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ABSTRACT

This paper discusses the value-at-risk (VaR) concept and assesses the financial adequacy of the price probability determined by frequency of trades at price *p*. We take the price definition as the ratio of executed trade value to volume and show that it leads to price statistical moments, which differ from those, generated by frequency price probability. We derive the price *n-th* statistical moments as ratio of *n-th* statistical moments of the value and the volume of executed transactions. We state that the price probability determined by frequency of trades at price *p* doesn't describe probability of executed trade prices and VaR based on frequency price probability may be origin of unexpected and excessive losses. We explain the need to replace frequency price probability by frequency probabilities of the value and the volume of executed transactions and derive price characteristic function. After 50 years of the VaR usage main problems of the VaR concept are still open. We believe that VaR commitment to forecast the price probability for the time horizon *T* seems to be one of the most tough and expensive puzzle of modern finance.

Keywords: value-at-risk, risk measure, price probability, market trades
JEL : C10, E37, G11, G32

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1. Introduction

The value-at-risk in current form as the risk measure of was proposed in the late 60s almost 50 years ago as respond to the request of JP Morgan's Chairman Dennis Weatherstone. "It was of JP Morgan, at the time the Chairman of JP Morgan, who clearly stated the basic question that is the basis for VaR as we know it today – "how much can we lose on our trading portfolio by tomorrow's close?""(Allen, Boudoukh and Saunders, 2004). The response of JP Morgan's team on Weatherstone's question results in development of VaR models by RiskMetrics Group and presented in numerous papers (Longerstaey and Spencer, 1996; CreditMetrics™, 1997; Duffie and Pan, 1997; Laubsch and Ulmer, 1999; Mina and Xiao, 2001; Holton, 2003; Allen, Boudoukh and Saunders, 2004; Mina, 2005; Choudhry, 2013; Auer, 2018).

Due to (Longerstaey and Spencer, 1996) "Value-at-Risk is a measure of the maximum potential change in value of a portfolio of financial instruments with a given probability over a pre-set horizon." Since then Value-at-Risk or VaR becomes standard tool for risk assessment and was studied in hundreds articles. As usual, the roots of any good concept like VaR can by found much early than it is noted by RiskMetrics "official mythology" and Holton (2002) takes the VaR back to 1922. We are not able to refer all those who contributed to VaR development as one of most effective and useful risk measures and mention here only few (Malkiel, 1981; Linsmeier and Pearson 1996; Marshall and Siegel, 1996; Simons, 1996; Duffie and Pan, 1997; Berkowitz and O'Brien, 2001; Manganelli and Engle, 2001; Kaplanski and Kroll, 2002; Holton, 2003; Jorion, 2006; Aramonte, Rodriguez and Wu 2011). Since RiskMetrics publications the VaR concept occupied permanent position in risk management monographs (Choudhry, 2013; Horcher, 2015). Various forms of the VaR were developed for risk assessment of market portfolios, corporate and credit risk, financial risk management (Sanders and Manfredo, 1999; Jondeau, Poon and Rockinger, 2007; Adrian and Brunnermeier, 2011; Aramonte, Rodriguez and Wu, 2011; Andersen et.al., 2012; Auer, 2018). VaR concept plays the important role in bank and security risk regulations (FRS, 1998; Amato and Remolona, 2005; CESR, 2010). Wide usage of VaR as a risk measure is explained by its clear and general concept. Let's take price probability density function $f(p)$:

$$\int dp\ f(p) = 1 \tag{1.1}$$

and choose small number ε<<1. Then one can derive the price $p(\varepsilon)$:

$$\int_0^{p(\varepsilon)} dp\ f(p) = \varepsilon \tag{1.2}$$

Price $p(\varepsilon)$ determines the bottom line of possible losses with probability $1- \varepsilon$

$$p(\varepsilon) \le p \ \ with\, probability \ \ 1-\varepsilon \tag{1.3}$$

Simple relations (1.1-1.3) give firm and clear ground for usage of VaR. Here are left only some "easy" problems: how to select, measure and forecast the price probability density function. In the late 60s RiskMetrics developed first approximations of the VaR.

