Judgement aggregation functions and ultraproducts

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Abstract

The relationship between propositional model theory and social decision making via premise-based procedures is explored. A one-to-one correspondence between ultrafilters on the population set and weakly universal, unanimity-respecting, systematic judgment aggregation functions is established. The proof constructs an ultraproduct of profiles, viewed as propositional structures, with respect to the ultrafilter of decisive coalitions. This representation theorem can be used to prove other properties of such judgment aggregation functions, in particular sovereignty and monotonicity, as well as an impossibility theorem for judgment aggregation in finite populations. As a corollary, Lauwers and Van Liedekerke's (1995) representation theorem for preference aggregation functions is derived.

Key words: Judgment aggregation function; ultraproduct; ultrafilter.

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1 Introduction

Ultrafilters have an almost four-decades long history of successful application in the theory of preference aggregation. Initiated by Fishburn (1970) and Hansson (1971, Postscript 1976), a seminal contribution was made by Kirman and Sondermann (1972) whose results motivated numerous other papers in this area; see Monjardet (1983) for a survey.

A quarter-century later, Lauwers and Van Liedekerke (1995) provided an axiomatic foundation for the ultrafilter method in the theory of preference aggregation: They constructed a one-to-one correspondence between preference aggregation functions satisfying Arrovian axioms and ultrafilters on the population set. The bijection is given by the restriction of the ultraproduct[1] — with respect to the ultrafilter of decisive coalitions — of the family of individual preference orderings to the original set of alternatives.

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[1]Ultraproducts were first studied extensively by Łoś (1955). Expositions of ultraproducts can be found in model-theoretic monographs such as Chang and Keisler (1973). The existence of non-principal ultrafilters (i.e. non-dictatorial large coalitions) on infinite sets was established by Ulam (1929) and Tarski (1930), under the assumption of the Axiom of Choice. (Strictly speaking, the Ultrafilter Existence Theorem only needs the Boolean Prime Ideal Theorem, which is a weaker set-theoretic axiom than the Axiom of Choice, as Halpern and Le! vy (1971) have shown. See also Halpern (1964) and Banaschewski (1983) for related results.)

Ultrafilters have now also entered the theory of judgment aggregation, through papers by Gärdenfors (2006), Daniëls (2006), Dietrich and Mongin (2007), Eckert and Klamler (2008),[2] and some of the results in this article have already been discovered, via different methods, by Dietrich and Mongin (2007) (see Remark 12 below). It is not surprising that the ultrafilter method is an appealing tool in judgment aggregation theory, since ultrafilters are a means of relating the propositional logical structure of the electorate's a! genda with the algebraic structure of the voting coalitions.

In this paper, we use the methodology of Lauwers and Van Liedekerke (1995) to find an axiomatisation of the relation between propositional model theory and social decision making via premise-based procedures in general, and between judgment aggregation functions and ultrafilters in particular. We prove that a given judgment aggregation function maps profiles to ultraproducts of profiles — with respect to the ultrafilter of decisive coalitions — if and only if it satisfies the axioms of weak universality, respect for unanimous decisions as well as systematicity. This correspondence between ultrafilters and certain judgment aggregation functions will be used to prove several other properties of these aggregation functions, as well as an impossibility theorem for judgment aggregation in finite populations. We show that the main theorem of Lauwers and Van Liedekerke (1995) is contained in our results.

In fact, even the converse is true: Many of our results could also be obtained in a lengthy indirect argument via suitable corollaries of the findings of Lauwers and Van Liedekerke (1995) through replacing the binary preference predicate on alternatives by a unary truth predicate on propositions. However, the technical translation effort required would be substantial, and the resulting arguments would therefore ultimately not be significantly shorter than the direct derivation in this paper; furthermore, it would conceal the simplicity of the ultraproduct construction for propositional structures and would thus detract from the very intuitive corresepondence between judgment aggregation functions (ultraproducts) and families of decisive coalitions (ultrafilters).

The paper is self-contained and only assumes basic familiarity with propositional logic on the part of the reader.

2 Judgment sets

Where possible, we follow the terminology of List and Puppe (2007) and the notation of Eckert and Klamler (2008).

Let N be a finite or infinite set, the *population set*. Its elements are also called *individuals*. Consider a set Y of at least two propositional variables, and let Z be the set of all propositions, in the sense of propositional calculus, with propositional variables from Y. Z is called *agenda* with *base* Y and should be conceived as the agenda of a premise-based procedure of social decision making. Let T be a consistent subset of Z (i.e. $T \not\vdash \perp$), called the population's *(unanimous) theory*. The set $X = \bigcup_{p \in Y} \{p, \neg p\}$ is called the *atomic agenda*.

A *fully rational judgment set* in X given T is a subset $A \subset X$ which is *complete* in X (i.e. for all $p \in Y$ either $p \in A$ or $\neg p \in A$) and *consistent* with T (i.e. $A \cup T \not\vdash \perp$).

The set of fully rational judgment sets in X given T shall be denoted by D.

[2] Eckert (2008) argues that this recent interest is only a rediscovery, as the very first application of the ultrafilter method in social choice theory is in fact due to Guilbaud (1952) (English translation (2008) by Monjardet), who proved an Arrow-like theorem for the aggregation of logically interconnected propositions.

Through Henkin's (1949) method[3], every $A \in D$ can be uniquely extended to a set $j(A)$ such that

- $j(A)$ extends A in Z, i.e. $A \subseteq j(A) \subset Z$,
- $j(A)$ is complete in Z, i.e. for all $p \in Z$ either $p \in j(A)$ or $\neg p \in j(A)$,
- $j(A)$ is consistent, i.e. $j(A) \nvdash \perp$, and
- $j(A)$ contains T, i.e. $T \subset j(A)$.

