How and where satellite cities form around a large city: Bifurcation mechanism of a long narrow economy

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How and where satellite cities form around a large city: Bifurcation mechanism of a long narrow economy

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Abstract

We investigate economic agglomerations in a long narrow economy, in which discrete locations are evenly spread over a line segment. The bifurcation mechanism of a monocentric city at the center is analyzed analytically to show how and where satellite cities form. This is an important step to elucidate the mechanism of the competition between a large central city and satellite cities, which is taking place worldwide. By the analysis of the Forslid & Ottaviano (J Econ Geo, 2003) model, we show that the larger the agglomeration forces, the farther from the monocentric city satellite cities emerge. As the trade freeness increases from a low value, there occurs a spatial period doubling in which every other city grows. Thereafter a central city with two satellite cities appears, en route to a complete agglomeration to the central city.

Keywords: Bifurcation; economic geography; replicator dynamics; satellite cities; spatial period doubling.

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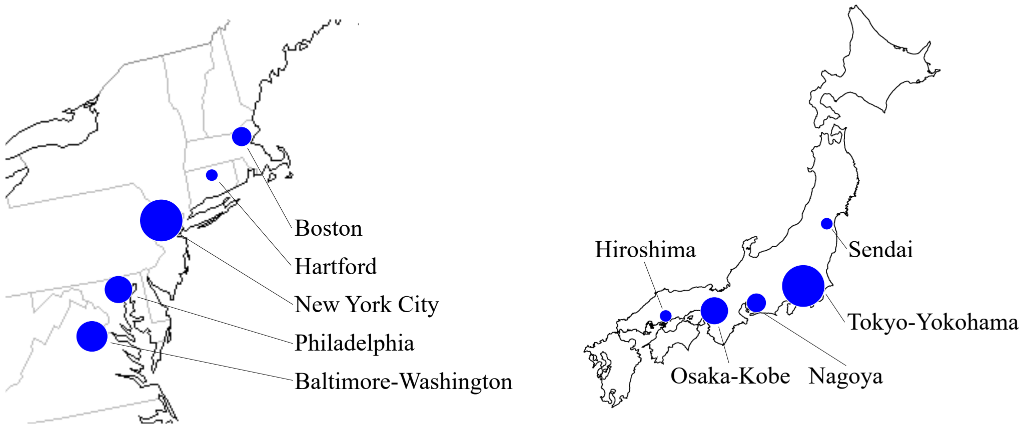


(a) East Atlantic (b) Main Island in Japan

Figure 1: A chain of cities in the world

1. Introduction

A chain of cities prospers in a closed narrow corridor between the Atlantic Ocean and the Appalachian Mountains (see Fig. 1(a)) and in the Main Island of Japan (see Fig. 1(b)). A megalopolis, such as New York City and Tokyo, is growing as a core of economic agglomeration. Chains of cities can be found at transnational scales, such as the Atlantic Axis (from Porto in Portugal to Coruña in Spain) and the STRING (from Hamburg in Germany to Oslo in Norway). We conduct a theoretical study on several characteristic agglomeration patterns, such as full agglomeration, twin cities, core–satellite pattern, and spatial period doubling pattern, as prototypes of diverse spatial agglomeration patterns of a chain of cities observed worldwide.

This paper models a chain of cities by a long narrow economy with equally spaced discrete places on a line segment. The literature reports several characteristic agglomeration patterns of this economy: the simplest core–satellite pattern for three places (Ago et al., 2006), a chain of spatially repeated core–periphery patterns *a la* Christaller and Lösch (e.g., Fujita and Mori, 1997) and a megalopolis which consists of large core cities that are connected by *an industrial belt*, i.e., *a continuum of cities* (Mori, 1997). These patterns were numerically observed by changing agglomeration forces and transport costs (Ikeda et al., 2017). Yet such patterns were investigated somewhat fragmentarily and in an ad hoc manner up to now.

That said, this paper aims to answer the question “How do satellite cities form around a large city?” As a novel theoretical contribution of the paper, we develop a bifurcation theory of the *sustain point*, applicable to any economic geography model with an arbitrary number of places. Although the sustain point in the two-place economy is not considered as a bifurcation point (e.g., Krugman, 1991; Fujita et al., 1999), the sustain point of the full agglomeration to a large single city at the center is shown to encounter bifurcation that produces satellite cities around the large city. We also demonstrate a historical and economical necessity that twin satellite cities are absorbed stably into a huge city, such as New York City, as the trade freeness increases to a certain level.

This paper is far more advanced than Ikeda et al. (2017), who observed several agglomeration patterns on a long narrow economy of economic interest but relied entirely on the numerical analysis to observe those patterns and considered only a specific number of locations. While it is customary to start from the uniform state shadowed by the great success of central place theory (Christaller, 1933), we place emphasis on the formation of a large central city and satellite cities, which is taking place worldwide. Nowadays it would be far more important to investigate the competition between central and satellite cities than to investigate the self-organization of cities in a flat land envisaged in central place theory.

As for the question “Where do satellite cities form around a large city?”, we present a general methodology to investigate which place is most suitable for the location of satellite cities. The bifurcation mechanism of the full agglomeration is theoretically investigated in detail with resort to a many-region version of the model (FO model) by Forslid and Ottaviano (2003) in favor of its analytical tractability and close resemblance to Krugman's (1991) seminal Core-Periphery model. We analyze analytically the existence and uniqueness of the sustain point for the state of the full agglomeration, and the existence and stability of bifurcating solutions from this point that engender satellite cities.

The location of satellite cities is actually obtained for the FO model, and is found to be dependent on the agglomeration forces that are a consequence of: (i) the global size of the industrial sector relative to the traditional sector, and (ii) the degree of scale economies in the industrial sector. When these forces are large, satellite cities appear far away from the primary city at the center. This would give an economic implication of *agglomeration shadow* (Arthur, 1990),[4] cast by cities with a large industry size over locations in vicinity, in which little or no settlement takes place because competition between neighboring regions is too intense to make them profitable for firms to settle. By contrast, sufficiently separated satellite cities and the central region can share industry. When agglomeration forces are very small, a large central place surrounded by two neighboring satellite places emerges, thus forming a hump-shaped megalopolis around the central city.

The progress of stable and sustainable equilibria as the trade freeness increases is of great economic interest as it captures the historical process of increasing economic integration and globalization. To observe this progress, we conduct extensive compara-

[4]See also Fujita et al. (1999), Ioannides and Overman (2004), and Fujita and Mori (2005).

tive statics analyses for various number of cities. There ubiquitously appear three stages called (1) *Dawn* stage, (2) *Core–satellite* stage, and (3) *Full agglomeration* stage, in this order, irrespective of the number of cities. In the Dawn stage, every other city grows, forming a chain of spatially repeated core–periphery patterns *a la* Christaller and Lösch.[5] The Core–satellite stage accommodates a *core–satellite pattern* comprising a central place with twin satellite places. This pattern is the main interest of this paper and its existence has come to be observed in the population data (Ikeda er al., 2019b). As the trade freeness increases further, the core place at the center grows and the twin satellite cities shrink, thereby leading to the Full agglomeration stage for a gigantic mono-center. Admittedly only for a specific spatial economic model, a scenario of historical progress of spatial agglomerations in a chain of cities is thus advanced. The novel bifurcation mechanism proposed in this paper is however potentially applicable to the study of many other models of economic geography.

This paper is organized as follows. The modeling of the spatial economy is presented in Section 2. Bifurcation mechanism of a long narrow economy is described in Section 3. Bifurcation mechanism from the full agglomeration of the FO model is studied analytically in Section 4 and is investigated numerically in Section 5. Section 6 is left for concluding remarks.

[5] Such a spatial alternation of a core place with a large population and a peripheral place with zero population was observed and studied for the racetrack economy in Tabuchi and Thisse (2011), Ikeda et al. (2012), and Akamatsu et al. (2012). Mossay and Picard (2011) and Ikeda et al. (2017) conducted comparative studies of long narrow and racetrack economies.

2. Modeling of the spatial economy

As a representative of spatial economic models, a multi-regional version of the analytically solvable core–periphery model (FO model) proposed by Forslid and Ottaviano (2003) is briefly introduced, whereas details are presented in Appendix A.

2.1. Basic assumptions for the FO model

The economy of this model comprises $K \geq 3$ cities labeled by the set $N = \{0, 1, ..., K-1\}$,[6] two factors of production (skilled and unskilled labor), and two sectors (manufacturing, M, and agriculture, A). The H skilled and L unskilled workers consume final goods of two types: manufacturing sector goods and an agricultural sector good. Workers supply one unit of each type of labor inelastically. Skilled workers are mobile among cities. The number of skilled workers in city $i \in N$ is denoted by λ_i under the constraint $\sum_{i \in N} \lambda_i = H$. The total number H of skilled workers is normalized as $H = 1$. Unskilled workers are immobile and distributed equally across all cities with L/K.

Preferences U over the M-sector and A-sector goods are identical across individuals. The utility of an individual in city i is

$$U(C_i^{\mathrm{M}}, C_i^{\mathrm{A}}) = \mu \ln C_i^{\mathrm{M}} + (1-\mu) \ln C_i^{\mathrm{A}} \qquad (0 < \mu < 1), \tag{1}$$

where μ is a constant parameter expressing the expenditure share of manufacturing sector goods, C_i^{A} stands for the consumption of the A-sector product in city i, and C_i^{M} represents the manufacturing aggregate in city i, defined as $C_i^{\mathrm{M}} \equiv \left(\sum_{j \in N} \int_0^{n_j} q_{ji}(\ell)^{(\sigma-1)/\sigma} d\ell\right)^{\sigma/(\sigma-1)}$, where $q_{ji}(\ell)$ represents the consumption in city $i \in N$ of a variety $\ell \in [0, n_j]$ produced in

[6] This labeling of a city $i \in N$ (which can go from $i = 0$ to $i = K-1$) will be very useful for the later introduction of the long narrow economy in Section 3.

city $j \in N$, n_j stands for the number of produced varieties at city j, and $\sigma > 1$ denotes the constant elasticity of substitution between any two varieties.

The transportation costs for M-sector goods are assumed to take the iceberg form. That is, for each unit of M-sector goods transported from city i to city $j \neq i$, only a fraction $1/\tau_{ij} < 1$ actually arrives ($\tau_{ii} = 1$). It is assumed that $\tau_{ij} = \exp(\tau\, m(i, j)\, \tilde{L})$ is a function of a transport cost parameter $\tau > 0$, where $m(i, j)$ is an integer expressing the road distance between cities i and j and $\tilde{L}$ is the distance unit. We further introduce the trade freeness

$$\phi = \exp[-\tau(\sigma - 1)\tilde{L}] \in (0, 1) \tag{2}$$

that is to be employed in the analysis. The spatial discounting factor $d_{ji} = \tau_{ji}^{1-\sigma} = \phi^{\,m(i,j)}$ represents friction between cities j and i that decays in proportion to the transportation distance. In our formulation, which relies on d_{ji}, the distance unit $\tilde{L}$ need not be specified.

The market equilibrium wage vector $\boldsymbol{w} = (w_i)$ can be obtained analytically ((A.10) in Appendix A). Indirect utility v_i is expressed in terms of w_i and $\Delta_i = \sum_{k \in N} d_{ki} \lambda_k$ as

$$v_i = \frac{\mu}{\sigma - 1} \ln \Delta_i + \ln w_i. \tag{3}$$

2.2. Spatial equilibrium and stability

We introduce a spatial equilibrium in which highly skilled workers are allowed to migrate among cities. A customary way of defining such an equilibrium is to consider the following problem: Find $(\boldsymbol{\lambda}^*, \hat{v})$ satisfying

$$(v_i - \hat{v})\lambda_i^* = 0, \qquad v_i - \hat{v} \leq 0, \qquad \lambda_i^* \geq 0, \qquad \sum_{i \in N} \lambda_i^* = 1, \tag{4}$$

where $\hat{v}$ is the highest (indirect) utility of the solution to this problem.

We consider the replicator dynamics (Sandholm, 2010): $\frac{d\lambda}{dt} = \boldsymbol{F}(\boldsymbol{\lambda}, \phi)$, where $\boldsymbol{\lambda} = (\lambda_i \mid i \in N)$, $\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = (F_i(\boldsymbol{\lambda}, \phi) \mid i \in N)$, and:

$$F_i(\boldsymbol{\lambda}, \phi) = (v_i(\boldsymbol{\lambda}, \phi) - \bar{v}(\boldsymbol{\lambda}, \phi))\lambda_i, \quad i \in N. \tag{5}$$

Here, $\bar{v} = \sum_{i \in N} \lambda_i v_i$ represents the weighted average utility. We restate the problem of obtaining a set of stable spatial equilibria by another problem to find a set of stable and sustainable stationary points of the replicator dynamics (Sandholm, 2010). Stationary points (rest points) $(\boldsymbol{\lambda}, \phi)$ are defined as solutions of the static governing equation

$$\boldsymbol{F}(\boldsymbol{\lambda}, \phi) = \boldsymbol{0}. \tag{6}$$

2.3. Classification of stationary points

Stationary points $(\boldsymbol{\lambda}, \phi)$ of the replicator dynamics are classified into *interior solutions*, for which all cities have positive populations, and *corner solutions*, for which some cities have zero population (i.e., skilled workers). We can appropriately permute the components of $\boldsymbol{\lambda}$, without loss of generality, to arrive at $\hat{\boldsymbol{\lambda}} = (\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0)$ with $\boldsymbol{\lambda}_+ = \{\lambda_i > 0 \mid i = 0, 1, \ldots, m\}$ and $\boldsymbol{\lambda}_0 = \boldsymbol{0}$. Whereas $\boldsymbol{\lambda}_0$ is present for a corner solution and is absent for an interior solution, $\boldsymbol{\lambda}_+$ is present for both solutions. The static governing equation (6) and associated Jacobian matrix can be rearranged, respectively, as (Ikeda et al., 2012)

$$\hat{\boldsymbol{F}} = \begin{pmatrix} \boldsymbol{F}_+(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0, \phi) \\ \boldsymbol{F}_0(\boldsymbol{\lambda}_+, \boldsymbol{\lambda}_0, \phi) \end{pmatrix}, \qquad \hat{J} = \frac{\partial \hat{\boldsymbol{F}}}{\partial \hat{\boldsymbol{\lambda}}} = \begin{pmatrix} J_+ & J_{+0} \\ O & J_0 \end{pmatrix}, \tag{7}$$

where $J_0 = \mathrm{diag}(v_{m+1} - \bar{v}, \ldots, v_{K-1} - \bar{v})$ and $\mathrm{diag}(\cdots)$ denotes a diagonal matrix with the entries in parentheses.

A stable spatial equilibrium is given by a stable and sustainable stationary solution, for which all eigenvalues of $\hat{J}$ are negative.[7] The conditions for stability and sustainability are given, respectively, as

$$\begin{cases} \text{Stability condition:} & \text{all eigenvalues of } J_+ \text{ are negative.} \\ \text{Sustainability condition:} & \text{all diagonal entries of } J_0 \text{ are negative.} \end{cases} \tag{8}$$

Critical points are those which have one or more zero eigenvalue(s) of the Jacobian matrix $\hat{J}$. Critical points are classified into a *bifurcation point*, with singular J_+ or J_0, and a *limit point* of ϕ, with singular J_+. We classify bifurcation points into a *break* bifurcation point with singular J_+ and a *corner* bifurcation point with singular J_0. The investigation of the mechanism of corner bifurcations is a major target of this paper.

A symmetric spatial platform with the replicator dynamics accommodates specific agglomeration patterns $\boldsymbol{\lambda} = \bar{\boldsymbol{\lambda}}$ for which $(\boldsymbol{\lambda}, \phi) = (\bar{\boldsymbol{\lambda}}, \phi)$ are stationary points of the replicator dynamics for any values of the trade freeness ϕ and microeconomic parameters, such as σ and μ. These patterns are called *invariant patterns* (Ikeda et al., 2012, 2018, 2019a),[8] and play an important role in the study of agglomeration patterns in a long narrow economy in this paper.

[7] Since the solution space of the governing equation is the $(K-1)$-dimensional simplex with a constant total population, the eigenvector $\boldsymbol{\eta}^* = (\eta_0, \dots, \eta_m)$ with $\sum_{i=0}^{m} \eta_i \neq 0$ and the associated eigenvalue e^* must be excluded in the investigation of stability and sustainability of the solutions (cf., Section 3.2).

