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Karmokar, Madhuparna and Roy, Souvik

Indian Statistical Institute, Kolkata

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Madhuparna Karmokar[1] and Souvik Roy[1]

[1]Economic Research Unit, Indian Statistical Institute, Kolkata

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Abstract

We explore the structure of locally ordinal Bayesian incentive compatible (LOBIC) random Bayesian rules (RBRs). We show that under lower contour monotonicity, for almost all prior profiles (with full Lebesgue measure), a LOBIC RBR is locally dominant strategy incentive compatible (LDSIC). We further show that for almost all prior profiles, a unanimous and LOBIC RBR on the unrestricted domain is random dictatorial, and thereby extend the result in Gibbard (1977) for Bayesian rules. Next, we provide sufficient conditions on a domain so that for almost all prior profiles, unanimous RBRs on it (i) are Pareto optimal, and (ii) are tops-only. Finally, we provide a wide range of applications of our results on single-peaked (on arbitrary graphs), hybrid, multiple single-peaked, single-dipped, single-crossing, multi-dimensional separable domains, and domains under partitioning. We additionally establish the marginal decomposability property for both random social choice functions and RBRs (for almost all prior profiles) on multi-dimensional domains, and thereby generalize Breton and Sen (1999). Since OBIC implies LOBIC by definition, all our results hold for OBIC RBRs.

KEYWORDS. random Bayesian rules; random social choice functions; (local) ordinal Bayesian incentive compatibility; (local) dominant strategy incentive compatibility

JEL CLASSIFICATION CODES. D71; D82

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1. Introduction

We consider social choice problems where a random social choice function (RSCF) selects a probability distribution over a finite set of alternatives at every collection of preferences of the agents in a society. It is dominant strategy incentive compatible (DSIC) if no agent can increase the probability of any upper contour set by misreporting her preference. A random Bayesian rule (RBR) consists of an RSCF and a prior belief of each agent about the preferences of the others. We assume that the prior of an agent is "partially correlated": her belief about the preference of one agent may depend on that about another agent, but it does not depend on her own preference. Ordinal Bayesian incentive compatibility (OBIC) is the natural extension of the notion of IC for RBRs. This notion is introduced in d'Aspremont and Peleg (1988) and it captures the idea of Bayes-Nash equilibrium in the context of incomplete information game. An RBR is OBIC if no agent can increase the expected probability (with respect to her belief) of any upper contour set by misreporting her preference.

The importance of Bayesian rules is well-established in the literature: on one hand, they model real life situations where agents behave according to their beliefs, on the other hand, they are significant weakening of the seemingly too demanding requirement of DSIC that leads to dictatorship (or random dictatorships) unless the domain is restricted. It is worth mentioning that the RBRs are particularly important as randomization has long been recognized as a useful device to achieve fairness in allocation problems.

Locally DSIC (LDSIC) or locally OBIC (LOBIC) are weaker versions of the corresponding notions. As the name suggests, they apply to deviations/misreports to only "local" preferences (the notion of which is fixed a priori). The importance of these local notions is well-established in the literature. They are useful in modeling behavioral agents (see Carroll (2012)). Furthermore, on many domains they turn out to be equivalent to their corresponding global versions, and thereby, they are used as a simpler way to check whether a given RSCF is DSIC (see Carroll (2012), Kumar et al. (2020), Sato (2013), Cho (2016), etc.).

The main objective of this paper is to explore the structure of LOBIC RBRs on different domains. The structure of DSIC RSCFs is well-explored in the literature. On the unrestricted domain, they turn out to be random dictatorial, and on restricted domains such as single-peaked or single-crossing or single-dipped, they are some versions of probabilistic fixed ballot rules. However, to the best of our knowledge, only thing known about the structure of LOBIC (or OBIC) RBRs is that if there are exactly two agents and at least four alternatives, then for almost all prior profiles (that is, for a set of prior profiles having full measure), a unanimous, neutral and OBIC RBR is random dictatorial (Majumdar and Roy (2018)).[1] Even for *deterministic* Bayesian rules (DBRs), not much is known. Majumdar and Sen (2004) show that for almost all prior profiles, a unanimous and OBIC DBR on the unrestricted domain is dictatorial, and later, Mishra (2016) shows that for almost all prior profiles, an "elementary monotonic" and OBIC DBR on a swap-connected domain is DSIC. Recently, Hong and Kim (2020) extend these results for weakly connected domains without restoration.[2]

We consider arbitrary notion of localness which we formulate by a graph over preferences. It is worth mentioning that our notion of neighbors (or local preference) is perfectly general. To the best of our knowledge, except in Kumar et al. (2020), all other papers in this area consider the notion of localness that is derived from Kemeny distance. According to this notion, two preferences are local if they differ by a swap of two adjacent alternatives. This notion has limitations: it does not apply to multi-dimensional domains, domains under partitioning, domains under categorization, sequentially dichotomous domains, etc. On the other hand, a general notion of localness is useful for each of the two purposes (as mentioned in Carroll (2012) and Sato (2013)) of considering local notions of incentive compatibility.

(i) Local notions of incentive compatibility makes it simpler for the designer to check if a given rule is DSIC. Naturally, which notion of localness will be suitable for this purpose totally depends on the device that the designer uses, moreover, it may vary over different domains/scenarios.

(ii) Due to social stigma or self-guilt or bounded rationality, some behavioral agents consider manipulations only for some particular deviations. Such deviations are captured

[1] A set of prior profiles is said to have full measure if its complement has Lebesgue measure zero.

[2] We provide a detailed discussion on the connection between our results and those in Hong and Kim (2020) in Section 10.3.

by the notion of local preferences. Clearly, such local deviations depend on the agents, as well as on the particular context.

We introduce the notion of lower contour monotonicity for an RBR and in Theorem 3.1 establish the equivalence between LOBIC and the much stronger (and well-studied) notion LDSIC on *any* domain for RBRs satisfying this property. The deterministic version of this result for the special case of swap-local domains is proved in Mishra (2016).[3]

We show that under LOBIC, unanimity implies lower contour monotonicity on the unrestricted domain. Therefore, it follows as a corollary of Theorem 3.1 that for almost all prior profiles, unanimous and LOBIC (and hence OBIC) RBRs on the unrestricted domain are random dictatorial. Next, we move to restricted domains. It turns out that unanimity is not strong enough to ensure lower contour monotonicity for LOBIC RBRs on most well-known restricted domains. Therefore, we proceed to explore the relation of unanimity to two other important properties of a rule, namely Pareto optimality and tops-onlyness, on such domains.

Pareto optimality is an efficiency requirement for a rule which ensures that the outcome cannot be modified in a way so that every agent is weakly better off and some agent is strictly better off. Clearly, it is much stronger than unanimity. However, it turns out that under DSIC, unanimity and Pareto optimality are equivalent for random rules on many restricted domains such as single-peaked, single-dipped, single-crossing, etc. (see Ehlers et al. (2002), Peters et al. (2017) and Roy and Sadhukhan (2019)). We show in Theorem 3.2 that similar result continues to hold for Bayesian rules for almost all prior profiles if we replace DSIC by OBIC (or LOBIC).

Tops-onlyness is quite a strong property for a rule as it says that the designer can ignore any information about a preference beyond the top-ranked alternative. On the positive side, this property makes the structure of a rule quite simple, however, on the negative side, this property is not quite desirable as it ignores most part of a preference and thereby significantly restricts the scope for designing incentive compatible rules. Interestingly, the negative side of the tops-only property does not play any role for

[3]A graph on a domain is swap-local if any two local preferences differ by a swap of consecutively ranked alternatives.

some domains as unanimity alone enforces it under DSIC. Chatterji and Sen (2011) provide a sufficient condition on a domain so that unanimity and DSIC imply tops-onlyness for DSCFs on it. Later, Chatterji and Zeng (2018) show that the same sufficient condition does not work for RSCFs, and consequently, they provide a stronger sufficient condition on a domain so that unanimity and DSIC imply tops-onlyness. We provide a sufficient condition on a domain so that for almost all prior profiles, unanimous and graph-LOBIC RBRs imply tops-onlyness. It is worth mentioning that establishing the tops-only property is a major (and crucial) step in characterizing unanimous and OBIC RBRs.

Finally, we establish our main equivalence result for weak preferences and provide a discussion explaining why none of these results can be extended for fully correlated priors (that is, when the prior of an agent depends on her own preference). It is worth emphasizing that all the existing results for LOBIC DBRs (Majumdar and Sen (2004) and Mishra (2016)) follow from our results. Furthermore, since every OBIC rule is LOBIC by definition, all our results hold for OBIC rules in particular.

Majumdar and Sen (2004) introduce the notion of generic priors, the particularity of which is that they have full measure. It is shown in Majumdar and Roy (2018) that a unanimous and OBIC RBR with respect to a generic prior profile need not be random dictatorial, and therefore, it seemed that the dictatorial result does not extend (almost surely) for OBIC RBRs. However, it follows from our results that in fact it does, only thing is that one needs to construct the right class of priors ensuring the full measure.

We provide a wide range of applications of our results. We introduce the notion of betweenness domains and establish the structure of RBRs that are LOBIC for almost all prior profiles on these domains. Well-known restricted domains such as single-peaked on arbitrary graphs, hybrid, multiple single-peaked, single-dipped, single-crossing, and domains under partitioning are important examples of betweenness domains. We introduce a weaker version of lower contour monotonicity and obtain a characterization of unanimous RBRs or DBRs (depending on what is known in the literature regarding the equivalence of LDSIC and DSIC) that are LOBIC on these domains for almost all prior profiles . Furthermore, we explain with the help of an example how our results can be utilized to construct the remaining RBRs (that is, the ones that do not satisfy lower contour monotonicity).

Our consideration of arbitrary notion of localness allows us to deal with multi-dimensional domains. Importance of such domains is well understood in the literature; we provide a discussion on this in Section 8. We provide the structure of LOBIC RBRs on full separable multi-dimensional domains when the marginal domains satisfy the betweenness property, for instance, when the marginal domains are unrestricted or single-peaked on graphs or hybrid or multiple single-peaked or single-dipped or single-crossing. Additionally, we establish an important property, called marginal decomposability, of RBRs that are OBIC for almost all prior profiles on multidimensional separable domains. The deterministic version of it, namely decomposability, is proved for DSCFs in Breton and Sen (1999) under DSIC. To the best of our knowledge, this property is not established for RSCFs (even under DSIC), which now follows from our general result about the same for RBRs.

As we have discussed, the results in this paper hold for RBRs for almost all priors profiles, that is, for each prior profile in a set of prior profiles having full measure. It is worth mentioning the economic motivation of such results. Firstly, if the designer thinks all prior profiles are equally likely (or she does not have any particular information about prior profiles), then she knows that except for some "rare" cases (with Lebesgue measure zero), an RBR is LOBIC (or OBIC) if and only if its RSCF component is LDSIC (or DSIC). Since the structure of LDSIC (or DSIC) RSCFs is much simpler, she can use her knowledge about the same in dealing with the RBRs for such prior profiles. Secondly, if the objective of the designer is to maximize the expected total welfare (with respect to any prior distribution over preference profiles and the uniform distribution over prior profiles) of a society over LOBIC (or OBIC) RBRs, then she can restrict her attention (that is, the feasible set) to the LDSIC (or DSIC) RSCFs. This is because a non-LDSIC RSCF can be part of a LOBIC (or OBIC) RBR only for a (Lebesgue) measure zero set of cases which will not contribute to the expected value.

The rest of the paper is organized as follows. Sections 2 introduces the notions of domains, RSCFs, RBRs, and their relevant properties. Sections 3 and 4 present our results for graph-connected and swap-connected domains. Sections 5, 6 and 8 present the applications of our results on unrestricted, betweenness and multi-dimensional domains. Section 9 presents our result for weak preferences. Finally, in Section 10 we provide a discussion on DBRs, (fully) correlated priors, and the relation of our paper with Hong and Kim (2020).

2. Preliminaries

We denote a finite set of alternatives by A and a finite set of n agents by N. A (strict) preference over A is defined as a linear order on A.[4] We deal with strict preferences throughout the paper, except in Section 9 where we provide the definition of weak preferences. The set of all preferences over A is denoted by $\mathscr{P}(A)$. A subset $\mathscr{D}$ of $\mathscr{P}(A)$ is called a domain. Whenever it is clear from the context, we do not use brackets to denote singleton sets.

The weak part of a preference P is denoted by R. Since P is strict, for any two alternatives x and y, xRy implies either xPy or $x=y$. The kth ranked alternative in a preference P is denoted by $P(k)$. The topset $\tau(\mathscr{D})$ of a domain $\mathscr{D}$ is defined as the set of alternatives $\cup_{P\in\mathscr{D}}P(1)$. A domain $\mathscr{D}$ is regular if $\tau(\mathscr{D})=A$. The upper contour set $U(x,P)$ of an alternative x at a preference P is defined as the set of alternatives that are strictly preferred to x in P, that is, $U(x,P)=\{a\in A\mid aPx\}$. A set U is called an upper contour set at P if it is an upper contour set of some alternative at P. The restriction of a preference P to a subset B of alternatives is denoted by $P|_B$, more formally, $P|_B\in\mathscr{P}(B)$ such that for all $a,b\in B$, $aP|_Bb$ if and only if aPb.

Each agent $i\in N$ has a domain $\mathscr{D}_i$ (of admissible preferences). We assume that each domain $\mathscr{D}_i$ is endowed with some graph structure $G_i=\langle\mathscr{D}_i,E_i\rangle$. The graph G_i represents the proximity relation between the preferences: an edge between two preferences implies that they are close in some sense. For instance, suppose $A=\{a,b,c\}$ and $\mathscr{D}_i$ is the set of all preferences over A. Suppose that two preferences are "close" if and only if they differ by a swap of two alternatives. The graph G_i that represents this proximity relation is given in Figure 1. The alternatives that swap between two preferences are mentioned

[4]A linear order is a complete, transitive, and antisymmetric binary relation.

on the edge between the two.

We denote by G_N a collection of graphs $(G_i)_{i\in N}$. Whenever we use some term involving the word "graph", we mean it with respect to a collection G_N. Two preferences P_i and P_i' of an agent i are graph-local if they form an edge in G_i, and a sequence of preferences $(P_i^1,\ldots,P_i^t)$ is a graph-local path if every two consecutive preferences in the sequence are graph-local. A domain $\mathscr{D}_i$ is graph-connected if there is a graph-local path between any two preferences in it. We denote by $\mathscr{D}_N$ the product set $\mathscr{D}_1\times\cdots\times\mathscr{D}_n$ of individual domains. An element of $\mathscr{D}_N$ is called a preference profile. All the domains we consider in this paper are assumed to be graph-connected.

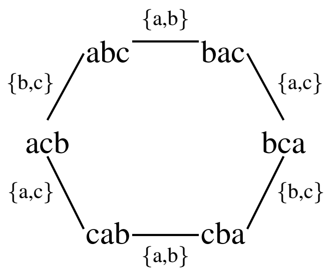

Figure 1

We use the following terminologies to ease the presentation: $P\equiv xy\cdots$ means $P(1)=x$ and $P(2)=y$; $P\equiv\cdots xy\cdots$ means x and y are consecutively ranked in P with xPy; $P\equiv\cdots x\cdots y\cdots$ means x is ranked above y. When the set of alternatives is precisely stated, say $A=\{a,b,c,d\}$, we write, for instance, $P=abcd$ to mean $P(1)=a$, $P(2)=b$, $P(3)=c$, and $P(4)=d$. We use similar notations without further explanations.

2.1 Random social choice functions and their properties

Let ΔA be the set of all probability distributions on A. A random social choice function (RSCF) is a mapping $\varphi:\mathscr{D}_N\to\Delta A$. We denote the probability of an alternative x at $\varphi(P_N)$ by $\varphi_x(P_N)$.

An RSCF $\varphi:\mathscr{D}_N\to\Delta A$ is **unanimous** if for all $P_N\in\mathscr{D}_N$ such that for all $i\in N$, $P_i(1)=x$ for some $x\in A$, we have $\varphi_x(P_N)=1$. An RSCF $\varphi:\mathscr{D}_N\to\Delta A$ is **Pareto optimal** if for all $P_N\in\mathscr{D}_N$ and all $x\in A$ such that there exists $y\in A$ with yP_ix for all $i\in N$, we have $\varphi_x(P_N)=0$. Clearly, Pareto optimality implies unanimity. An RSCF

$\varphi : \mathscr{D}_N \to \Delta A$ is **tops-only** if for all $P_N, P'_N \in \mathscr{D}_N$ such that $P_i(1) = P'_i(1)$ for all $i \in N$, we have $\varphi(P_N) = \varphi(P'_N)$.

A probability distribution ν stochastically dominates another probability distribution $\hat{\nu}$ at a preference P, denoted by $\nu P^{\text{sd}} \hat{\nu}$, if $\nu_{U(x,P_i)} \geq \hat{\nu}_{U(x,P_i)}$ for all $x \in A$ and $\nu_{U(y,P_i)} > \hat{\nu}_{U(y,P_i)}$ for some $y \in A$. We write $\nu R^{sd} \hat{\nu}$ to mean either $\nu P^{sd} \hat{\nu}$ or $\nu = \hat{\nu}$. An RSCF $\varphi : \mathscr{D}_N \to \Delta A$ is *dominant strategy incentive compatible (DSIC) on a pair of preference* (P_i, P'_i) of an agent $i \in N$, if $\varphi(P_i, P_{-i}) R_i^{\text{sd}} \varphi(P'_i, P_{-i})$ for all $P_{-i} \in \mathscr{D}_{-i}$. An RSCF is **graph-locally dominant strategy incentive compatible (graph-LDSIC)** if it is DSIC on every pair of graph-local preferences of each agent, and it is called **dominant strategy incentive compatible (DSIC)** if it is DSIC on *every* pair of preferences of each agent.