$j(A)$ is obtained as the maximally consistent superset of $A \cup T$. In particular, A itself is consistent. $j(A)$ will be called the *judgment completion* (or *conclusive completion*) of A. From the perspective of propositional model theory, $j(A)$ corresponds to an interpretation of Z.

The following observations about j are almost trivial, but will be helpful later on. Herein, we denote by $-$ the negation operator on X, defined via $-p = \neg p$ and $-\neg p = p$ for all $p \in Y$.

Remark 1 (Inverse of j)**.** *$A = j(A) \cap X$ for all $A \in D$. Hence, j is injective.*

Proof. By definition, $A \subseteq j(A)$ and $A \subseteq X$, hence $A \subseteq j(A) \cap X$. If there existed some $p \in j(A) \cap X \setminus A$, then also $-p \in A \subseteq j(A)$ since A is complete in X. This contradicts the consistency of $j(A)$. □

Remark 2 (Decuctive closedness)**.** *For every $A \in D$, the set $j(A)$ is decuctively closed, i.e. $\forall p \in Z\, (j(A) \vdash p \Rightarrow p \in j(A))$.*

Proof by contraposition. If $p \notin j(A)$, then $j(A) \ni \neg p$ as $j(A)$ is complete. But $j(A)$ is also onsistent. Therefore, $j(A) \nvdash p$. □

The following Remark 3 corresponds to Tarski's (1933) definition of truth:

Remark 3 (à la Tarski)**.** *For all $p, q \in Z$ and $A \in D$:*

1. *$\neg p \in j(A)$ if and only if $p \notin j(A)$.*
2. *$p \wedge q \in j(A)$ if and only if both $p \in j(A)$ and $q \in j(A)$.*
3. *$p \vee q \in j(A)$ if and only if $p \in j(A)$ or $q \in j(A)$.*

Proof. Let $p, q \in Z$ and $A \in D$.

1. "$\Rightarrow$". $j(A)$ is consistent. "$\Leftarrow$". $j(A)$ is complete.
2. $j(A)$ is deductively closed by Remark 2.
3. Combine De Morgan's laws with the first two parts of the Remark.

□

Remark 4. *Through writing $A(p) = 1$ instead of $p \in A$, one could also view A as a map $A : Y \to 2$ such that*

$$T \cup A^{-1}\{1\} \cup \left(\neg A^{-1}\{0\}\right) \nvdash \perp$$

(wherein $\neg B$ is shorthand for $\left\{\neg p \;:\; p \in A^{-1}\{0\}\right\}$ for any $B \subseteq Z$, and $2 = \{0, 1\}$). Every such map A can be extended to a homomorphism of Boolean algebras $j(A) : Z \to 2$ such that $T \subseteq j(A)^{-1}\{1\}$.

[3] The method of extending a consistent set of propositions to a maximally consistent set was used in Henkin's (1949) famous proof of the completeness proof of first-order logic.

3 Coalitions and judgment aggregation functions

The elements of D^N are referred to as *profiles*. For any $p \in Z$ and $\underline{A} \in D^N$, the *p-supporting coalition in $\underline{A}$* is denoted by $\underline{A}(p) := \{i \in N \ : \ p \in j\left(\underline{A}(i)\right)\}$.[4] In this section, we study the properties of the map $\underline{A} : p \mapsto \underline{A}(p)$.

First, the map $\underline{A} : p \mapsto \underline{A}(p)$ allows us to translate the Boolean operations on coalitions into logical operations on propositions in Z:

Remark 5. *For all $\underline{A} \in D^N$, the map $\underline{A} : D^N \to \mathcal{P}(N), \quad p \mapsto \underline{A}(p)$ is a Boolean algebra homomorphism:*

1. $\underline{A}(\top) = N$ *and* $\underline{A}(\bot) = \emptyset$.
2. $\forall p \in Z \qquad \underline{A}(\neg p) = N \setminus \underline{A}(p)$.
3. $\forall p, q \in Z \qquad \underline{A}(p \wedge q) = \underline{A}(p) \cap \underline{A}(q), \quad \underline{A}(p \vee q) = \underline{A}(p) \cup \underline{A}(q)$.

Proof.

1. For all $i \in N$, $j\left(\underline{A}(i)\right)$ is complete and consistent, hence $\top \in j\left(\underline{A}(i)\right)$ and $\bot \notin j\left(\underline{A}(i)\right)$.
2. For all $i \in N$, $j\left(\underline{A}(i)\right)$ is complete and consistent, hence $\neg p \in j\left(\underline{A}(i)\right)$ if and only if $p \notin j\left(\underline{A}(i)\right)$.
3. For all $i \in N$, $j\left(\underline{A}(i)\right)$ is deductively closed (Remark 2), hence $p, q \in j\left(\underline{A}(i)\right)$ if and only if $p \wedge q \in j\left(\underline{A}(i)\right)$. This proves $\underline{A}(p \wedge q) = \underline{A}(p) \cap \underline{A}(q)$. The formula $\underline{A}(p \vee q) = \underline{A}(p) \cup \underline{A}(q)$ follows via De Morgan's laws combined with the already established parts 1 and 2 of the Remark.

$\square$

Almost needless to say, the whole population supports its unanimous theory, regardless of the population's profile.