[8] In Ikeda et al. (2012), such patterns are called trivial equilibria.

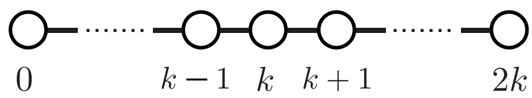


Figure 2: A long narrow economy

3. Bifurcation mechanism of a long narrow economy

To answer the question "How and where do satellite cities form around a large city?", we study the bifurcation mechanism of full agglomeration to a single city and of twin cities in a long narrow economy. The results are general and applicable to any spatial economic models with a single scalar independent variable at each city.

As depicted in Fig. 2, there are $K = 2k + 1$ ($k \in \mathbb{Z} : k \geq 1$) cities labeled $i \in N = \{0, ..., k, ..., 2k\}$, equally spread on a line segment. The kth city is located at the center, and a city $i \neq k$ is said to be $\delta \equiv m(k, i) = |i - k|$ steps away from the center. In other words, $\delta \in \{1, ..., k\} = N_\delta$ is the integer expressing the road distance between a satellite city and the central city.

3.1. Invariant patterns

Full agglomerations and twin cities are invariant patterns of the long narrow economy that are stationary points for any ϕ (cf., Section 2.3).

Proposition 1. *There are two kinds of invariant patterns:*

(i) The full agglomeration (FA) $\boldsymbol{\lambda} = \boldsymbol{\lambda}_\delta^{\mathrm{FA}}$ *located* δ *steps away from the center, i.e.,* $\lambda_{k-\delta} = 1$ *for some* δ $(0 \leq \delta \leq k)$ *and no population elsewhere.*

(ii) The twin cities $\boldsymbol{\lambda}_\delta^{\mathrm{Twin}}$ *with two identical agglomerated places located* δ *steps away from the center, i.e.,* $\lambda_{k\pm\delta} = 1/2$ *(for some* $\delta \in N_\delta$*) and no population elsewhere.*

Proof. See Appendix B.1 for the proof. □

3.2. Bifurcation from a full agglomeration at the center

We are interested in the branches from a state of full agglomeration (FA) at the center $\boldsymbol{\lambda}^{\mathrm{FA}} = \boldsymbol{\lambda}^{\mathrm{FA}}_{\delta=0}$, which turns out to be much superior in sustainability to full agglomerations elsewhere (refer to Section 5). Since this state has the bilateral symmetry about the center, the indirect utility satisfies $v_{k-\delta}(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi) = v_{k+\delta}(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi)$ $(\delta \in N_\delta)$.

Since a break bifurcation point is absent for the full agglomeration,[9] we focus hereafter on a corner bifurcation point, at which the matrix J_0 in (7) becomes singular (cf., Section 2.3). The matrix J_0 at $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathrm{FA}}$ is a diagonal matrix with the diagonal entries: $\{v_{k\pm\delta} - \bar{v} \mid \delta \in N_\delta\}$, and each entry is repeated twice because $v_{k-\delta} - \bar{v} = v_{k+\delta} - \bar{v}$. Thus there possibly exists a series of critical points $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi^{\mathrm{c}}_{\delta})$ $(\delta \in N_\delta)$.

Lemma 1. *The critical point $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi^{\mathrm{c}}_{\delta})$ (for some $\delta \in N_\delta$) of the full agglomeration $\boldsymbol{\lambda}^{\mathrm{FA}}$ at the center is located where $v_{k\pm\delta} - v_k = 0$ is satisfied.*

In the analysis of bifurcating solutions at a critical point, the so-called bifurcation equation is employed (e.g., Golubitsky et al., 1988 and Ikeda and Murota, 2019). In the neighborhood of the present critical point $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi^{\mathrm{c}}_{\delta})$, the governing equation $\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \boldsymbol{0}$ in (6) can be reduced to a two-dimensional bifurcation equation $\tilde{F}_i = 0$ $(i = k - \delta,\ k + \delta)$ in two independent variables $v_{k-\delta}$ and $v_{k+\delta}$ (see Lemma 5 in Appendix B.2). By solving this bifurcation equation, we can show the emergence of one or two satellite cities, δ steps away from the central region, branching from this critical point:

Proposition 2. *The critical point $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi^{\mathrm{c}}_{\delta})$ is a corner bifurcation point with two kinds of*

[9]Since eigenvector associated with $J_+ = -v_k < 0$ for the full agglomeration is not in the $(K-1)$-dimensional simplex, there is no break bifurcation by Footnote 7.

bifurcating solutions that have either two satellite cities ($\lambda_i > 0$ *at* $i = k,\ k \pm \delta$) *or one satellite city* ($\lambda_i > 0$ *at* $i = k,\ k - \delta$ *or* $i = k,\ k + \delta$).

Proof. See Lemma 6 in Appendix B.2 for the proof. □

The full agglomeration is sustainable if $v_{k-\delta} - \bar{v} = v_{k-\delta} - v_k < 0$ ($\forall \delta \in N_\delta$), that is, $(\max_{\delta \in N_\delta} v_{k-\delta}) - v_k < 0$. In other words, agglomeration at the central city is economically sustainable if the indirect utility there is higher than the highest indirect utility across all potential satellite cities with zero population. We use the following assumption on sustainability, with reference to the behavior of the FO model (Proposition 7 in Section 4).

Assumption 1. *Among the corner bifurcation points, there is a sustain bifurcation point* $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi^{\mathrm{c}}_{\delta})$ *(for some* $\delta \in N_\delta$*) with a sustain point* $\phi^{\mathrm{s}} = \phi^{\mathrm{c}}_{\delta}$ *and* $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi)$ *is sustainable for* $\phi > \phi^{\mathrm{c}}_{\delta}$ *and is unsustainable for* $\phi < \phi^{\mathrm{c}}_{\delta}$.

The stability and sustainability of bifurcating equilibria are described as follows.

Proposition 3. *At most one of the two bifurcating paths, just after bifurcation from the sustain point, is stable and sustainable. A stable and sustainable bifurcating path, if it exists, branches in the direction of decreasing trade freeness* ($\phi < \phi^{\mathrm{c}}_{\delta}$).

Proof. See Lemma 8 in Appendix B.2 for the proof. □

Proposition 4. *Bifurcating paths of the corner bifurcation points (other than the sustain bifurcation point) are all unsustainable just after bifurcation.*

Proof. See Appendix B.3 for the proof. □

By Propositions 3–4, the sustain bifurcation point plays an important role in the discussion of stable and sustainable bifurcating equilibria just after bifurcation. For the sustain point, Proposition 3 indicates two mathematical possibilities: (1) either a bifurcating path with two satellite cities or the path with one satellite city is stable and unsustainable and (2) both of them are unstable and/or unsustainable. For the FO model (Section 5.3), both possibilities are shown to exist; stable bifurcating equilibria with the twin satellite cities are observed, but those with a single satellite city are never observed.

By Proposition 3, a stable bifurcating equilibrium, just after bifurcation from the sustain bifurcation point, exists only in the direction of decreasing trade freeness ϕ, engendering a stable state of one or two satellite cities. If we observe this bifurcation behavior conversely, following a historical trend of increasing trade freeness ϕ, we see an emergence of a sustainable state of full agglomeration at the center by steadily absorbing and finally nullifying the population of satellite cities.

3.3. Bifurcation from the twin cities

We investigate the bifurcation from the twin cities $\boldsymbol{\lambda}_{\delta}^{\mathrm{Twin}}$ with $\lambda_{k\pm\delta} = 1/2$ for some $\delta \in N_{\delta}$ and no population elsewhere. The twin cities have both break and corner bifurcation points. We hereafter focus on a sustain bifurcation point, which is the most important corner point in the analysis of stable equilibria, whereas a break bifurcation point does not play an important role at least in the analysis of the FO model (Section 5.3).[10]

The matrix $J_0(\boldsymbol{\lambda}_{\delta}^{\mathrm{Twin}}, \phi)$ for the sustain bifurcation at $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\delta}^{\mathrm{Twin}}$ is a diagonal matrix with the diagonal entries of $v_k - \bar{v}$ (repeated once) and $v_{k-\delta'} - \bar{v}$ (repeated twice; $v_{k-\delta'} = v_{k+\delta'}$; $\bar{v} = v_{k-\delta} = v_{k+\delta}$; $\delta' \in N_{\delta}$, $\delta' \neq \delta$). There, accordingly, are two kinds of sustain bifurcation points associated with either (1) $v_k - \bar{v} = 0$ or (2) $v_{k\pm\delta'} - \bar{v} = 0$.

[10]A break bifurcation point possibly exists where the 2×2 matrix J_+ in (7) becomes singular for an eigenvector $(1, -1)$. This is a pitchfork bifurcation with the same bifurcation mechanism as that of the two-place economy. This bifurcation engenders a core–periphery pattern at the $(k \pm \delta)$th cities.

A sustain bifurcation point with $v_k - \bar{v} = 0$ produces a central city between the twin cities, which grows into a core–satellite pattern, as actually observed for the FO model (Section 5.3).

Proposition 5. *A sustain bifurcation point* $(\boldsymbol{\lambda}_\delta^{\mathrm{Twin}}, \phi_{\delta=0}^{\mathrm{c}})$ *with* $v_k - \bar{v} = 0$ *has the following properties: (i) On a bifurcating path, the city at the center gains mobile population leading to agglomeration to three cities at* $i = k, k \pm \delta$ *with* $\lambda_{k-\delta} = \lambda_{k+\delta}$*. (ii) When the pre-bifurcation state of the twin cities is stable and sustainable for* $\phi < \phi_{\delta=0}^{\mathrm{c}}$ *(resp.,* $\phi > \phi_{\delta=0}^{\mathrm{c}}$*), the bifurcating path is stable and sustainable just after bifurcation if it resides on* $\phi > \phi_{\delta=0}^{\mathrm{c}}$ *(resp.,* $\phi < \phi_{\delta=0}^{\mathrm{c}}$*).*

Proof. See Appendix B.4 for the proof. □

Another sustain bifurcation point associated with $v_{k\pm\delta'} - \bar{v} = 0$ bridges the stable equilibrium of twin cities with that of four cities for the FO model (Section 5.3). We have the following proposition for this point, similarly to Propositions 2 and 3 for the full agglomeration.

Proposition 6. *A sustain bifurcation point* $(\boldsymbol{\lambda}_\delta^{\mathrm{Twin}}, \phi_{\delta'}^{\mathrm{c}})$ *with* $v_{k\pm\delta'} - \bar{v} = 0$ *has the following properties: (i) There emerge (1) a bifurcating solution with nonezero population only at a pair of twin cities at* $i = k \pm \delta,\ k \pm \delta'$ $(\delta' \in N_\delta;\ \delta' \neq \delta)$ *and (2) that at three cities at* $i = k \pm \delta,\ k - \delta'$ *or* $i = k \pm \delta,\ k + \delta'$*. (ii) When the pre-bifurcation state of the twin cities is stable and sustainable for* $\phi > \phi_{\delta'}^{\mathrm{c}}$ *(resp.,* $\phi < \phi_{\delta'}^{\mathrm{c}}$*), there is at most one stable bifurcating equilibrium path just after bifurcation that resides on* $\phi < \phi_{\delta'}^{\mathrm{c}}$ *(resp.,* $\phi > \phi_{\delta'}^{\mathrm{c}}$*).*

4. Theoretical bifurcation analysis of the full agglomeration for the FO model

The general bifurcation mechanism of a long narrow economy was presented in the previous section. In this section, using the FO model (Sectin 2.1), we investigate bifurcation mechanism of the full agglomeration $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathrm{FA}}$ at the center in more detail. We focus on this full agglomeration because it turns out to be much superior in sustainability to full agglomerations elsewhere (Fig. 3 in Section 5.1). By virtue of the analytical solvability of the FO model, the indirect utility at each city is expressed explicitly as follows:[11]

Lemma 2. *The indirect utility at each city for $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathrm{FA}}$ is expressed as* ($\delta = k - i \in N_\delta$)

$$v_k = \ln \frac{\theta}{1-\theta}(2k+1), \qquad \theta = \frac{\mu}{\sigma} \in (0,1); \tag{9}$$

$$v_{k\pm\delta} = \ln \frac{\theta}{1-\theta} + \frac{\delta\mu}{\sigma-1}\ln\phi + \ln\left((\theta k + k + 1)\phi^{\delta} + (1-\theta)\left[(k-\delta)\phi^{-\delta} + \sum_{p=1}^{\delta}\phi^{\delta-2p}\right]\right). \tag{10}$$

Proof. See Appendix C.1 for the proof. □

By bilateral symmetry of the full agglomeration, we have $v_{k-\delta} = v_{k+\delta}$. We, therefore, consider only the cities on the left hand side of the economy labeled by $i = 0, \dots, k$ in the discussion below.

4.1. Limit behaviors when trade freeness is very low or very high

We consider the limit behaviors when the trade freeness ϕ is either very low or very high. As shown in the following lemma, in an extreme case of $\phi \to 1$ with no transport costs, the central city has a locational advantage due to a higher market access and a wider array of varieties for consumers. Firms in the central region can avoid costly

[11] The choice of the total population L of low skilled workers is not influential on the results as the payoff is linear in L (see also Gaspar et al. (2019, pp. 9) for a detailed explanation). For simplicity, we set $L = K$.

transportation while consumers consume more varieties and enjoy a lower cost of living (lower regional price index). Thus, the central city has a better trade environment and workers living there are endowed with a larger indirect utility.

Lemma 3. *As $\phi \to 1$, we have $v_k > v_{k-1} > \cdots > v_0$.*

Proof. See Appendix C.2 for the proof. □

In another extreme case of $\phi \to +0$ with extremely high transport costs, the limit behavior of v_i depends on whether the no-black-hole condition $\mu < \sigma - 1$ (Forslid and Ottaviano, 2003) is satisfied or not.[12] We hereafter assume this condition, since its violation is quite exceptional and empirically unrealistic.[13]

Assumption 2. *The no-black-hole condition $\mu < \sigma - 1$ is satisfied.*

Under this assumption, a city at an outer location has a larger indirect utility when transport costs are extremely high as explained in the lemma below. This is because price competition in the central region is fiercer which induces firms to locate at cities near the border where competition is softer.

Lemma 4. *As $\phi \to +0$, we have $v_k < v_{k-1} < \cdots < v_0$.*

Proof. See Appendix C.3 for the proof. □

4.2. Corner bifurcation point

We march on to investigate the bifurcation from the state of full agglomeration $(\lambda^{\mathrm{FA}}, \phi)$. Based on Lemmas 3 and 4 and on the Intermediate Value Theorem, the sustainability of

[12]As shown in (C.8), $v_i \to +\infty$ for $\mu < \sigma - 1$ and $v_i \to -\infty$ for $\mu > \sigma - 1$ $(i = 0, \ldots, k-1)$.

[13] Anderson and Wincoop (2004), for instance, find that the elasticity of substitution σ is likely to range between 5 and 10.

full agglomeration $(\lambda^{\mathrm{FA}}, \phi)$ is described in the following proposition, which underpins Assumption 1 in Section 3.

Proposition 7. *The full agglomeration* $(\lambda^{\mathrm{FA}}, \phi)$ *is unsustainable as* $\phi \to +0$ *and is sustainable as* $\phi \to 1$. *There exists a sustain point* $\phi^{\mathrm{s}} \in (0, 1)$.

The existence of a corner bifurcation point engendering bifurcating solutions with one or two satellite cities can be shown by the following proposition.

Proposition 8. *There is a corner bifurcation point satisfying* $v_{k-\delta} - v_k = 0$ *for each* $\delta \in N_\delta$, *from which two satellite cities emerge at* $i = k \pm \delta$ *or one satellite city emerges at* $i = k - \delta$.

Proof. See Appendix C.4 for the proof. □

The uniqueness of the corner bifurcation point is dependent on the distance δ to the central city as explained in the following result, which holds for any number of cities $K \geq 2\delta + 1$ $(\delta \in \{1, 2, \ldots, 6\})$.

Proposition 9. *There exists one unique corner bifurcation point* $\phi^{\mathrm{c}}_{\delta} \in (0, 1)$ *(possibly a sustain bifurcation point) for each of* $\delta \in \{1, 2, \ldots, 6\}$ *and for any number of cities* $(K \geq 2\delta + 1)$.