A set of alternatives B is a block in a pair of preferences (P, P') if it is a minimal non-empty set satisfying the following property: for all $x \in B$ and $y \notin B$, $P|_{\{x,y\}} = P'|_{\{x,y\}}$. For instance, the blocks in the pair of preferences $(abcdefg, bcadegf)$ are $\{a,b,c\}, \{d\}, \{e\}$, and $\{f,g\}$. The lower contour set $L(x,P)$ of an alternative x at a preference P is $L(x,P) = \{a \in A \mid xPa\}$. A set L is a lower contour set at a preference P if it is a lower contour set of some alternative at P. Lower contour monotonicity says that whenever an agent i unilaterally deviates from P_i to a graph-local preference P'_i, the probability of each lower contour set at P_i restricted to any non-singleton block in (P_i, P'_i) will weakly increase. For instance, consider our earlier example $P_i = abcdefg$ and $P'_i = bcadegf$ with non-singleton blocks $\{a,b,c\}$ and $\{f,g\}$. The lower contour sets at P_i restricted to $\{a,b,c\}$ are $\{c\}$ and $\{b,c\}$, and that restricted to $\{f,g\}$ is $\{g\}$. Lower contour monotonicity says that the probability of each of the sets $\{c\}$, $\{b,c\}$, and $\{g\}$ will weakly increase if agent i unilaterally deviates from P_i to P'_i.

Definition 2.1. An RSCF $\varphi : \mathscr{D}_N \to \Delta A$ is called **lower contour monotonic** if for all $i \in N$, all graph-local preferences $P_i, P'_i \in \mathscr{D}_i$, all non-singleton blocks B in (P_i, P'_i), and all $P_{-i} \in \mathscr{D}_{-i}$, we have $\varphi_L(P_i, P_{-i}) \leq \varphi_L(P'_i, P_{-i})$ for each lower contour set L of $P_i|_B$.

2.2 Random Bayesian rules and their properties

A prior μ_i of an agent i is a probability distribution over $\mathscr{D}_{-i}$ which represents her belief about the preferences of the others, and a prior profile $\mu_N := (\mu_i)_{i \in N}$ is a collection of priors, one for each agent. A pair (φ, μ_N) consisting of an RSCF $\varphi : \mathscr{D}_N \to \Delta A$ and a prior profile μ_N is called a random Bayesian rule (RBR) on $\mathscr{D}_N$. When the RSCF φ is a DSCF, then it is called a deterministic Bayesian rule (DBR).

The expected outcome with respect to the belief of an agent is called her interim expected outcome. More formally, the *interim expected outcome* $\varphi(P_i, \mu_i)$ for an agent $i \in N$ at a preference $P_i \in \mathscr{D}_i$ from an RBR (φ, μ_N) on $\mathscr{D}_N$ is defined as the following probability distribution on A: for all $x \in A$,

$$\varphi_x(P_i, \mu_i) = \sum_{P_{-i} \in \mathscr{D}_{-i}} \mu_i(P_{-i}) \varphi_x(P_i, P_{-i}).$$

Example 2.1. Let $N = \{1, 2\}$ and $A = \{a, b, c\}$. Consider the RBR (φ, μ_N) given in Table 1. Agent 1's belief μ_1 about agent 2's preferences is given in the top row and agent 2's belief μ_2 about agent 1's preferences in the leftmost column of the table. The outcomes of φ at different profiles are presented in the corresponding cells. Here, for instance, $(0.7, 0, 0.3)$ denotes the outcome where a, b, and c are given probabilities 0.7, 0, and 0.3, respectively. The rest of the table is self-explanatory. Consider the preference $P_1 = abc$ of agent 1. In what follows, we show how to compute her interim expected outcome $\varphi(P_1, \mu_1)$ at this preference: $\varphi_a(P_1, \mu_1) = 0.2 \times 1 + 0.1 \times 1 + 0.05 \times 1 + 0.3 \times 0.5 + 0.15 \times 1 + 0.2 \times 1 = 0.85$. Similarly, one can calculate that $\varphi_b(P_1, \mu_1) = 0.15$, and $\varphi_c(P_1, \mu_1) = 0$, and for agent 2's preference $P_2 = bca$, $\varphi_b(P_2, \mu_2) = 0.575$, $\varphi_c(P_2, \mu_2) = 0.06$, and $\varphi_a(P_2, \mu_2) = 0.365$.

	μ_1	0.2	0.1	0.05	0.3	0.15	0.2
μ_2	1 \ 2	abc	acb	bac	bca	cba	cab
0.25	abc	(1,0,0)	(1,0,0)	(1,0,0)	(0.5,0.5,0)	(1,0,0)	(1,0,0)
0.2	acb	(1,0,0)	(1,0,0)	(1,0,0)	(0.7,0,0.3)	(1,0,0)	(1,0,0)
0.15	bac	(1,0,0)	(1,0,0)	(0,1,0)	(0,1,0)	(0,1,0)	(1,0,0)
0.1	bca	(0,1,0)	(1,0,0)	(0,1,0)	(0,1,0)	(0,1,0)	(0,1,0)
0.2	cba	(1,0,0)	(0,0,1)	(0,0.4,0.6)	(0,1,0)	(0,0,1)	(0,0,1)
0.1	cab	(1,0,0)	(0,0.4,0.6)	(1,0,0)	(1,0,0)	(0,0,1)	(0,0,1)

Table 1

The notion of ordinal Bayesian incentive compatibility (OBIC) captures the idea of DSIC for an RBR by ensuring that no agent can improve her interim expected outcome by misreporting her preference.

Definition 2.2. An RBR (φ, μ_N) on $\mathscr{D}_N$ is *ordinal Bayesian incentive compatible (OBIC) on a pair of preferences* (P_i, P'_i) of an agent $i \in N$ if $\varphi_{\mu_i}(P_i) R_i^{\text{sd}} \varphi_{\mu_i}(P'_i)$. An RBR (φ, μ_N) is **graph-locally ordinal Bayesian incentive compatible (graph-LOBIC)** if it is OBIC on every pair of graph-local preferences in the domain of each agent, and it is **ordinal Bayesian incentive compatible (OBIC)** if it is OBIC on *every* pair of preferences in the domain of each agent.

Note that OBIC is a weaker requirement than DSIC since if an RSCF φ is DSIC, then (φ, μ_N) is OBIC for all prior profiles μ_N.

For ease of presentation, given a property defined for an RSCF, we say an RBR (φ, μ_N) satisfies it, if φ satisfies the property.

3. Results on graph-connected domains

In this section, we explore the structure of graph-LOBIC Bayesian rules on graph-connected domains. Since OBIC implies graph-LOBIC (by definition), all these results hold for OBIC RBRs as well.

Recall the definition of a block given in Page 9. The block preservation property says that if an agent unilaterally changes her preference to a graph-local preference, the total probability of any block in the two preferences will remain unchanged.

Definition 3.1. An RSCF $\varphi : \mathscr{D}_N \to \Delta A$ satisfies the **block preservation property** if for all $i \in N$, all graph-local preferences $P_i, P'_i \in \mathscr{D}_i$ of agent i, all blocks B in (P_i, P'_i), and all $P_{-i} \in \mathscr{D}_{-i}$, we have $\varphi_B(P_i, P_{-i}) = \varphi_B(P'_i, P_{-i})$.

For two preferences P and P', $P \triangle P' = \{x \in A \mid U(x,P) \neq U(x,P')\}$ denotes the set of alternatives that change their relative ordering with some other alternative from P to P'. Note that the block preservation property implies $\varphi_x(P_i, P_{-i}) = \varphi_x(P'_i, P_{-i})$ for all $x \notin P_i \triangle P'_i$ as such an alternative forms a singleton block in (P_i, P'_i).

Our next proposition says that graph-LOBIC implies the block-preservation property almost surely (with probability one). In other words, for each RSCF φ, there is a set of prior profiles with full measure such that if it is graph-LOBIC with respect to any of the prior profiles in the set, it will satisfy the block-preservation property. The economic interpretation of this result is that if the designer thinks that all the priors of an agent are equally likely and wants to ensure that no agent can manipulate her RBR, then "almost surely" she needs to make the RSCF component of the RBR satisfy the block-preservation property.

Proposition 3.1. *For every RSCF $\varphi : \mathscr{D}_N \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is graph-LOBIC implies that φ satisfies the block-preservation property.*

The proof of this proposition is relegated to Appendix B.

3.1 Equivalence of graph-LOBIC and graph-LDSIC under lower contour monotonicity

The following theorem says that under lower contour monotonicity, the notion of graph-LDSIC becomes almost surely equivalent to the much weaker notion of graph-LOBIC.

Theorem 3.1. *For every lower contour monotonic RSCF $\varphi : \mathscr{D}_N \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is graph-LOBIC if and only if φ is graph-LDSIC.*

The proof of this theorem is relegated to Appendix C.1.

The economic interpretation of Theorem 3.1 is that if the designer wants to construct a graph-LOBIC RBR satisfying lower contour monotonicity, then for almost all prior profiles (that is, with full measure) she can restrict her attention to graph-LDSIC RSCF only.

Even though there is a measure zero set of prior profiles such that the RBR (φ, μ_N) is graph-LOBIC but φ is not graph-LDSIC, it is important to know the exact structure of that (measure zero) set. The structure of the set depends on the RSCF φ: it contains the prior profiles that satisfy a particular system of linear equations involving the outcomes of the RSCF φ. We present this system of equations in Appendix A.

It is worth emphasizing that Theorem 3.1 holds for *any* domain and for *any* graph structure on it (as long as it is connected). In Sections 5, 6 and 8, we discuss its applications on unrestricted, single-peaked on a graph (and on a tree or a line as special cases), multiple single-peaked, hybrid, multiple single-peaked, intermediate, single-dipped, single-crossing and multi-dimensional separable domains. One can also apply the theorem on domains under categorization, sequentially dichotomous domains, etc.

3.2 Sufficient condition for the equivalence of unanimity and Pareto optimality

Pareto optimality is much stronger than unanimity. However, under DSIC, these two notions turn out to be equivalent for RSCFs on many domains such as the unrestricted, single-peaked, single-dipped, single-crossing, etc. In this section, we show that similar results hold with probability one if we replace DSIC by its weaker version OBIC. We introduce the notion of upper contour preservation property for our result.

Definition 3.2. A domain $\mathscr{D}$ satisfies the **upper contour preservation property** if for all $x,y \in A$ and all $P \in \mathscr{D}$ with xPy, there exists a graph-local path from P to a preference $\hat{P} \in \mathscr{D}$ with $\hat{P}(1) = x$ such that $U(P,y) = U(\hat{P},y)$.

Our next theorem says that if a domain satisfies the upper contour preservation property then for almost all prior profiles, a unanimous and graph-LOBIC RBRs on it will be Pareto optimal.

Theorem 3.2. *Suppose $\mathscr{D}_i$ satisfies the upper contour preservation property for all $i \in N$. For every unanimous RSCF $\varphi : \mathscr{D}_N \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is graph-LOBIC implies that φ is Pareto optimal.*

The proof of this theorem is relegated to Appendix C.2.

3.3 Relation between unanimity and tops-onlyness

We use the notion of path-richness in our result. A domain satisfies the path-richness property if for every two preferences P and P' having the same top-ranked alternative, say x, the following happens: (i) if P and P' are not graph-local then there is graph-local path from P to P' such that x appears as the top-ranked alternative in each preference in the path, and (ii) if P and P' are graph-local, then from any preference $\hat{P}$ there is a path to some preference $\bar{P}$ with x as the top-ranked alternative such that for any two alternatives a,b that change their relative ranking from P to P' and for any two consecutive preferences in the path, there is a common upper contour set of the preferences such that exactly one of a and b belongs to it. For an illustration of Part (ii) of the path-richness property, suppose $A = \{a,b,c,d\}$, $P = abcd$ and $P' = adcb$, and assume that P and P' are graph-local. Consider a preference $\hat{P} = dbca$. Path-richness requires that a path of the following type must be present in the domain: $(dbca, dbac, dabc, adbc)$. To see that this path satisfies (ii), consider two alternatives that change their relative ordering from P to P', say b and c. Note that the upper contour set $\{d,b\}$ in P^1 and P^2 contains b but not c, the upper contour set $\{d,b,a\}$ in P^2 and P^3 contains b but not c, and so on. Path-richness requires that such a path must exist for every preference $\hat{P}$ in the domain.

Definition 3.3. A domain $\mathscr{D}$ satisfies the **path-richness property** if for all preferences $P, P' \in \mathscr{D}$ such that $P(1) = P'(1)$,

(i) if P and P' are not graph-local, then there is a graph-local path $(P^1 = P, \ldots, P^t = P')$ such that $P^l(1) = P(1)$ for all $l = 1, \ldots, t$, and

(ii) if P and P' are graph-local, then for each preference $\hat{P} \in \mathscr{D}$, there exists a graph-local path $(P^1 = \hat{P}, \ldots, P^t)$ with $P^t(1) = P(1)$ such that for all $l < t$ and all distinct $y, z \in P \triangle P'$, there is a common upper contour set U of P^l and $\bar{P}^{l+1}$ such that exactly one of y and z is contained in U.

Example 3.1. Consider the domain in Table 2. We explain that this domain satisfies the path-richness property. Suppose that two preferences are graph-local if and only if they differ by a swap of two alternatives. Consider the preferences P^1 and P^3 having the same top-ranked alternative. Note that they are not graph-local. The path (P^1, P^2, P^3) is graph-local and a appears as the top-ranked alternative in each preference in the path. So, the path satisfies the requirement of (i). It can be verified that for other non graph-local preferences with the same top-ranked alternative (such as P^4 and P^7, or P^8 and P^{11}, etc.) such a path lies in the domain. Now, consider the preferences P^1 and P^2. Note that they are graph-local and the alternatives b and c are swapped in the two preferences (that is, $P^1 \triangle P^2 = \{a,b\}$). Consider any other preference, say P^7. The path $(P^7, P^6, P^5, P^4, P^3)$

has the property that (a) it ends with a preference that has the same top-ranked alternative a as P^1 and P^2, and (b) for every two consecutive preferences in the path, there is a common upper contour set of the two preferences that contains exactly one of b and c (for instance, the common upper contour set $\{a,c\}$ of P^3 and P^4 contains c but not b, and so on). It can be verified that such a path exists for every pair of graph-local preferences P and P' having the same top-ranked alternative and for every preference $\hat{P}$. It is worth mentioning that for the kind of graph-localness we consider in this example, the requirement of (b) boils down to requiring that the swapping alternatives in the graph-local preferences maintain their relative ranking throughout the path.

P^1	P^2	P^3	P^4	P^5	P^6	P^7	P^8	P^9	P^{10}	P^{11}
a	a	a	c	c	c	c	e	e	e	e
b	c	c	a	b	b	e	c	c	c	d
c	b	b	b	a	e	b	b	b	d	c
d	d	e	e	e	a	a	a	d	b	b
e	e	d	d	d	d	d	d	a	a	a

Table 2

The path-richness property may seem to be somewhat involved but we show in Section 6, most restricted domains of practical importance satisfy this property.

Our next theorem says that if the designer wants construct a unanimous and graph-LOBIC RBR on a domain satisfying the path-richness property, then for almost all prior profiles she can restrict her attention to tops-only RSCFs. Clearly, this makes the construction considerably simpler. As we have mentioned in case of Theorem 3.1, the economic implication of this theorem is that if the designer thinks all the priors of an agent are equally likely, then she can be assured that a unanimous and graph-LOBIC RBR on a path-rich domain will be tops-only with probability one.

Theorem 3.3. *Suppose $\mathscr{D}$ satisfies the path-richness property. For every unanimous RSCF $\varphi : \mathscr{D}_N \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is graph-LOBIC implies that φ is tops-only.*

The proof of this theorem is relegated to Appendix C.3.

Remark 3.1. Lower contour monotonicity can be weakened in a straightforward way under tops-onlyness. Let us say that an RSCF satisfies top lower contour monotonicity if it satisfies lower contour monotonicity only over (unilateral) deviations to graph-local preferences where the top-ranked alternative is changed. Thus, top lower contour monotonicity does not impose any restriction for graph-local preferences P and P' with $\tau(P) = \tau(P')$. Clearly, under tops-onlyness, lower contour monotonicity will be automatically guaranteed in all other cases, and hence, top lower contour monotonicity will be equivalent to lower contour monotonicity. Since under graph-LOBIC, unanimity implies tops-onlyness on a large class of domains, this simple observation is of great help for practical applications. □

4. THE CASE OF SWAP-CONNECTED DOMAINS

In this section, we consider graphs where two preferences are local if and only if they differ by a swap of two consecutively ranked alternatives. Formally, two preferences P and P' are swap-local if $P \triangle P' = \{x, y\}$ for some $x, y \in A$. For two swap-local preferences P and P', we say x overtakes y from P to P' if yPx and $xP'y$. A domain $\mathscr{D}_i$ is swap-connected if there is a swap-local path between any two preferences in it. We use terms like swap-LOBIC, swap-LDSIC, etc. (instead of graph-LOBIC, graph-LDSIC, etc.) to emphasize the fact that the graph is based on the swap-local structure.