Remark 6. *For all $p \in Z$ with $T \vdash p$ and every $\underline{A} \in D^N$, one has $\underline{A}(p) = N$.*

Proof. Let $p \in Z$ with $T \vdash p$, let $\underline{A} \in D^N$, and let $i \in N$. Since $j\left(\underline{A}(i)\right)$ is deductively closed (Remark 2) and contains T, we must have $p \in j\left(\underline{A}(i)\right)$. $\square$

This implies that the map $\underline{A} : p \mapsto \underline{A}(p)$ is well-defined on equivalence classes with respect to provable logical equivalence under T:

Remark 7. *For all $p, q \in Z$ with $T \vdash (p \leftrightarrow q)$ and every $\underline{A} \in D^N$, one has $\underline{A}(p) = \underline{A}(q)$.*

Proof. Let $p, q \in Z$ with $T \vdash (p \leftrightarrow q)$ and $\underline{A} \in D^N$. By Remark 6, $\underline{A}(p \leftrightarrow q) = N$. Combining this with the definition of $p \leftrightarrow q$ (i.e. $p \leftrightarrow q = (\neg p \vee q) \wedge (p \vee \neg q)$) and Remark 5, we obtain

$$N = \underline{A}((\neg p \vee q) \wedge (p \vee \neg q)) = ((N \setminus \underline{A}(p)) \cup \underline{A}(q)) \cap (\underline{A}(p) \cup (N \setminus \underline{A}(q))),$$

which via De Morgan's laws can be simplified to

$$\emptyset = (\underline{A}(p) \setminus \underline{A}(q)) \cup (\underline{A}(q) \setminus \underline{A}(p)).$$

The right-hand side is the symmetric difference between $\underline{A}(p)$ and $\underline{A}(q)$ This proves $\underline{A}(p) = \underline{A}(q)$. $\square$

[4] We may assume $N \cap Z = \emptyset$ to avoid amiguity.

An *aggregation function* is a map f from a subset $\mathbb{D}_f$ of D^N to D.

Consider now the following axioms:

(A1) *Universality*. $\mathbb{D}_f = D^N$.

(A1') *Weak Universality*. There exist $p, q \in Y$ and $A_1, A_2, A_3 \in D$ such that

- $p, q \in A_1$,
- $p, \neg q \in A_2$,
- $\neg p, q \in A_3$, and
- $\{A_1, A_2, A_3\}^N \subseteq \mathbb{D}_f$.

(A2) *Respect for Unanimity*. For all $\underline{A} \in \mathbb{D}_f$ and for all $p \in Z$, if $p \in j \circ f(\underline{A})$, then $\underline{A}(p) \neq \emptyset$.

(A3) *Systematicity*. For all $\underline{A}, \underline{A}' \in \mathbb{D}_f$ and for all $p, p' \in Z$ with $\underline{A}(p) = \underline{A}'(p')$, one has $p \in j \circ f(\underline{A})$ if and only if $p' \in j \circ f\left(\underline{A}'\right)$.

Remark 8. *If there exist $p, q \in Y$ such that $p \wedge q$, $p \wedge \neg q$ and $\neg p \wedge q$ are each consistent with T, then (A1) implies (A1').*

Finally, the set of *decisive coalitions* is

$$\mathcal{F}_f := \{\underline{A}(p) \ : \ \underline{A} \in \mathbb{D}_f, \quad p \in Z, \quad p \in j \circ f(\underline{A})\}.$$

Remark 9. *Let $\underline{A} \in \mathbb{D}_f$ and $p \in Z$, and suppose f satisfies (A3). Then, $\underline{A}(p) \in \mathcal{F}_f$ if and only if $p \in j \circ f(\underline{A})$.*

Proof. "$\Rightarrow$". If $p \notin j \circ f(\underline{A})$, then (A3) yields that $p' \notin j \circ f\left(\underline{A}'\right)$ for all $p' \in Z$ and $\underline{A}' \in \mathbb{D}_f$ satisfying $\underline{A}(p) = \underline{A}'(p')$. Hence $\underline{A}(p) \notin \mathcal{F}_f$. "$\Leftarrow$". Definition of $\mathcal{F}_f$. ☐

4 Ultrafilters and ultraproducts

In this section, we review ultrafilters and define ultraproducts of profiles. We define an *ultrafilter* on N as a collection $\mathcal{G}$ of subsets of N which is

- non-trivial, i.e. $\emptyset \notin \mathcal{G}$,
- maximal, i.e. for all $U \subseteq N$, either $U \in \mathcal{G}$ or $N \setminus U \in \mathcal{G}$, and
- closed under finite intersections, i.e. $U \cap U' \in \mathcal{G}$ for all $U, U' \in \mathcal{G}$.

These properties ensure that there is a one-to-one correspondence between ultrafilters on N and $\{0,1\}$-valued finitely additive measures on $\mathcal{P}(N)$: Given any such measure μ, the corresponding ultrafilter is just the collection of sets of μ-measure 1.

Often, ultrafilters are defined as being also closed under supersets, thus being special filters[5] *per definitionem*. This part of the definition is, in fact, redundant, as was stressed e.g. by Lauwers and Van Liedekerke (1995):

Remark 10. *Every ultrafilter $\mathcal{G}$ is closed under supersets, i.e. if $U' \supseteq U \in \mathcal{G}$, then $U' \in \mathcal{G}$. Hence, ultrafilters are filters.*

[5] A filter on N is a non-trivial collection of subsets of N which is closed under finite intersections and supersets.