Proof. See Appendix D for the proof. □

Propositions 8 and 9 guarantee the existence and uniqueness of a corner bifurcation point (which may or may not be a sustain point) that leads to twin cities located at $i = k \pm \delta$ $(\delta = 1, 2, \ldots, 6)$, irrespective of the total number K of cities in the economy (of course, we must have $K \geq 2\delta + 1$). For $\delta \geq 7$, we are yet to analytically prove the uniqueness of a bifurcation point (although its existence is assured by Proposition 8).

4.3. Sustain bifurcation point

Denote by $\phi^{\mathrm{c}}_{\delta}$ the largest ϕ satisfying $v_{k-\delta} - v_k = 0$ for each $\delta \in N_\delta$ and set: $\phi^{\mathrm{s}} = \max_{\delta \in N^\delta} \phi^{\mathrm{c}}_{\delta}$. Similarly to the two-place economy for most spatial economic models, a sustain point ϕ^{s} always exists as shown below.

Proposition 10. *There exists a sustain point at* ϕ^{s} $(= \max_{\delta \in N_\delta} \phi^{\mathrm{c}}_{\delta})$ *and the full agglomeration is sustainable for* $\phi > \phi^{\mathrm{s}}$.

Proof. See Appendix C.5 for the proof. □

Corollary 1. *For* $K = 3$ *places, the corner bifurcation point is the unique sustain bifurcation point.*

Proof. See Appendix C.6 for the proof. □

The sustain bifurcation point ϕ^{s} is dependent on σ and μ as explained below, displaying the same tendency as the two-place economy (e.g., Fujita et al., 1999).

Proposition 11. $\dfrac{\mathrm{d}\phi^{\mathrm{s}}}{\mathrm{d}\sigma} > 0$ *and* $\dfrac{\mathrm{d}\phi^{\mathrm{s}}}{\mathrm{d}\mu} < 0$.

Proof. See Appendix C.7 for the proof. □

As σ increases (μ decreases), the sustain point increases. This is in line with the intuition that, for a larger σ, scale economies become weaker as goods become more substitutable, which mitigates the agglomeration forces that promote the full agglomeration of industry. On the other hand, an increase of the expenditure share μ on manufactured goods expands the relative size of the industrial sector, which favors full agglomeration.

5. Progress of stable and sustainable equilibria for the FO model

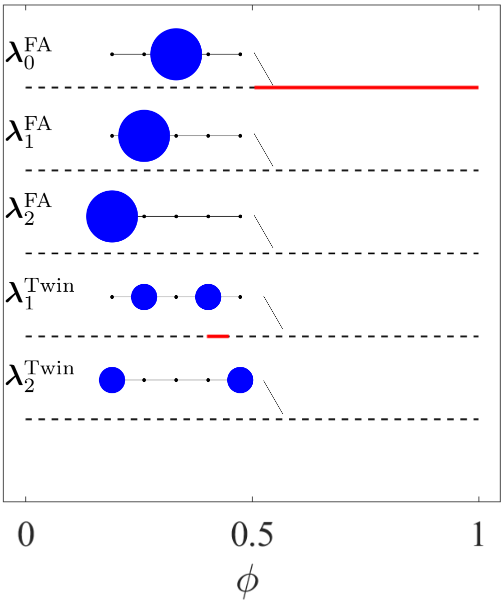
We observe the agglomeration mechanism of the FO model in a long narrow economy as the trade freeness increases. The economy with five cities is employed as the standard model of a chain of cities, such as (1) Boston, Hartford, New York City, Philadelphia, and Baltimore–Washington in East Atlantic and (2) Sendai, Tokyo–Yokohama, Nagoya, Osaka–Kobe, and Hiroshima in the Main Island in Japan. The former is closer to a full agglomeration at the center, while the latter to twin cities. The economy with more than five cities is also analyzed to provide insights on how megalopolises, along narrow corridors with an increased number of cities, behave.

5.1. Stability and sustainability of full agglomerations and twin cities

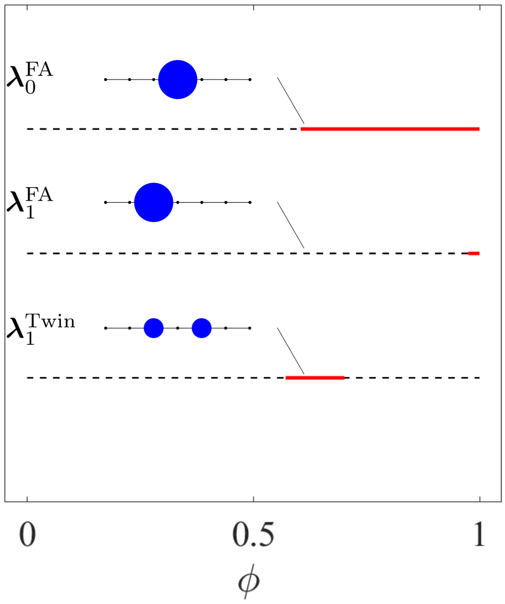
We investigate the stability and sustainability of the states of the full agglomerations $\boldsymbol{\lambda} = \boldsymbol{\lambda}_\delta^{\mathrm{FA}}$ and of the twin cities $\boldsymbol{\lambda} = \boldsymbol{\lambda}_\delta^{\mathrm{Twin}}$, which are invariant patterns and are stationary points for any ϕ (Proposition 1). The ranges of ϕ in which these patterns are stable and sustainable are depicted by red solid lines in Fig. 3.

First, we consider the full agglomerations. For the economy with a relatively small number of cities ($K = 5$ shown in Fig. 3(a)), the full agglomeration $\boldsymbol{\lambda}^{\mathrm{FA}} = \boldsymbol{\lambda}_0^{\mathrm{FA}}$ at the center has a long range of sustainable state $\phi \in (\phi^{\mathrm{s}}_{\delta=0}, 1)$ ($\phi^{\mathrm{s}}_{\delta=0} \approx 0.5$), whereas the full agglomerations elsewhere ($\boldsymbol{\lambda} = \boldsymbol{\lambda}_\delta^{\mathrm{FA}}$ with $\delta > 0$) are always unsustainable for any ϕ. For more cities ($K = 7, 11, 15$ in Figs. 3(b), (c) and (d)), the full agglomeration at the center $\boldsymbol{\lambda}_0^{\mathrm{FA}}$ is most superior in sustainability,[14] the full agglomeration $\boldsymbol{\lambda}_1^{\mathrm{FA}}$ to the city one step

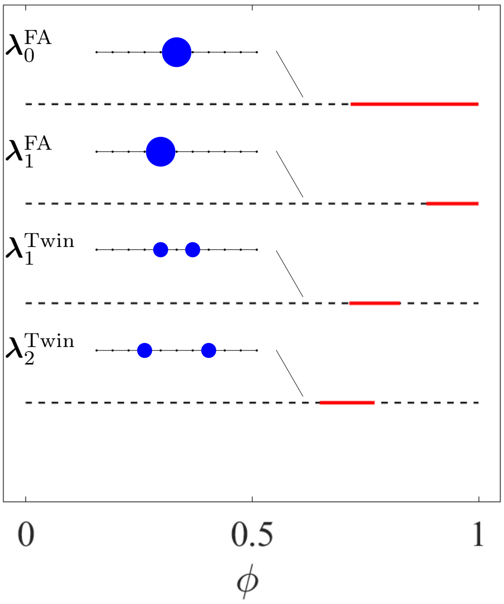
[14]For a larger K, the range of $\phi \in (\phi^{\mathrm{s}}_{\delta=0}, 1)$ becomes shorter and, accordingly, the full agglomeration to the center is less predominant.



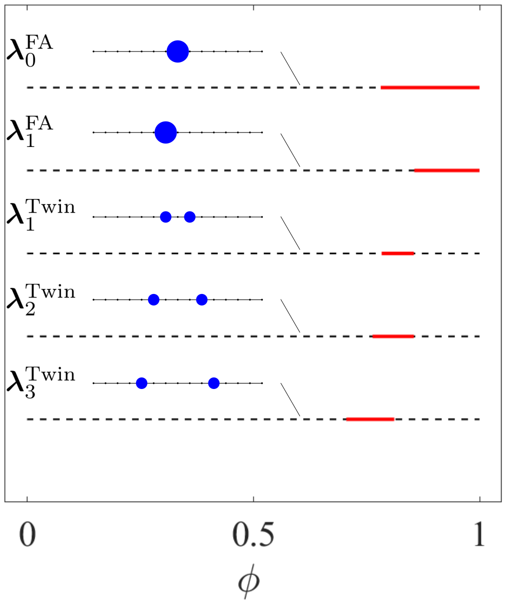
(a) $K = 5$



(b) $K = 7$



(c) $K = 11$



(d) $K = 15$

Figure 3: The range of ϕ of stable and sustainable invariant patterns: full agglomerations $\lambda = \lambda_\delta^{\mathrm{FA}}$ and twin cities $\lambda = \lambda_\delta^{\mathrm{Twin}}$ (unstable and/or unsustainable patterns are included only for $K = 5$; $(\sigma, \mu) = (6.0, 0.4)$; red solid line: stable and sustainable; broken line: unstable and/or unsustainable)

away from the center is sustainable for a shorter range $\phi \in (\phi^{\mathrm{s}}_{\delta=1}, 1)$ $(\phi^{\mathrm{s}}_{\delta=1} > \phi^{\mathrm{s}}_{\delta=0})$, while the full agglomerations to the city further away ($\delta > 1$) are all unsustainable for any ϕ. In addition, the full agglomeration at the center $\boldsymbol{\lambda}^{\mathrm{FA}}_0$ is the one which becomes sustainable first, when the trade freeness increases from a low value. We, accordingly, specifically examine this full agglomeration to the center, which is the most advantageous location of economic activity, in the following subsections.

Next, we investigate twin cities. For the economy with a relatively small number of cities ($K = 5, 7$ shown in Figs. 3(a) and (b)), only the twin cities $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathrm{Twin}}_1$ located one step away from the center have stable equilibria with a short range (near $\phi = 0.44$ for $K = 5$ and $\phi = 0.65$ for $K = 7$). For more cities ($K = 11, 15$ in Figs. 3(c) and (d)), the location of stable and sustainable twin cities extends outwards ($1 \leq \delta \leq 2$ for $K = 11$ and $1 \leq \delta \leq 3$ for $K = 15$). Although all of these twin cities have only short ranges of stable equilibria, some of their ranges cover smaller values of ϕ that the full agglomeration $\boldsymbol{\lambda}^{\mathrm{FA}}_0$ at the center cannot cover. This demonstrates the importance of the state of twin cites for an intermediate value of ϕ, whereas the full agglomeration state dominates for a large ϕ. It implies an inevitable transition from the twin cities to the full agglomeration as ϕ increases from an intermediate to a large value, as will be observed in the comparative static analysis in the next subsection.

5.2. A network of equilibrium paths observed by comparative static analysis

We conduct comparative static analysis with respect to the trade freeness ϕ to observe the progress of stable equilibria as ϕ increases, which is of great economic interest as it captures the historical process of increasing economic integration and globalization. In this analysis, we employ the following innovative strategy that exploits the existence of invariant patterns and the bifurcation mechanism of the full agglomeration and twin cities (Section 3):

1. Stability analysis: Obtain the ranges of the trade freeness ϕ for stable and sustainable full agglomerations and twin cities, which are invariant patterns (Section 3.1).
2. Comparative static analysis: Obtain the equilibrium path connected to the almost uniform state at $\phi = 0$ and bifurcating equilibria from those invariant patterns to find a network of equilibrium paths.
3. Stability analysis: Find stable equilibrium paths on this network.

This strategy is to be employed for given values of microeconomic parameters. For the standard case (Sections 5.1–5.5), we use $(\sigma, \mu) = (6.0, 0.4)$, which satisfies the no-black-hole condition ($\mu < \sigma - 1$) and follows Footnote 13. The influence of the values of σ and μ on the location of satellite cities is studied in Section 5.6.

Using this strategy, we obtained the equilibrium paths and associated spatial distributions of mobile population. We discuss here in detail those for $K = 5, 7, 11, 15$ cities shown respectively in Figs. 4–7, and refer to those for $K = 9, 13$ cities (Figs. E.1 and E.2 in Appendix E) from time to time.[15] Note that the central city exists only for K odd.

5.3. Bifurcating equilibria from the full agglomeration and twin cities

First, we investigate the bifurcating paths from the full agglomeration $\boldsymbol{\lambda}^{\mathrm{FA}}$ at the center. The path of equilibria for this full agglomeration is shown by the horizontal line at

[15]These figures include stable paths that play a key role in the progress of agglomeration, as well as associated unstable paths. Full agglomeration to a city other than the center is not considered in this section, since such an agglomeration is much inferior in sustainability as demonstrated in Section 5.1. Some other paths for $K = 15$ are presented in Fig. E.3 in Appendix E.

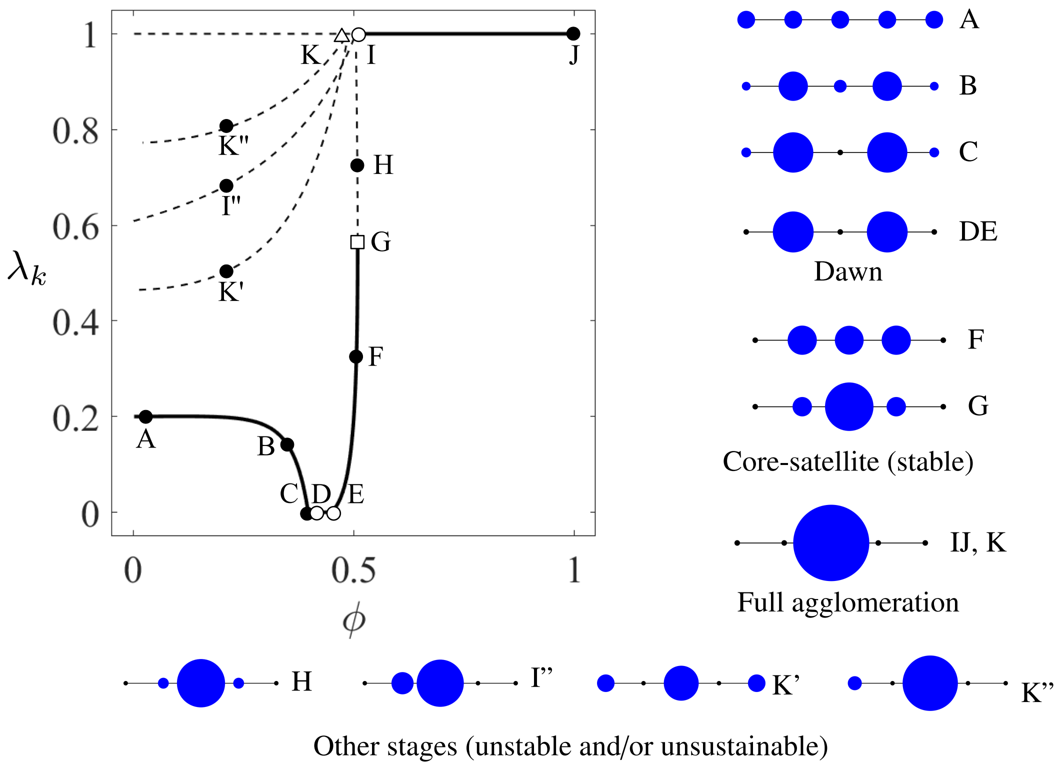


Figure 4: Paths of equilibria for $K = 5$ cities for $(\sigma, \mu) = (6.0, 0.4)$ (solid line: stable and sustainable; broken line: unstable and/or unsustainable; △: bifurcation point; ○: sustain point; □: maximum point of ϕ)