When graphs are swap-connected, lower contour monotonicity boils down to the following condition called elementary monotonicity. An RSCF $\varphi : \mathscr{D}_N \to \Delta A$ is called **elementary monotonic** if for every $i \in N$, all swap-local preferences $P_i, P_i' \in \mathscr{D}_i$ of agent i, and all $P_{-i} \in \mathscr{D}_{-i}$, x overtakes some alternative from P_i to P_i' implies $\varphi_x(P_i, P_{-i}) \leq \varphi_x(P_i', P_{-i})$.

As we have mentioned in Example 3.1, under swap-connectedness, Condition (ii) of the path-richness property (Definition 3.3) simplifies to the following condition: if there are two swap-local preferences having the same top-ranked alternative, say x, where two alternatives, say y and z, are swapped, then from every preference in the domain there must be a swap-local path to some preference with x as the top-ranked alternative such that the relative ranking of y and z remains the same along the path.

4.1 Equivalence of swap-LDSIC and weak elementary monotonicity under tops-onlyness

Weak elementary monotonicity (Mishra (2016)) is a restricted version of elementary monotonicity where the latter is required to be satisfied only for a particular type of profiles where all the agents agree on the ranking of alternatives from rank three onward.

Definition 4.1. An RSCF $\varphi : \mathscr{D}^n \to \Delta A$ satisfies **weak elementary monotonicity** if for all $i \in N$, and all (P_i, P_{-i}) and (P'_i, P_{-i}) such that $P_i(k) = P'_i(k) = P_j(k)$ for all $j \in N \setminus i$ and all $k > 2$, we have $\varphi_{P_i(1)}(P_i, P_{-i}) \geq \varphi_{P_i(1)}(P'_i, P_{-i})$.

Our next result says that under tops-onlyness, for almost all priors, weak elementary monotonic and swap-LOBIC RBRs are swap-LDSIC.

Theorem 4.1. *For every tops-only and weak elementary monotonic RSCF $\varphi : \mathscr{D}_N \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is swap-LOBIC if and only if φ is swap-LDSIC.*

The proof of this theorem is relegated to Appendix C.4.

We obtain the following corollary from Theorem 3.3 and Theorem 4.1.

Corollary 4.1. *Suppose $\mathscr{D}$ satisfies the path-richness property. For every unanimous and weak elementary monotonic RSCF $\varphi : \mathscr{D}_N \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is swap-LOBIC if and only if φ is swap-LDSIC.*

5. Application on the unrestricted domain

The domain $\mathscr{P}(A)$ containing all preferences over A is called the **unrestricted domain** (over A). Since, the unrestricted domain satisfies both the upper contour preservation property and the path-richness property, it follows from Theorem 3.2 and Theorem 3.3 that for almost all prior profiles, unanimity and swap-LOBIC imply both Pareto optimality and tops-only. The following theorem further establishes that for almost all prior profiles, swap-LOBIC RBRs are in fact swap-LDSIC.

Theorem 5.1. *For every unanimous RSCF $\varphi : \mathscr{P}^n \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is swap-LOBIC if and only if φ is swap-LDSIC.*

Gibbard (1977) shows that every unanimous and DSIC RSCF on the unrestricted domain is *random dictatorial*. An RSCF is random dictatorial if it is convex combination of the dictatorial rules, that is, for each agent there is a fixed probability such that the agent is the dictator with that probability.

Definition 5.1. An RSCF $\varphi : \mathscr{D}_N \to \Delta A$ is **random dictatorial** if there exist non-negative real numbers β_i; $i \in N$, with $\sum\limits_{i \in N} \beta_i = 1$, such that for all $P_N \in \mathscr{D}_N$ and $a \in A$, $\varphi_a(P_N) = \sum\limits_{\{i | P_i(1) = a\}} \beta_i$.

Let us call a domain **swap random local-global equivalent (swap-RLGE)** if every swap-LDSIC RSCF on it is DSIC. It follows from Cho (2016) that the unrestricted domain is swap-RLGE. Since every OBIC RBR is swap-LOBIC by definition, it follows from Theorem 5.1 that the same result as Gibbard (1977) holds for almost all prior profiles even if we replace DSIC with the much weaker notion OBIC.

Corollary 5.1. *Let $|A| \geq 3$. For every unanimous RSCF $\varphi : \mathscr{P}^n \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is swap-LOBIC if and only if φ is random dictatorial.*

6. Applications on domains satisfying the betweenness property

A **betweenness relation** β maps every pair of distinct alternatives (x, y) to a subset of alternatives $\beta(x, y)$ including x and y. We only consider betweenness relations β that are rational: for every $x \in A$, there is a preference P with $P(1) = x$ such that for all $y, z \in A$, $y \in \beta(x, z)$ implies yRz. Such a preference P is said to respect the betweenness relation β. A domain $\mathscr{D}$ respects a betweenness relation β if it contains all preferences respecting β. We denote such a domain by $\mathscr{D}(\beta)$. For a collection of betweenness relations $\mathscr{B} = \{\beta_1, \ldots, \beta_r\}$, we denote the domain $\cup_{l=1}^{r} \mathscr{D}(\beta_l)$ by $\mathscr{D}(\mathscr{B})$.

A pair of alternatives (x, y) is adjacent in β if $\beta(x, y) = \{x, y\}$. A betweenness relation β is **weakly consistent** if for all $x, \bar{x} \in A$, there is a sequence $(x^1 = x, \ldots, x^t = \bar{x})$

of adjacent alternatives in $\beta(x,\bar{x})$ such that for all $l<t$, we have $\beta(x^{l+1},\bar{x}) \subseteq \beta(x^l,\bar{x})$. A betweenness relation β is **strongly consistent** if for all $x,\bar{x} \in A$, there is a sequence $(x^1 = x,\ldots,x^t = \bar{x})$ of adjacent alternatives in $\beta(x,\bar{x})$ such that for all $l<t$ and all $w \in \beta(x^l,\bar{x})$, we have $\beta(x^{l+1},w) \subseteq \beta(x^l,\bar{x})$. A collection $\mathscr{B} = \{\beta_1,\ldots,\beta_r\}$ or a betweenness domain $\mathscr{D}(\mathscr{B})$ is strongly/weakly consistent if β_l is strongly/weakly consistent for all $l = 1,\ldots,r$.

Two betweenness relations β and β' are swap-local if for every $x \in A$, there are $P \in \mathscr{D}(\beta)$ and $P' \in \mathscr{D}(\beta')$ such that $P(1) = P'(1)$ and P and P' are swap-local. A collection $\mathscr{B}$ of betweenness relations is called swap-connected if for all $\beta,\beta' \in \mathscr{B}$, there is a sequence $(\beta^1 = \beta,\ldots,\beta^t = \beta')$ in $\mathscr{B}$ such that β^l and β^{l+1} are swap-local for all $l<t$.

We now define the local structure on a betweenness domain $\mathscr{D}(\mathscr{B})$ in a natural way. A preference P' is graph-local to another preference P if there is no preference $P'' \in \mathscr{D}(\mathscr{B}) \setminus \{P,P'\}$ that is "more similar" to P than P' is to P, that is, there is no P'' such that for all $x,y \in A$, $P|_{\{x,y\}} = P'|_{\{x,y\}}$ implies $P|_{\{x,y\}} = P''|_{\{x,y\}}$. Our next corollary follows from Theorem 3.3.

Corollary 6.1. *Let $\mathscr{B}$ be a collection of strongly consistent and swap-connected betweenness relations. For every unanimous RSCF $\varphi : \mathscr{D}(\mathscr{B})^n \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ,μ_N) is graph-LOBIC implies that φ is tops-only.*

The proof of this corollary is relegated to Appendix C.6.

A domain is called graph deterministic local-global equivalent (graph-DLGE) if every graph-LDSIC DSCF on it is DSIC.

Theorem 6.1. *Let $\mathscr{B}$ be a collection of weakly consistent and swap-connected betweenness relations. Then, $\mathscr{D}(\mathscr{B})$ is a graph-DLGE domain.*

The proof of this corollary is relegated to Appendix C.7.

In what follows, we apply our results to explore the structure of LOBIC RBRs on well-known betweenness domains.

6.1 SINGLE-PEAKED DOMAINS ON GRAPHS

Peters et al. (2019) introduce the notion of single-peaked domains on graphs and characterize all unanimous and DSIC RSCFs on these domains. We assume that the set of alternatives is endowed with an (undirected) graph $\mathscr{G} = \langle A, E \rangle$. For $x, \bar{x} \in A$ with $x \neq \bar{x}$, a path $(x^1 = x, \ldots, x^t = \bar{x})$ from x to $\bar{x}$ in $\mathscr{G}$ is a sequence of distinct alternatives such that $\{x^i, x^{i+1}\} \in E$ for all $i = 1, \ldots, t-1$. If it is clear which path is meant, we also denote it by $[x, \bar{x}]$. We assume that $\mathscr{G}$ is connected, that is, there is a path from x to $\bar{x}$ for all distinct $x, \bar{x} \in A$. If this path is unique for all $x, \bar{x} \in A$, then $\mathscr{G}$ is called a tree. A spanning tree of $\mathscr{G}$ is a tree $T = \langle A, E_T \rangle$ where $E_T \subseteq E$. In other words, spanning tree of $\mathscr{G}$ is a tree that can be obtained by deleting some edges of $\mathscr{G}$.

Definition 6.1. A preference P is single-peaked on $\mathscr{G}$ if there is a spanning tree T of $\mathscr{G}$ such that for all distinct $x, y \in A$ with $P(1) \neq y$, $x \in [P(1), y] \implies xPy$, where $[P(1), y]$ is the path from $P(1)$ to y in T. A domain is called **single-peaked on** $\mathscr{G}$ if it contains all single-peaked preferences on $\mathscr{G}$.

In what follows, we argue that a single-peaked domain on a graph satisfies the upper contour preservation property. Since a single-peaked domain on a graph is a union of single-peaked domains on trees, it is enough to show that a single-peaked domain on a tree satisfies the upper contour preservation property. Consider a single-peaked domain $\mathscr{D}_T$ on a tree T. Let P be a preference with xPy for some $x, y \in A$. Suppose $P(1) = a$. Consider the path $[a, x]$ in T. Since xPy, it must be that $y \notin [a, x]$. Suppose $[a, x] = (x^1 = a, \ldots, x^k = x)$. By the definition of single-peaked domain on a tree, one can go from P to a preference with x^2 at the top through a swap-local path maintaining the upper contour set of y. Continuing in this manner, one can go to a preference with x at the top maintaining the upper contour set of y. This concludes that $\mathscr{D}_T$ satisfies the upper contour preservation property, and hence, we obtain the following corollary from Theorem 3.2.

Corollary 6.2. *Let $\mathscr{D}$ be the single-peaked domain on a graph. For every unanimous RSCF $\varphi : \mathscr{D}^n \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is swap-LOBIC implies that φ is Pareto optimal.*

It follows from the definition that a single-peaked domain $\mathscr{D}_T$ on a tree T can be represented as a betweenness domain $\mathscr{D}(\beta^T)$ where β^T is defined as follows: $\beta^T(x,y) = [x,y]$. Single-peaked domains on graphs are well-known for the cases when the graph $\mathscr{G}$ is a line or a tree.[5] When the graph $\mathscr{G}$ is a line, then the corresponding domain is known in the literature as the **single-peaked domain**.[6]

We now argue that the betweenness relation β^T is strongly consistent. To see that β^T is strongly consistent consider two alternatives x and $\bar{x}$, and consider the unique path $[x,\bar{x}]$ between them in T. Let $[x,\bar{x}] = (x^1 = x, \ldots, x^t = \bar{x})$. By the definition of β^T, the path $[x,\bar{x}]$ lies in (in fact, is equal to) $\beta^T(x,\bar{x})$. Consider $x^l \in \beta^T(x,\bar{x})$ and $w \in \beta^T(x^l,\bar{x})$. Since both w and x^{l+1} lie on the path $[x^l,\bar{x}]$, it follows that $[x^{l+1},w] \subseteq [x^l,\bar{x}]$, and hence $\beta^T(x^{l+1},w) \subseteq \beta^T(x^l,\bar{x})$. This proves that β^T is the strongly consistent (and hence is also weakly consistent). Since a betweenness relation that generates a single-peaked domain on a tree is strongly consistent, it follows from the definition of a single-peaked domain on a graph that the betweenness relation that generates such a domain also satisfies the property. It is shown in Peters et al. (2019) (see Lemma A.1 for details) that for all $x \in A$, the (sub)domain of $\mathscr{D}_{\mathscr{G}}$ containing all preferences with x as the top-ranked alternative is swap-connected, which implies that the betweenness relations generated by the spanning trees of a graph are swap-connected. Therefore, it follows from Corollary 6.1 that for almost all prior profiles, unanimous and swap-LOBIC RBRs on the single-peaked domain on a graph are tops-only. Consequently, we obtain the following corollary from Corollary 4.1.

Corollary 6.3. *Let $\mathscr{D}$ be the single-peaked domain on a graph. For every unanimous and weak elementary monotonic RSCF $\varphi : \mathscr{D}^n \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is swap-LOBIC if and only if φ is swap-LDSIC.*

Remark 6.1. It follows from Theorem 6.1 that the single-peaked domain on a graph is swap-DLGE. It is shown in Peters et al. (2019) that a DSCF on the single-peaked domain on a graph is unanimous and DSIC if and only if it is a *monotonic collection of*

[5]A tree is called a line if it has exactly two nodes with degree one (such nodes are called leafs).

[6]A line graph can be represented by a linear order $\prec$ over the alternatives in an obvious manner: if the edges in a line graph are $\{(a_1,a_2),\ldots,(a_{m-1},a_m)\}$, then one can take the linear order $\prec$ as $a_1 \prec \cdots \prec a_m$.

parameters based rule (see Theorem 5.5 in Peters et al. (2019) for details). Therefore, it follows as a corollary of Theorem 6.1 that for almost all prior profiles, unanimous and weak elementary monotonic swap-LOBIC RBRS on the single-peaked domain on a graph are monotonic collection of parameters based rule.[7] □

Remark 6.2. Cho (2016) shows that the single-peaked domain is swap-RLGE. Moreover, Peters et al. (2014) show that every unanimous and DSIC RSCF on the single-peaked domain is a *probabilistic fixed ballot rule (PFBR)*. Therefore, for almost all prior profiles, unanimous and weak elementary monotonic swap-LOBIC RBRS on the single-peaked domain are PFBRs. □

In what follows, we provide a discussion on the structure of unanimous and swap-LOBIC RBRs on the single-peaked domain that do not satisfy weak elementary monotonicity. The structure of such RBRs depends on the specific prior profile. In the following example, we present an RSCF for three agents that is unanimous and OBIC with respect to any independent prior profile (μ_1,μ_2,μ_3) where $\mu_2(abc) \geq \frac{1}{6}$.[8] By Corollary 6.1, we know that such an RSCF will be tops-only. In Table 3, the preferences in rows and columns belong to agents 1 and 2, respectively, and the preferences written at the top-left corner of any table belong to agent 3. Note that agent 3 is the dictator for this RSCF except when she has the preference *abc*. When she has the preference *abc*, the rule violates weak elementary monotonicity over the profiles (abc,bac,abc) and (bac,bac,abc). Note that except from such violations, the rule behaves like a PFBR.

abc	*abc*	*bac*	*bca*	*cba*
abc	(1,0,0)	(0.4,0.6,0)	(0.4,0.6,0)	(0.4,0.6,0)
bac	(0.5,0.5,0)	(0.5,0.5,0)	(0.5,0.5,0)	(0.5,0.5,0)
bca	(0.5,0.5,0)	(0.5,0.5,0)	(0.5,0.5,0)	(0.5,0.5,0)
cba	(0.5,0.5,0)	(0.5,0.5,0)	(0.5,0.5,0)	(0.5,0.5,0)

bac	*abc*	*bac*	*bca*	*cba*
abc	(0,1,0)	(0,1,0)	(0,1,0)	(0,1,0)
bac	(0,1,0)	(0,1,0)	(0,1,0)	(0,1,0)
bca	(0,1,0)	(0,1,0)	(0,1,0)	(0,1,0)
cba	(0,1,0)	(0,1,0)	(0,1,0)	(0,1,0)

[7] Although Peters et al. (2019) provide the said characterization (Theorem 5.5) for RSCFs, we cannot apply it to obtain a characterization of LOBIC RSCFs as it is not known whether the single-peaked domain on a graph is RLGE or not.

[8] The rule is OBIC for dependent priors if: $5\mu_1(abc,abc) \geq \mu_1(bac,abc) + \mu_1(bca,abc) + \mu_1(cba,abc)$, where the first and the second preference in μ_1 denote the preferences of agents 2 and 3, respectively.

bca	abc	bac	bca	cba
abc	$(0,1,0)$	$(0,1,0)$	$(0,1,0)$	$(0,1,0)$
bac	$(0,1,0)$	$(0,1,0)$	$(0,1,0)$	$(0,1,0)$
bca	$(0,1,0)$	$(0,1,0)$	$(0,1,0)$	$(0,1,0)$
cba	$(0,1,0)$	$(0,1,0)$	$(0,1,0)$	$(0,1,0)$

cba	abc	bac	bca	cba
abc	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$
bac	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$
bca	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$
cba	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$

Table 3

6.2 Hybrid domains

Chatterji et al. (2020) introduce the notion of hybrid domains and discuss its importance. These domains satisfy single-peaked property only over a subset of alternatives. Let us assume that $A = \{1,\ldots,m\}$. Throughout this subsection, we assume that two alternatives $\underline{k}$ and $\overline{k}$ with $\underline{k} < \overline{k}$ are arbitrary but fixed.