Proof by contraposition. If $U' \notin \mathcal{G}$, then $N \setminus U' \in \mathcal{G}$ as $\mathcal{G}$ is maximal, and thus $\emptyset = U \cap (N \setminus U') \in \mathcal{G}$ since $\mathcal{G}$ is $\cap$-closed. Hence $\mathcal{G}$ is trivial, contradiction. $\square$

Another useful property of filters and hence ultrafilters is the following:

Remark 11. *Let $\mathcal{G}$ be a filter. For all $U, U' \subseteq N$, one has $U, U' \in \mathcal{G}$ if and only if $U \cap U' \in \mathcal{G}$.*

Proof. "$\Rightarrow$". $\mathcal{G}$ is closed under intersections. "$\Leftarrow$". $\mathcal{G}$ is closed under supersets. $\square$

In the context of social choice, filters formalise the notion of a large coalition. Ultrafilters do this in a maximal way, by postulating that every set of individuals is either a large coalition or the complement of a large coalition. For this reason, every ultrafilter on a set N can be used to define, starting from a sequence of models (referred to as *factors*), a new model with the property that a proposition holds in that model if and only if it holds for a large coalition of factors. This new model is known as an *ultraproduct*. In propositional logic, models can be identified with maximally consistent sets. Given that fullly rational judgment sets can be completed to maximally consistent sets via j, we will define ultraproducts for N-sequences of fully rational judgment sets — in other words profiles.

The ultraproduct $\prod \underline{A}/\mathcal{G}$ of a profile $\underline{A} \in D^N$ with respect to an ultrafilter $\mathcal{G}$ on N is defined as

$$\prod \underline{A}/\mathcal{G} := \prod_{i \in N} \underline{A}(i)/\mathcal{G} := \{p \in X \ : \ \{i \in N \ : \ p \in \underline{A}(i)\} \in \mathcal{G}\}\,.$$

The following lemma corresponds to the famous Łoś's (1955) theorem on ultraproducts in model theory of first-order predicate logic:

Lemma 1 (à la Łoś)**.** *For all $\underline{A} \in D^N$ and every ultrafilter $\mathcal{G}$ on N:*

1. $\prod \underline{A}/\mathcal{G} = \{p \in X \ : \ \underline{A}(p) \in \mathcal{G}\}$.
2. $\prod \underline{A}/\mathcal{G} \in D$. *In particular,* $T \subseteq j\left(\prod \underline{A}/\mathcal{G}\right)$.
3. $j\left(\prod \underline{A}/\mathcal{G}\right) = \{p \in Z \ : \ \underline{A}(p) \in \mathcal{G}\}$.

Proof. 1. Let $p \in X$ and $i \in N$. Since $\underline{A}(i) = j\left(\underline{A}(i)\right) \cap X$ by Remark 1, we have

$$p \in \underline{A}(i) \Leftrightarrow p \in j\left(\underline{A}(i)\right) \cap X \Leftrightarrow p \in j\left(\underline{A}(i)\right). \tag{1}$$

Hence $\{i \in N \ : \ p \in \underline{A}(i)\} = \{i \in N \ : \ p \in j\left(\underline{A}(i)\right)\} = \underline{A}(p)$. Therefore, $p \in \prod \underline{A}/\mathcal{G}$ if and only if $\underline{A}(p) \in \mathcal{G}$.

2. We verify:

   - *Completeness in X.* If $p \in Y$, the maximality of the ultrafilter ensures that either $\{i \in N \ : \ p \in \underline{A}(i)\} \in \mathcal{G}$ or $\{i \in N \ : \ p \notin \underline{A}(i)\} \in \mathcal{G}$. In the former case, $p \in \prod \underline{A}/\mathcal{G}$ amd we are done. In the latter case, note that $p \notin \underline{A}(i)$ holds if and only if $\neg p \in \underline{A}(i)$ by the consistency and completeness of $\underline{A}(i)$, hence $\{i \in N \ : \ \neg p \in \underline{A}(i)\} \in \mathcal{G}$, and therefore $\neg p \in \prod \underline{A}/\mathcal{G}$.

- *Consistency with T.* Suppose $\prod \underline{A}/\mathcal{G} \cup T \vdash \perp$. Since proofs of propositional logic have finite length, there is a finite set $I \subseteq \prod \underline{A}/\mathcal{G} \cup T$ such that already $I \vdash \perp$. Now, $I \cap T \subseteq T \subseteq j(\underline{A}(i))$ for all $i \in N$, hence $\{i \in N : p \in j(\underline{A}(i))\} = N \in \mathcal{G}$ for all $p \in I \cap T$. On the other hand, for all $p \in I \cap \prod \underline{A}/\mathcal{G}$, one has $\{i \in N : p \in j(\underline{A}(i))\} = \{i \in N : p \in \underline{A}(i)\} \in \mathcal{G}$ by equivalence (1), so $\{i \in N : p \in j(\underline{A}(i))\} \in \mathcal{G}$.

  In summary, $\{i \in N : p \in j(\underline{A}(i))\} \in \mathcal{G}$ holds for all $p \in I$. However, $\mathcal{G}$ is closed under finite intersections, hence $\bigcap_{p\in I} \{i \in N : p \in j(\underline{A}(i))\} \in \mathcal{G}$. Since $\mathcal{G}$ is non-trivial, there exists some $i \in N$ such that $I \subseteq j(\underline{A}(i))$ and thus, by choice of I, $j(\underline{A}(i)) \vdash \perp$, a contradiction.

Thus, $\prod \underline{A}/\mathcal{G} \in D$ and therefore $T \subseteq j(\prod \underline{A}/\mathcal{G})$.

3. We have to prove that for all $p \in Z$,

$$p \in j\left(\prod \underline{A}/\mathcal{G}\right) \Leftrightarrow \underline{A}(p) \in \mathcal{G}.$$

We shall give an inductive proof in the complexity of $p \in Z$.

