission parameters are reversed ($\kappa < 0$, $\gamma < 0$), then the stability triangle is in the quadrant such that $F_x > 0$ and $F_\pi > 0$ (Figure 4).

4.2.2 First Equilibrium: Fiscal Theory of the Price Level

In this equilibrium, fiscal policy is "active" $|\lambda_b(\mathbf{F})| > 1$. Monetary policy is "passive" in the sense that the inflation Taylor rule parameter does not satisfy the Taylor principle ($F_\pi < 1$) see Woodford (1996).

The parameter F_π can also be negative without finite bound. Output gap Taylor rule parameter is positive without finite upper bound or negative with a finite lower bound.

This corresponds to region 1 of the top-left of Figure 1 which corresponds to these conditions on eigenvalues $-1 < \lambda_{\pi x}(\mathbf{F}) < 1 < \lambda_{x\pi}(\mathbf{F})$. The Taylor rule parameters are such that $\{\mathbf{F} \;/\; |\lambda_{\pi x}(\mathbf{F})| < 1 < \min(|\lambda_b(\mathbf{F})|, |\lambda_{x\pi}(\mathbf{F})|)\}$.

Region 1 is limited by two lines. The first line includes the segment AC. It defines the new-Keynesian border of the Taylor principle, including the point: ($F_\pi = 1$, $F_x = 0$). It is a nearly vertical line such that at least one of the two roots is equal 1: $p(1) = 0$. The second line includes the segment BC. It is a negative slope line where at least one of the two roots is equal to -1: $p(-1) = 0$.

Region 1 includes the origin in the plane (F_π,F_x). This point corresponds to the laissez-faire open-loop equilibrium ($\mathbf{F} = \mathbf{0}$) where the policy instruments are pegged to their long run values. For our benchmark calibration $\beta = 0.99$, $\kappa = 0.1$, $\gamma = 0.5$, the values of the eigenvalues of matrix $\mathbf{A}$ (not multiplied by $\sqrt{\beta}$) are:

$$\Lambda_{\mathbf{A}} = \begin{pmatrix} \lambda_{x\pi} = 0.803 & \lambda_{\pi x} = 1.257 & \lambda_b = 1.01 \end{pmatrix} \tag{4}$$

The projection of inflation and output gap on the stable eigenvectors implies the following dynamics in a stable subspace of dimension 1, with the following eigenvector times its eigenvalue power t times the initial condition for predetermined public debt b_0:

$$\begin{pmatrix} x_t \\ \pi_t \\ b_t \end{pmatrix} = \begin{pmatrix} 0.418 \\ 0.204 \\ 1 \end{pmatrix} \cdot 0.803^t \cdot b_0$$

4.2.3 Unpleasant Second Equilibrium

In this equilibrium, fiscal policy is "active" $|\lambda_b(\mathbf{F})| > 1$. Monetary policy is "active" in the sense that the inflation Taylor rule parameter satisfies the Taylor principle ($F_\pi > 1$) without a finite upper bound. However, the output gap rule parameter is always strictly negative without a finite lower bound, which is the implausible or unpleasant property of this equilibrium.

This equilibrium corresponds to region 3 on the bottom-right of figure 1 such that: $\lambda_{\pi x}(\mathbf{F}) < -1 < \lambda_{x\pi}(\mathbf{F}) < 1$. Region 3 is limited by two lines. The first line defines the new-Keynesian border of the Taylor principle. It is a nearly vertical line such that at least one of the two roots is equal 1: $p(1) = 0$. The second line includes the segment BC. It is a negative slope line where at least one of the two roots is equal to -1: $p(-1) = 0$. The segment BC is Barnett and Duzhak's (2010) border of the flip bifurcation of the discrete time new-Keynesian model.