Standard treatment of Value-at-Risk (Longerstaey and Spencer, 1996) is based on price probability *f(p)* determined by number (frequency) of trades *m(p)* at price *p*. In simple words, RiskMetrics takes price probability *f(p)* of particular asset *A* equals number *m(p)* of trades at price *p* normalized to unit. More accurately, one should collect all trades *N* with asset *A* during time interval *Δ* and count the number *m(p)* of trades at price *p*. Investor may choose time interval *Δ* to be equal hour, day or whatever and the choice of interval *Δ* impact the properties of distribution *f(p)*. Then price probability distribution *f(p)* of number of trades at price *p* at moment *t* with asset *A* during the interval *Δ* equals

$$f(p) = \frac{1}{N} m(p) \ ; \quad \int dp \ f(p) = 1 \tag{1.4}$$

If one choose ε=5% then with probability 95% (1.2; 1.3) all trade prices *p* during interval *Δ* will be higher than *p(5%)* and hence the portfolio value *C(M)* of *M* shares of asset *A* with probability 95% will have value more or equal than *p(5%)M.*

Instead of the benchmark 5% investor may choose 1%, 3% or whatever he expects and obtain the lower estimate of his portfolio's value or possible losses – with probability 99%, 97% etc.

As the first approximation RiskMetrics Group (Longerstaey and Spencer, 1996) assumed that price probability distribution (1.1; 1.4) of frequencies *f(p)* of trades at price *p* takes form of standard Normal distribution. "A standard property of the Normal distribution is that outcomes less than or equal to 1,65 standard deviations below the mean occur only 5 percent of the time" (Longerstaey and Spencer, 1996). This result for years was widely used by investors as risk assessment of portfolio losses. Further researchers investigate the way to forecast the price frequency distribution *f(p)*, estimate the deviation of frequency probability *f(p)* (1.4) from normal distribution, explain the "fat tails" of the observed frequency price probability and etc. These problems are difficult and till now are far from final solution.

We propose to take one more look at the value-at-risk concept and discus "simple" notions in the base of the VaR concept – the notion of price and the notion of "price probability".

2. Price probability

Price is the core notion of economics and finance and references (Fetter, 1912; Hall and Hitch, 1939; Muth, 1961; Friedman, 1990; Weyl, 2019) indicate that the price theory attracts permanent interest during the century. As any common and "simple" notion, price has numerous meaning and definitions. We notice Fetter (1912) who almost a century ago mentioned 117 price definitions and we believe that since then the number of price definitions may grow up. Within this paper we take the single price definition: "Ratio-of-exchange definitions of price in terms of value in the sense of a mere ratio of exchange" (Fetter, 1912) as result of market transaction. Each market transaction *D(t)* performed at moment *t* with particular asset *A* is described by its value *C(t)* and volume *U(t)*:

$$D(t) = (U, C) \;\; ; \;\; C(t) = p(t)U(t) \tag{2.1}$$

(2.1) determines the price *p(t)* of a single executed market trade *D(t)*. We propose consistently use (2.1) to discuss the reasons for value-at-risk measure. It is obvious that all other 116 or more price definitions may have economic meaning. We propose regard all price definitions different from (2.1) as agents expectations of price *p*. Agents expectations drive agents decisions to take or reject market transactions (2.1) and hence agents expectations impact the price *p* of the executed trade (2.1). Nevertheless relations (2.1) remain the only and single source that establishes the historical time-series data of the executed trades at price *p* (2.1) and that eventually impact formation of agents price expectations. Below we describe how definition of price (2.1) impacts the properties of the price probability density function as ground notion of VaR.