Definition 6.2. A preference P is called $(\underline{k},\overline{k})$-hybrid if the following two conditions are satisfied:

(i) For all $r,s \in A$ such that either $r,s \in [1,\underline{k}]$ or $r,s \in [\overline{k},m]$, $\big[r < s < P(1)$ or $P(1) < s < r\big] \Rightarrow [sPr]$.

(ii) $\big[P(1) \in [1,\underline{k}]\big] \Rightarrow \big[\underline{k}Pr$ for all $r \in (\underline{k},\overline{k}]\big]$ and
$\big[P(1) \in [\overline{k},m]\big] \Rightarrow \big[\overline{k}Ps$ for all $s \in [\underline{k},\overline{k})\big]$.[9]

A domain is $(\underline{k},\overline{k})$-**hybrid** if it contains all $(\underline{k},\overline{k})$-hybrid preferences. The betweenness relation β that generates a $(\underline{k},\overline{k})$-hybrid domain is as follows: if $x < y$ then $\beta(x,y) = \{x,y\} \cup \big((x,y) \setminus (\underline{k},\overline{k})\big)$ and if $y < x$ then $\beta(x,y) = \{x,y\} \cup \big((y,x) \setminus (\underline{k},\overline{k})\big)$. In other words, an alternative other than x and y lies between x and y if and only if it lies in the interval $[x,y]$ or $[y,x]$ but not in the interval $(\underline{k},\overline{k})$.

In what follows, we argue that a hybrid domain satisfies the upper contour preservation property. Consider a preference P in a $(\underline{k},\overline{k})$-hybrid domain. Suppose xPy for some $x,y \in A$. Assume without loss of generality that $x < a$. Let $P(1) = a$ and let $U(x,P) \cap [x,a] = \{x^1 = a,\ldots,x^k = x\}$ where $x^1Px^2P\cdots Px^k$. Note that by the definition of the $(\underline{k},\overline{k})$-hybrid domain, from P one can go to a preference with x^2 at the top though a swap-local path by maintaining the upper contour set of y. Therefore, by repeated

[9] For two alternatives x and y, by $(x,y]$ we denote the alternatives z such that $x < z \leq y$. The interpretation of the notation $[x,y)$ is similar.

application of this fact, one can go to a preference with x at the top by maintaining the upper contour set of y. This shows that a hybrid domain satisfies the upper contour preservation property. Therefore, we obtain the following corollary from Theorem 3.2.

Corollary 6.4. *Let $\mathscr{D}$ be the the $(\underline{k},\overline{k})$-hybrid domain. For every unanimous RSCF $\varphi : \mathscr{D}^n \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is swap-LOBIC implies that φ is Pareto optimal.*

Using similar logic as we have used in the case of a single-peaked domain on a tree, it follows that the betweenness relation that generates a hybrid domain is strongly consistent. Therefore, Corollary 6.1 implies that for almost all prior profiles, unanimous and swap-LOBIC RBRs on the $(\underline{k},\overline{k})$-hybrid domain are tops-only. Therefore, by Corollary 4.1, we obtain the following corollary.

Corollary 6.5. *Let $\mathscr{D}$ be the $(\underline{k},\overline{k})$-hybrid domain. For every unanimous and weak elementary monotonic RSCF $\varphi : \mathscr{D}^n \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is swap-LOBIC if and only if φ is swap-LDSIC.*

Remark 6.3. Chatterji et al. (2020) show that every unanimous and DSIC RSCF on the hybrid domain is a $(\underline{k},\overline{k})$-*restricted probabilistic fixed ballot rule ($(\underline{k},\overline{k})$-RPFBR)*. Since the hybrid domain is swap-RLGE (see Chatterji et al. (2020) for details), Corollary 6.5 implies that for almost all prior profiles, unanimous and weak elementary monotonic swap-LOBIC RBRS on the $(\underline{k},\overline{k})$-hybrid domain are $(\underline{k},\overline{k})$-RPFBR. □

6.3 Multiple single-peaked domains

The notion of multiple single-peaked domains is introduced in Reffgen (2015). As the name suggests, these domains are union of several single-peaked domains. It is worth mentioning that these domains are different from hybrid domains–neither of them contains the other. For ease of presentation, we denote a single-peaked domain with respect to a prior ordering $\prec$ over A by $\mathscr{D}_\prec$.

Definition 6.3. Let $\Omega \subseteq \mathscr{P}(A)$ be a swap-connected collection of prior orderings over A. A domain $\mathscr{D}$ is called **multiple single-peaked** with respect to Ω if $\mathscr{D} = \cup_{\prec \in \Omega} \mathscr{D}_\prec$.

Since the prior orders in a multiple single-peaked domain are assumed to be swap-connected, it follows that preferences with the same top-ranked alternative are swap-connected. This implies that the collection $\mathscr{B}$ of betweenness relations that generate a multiple single-peaked domain is swap-connected. Using similar logic as we have used in the case of a single-peaked domain on a tree, it follows that multiple single-peaked domains are both weakly and strongly consistent betweenness domains. Therefore, Corollary 6.1 implies that for almost all prior profiles, unanimous and swap-LOBIC RBRs on the multiple single-peaked domain are tops-only. Using similar argument as we have used in the case of a single-peaked domain on a tree, it follows that multiple single-peaked domains satisfy the upper contour preservation property. In view of these observations, we obtain the following corollaries from Theorem 3.2 and Corollary 4.1.

Corollary 6.6. *Let $\mathscr{D}$ be the multiple single-peaked domain. For every unanimous RSCF $\varphi : \mathscr{D}^n \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is swap-LOBIC implies that φ is Pareto optimal.*

Corollary 6.7. *Let $\mathscr{D}$ be the multiple single-peaked domain. For every unanimous and weak elementary monotonic RSCF $\varphi : \mathscr{D}^n \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is swap-LOBIC if and only if φ is swap-LDSIC.*

Let us assume without loss of generality that Ω contains the integer ordering $<$ over $A = \{1, \ldots, m\}$. For a class of prior ordering Ω over A, the left cut-off $\underline{k}$ is defined as the maximum (with respect to $<$) alternative with the property that $1 \prec 2 \prec \cdots \prec \underline{k} \prec x$ for all $x \notin \{1, \ldots, \underline{k}\}$ and all $\prec \in \Omega$. Similarly, define the right cut-off as the minimum alternative $\overline{k}$ such that $x \prec \overline{k} \prec \cdots \prec m-1 \prec m$ for all $x \notin \{\overline{k}, \ldots, m\}$ and all $\prec \in \Omega$.

Remark 6.4. Reffgen (2015) shows that a DSCF is unanimous and DSIC on a multiple single-peaked domain with left cut-off $\underline{k}$ and right cut-off $\overline{k}$ if and only if it is a $(\underline{k}, \overline{k})$-*partly dictatorial generalized median voter scheme* ($(\underline{k}, \overline{k})$*-PDGMVS)*. Moreover, by Theorem 6.1, a multiple single-peaked domain is a swap-DLGE domain. Combining all these results with Corollary 6.7, we obtain that for almost all prior profiles, unanimous and weak elementary monotonic swap-LOBIC RBRs on the multiple single-peaked domain are $(\underline{k}, \overline{k})$-PDGMVS. □

6.4 Domains under partitioning

The notion of domains under partitioning is introduced in Mishra and Roy (2012). Such domains arise when a group of objects are to be partitioned based on the preferences of the agents over different partitions.

Let X be a finite set of objects and let A be the set of all partitions of X.[10] For instance, if $X = \{x,y,z\}$, then elements of A are $\big\{\{x\},\{y\},\{z\}\big\}$, $\big\{\{x\},\{y,z\}\big\}$, $\big\{\{y\},\{x,z\}\big\}$, $\big\{\{z\},\{x,y\}\big\}$, and $\big\{\{x,y,z\}\big\}$. We say that two objects are together in a partition if they are contained in a common element (subset of X) of the partition. For instance, objects x and y are together in the partition $\big\{\{z\},\{x,y\}\big\}$. If two objects are not together in a partition, we say they are separated. For three distinct partitions $X_1,X_2,X_3 \in A$, we say X_2 lies between X_1 and X_3 if for every two objects x and y, x and y are together in both X_1 and X_3 implies they are also together in X_2, and x and y are separate in both X_1 and X_3 implies they are also separate in X_2. For instance, any of the partitions $\big\{\{x\},\{y,z\}\big\}$ or $\big\{\{y\},\{x,z\}\big\}$ or $\big\{\{z\},\{x,y\}\big\}$ lies between $\big\{\{x\},\{y\},\{z\}\big\}$ and $\big\{\{x,y,z\}\big\}$. This follows from the fact that no two objects are together (or separated) in both $\big\{\{x\},\{y\},\{z\}\big\}$ and $\big\{\{x,y,z\}\big\}$, so the betweenness condition is vacuously satisfied. For another instance, consider the partitions $\big\{\{x,y\},\{z\}\big\}$ and $\big\{\{x,z\},\{y\}\big\}$. The only partition that lies between these two partitions is $\big\{\{x\},\{y\},\{z\}\big\}$. To see this, note that y are z are separate in both the partitions (and no two objects are together in both), and $\big\{\{x\},\{y\},\{z\}\big\}$ is the only partition (other than the two) in which y and z are separated.

Definition 6.4. A domain $\mathscr{D}$ is **intermediate** if for all $P \in \mathscr{D}$ and every two partitions $X_1,X_2 \in A$, X_1 lies between $P(1)$ and X_2 implies X_1PX_2.

By definition, intermediate domains are betweenness domains. In Table 4, we present three preferences (having have different structures of the top-ranked partition) in an intermediate domain over three objects. Note that the betweenness relation does not specify the ordering of $\{\{a,b\},\{c\}\}$, $\{\{a,c\},\{b\}\}$, and $\{\{a\},\{b,c\}\}$ when $\{\{a\},\{b\},\{c\}\}$ is the top-ranked partition. Therefore, there are six preferences with $\{\{a\},\{b\},\{c\}\}$ as the top-ranked partition, P^1 is one of them. It is worth noting that an intermediate domain

[10] A partition of a set is a set of subsets of that set that are mutually exclusive and exhaustive.

is not swap-connected. For instance, the preferences P^2 and P^3 are graph-local but not swap-local.

P^1	P^2	P^3
$\{\{a\},\{b\},\{c\}\}$	$\{\{a,b\},\{c\}\}$	$\{\{a,b,c\}\}$
$\{\{a,b\},\{c\}\}$	$\{\{a,b,c\}\}$	$\{\{a,b\},\{c\}\}$
$\{\{a,c\},\{b\}\}$	$\{\{a\},\{b\},\{c\}\}$	$\{\{a,c\},\{b\}\}$
$\{\{a\},\{b,c\}\}$	$\{\{a,c\},\{b\}\}$	$\{\{a\},\{b,c\}\}$
$\{\{a,b,c\}\}$	$\{\{a\},\{b,c\}\}$	$\{\{a\},\{b\},\{c\}\}$

Table 4

Proposition 6.1. *The intermediate domain is strongly consistent.*

The proof of this proposition is relegated to Appendix C.8.

By Corollary 6.1 and Proposition 6.1, it follows that for almost all prior profiles, unanimous and DSIC RBRs on the intermediate domain are tops-only. This is a major step towards characterizing unanimous and OBIC RBRs for almost all prior profiles on the intermediate domain. It is worth mentioning that the structure of unanimous and DSIC RSCFs are yet not explored on the intermediate domain and it follows from Corollary 6.1 that every such rule is tops-only.

Remark 6.5. It is shown in Mishra and Roy (2012) that a DSCF is unanimous and DSIC on the intermediate domain if and only if it is a *meet aggregator*. Moreover, by Theorem 6.1 and Proposition 6.1, every intermediate domain is graph-DLGE. Combining these results with Remark 3.1, we obtain that for almost all prior profiles, unanimous and weak elementary monotonic swap-LOBIC RBRS on the intermediate domain are meet aggregators. □

7. Applications on non-regular domains

In this section, we consider two important non-regular domains, namely single-dipped and single-crossing domains. Let the alternatives be $A = \{1, \ldots, m\}$.

7.1 Single-dipped domains

A preference is single-dipped if there is a "dip" (the worst alternative) of it so that as one moves farther away from it, preference increases. These domains arise in the context of locating a "public bad" (such as garbage dump, nuclear plant, wind mill, etc.).

Definition 7.1. A preference P is **single-dipped** if it has a unique minimal element $d(P)$, the *dip* of P, such that for all $x,y \in A$, $[d(P) \leq x < y$ or $y < x \leq d(P)] \Rightarrow yPx$. A domain is single-dipped if it contains all single-dipped preferences.

In what follows, we argue that the single-dipped domain satisfies the path-richness property. Consider two preferences of the form $a \cdots xy \cdots$ and $a \cdots yx \cdots$. We need to show that from every preference of the form $b \cdots$, we can reach a preference with a as the top-ranked alternative through a swap-local path such that the relative ranking of x and y does not change along the path. Since only one of the alternatives 1 and m can be a top-ranked alternative in the single-dipped domain and the domain is symmetric with respect to 1 and m, it is sufficient to show this for $a = 1$ and $b = m$.

First note that for any $x,y \in \{2,\dots,m\}$, there are preferences of the form 1 and $1 \cdots xy \cdots$ in the $1 \cdots yx \cdots$ domain. Consider arbitrary $x,y \in \{2,\dots,m\}$ and a preference $P \equiv m \cdots$. Suppose xPy. We can construct a swap-local path to a preference $P' \equiv m \cdots y$ such that no alternative overtakes y along the path. This can be done by shifting the dip of the preferences to y along the path, which is always possible by the definition of the single-dipped domain. Next, we go to a preference $P'' \equiv m\,1 \cdots y$ through a swap-local path such that y remains as the bottom ranked alternative in each preference in the path. Finally, we swap m and 1 to obtain a preference with 1 as the top-ranked alternative. By the construction of the whole path, no alternative overtakes y along the path. Since x is ranked above y in P, this, in particular, implies the relative ranking of x and y does not change along the path. Hence we obtain the following corollaries from Theorem 3.3 and Corollary 4.1.

Corollary 7.1. *Let $\mathscr{D}$ be the single-dipped domain. For every unanimous RSCF $\varphi : \mathscr{D}^n \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is swap-LOBIC implies that φ is tops-only.*

Corollary 7.2. *Let $\mathscr{D}$ be the single-dipped domain. For every unanimous and weak elementary monotonic RSCF $\varphi : \mathscr{D}^n \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is swap-LOBIC if and only if φ is swap-LDSIC.*

Remark 7.1. It is shown in Peters et al. (2017) that an RSCF on the single-dipped domain is unanimous and DSIC if and only if it is a *random committee rule*.By combining this result with Corollary 7.2 and the fact that every swap-LDSIC RSCF on the single-dipped domain is DSIC (see Cho (2016) for details) we obtain that for almost all prior profiles, unanimous and weak elementary monotonic swap-LOBIC RBRs on the single-dipped domain are random committee rules. □

7.2 SINGLE-CROSSING DOMAINS

A domain is single-crossing if its preferences can be ordered in a way so that no two alternatives change their relative ranking more than once along that ordering. Such domains are used in models of income taxation and redistribution, local public goods and stratification, and coalition formation (see Saporiti (2009) for details).

Definition 7.2. A domain $\mathscr{D}$ is **single-crossing** if there is an ordering $\lhd$ over $\mathscr{D}$ such that for all $x,y \in A$ and all $P,P' \in \mathscr{D}$, $[x < y, P \lhd P', \text{ and } yPx] \implies yP'x$.

To see that a single-crossing domain satisfies the path-richness property, consider an alternative a and suppose that there are two swap-local preferences $P \equiv a\cdots xy \cdots$ and $P' \equiv a \cdots yx \cdots$. Since P and P' are swap-local, they must be consecutive in the ordering $\lhd$. Assume without loss of generality that $P \lhd P'$. This means $x\hat{P}y$ for all $\hat{P}$ with $\hat{P} \lhd P$ and $y\bar{P}x$ for all $\bar{P}$ with $P' \lhd \hat{P}$. Consider any preference $\tilde{P}$. If $x\tilde{P}y$, then $\tilde{P} \lhd P$, and hence from $\tilde{P}$ one can go to the preference P following the path given by $\lhd$ maintaining the relative ordering between x and y. On the other hand, if $y\tilde{P}x$, then one can go from $\tilde{P}$ to the preference P' following the path given by $\lhd$. This shows that a single-crossing domain satisfies the path-richness property, and hence we obtain the following corollaries from Theorem 3.3 and Corollary 4.1.

Corollary 7.3. *Let $\mathscr{D}$ be the single-crossing domain. For every unanimous RSCF $\varphi : \mathscr{D}^n \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is graph-LOBIC implies that φ is tops-only.*

Corollary 7.4. *Let $\mathscr{D}$ be the single-crossing domain. For every unanimous and weak elementary monotonic RSCF $\varphi : \mathscr{D}^n \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with* *full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is swap-LOBIC if and only if φ is swap-LDSIC.*

Remark 7.2. Roy and Sadhukhan (2019) show that an RSCF on the single-crossing domain is unanimous and DSIC if and only if it is a *tops-restricted probabilistic fixed ballot rules (TPFBRs)*. Moreover, Cho (2016) shows that every swap-LDSIC RSCF on the single-crossing domain is DSIC. Combining these results with Corollary 7.4, we obtain that for almost all prior profiles, unanimous and weak elementary monotonic swap-LOBIC RBRs on the single-crossing domain are TPFBRs. □

8. Applications on multi-dimensional separable domains

Multi-dimensional separable domains comprise the main application of our general model. Multi-dimensional models are used in political economy, as well as in public good location problems where an alternative represents the location of a political party/public good in the multi-dimensional political spectrum/Euclidean space (see Breton and Sen (1999) and Border and Jordan (1983) for details). Such models are also used to deal with the problem of forming a committee by taking members from a given set of candidates (see Barberà et al. (1991)). In a different context, this model is used in formulating the model of externalities in the context of the debate on liberalism (see Sen (1970) and Wriglesworth (1985)). In this setting, a social alternative has several components. Each component represents some aspect of the alternative. There is no dependence between the components, that is, the set of alternatives is a product set (of the alternatives available in different components). Separability implies that there is no interaction between the preferences of an agent (over the alternatives) in different components.