For our benchmark calibration $\beta = 0.99$, $\kappa = 0.1$, $\gamma = 0.5$, the policy rule parameters ($F_x = -4.5$, $F_\pi = 2$, $\mathbf{G} = \mathbf{0}$) corresponds to a point located in region 3, with the following eigenvalues:

$$\Lambda_{\mathbf{A+BF}} = \begin{pmatrix} \lambda_{\pi x} & \lambda_{x\pi} & \lambda_b \end{pmatrix} = \begin{pmatrix} -1.17 & 0.98 & 1.01 \end{pmatrix} \tag{5}$$

The projection of inflation and output gap on the stable eigenvectors implies the following dynamics in a stable subspace of dimension 1 given by the following eigenvector of the eigenvalue $\lambda_{x\pi}$:

$$\begin{pmatrix} x_t \\ \pi_t \\ b_t \end{pmatrix} = \begin{pmatrix} 0.18068 \\ 0.79828 \\ 1 \end{pmatrix} \cdot 0.987^t \cdot b_0$$

This regime is not mentioned by Woodford (1996). It is an even more unpleasant equilibrium than Ramsey optimal policy, because the response of interest to the output gap or to inflation (F_x) is not only strictly negative, it is also always strictly large in absolute value ("more negative") than the Ramsey optimal interest rule parameter of the output gap.

4.2.4 New-Keynesian Third Equilibrium

In this equilibrium, fiscal policy is passive $|\lambda_b(\mathbf{F})| < 1$. Monetary policy is "active" in the sense that both the eigenvalues of inflation and output gap are larger than one. This equilibrium corresponds to this set of policy rule parameters:

$$\{\mathbf{F} \ / \ |\lambda_b(\mathbf{F})| < 1 < \min(|\lambda_{\pi x}(\mathbf{F})|, |\lambda_{x\pi}(\mathbf{F})|)\}.$$

Assuming $F_b = 0$, the condition $|\lambda_{\pi x}(\mathbf{F})| > 1$, $|\lambda_{x\pi}(\mathbf{F})| > 1$ corresponds to the regions 4.3, 4.4, 4.5 on the top right of Figure 1 and region 2 on the bottom left of Figure 1. Region 4.3 is for complex conjugate eigenvalues such that $|\lambda_{\pi x}(\mathbf{F})| = |\lambda_{x\pi}(\mathbf{F})| > 1$. Regions 4.4 and 4.5 are for real eigenvalues: $1 < \lambda_{\pi x}(\mathbf{F}) < \lambda_{x\pi}(\mathbf{F})$.

For the regions 4.3, 4.4, 4.5 on the top right of Figure 1, the inflation Taylor rule parameter satisfies the Taylor principle ($F_\pi > 1$) without a finite upper bound. However, the output gap rule parameter can be positive or implausibly negative without a finite upper bound.

Region 2 is for real eigenvalues such that:: $\lambda_{\pi x}(\mathbf{F}) < \lambda_{x\pi}(\mathbf{F}) < -1$ on the left of the Taylor principle line ($p(1) = 0$) and below the line ($p(-1) = 0$) including the segment CB. The inflation Taylor rule parameter does not satisfy the Taylor principle including negative value without bound. The output gap rule parameter is often negative, or the inflation Taylor rule parameter is negative when the output gap rule parameter is positive: region 2 is an unpleasant new-Keynesian equilibrium.

For our benchmark calibration, we select an example with two policy rule parameters in region 4.4 ($F_x = 1$, $F_\pi = 1$) along with ($F_b = 0$, $G_b = 1.5$), with the following eigenvalues:

$$\Lambda_{\mathbf{A}+\mathbf{BF}} = \begin{pmatrix} \lambda_{\pi x} & \lambda_{x\pi} & \lambda_b \end{pmatrix} = \begin{pmatrix} 1.009 & 1.551 & 0.995 \end{pmatrix} \tag{6}$$

The projection of inflation and output gap on the stable eigenvectors implies the following dynamics in a stable subspace of dimension 1 given by the following eigenvector of the eigenvalue $\lambda_{x\pi}$:

$$\begin{pmatrix} x_t \\ \pi_t \\ b_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot 0.995^t \cdot b_0$$