What are the reasons to take the price definition (2.1) and how this choice impacts the standard approach to value-at-risk (1.1-1.4)? Let's remind that due to (Longerstaey and Spencer, 1996) price probability *f(p)* (1.4) is formed by numbers *m(p)* of transactions during interval Δ at the price *p*. It is implicitly assumed that probability that investor may trade at price *p* is proportional to frequency of trades. These obvious considerations explicitly or implicitly justify the usage of frequency of trades *f(p)* (1.4) at price *p* as price probability that define the VaR.

Meanwhile obvious and generally accepted statements are not always correct. In this paper we present definite reasons to state that the frequency trading statistics formed by *f(p)* (1.4) function has no meaning as trading price *p* (2.1) probability distribution. We show that price probability density function that match relations (2.1) should have different form and its measurement and forecasting is a really tough problem.

To explain our exotic statement in details, let's consider the trade price definition (2.1). Indeed, all investors take their decisions and follow their strategies taking into account market trade records and statistics. Financial profits and losses are results of executed market trades. Only executed trade price (2.1) determine the real value of the portfolio and final profits received. Thus relations and price probability distributions those match the executed trade price definition (2.1) determine the benefits and adequacy of the VaR concept usage. To go further let's introduce some formal notions.

3. Transactions and price statistical moments

The description of the trading price probability problem follows (Olkhov; 2020a; 2020b). Relations (2.1) define price of the single trade *D(t)* at moment *t*. To derive probability distribution that corresponds (2.1) one should select time averaging interval *Δ* that can be equal hour, day, week or whatever. The choice of averaging interval *Δ* defines number of transactions, their values, volumes, price of separate transactions and the price probability distribution during this particular interval *Δ*. On the other hand in economics each given probability distribution implicitly define the averaging interval *Δ*. Thus the choice of averaging interval *Δ* plays important role for determining and forecasting of probability distributions trading data. For the given averaging interval *Δ* define the number *N(t)* of transactions at moment t_i with particular asset *A* executed during interval *Δ* near moment *t*:

$$t-\frac{\Delta}{2}\le t_i\le t+\frac{\Delta}{2}\ ;\quad i=1,\dots N(t) \tag{3.1}$$

Relations (3.1) denote *N(t)* transactions at moments t_i. Then the mean price *p(1;t)* of *N(t)* transactions (3.1) executed during interval *Δ* equals the ratio of the total value *C(1;t)* to the total volume *U(1;t)* of trades performed during interval *Δ*:

$$C(1;t)=\sum_{i=1}^{N(t)}C(t_i)\ \ ;\ \ U(1;t)=\sum_{i=1}^{N(t)}U(t_i)\ \ ;\ C(1;t)=\ p(1;t)U(1;t) \tag{3.2}$$

Index 1 in the notions of the total value *C(1;t)* an total volume *U(1;t)* determine that sum (3.2) was taken by the values $C(t_i)$ and the volumes $U(t_i)$ to the first power. Relations (3.2) are well known for at least 30 years and are widely used by financial markets as Volume Weighted Average Price – VWAP (Berkowitz et.al 1988; Buryak and Guo, 2014; Guéant and Royer, 2014; Busseti and Boyd, 2015; Padungsaksawasdi and Daigler, 2018; CME Group, 2020). Other equal form of VWAP *p(1;t)* can be presented as:

$$p(1;t)=\frac{1}{U(1;t)}\sum_{i=1}^{N(t)}p(t_i)\,U(t_i)=\frac{C(1;t)}{U(1;t)} \tag{3.3}$$