We assume that the alternative set can be decomposed as a Cartesian product, i.e., $A = A^1 \times \cdots \times A^k$, where $1, \ldots, k$ are the components/dimensions with $k \geq 2$, and for each component $l \in K$, the component set A^l contains at least two elements. Thus, an alternative x is a vector of k elements, and hence we denote it $(x^1, \ldots, x^k)$. For $l \in K$, we denote by A^{-l} the set $A^1 \times \cdots \times A^{l-1} \times A^{l+1} \times \cdots \times A^k$ and by x^{-l} an element of A^{-l}.

A preference $P \in \mathscr{P}(A)$ is *separable* if there exists a (unique) marginal preference P^l for each $l \in K$ such that for all $x, y \in A$, we have $[x^l P^l y^l$ for some $l \in K$ and $x^{-l} =$

$y^{-l}] \Rightarrow [xPy]$. A domain is called separable if each preference in it is separable.

For a collection of marginal preferences $(P^1,\dots,P^k)$, the collection of all separable preferences with marginals as $(P^1,\dots,P^k)$ is denoted by $\mathscr{S}(P^1,\dots,P^k)$. Similarly, for a collection of marginal domains $(\mathscr{D}^1,\dots,\mathscr{D}^k)$, the set of all separable preferences with marginals in $(\mathscr{D}^1,\dots,\mathscr{D}^k)$ is denoted by $\mathscr{S}(\mathscr{D}^1,\dots,\mathscr{D}^k)$, that is, $\mathscr{S}(\mathscr{D}^1,\dots,\mathscr{D}^k) = \cup_{(P^1,\dots,P^k)\in(\mathscr{D}^1,\dots,\mathscr{D}^k)}\mathscr{S}(P^1,\dots,P^k)$. A separable domain of the form $\mathscr{S}(\mathscr{D}^1,\dots,\mathscr{D}^k)$ is called a full separable domain. Throughout this subsection, we assume that the marginal domains are betweenness domains satisfying swap-connectedness and consistency, for instance, they can be any domain we have discussed so far except the intermediate domain. For $P_N \in \mathscr{S}(\mathscr{D}^1,\dots,\mathscr{D}^k)$, we denote its restriction to a component $l \in K$ by P_N^l, that is, $P_N^l = (P_1^l,\dots,P_n^l)$. We introduce the local structure in a full separable domain in a natural way.

Definition 8.1. Let $\mathscr{D}^l$ be swap-connected for all $l \in K$. Two preferences $P,\bar{P} \in \mathscr{S}(\mathscr{D}^1,\dots,\mathscr{D}^k)$ are **sep-local** if one of the following two holds:

(i) $P\triangle\bar{P} = \{x,y\}$ where x,y are such that $|\{l \mid x^l \neq y^l\}| \geq 2$.

(ii) $P\triangle\bar{P} = \{((a^{-l},x^l),(a^{-l},y^l)) \mid a^{-l} \in A^{-l}\}$, where $l \in K$ and $x^l,y^l \in A^l$ swap from P^l to $\bar{P}^l$.

Thus, (i) in Definition 8.1 says that exactly one pair of alternatives (x,y), that vary over at least two components, swap from P to P', and (ii) in Definition 8.1 says that multiple pairs of alternatives of the form $((a^{-l},x^l),(a^{-l},y^l))$, where $a^{-l} \in A^{-l}$, swaps from P to P'. This structure makes the lower contour monotonicity property simpler: it imposes elementary monotonicity to every pair of swapping alternatives. We call it **sep-monotonicity**.

For notational convenience, we denote a domain $\mathscr{S}(\mathscr{D}^1,\dots,\mathscr{D}^k)$ by $\mathscr{S}$ in the following results. The following corollary is obtained from Theorem 3.1.

Corollary 8.1. *For every unanimous and weak elementary monotonic RSCF $\varphi : \mathscr{S}^n \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ,μ_N) is sep-LOBIC if and only if φ is sep-LDSIC.*

It is worth mentioning that Corollary 8.1 holds as long as the marginal domains are swap-connected.

Our next two propositions are derived by using Theorem 3.3. An RSCF $\varphi : \mathscr{S}^n \to \Delta A$ satisfies **component-unanimity** if for each component $l \in K$ and each $P_N \in \mathscr{S}^n$ such that $P_i^l(1) = x^l$ for all $i \in N$ and some $x^l \in A^l$, we have $\varphi^l_{x^l}(P_N) = 1$.

Proposition 8.1. *For every unanimous RSCF $\varphi : \mathscr{S}^n \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is sep-LOBIC implies that φ satisfies component-unanimity.*

The proof of this proposition is relegated to Appendix C.9.

Proposition 8.2. *For every unanimous RSCF $\varphi : \mathscr{S}^n \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, (φ, μ_N) is sep-LOBIC implies that φ is tops-only.*

The proof of this proposition is relegated to Appendix C.10.

For random rules, to the best of our knowledge, it is still not known whether sep-LDSIC implies DSIC or not. However, the same is shown for DSCFs on domains having unrestricted marginals (see Kumar et al. (2020) for details). Thus, it follows from Corollary 8.1 that for almost all prior profiles, sep-monotonic DSCFs, OBIC and DSIC are equivalent on such domains.

8.1 Marginal decomposability of random rules

Breton and Sen (1999) show that every unanimous and DSIC DSCF on a multi-dimensional (full) separable domain is decomposable: its outcome in a particular dimension depends only on the (marginal) preferences of agents in that dimension. In our view, a suitable version of decomposability for random rules is marginal decomposability, which we investigate for sep-LOBIC rules in this section.

We define the notion of marginal distribution for an RSCF or a prior. The notion is the same as the standard notion of marginal distribution of a multivariate/joint distribution in statistics. The marginal distribution of an RSCF $\varphi : \mathscr{S}^n \to \Delta A$ over component $l \in K$ at a preference profile P_N, denoted by $\varphi^l(P_N)$, is defined as $\varphi^l_{x^l}(P_N) = \sum\limits_{x^{-l} \in A^{-l}} \varphi_{(x^l, x^{-l})}(P_N)$

for all $x^l \in A^l$. Similarly, marginal distribution of a prior μ_i over component $l \in K$, denoted by μ_i^l, is defined as $\mu_i^l(\hat{P}_{-i}^l) = \sum_{P_{-i}|P_{-i}^l=\hat{P}_{-i}^l} \mu_i(P_{-i})$ for all $\hat{P}_{-i}^l \in \mathscr{D}_{-i}^l$.

An OBIC RBR is marginally decomposable if it can be viewed as a collection of OBIC RBRs, one for each component. Clearly, it is easier for the designer to work with the problem of analyzing marginally decomposable RBR: she does not need to deal with the multi-dimensional objects (alternatives or preferences), instead she can reduce the whole problem to a collection of one-dimensional problems and solve it solely by using her knowledge about the same.

Definition 8.2. An OBIC RBR (φ, μ_N) on a domain $\mathscr{S}^n$ is **marginally decomposable** if for all $l \in K$, there is an OBIC RBR $(\hat{\varphi}^l, \mu_N^l)$ on $(\mathscr{D}^l)^n$ such that for all $P_N \in \mathscr{S}^n$, we have $\varphi^l(P_N) = \hat{\varphi}^l(P_N^l)$.

Similarly, we define the notion of marginal decomposability for a DSIC RSCF.

Definition 8.3. A DSIC RSCF φ on a domain $\mathscr{S}^n$ is **marginally decomposable** if for all $l \in K$, there is a DSIC RSCF $\hat{\varphi}^l$ on $(\mathscr{D}^l)^n$ such that for all $P_N \in \mathscr{S}^n$, we have $\varphi^l(P_N) = \hat{\varphi}^l(P_N^l)$.

Remark 8.1. Note that for a DSCF $f : \mathscr{S}^n \to A$, marginal decomposability is equivalent to decomposability defined in Breton and Sen (1999) as follows: a DSCF $f : \mathscr{S}^n \to A$ is decomposable if for all $l \in K$, there is a DSIC DSCF $\hat{f}^l$ on $(\mathscr{D}^l)^n$ such that for all $P_N \in \mathscr{S}^n$, we have $f^l(P_N) = \hat{f}^l(P_N^l)$. Thus, the notion of marginal decomposability for RSCFs indeed generalizes the same for DSCFs. □

Theorem 8.1. *For every unanimous RSCF $\varphi : \mathscr{S}^n \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is OBIC implies that (φ, μ_N) is marginally decomposable.*

The proof of this theorem is relegated to Appendix C.11.

It can be noted that in the proof of Theorem 8.1, only the block preservation property of φ is used in the proof, which is derived from the fact that (φ, μ_N) OBIC for all $\mu_N \in \mathscr{M}(\varphi)$. Consider a unanimous and DSIC RSCF $\varphi : \mathscr{S}^n \to \Delta A$. Since φ is DSIC, it is straightforward that φ satisfies the block preservation property. Therefore, by using similar arguments as in the proof of Theorem 8.1, we obtain the following result.

Theorem 8.2. *Every unanimous and DSIC RSCF $\varphi : \mathscr{S}^n \to \Delta A$ is marginally decomposable.*

9. THE CASE OF WEAK PREFERENCES

A weak preference is a complete and transitive binary relation. We denote a weak preference by R and the set of all weak preferences by $\mathscr{R}(A)$. For a weak preference R, we denote its strict part by P and indifference part by I. An indifference class of a preference is the maximal set of alternatives that are indifferent to each other.

As in the case of strict preferences, we assume that each domain $\mathscr{D}_i \subseteq \mathscr{R}_i$ is endowed with a graph structure with respect to which it is connected. We generalize the definition of a block for weak preferences in the following way. A set of alternatives B is a block in a pair of preferences (R,R') if it is a minimal non-empty set satisfying the following properties: (i) for all $x \in B$ and $y \notin B$, $P|_{\{x,y\}} = P'|_{\{x,y\}}$, and (ii) B is not a strict subset of an indifference class of R and an indifference class of R'.

Note that the technical definition of lower contour monotonicity and block preservation property (Proposition 3.1 and Theorem 3.1) do not involve the assumption of strict preferences, therefore we continue to use the same definitions for weak preferences. Our next two results say that Proposition 3.1 and Theorem 3.1 continue to hold in this scenario.

Proposition 9.1. *For every RSCF $\varphi : \mathscr{D}_N \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is graph-LOBIC implies that φ satisfies the block-preservation property.*

The proof of this proposition is relegated to Appendix B.

Theorem 9.1. *For every lower contour monotonic RSCF $\varphi : \mathscr{D}^n \to \Delta A$, there is a set of prior profiles $\mathscr{M}(\varphi)$ with full measure such that for each $\mu_N \in \mathscr{M}(\varphi)$, the RBR (φ, μ_N) is graph-LOBIC if and only if φ is graph-LDSIC.*

The proof of this theorem is relegated to Appendix C.1.

10. DISCUSSION

10.1 THE CASE OF DBRS

A probability distribution ν on a finite set S is generic if for all subsets U and V of S, $\nu(U) = \nu(V)$ implies $U = V$. Majumdar and Sen (2004) show that on the unrestricted domain, every unanimous DBR that is OBIC with respect to a generic prior is dictatorial, and Mishra (2016) shows that under elementary monotonicity, the notions DSIC and OBIC with respect to generic priors are equivalent. It can be verified that all our results hold for generic priors if we restrict our attention to DBRs. Additionally, our results establish tops-onlyness and decomposability of OBIC DBRs with respect to generic priors.

10.2 FULLY CORRELATED PRIORS

Note that the priors we consider in this paper are partially correlated: prior of an agent is independent of her own preference, while it may be correlated over the preferences of other agents. The natural question arises here as to what will happen if the prior of an agent depends on her own preferences too. Firstly, our proof technique for Theorem 3.1 will fail, but more importantly, Theorem 3.1 will not even hold anymore. It can be verified from the proof of Proposition 3.1 that if an RSCF is graph-LOBIC but not graph-LDSIC then it must satisfy a system of equations . The proof follows from the fact that the set of prior profiles that satisfy such a system of equations has Lebesgue measure zero. However, if an agent has two different priors for two local preferences, then we cannot obtain such a system of equations on a given prior (what we obtain are equations involving different priors), and consequently, nothing can be concluded about the Lebesgue measure of such priors. We illustrate this with the following example.

Suppose that there are two agents 1 and 2, and three alternatives a,b, and c. Consider two swap-local preferences bac and bca of agent 1. Consider the anti-plurality rule with the tie-breaking criteria as $a \succ b \succ c$. In Table 5, we present this rule when agent 1 has preferences bac and bca, and 2 has any preference. It is well-known (and also can be verified from the example) that anti-plurality rule is not swap-LDSIC. However, it is swap-LOBIC over the mentioned preferences of agent 1 if her prior satisfies the following conditions: $\mu_1(bca|cab)+\mu_1(cba|cab)-\mu_1(acb|cab)-\mu_1(cab|cab)\geq 0$ and $\mu_1(acb|cba)+\mu_1(cab|cba)-\mu_1(bca|cba)-\mu_1(cba|cba)\geq 0$. It is clear that the Lebesgue measure of such priors is not zero (this is because, as we have argued, the inequalities are imposed on two different priors $\mu_1(\cdot|cab)$ and $\mu_1(\cdot|cba)$). In a similar way, it follows that if one considers all possible restrictions arising from all possible swap-local preferences of each agent, the resulting priors for which the rule is LOBIC can have Lebesgue measure strictly bigger than zero.

1 \ 2	abc	acb	bac	bca	cba	cab
cab	a	a	a	c	a	c
cba	b	c	b	b	c	b

Table 5

10.3 Relation with Hong and Kim (2020)

Hong and Kim (2020) explore the structure of LOBIC DBRs with respect to generic priors (as defined in Majumdar and Sen (2004)) on weakly connected domains without restoration. They show that if a unanimous DBR on a weakly connected domains without restoration domain is LOBIC with respect to generic priors, then it will be tops-only. Since they consider weakly connected domains, even the deterministic versions of our results for multi-dimensional domains and intermediate domains do not follow from their result. Coming to the unrestricted domain and single-peaked domains, which are weakly connected without restoration (see Sato (2013) for details), Example 1 of Majumdar and Roy (2018) already shows that their results do not extend for RBRs on the unrestricted domain. Below, we provide an example to show that it does not extend for RBRs on single-peaked domains either.

Example 10.1. Consider the RSCF in Table 6.[11] The priors of agents 1 and 2, μ_1 and μ_2 are generic. For instance, $\mu_1(abc)(=0.1)$ is different from $\mu_1(S)$ for any set of preferences S other than $\{abc\}$, $\mu_1(abc)+\mu_1(bac)(=.4)$ is different from $\mu_1(S)$ for

[11] See Example 2.1 for an explanation of the table.

any set of preferences S other than $\{abc, bac\}$, etc. Preferences of agents 1 and 2 are depicted in the second column and the second row, and the outcome of the RSCF, say φ, is given by the corresponding cells. Clearly, the rule φ is unanimous. To see that φ is OBIC with respect to the given priors, consider, for instance, agent 1. Suppose her sincere preference is abc. If she reports this preference, she receives interim expected outcome $\varphi(abc, \mu_1) = (0.514, 0.3856, 0.1004)$. If she misreports, say as the preference bac, then she receives interim expected outcome $\varphi(bac, \mu_1) = (0.04, 0.8596, 0.1004)$. Since $\varphi(abc, \mu_1)$ stochastically dominates $\varphi(bac, \mu_1)$ at abc, agent 1 cannot manipulate by misreporting the preference abc as bac. In a similar fashion, it can be verified that no agent can manipulate φ. Now, consider the profiles (abc, bac) and (abc, bca). Each agent has the same top-ranked alternative in these two profiles. However, $\varphi(abc, bac) \neq \varphi(abc, bca)$, which means φ is not tops-only.

	μ_1	0.1	0.3	0.44	0.16
μ_2	1 \ 2	abc	bac	bca	cba
0.2	abc	(1,0,0)	(0.5,0.4,0.1)	(0.44,0.4,0.16)	(0.44,0.56,0)
0.24	bac	(0.4,0.3,0.3)	(0,1,0)	(0,1,0)	(0,0.56,0.44)
0.34	bca	(0.4,0.3,0.3)	(0,1,0)	(0,1,0)	(0,0.56,0.44)
0.22	cba	(0.4,0.3,0.3)	(0,0,1)	(0,0,1)	(0,0,1)

Table 6

APPENDIX

A. PRELIMINARIES FOR THE PROOFS

Consider an RSCF $\varphi : \mathscr{D}_N \to \Delta A$. A prior profile μ_N is called compatible with φ if for all $i \in N$, all $R_i, R_i' \in \mathscr{D}_i$, and all $X \subsetneq A$,

$$\sum_{R_{-i}} \mu_i(R_{-i})(\varphi_X(R_i, R_{-i}) - \varphi_X(R_i', R_{-i})) = 0 \tag{1}$$
$$\implies \varphi_X(R_i, R_{-i}) - \varphi_X(R_i', R_{-i}) = 0 \text{ for all } R_{-i}.$$

Let $\mathscr{M}(\varphi)$ denote the set of all prior profiles that are compatible with φ.