(a) $p \in X$. By Remark 1, one has $p \in j(\prod \underline{A}/\mathcal{G}) \Leftrightarrow p \in \prod \underline{A}/\mathcal{G}$, but we also know that $p \in \prod \underline{A}/\mathcal{G} \Leftrightarrow \underline{A}(p) \in \mathcal{G}$ by part 1 of the present Lemma.

(b) $p = \neg q$. By applying Remark 3, the induction hypothesis, the maximality of the ultrafilter $\mathcal{G}$, and Remark 5 successively, we obtain the following chain of equivalences:

$$\begin{aligned} & \neg q \in j\left(\prod \underline{A}/\mathcal{G}\right) \Leftrightarrow q \notin j\left(\prod \underline{A}/\mathcal{G}\right) \Leftrightarrow \underline{A}(q) \notin \mathcal{G} \\ \Leftrightarrow \quad & N \setminus \underline{A}(q) \in \mathcal{G} \Leftrightarrow \underline{A}(\neg q) \in \mathcal{G}. \end{aligned}$$

(c) $p = q \wedge r$. By applying Remark 3, the induction hypothesis, Remark 11, and Remark 5 successively, we obtain

$$\begin{aligned} & q \wedge r \in j\left(\prod \underline{A}/\mathcal{G}\right) \Leftrightarrow q, r \in j\left(\prod \underline{A}/\mathcal{G}\right) \Leftrightarrow \underline{A}(q), \underline{A}(r) \in \mathcal{G} \\ \Leftrightarrow \quad & \underline{A}(q) \cap \underline{A}(r) \in \mathcal{G} \Leftrightarrow \underline{A}(q \wedge r) \in \mathcal{G}. \end{aligned}$$

□

5 Representation of judgment aggregation functions

This section contains the main results of this article. Lemma 2, Theorem 1 and Theorem 2 translate the findings by Lauwers and Van Liedekerke (1995) into the context of judgment aggregation.

For the rest of this paper, let f be a judgment aggregation function $f : \mathbb{D}_f \to D$, where $\mathbb{D}_f \subseteq D^N$.

Lemma 2. *If f satisfies the axioms (A1'), (A2) and (A3), then $\mathcal{F}_f$ is an ultrafilter.*

In abstract terms, we construct a map $h : \mathcal{P}(N) \to 2$ such that

$$h : U \mapsto \begin{cases} 1, & \exists \underline{A} \in D^N \quad \exists p \in Z \qquad U = \underline{A}(p) \\ 0, & \text{else} \end{cases}$$

and prove that h is a Boolean algebra homomorphism whilst $h^{-1}\{1\} = \mathcal{F}_f$.

In our proof, we have to verify all filter and ultrafilter axioms, i.e. including superset closedness, because the proof of the intersection closedness depends on superset closedness. Lauwers and Van Liedekerke (1995) omitted this part of the proof — not because of Remark 10 (since their proof of the intersection closedness is dependent on superset closedness), but apparently as an exercise for the reader.

Proof of Lemma 2. Let us write $\mathcal{F}$ for $\mathcal{F}_f$.

1. *Non-triviality* (i.e. $\emptyset \notin \mathcal{F}$). For all $U \in \mathcal{F}$, axiom (A2) yields $U \neq \emptyset$.

2. *Maximality* (i.e. $U \in \mathcal{F}$ or $N \setminus U \in \mathcal{F}$ for all $U \subseteq N$). Let $U \subseteq N$. Due to axiom (A1') and deductive closedness, there exists some $\underline{A} \in \mathbb{D}_f$ such that

   - $p \wedge \neg q \in j\left(\underline{A}(i)\right)$ for all $i \in U$, and
   - $\neg p \wedge q \in j\left(\underline{A}(i)\right)$ for all $i \in N \setminus U$.

   Then, for all $i \in N$, one has $\neg\left((p \vee q) \wedge \neg(p \wedge q)\right) \notin j\left(\underline{A}(i)\right)$ as $j\left(\underline{A}(i)\right)$ is consistent. Therefore, $\underline{A}\left(\neg\left((p \vee q) \wedge \neg(p \wedge q)\right)\right) = \emptyset$ and thus by axiom (A2) one gets $\neg\left((p \vee q) \wedge \neg(p \wedge q)\right) \notin j \circ f\left(\underline{A}\right)$. Hence, by completeness of $j \circ f\left(\underline{A}\right)$, we arrive at $(p \vee q) \wedge \neg(p \wedge q) \in j \circ f\left(\underline{A}\right)$. Therefore,

   $$j \circ f\left(\underline{A}\right) \supseteq \{(p \vee q) \wedge \neg(p \wedge q)\} \vdash (p \wedge \neg q) \vee (!\neg p \wedge q),$$

   which through deductive closedness yields $(p \wedge \neg q) \vee (\neg p \wedge q) \in j \circ f\left(\underline{A}\right)$. By Remark 3, we obtain that either $p \wedge \neg q \in j \circ f\left(\underline{A}\right)$ or $\neg p \wedge q \in j \circ f\left(\underline{A}\right)$, in other words, either $\underline{A}\left(p \wedge \neg q\right) \in \mathcal{F}$ or $\underline{A}\left(\neg p \wedge q\right) \in \mathcal{F}$. Hence, by the choice of $\underline{A}$, either $U = \{i \in N \,:\, p \wedge \neg q \in j\left(\underline{A}(i)\right)\} = \underline{A}(p \wedge \neg q) \in \mathcal{F}$ or $N \setminus U = \{i \in N \,:\, \neg p \wedge q \in j\left(\underline{A}(i)\right)\} = \underline{A}(\neg p \wedge q) \in \mathcal{F}$.