Mean price *p(1;t)* is noted as price statistical moment of the first order. Market data of transactions *D(t)* (2.1) executed during time interval *Δ* permit derive all price statistical moments *p(n;t)*. For *n=1,…* relations (2.1) for the single trade *D(t)=(U,C)* and price *p(t)* at moment *t* take form:

$$C^n(t)=p^n(t)\,U^n(t) \tag{3.4}$$

Relations (3.4) are trivial consequences of (2.1). However (3.4) permit derive price *n-th* statistical moments *p(n;t)* similar to (3.2; 3.3). The mean value *p(n;t)* that match (3.4) equals ratio of sum of values C^n and volumes U^n of *N(t)* (3.1) trades executed during interval *Δ*:

$$C(n;t) = \sum_{i=1}^{N(t)} C^n(t_i) \quad ; \quad U(n;t) = \sum_{i=1}^{N(t)} U^n(t_i) \quad ; \; C(n;t) = \; p(n;t)U(n;t) \qquad (3.5)$$

or in the form alike to VWAP (3.3):

$$p(n;t) = \frac{1}{U(n;t)} \sum_{i=1}^{N(t)} p^n(t_i) \, U^n(t_i) = \frac{C(n;t)}{U(n;t)} \qquad (3.6)$$

Relations (3.5; 3.6) define *n-th* price statistical moment for all *n=1,…* and thus determine the price probability density function *ϕ(t;p)* at moment *t* that should match relations (3.5; 3.6):

$$p(n;t) = \int dp \;\; p^n \; \phi(t;p) \qquad (3.7)$$

It is easy to show that price statistical moments *p(n;t)* (3.5; 3.6) determine price characteristic function *Φ(t;x)* (Klyatskin, 2005; Jondeau, Poon and Rockinger, 2007; Klyatskin, 2015) that is very useful for studies of stochastic systems. Characteristic function *Φ(t;x)* is a Fourier transform of the probability density function:

$$\Phi(t;x) = \int dp \;\; \phi(t;p) \exp ipx \qquad (3.8)$$

Price characteristic function *Φ(t;x)* determines price statistical moments *p(n;t)* as

$$p(n;t) = \int dp \;\; p^n \; \phi(t;p) = i^{-n} \frac{d^n}{dx^n} \; \Phi(t;x)|_{x=0} \qquad (3.9)$$

and hence price characteristic function *Φ(t;x)* can be presented as power series:

$$\Phi(t;x) = \sum_{n=1}^{\infty} \frac{i^n}{n!} \; p(n;t) \; x^n \qquad (3.10)$$

Market data for transactions *D(t)* (2.1) performed during averaging interval *Δ* at moment *t* determine price statistical moments *p(n;t)* (3.5; 3.6) for all *n=1,…* and hence determine price characteristic function *Φ(t;x)* (3.8) as power series (3.10) and price probability density function *ϕ(t;p)* (3.8). It is clear that price statistical moments *p(n;t)* (3.5; 3.6) are different from statistical moments generated by price frequency probability distribution (1.4).

$$\pi(n;t) = \frac{1}{N(t)} \int dp \;\; p^n \; m(t;p) \qquad (3.11)$$

Here *m(t;p)* – number of trades at price *p* at moment *t* during interval *Δ* and *N(t)* (3.1) – the total number of trades during *Δ*.

4. Market trades and probabilities

Now let's discuss the origin of the relations (3.2; 3.5). Why do they have economic sense and why (3.2; 3.5) don't use the frequency trade data to introduce the price probability distribution? Our respond is simple and clear – relations (3.2; 3.5) completely match the frequency-based notion of probability. But they correspond not to the price frequency based function *f(p)* (1.4) but to the frequency based probability density functions for the value *C(t)* and the volume *U(t)* of the transactions *D(t) =(U,C)* executed during interval *Δ*. To show that let's take the *q(U;t)* as the number of trades at moment *t* during interval *Δ* with the trade volume equals *U* and *s(C;t)* as the number of trades with the trade value equals *C*. As we mentioned above the total number of trades at moment *t* during interval *Δ* equals *N(t)* (3.1):

$$\sum_U q(U;t) = \sum_C s(C;t) = N(t)$$

Thus functions *φ(U;t)* and *ψ(C;t)* define probability distributions at time *t* for averaging interval *Δ* for the volume *U(t)* and the value *C(t)*:

$$\varphi(U;t) = \frac{1}{N(t)} q(U;t) \ ; \quad \int dU \ \varphi(U;t) = 1 \tag{4.1}$$

$$\psi(C;t) = \frac{1}{N(t)} s(C;t) \ ; \quad \int dC \ \psi(C;t) = 1 \tag{4.2}$$