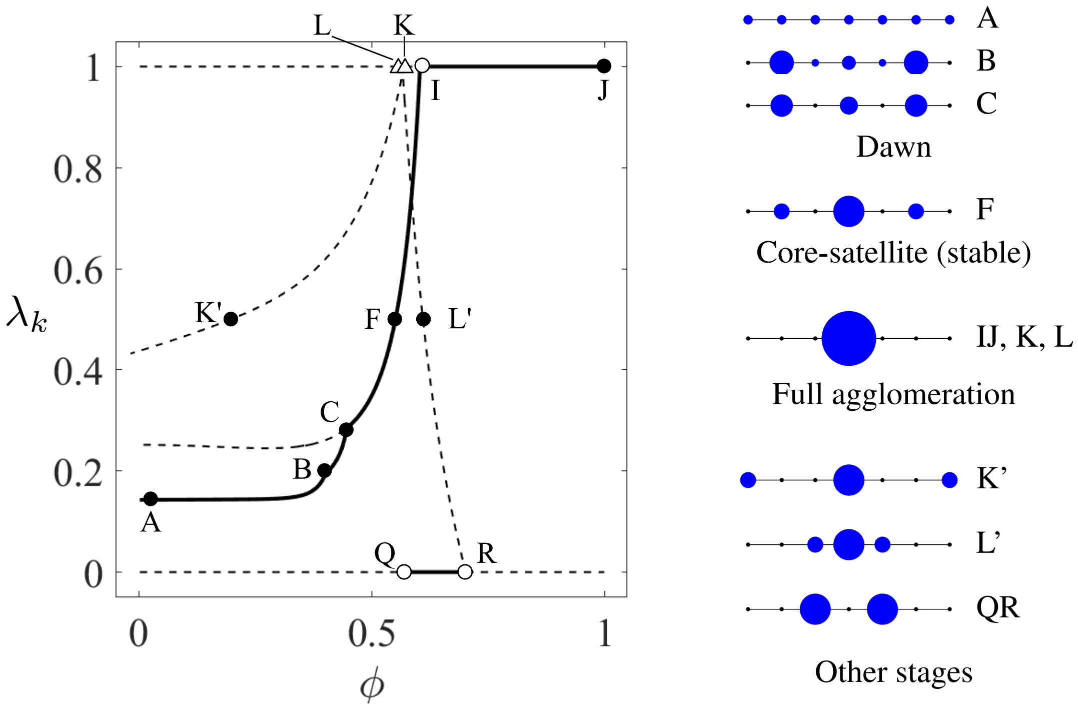


Figure 5: Paths of equilibria for $K = 7$ cities for $(\sigma, \mu) = (6.0, 0.4)$ (solid line: stable and sustainable; broken line: unstable and/or unsustainable; △: bifurcation point; ○: sustain point)

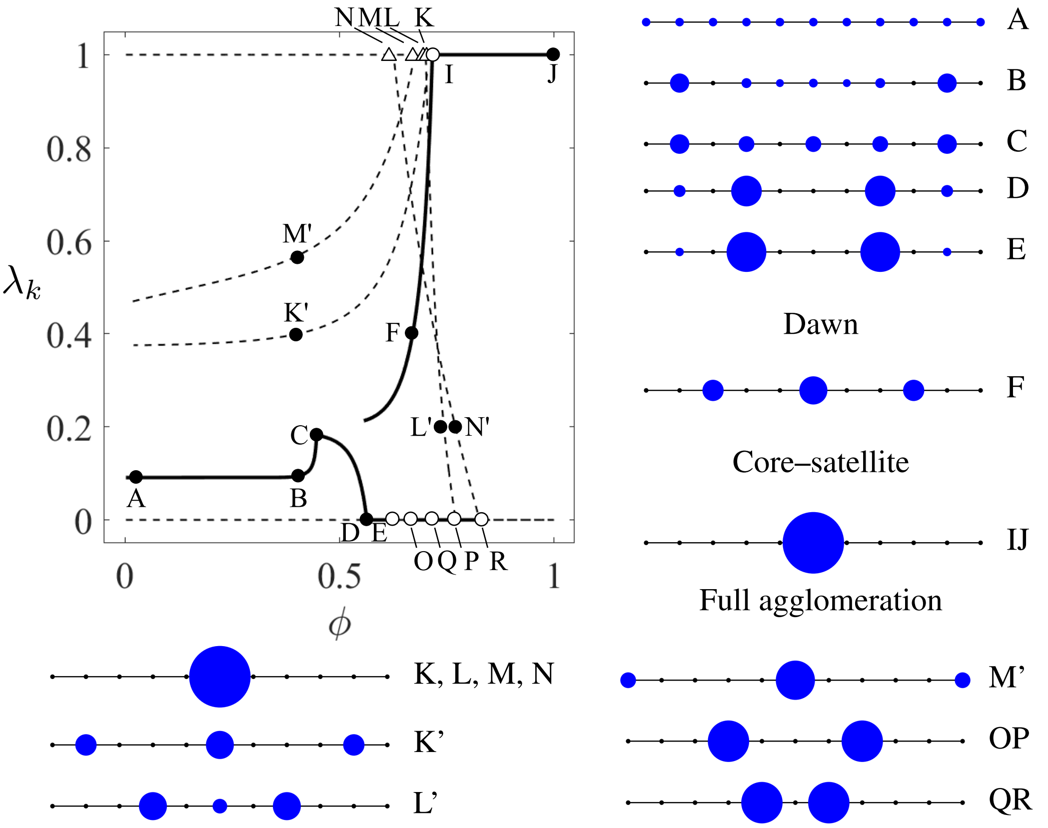


Figure 6: Paths of equilibria for $K = 11$ cities for $(\sigma, \mu) = (6.0, 0.4)$ (solid line: stable and sustainable; broken line: unstable and/or unsustainable; $\triangle$: bifurcation point; $\circ$: sustain point)

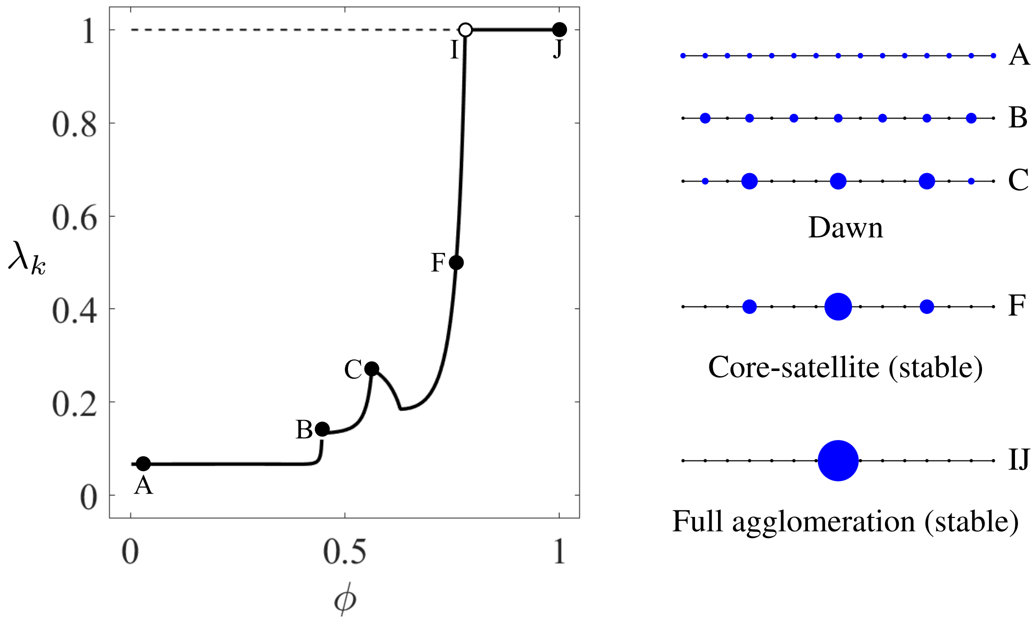


Figure 7: Paths of equilibria for $K = 15$ cities for $(\sigma, \mu) = (6.0, 0.4)$ (solid line: stable and sustainable; broken line: unstable and/or unsustainable; $\circ$: sustain point)

$\lambda_k = 1$ $(0 < \phi < 1;\ k = (K-1)/2)$ of each of Figs. 4–7 and E.1–E.2. The economy with $K = 5$ cities has a unique critical (sustain or bifurcation) point for each δ ($\delta = 1$ for the Point I and $\delta = 2$ for the Point K). Such uniqueness holds also for $K = 7, 9, 11, 13, 15$, which have as many as k $(= 3, \ldots, 7)$ bifurcation points corresponding one to one with each $\delta \in N_\delta = \{1, \ldots, k\}$. Although the uniqueness is proved in Proposition 9 for $K \leq 13$, it is found here to hold also for $K = 15$ (Fig. E.3), implying possible extendability of this proposition to $K \geq 15$. For each K, one of these bifurcation points is the sustain bifurcation point I and the full agglomeration is sustainable during the Path IJ ($\phi \in (\phi^{s}, 1)$) (Proposition 7).

We investigate the stability and sustainability of the two bifurcating paths just after bifurcation for each corner bifurcation point (Proposition 2). All bifurcating paths other than those of the sustain point are found to be unsustainable (Proposition 4). As for the sustain point, the bifurcation behaviors are in accordance with the scenario presented in Proposition 3 that at most one bifurcating path is stable and sustainable: both of these paths are unstable and/or unsustainable for $K = 5$ cities,[16] whereas only one path is stable and sustainable for more cities ($7 \leq K \leq 15$). The stable and sustainable paths always have twin satellite cities, whereas the paths with one satellite city are always unstable and/or unsustainable. Thus, bifurcating paths with twin satellite cities, which are superior in stability, are of most economical importance. Such superior stability might be due to balanced economic activities in both sides of the economy.

Next, we turn our eyes to the twin cities. The path for the twin cities resides on

[16]The bifurcating path IHG is unstable just after bifurcation but regains stability at the limit point of ϕ (Point G shown as □), where satellite cities grow to have significant population size (Fig. 4).

the horizontal line at $\lambda_k = 0$ in each figure.[17] For $K = 5$ cities, for instance, there is a stable and sustainable Path DE for twin cities $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathrm{Twin}}_{\delta=1} = (0, 1, 0, 1, 0)$, enclosed by the two sustain points D and E. The sustain point D corresponds to the case treated in Proposition 6; there is a stable bifurcating equilibrium Path DCBA,[18] on which the population of the two border cities ($i = 0$ and 4, i.e., $\delta = 2$) becomes non-zero, thereby engendering a stable state with a pair of twin cities.[19] Another sustain point E has the stable bifurcating equilibrium Path EFG, on which the central city regains population leading to an agglomeration to three cities (Proposition 5). This demonstrates a vital role in the progress of agglomeration played by sustain bifurcations on the twin cities, which connect the state of the twin cities to other agglomeration patterns.

5.4. Agglomeration to every other city: Dawn stage

As the trade freeness ϕ increases from a very low value, irrespective of the number K of cities, we can observe the following three characteristic and distinctive stages of stable equilibria: (1) Dawn stage, (2) Core–satellite stage, and (3) Full agglomeration stage, in this order.

In the Dawn stage, almost uniform population distribution prevails for a very low value of ϕ. As ϕ increases, odd numbered cities ($i = 1, 3, \ldots, 2k - 1$) grow, while even numbered cities ($i = 0, 2, \ldots, 2k$) that include border cities shrink. For instance, for the Path DE for $K = 5$ in Fig. 4, there appear an agglomeration to every other city at $i = 1, 3$ and no population at another every other city at $i = 0, 2, 4$. This looks like a chain of

[17] This horizontal line contains paths other than those of the twin cities (e.g., Point C for $K = 5$ in Fig 4).

[18] The kink at the Point C is due to the vanishing of the population at the central city ($i = k = 2$).

[19] Another bifurcating path for which the population of one satellite city at $i = 0$ or $i = 4$ becomes none-zero is unstable and is not included in Fig. 4.

spatially repeated core–periphery patterns *a la* Christaller and Lösch (e.g., Fujita and Mori, 1997). This behavior is called the *spatial period doubling* since the spatial period between agglomerated places is doubled from 1 to 2 distance units.[20]

Such doubling also exists in the economy with more cities ($K = 7, 11, 15$ in Figs. 5–7) and is characterized by a decrease of the number K_{agg} of agglomerated places as $K_{\text{agg}} = 2p - 1 \longrightarrow p$ for some integer p (≥ 2),[21] where $\longrightarrow$ denotes the occurrence of spatial period doubling. For $K = 7$ (Fig. 5), the doubling occurs only once as $K_{\text{agg}} = 7 \longrightarrow 3$. For $K = 11$ and 15 (Figs. 6 and 7), the doubling occurs twice as

$$K_{\text{agg}} = K = \begin{cases} 11 \quad \longrightarrow \quad 5 \quad \longrightarrow \quad 2, & \text{for } K = 11, \\ 15 \quad \longrightarrow \quad 7 \quad \longrightarrow \quad 3, & \text{for } K = 15. \end{cases}$$

Spatial period doubling cascade is expected to be observed for $K = 2^p - 1$ cities as

$$K_{\text{agg}} = 2^p - 1 \longrightarrow 2^{p-1} - 1 \longrightarrow \cdots \longrightarrow 3. \tag{11}$$

We would like to advance a formula for the value of the trade freeness at the occurrence of the mth doubling as[22]

$$\phi^m_{\text{doubling}} \approx \exp\left(-\sqrt{\frac{8\mu}{m!(\sigma - 1)}} \right) \qquad (m = 1, 2). \tag{12}$$

Note that ϕ^m_{doubling} is not dependent on the number K of cities and, accordingly, has an objectivity as an index for the description of an early stage of agglomeration. As can be

[20]Such doubling is studied for a long narrow economy (Ikeda et al., 2017) and for a racetrack economy (Tabuchi and Thisse, 2011; Ikeda et al., 2012; and Akamatsu et al., 2012).

[21]Note that $K_{\text{agg}} = 3 \longrightarrow 1$ for $p = 1$ is exceptional because the population is completely agglomerated to the central city and the spatial period between agglomerated places cannot be defined.

[22]Ikeda et al. (2017) introduced a *racetrack economy analogy*, and proposed a formula for the value of the transport cost τ at the occurrence of the mth doubling for a long narrow economy. We have rewritten this formula using the trade freeness $\phi = \exp[-\tau(\sigma - 1)\tilde{L}]$ (cf, (2)) to arrive at (12).

Table 1: Comparison of the values of ϕ^m_{doubling} by the formula (12) with numerical ones.

	Formula (12)	Numerical values			
		$K = 5$	$K = 7$	$K = 11$	$K = 15$
ϕ^1_{doubling}	0.449	0.400	0.446	0.447	0.447
ϕ^2_{doubling}	0.568			0.564	0.564

seen from Table 1, the values of ϕ^m_{doubling} computed by the formula correlate well with the numerically observed first doubling ($m = 1$) for $K = 7$, 11, and 15 and the second doubling ($m = 2$) for $K = 11$ and 15.[23]

5.5. Emergence of the core–satellite pattern and the full agglomeration

After the Dawn stage, the economy evolves to the Core–satellite stage with a large central city and twin satellite cities (Point F in Figs. 4–7). As ϕ increases further, the core city at the center grows and the twin satellite cities shrink, eventually leading to the Full agglomeration stage for sustainable $\boldsymbol{\lambda}^{\text{FA}}$ (Path IJ in each figure).

As can be seen from the agglomeration patterns of the Point F depicted at the right of each figure, the number of satellite cities is two for each K, thereby demonstrating the vital role of the core–satellite pattern as a transient state. This arises from the bifurcation mechanism of a corner bifurcation point that the number of satellite cities on a bifurcating path is at most two (Proposition 2) and an observed fact that a bifurcating state with a single satellite city is always unstable and/or unsustainable, whereas that with twin satellite cities is mostly stable and sustainable.

The way of continuation from the Dawn stage to the Core–satellite stage is dependent

[23]The formula is less accurate for $K = 5$ with a few cities possibly due to the boundary effect.

Table 2: The values of δ_{sat}.

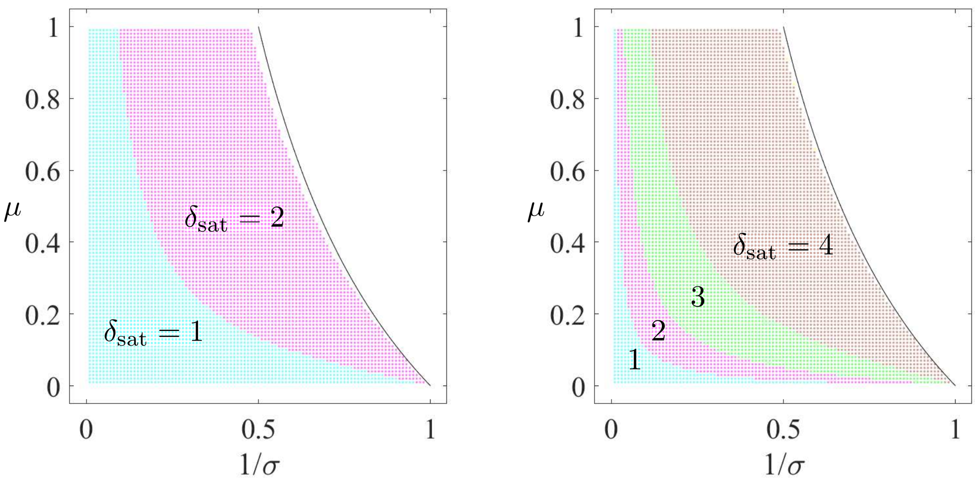
K	5	7	9	11	15	$\cdots$	99
δ_{sat}	1	2	3	3	4	$\cdots$	28
δ_{sat}/k	0.5	0.667	0.75	0.6	$4/7 \approx 0.57$	$\cdots$	$28/49 \approx 0.57$

on the number K of cities. Smooth continuation is observed for $K = 7 = 2^3 - 1$ and $K = 15 = 2^4 - 1$ cities, which follow the spatial period doubling cascade in (11). By contrast, for $K = 11$ cities, for which the central city has no population at the end of the Dawn stage (Path OP in Fig. 6), such smooth continuation is infeasible and there is a dynamical jump to continue to the subsequent stages of Core–satellite and Full agglomeration.