Claim A.1. *For every RSCF φ, the Lebesgue measure of the complement of $\mathscr{M}(\varphi)$ is zero.*

Proof of Claim A.1. The proof of this claim follows from elementary measure theory; we provide a sketch of it for the sake of completeness. First note that for a given RSCF φ and for all $i \in N$, all $R_i, R_i' \in \mathscr{D}_i$, and all $X \subsetneq A$, (1) is equivalent to an equation of the form:

$$x_1\alpha_1 + \cdots + x_k\alpha_k = 0, \tag{2}$$

where α's are some constants and x's are non-negative variables summing up to 1 (that is, probabilities). The question is if x's are drawn randomly (uniformly) from the space $\{(x_1,\ldots,x_k) \mid x_l \geq 0 \text{ for all } l \text{ and } \sum_l x_l = 1\}$, what is the Lebesgue measure of the priors for which (2) will be satisfied? Clearly, if α's are all zeros, (2) will be satisfied for all prior profiles. We argue that if α's are not all zeros, then (2) can be satisfied only for a set of prior profiles with Lebesgue measure zero, which will complete the proof by means of the fact that the number of agents, preferences, and alternatives are all finite. However, this follows from the facts that the solutions of (2) form a hyperplane and that the Lebesgue measure of a hyperplane is zero (because of dimensional reduction, such as the Lebesgue measure of a line in a plane is zero, that of a plane in a cube is zero, etc.).[12] ■

B. Proof of Proposition 3.1 and Proposition 9.1

Proof. Let (φ, μ_N) be a graph-LOBIC RBR. Since we prove the claim for a set of prior profiles with full measure, in view of Claim A.1, we assume that μ_N is compatible with φ. Consider graph-local preferences $R_i, R_i' \in \mathscr{D}_i$ and $R_{-i} \in \mathscr{D}_{-i}$. Suppose that B is a block in (R_i, R_i'). Let $U_B(R_i) = \{x \in A \mid xP_ib \text{ for all } b \in B\}$ be the set of alternatives that are strictly preferred to each element of B according to R_i. By the definition of a block in (R_i, R_i'), it follows that both $U_B(R_i)$ and $U_B(R_i) \cup B$ are upper contour sets in each of

[12] For a detailed argument, suppose that exactly one α, say α_1 is not zero. Note that this assumption gives maximum freedom for the values of x's and thereby maximize the Lebesgue measure of the solution space of (2). However, this means in any solution x_1 must be zero, the measure of which in the solution space is zero.

the preferences R_i and R'_i. Since R_i and R'_i are graph-local, by graph-LOBIC,

$$\sum_{R_{-i}\in\mathscr{D}_{-i}} \mu_i(R_{-i})\varphi_{U_B(R_i)}(R_i,R_{-i}) = \sum_{R_{-i}\in\mathscr{D}_{-i}} \mu_i(R_{-i})\varphi_{U_B(R_i)}(R'_i,R_{-i}) \tag{3}$$

and

$$\sum_{R_{-i}\in\mathscr{D}_{-i}} \mu_i(R_{-i})\varphi_{U_B(R_i)\cup B}(R_i,R_{-i}) = \sum_{R_{-i}\in\mathscr{D}_{-i}} \mu_i(R_{-i})\varphi_{U_B(R_i)\cup B}(R'_i,R_{-i}). \tag{4}$$

Subtracting (3) from (4), we have

$$\sum_{R_{-i}\in\mathscr{D}_{-i}} \mu_i(R_{-i})(\varphi_B(R_i,R_{-i}) - \varphi_B(R'_i,R_{-i})) = 0. \tag{5}$$

Since μ_N is compatible with φ, this means $\varphi_B(R_i,R_{-i}) = \varphi_B(R'_i,R_{-i})$ for all $R_{-i}\in\mathscr{D}_{-i}$, which completes the proof. ■

Remark B.1. It is worth noting from the proof that an RBR (φ,μ_N) must satisfy (5) in order to be graph-LOBIC. If the RSCF φ is not LDSIC, then there will be at least one B such that $\varphi_B(R_i,R_{-i}) - \varphi_B(R'_i,R_{-i}) \neq 0$, in which case (5) can only be satisfied for set of prior profiles with measure zero. □

C. Other proofs

In view of Proposition 3.1, whenever we prove some statement for a class of RBRs (φ,μ_N) where μ_N belongs to a set with full measure, we assume that φ is satisfies the block preservation property.

C.1 Proof of Theorem 3.1 and Theorem 9.1

Proof. If part of the theorem follows from the definitions of graph-LDSIC and graph-LOBIC. We proceed to prove the only-if part. Let $\varphi : \mathscr{D}_N \to \Delta A$ be an RSCF satisfying lower contour monotonicity and the block preservation property. We show that φ is graph-LDSIC. Consider graph-local preferences $R_i, R'_i \in \mathscr{D}_i$, $R_{-i}\in\mathscr{D}_{-i}$, and $x\in A$. We show $\varphi_{U(x,R_i)}(R_i,R_{-i}) \geq \varphi_{U(x,R_i)}(R'_i,R_{-i})$. Let $B_1,\dots,B_t$ be the blocks in (R_i,R'_i) such that for all $l < t$ and all $b \in B_l$ and $b' \in B_{l+1}$, we have bP_ib'. Suppose that $x \in B_l$ for some $l \in \{1,\dots,t\}$.

Let $\hat{B}_l = \{b \in B_l \mid bP_ix\}$ be the set of alternatives (possibly empty) in B_l that are (strictly) preferred to x. Note that the set $B_l \setminus \hat{B}_l$ is lower contour set of $R_i|_{B_l}$. Therefore, by lower contour monotonicity,

$$\varphi_{B_l \setminus \hat{B}_l}(R'_i, R_{-i}) \geq \varphi_{B_l \setminus \hat{B}_l}(R_i, R_{-i}). \tag{6}$$

Furthermore, by the block preservation property, we have

$$\varphi_{B_l}(R'_i, R_{-i}) = \varphi_{B_l}(R_i, R_{-i}). \tag{7}$$

Subtracting (6) from (7), we have

$$\varphi_{\hat{B}_l}(R_i, R_{-i}) \geq \varphi_{\hat{B}_l}(R'_i, R_{-i}). \tag{8}$$

Note that $U(x,R_i) = B_1 \cup \dots \cup B_{l-1} \cup \hat{B}_l$. This means $\varphi_{U(x,R_i)}(R_i,R_{-i}) = \varphi_{B_1 \cup \dots \cup B_{l-1}}(R_i,R_{-i}) + \varphi_{\hat{B}_l}(R_i,R_{-i})$ and $\varphi_{U(x,R_i)}(R'_i,R_{-i}) = \varphi_{B_1 \cup \dots \cup B_{l-1}}(R'_i,R_{-i}) + \varphi_{\hat{B}_l}(R'_i,R_{-i})$. By the block preservation property, $\varphi_{B_1 \cup \dots \cup B_{l-1}}(R_i,R_{-i}) = \varphi_{B_1 \cup \dots \cup B_{l-1}}(R'_i,R_{-i})$, and by (8) , $\varphi_{\hat{B}_l}(R_i,R_{-i}) \geq \varphi_{\hat{B}_l}(R'_i,R_{-i})$. Combining these observations, we have $\varphi_{U(x,R_i)}(R_i,R_{-i}) \geq \varphi_{U(x,R_i)}(R'_i,R_{-i})$, which completes the proof. ■

C.2 Proof of Theorem 3.2

Proof. Let $\mathscr{D}_i$ satisfy upper contour preservation property for all $i \in N$ and suppose that $\varphi : \mathscr{D}_N \to \Delta A$ is an RSCF satisfying unanimity and the block preservation property. We show that φ is Pareto optimal. Consider $P_N \in \mathscr{D}_N$ such that xP_iy for all $i \in N$ and some $x,y \in A$. We show that $\varphi_y(P_N) = 0$. Assume for contradiction $\varphi_y(P_N) > 0$. Consider $i \in N$. By the upper contour preservation property there exists a graph-local path $(P_i^1 = P_i, \dots, P_i^t)$ such that $P_i^t(1) = x$ and $U(P_i,y) = U(P_i^l,y)$ for all $l = 1,\dots,t$. Since $U(y,P_i^1) = U(y,P_i^2)$, we have $y \notin P_i^1 \triangle P_i^2$, which implies that $\{y\}$ is a singleton block in (P_i^1,P_i^2). By the block preservation property, this implies $\varphi_y(P_i^2,P_{-i}) = \varphi_y(P_i,P_{-i})$. Continuing in this manner, we reach a preference profile (P_i^t,P_{-i}) such that $P_i^t(1) = x$

and $\varphi_y(P_i^t, P_{-i}) > 0$. By applying the same argument to the agents $j \in N \setminus \{i\}$ we can construct a preference profile P'_N such that $P'_j(1) = x$ for all $j \in N$ and $\varphi_y(P'_N) > 0$. Since $P'_j(1) = x$ for all $j \in N$, by unanimity we have $\varphi_x(P'_N) = 1$, which contradicts that $\varphi_y(P'_N) > 0$. ■

C.3 Proof of Theorem 3.3

We use the following lemma in our proof.

Lemma C.1. *Suppose an RSCF $\varphi : \mathscr{D}^n \to \Delta A$ satisfies unanimity and the block preservation property. Let $P_i, P'_i \in \mathscr{D}$ be graph-local and let $P_{-i} \in \mathscr{D}^{n-1}$ be such that $\varphi_x(P_i, P_{-i}) \neq \varphi_x(P'_i, P_{-i})$ for some $x \in P_i \triangle P'_i$. Consider an agent $j \neq i$ and suppose that there is a graph-local path $(P_j^1 = P_j, \dots, P_j^t = \bar{P}_j)$ such that for all $l < t$ and for every two alternatives $a, b \in P_i \triangle P'_i$, there is a common upper contour set U of both P_j^l and P_j^{l+1} such that exactly one of a and b is contained in U. Then $\varphi_x(P_i, \bar{P}_j, P_{-\{i,j\}}) \neq \varphi_x(P'_i, \bar{P}_j, P_{-\{i,j\}})$.*

Proof of Lemma C.1. Suppose $\varphi_x(P_i, P_j^l, P_{-\{i,j\}}) \neq \varphi_x(P'_i, P_j^l, P_{-\{i,j\}})$ for some $l < t$ and some $x \in P_i \triangle P'_i$. It is enough to show that $\varphi_x(P_i, P_j^{l+1}, P_{-\{i,j\}}) \neq \varphi_x(P'_i, P_j^{l+1}, P_{-\{i,j\}})$. Let a and $\bar{a}$ be the alternatives, if exist, that are ranked just above and just below x, respectively, in $P_j^l|_{P_i \triangle P'_i}$. More formally, let $a \in P_i \triangle P'_i$ be such that $a P_j^l x$ and no alternative in $P_i \triangle P'_i$ is ranked between a and x, and let $\bar{a} \in P_i \triangle P'_i$ be such that $x P_j^l \bar{a}$ and no alternative in $P_i \triangle P'_i$ is ranked between x and a. Let U be the common upper contour set of P_j^l and P_j^{l+1} such that $U \cap \{a, x\} = a$, and $\widehat{U}$ be the common upper contour set of P_j^l and P_j^{l+1} such that $\widehat{U} \cap \{x, \bar{a}\} = x$. Here, U might be empty and $\widehat{U}$ might be A. Consider the set of alternatives $B = U \setminus \widehat{U}$. Note that B can be expressed as a union of blocks in (P_j^l, P_j^{l+1}). Therefore, by applying the block preservation property to each block in B, we obtain $\varphi_B(P_i, P_j^l, P_{-\{i,j\}}) = \varphi_B(P_i, P_j^{l+1}, P_{-\{i,j\}})$ and $\varphi_B(P'_i, P_j^l, P_{-\{i,j\}}) = \varphi_B(P'_i, P_j^{l+1}, P_{-\{i,j\}})$. Moreover, since each $c \in B \setminus x$ is a block in (P_i, P'_i), we have by the block preservation property, $\varphi_c(P_i, P_j^l, P_{-\{i,j\}}) = \varphi_c(P'_i, P_j^l, P_{-\{i,j\}})$ and $\varphi_c(P_i, P_j^{l+1}, P_{-\{i,j\}}) = \varphi_c(P'_i, P_j^{l+1}, P_{-\{i,j\}})$ for all $c \in B \setminus x$. Combining these observations, it follows that $\varphi_x(P_i, P_j^{l+1}, P_{-\{i,j\}}) \neq \varphi_x(P'_i, P_j^{l+1}, P_{-\{i,j\}})$. ■

Proof of Theorem 3.3. Let $\mathscr{D}$ satisfy the path-richness property (see Definition 3.3) and suppose that $\varphi : \mathscr{D}^n \to \Delta A$ is an RSCF satisfying unanimity and the block preservation property. We show that φ is tops-only. Assume for contradiction that $\varphi(P_i, P_{-i}) \neq \varphi(P'_i, P_{-i})$ for some $P_i, P'_i \in \mathscr{D}$ with $P_i(1) = P'_i(1)$ and some $P_{-i} \in \mathscr{D}^{n-1}$. By means of Condition (i) of the path-richness property, it is enough to assume that P_i and P'_i are graph-local. Therefore, by the block preservation property, it follows that $\varphi_x(P_i, P_{-i}) \neq \varphi_x(P'_i, P_{-i})$ for some $x \in P_i \triangle P'_i$.

Consider $j \in N \setminus \{i\}$. By Condition (ii) of the path-richness property, there is a path $(P^1_j = P_j, \ldots, P^t_j = P'_j)$ with $P'_j(1) = P_i(1)$ such that for all $l < t$ and for every two alternatives $a, b \in P_i \triangle P'_i$, there is a common upper contour set U of both P^l_j and P^{l+1}_j such that exactly one of a and b is contained in U. By applying Lemma C.1, it follows that $\varphi_x(P_i, P'_j, P_{-i}) \neq \varphi_x(P'_i, P'_j, P_{-i})$. By applying this logic to all agents except i, we construct $P'_{-i}\mathscr{D}^{n-1}$ such that $P'_j(1) = P_i(1)$ for all $j \neq i$ and $\varphi_x(P_i, P'_{-i}) \neq \varphi_x(P'_N)$. However, since (P_i, P'_{-i}) and (P'_i, P'_{-i}) are unanimous preference profiles with the top-ranked alternative different from x, $\varphi_x(P_i, P'_{-i}) = \varphi_x(P'_i, P'_{-i}) = 0$, a contradiction. ■

C.4 PROOF OF THEOREM 4.1

Proof. Let $\mathscr{D}$ be swap-connected and suppose that $\varphi : \mathscr{D}_N \to \Delta A$ is a tops-only RSCF satisfying weak elementary monotonicity and the block preservation property. We show that φ is swap-LDSIC.

Let P_i and P'_i be two swap-local preferences. If $\tau(P_i) = \tau(P'_i)$, then by tops-onlyness, $\varphi(P_i, P_{-i}) = \varphi(P'_i, P_{-i})$, and we are done. So, suppose $P_i \equiv ab\cdots$ and $P'_i \equiv ba\cdots$. Assume for contradiction that $\varphi_a(P_i, P_{-i}) < \varphi_a(P'_i, P_{-i})$. By the block preservation property, $\varphi_{\{a,b\}}(P_i, P_{-i}) = \varphi_{\{a,b\}}(P'_i, P_{-i})$, and hence our assumption for contradiction means $\varphi_b(P_i, P_{-i}) > \varphi_b(P'_i, P_{-i})$. Consider an agent $j \in N \setminus i$ such that $\tau(P_j) \notin \{a, b\}$. Note that since $\mathscr{D}_j$ is swap-connected one of the following two cases must hold for P_j: (i) there is a swap-local path from P_j to a preference $P'_j \equiv a\cdots$ such that b does not appear as the top-ranked alternative in any preference in the path, (ii) there is a swap-local path from P_j to a preference $P'_j \equiv b\cdots$ such that a does not appear as the top-ranked alternative in any preference in the path.

Suppose Case (i) holds. Let B be the set of alternative that appear as the top-ranked alternative in some preference in the mentioned path. Consider the outcomes of φ when agent j changes her preferences along the path, while all other agents keep their preferences unchanged. By tops-onlyness, the outcome can change only when the top-ranked alternative changes along the path. Moreover, by the definition of swap-local path, the top-ranked alternative can change along the path only through a swap between two alternatives in B. By block preservation, this implies that the probability of the two swapping alternatives can only change in any such situations, and hence, the probability of the alternatives outside B will remain unchanged at the end of the path. Since $b \notin B$, this yields $\varphi_b(P_i, P_j, P_{-\{i,j\}}) = \varphi_b(P_i, P'_j, P_{-\{i,j\}})$ and $\varphi_b(P'_i, P_j, P_{-\{i,j\}}) = \varphi_b(P'_i, P'_j, P_{-\{i,j\}})$. This, together with our assumption for contradiction that $\varphi_b(P_i, P_{-i}) > \varphi_b(P'_i, P_{-i})$, implies $\varphi_b(P_i, P'_j, P_{-\{i,j\}}) > \varphi_b(P'_i, P'_j, P_{-\{i,j\}})$. Now, since $P_i \triangle P'_i = \{a, b\}$, we have by block preservation, $\varphi_{\{a,b\}}(P_i, P'_j, P_{-\{i,j\}}) = \varphi_{\{a,b\}}(P'_i, P'_j, P_{-\{i,j\}})$. Because $\varphi_b(P_i, P'_j, P_{-\{i,j\}}) > \varphi_b(P'_i, P'_j, P_{-\{i,j\}})$, this yields $\varphi_a(P_i, P'_j, P_{-\{i,j\}}) < \varphi_a(P'_i, P'_j, P_{-\{i,j\}})$. Using similar logic, we can conclude for Case (ii) that $\varphi_a(P_i, P'_j, P_{-\{i,j\}}) < \varphi_a(P'_i, P'_j, P_{-\{i,j\}})$.