It is obvious that relations (3.5) identically match the probabilities (4.1; 4.2) for interval *Δ*:

$$C(n;t) = \sum_{i=1}^{N(t)} C^n(t_i) = \sum_C s(C;t)\, C^n = N(t) \int dC \ C^n \, \psi(C;t) \tag{4.3}$$

$$U(n;t) = \sum_{i=1}^{N(t)} U^n(t_i) = \sum_U q(U;t)\, U^n = N(t) \int dU \ U^n \, \varphi(U;t) \tag{4.4}$$

Thus price probability density function *ϕ(t;p)* and price characteristic function *Φ(t;x)* (3.7-3.10) are direct consequences of the volume and the value probability density functions *φ(U;t)* and *ψ(C;t)*. Hence usage of VaR as a risk measure requires forecasting volume and value probability density functions *φ(U;t)* and *ψ(C;t)* or what is the same - forecasting price characteristic function *Φ(t;x)*. As one may see from (3.5; 3.6; 3.10) forecasting price characteristic function *Φ(t;x)* requires forecasting the price statistical moments *p(n;t)* for all *n=1,…* and that require forecasting total volume *U(n;t)* (4.4) and total value *C(n;t)* (4.3) of the market trades during averaging interval *Δ*. In other words – to predict VaR for time horizon *T* equals days, weeks or months one should forecast evolution of total sum of *n-th* degree volume *U(n;t)* (4.4) and *n-th* degree value *C(n;t)* (4.3) of trades during interval *Δ*. It is clear that forecasting dynamics of market trades with particular asset *A* and market diversified portfolios requires modeling wide range of macroeconomic and macro financial variables and transactions those impact market trends, investment priorities, prospect inflation and etc… Moreover, as we discussed in (Olkhov, 2020b), current economic models describe relations between macroeconomic and financial variables and transactions of the first order. Macroeconomic investment, demand, consumption, trade volumes and etc., are formed as sum of the first order investment, consumption and trade of economic agents that can be presented as relations of the volume *U(1;t)* (4.4) and the value *C(1;t)* (4.3) of the first degree. To forecast evolution of the *second* degree volume *U(2;t)* (4.4) and the *second* degree value *C(2;t)* (4.3) one should develop relations that involve macroeconomic variables formed as sum of the *second* degree investment, demand, consumption and etc., of economic agents. Forecasting of the *n-th* degree volume *U(n;t)* (4.4) and *n-th* degree value *C(n;t)* (4.3) require relations that involve macroeconomic variables defined as sum of the *n-th* degree investment, demand, consumption and etc. To avoid here excess complexity we refer (Olkhov, 2020b) for details.

On the other hand attempts to forecast price probability density function in the form of characteristic function (3.10) require description of the first and the second statistical moments *p(1;t)* and *p(2;t)* (3.5; 3.6) at least. As we describe in (Olkhov, 2020a; 2020b) forecasting of *p(1;t)* and *p(2;t)* (3.5; 3.6) equals forecasting the price volatility of the asset *A* on the time horizon *T*. Indeed, price volatility $\sigma_p{}^2(t)$ for averaging interval *Δ* is expressed by the first two price statistical moments *p(1;t)* and *p(2;t)* (3.5; 3.6) in a usual form:

$$\sigma_p^2(t) = p(2;t) - p^2(1,t) \tag{4.5}$$

Thus the solution of the VaR problem and forecasting of the price probability density function as the first step requires forecasting of price volatility at the time horizon *T*. It is well known that volatility establish the core problem of options and derivatives markets and volatility trading (Black and Scholes, 1973; Whaley, 1993; Hull, 2009; Sinclair, 2013; Bennett, 2014). Volatility modeling and forecasting are among the most important subjects of financial theory (Poon and Granger, 2003; Andersen et.al., 2005; Brownlees, Engle and Kelly, 2011). We refer only few of hundreds studies of these important issues. In Olkhov (2020c) we show that the trading price definition (2.1) leads to the 2-dimensional Black-Scholes-Merton (Black and Scholes, 1973; Merton, 1973) like equation with two constant volatilities, impacts Heston (1993) stochastic volatility model, influences on non-linear option pricing and etc.

Any progress in VaR should have immediate impact on major financial and economic problems. It can be said that the VaR problem and price probability forecasting collect in one point almost all complexities associated with macroeconomics and finance.

5. Conclusion

The Value-at-Risk method is successfully used for almost half a century and we hope it may serve further. However the problems with effective usage of VaR are really tough. The VaR concept is perfect but economic reality is too complex. Thus the VaR requirement to forecast the price probability density function to assess 5% tail seems to be almost impossible. Moreover, the interrelations and collisions between "rigorous and accurate" derivation of the price probability density function (3.5-3.10) according to market trade price definition (2.1) on one hand and collective agents expectations, beliefs and preferences those ultimately impact decisions on market transactions and trade price (2.1) establish a horror and nightmare both for theoretical economics and sufficient econometric statistics and observations. Any reasonable economic theory and predictions can be completely disturbed by impact of unpredictable and sudden agents expectations those determine trade decisions and those impact the market trade price statistics. Thus investors may use the VaR model to which they long accustomed and in such a case price frequency probability function (1.4) or its modifications "calibrated" by market statistics may serve further. But smart investors definitely should keep in mind that the probability that is based on frequencies of trades at price p (1.4) doesn't describe probability of the executed trade prices (2.1). This distinction for sure will be origin of unexpected losses and disappointments.

Simplicity and generality of the VaR method force pay the high price for this. The VaR requires use price probability density function $\phi(t;p)$ (3.7; 3.8) now, at moment t and requires forecast it on time horizon T up to moment $(t+T)$. As we mentioned above, price probability that match trade relations (2.1; 3.5; 3.6) during averaging interval Δ depends upon the probability density functions (4.1-4.4) for the volume and the value of market transactions. Additivity of the volume and the value of market trade makes it possible to define their probabilities as frequencies of trades during averaging interval Δ with particular volume and value. Statistical moments $C(n;t)$ and $U(n;t)$ (4.3; 4.4) for $n=1,\ldots$ determine the statistical moments of price $p(n;t)$ (3.5; 3.6). It should be mentioned that forecasting the second moments of value $C(2;t)$ and volume $U(2;t)$ predict the price volatility (4.5) and hence impact core problem of option pricing. Thus the VaR problem interrelates to the option pricing problem as a whole and that doesn't simplify the possible solutions.

The choice of averaging interval Δ plays important role for determining the properties of the probability distributions $\varphi(U;t)$ and $\psi(C;t)$ for volume and value (4.1; 4.2) and for price $\phi(t;p)$ (3.7; 3.8). Moreover, interval Δ determine the averaging scale for the macroeconomic and financial forecasting at time horizon T that should deliver projections for the trade properties and the volume, value and price probabilities. The interval Δ defines the internal scale of smoothness for economic fluctuations and disturbances that should be taken into account. Relations between the interval Δ and forecasting horizon T determine internal and external scales for macroeconomic modeling and their ratio should define different approximations of forecasting models.

We outline that the ground elements of the VaR concept – choice, determining and forecasting of the price probability – are in the heart of the advanced economic and financial studies. After 50 years of the VaR usage main problems in the base of the VaR concept are still open. One who succeeds in market trade price probability forecasting could manage the world markets alone. This is not the worst incentive to solve the VaR problem.

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