Recall that a major target of this paper is to answer the question "where do satellite cities form?" As an index for the location of a satellite city for the sustain bifurcation point, we denote by δ_{sat} the integer δ that maximizes $\phi^{\mathrm{c}}_{\delta}$; then we have $\phi^{\mathrm{s}} = \phi^{\mathrm{c}}_{\delta_{\mathrm{sat}}}$. As listed in Table 2, the value of δ_{sat} increases as the number K of cities increases. As an index for the optimal location of the satellite cities, we consider a normalized length from the center, being defined as δ_{sat}/k. As K increases to a very large value, such as $K = 99$, the optimal location becomes a little beyond half way from the center to the border ($\delta_{\mathrm{sat}}/k = 28/49 \approx 0.57$).

5.6. Parameter dependence of the location of satellite city

We investigate the dependence of the distance δ_{sat} of satellite cities from the center on the values of the model parameters σ and μ. Figure 8 depicts the contour of δ_{sat} in the space of μ and $1/\sigma$ in the range $(0, 1) \times (0, 1)$ for $K = 5$ and 11 cities. There is a white zone ($\mu > \sigma - 1$) at the upper right corner, where the full agglomeration is always



$K = 5$ $K = 11$

Figure 8: Dependence of the location of satellite cities emerging from the sustain point on the values of the parameters σ and μ (solid line: $\mu = \sigma - 1$)

sustainable and no satellite city emerges. It is to be noted that, the parameter zone for the the border city ($\delta_{\text{sat}} = 5$) is not discernible for $K = 11$. Thus, locations too far from the center are not suitable for the accommodation of satellite cities.

As $1/\sigma$ and/or μ increases, δ_{sat} increases one by one from the smallest value of $\delta_{\text{sat}} = 1$. That is, in association with an increase of agglomeration forces due to stronger scale economies or a larger size of the manufacturing sector (resp., a decrease in σ and/or an increase in μ), the satellite cities tend to form away from the primary city at the center, thereby forming an *agglomeration shadow* (Arthur, 1990; Ikeda et al., Fig. 5, 2017). By contrast, as agglomeration forces decrease, the satellite cities tend to locate closer to the primary city, thereby forming a hump-shaped megalopolis around this city for $\delta_{\text{sat}} = 1$. Thus we have observed the dependence of agglomeration patterns on the values of microeconomic parameters, which possibly is a source of the diversity of the population distribution of a chain of cities observed worldwide.

6. Conclusion

We have conducted a theoretical study on several characteristic agglomeration patterns, such as full agglomeration, twin cities, core–satellite pattern, and spatial period doubling pattern, as prototypes of diverse spatial agglomeration patterns of a chain of cities observed worldwide. We have elucidated the bifurcation mechanism for the full agglomeration at a single big city and twin cities in a long narrow economy in a manner readily applicable to many NEG models. In particular, a sustain bifurcation from the full agglomeration is highlighted as a mechanism to engender a core–satellite pattern with twin satellite cities around a large city. There is a budding of a search of core–satellite patterns in the real population data in Western Germany and Eastern USA (Ikeda et al., 2019b).

A remark is on the standpoint of this paper. While it is customary to start from the uniform state, we place emphasis on agglomeration patterns emanating from the completely agglomerated state. Nowadays it would be far more important to investigate the competition between a large central city and satellite cities than to investigate the self-organization of cities in a flat land envisaged in central place theory. Future work will extend this theory to different spatial topologies, such as a two-dimensional space.

A pertinent combination of model-independent general bifurcation mechanism with model-dependent properties, such as stability/sustainability and parameter dependency, is vital in the successful elucidation of the agglomeration mechanism. For the FO model, we have conducted comparative static analysis with respect to the trade freeness ϕ to observe the progress of stable equilibria. This analysis is of great economic interest as it captures the historical process of increasing economic integration and globalization.

When the trade freeness ϕ increases from a small value to a large value, we have observed the following three characteristic stages regardless the number of cities. It starts with the Dawn stage with a chain of spatially repeated core–periphery patterns *a la* Christaller and Lösch. As the trade freeness increases, a central city with twin satellite cities emerges in the Core–satellite stage. Thereafter, the central city grows and the twin satellite cities shrink, en route to the Full agglomeration stage, in which the population is completely agglomerated in the central city. Admittedly only for a spatial economic model, this paper has demonstrated a scenario of historical progress of spatial agglomerations in a chain of cities. It will be a topic in the future to investigate the progress of agglomerations for other NEG models in the light of the bifurcation mechanism proposed in this paper.

The higher the expenditure share of manufactured goods on income is and/or the lower the elasticity of substitution is, the farther satellite cities emerge from the central city. Conversely, if the size of the industrial sector relative to the traditional sector is very low and/or scale economies are weak, there emerges a hump-shaped megalopolis with satellite cities located side-by-side with the primary central city.

We have thus observed diverse agglomeration patterns dependent on the values of trade freeness and on microeconomic parameters. Such dependence possibly is a source of the diversity of the population distribution of a chain of cities observed worldwide.

Acknowledgments

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Appendix A. Details of modeling of the spatial economy

The fundamental logic and the governing equation of a multi-regional version of the model by Forslid and Ottaviano (2003) are presented (Akamatsu et al., 2016). The budget constraint is given as

$$p_i^{\mathrm{A}} C_i^{\mathrm{A}} + \sum_{j \in N} \int_0^{n_j} p_{ji}(\ell) q_{ji}(\ell) d\ell = Y_i, \tag{A.1}$$

where p_i^{A} represents the price of the A-sector good in place i, C_i^{A} is the consumption of A-sector goods in place i, $N = \{0, 1, ..., K-1\}$, n_j is the number of varieties produced in region j, $p_{ji}(\ell)$ denotes the price of a variety ℓ in place i produced in place j, $q_{ji}(\ell)$ is the consumption of variety $\ell \in [0, n_j]$ in place i produced in place j, and Y_i is the income of an individual in place i. The incomes (wages) of skilled workers and unskilled workers are represented respectively by w_i and w_i^{L}.

An individual at place i maximizes the utility in (1) subject to the budget constraint in (A.1). This maximization yields the following demand functions

$$C_i^{\mathrm{A}} = (1-\mu)\frac{Y_i}{p_i^{\mathrm{A}}}, \qquad C_i^{\mathrm{M}} = \mu \frac{Y_i}{\rho_i}, \qquad q_{ji}(\ell) = \mu \frac{\rho_i^{\sigma-1} Y_i}{p_{ji}(\ell)^{\sigma}},$$

where ρ_i denotes the price index of the differentiated products in place i, and is given by

$$\rho_i = \left(\sum_{j \in N} \int_0^{n_j} p_{ji}(\ell)^{1-\sigma} d\ell \right)^{1/(1-\sigma)}. \tag{A.2}$$

Because the total income in place i is $w_i \lambda_i + w_i^{\mathrm{L}}$, the total demand $Q_{ji}(\ell)$ in place i for a variety ℓ produced in place j is given as

$$Q_{ji}(\ell) = \mu \frac{\rho_i^{\sigma-1}}{p_{ji}(\ell)^{\sigma}} (w_i \lambda_i + w_i^{\mathrm{L}}). \tag{A.3}$$

The A-sector is perfectly competitive and produces homogeneous goods under constant-returns-to-scale, and requires one unit of unskilled labor per unit of output. The A-sector good is traded freely across locations and is chosen as the numéraire. In equilibrium, $p_i^{\mathrm{A}} = w_i^{\mathrm{L}} = 1$ for each i.

The M-sector output is produced under increasing-returns-to-scale and Dixit–Stiglitz monopolistic competition. A firm incurs a fixed input requirement of α units of skilled labor and a marginal input requirement of β units of unskilled labor. An M-sector firm located in place i chooses $(p_{ij}(\ell) \mid j \in N)$ that maximizes its profit

$$\Pi_i(\ell) = \sum_{j \in N} p_{ij}(\ell) Q_{ij}(\ell) - (\alpha w_i + \beta x_i(\ell)), \tag{A.4}$$

where $x_i(\ell)$ denotes the total supply of variety ℓ produced in place i and $\alpha w_i + \beta x_i(\ell)$ signifies the cost function introduced by Flam and Helpman (1987).

With the use of the iceberg form of the transport cost, we have

$$x_i(\ell) = \sum_{j \in N} \tau_{ij} Q_{ij}(\ell). \tag{A.5}$$

Then the profit function of the M-sector firm in place i, given in (A.4) above, can be rewritten as

$$\Pi_i(\ell) = \sum_{j \in N} p_{ij}(\ell) Q_{ij}(\ell) - \left(\alpha w_i + \beta \sum_{j \in N} \tau_{ij} Q_{ij}(\ell) \right),$$

which is maximized by the firm. The first-order condition for this profit maximization yields the following optimal price

$$p_{ij}(\ell) = \frac{\sigma \beta}{\sigma - 1} \tau_{ij}. \tag{A.6}$$

This result implies that $p_{ij}(\ell)$, $Q_{ij}(\ell)$, and $x_i(\ell)$ are independent of ℓ. Therefore, the argument ℓ is suppressed subsequently.

In the short run, skilled workers are immobile between places, i.e., their spatial distribution $\boldsymbol{\lambda} = (\lambda_i \mid i \in N)$ is assumed to be given. The market equilibrium conditions consist of three conditions: the M-sector goods market clearing condition, the zero-profit condition attributable to the free entry and exit of firms, and the skilled labor market clearing condition. The first condition is written as (A.5) above. The second one requires that the operating profit of a firm, given in (A.4), be absorbed entirely by the wage bill of its skilled workers. This gives

$$w_i = \frac{1}{\alpha}\left\{\sum_{j\in N} p_{ij}Q_{ij} - \beta x_i\right\}. \tag{A.7}$$

The third condition is expressed as $\alpha n_i = \lambda_i$ and the price index ρ_i in (A.2) can be rewritten using (A.6) as

$$\rho_i = \frac{\sigma\beta}{\sigma-1}\left(\frac{1}{\alpha}\sum_{j\in N}\lambda_j d_{ji}\right)^{1/(1-\sigma)}. \tag{A.8}$$

The market equilibrium wage w_i in (A.7) can be represented as

$$w_i = \frac{\mu}{\sigma}\sum_{j\in N}\frac{d_{ij}}{\Delta_j}(w_j\lambda_j + 1) \tag{A.9}$$

using $d_{ji} = \tau_{ji}^{1-\sigma} = \phi^{m(i,j)}$, (A.3), (A.5), (A.6), and (A.8). Here, $\Delta_j = \sum_{k\in N} d_{kj}\lambda_k$. Equation (A.9) can be rewritten, using $\boldsymbol{w} = (w_i)$, as $\boldsymbol{w} = \frac{\mu}{\sigma} D\Delta^{-1}(\Lambda\boldsymbol{w} + \mathbf{1})$, which is solved for $\boldsymbol{w}$ as

$$\boldsymbol{w} = \frac{\mu}{\sigma}\left(I - \frac{\mu}{\sigma}D\Delta^{-1}\Lambda\right)^{-1} D\Delta^{-1}\mathbf{1} \tag{A.10}$$

with I being the identity matrix, $\mathbf{1} = (1,\ldots,1)^\top$, and

$$D = (d_{ij}),\quad \Delta = \mathrm{diag}(\Delta_0,\ldots,\Delta_{K-1}),\quad \Lambda = \mathrm{diag}(\lambda_0,\ldots,\lambda_{K-1}). \tag{A.11}$$

Appendix B. Theoretical details of Section 3

Appendix B.1. Proof of Proposition 1

For $\boldsymbol{\lambda} = \boldsymbol{\lambda}_\delta^{\mathrm{FA}}$, we have $\lambda_i = 0$ $(i \neq k-\delta)$ and $v_{k-\delta} - \bar{v} = 0$ since $\bar{v} = v_{k-\delta}$; accordingly, the governing equation (6) with (5) is satisfied for any $i \in N$. For $\boldsymbol{\lambda}_\delta^{\mathrm{Twin}}$, we have $v_{k-\delta} = v_{k+\delta}$ by symmetry, $v_{k-\delta} - \bar{v} = v_{k-\delta} - \frac{1}{2}v_{k-\delta} - \frac{1}{2}v_{k+\delta} = 0$ and similarly $v_{k+\delta} - \bar{v} = 0$. We also have $\lambda_i = 0$ $(i \neq k \pm \delta)$; accordingly, the governing equation is satisfied.

Appendix B.2. Proof of Proposition 2

We can derive a two-dimensional bifurcation equation in incremental variables $(x, y, \psi) = (\lambda_{k-\delta}, \lambda_{k+\delta}, \psi)$ at the critical point $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi_\delta^{\mathrm{c}})$, using $\psi = \phi - \phi_\delta^{\mathrm{c}}$, as follows.

Lemma 5. *The bifurcation equation at the critical point* $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi_\delta^{\mathrm{c}})$ *is expressed as*

$$\begin{aligned} \tilde{F}_{k-\delta}(x, y, \psi) &= x\,(a\psi + bx + cy + \text{higher order terms}) = 0, \\ \tilde{F}_{k+\delta}(x, y, \psi) &= y\,(a\psi + by + cx + \text{higher order terms}) = 0 \end{aligned} \tag{B.1}$$

with the symmetry condition $\tilde{F}_{k+\delta}(x, y, \psi) = \tilde{F}_{k-\delta}(y, x, \psi)$ *and expansion coefficients:*

$$(a, b, c) = \left(\frac{\partial g}{\partial \phi}, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)\Bigg|_{(x,y,\psi)=(0,0,\phi_\delta^{\mathrm{c}})}, \qquad g(x, y, \psi) = v_{k-\delta}(\tilde{\boldsymbol{\lambda}}) - v_k(\tilde{\boldsymbol{\lambda}});$$

$$\tilde{\boldsymbol{\lambda}} = (\mathbf{0}_{k-\delta-1}, x, \mathbf{0}_\delta, 1 - x - y, \mathbf{0}_\delta, y, \mathbf{0}_{k-\delta-1}, \phi_\delta^{\mathrm{c}} + \psi), \qquad \mathbf{0}_p = (\underbrace{0, \ldots, 0}_{p \text{ times}}).$$

Proof. In the neighborhood of the critical point $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi_\delta^{\mathrm{c}})$, $\boldsymbol{F}(\boldsymbol{\lambda}, \tau) = \mathbf{0}$ in (6) reduces to three equations $F_j = 0$ with three variables v_j $(j = k, k \pm \delta)$, while the other variables are equal to 0. Then $F_{k-\delta} + F_k + F_{k+\delta} = 0$ gives the conservation law: $\lambda_{k-\delta} + \lambda_k + \lambda_{k+\delta} = 0$. The variable λ_k can be eliminated from $F_{k-\delta}$ and $F_{k+\delta}$ to arrive at (B.1). The symmetry condition arises from the bilateral symmetry of the long narrow economy. □

The bifurcation equation (B.1) with the symmetry condition has solutions $(x, y) = (\lambda_{k-\delta}, \lambda_{k+\delta}) = (0, 0)$, $(w, 0)$, $(0, w)$, and (w, w); $(x, y) = (0, 0)$ corresponds to the pre-bifurcation solution $(\boldsymbol{\lambda}^{\text{FA}}, \phi)$ and others to bifurcating solutions. Since the solutions $(w, 0)$ and $(0, w)$ are identical, we hereafter consider only the former solution.