Because there is no effect of public debt on future output gap (no wealth effect: $A_{xb} = 0$) nor on future inflation ($A_{\pi b} = 0$) in the transmission mechanism, if the interest rate rule does not respond to public debt ($F_b = 0$), the dynamical system $\mathbf{A} + \mathbf{BF}$ is block triangular, with an upper block for inflation and output gap. The eigenvectors of the eigenvalue of public debt are proportional to $(0, 0, 1)^T$. They are orthogonal to the space of inflation and output gap. This implies that the initial jump of inflation and output gap is always back to its long run equilibrium: $\pi_0 = 0.b_0$ and $x_0 = 0.b_0$. This outcome is the same as the one of degenerate rational expectations equilibrium where there is no predetermined variable.

Therefore, it is necessary to add the unpleasant and arbitrary assumptions of exogenous auto-regressive forcing variables for inflation and/or for the output gap so that both variables have transitory dynamics proportional to these forcing variables.

4.3 Controlling Type II Multiple Equilibria Indeterminacy by Ramsey Optimal Policy

We define type III determinacy in the Ramsey optimal policy case:

Definition 5 *Ramsey optimal policy type III determinacy arises for $n \geq 1$ predetermined controllable policy targets and for $m \geq 1$ non-predetermined controllable policy targets, for $1 \leq k \leq n+m$ optimally predetermined policy instruments, for m initial transversality conditions on Lagrange multipliers anchoring the m initial values of non-predetermined policy targets and for policy makers deciding the optimal values of policy rule parameters $\mathbf{F}$ in $\mathbb{R}^k \times \mathbb{R}^{n+m}$ using a quadratic loss function with non-zero costs of changing the policy instruments. The number of eigenvalues of $\mathbf{A} + \mathbf{BF}$ strictly inside the unit circle is equal to the number $n + m$ which is the sum of n predetermined and controllable policy targets and of m predetermined Lagrange multipliers (set to zero) of each non-predetermined policy target (Hansen and Sargent (2008)). There is a unique equilibrium because $\binom{n+m}{n+m} = 1$: there is only one way to select $n + m$ eigenvalues to be controlled to be into the unit disk in the set of $n + m$ eigenvalues.*

With Ramsey optimal policy, optimal initial conditions are derived for inflation and output gap with co-state variables of inflation and output gap optimally predetermined at zero (transversality conditions). The negative-feedback values of the policy rules parameters implies that the number of stable eigenvalues inside the unit circle (three in this model) is equal to the number of forward-looking variables (output gap and inflation) and predetermined variable (public debt).

Chatelain and Ralf (2020) using Table 2 simulation grid show that the locus of Ramsey optimal policy rule parameter is a smaller triangle included into the larger stability triangle where both eigenvalues are inside the unit circle. In particular, negative eigenvalues $-1 < \lambda_{\pi x} \leq \lambda_{x\pi} < 0$ cannot be eigenvalues of optimal policy (figure 3 and a zoom in figure 2). Because Ramsey optimal policy parameter are located within the stability triangle, this implies that its Taylor rule parameter of inflation satisfies the Taylor principle. But it also implies that its Taylor rule parameter of output gap is strictly negative, because of the intertemporal substitution effect of the Euler equation.

Crossing the border AB corresponds to a shift from Ramsey optimal policy to new-Keynesian equilibrium corresponds to a Barnett and Duzhak (2008) Hopf bifurcation.

Crossing the border BC corresponds to a shift from Ramsey optimal policy to the very unpleasant equilibrium corresponds to a Barnett and Duzhak (2010) flip bifurcation.

Crossing the border AC corresponds to a shift from Ramsey optimal policy to FTPL equilibrium corresponds to a saddle-node bifurcation, reneging the Taylor principle.

Finally, as a thought experiment, if the sign of the transmission parameters are reversed, in particular if the intertemporal elasticity of substitution turns to be negative ($\kappa < 0$, $\gamma < 0$), then the Ramsey triangle is within the stability triangle which is this time in the quadrant with plausible positive rule parameters: $F_x > 0$ and $F_\pi > 0$ (Figure 4).