3. *Superset Closedness* (i.e. $U' \in \mathcal{F}$ whenever $U' \supseteq U$ for some $U \in \mathcal{F}$). Suppose $U' \supseteq U \in \mathcal{F}$. By axiom (A1') and Remark 2 (deductive closedness), there exists some $\underline{A} \in \mathbb{D}_f$ such that for all $i \in N$,

   - $p \wedge q \in j\left(\underline{A}(i)\right)$ if $i \in U$,
   - $p \wedge \neg q \in j\left(\underline{A}(i)\right)$ if $i \in U' \setminus U$, and
   - $\neg p \wedge q \in j\left(\underline{A}(i)\right)$ if $i \in N \setminus U'$.

   Then, $\underline{A}\left(p \wedge q\right) \in \mathcal{F}$, hence $p \wedge q \in j \circ f\left(\underline{A}\right)$ and therefore $p \in j \circ f\left(\underline{A}\right)$ by Remark 2 (deductive closedness). By definition of $\mathcal{F}$, $U = \underline{A}(p) \in \mathcal{F}$.

4. *Intersection Closedness* (i.e. $U \cap U' \in \mathcal{F}$ for all $U, U' \in \mathcal{F}$). Suppose $U, U' \in \mathcal{F}$. By axiom (A1'), there exists some $\underline{A} \in \mathbb{D}_f$ such that for all $i \in N$,

   - $p \wedge q \in j\left(\underline{A}(i)\right)$ if $i \in U \cap U'$,
   - $p \wedge \neg q \in j\left(\underline{A}(i)\right)$ if $i \in U \setminus U'$, and
   - $\neg p \wedge q \in j\left(\underline{A}(i)\right)$ if $i \in N \setminus U$.

   Then,

   - $\mathcal{F} \ni U = (U \cap U') \cup (U \setminus (U \cap U')) = \underline{A}(p)$, and
   - $\mathcal{F} \ni U' \subseteq (U \cap U') \cup (N \setminus U) = \underline{A}(q)$, which by superset closedness of $\mathcal{F}$ means $\mathcal{F} \ni \underline{A}(q)$.

Therefore, due to Remark 9, we arrive at $p, q \in j \circ f\left(\underline{A}\right)$ and thus $p \wedge q \in j \circ f\left(\underline{A}\right)$ by Remark 2. Hence $U \cap U' = \underline{A}(p \wedge q) \in \mathcal{F}$.

$\square$

Theorem 1. *If f satisfies the axioms (A1'), (A2) and (A3), then $f\left(\underline{A}\right) = \prod \underline{A}/\mathcal{F}_f$ for all $\underline{A} \in \mathbb{D}_f$.*

Proof. Let $\underline{A} \in \mathbb{D}_f$. By Lemma 1, $\prod \underline{A}/\mathcal{F}_f = \{p \in X \,:\, \underline{A}(p) \in \mathcal{F}_f\}$. Therefore, we need to prove that

$$p \in f\left(\underline{A}\right) \Leftrightarrow \underline{A}(p) \in \mathcal{F}_f$$

for all $p \in X$. However, by Remark 9, $\underline{A}(p) \in \mathcal{F}_f \Leftrightarrow p \in j \circ f\left(\underline{A}\right)$, hence due to Remark 1, finally $\underline{A}(p) \in \mathcal{F}_f \Leftrightarrow p \in f\left(\underline{A}\right)$ for all $p \in X$. $\square$

Theorem 2. *If $\mathcal{G}$ is an ultrafilter on N, then the map $g : D^N \to D,\ \underline{A} \mapsto \prod \underline{A}/\mathcal{G}$ satisfies the axioms (A1), (A2) and (A3).*

Proof. We verify:

(A1). $\mathbb{D}_g = D^N$ holds by definition, and $g\left(D^N\right) \subseteq D$ by Lemma 1.

(A2). Let $\underline{A} \in D^N$. By Lemma 1, $j \circ g\left(\underline{A}\right) = j\left(\prod \underline{A}/\mathcal{G}\right) = \{p \in Z \,:\, \underline{A}(p) \in \mathcal{G}\}$. Hence, for all $p \in Z$ such that $p \in j \circ g\left(\underline{A}\right)$, one has $\underline{A}(p) \in \mathcal{G}$ and thus $\underline{A}(p) \neq \emptyset$ since $\mathcal{G}$ is non-trivial.

(A3). Let $\underline{A}, \underline{A}' \in \mathbb{D}_f$ and $p, p' \in Z$ with $\underline{A}(p) = \underline{A}'(p')$. Again, by Lemma 1, $j \circ g\left(\underline{A}\right) = \{q \in Z \,:\, \underline{A}(q) \in \mathcal{G}\}$ and $j \circ g\left(\underline{A}'\right) = \left\{q \in Z \,:\, \underline{A}'(q) \in \mathcal{G}\right\}$. Hence $p \in j \circ g\left(\underline{A}\right)$ if and only if $\underline{A}(p) \in \mathcal{G}$, and $p' \in j \circ g\left(\underline{A}'\right)$ if and only if $\underline{A}'(p') \in \mathcal{G}$. But on the other hand, $\underline{A}(p) = \underline{A}'(p')$, so $\underline{A}(p) \in \mathcal{G}$ if and only if $\underline{A}'(p') \in \mathcal{G}$. Therefore, indeed, $p \in j \circ g\left(\underline{A}\right)$ if and only if $p' \in j \circ g\left(\underline{A}'\right)$.