Lemma 6. *The critical point $(\boldsymbol{\lambda}^{\text{FA}}, \phi^{\text{c}}_{\delta})$ is a bifurcation point with two kinds of branches:*

$$(\boldsymbol{\lambda}, \phi) = (\boldsymbol{\lambda}^{\text{FA}}, \phi^{\text{c}}_{\delta}) + (\Delta\boldsymbol{\lambda}_p, \psi_p), \quad p = 1, 2;$$

$$\Delta\boldsymbol{\lambda}_1 = w(\boldsymbol{e}^1_{\delta}, -2, \boldsymbol{e}^2_{\delta}), \quad \psi_1 \approx -(b + c)w/a; \quad \boldsymbol{e}^1_{\delta} = (\mathbf{0}_{k-\delta-1}, 1, \mathbf{0}_{\delta}), \quad 0 < w \ll 1, \tag{B.2}$$

$$\Delta\boldsymbol{\lambda}_2 = w(\boldsymbol{e}^1_{\delta}, -1, \mathbf{0}_k), \quad \psi_2 \approx -bw/a; \qquad \boldsymbol{e}^2_{\delta} = (\mathbf{0}_{\delta}, 1, \mathbf{0}_{k-\delta-1}). \tag{B.3}$$

Proof. We see that $(x, y) = (\lambda_{k-\delta}, \lambda_{k+\delta}) = (w, w)$ corresponds to $\Delta\boldsymbol{\lambda}_1 = w(\boldsymbol{e}^1_{\delta}, -2, \boldsymbol{e}^2_{\delta})$ and satisfies (B.1) in Lemma 5 for $\psi = \psi_1 \approx -(b + c)w/a$. Also, $(x, y) = (w, 0)$ corresponds to $\Delta\boldsymbol{\lambda}_2 = w(\boldsymbol{e}^1_{\delta}, -1, \mathbf{0}_k)$ and satisfies (B.1) for $\psi = \psi_2 \approx -bw/a$. □

The Jacobian matrix for the bifurcation equation (B.1) reads

$$\hat{J} \approx \begin{pmatrix} a\psi + 2bx + cy & cx \\ cy & a\psi + 2by + cx \end{pmatrix}.$$

The use of $(x, y) = w(1, 1)$ and $\psi = \psi_1 \approx -(b + c)w/a$ (cf., (B.2)) in $\hat{J}$ leads to $\hat{J}_1$ and the use of $(x, y) = w(1, 0)$ and $\psi = \psi_2 \approx -bw/a$ (cf, (B.3)) leads to $\hat{J}_2$ as follows:

$$\hat{J}_1 \approx w\begin{pmatrix} b & c \\ c & b \end{pmatrix}, \qquad \hat{J}_2 \approx w\begin{pmatrix} b & c \\ 0 & c - b \end{pmatrix}.$$

Lemma 7. *The bifurcating solution $(\Delta\boldsymbol{\lambda}_1, \psi_1)$ has the eigenvalues: $e_1 \approx (b+c)w$ and $e_2 \approx (b - c)w$. On the other hand, $(\Delta\boldsymbol{\lambda}_2, \psi_2)$ has the eigenvalues: $e_1 \approx bw$ and $e_2 \approx (c - b)w$.*

Lemma 8. *Under Assumption 1, there are three cases: (i) If $-b > |c|$, only the first bifurcating path $(\Delta\boldsymbol{\lambda}_1, \psi_1)$ is stable and sustainable. (ii) If $c < b < 0$, only the second* *bifurcating path* $(\Delta\boldsymbol{\lambda}_2, \psi_2)$ *is stable and sustainable. (iii) Otherwise, both paths are unstable and/or unsustainable. A stable and sustainable bifurcating path branches in the direction of* $\psi < 0$.

Proof. For the fully agglomerated state $(x, y) = (0, 0)$, we have $\hat{J} = a\psi I$ with the eigenvalue $a\psi$ (twice repeated). Since, by Assumption 1, this state is sustainable for $\psi > 0$, we have $a < 0$. (i) The first bifurcating solution $(\Delta\boldsymbol{\lambda}_1, \psi_1)$ with $e_1 \approx (b + c)w$ and $e_2 \approx (b - c)w$ (cf., Lemma 7) is stable if $-b > |c|$. Since $b + c < 0$, $a < 0$, and $w > 0$, $\psi = \psi_1 \approx -(b + c)w/a$ in (B.2) gives $\psi = \psi_1 < 0$. (ii) The second bifurcating solution $(\Delta\boldsymbol{\lambda}_2, \psi_2)$ with $e_1 \approx bw$ and $e_2 \approx (c - b)w$ $(w > 0)$ is stable if $c < b < 0$. Since $b < 0$, $a < 0$ and $w > 0$, $\psi = \psi_2 \approx -bw/a$ in (B.3) gives $\psi = \psi_2 < 0$. The two bifurcating solutions cannot be stable simultaneously since $-b > |c|$ and $c < b < 0$ are contradictory. □

Appendix B.3. Proof of Proposition 4

Let a corner bifurcation point $\phi^{\mathrm{c}}_{\delta}$ not be the sustain point. Then there exists δ' $(\delta' \neq \delta)$ such that $v_{k-\delta'} - v_k > 0$ at this point. By continuity of $v_{k-\delta'}$ and v_k as functions in ϕ, $v_{k-\delta'} - v_k > 0$ is satisfied in a neighborhood of $(\boldsymbol{\lambda}^{\mathrm{FA}}, \phi^{\mathrm{c}}_{\delta})$. Therefore, the bifurcation solution is unsustainable just after bifurcation.

Appendix B.4. Proof of Proposition 5

(i) The critical eigenvector of the Jacobian matrix J associated with the bifurcation point is given by $(-\boldsymbol{e}^1_{\delta}, 2, -\boldsymbol{e}^2_{\delta})$ (see (B.2) and (B.3) for the notations), which has three nonzero components $i = k,\ k \pm \delta$. This suffices for the proof. (ii) This is a so-called transcritical bifurcation point and its stability is studied in the literature (e.g., Ikeda and Murota, 2019, Section 2.5.2).

Appendix C. Theoretical details of Section 4

Appendix C.1. Proof of Lemma 2

For the full agglomeration $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{\mathrm{FA}}$, we rearrange the components of the variable $\boldsymbol{\lambda}$ using the permutation of place numbers:

$$\begin{pmatrix} 0 & \cdots & k-1 & k & k+1 & \cdots & 2k \\ k & \cdots & 1 & 0 & k+1 & \cdots & 2k \end{pmatrix}.$$

Then the variables used to define $\boldsymbol{w}$ in (A.10) are expressed using $\boldsymbol{d} = (\phi, \phi^2, \dots, \phi^k)$ as

$$\Lambda = \mathrm{diag}(1, \mathbf{0}_{2k}), \qquad \Delta_i = \sum_{j=0}^{2k} d_{ji}\lambda_j = d_{0i}, \qquad (\Delta_0, \Delta_1, \dots, \Delta_{2k}) = (1, \boldsymbol{d}, \boldsymbol{d}), \tag{C.1}$$

$$\Delta = \mathrm{diag}(\Delta_0, \dots, \Delta_{2k}) = \mathrm{diag}\,(1, \boldsymbol{d}, \boldsymbol{d}), \qquad \Delta^{-1} = \mathrm{diag}(1, \Theta, \Theta), \qquad \Theta = \mathrm{diag}(\boldsymbol{d})^{-1},$$

$$D = \left(\begin{array}{c|c|c} 1 & \boldsymbol{d} & \boldsymbol{d} \\ \hline \boldsymbol{d}^\top & D_1 & D_2 \\ \hline \boldsymbol{d}^\top & D_2 & D_1 \end{array}\right), \quad D\Delta^{-1} = \left(\begin{array}{c|c|c} 1 & \boldsymbol{d}\Theta & \boldsymbol{d}\Theta \\ \hline \boldsymbol{d}^\top & D_1\Theta & D_2\Theta \\ \hline \boldsymbol{d}^\top & D_2\Theta & D_1\Theta \end{array}\right), \quad D\Delta^{-1}\Lambda = \left(\begin{array}{c|c|c} 1 & O & O \\ \hline \boldsymbol{d}^\top & O & O \\ \hline \boldsymbol{d}^\top & O & O \end{array}\right),$$

$$D\Delta^{-1}\mathbf{1} = (2k+1, \boldsymbol{g}, \boldsymbol{g})^\top\,;\; D_1 = \{\phi^{|j-i|} \mid 1 \le i, j \le k\},\; D_2 = \{\phi^{i+j} \mid 1 \le i, j \le k\}, \tag{C.2}$$

$$\boldsymbol{d}\Theta = \mathbf{1}^\top; \qquad D_1\Theta = \{\phi^{|j-i|-j} \mid 1 \le i, j \le k\}, \qquad D_2\Theta = \{\phi^i \mid 1 \le i, j \le k\};$$

$$\boldsymbol{g} = \{g_i \mid 1 \le i \le k\}, \qquad g_i = \phi^i + \sum_{j=1}^{k} (\phi^i + \phi^{|i-j|-j}) = (k+1)\phi^i + (k-i)\phi^{-i} + \sum_{p=1}^{i} \phi^{i-2p}.$$

Hence we have (I_k being $k \times k$ identity matrix)

$$(I - \theta D\Delta^{-1}\Lambda)^{-1} = \left(\begin{array}{c|c|c} 1-\theta & & \\ \hline -\theta\boldsymbol{d}^\top & I_k & \\ \hline -\theta\boldsymbol{d}^\top & & I_k \end{array}\right)^{-1} = \frac{1}{1-\theta}\left(\begin{array}{c|c|c} 1 & & \\ \hline \theta\boldsymbol{d}^\top & (1-\theta)I_k & \\ \hline \theta\boldsymbol{d}^\top & & (1-\theta)I_k \end{array}\right),$$

$$(I - \theta D\Delta^{-1}\Lambda)^{-1} D\Delta^{-1}\mathbf{1} = \frac{\theta}{1-\theta}(2k+1, \boldsymbol{z}, \boldsymbol{z})^\top, \qquad \boldsymbol{z} = \theta(2k+1)\boldsymbol{d} + (1-\theta)\boldsymbol{g}.$$

The use of (C.2) and this equation in (A.10) leads to the expressions of the wage as

$$w_0 = \frac{\theta(2k+1)}{1-\theta}, \quad w_i = w_{i+k} = \frac{\theta}{1-\theta}\left\{(\theta k + k + 1)\phi^i + (1-\theta)\left[(k-i)\phi^{-i} + \sum_{p=1}^{i} \phi^{i-2p}\right]\right\}$$

$(1 \le i \le k)$. In the original place numbers $i \mapsto k - i = \delta$, these equations are rewritten as

$$w_k = \frac{\theta(2k+1)}{1-\theta}, \quad w_{k\pm\delta} = \frac{\theta}{1-\theta}\left\{(\theta k + k + 1)\phi^{\delta} + (1-\theta)\left[(k-\delta)\phi^{-\delta} + S_{\delta}\right]\right\} \quad (1 \le \delta \le k)$$

with $S_{\delta} = \sum_{p=1}^{\delta} \phi^{\delta-2p}$. The use of (C.1) and these expressions in (3) proves (9) and (10).

Appendix C.2. Proof of Lemma 3

To prove $v_i > v_{i-1}$ $(1 \le i \le k)$ for $\phi \to 1$, we put $\phi = 1 - \epsilon$ $(0 < \epsilon \ll 1)$ and consider a limit of $\epsilon \to +0$. Then $v_i = v_{k-\delta}$ in (10) $(0 \le i \le k-1)$ can be expanded as

$$\begin{aligned} v_i &= \ln\frac{\theta}{1-\theta} + \frac{\delta(i)\mu}{\sigma-1}\ln\phi + \ln\hat{v}_i \\ &= \ln\frac{\theta}{1-\theta} + \left(-\frac{\delta(i)\mu}{\sigma-1}\epsilon + \text{h.o.t.}\right) + \left((\ln\hat{v}_i)|_{\epsilon=0} + \left.\frac{\partial(\ln\hat{v}_i)}{\partial\epsilon}\right|_{\epsilon=0}\epsilon + \text{h.o.t.}\right) \\ &= \ln\frac{\theta}{1-\theta} - \frac{\delta(i)\mu}{\sigma-1}\epsilon + \ln(2k+1) + \left.\frac{\partial(\ln\hat{v}_i)}{\partial\epsilon}\right|_{\epsilon=0}\epsilon + \text{h.o.t.} \end{aligned}$$

with $\delta = \delta(i) = k - i$ $(1 \le \delta \le k)$ and

$$\hat{v}_i = (\theta k + k + 1)(1-\epsilon)^{\delta} + (1-\theta)\left[(k-\delta)(1-\epsilon)^{-\delta} + \sum_{p=1}^{\delta}(1-\epsilon)^{\delta-2p}\right], \tag{C.3}$$

$$\begin{aligned} \frac{\partial(\ln\hat{v}_i)}{\partial\epsilon} &= -\frac{1}{\hat{v}_i}\{\delta(\theta k + k + 1)(1-\epsilon)^{\delta-1} + (1-\theta)[\delta(\delta-k)(1-\epsilon)^{-(\delta+1)} + \hat{S}]\}, \\ \left.\frac{\partial(\ln\hat{v}_i)}{\partial\epsilon}\right|_{\epsilon=0} &= -\left(\theta\delta + \frac{(1-\theta)\delta^2}{2k+1}\right); \quad \hat{S} = \sum_{p=1}^{\delta}(\delta-2p)(1-\epsilon)^{\delta-2p-1}. \end{aligned}$$

We can express v_i $(i \ne k)$ asymptotically as

$$\begin{aligned} v_i &\approx \ln\frac{\theta}{1-\theta} + \ln\,(2k+1) - \left[\left(\theta + \frac{\mu}{\sigma-1}\right)\delta + \frac{1-\theta}{2k+1}\delta^2\right]\epsilon \\ &= v_k - \left[\left(\theta + \frac{\mu}{\sigma-1}\right)(k-i) + \frac{1-\theta}{2k+1}(k-i)^2\right]\epsilon. \end{aligned} \tag{C.4}$$

We have $v_k > v_i$ $(i \ne k)$ because $k - i > 0$ and $0 < \theta < 1$. Furthermore,

$$v_i - v_{i-1} \approx \left[\theta + \frac{\mu}{\sigma-1} + \frac{(1-\theta)(2(k-i)+1)}{2k+1}\right]\epsilon > 0 \qquad (1 \le i \le k-1). \tag{C.5}$$

Hence we have $v_k > v_{k-1} > \cdots > v_0$ $(\phi \to 1)$.

Appendix C.3. Proof of Lemma 4

Using $\sum_{p=1}^{\delta} \phi^{\delta-2p} = \frac{\phi^{\delta}-\phi^{-\delta}}{\phi^2-1}$, we rewrite $\hat{v}_i$ in (C.3) with $1-\epsilon=\phi$ and evaluate $\ln \hat{v}_i$ as

$$\begin{aligned} \hat{v}_i &= A\phi^{\delta} + B\left(i\phi^{-\delta} + \frac{\phi^{\delta}-\phi^{-\delta}}{\phi^2-1}\right); \quad A \equiv \theta k + k + 1 > 0, \quad B \equiv 1-\theta > 0. \quad \text{(C.6)} \\ \ln \hat{v}_i &= -\delta \ln \phi + \ln \left[\left(A + \frac{B}{\phi^2-1}\right)\phi^{2\delta} + B\left(i + \frac{1}{1-\phi^2}\right)\right]. \end{aligned}$$

Using this equation, we can rewrite $v_i = v_{k-\delta}$ in (10) and evaluate its limit behavior as

$$\begin{aligned} v_i &= \ln \frac{\theta}{1-\theta} + \delta\left(\frac{\mu}{\sigma-1} - 1\right)\ln\phi + \ln\left[\left(A + \frac{B}{\phi^2-1}\right)\phi^{2\delta} + B\left(i + \frac{1}{1-\phi^2}\right)\right], \quad \text{(C.7)} \\ \lim_{\phi\to+0} v_i &= \ln\frac{\theta}{1-\theta} + \delta\left(\frac{\mu}{\sigma-1} - 1\right)\left(\lim_{\phi\to+0} \ln\phi\right) + \ln(1-\theta)(i+1). \end{aligned}$$

Since $\lim_{\phi\to+0}(\ln\phi) = -\infty$, the limit behavior of v_i depends on the magnitude relation between $\frac{\mu}{\sigma-1}$ and 1; accordingly, $\lim_{\phi\to+0} v_i$ is given as

$$\lim_{\phi\to+0} v_i = \begin{cases} +\infty, & (\mu < \sigma-1), \\ -\infty, & (\mu > \sigma-1). \end{cases} \quad \text{(C.8)}$$

We consider the case $\mu < \sigma - 1$ and prove $v_k < v_{k-1} < \cdots < v_0$ $(\phi \to +0)$ by showing $v_k < v_i$ $(i \neq k)$ and $v_i < v_{i-1}$ $(1 \leq i \leq k-1)$. First, since v_k is constant and $\lim_{\phi\to+0} v_i = +\infty$ in (C.8) $(i \neq k)$, we have $v_k < v_i$ $(\phi \to +0, i \neq k)$. Next, using (C.7) with $\delta = k - i$ and $B = 1-\theta$, we have

$$v_{i-1} - v_i = (\rho-1)\ln\phi + V(\phi); \quad V(\phi) = \ln\left(\frac{\left(A + \frac{1-\theta}{\phi^2-1}\right)\phi^{2(k-i+1)} + (1-\theta)\left(i - 1 - \frac{1}{\phi^2-1}\right)}{\left(A + \frac{1-\theta}{\phi^2-1}\right)\phi^{2(k-i)} + (1-\theta)\left(i - \frac{1}{\phi^2-1}\right)}\right)$$

$(\rho = \frac{\mu}{\sigma-1} < 1)$. Since $\lim_{\phi\to+0} V(\phi) = \ln\left(\frac{i}{i+1}\right)$ and $\lim_{\phi\to+0}[(\rho-1)\ln\phi] = +\infty$, we have $\lim_{\phi\to+0}(v_{i-1} - v_i) = +\infty$. This shows $v_i < v_{i-1}$ $(\phi \to +0)$.