4.4 The Case for Controlling Type II Multiple Equilibria Indeterminacy

As seen in the previous section, it is feasible to control type II multiple equilibria determinacy using Ramsey optimal policy. This section emphasizes why it is worth it. Type II multiple equilibria have implausible and unpleasant properties:

(1) Because the policy maker only considers simple rules, he has no loss function to decide among these equilibria. Conversely, those multiple equilibria only exists because there is no loss function leading to the unique Ramsey equilibrium.

(2) There is a curse of dimensionality: the number of equilibria increases faster than the number of variables $n+m$. Increasing the number k of optimizing private agents leads to at least k predetermined state variables and k non-predetermined co-state variables. For a dynamic stochastic general equilibrium model investigating macro-prudential policy, it usually includes at least two private sector agents, such as a non-financially constrained household and a bank (or a financially constrained household) deciding their own optimal non-predetermined saving in order to optimize their own predetermined wealth dynamics. In this case, the number of equilibria is $\binom{n+m}{n} = \binom{2+2}{2} = 6 > 4 = n + m$.

(3) Building epicycles on epicycles, getting rid of type II multiple equilibria in setting, ex post, after the computation of households optimal behavior, that households' state variables are forced to be equal to zero at all periods (zero net supply of public debt in Gali (2015), footnote 3, $b_{t+1} = b_t = 0$) and replace it by non-controllable autoregressive shocks is implausible and logically inconsistent. The households does not observe during an infinite horizon that his state variable is exogenously set to zero (or its equilibrium set point) at all dates. In this case, the dynamic equation of the state variable vanishes because $b_{t+1} = b_t = 0$. In this case, the household does not need to do intertemporal optimization of his wealth dynamics and the Euler equation is an irrelevant equation.

(4) Building epicycles on epicycles, these simple rule multiple equilibria require the implausible extreme assumption of the knowledge by the policy maker with an infinite precision of the values of the slope of the new-Keynesian Phillips curve κ and of the intertemporal elasticity of substitution γ, as well are zero measurement errors for initial

inflation π_0, output gap x_0 and public debt b_0. Else, because of the local instability of these simple rule equilibria in the three dimensions space of inflation, output gap and public debt, the probability to follow an unstable off-equilibrium path is equal to one.

For example, Cochrane (2019, p.345) has a concern related to the local instability of inflation for this monetary and fiscal regime:

> "*To produce those equilibria, the central bank commits that if inflation gets going, the bank will increase interest rates, and by doing so it will increase subsequent inflation, without bound. Likewise, should inflation be less than the central bank wishes, it will drive the economy down to the liquidity trap... No central bank on this planet describes its inflation-control efforts this way. They uniformly explain the opposite. Should inflation get going, the bank will increase interest rates in order to reduce subsequent inflation. It will induce stability into an unstable economy, not the other way around. I have not seen selecting among multiple equilibria on any central bank's descriptions of what it does... That the central bank will react to inflation by pushing the economy to hyperinflation seems an even more tenuous statement about people's beliefs, today and in any sample period we might study, than it is about actual central bank behavior.*"

This concern allows Cochrane (2019) to promote the following fiscal theory of the price level equilibrium. However, Benassy (2009) has a symmetric concern about the local instability of off-equilibrium paths of public debt dynamics $(|\lambda_b(\mathbf{F})| > 1)$ in the FTPL equilibrium:

> "*The basic idea behind the FTPL is that the government pursues fiscal policies such that, in off-equilibrium paths, it will not satisfy its intertemporal budget constraint, and run an explosive debt policy. This leaves only one feasible equilibrium path. Now although such off-equilibrium paths are not observed in the model's equilibrium, it would be extremely optimistic to assume that in real life situations the economy would follow at every instant the equilibrium path while the government pursues such policies. As a result many people would be reluctant to advise such policies to a real life government.*"

(5) The local instability in the space of policy targets may give rise to indeterminacy related to Shilnikov bifurcation in Barnett et al. (2020) in continuous time, and, in discrete time, to Hopf bifurcation (Barnett and Duzhak (2008)) and to Flip bifurcation in Barnett and Duzhak (2010)). Point A where $\lambda_{x\pi}(\mathbf{F}) = \lambda_{x\pi}(\mathbf{F}) = 1$ and $|\lambda_b(\mathbf{F})| < 1$ may be analogous to a Bodganov Takens bifurcation in continuous time with two eigenvalues set to zero.