$\square$

Theorem 3. *Suppose f and f' are judgment aggregation functions which satisfy axiom (A3). If $E := \mathbb{D}_f \cap \mathbb{D}_{f'} \neq \emptyset$ and $f \restriction E \neq f' \restriction E$, then $\mathcal{F}_f \neq \mathcal{F}_{f'}$.*

Proof. There exists some $\underline{A} \in E$ such that $f\left(\underline{A}\right) \neq f'\left(\underline{A}\right)$. By Remark 1, this implies $j \circ f\left(\underline{A}\right) \neq j \circ f'\left(\underline{A}\right)$ hence there exists some $p \in Z$ such that $p \in j \circ f\left(\underline{A}\right) \setminus j \circ f'\left(\underline{A}\right)$ or $p \in j \circ f'\left(\underline{A}\right) \setminus j \circ f\left(\underline{A}\right)$. Without loss of generality, we may assume the former, i.e. $p \in j \circ f\left(\underline{A}\right)$ and $p \notin j \circ f'\left(\underline{A}\right)$. Thanks to axiom (A3), we can now apply Remark 9 and obtain both $\underline{A}(p) \in \mathcal{F}_f$ and $\underline{A}(p) \notin \mathcal{F}_{f'}$. Thus, $\mathcal{F} \neq \mathcal{F}_{f'}$. $\square$

Remark 12. *After writing the first draft of the technical part of this paper, the author read through the recent paper by Dietrich and Mongin (2007) and discovered that a special case of Theorem 1 (viz. where $f : D^N \to D$) is already contained in Dietrich and Mongin's (2007) Theorem 1', and that a weak form of Theorem 2 — which replaces (A2) by unanimity preservation for formulae in X — is contained in Dietrich and Mongin's (2007) Theorem 2. (For, what Dietrich and Mongin (2007) refer to as an "ultrafilter rule" is nothing else than an ultraproduct as defined in the present paper.)*

Wherever the two papers overlap, priority belongs to Dietrich and Mongin (2007). The novelty of the present paper is its systematic model-theoretic approach in translating judgment aggregation functions into propositional ultraproducts and vice versa, and its connection with related work on preference aggregation. Furthermore, we study judgment aggregation for the more general case where the population may be assumed to share a common theory T.

6 Applications

We begin with corollaries to Theorem 1, the first of which mirrors a result by Lauwers and Van Liedekerke (1995) for preference aggregation functions. Again, we adopt the terminology and of List and Puppe (2007) and Eckert and Klamler (2008).

Corollary 1 (Sovereignty)**.** *If f satisfies (A1), (A2) and (A3), then $f : D^N \to D$ is surjective.*

Proof. Let $A \in D$ and set $\underline{A}(i) = A$ for all $i \in N$. Then, for all $p \in A$, $\{i \in N \,:\, p \in \underline{A}(i)\} = N \in \mathcal{G}$ and for all $p \in X \setminus A$, $\{i \in N \,:\, p \in \underline{A}(i)\} = \emptyset \notin \mathcal{G}$. Theorem 1 therefore tells us that

$$\begin{aligned} f\left(\underline{A}\right) &= \prod \underline{A}/\mathcal{F}_f = \{p \in X \,:\, \{i \in N \,:\, p \in \underline{A}(i)\} \in \mathcal{F}_f\} \\ &= \{p \in X \,:\, p \in A\} = A. \end{aligned}$$

□

Corollary 2 (Monotonicity)**.** *Let f satisfy (A1'), (A2) and (A3). Then,*

$$\forall \underline{A}, \underline{A}' \in \mathbb{D}_f \quad \forall p \in f\left(\underline{A}\right) \qquad \left(\underline{A}(p) \subseteq \underline{A}'(p) \Rightarrow p \in f\left(\underline{A}'\right)\right) \tag{2}$$

Proof. Let $\underline{A} \in \mathbb{D}_f$. By Theorem 1 and Lemma 1,

$$f\left(\underline{A}\right) = \prod \underline{A}/\mathcal{F}_f = \{q \in X \,:\, \underline{A}(q) \in \mathcal{F}_f\},$$

whence $\underline{A}(p) \in \mathcal{F}_f$ for all $p \in f\left(\underline{A}\right)$, and therefore $\underline{A}'(p) \in \mathcal{F}_f$ for all $\underline{A}' \in \mathbb{D}_f$ with $\underline{A}(p) \subseteq \underline{A}'(p)$. But analogously, $f\left(\underline{A}'\right) = \left\{q \in X \,:\, \underline{A}'(q) \in \mathcal{F}_f\right\}$ holds. Therefore $p \in f\left(\underline{A}'\right)$. □

A judgment aggregation function f satisfying Formula (2) is called *monotone*.

As another application, we state an impossibility theorem for finite electorates. It is based on the well-known fact that on a finite set, every ultrafilter (decisive coalition) is *principal*, i.e. is the family of supersets of some singleton:

Remark 13. *If $\mathcal{G}$ is an ultrafilter on a finite set N, then there exists some $\ell \in N$ such that $\mathcal{G} = \{U \subseteq N \,:\, \ell \in U\}$.*

Proof. By Remark 10, it is enough to prove that there is some $\ell \in N$ such that $\{\ell\} \in \mathcal{G}$. Suppose otherwise. Then, $N \setminus \{\ell\} \in \mathcal{G}$ for all $\ell \in N$, and since $\mathcal{G}$ is closed under finite intersections, $\emptyset = N \setminus \bigcup_{\ell=1}^{N} \{\ell\} = \bigcap_{\ell \in N} N \setminus \{\ell\} \in \mathcal{G}$. Contradiction. □