Note that the preceding argument holds no matter what the preferences of the agents in $N \setminus \{i, j\}$ are. Therefore, by repeated application of this argument for each agent $j \in N \setminus i$ with $\tau(P_j) \notin \{a, b\}$, we obtain $P'_{-i} \in \mathscr{D}_{-i}$ of the agents in $N \setminus i$ such that (i) $\tau(P'_j) \in \{a, b\}$ for each $j \in N \setminus i$, and (ii) $\varphi_a(P_i, P'_{-i}) < \varphi_a(P'_i, P'_{-i})$.

We now complete the proof by means of tops-onlyness. If $P'_j \equiv a \cdots$ then let $P''_j = P_i$, and if $P'_j \equiv b \cdots$ then let $P''_j = P'_i$. By tops-onlyness, $\varphi(P_i, P'_{-i}) = \varphi(P_i, P''_{-i})$ and $\varphi(P'_i, P'_{-i}) = \varphi(P'_i, P''_{-i})$, and hence, $\varphi_a(P_i, P''_{-i}) < \varphi_a(P'_i, P''_{-i})$. However, since for each $j \in N$, either $P''_j \equiv P_i$ or $P''_j \equiv P'_i$, this violates weak elementary monotonicity, a contradiction. ■

C.5 PROOF OF THEOREM 5.1

Proof. If part of the theorem follows from the definitions of swap-LDSIC and swap-LOBIC. We proceed to prove the only-if part. Let $\varphi : \mathscr{P}(A)^n \to \Delta A$ be a unanimous RSCF satisfying the block preservation property. We show that φ is swap-LDSIC. By Theorem 3.2 and Theorem 3.3, φ is Pareto optimal and tops-only. To show that φ is swap-LDSIC, by Theorem 4.1, it is sufficient to show that φ is weak elementary monotonic. Consider swap-local preferences $P_i, \bar{P}_i \in \mathscr{P}(A)$ such that $P_i \equiv ab \cdots$ and $\bar{P}_i \equiv ba \cdots$.

Assume for contradiction that $\varphi_b(P_i, P_{-i}) > \varphi_b(\bar{P}_i, P_{-i})$ for some $P_{-i} \in \mathscr{P}(A)^{n-1}$ such that $P_i(k) = \bar{P}_i(k) = P_j(k)$ for all $j \in N \setminus i$ and all $k > 2$. Let c be the alternative such that $P_i \equiv abc\cdots$. Because P_i and $\bar{P}_i$ are swap-local, this means $\bar{P}_i \equiv bac\cdots$. Consider $P_i^1 \in \mathscr{P}(A)$ such that $P_i^1 = acb\cdots$ and P_i^1 and P_i are swap-local, that is $P_i^1 \triangle P_i = \{b, c\}$. By tops-onlyness of φ, $\varphi(P_i^1, P_{-i}) = \varphi(P_i, P_{-i})$. Next, consider $P_i^2 \in \mathscr{P}(A)$ such that $P_i^2 \equiv cab\cdots$ and P_i^2 and P_i^1 are swap-local. By the block preservation property, $\varphi_b(P_i^2, P_{-i}) = \varphi_b(P_i^1, P_{-i})$. Now, consider $P_i^3 \in \mathscr{P}(A)$ such that $P_i^3 \equiv cba\cdots$ and P_i^3 and P_i^2 are swap-local. By tops-onlyness of φ, $\varphi(P_i^3, P_{-i}) = \varphi(P_i^2, P_{-i})$. Finally, consider $P_i^4 \in \mathscr{P}(A)$ such that $P_i^4 \equiv bca\cdots$ and P_i^4 and P_i^3 are swap-local. Since bP_i^4c and bP_jc for all $j \in N \setminus i$, we have by Pareto optimality, $\varphi_c(P_i^4, P_{-i}) = 0$. Moreover, by the block preservation property, we have $\varphi_b(P_i^4, P_{-i}) = \varphi_b(P_i^3, P_{-i}) + \varphi_c(P_i^3, P_{-i})$. This, together with the fact that $\varphi_b(P_i^3, P_{-i}) = \varphi_b(P_i, P_{-i})$, implies $\varphi_b(P_i^4, P_{-i}) \geq \varphi_b(P_i, P_{-i})$. By our assumption, this means that $\varphi_b(P_i^4, P_{-i}) > \varphi_b(\bar{P}_i, P_{-i})$. Since $P_i^4(1) = \bar{P}_i(1)$ which contradicts that φ is tops-only. ■

C.6 Proof of Corollary 6.1

First, we state some important observations about betweenness domains which we will use in the proof.

Observation C.1. *Consider an alternative $x \in A$ and let $\mathscr{D}^x(\beta)$ be the set of all preferences with top-ranked alternative x and satisfying the betweenness condition β. Then, the domain $\mathscr{D}^x(\beta)$ is swap-connected.*

Observation C.2. *Let $x, y \in A$ and let $P \in \mathscr{D}(\beta)$ be such that $P(1) = x$ and $U(y, P) \cup y = \beta(x, y)$. Then, for all $\hat{P} \equiv x\cdots$, there is a swap-local path from $\hat{P}$ to P such that no alternative overtakes y along the path.*

Observation C.3. *Let $\mathscr{D}(\beta)$ be strongly consistent. Let $x, \bar{x} \in A$ and let $(x^1 = x, \ldots, x^t = \bar{x})$ be a sequence of adjacent alternatives in $\beta(x, \bar{x})$ such that for all $l < t$ and all $w \in \beta(x^l, \bar{x})$, we have $\beta(x^{l+1}, w) \subseteq \beta(x^l, \bar{x})$. Then, for all $l < t$, there exist $P \equiv x^l\cdots$ and $P' \equiv x^{l+1}\cdots$ such that $\beta(x^l, x^t)$ is an upper contour set in both P and P'. To see this, consider x^l. Since $\mathscr{D}(\beta)$ is strongly consistent, there is a preference $P \in \mathscr{D}(\beta)$*

such that $\beta(x^l,x^t)$ is an upper contour set of P. Name the alternatives in $\beta(x^l,x^t)$ as $w_1,\ldots,w_u$ such that $\beta(x^{l+1},w_r) \subsetneq \beta(x^{l+1},w_s)$ implies $r<s$. Since $\mathscr{D}(\beta)$ is strongly consistent, we have $\beta(x^{l+1},w) \subseteq \beta(x^l,x^t)$ for all $w \in \beta(x^l,x^t)$, and hence there is a preference P', graph-local to P, satisfying the betweenness relation β such that $P' \equiv w_1w_2\cdots w_{u-1}w_u\cdots$. Therefore, $U(w_u,P')\cup w_u = \beta(x^l,x^t)$.

We are now ready to start the proof. To ease the presentation, for a path π, we denote by π^{-1} the path π in the reversed direction, that is, if $\pi = (P^1,P^2,\ldots,P^t)$, then $\pi^{-1} = (P^t,P^{t-1},\ldots,P^1)$.

Proof of Corollary 6.1. Let $\mathscr{B}$ be a collection of strongly consistent and swap-connected betweenness relations. We show that $\mathscr{D}(\mathscr{B})$ satisfies the path-richness property.

First, we show $\mathscr{D}(\mathscr{B})$ satisfies Condition (i) of the path-richness property (see Definition 3.3). Consider P and P' with $P(1)=P'(1)$ that are not graph-local. If $P,P' \in \mathscr{D}(\beta)$ for some $\beta \in \mathscr{B}$, then by Observation C.1 there is a swap-local path from P to P' such that the top-ranked alternative does not change along the path. Suppose $P \in \mathscr{D}(\beta)$ and $P' \in \mathscr{D}(\hat{\beta})$ for some $\beta,\hat{\beta} \in \mathscr{B}$. Let $P(1)=P'(1)=x$ and let $(\beta^1=\beta,\ldots,\beta^t=\hat{\beta})$ be a swap-local path. By the swap-connectedness of $\mathscr{B}$, there are swap-local preferences $P^1 \in \mathscr{D}(\beta^1)$ and $P^2 \in \mathscr{D}(\beta^2)$ with $P^1(1)=P^2(1)=x$. By Observation C.1, there is a swap-local path π^1 from P to P^1 in $\mathscr{D}(\beta^1)$ such that x remains at the top-position in all the preferences in the path. Thus, the path (π^1,P^2) from P to P^2 satisfies Condition (i) of the path-richness property. Continuing in this manner, we can construct a path from P to P' that satisfies Condition (i) of the path-richness property.

Now, we show $\mathscr{D}(\mathscr{B})$ satisfies Condition (ii) of the path-richness property, that is, for all $P,P' \in \mathscr{D}(\mathscr{B})$ with $P(1)=P'(1)$, if P and P' are graph-local, then for each preference $\hat{P} \in \mathscr{D}(\mathscr{B})$, there exists a graph-local path $(P^1=\hat{P},\ldots,P^v)$ with $P^v(1)=P(1)$ such that for all $l<v$ and all distinct $a,b \in P\triangle P'$, there is a common upper contour set U of both P^l and P^{l+1} such that exactly one of a and b is contained in U. Since P and P' are graph-local with $P(1)=P'(1)$, by means of the fact that the collection $\mathscr{B}$ is swap-connected, it follows that P and P' are swap-local. So assume that $P \equiv w\cdots yz\cdots$ and $P' \equiv w\cdots zy\cdots$. Consider $\hat{P} \in \mathscr{D}(\mathscr{B})$. Suppose $\hat{P}(1)=x$ and $y\hat{P}z$. Let $\hat{P} \in \mathscr{D}(\beta)$ for some $\beta \in \mathscr{B}$. We construct a path from $\hat{P}$ to a preference with w as the top-ranked

alternative maintaining Condition (ii) of the path-richness property with respect to y and z in two steps. For ease of presentation, we denote $\hat{P}$ by P^1.

Step 1: Since β is strongly consistent, there is a sequence $(x^1 = x, \ldots, x^t = y)$ of adjacent alternatives in $\beta(x^1, x^t)$ such that for all $l < t$ and all $u \in \beta(x^l, x^t)$, $\beta(x^l, x^t) \supseteq \beta(x^{l+1}, u)$. By Observation C.2, there is a path π^1 from P^1 to a preference $\bar{P}^1$ with $\bar{P}^1(1) = x^1$ such that $U(x^t, \bar{P}^1) \cup x^t = \beta(x^1, x^t)$ and no alternative overtakes x^t along the path. Consider x^2. By Observation C.3, there is a preference P^2 with $P^2(1) = x^2$ such that P^2 is graph-local to $\bar{P}^1$ and $\beta(x^1, x^t)$ is an upper contour set in P^2. Since $z \notin \beta(x^1, x^t)$ and $\beta(x^1, x^t)$ is a common upper contour set of $\bar{P}^1$ and P^2, Condition (ii) of the path-richness property is satisfied with respect to x^t and z on the path $(\bar{P}^1, P^2)$. As in the case for P^1 and $\bar{P}^1$, by Observation C.2, we can construct a swap-local path π^2 from P^2 to some preference $\bar{P}^2$ with $\bar{P}^2(1) = x^2$ such that $U(x^t, \bar{P}^2) \cup x^t = \beta(x^2, x^t)$ and no alternative overtakes x^t along the path. As in the case for $\bar{P}^1$ and P^2, by Observation C.3, there is a preference P^3 with $P^3(1) = x^3$ such that P^3 is graph-local to $\bar{P}^2$ and $\beta(x^2, x^t)$ is an upper contour set in P^3. It follows that the path (π^1, π^2, P^3) from P^1 to the preference P^3 satisfies Condition (ii) of the path-richness property with respect to x^t and z. Continuing in this manner, we can construct a path $\hat{\pi}$ in $\mathscr{D}(\beta)$ from $\hat{P}$ to a preference $\hat{\hat{P}}$ with $\hat{\hat{P}}(1) = y$ such that Condition (ii) of the path-richness property is satisfied along the path.

Step 2: Consider the preference $P \equiv w \cdots yz \cdots$. Let $P \in \mathscr{D}(\tilde{\beta})$ for some $\tilde{\beta} \in \mathscr{B}$. Using similar argument as in Step 1, we can construct a path $\tilde{\pi}$ in $\mathscr{D}(\tilde{\beta})$ from P to some $\tilde{P}$ with $\tilde{P}(1) = y$ such that Condition (ii) of the path-richness property is satisfied with respect to y and z.

Step 3: Since $\hat{\hat{P}}(1) = \tilde{P}(1) = y$ and the collection $\mathscr{B}$ is swap-connected, there is a swap-local path $\bar{\pi}$ in $\mathscr{D}(\mathscr{B})$ from $\hat{\hat{P}}$ to $\tilde{P}$ such that y stays as the top-ranked alternative in each preference of the path. Clearly, such a path will satisfy Condition (ii) of the path-richness property with respect to y and z.

Consider the path $(\hat{\pi}, \bar{\pi}, \tilde{\pi}^{-1})$ from $\hat{P}$ to P. By construction, this path satisfies Condition (ii) of the path-richness property with respect to y and z, which completes the proof. ■

C.7 Proof of Theorem 6.1

Proof. Kumar et al. (2020) show that a domain $\mathscr{D}$ is graph-DLGE if and only if it satisfies the following property: for all distinct $P,P' \in \mathscr{D}$ and all $a \in A$, there exists a path π from P to P' with no (a,b)-restoration for all $b \in L(a,P)$. Here, a path is said to have no (a,b)-restoration if the relative ranking of a and b is reversed at most once along π. In what follows, we show that $\mathscr{D}(\mathscr{B})$ satisfies the above-mentioned property when $\mathscr{B}$ is weakly consistent and swap-connected. Consider two preferences $P \in \mathscr{D}(\beta)$ and $P' \in \mathscr{D}(\beta')$ for some $\beta,\beta' \in \mathscr{B}$ and $a \in A$. We show that there is a path π from P to P' that has no (a,x)-restoration for all $x \in L(a,P)$. By Observation C.3, from P and P' there are graph-local paths $\hat{\pi}$ and $\bar{\pi}$, respectively, to some preferences $\hat{P}$ and $\bar{P}$ with a as the top-ranked alternatives such that no alternative overtakes a along each of the paths. Let $\tilde{\pi}$ be a swap-local path joining $\hat{P}$ and $\bar{P}$ such that a remains the top-ranked alternative throughout the path. Consider the path $(\hat{\pi},\tilde{\pi},\bar{\pi}^{-1})$. No alternative in $L(a,P)$ overtakes a along the path $\hat{\pi}$. So, if there is an (a,x)-restoration for some $x \in L(a,P)$ in the path $(\hat{\pi},\tilde{\pi},\bar{\pi}^{-1})$, then it must be that the restoration happens in the path $\bar{\pi}^{-1}$. However, then a must overtake x in this path, which means x overtakes a in the reversed path $\bar{\pi}$, which is not possible by the construction of the path $\bar{\pi}$. This completes the proof. ■

C.8 Proof of Proposition 6.1

Proof. Consider $X,\bar{X} \in A$. We show that there is a sequence $(X^1 = X,\ldots,X^t = \bar{X})$ of adjacent alternatives in $\beta(X,\bar{X})$ such that for all $l < t$ and all $W \in \beta(X^l,X^t)$, we have $\beta(X^{l+1},W) \subseteq \beta(X^l,X^t)$. Let $l < t$ and consider $W \in \beta(X^l,X^t)$. We show $\beta(X^{l+1},W) \subseteq \beta(X^l,X^t)$. Take $Z \notin \beta(X^l,X^t)$. Because Z does not lie in $\beta(X^l,X^t)$, there must be a pair (a,b) of objects such that either (i) a and b are together in both X^l and X^t, but separate in Z, or (ii) a and b are separate in both X^l and X^t, but together in Z. Because both X^{l+1} and W are in $\beta(X^l,X^t)$, it must hold that in case (i) a and b are together in both X^{l+1} and W, and in case (ii) they are separate in both X^{l+1} and W. In case (i), a and b are together in both X^{l+1} and W but they are separate in Z. Therefore, Z cannot lie in $\beta(X^{l+1},W)$. On the other hand, in case (ii) a and b are separate in both X^{l+1} and W, but they are together in Z. Therefore, Z cannot lie in $\beta(X^{l+1},W)$. This completes the proof. ■

C.9 Proof of Proposition 8.1

We first prove some lemmas which we later use in the proof of the proposition. We use the following notions in the proofs. A preference P is **lexicographically separable** if there exists a (unique) component order $P^0 \in \mathscr{P}(K)$ and a (unique) marginal preference $P^j \in \mathscr{P}(A^j)$ for each $j \in K$ such that for all $x, y \in A$, we have $\left[x^l P^l y^l \text{ for some } l \in K \text{ and } x^j = y^j \text{ for all } jP^0 l\right] \Rightarrow [xPy]$. A lexicographically separable preference P can be uniquely represented by a $(k+1)$-tuple consisting of a lexicographic order P^0 over the components and marginal preferences $P^1, \dots, P^k$.