Appendix C.4. Proof of Proposition 8

By Lemmas 3 and 4, $v_{k-\delta} - v_k > 0$ as $\phi \to +0$ and $v_{k-\delta} - v_k < 0$ as $\phi \to 1$ for each $\delta \in N_{\delta}$. By the Intermediate Value Theorem, there is a critical point satisfying

$v_{k-\delta} - v_k = 0$ for each $\delta \in N_\delta$. By Proposition 2, this critical point is a bifurcation point, at which two satellite cities emerges at the $(k \pm \delta)$th cities or a satellite city emerge at the $(k - \delta)$th city.

Appendix C.5. Proof of Proposition 10

By Proposition 8, ϕ^{c}_δ exists for each δ and, accordingly, ϕ^{s} $(= \max_{\delta \in N^\delta} \phi^{\mathrm{c}}_\delta)$ can be defined. Hence $v_{k-\delta} - v_k$ does not change its sign in $\phi \in (\phi^{\mathrm{s}}, 1)$. By Lemma 3, $v_{k-\delta} - v_k < 0$ as $\phi \to 1$ for any δ. Accordingly, $v_{k-\delta} - v_k < 0$ in $\phi \in (\phi^{\mathrm{s}}, 1)$ for any δ and the full agglomeration is sustainable for $\phi > \phi^{\mathrm{s}}$.

Appendix C.6. Proof of Corollary 1

For $K = 3$, there is only one bifurcation point, and it is necessarily the sustain point.

Appendix C.7. Proof of Proposition 11

We consider a sustain point $(\phi^{\mathrm{s}}, \sigma, \mu)$, at which $v_i = v_k$ with $v_i = \max(v_0, \dots, v_{k-1})$ is satisfied. We put $g(\phi, \sigma, \mu) \equiv v_i - v_k$ $(i = k - \delta)$ and employ (9) and (10) to obtain

$$g(\phi, \sigma, \mu) = \frac{\delta\mu}{\sigma - 1} \ln \phi + \ln X - \ln(2k + 1)$$

with $X = \left(\frac{\mu}{\sigma}k + k + 1\right)\phi^\delta + \left(1 - \frac{\mu}{\sigma}\right)\left[(k - \delta)\phi^{-\delta} + \sum_{p=1}^{\delta} \phi^{\delta - 2p}\right] > 0$.

At the sustain point, we have $g(\phi^{\mathrm{s}}, \sigma, \mu) = 0$ and, at a perturbed sustain point, we have $g(\phi^{\mathrm{s}} + \mathrm{d}\phi^{\mathrm{s}}, \sigma + \mathrm{d}\sigma, \mu + \mathrm{d}\mu) = 0$. The total derivative of this equation is given by

$$\mathrm{d}g = \frac{\partial g}{\partial \phi}(\phi^{\mathrm{s}}, \sigma, \mu)\,\mathrm{d}\phi^{\mathrm{s}} + \frac{\partial g}{\partial \sigma}(\phi^{\mathrm{s}}, \sigma, \mu)\,\mathrm{d}\sigma + \frac{\partial g}{\partial \mu}(\phi^{\mathrm{s}}, \sigma, \mu)\,\mathrm{d}\mu = 0. \tag{C.9}$$

Concrete forms of partial derivatives of $g(\phi, \sigma, \mu)$ to be used later are

$$\frac{\partial g}{\partial \sigma} = -\frac{\delta\mu}{(\sigma - 1)^2}\ln\phi + \frac{1}{X}\frac{\partial X}{\partial \sigma}, \qquad \frac{\partial g}{\partial \mu} = \frac{\delta}{\sigma - 1}\ln\phi + \frac{1}{X}\frac{\partial X}{\partial \mu}; \tag{C.10}$$

$$\frac{\partial X}{\partial \sigma} = -\frac{1}{\sigma^2}E, \quad \frac{\partial X}{\partial \mu} = \frac{1}{\sigma}E; \qquad E = k\phi^\delta - (k - \delta)\phi^{-\delta} - \sum_{p=1}^{\delta} \phi^{\delta - 2p}. \tag{C.11}$$

First, we investigate the sign of $\mathrm{d}\phi^{\mathrm{s}}/\mathrm{d}\sigma(\phi^{\mathrm{s}}, \sigma, \mu)$ for a constant μ. We generically have $\frac{\partial g}{\partial \phi^{\mathrm{s}}}(\phi^{\mathrm{s}}, \sigma, \mu) < 0$ because, by the definition of the sustain point, we have $g(\phi^{\mathrm{s}}, \sigma, \mu) = 0$ and $g(\phi^{\mathrm{s}} + \mathrm{d}\phi^{\mathrm{s}}, \sigma, \mu) < 0$ $(0 < \mathrm{d}\phi^{\mathrm{s}} \ll 1)$. We also have $\frac{\partial g}{\partial \sigma}(\phi^{\mathrm{s}}, \sigma, \mu) > 0$ because, in (C.10), we have $\ln \phi < 0$, $X > 0$, and $\partial X/\partial \sigma|_{\phi=\phi^{\mathrm{s}}} > 0$ from (C.11) with $E|_{\phi=\phi^{\mathrm{s}}} < 0$:

$$E|_{\phi=\phi^{\mathrm{s}}} = k(\phi^{\mathrm{s}})^{\delta} - (k-\delta)(\phi^{\mathrm{s}})^{-\delta} - \sum_{p=1}^{\delta}(\phi^{\mathrm{s}})^{\delta-2p} < k(\phi^{\mathrm{s}})^{\delta} - (k-\delta)(\phi^{\mathrm{s}})^{\delta} - \sum_{p=1}^{\delta}(\phi^{\mathrm{s}})^{\delta} = 0.$$

Then by setting $\mathrm{d}\mu = 0$ in (C.9), we obtain $\frac{\mathrm{d}\phi^{\mathrm{s}}}{\mathrm{d}\sigma}(\phi^{\mathrm{s}}, \sigma, \mu) = -\frac{\partial g}{\partial \sigma}(\phi^{\mathrm{s}}, \sigma, \mu) \,/\, \frac{\partial g}{\partial \phi^{\mathrm{s}}}(\phi^{\mathrm{s}}, \sigma, \mu) > 0$.

Next, we investigate the sign of $\mathrm{d}\phi^{\mathrm{s}}/\mathrm{d}\mu(\phi^{\mathrm{s}}, \sigma, \mu)$ for a constant σ. By setting $\mathrm{d}\sigma = 0$ in (C.9), we obtain a relation $\frac{\mathrm{d}\phi^{\mathrm{s}}}{\mathrm{d}\mu}(\phi^{\mathrm{s}}, \sigma, \mu) = -\frac{\partial g}{\partial \mu}(\phi^{\mathrm{s}}, \sigma, \mu) \,/\, \frac{\partial g}{\partial \phi^{\mathrm{s}}}(\phi^{\mathrm{s}}, \sigma, \mu) < 0$ as we already know $\frac{\partial g}{\partial \phi^{\mathrm{s}}}(\phi^{\mathrm{s}}, \sigma, \mu) < 0$ and have $\frac{\partial g}{\partial \mu}(\phi^{\mathrm{s}}, \sigma, \mu) < 0$ in (C.10) ($\frac{\partial X}{\partial \mu}|_{\phi=\phi^{\mathrm{s}}} = \frac{1}{\sigma} E|_{\phi=\phi^{\mathrm{s}}} < 0$).

Appendix D. Uniqueness of bifurcation points (Proof of Proposition 9)

In preparation for the discussion regarding the uniqueness of bifurcation points in subsequent proofs, we have the following essential limits:

$$\lim_{\phi\to 1}(v_i - v_k) = 0, \qquad \lim_{\phi\to 0}(v_i - v_k) = +\infty, \tag{D.1}$$

$$\lim_{\phi\to 1}\frac{\partial(v_i - v_k)}{\partial \phi} = \delta\left[\frac{\delta(\sigma-\mu)}{2k\sigma+\sigma} + \mu\left(\frac{1}{\sigma} + \frac{1}{\sigma-1}\right)\right] > 0, \tag{D.2}$$

where (D.1) is apparent from (C.4) and Lemma 4 with (C.8) and (D.2) is given by a straightforward calculation. In the following discussion, the sign of $\frac{\partial^2(v_i-v_k)}{\partial\phi^2}$ plays an important role. We express $\frac{\partial^2(v_i-v_k)}{\partial\phi^2}$ such that its denominator is positive; accordingly, its sign is given by the sign of its numerator, being defined as $P_\delta(\phi)$ for a given δ. Then, we have the following important result.

Lemma 9. *If $\frac{\partial^2(v_i-v_k)}{\partial\phi^2}$ has at most one root for $\phi > 0$, there is a unique bifurcation point satisfying $v_i - v_k = 0$ for $\phi \in (0, 1)$.*

Proof. We have: $P_\delta(0) = -\delta(k+1-\delta)^2(\mu-\sigma)^2(\mu-\sigma+1) > 0$. This means that $v_i - v_k$ is convex for $\phi = 0$. If $P_\delta(\phi)$ has at most one root for $\phi > 0$, then $v_i - v_k$ may become concave for some $\phi > 0$. This implies that $v_i - v_k$ may have either one zero or three zeros for $\phi \in (0, 1)$. However, the limits in (D.1) and (D.2) rule out the latter case and establish that there exists exactly one root of $v_i - v_k = 0$ for $\phi \in (0, 1)$. Therefore, there exists a unique bifurcation point satisfying $v_i - v_k = 0$ for $\phi \in (0, 1)$. □

We would like to show the following lemma for $\delta \in \{1, 2, \dots, 6\}$. Then by Lemma 9 and Descartes' rule of sings, Proposition 9 can be proven in a straightforward manner.

Lemma 10. *$P_\delta(\phi)$ takes a polynomial form of*

$$P_\delta(\phi) = a_1\phi^{4\delta} + a_2\phi^{4\delta-2} + \cdots + a_{2\delta}\phi^2 + a_{2\delta+1} \qquad (\delta = 1, 2, \dots, 6)$$

and the sign of a series of coefficients a_1, a_2, $\dots$, $a_{2\delta+1}$ changes once for $\mu < \sigma - 1$.

Proof. The sign of the series of coefficients changes once for each $\delta = 1, 2, \dots, 6$ as expressed by the explicit forms of these coefficients listed below:

For $\delta = 1$ and for any $k \geq 1$, we have $P_1(\phi) = a_1\phi^4 + a_2\phi^2 + a_3$ with

$$a_1 = -(\mu+\sigma-1)\left[k(\mu+\sigma)+\sigma\right]^2 < 0,$$

$$a_2 = 2k(\mu-2\sigma+2)(\mu-\sigma)\left[k(\mu+\sigma)+\sigma\right] > 0, \quad a_3 = -k^2(\mu-\sigma)^2(\mu-\sigma+1) > 0.$$

For $\delta = 2$ and for any $k \geq 2$, we have $P_2(\phi) = a_1\phi^8 + \cdots + a_4\phi^2 + a_5$ with

$$a_1 = -(\mu+\sigma-1)\left[k(\mu+\sigma)+\sigma\right]^2 < 0, \quad a_2 = (\mu-\sigma)(2\mu-\sigma+1)\left[k(\mu+\sigma)+\sigma\right],$$

$$a_3 = (\mu-\sigma)\Big[2k\{k(\mu+\sigma)-\mu\}(\mu-4\sigma+4) - \mu^2 - \mu\sigma + 8(\sigma-1)\sigma\Big] > 0,$$

$$a_4 = -(k-1)(2\mu-3\sigma+3)(\mu-\sigma)^2 > 0, \ \ a_5 = -(k-1)^2(\mu-\sigma)^2(\mu-\sigma+1) > 0.$$

Note that the sign of the series of coefficients changes once, irrespective of the sign of a_2. Such is also the cases for $\delta = 5, 6$ below.

For $\delta = 3$ and for any $k \geq 3$, we have $P_3(\phi) = a_1\phi^{12} + \cdots + a_6\phi^2 + a_7$ with

$$a_1 = -3(\mu + \sigma - 1)(k(\mu + \sigma) + \sigma)^2 < 0, \quad a_2 = 6\mu(\mu - \sigma)(k(\mu + \sigma) + \sigma) < 0,$$

$$a_3 = (\mu - \sigma)\left((6k - 3)\mu^2 + k\mu(14 - 8\sigma) - (14k + 13)(\sigma - 1)\sigma + 8\mu\sigma + \mu\right) > 0,$$

$$a_4 = 2(\mu - \sigma)\Big[3((k - 2)k - 1)\mu^2 - \mu(3k(k(5\sigma - 6) - 11\sigma + 12) + \sigma + 2)$$
$$- 2(9(k - 1)k - 17)(\sigma - 1)\sigma] > 0,$$

$$a_5 = -(\mu - \sigma)^2(6k(\mu - 3\sigma + 3) - 9\mu + 35(\sigma - 1)) > 0$$

$$a_6 = -2(k - 2)\phi^2(3\mu - 4\sigma + 4)(\mu - \sigma)^2 > 0, \quad a_7 = -3(k - 2)^2(\mu - \sigma)^2(\mu - \sigma + 1) > 0.$$

For $\delta = 4$ and for any $k \geq 4$, we have $P_4(\phi) = a_1\phi^{16} + \cdots + a_8\phi^2 + a_9$ with

$$a_1 = -2(\mu + \sigma - 1)\left[k(\mu + \sigma) + \sigma\right]^2 < 0, \quad a_2 = (\mu - \sigma)(4\mu + \sigma - 1)\left[k(\mu + \sigma) + \sigma\right] < 0,$$

$$a_3 = (\mu - \sigma)\left[(4k - 2)\mu^2 - 2k\mu(\sigma - 3) - (6k + 5)(\sigma - 1)\sigma + 5\mu\sigma + \mu\right] > 0,$$

$$a_4 = (\mu - \sigma)\left\{4(k - 1)\mu^2 + \mu\left[k(17 - 13\sigma) + 9\sigma - 1\right] - (17k + 18)(\sigma - 1)\sigma\right\} > 0,$$

$$a_5 = 2(\mu - \sigma)\Big(\left[2(k - 3)k - 3\right]\mu^2 + \mu\left\{2k\left[k(8 - 7\sigma) + 22\sigma - 24\right] + \sigma - 4\right\}$$
$$-4\left[4(k - 2)k - 11\right](\sigma - 1)\sigma) > 0,$$

$$a_6 = -(\mu - \sigma)^2\left\{k\left[4\mu - 19\sigma + 19\right] - 8\mu + 54(\sigma - 1)\right\} > 0,$$

$$a_7 = -(\mu - \sigma)^2\left[2k(2\mu - 5\sigma + 5) - 10\mu + 29(\sigma - 1)\right] > 0,$$

$$a_8 = -(k - 3)(4\mu - 5\sigma + 5)(\mu - \sigma)^2 > 0, \quad a_9 = -2(k - 3)^2(\mu - \sigma)^2(\mu - \sigma + 1) > 0.$$