The hypothesis of the perfect knowledge with infinite precision of parameters is required in order that Barnett and Duzhak (2008, 2010) bifurcations shifting from stable to unstable off-equilibrium paths do not arise.

One way to reconcile Benassy (2009) and Cochrane (2019) is to use Ramsey optimal policy equilibrium with the local stability of off-equilibrium paths. After all, not only Svensson (2003) but also Cochrane (2011) himself describe Ramsey optimal policy as a relevant benchmark for stabilization policy instead of "simple rule":

> "*In most (Ramsey) analysis of policy choices,..., we think of governments choosing policy configurations while taking first order conditions as constraints; we think of governments acting in markets.*"

5 CONCLUSION

This paper highlighted the properties of Ramsey optimal stabilizing policy for the new-Keynesian model with public debt where the intertemporal substitution effect of the interest rate on consumption is a key driver of the policy transmission mechanism. Ramsey optimal policy favors negative feedback of monetary policy. If public debt is included in the welfare and policy maker's loss function, the interest rate should optimally respond to the deviation of public debt from its long-run equilibrium value, as well as to inflation and output gap.

A key property of Ramsey optimal policy in the new-Keynesian model with public debt is that it eliminates three equilibria obtained with simple rules. Using Leeper (1991) terminology where passive policy corresponds to the range of values of rule parameters allowing negative feedback mechanism, Ramsey optimal policy is a passive monetary policy and passive fiscal policy. It eliminates the active monetary policy and passive fiscal policy equilibrium (new-Keynesian model) versus the passive monetary policy and active fiscal policy (the fiscal theory of the price level).

Ramsey optimal monetary policy in the new-Keynesian model with or without public debt suggest that the interest rate should decrease for a positive output gap, because the policy transmission mechanism is such that an increase of the interest rate is correlated with a fall of future output gap due to the intertemporal substitution effect of the interest rate on consumption. This transmission channel has the opposite sign of the correlation between output and interest rate with respect to the investment saving (IS) curve effect of Keynesian macroeconomics where investment decreases because of the cost of capital. Because the production function does not depend on the stock of capital and because investment and saving are equal to zero in the reference new-Keynesian model, the cost of capital effect of the interest rate on investment is excluded by assumption.

Further research will consider alternative policy transmission mechanisms (Cardani et al. (2018), Gomis-Porqueras and Zhang (2019), Jia (2020), Drygalla et al. (2020)). For example, alternative micro-foundations can restore the income effect of interest rate so that it offsets the intertemporal substitution effect. If one assumes that there is capital in the production function, then, there can be a cost of capital effect decreasing future output. An effect of the cost of working capital on future inflation can also be introduced in the new-Keynesian Phillips curve. Credit constrained households, limited asset market participation also increase the magnitude of the income effect.

Reversing the sign of the transmission mechanism of interest rate, so that the income effect dominates the substitution effect, reverts the sign of the response of interest rate to a positive output gap. In this case, the interest rate increases in proportion to a positive output gap in Ramsey optimal policy. This corresponds to the negative feedback mechanism of stabilization policy for a Keynesian transmission mechanism.