Theorem 4 (Dictatorship)**.** *If N is finite and f satisfies the axioms (A1'), (A2) and (A3), then there exists some $\ell \in N$ such that $f\left(\underline{A}\right) = \underline{A}(\ell)$ for all $\underline{A} \in \mathbb{D}_f$.*

Proof. Theorem 1 and the definition of the ultrapower yield that for all $\underline{A} \in \mathbb{D}_f$,

$$f\left(\underline{A}\right) = \prod \underline{A}/\mathcal{F}_f = \{p \in X \,:\, \{i \in N \,:\, p \in \underline{A}(i)\} \in \mathcal{F}_f\}.$$

However, by Remark 13 and Lemma 2, there exists some $\ell \in N$ such that $\mathcal{F}_f = \{U \subseteq N \,:\, \ell \in U\}$. Thus, we arrive at

$$f\left(\underline{A}\right) = \{p \in X \,:\, p \in \underline{A}(\ell)\} = \underline{A}(\ell).$$

□

Remark 14. *The axiom (A1') is, albeit substantially weaker than the usual universality axiom (A1), still strong enough to demand that no weakly value-restricted profiles — in the sense of Dietrich and List (2007), who generalised Sen's (1966) original notion of triplewise value restriction — are in the domain of the judgment aggregation function. Therefore, Theorem 4 is compatible with the results of both Dietrich and List (2007) and Sen (1966).*

Preference aggregation can be studied as a special case of judgment aggregation, simply by interpreting the atomic agenda as a set of atomic preference relations among alternatives and their negations. In this spirit, we shall now show how one can obtain the findings of Lauwers and Van Liedekerke (1995) as a corollary to Theorems 1 and 2.

Let A be a set of at least three elements, called *alternatives*, let P be a binary relation on A and let $Y = \{P(a,b) \,:\, a,b \in A\}$. Then, $X = \bigcup_{a,b\in A} \{P(a,b), \neg P(a,b)\}$ and Z is the Boolean closure of Y, i.e. the smallest superset of Y which is closed under negation, conjunction and disjunction. In the terminology of Lauwers and Van Liedekerke (1995), Z is the set of *test sentences with base* Y. Denote the first-order language with one binary relation symbol P and a constant symbol for every element of A by $\mathcal{L}(A,P)$.

Note:

1. Every universal sentence $\forall x_1,\ldots,x_n \quad p\,(x_1,\ldots,x_n)$, wherein p is a quantifier-free formula of the language $\mathcal{L}(A,P)$ is true in the restriction $\mathcal{M}$ of an $\mathcal{L}(A,P)$-structure to A if and only if $\mathcal{M} \models \{p\,(a_1,\ldots,a_n) \,:\, a_1,\ldots,a_n \in A\}$.

   Therefore, given any consistent set S of universal $\mathcal{L}(A,P)$-sentences, one can find a consistent set $T \subseteq Z$ such that for all restrictions $\mathcal{M}$ of $\mathcal{L}(A,P)$-structures to A, one has $\mathcal{M} \models S$ if and only if $\mathcal{M} \models T$. For this T, let D be the set of fully rational judgment sets in X given T.

2. Denote the set of restrictions of models of S to A by Ω. There is a one-to-one correspondence between Ω and D.

   Given $\omega \in \Omega^N$ and $p \in Z$, the p-supporting coalition in ω is denoted

   $$\omega(p) := \{i \in N \,:\, \omega(i) \models p\}\,.$$

3. The one-to-one correspondence between Ω and D entails a one-to-one correspondence between

   - restrictions of ultraproducts of N-sequences of models of S with respect to an ultrafilter on N, and
   - ultraproducts in the sense of Section 4 of the present paper.

In view of these observations, Theorems 1 and 2 of the present paper imply the main theorem of Lauwers and Van Liedekerke (1995, Theorem 1):

Theorem 5. *A map $\mathcal{A}$ is given by*

$$\forall \omega \in \Omega^N \qquad \mathcal{A}(\omega) = \prod_{i\in N} \omega(i)/\mathcal{F}_{\mathcal{A}}$$

if and only if it satisfies the following axioms:

(L1) $\mathcal{A} : \Omega^N \to \Omega$.

(L2) $\forall \omega \in \Omega^N \quad \forall p \in Z \quad (\mathcal{A}(\omega) \models p \Rightarrow \omega(p) \neq \emptyset)$

(L2) $\forall \omega, \omega' \in \Omega^N \quad \forall p, p' \in Z \quad (\omega(p) = \omega'(p') \Rightarrow (\mathcal{A}(\omega) \models p \Leftrightarrow \mathcal{A}(\omega') \models p'))$.

7 Conclusion

Ideas from model theory can be fruitfully applied in the theory of judgment aggregation. In particular, the notion of an ultraproduct in the sense of propositional model theory is the same as the notion of a weakly universal, unanimity-respecting, systematic judgment aggregation function. This representation result can be employed to prove other properties of such judgment aggregation functions, as well as an impossibility theorem for finite electorates. A special case is Lauwers and Van Liedekerke's (1995) representation theorem for preference aggregation functions.

Kirman and Sondermann (1972) as well as Armstrong (1980, 1985) have shown that even non-principal ultrafilters on infinite populations may be interpreted as "invisible dictators", provided the population is endowed with some measure-theoretic or topological structure. Hence, our results show that judgment aggregation functions are always, even for infinite populations, dictatorial in some weak sense. However, as Hansson (1976) elaborated, the strength of this concept of dictatorship is highly dependent on the actual topology imposed on the population set, and for some topologies, the "dictators" provided by the Kirman-Sondermann (1972) construction are non-unique and therefore hardly deserve this name.

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