For $\delta = 5$ and for any $k \geq 5$, we have $P_5(\phi) = a_1\phi^{20} + \cdots + a_{10}\phi^2 + a_{11}$ with

$$a_1 = -5(\mu + \sigma - 1)\left[k(\mu + \sigma) + \sigma\right]^2 < 0, \quad a_2 = 2(\mu - \sigma)(5\mu + 2\sigma - 2)\left[k(\mu + \sigma) + \sigma\right] < 0,$$

$$a_3 = (\mu - \sigma)\left[5(2k - 1)\mu^2 + \mu(10k + 12\sigma + 3) - (10k + 7)(\sigma - 1)\sigma\right],$$

$$a_4 = 2(\mu - \sigma)\left[k(\mu + \sigma)(5\mu - 16\sigma + 16) - 5\mu^2 + 10\mu\sigma - 16(\sigma - 1)\sigma\right] > 0,$$

$$a_5 = (\mu - \sigma)\left\{5(2k - 3)\mu^2 + \mu\,[k(62 - 52\sigma) + 38\sigma - 13] - (62k + 75)(\sigma - 1)\sigma\right\} > 0,$$

$$a_6 = 10(\mu - \sigma)\Big([(k - 4)k - 2]\,\mu^2 + \mu\,\{k\,[k(10 - 9\sigma) + 37\sigma - 40] + 2(\sigma - 2)\}$$

$$-2(5(k - 3)k - 18)(\sigma - 1)\sigma) > 0,$$

$$a_7 = (\mu - \sigma)^2\,[-2k(5\mu - 33\sigma + 33) + 5(5\mu - 49\sigma + 49)] > 0,$$

$$a_8 = -\,2(\mu - \sigma)^2\,[5k(\mu - 4\sigma + 4) - 15\mu + 76(\sigma - 1)] > 0,$$

$$a_9 = -\,(\mu - \sigma)^2\,[2k(5\mu - 11\sigma + 11) - 35\mu + 85(\sigma - 1)] > 0,$$

$$a_{10} = -\,2(k - 4)\phi^2(5\mu - 6\sigma + 6)(\mu - \sigma)^2 > 0, \quad a_{11} = -5(k - 4)^2(\mu - \sigma)^2(\mu - \sigma + 1) > 0.$$

For $\delta = 6$ and for any $k \geq 6$, we have $P_6(\phi) = a_1\phi^{24} + \cdots + a_{12}\phi^2 + a_{13}$ with

$$a_1 = -\,3(\mu + \sigma - 1)\,[k(\mu + \sigma) + \sigma]^{\,2} < 0, \quad a_2 = \ 3(\mu - \sigma)(2\mu + \sigma - 1)\,[k(\mu + \sigma) + \sigma] < 0,$$

$$a_3 = (\mu - \sigma)\left\{(6k - 3)\mu^2 + \mu\,[2k(\sigma + 2) + 7\sigma + 2] - 2(2k + 1)(\sigma - 1)\sigma\right\},$$

$$a_4 = (\mu - \sigma)\left[6(k - 1)\mu^2 + k\mu(15 - 9\sigma) - (15k + 14)(\sigma - 1)\sigma + 11\mu\sigma + \mu\right] > 0,$$

$$a_5 = (\mu - \sigma)\left\{(6k - 9)\mu^2 + \mu\,[6k(5 - 4\sigma) + 20\sigma - 5] - 5(6k + 7)(\sigma - 1)\sigma\right\} > 0,$$

$$a_6 = (\mu - \sigma)\left\{6(k - 2)\mu^2 + \mu\,[k(49 - 43\sigma) + 36\sigma - 18] - (49k + 67)(\sigma - 1)\sigma\right\} > 0,$$

$$a_7 = (\mu - \sigma)\big(3\,[2(k - 5)k - 5]\,\mu^2 + \mu\,\{6k\,[k(12 - 11\sigma) + 56\sigma - 60] + 5(5\sigma - 8)\}$$

$$-8(9(k - 4)k - 40)(\sigma - 1)\sigma) > 0,$$

$$a_8 = -\,(\mu - \sigma)^2\,\{k\,[6\mu - 51\sigma + 51] - 18\mu + 233(\sigma - 1)\} > 0,$$

$$a_9 = -\,(\mu - \sigma)^2\,[k(6\mu - 34\sigma + 34) - 3(7\mu - 53\sigma + 53)] > 0,$$

$$a_{10} = -\,(\mu - \sigma)^2\,[3k(2\mu - 7\sigma + 7) - 4(6\mu - 25\sigma + 25)] > 0,$$

$$a_{11} = -\,(\mu - \sigma)^2\,[6k(\mu - 2\sigma + 2) - 27\mu + 58(\sigma - 1)] > 0,$$

$$a_{12} = -\,(k - 5)(6\mu - 7\sigma + 7)(\mu - \sigma)^2, \quad a_{13} = -3(k - 5)^2(\mu - \sigma)^2(\mu - \sigma + 1) > 0.$$

□

Appendix E. Equilibrium paths for $K = 9, 13, 15$ cities

Figures E.1 and E.2 show equilibrium paths for $K = 9$ and 13 cities, respectively. Figure E.3 shows other equilibrium paths for $K = 15$ cities that are not contained in Fig. 7.

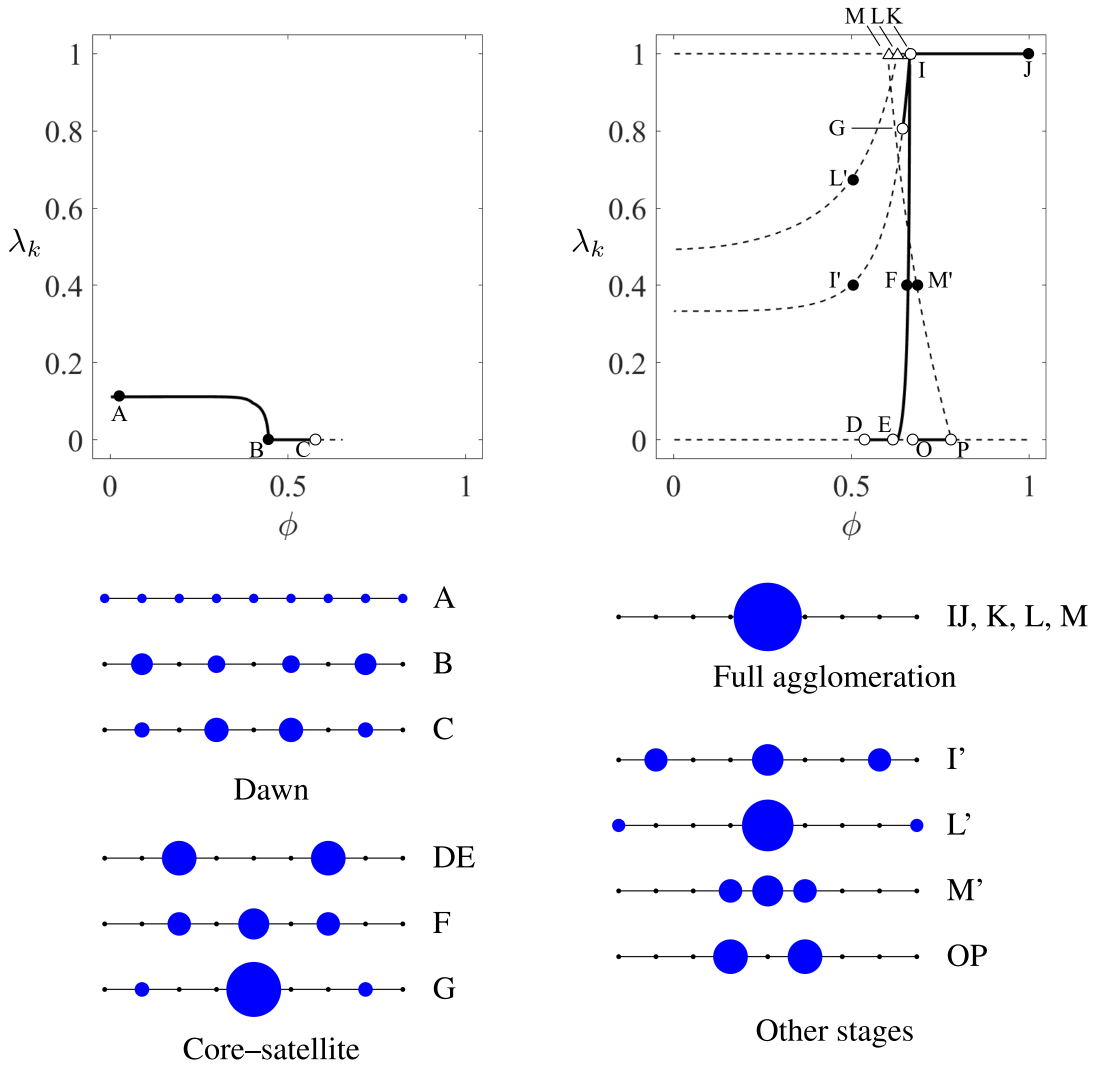

Figure E.1: Paths of equilibria for $K = 9$ cities for $(\sigma, \mu) = (6.0, 0.4)$ (the sustain point I with a branch IG is located closely to a bifurcation point K with a branch KF) (solid line: stable and sustainable; broken line: unstable and/or unsustainable; △: bifurcation point; ∘: sustain point)

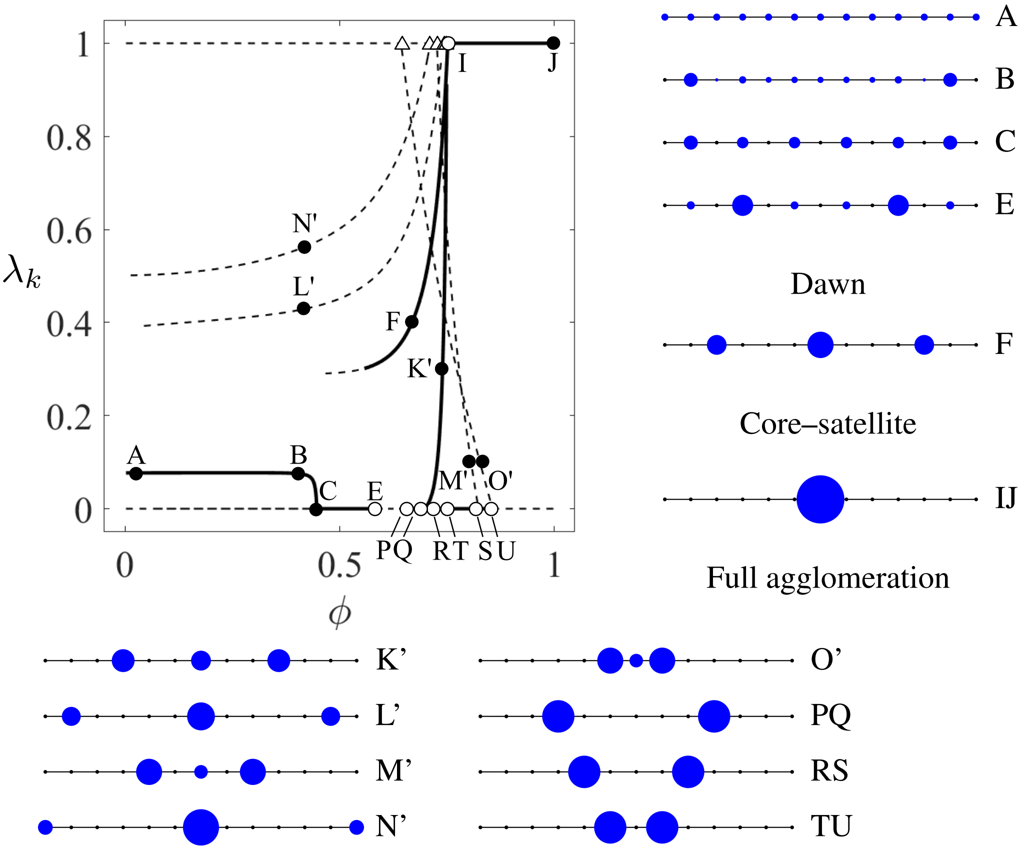

Figure E.2: Paths of equilibria for $K = 13$ cities for $(\sigma, \mu) = (6.0, 0.4)$ (solid line: stable and sustainable; broken line: unstable and/or unsustainable; $\triangle$: bifurcation point; $\circ$: sustain point)

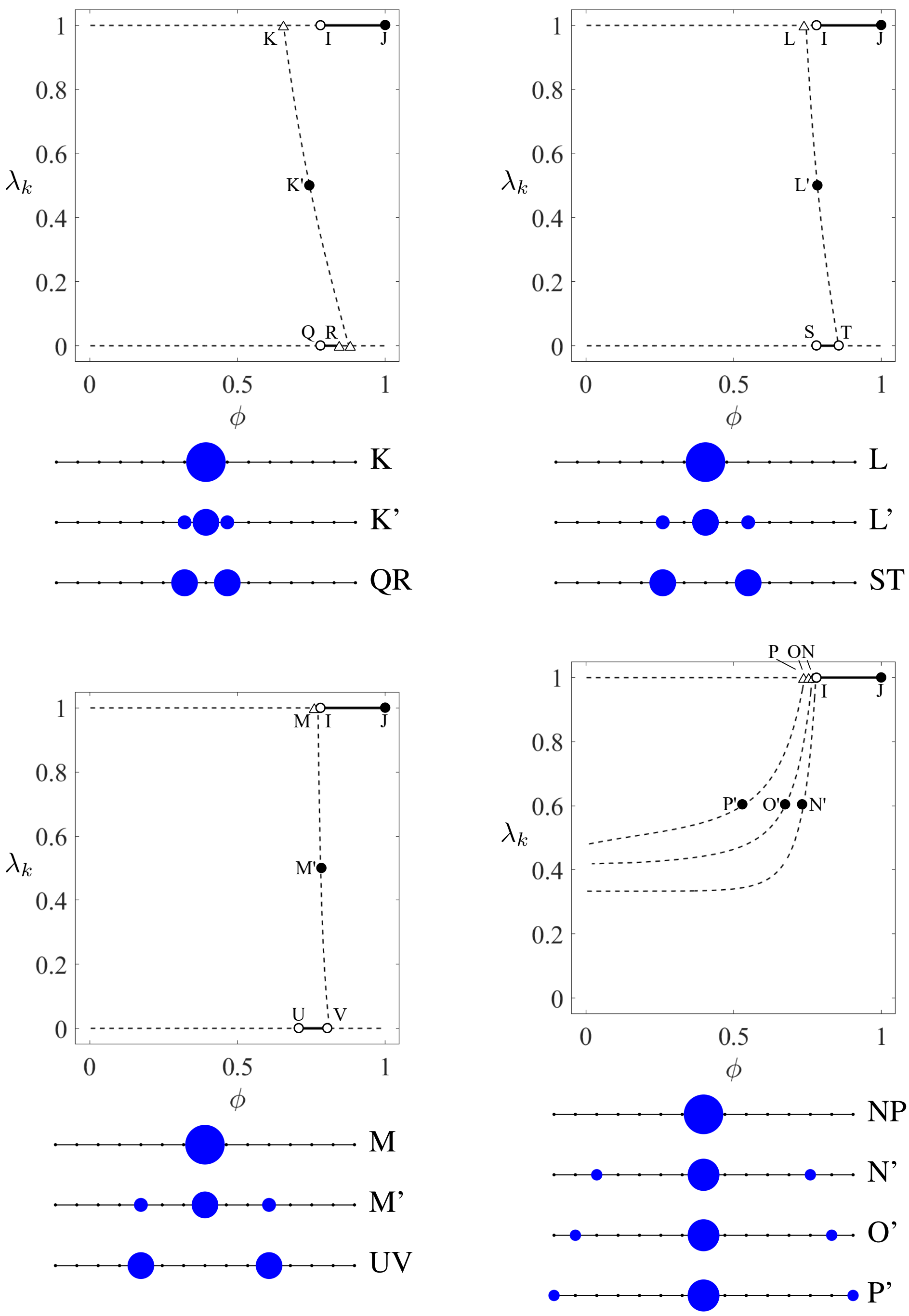

Figure E.3: Other paths of equilibria for $K = 15$ places for $(\sigma, \mu) = (6.0, 0.4)$ (solid line: stable and sustainable; broken line: unstable and/or unsustainable; $\triangle$: bifurcation point; $\circ$: sustain point)

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