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6 Appendix A: SCILAB code

Remark: SCILAB code computes this transformation of the preferences of the policy maker, to be called in the LQR instruction. In our case, **S** = **0** and **Q** and **R** are diagonal, so that **C** and **D** elements includes square roots of these diagonal elements of **Q** and **R**.

$$\begin{pmatrix} C^T \\ D^T \end{pmatrix} \begin{pmatrix} C & D \end{pmatrix} = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix}$$

```
Qx=0;
Qpi=0;
Qb=0 ;
Ri=1;
Rt=1;
beta1=0.99; gamma1=0.5; kappa=0.1;
A1=[1+(kappa*gamma1/beta1) -gamma1/beta1 0 ; -kappa/beta1 1/beta1 0 ; 0 -1/beta1 1/beta1]
spec(A1)
A=sqrt(beta1)*A1
B1=[gamma1 0 ; 0 0 ; 1 1-(1/beta1) ]
B=sqrt(beta1)*B1
Q=[Qx 0 0 ; 0 Qpi 0 ; 0 0 Qb ]
R=[Ri 0 ; 0 Rt ]
Big=sysdiag(Q,R)
[w,wp]=fullrf(Big)
C1=wp(:,1:3)
D12=wp(:,4:$)
M=syslin('d',A,B,C1,D12)
[Fy,Py]=lqr(M)
spec(A)
spec(A+B*Fy)
Px=Py(1:2,1:2)
Pd=Py(1:2,3)
T0=-inv(Px)*Pd
T0(3,1)=1
T0(1:2,1)
Ins0=Fy*T0
B(:,1)*Fy(1,:)
B(:,2)*Fy(2,:)
```

Appendix B

Figure 1: Three simple rules equilibria outside the stability triangle in the plane of Taylor rule parameters, γ=0.5, κ=0.1, β=0.99

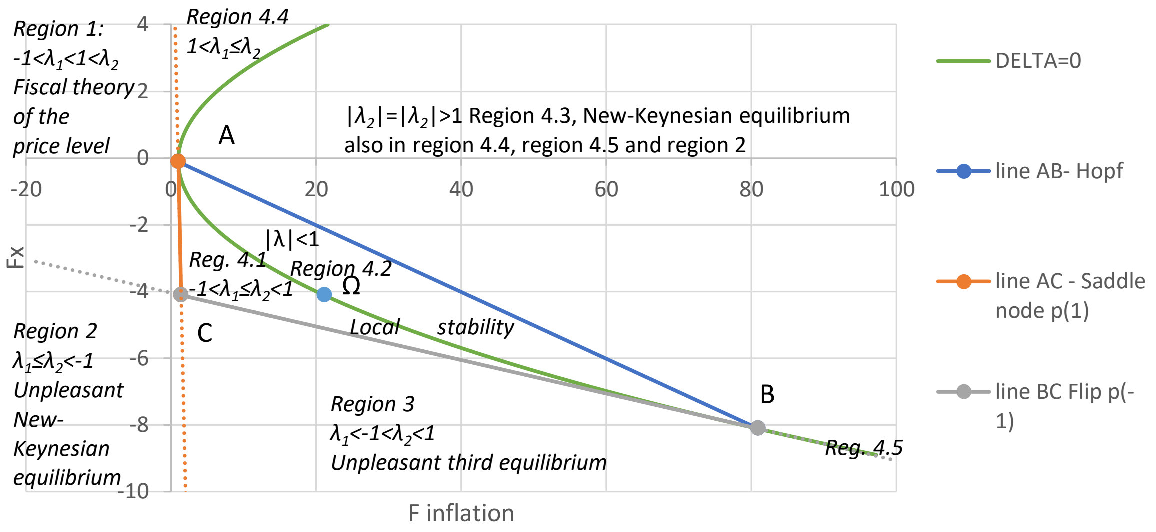

Figure 2: Fiscal Theory of the Price Level vs New-Keynesian vs Ramsey optimal policy, γ=0.5, κ=0.1, β=0.99.