Lemma C.2. *Let $P \in \mathscr{S}$, $l \in K$, and $x, y \in A$ be such that $x^l P^l y^l$ and xPy. Then, for every component $j \neq l$ there is a sep-local path from P to a lexicographically separable preference $\bar{P} \in \mathscr{S}$ having same marginal preferences as P, and l and j as the lexicographically best and worst components, respectively, such that the x and y do not swap along the path.*

Proof. Assume without loss of generality, $l = 1$ and $j = m$. First, make the component 1 lexicographically best (without changing the marginal preferences of P) by swapping consecutively ranked alternatives multiple times in the following manner: each time swap a pair of consecutively ranked alternatives a and b where $a^1 P^1 b^1$ and bPa. Note that since $x^1 P^1 y^1$ and xPy, x and y are never swapped in this step. Having made 1 the lexicographically best component, the component 2 can be made lexicographically second-best in the following manner: each time swap a pair of consecutively ranked alternatives a and b in P where $a^1 = b^1$, $a^2 P^2 b^2$, and bPa. As we have explained for the case of component 1, alternatives x and y will not swap in this process. Continuing in this manner, we can finally obtain a preference $\bar{P}$ with lexicographic ordering over the components as $1\bar{P}^0 \cdots \bar{P}^0 k$ through a sep-local path along which the alternatives x and y are not swapped. ■

Lemma C.3. *Let $P \in \mathscr{S}$ be a preference such that xPy for some alternatives x and y that differ in at least two components. Then, there is a sep-local path $(P^1 = P, \dots, P^t = \hat{P})$ with $\hat{P}(1) = x$ such that $xP^l y$ for all $l < t$.*

Proof. Since xPy, there is a component l such that $x^l P^l y^l$. Assume without loss of generality $l = 1$. Consider component 2 . By Lemma C.2, there is a sep-local path π^1 from P to a preference $\bar{P}$ having components 1 and 2 as the lexicographically best and the worst components, respectively, such that x and y do not swap along the path. Since 2 is the lexicographically worst component of $\bar{P}$, we can construct a sep-local path from $\bar{P}$ to a preference $\bar{\bar{P}}$ such that (i) the marginal preferences in each component other than 2 and the lexicographic ordering over the components of each preference in the path remains the same as $\bar{P}$, and (ii) x^2 appears at the top-position of $\bar{\bar{P}}^2$. Since component 1 is the lexicographically best component in all these preferences and x^1 is preferred to y^1 in the marginal preference in component 1 for all these preferences, it follows that x remains ranked above y along the path. Repeating this process for all the components $3,\dots,k$, we can construct a path having no swap between x and y from P to a preference $\tilde{P}$ having (i) the same marginal preference as P in component 1, and (ii) x^t at the top-position of the marginal preference in component t for all $t > 1$.

Starting from the preference $\tilde{P}$, make component 1 lexicographically worst through a sep-local path without changing the marginal preferences. Since x^l is weakly preferred to y^l in each component l in each preference of this path, x will remain ranked above y throughout the path. Finally, move x^1 to the top-position in the marginal preference in component 1 thorough a(ny) swap-local path. Since x and y are different in at least two components, there is a component j lexicographically dominating component 1 (as it is the worst component) such that x^j is preferred to y^j in its marginal preference. Therefore, x will be ranked above y throughout the path. Note that in the final preference, for each component t, x^t appears at the top-position in the marginal preference in component t, and hence the alternative x appears at the top-position in it. ■

Lemma C.4. *Let $\varphi : \mathscr{S}^n \to \Delta A$ be a unanimous RSCF satisfying the block preservation property. Then $\varphi(P_N) = \varphi(\bar{P}_N)$ for all $P_N, \bar{P}_N$ such that $P^l_N = \bar{P}^l_N$ for all $l \in K$.*

Proof. It is enough to show that $\varphi(P_i, P_{-i}) = \varphi(\bar{P}_i, P_{-i})$ where $P^l_i = \bar{P}^l_i$ for all $l \in K$. Since preferences with the same marginals are swap-connected, we can assume without loss of generality that P_i and $\bar{P}_i$ are swap-local with the swap of alternatives x and y. Assume for contradiction $\varphi(P_i, P_{-i}) \neq \varphi(\bar{P}_i, P_{-i})$. By the block preservation property, this means $\varphi_x(P_i, P_{-i}) \neq \varphi_x(\bar{P}_i, P_{-i})$. By Lemma C.3, for all $j \in N \setminus i$, there is a sep-local path $(P_j^1 = P_j, \ldots, P_j^t = \bar{P}_j)$ with $\bar{P}_j(1) = P_i(1)$ satisfying the property that for all $l < t$ there is a common upper contour set U of both P_j^l and P_j^{l+1} such that exactly one of x and y is contained in U.[13] By Lemma C.1, we have $\varphi_x(P_i, \bar{P}_j, P_{-\{i,j\}}) \neq \varphi_x(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}})$. Continuing in this manner, we can construct $\bar{P}_{-i} \in \mathscr{S}^{n-1}$ such that $\bar{P}_j(1) = P_i(1)$ for all $j \neq i$ and $\varphi_x(P_i, \bar{P}_{-i}) \neq \varphi_x(\bar{P}_i, \bar{P}_{-i})$. However, since $(P_i, \bar{P}_{-i})$ and $(\bar{P}_i, \bar{P}_{-i})$ are unanimous preference profiles with the top-ranked alternative different from x, $\varphi_x(P_i, \bar{P}_{-i}) = \varphi_x(\bar{P}_i, \bar{P}_{-i}) = 0$, a contradiction. ■

Proof of Proposition 8.1. Let $\varphi : \mathscr{S}^n \to \Delta A$ be a unanimous RSCF satisfying the block preservation property. We show that φ satisfies component-unanimity. Consider $P_N \in \mathscr{S}^n$ such that $P_i^l(1) = x^l$ for all $i \in N$, some $l \in K$, and some $x^l \in A^l$. Assume for contradiction $\varphi_{x^l}^l(P_N) \neq 1$. Without loss of generality assume $l = 1$. By Lemma C.2 and Lemma C.4, we can assume that P_N is a profile of lexicographically separable preferences with each agent i having the component ordering $1P_i^0 \cdots P_i^0 k$. Fix some alternative x^k in component k and consider some agent i. As we have argued in the proof of Lemma C.3, there is a sep-local path from P_i to a preference $\bar{P}_i$ such that each preference in the path has the same lexicographic ordering over the components as P_i, $\bar{P}_i^k(1) = x^k$, and $\bar{P}_i^l = P_i^l$ for all $l \neq k$. By construction, for all $x^{-k} \in A^{-k}$ and $y^k, z^k \in A^k$, each pair of alternatives $((x^{-k}, y^k), (x^{-k}, z^k))$ forms a block for any two consecutive (sep-local) preferences in the path. This in particular implies $\varphi_{x^1}^1(\bar{P}_i, P_{-i}) = \varphi_{x^1}^1(P_N)$. Continuing this way, we can construct $\bar{P}_N \in \mathscr{S}^n$ such that $\bar{P}_i^k(1) = x^k$ for all $i \in N$ and $\varphi_{x^1}^1(\bar{P}_N) = \varphi_{x^1}^1(P_N)$.

Let $\bar{\bar{P}}_N$ be the profile of lexicographically separable preferences that has same marginal preferences as $\bar{P}$ and has lexicographic ordering over the components as $1\bar{\bar{P}}_i^0 \ldots \bar{\bar{P}}_i^0 k \bar{\bar{P}}_i^0 k - 1$ for all $i \in N$. That is, the components $k-1$ and k are swapped from $\bar{P}_i^0$ to $\bar{\bar{P}}_i^0$. By Lemma C.4, $\varphi(\bar{\bar{P}}_N) = \varphi(\bar{P}_N)$. Now, by using similar logic as for component k, we can construct $\hat{P}_N \in \mathscr{S}^n$ such that $\hat{P}_i^{k-1}(1) = x^{k-1}$ for all $i \in N$ and $\varphi_{x^1}^1(\hat{P}_N) = \varphi_{x^1}^1(P_N)$. Continuing in this manner, we can arrive at $\tilde{P}_N \in \mathscr{S}^n$ such that $\tilde{P}_i^t(1) = x^t$ for all $t \in K$ and all $i \in N$ and $\varphi_{x^1}^1(\tilde{P}_N) = \varphi_{x^1}^1(P_N)$. However, since $\tilde{P}_N$ is unanimous with $\tilde{P}_i(1) = x$ for all $i \in N$, we have $\varphi_x(\tilde{P}_N) = 1$, which in particular implies $\varphi_{x^1}^1(\tilde{P}_N) = 1$, a contradiction. ■

[13] Note that the statement of Lemma C.3 is slightly different from what we mention here. Since any two consecutive preferences in a sep-local path differ by swaps of multiple pairs of consecutively ranked alternatives, these two statements are equivalent.

C.10 Proof of Proposition 8.2

We use the following observation in the proof of Proposition 8.2.

Observation C.4. *Let $l \in K$ and let $\pi^l = (\pi^l(1),\dots,\pi^l(t))$ be a swap-local path in $\mathscr{D}^l$ such that the relative ordering of two alternatives $x^l, y^l \in A^l$ remains the same along the path. Then, for every component ordering $P^0 \in \mathscr{P}(K)$ having l as the worst component, and for every collection of marginal preferences $(P^1,\dots,P^{l-1},P^{l+1},\dots,P^k)$ over components other than l, the relative ordering of any two alternatives in the set $\{a \in A \mid a^l \in \{x^l,y^l\}\}$ will remain the same along the sep-local path $((P^0,P^1,\dots,P^{l-1},\pi^l(1),P^{l+1},\dots,P^k),\dots,(P^0,P^1,\dots,P^{l-1},\pi^l(t),P^{l+1},\dots,P^k))$ in the domain $\mathscr{S}(\mathscr{D}^1,\dots,\mathscr{D}^k)$.*

Proof. Let $\varphi : \mathscr{S}^n \to \Delta A$ be a unanimous RSCF satisfying the block preservation property. We show that φ is tops-only. Consider $P_N, \bar{P}_N \in \mathscr{S}^n$ with $P_i(1) = \bar{P}_i(1)$ for all $i \in N$. If $P_N^l = \bar{P}_N^l$ for all $l \in K$, then we are done by Lemma C.4. It is sufficient to assume that only one agent, say i, changes her marginal preference to a swap-local preference in exactly one component, say t, and nothing else changes from P_N to $\bar{P}_N$. That is, P_i^t and $\bar{P}_i^t$ are swap-local with the swap of some y^t and z^t, $P_j^t = \bar{P}_j^t$ for all $j \in N \setminus i$, and $P_N^l = \bar{P}_N^l$ for all $l \neq t$. Assume without loss of generality, $t = k$. Furthermore, in view of Lemma C.4, let us assume that all agents have the same component ordering Q^0 in both P_N and $\bar{P}_N$ where Q^0 is given by $1Q^0\dots Q^0k$. We need to show $\varphi(P_N) = \varphi(\bar{P}_N)$. Assume for contradiction $\varphi(P_N) \neq \varphi(\bar{P}_N)$. Since k is the worst component in P_i^0, by block preservation property, this implies $\varphi_{(x^{-k},y^k)}(P_N) \neq \varphi_{(x^{-k},y^k)}(\bar{P}_N)$ for some (x^{-k},y^k).

Consider P_j^k for some $j \neq i$. By our assumption on the marginal domains, there is a swap-local path $\pi^k = (\pi^k(1) = P_j^k,\dots,\pi^k(t) = \hat{P}_j^k)$ in $\mathscr{D}^k$ with $\hat{P}_j^k(1) = P_i^k(1)$ such that for any two consecutive preferences in the path there is a common upper contour set U such that exactly one of y^k and z^k is contained in U. By Observation C.4, the path $((P_j^0,P_j^1,\dots,P_j^{k-1},\pi^k(1)),\dots,(P_j^0,P_j^1,\dots,P_j^{k-1},\pi^k(t)))$ satisfies the property that for all $l < t$ and all $u,v \in P_i \triangle \bar{P}_i$ there is a common upper contour set U of both $(P_j^0,P_j^1,\dots,P_j^{k-1},\pi^k(l))$ and $(P_j^0,P_j^1,\dots,P_j^{k-1},\pi^k(l+1))$ such that exactly one of u and v is contained in U, and hence by Lemma C.1, we have $\varphi_{(x^{-k},y^k)}(P_i,\hat{P}_j,P_{-\{i,j\}}) \neq \varphi_{(x^{-k},y^k)}(\bar{P}_i,\hat{P}_j,P_{-\{i,j\}})$, where $\hat{P}_j = (P_j^0,P_j^1,\dots,P_j^{k-1},\hat{P}_j^k)$. Continuing in this manner, we can construct $\hat{P}_{-i} \in \mathscr{S}^{n-1}$ such that for all $j \in N \setminus i$, $\hat{P}_j^k(1) = P_i^k(1)$ and $\hat{P}_j^l =$

P_j^l for all $l \neq k$, and $\varphi_{(x^{-k},y^k)}(P_i,\hat{P}_{-i}) \neq \varphi_{(x^{-k},y^k)}(\bar{P}_i,\hat{P}_{-i})$. Note that the preference profiles $(P_i,\hat{P}_{-i})$ and $(\bar{P}_i,\hat{P}_{-i})$ are component-unanimous for component k, and hence by Proposition 8.1, $\varphi^k_{P_i^k(1)}(P_i,\hat{P}_{-i}) = \varphi^k_{P_i^k(1)}(\bar{P}_i,\hat{P}_{-i}) = 1$. This implies $\varphi_{(x^{-k},y^k)}(P_i,\hat{P}_{-i}) = \varphi_{(x^{-k},y^k)}(\bar{P}_i,\hat{P}_{-i}) = 0$, which contradicts $\varphi_{(x^{-k},y^k)}(P_i,\hat{P}_{-i}) \neq \varphi_{(x^{-k},y^k)}(\bar{P}_i,\hat{P}_{-i})$. ■

C.11 Proof of Theorem 8.1

Proof. We prove the theorem in two steps. In Step 1, we show that for all $l \in K$ and all $P_N,\bar{P}_N \in \mathscr{S}^n$ with $P_N^l = \bar{P}_N^l$, $\varphi^l(P_N) = \varphi^l(\bar{P}_N)$, and in Step 2, we use this fact to complete the proof.

Step 1: Since OBIC implies LOBIC, by means of Proposition 8.2, it is enough to prove Theorem 8.1 for every RSCF satisfying tops-onlyness and block preservation. Let $\varphi : \mathscr{S}^n \to \Delta A$ be a tops-only RSCF satisfying the block preservation property. We show that φ is marginally decomposable. Let $P_N,\bar{P}_N \in \mathscr{S}^n$ be such that $P_N^l = \bar{P}_N^l$ for some $l \in K$. Since φ is tops-only we assume without loss of generality that the l-th component is top-ranked according to the lexicographic ordering over the components in P_i and $\bar{P}_i$ for all $i \in N$. Consider an agent $j \in N$. Since component l is the lexicographically best component in both P_j and $\bar{P}_j$, for each $a^l \in A^l$, the set of alternatives $B(a^l) = \{(x^{-l},a^l) \mid x^{-l} \in A^{-l}\}$ can be expressed as a union of blocks in $(P_j,\bar{P}_j)$. Therefore, by applying the block preservation property to each block in $B(a^l)$, we obtain $\varphi_{B(a^l)}(P_j,P_{-j}) = \varphi_{B(a^l)}(\bar{P}_j,P_{-j})$ for all $a^l \in A^l$. Continuing in this manner, it follows that $\varphi_{B(a^l)}(P_N) = \varphi_{B(a^l)}(\bar{P}_N)$ for all $a^l \in A^l$. By the definition of marginal distribution, this means $\varphi^l(P_N) = \varphi^l(\bar{P}_N)$ which completes the proof.

Step 2: Consider an OBIC RBR (φ,μ_N). By Step 1, for each component $l \in K$, there exists $\hat{\varphi}^l : (\mathscr{D}^l)^n \to \Delta A^l$ such that for all $P_N \in \mathscr{S}^n$, $\varphi^l(P_N) = \hat{\varphi}^l(P_N^l)$. It remains to show that $(\hat{\varphi}^l,\mu_N^l)$ is unanimous and OBIC for each $l \in K$. Fix $l \in K$. Unanimity of $\hat{\varphi}^l$ follows from the unanimity property of φ. Assume for contradiction that the RBR $(\hat{\varphi}^l,\mu_N^l)$ is not OBIC. Then, there is an agent $i \in N$ and preferences $P_i^l,\hat{P}_i^l \in \mathscr{D}^l$ such that agent i's interim expected outcome at $(\bar{P}_i^l,\hat{P}_{-i}^l)$ is better than that at $(\hat{P}_i^l,\hat{P}_{-i}^l)$ according to the preference $\hat{P}_i^l$. Consider two profiles $(P_i,P_{-i}),(\tilde{P}_i,P_{-i}) \in \mathscr{S}^n$ such that $P_j^l = \hat{P}_j^l$ for all $j \in N$, $\tilde{P}_i^l = \bar{P}_i^l$ and the l is the lexicographically best component in P_i. Since the interim expected outcome at $(\bar{P}_i^l, \hat{P}_{-i}^l)$ is higher than that at $(\hat{P}_i^l, \hat{P}_{-i}^l)$ according to $\hat{P}_i^l$ and the l is the lexicographically best component in P_i, it follows that the interim expected outcome at (P_i, P_{-i}) will be higher than that at $(\tilde{P}_i, P_{-i})$ according to the preference P_i. However, this contradicts the fact that the RBR (φ, μ_N) is OBIC. ■

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