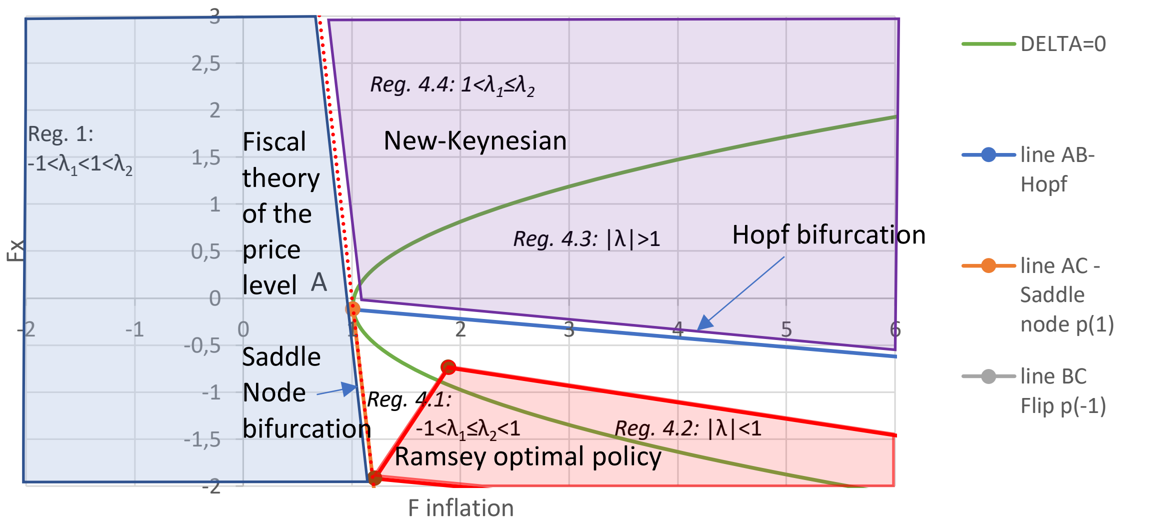

Figure 3: Ramsey LQR triangle in the stability triangle in the plane of Taylor rule parameters, γ=0.5, κ=0.1, β=0.99

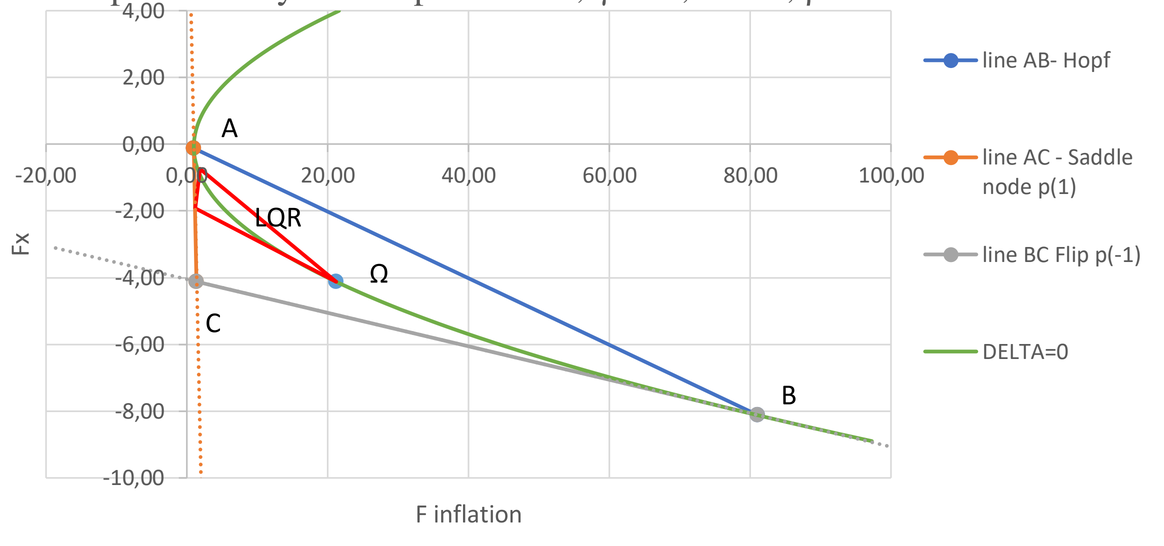

Figure 4: Ramsey LQR triangle in the stability triangle in the plane of Taylor rule parameters, γ=-0.5, κ=-0.1, β=0.99