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Abstract

There are multiple situations in which bilateral interaction between agents results in considerable cost reductions. Such interaction can occur in settings where agents are interested in sharing resources, knowledge or infrastructures. Their common purpose is to obtain individual advantages, e.g. by reducing their respective individual costs. Achieving this pairwise cooperation often requires the agents involved to make some level of effort. It is natural to think that the amount by which one agent could reduce the costs of the other may depend on how much effort the latter exerts. In the first stage, agents decide how much effort they are to exert, which has a direct impact on their pairwise cost reductions. We model this first stage as a non-cooperative game, in which agents determine the level of pairwise effort to reduce the cost of their partners. In the second stage, agents engage in a bilateral interaction between independent partners. We study this bilateral cooperation as a cooperative game in which agents reduce each other's costs as a result of cooperation, so that the total reduction in the cost of each agent in a coalition is the sum of the reductions generated by the rest of the members of that coalition. In the non-cooperative game that precedes cooperation with pairwise cost reduction, the agents anticipate the cost allocation that results from the cooperative game in the second stage by incorporating the effect of the effort exerted into their cost functions. Based on this model, we explore the costs, benefits, and challenges associated with setting up a pairwise effort network. We identify a family of cost allocations with weighted pairwise reduction which are always feasible in the cooperative game and contain the Shapley value. We show that there are always cost allocations with weighted pairwise reductions that generate an optimal level of efficient effort and provide a procedure for finding the efficient effort equilibrium.

Keywords Allocation, Cost models, Efficiency, Game Theory.

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1 Introduction

The search for greater efficiency, access to new markets and greater competitiveness are some of the main factors that result in inter-organizational or inter-corporate cooperation structures. Depending on the degree of integration or interdependence between partners and on the intended goals of agreements, there are different forms of cooperation. These forms have been widely studied in economic literature (see e.g. Todeva and Knoke (2005) for a survey). There is one specific type of cooperation whose properties and characteristics differentiate it from the rest. It can occur between agents that share for example resources, knowledge or infrastructure. The common purpose is to obtain individual advantages such as reducing their respective individual costs. The particularity of this form of cooperation lies in the fact that the cost reduction is based on bilateral interactions. In particular, given any pair of cooperating agents, one agent reduces the cost of the other agent by a certain amount which is independent of cooperation with other agents. This means that if there are more agents in the coalition the amount of the cost reduction does not change. This pairwise cost reduction remains constant for any possible coalition to which the pair of agents may belong. Therefore, for any agent, the total cost reduction in any coalition can easily be calculated as the sum of the reductions obtained from each bilateral interaction with the other members of the coalition.

There are several situations where this kind of cooperation with pairwise cost reduction occurs and is profitable. For example, the strategic collaboration agreements between firms to reduce logistical operational costs. The need to increase market share requires logistics firms to expand their radius of action as far as possible. This means major investments in expensive infrastructures at new sites, which increase operational costs. To reduce those costs while maintaining control of their respective markets and hindering access by new competitors, agreements are established between companies. They offer the resources held by each firm in its respective area of influence under advantageous conditions. This enables them to expand their operating ranges with significant cost savings. Interactions occur bilaterally, with each company using the resources of the other. These cost reductions are independent of any cost reductions that can also be obtained by interacting with other agents in larger coalitions.

The second situation is that of free trade agreements between countries. In a globalized economy, free trade agreements are quite common. They facilitate trade in goods and services between countries, reducing trade barriers and consequently the cost of trade. These cost reductions are specific to each pair of countries, and are independent of any other agreements that either may decide to establish with other countries. A third situation is the sharing of market data. Currently, information on customers and their purchase patterns is vitally important for firms. It enables them to maximize returns on advertising costs and focus on their ideal target markets. Cooperation between firms (usually from complementary sectors) consists of sharing information about their respective customers. This reduces the costs of each of the firms involved. The information that a particular firm provides is specific to it, so the value of the information that it receives from another specific firm is independent of information from other firms. Even if two firms provide information about the same customer, the information itself is different because it describes the purchase of a different good or service. This can increase the value of that particular customer as a target, which again boosts the value of this particular kind of cooperation.

The last situation considered here is that of inter-firm cooperation agreements to reduce costs by increasing the range of their respective telecommunication networks. In eminently competitive sectors such as mobile telephony and online services, cooperation between operators has become quite common. For example, they share the locations of their respective antennas, which enables them to expand the reach of their networks. This means greater benefits thanks to the offering of a broader service, while avoiding the costs that would be entailed by each company installing its own structures. Here again, cost reduction is bilateral when two agents decide to share and use each other's antennas. These cost savings are independent of any collaboration agreements that each firm may have with other agents to share antennas in larger coalitions.

The cost reduction between agents may be highly asymmetric when they cooperate in pairs. For example, if two agents A and B decide to cooperate, agent A could provide a major reduction for agent B, while the reduction provided in the opposite direction could be more modest. These asymmetries could induce imbalances or discriminations that may jeopardize cooperation. A fair distribution mechanism for the costs generated by cooperation is undoubtedly needed to ensure the stability of any strategic partnership (see Thomson, 2010).

In addition, it is quite common for this kind of cooperation to require the agents involved to make a set level of effort. It is natural to think that the amount by which agent can reduce the costs of the other (if they decide to cooperate) could depend on the effort that the agent exerts. For example, if one country can obtain information relevant to another (e.g. information on tax evasion and the flight of capital involving its citizens), the amount and quality of the specific information may depend of the effort that the first country exerts in gathering it. This extends the situation beyond a cooperative model. We model the sequence of decisions as a two-stage bi-form game (Brandenburger and Stuart, 2007). In the first stage, agents decide how much (costly) effort they are willing to exert. This has a direct impact on their pairwise cost reductions. This first stage is modeled as a non-cooperative game in which agents determine the level of pairwise effort to reduce the costs of their partners. In the second stage, agents engage in bilateral interaction with multiple independent partners where the cost reduction brought by each agent to another agent remains constant in any possible coalition. We study this bilateral cooperation as a cooperative game in which cooperation leads agents to reduce their respective costs, so that the total reduction in costs for each agent in a coalition is the sum of the reductions generated by the rest of the members of that coalition. In the non-cooperative game that precedes cooperation, the agents anticipate the cost allocation that will result from the cooperative game in the second stage by incorporating the effect of the effort made into their cost functions. Based on this model, we explore costs, benefits, and challenges associated with setting up a pairwise effort network.

We investigate the impact of pairwise efforts on cost reductions and the resulting cost structure for the network. In particular, we explore the design of a cost-allocation mechanism that efficiently allocates the gains from pairwise effort to all parties. To that end, we first compute the optimal level of cost reduction, taking into account the pairwise cost reductions collectively accrued by all agents. An ideal allocation scheme should encourage agents to participate in it and, at the same time, establish proper incentives to make efforts prior to cooperation. Specifically, we first show that it is profitable for all agents to participate in a pairwise effort network. Then we study how the total reduction in costs should be allocated to the members of the network. We do this by modeling the pairwise cost reduction between agents that takes place in the second stage as a cooperative game, which we refer to as the pairwise effort game or "PE-game".

We prove that PE-games are concave (i.e. the marginal contribution of an agent diminishes as a coalition grows) and thus totally balanced, i.e. the core of every subgame is non-empty. We interpret a non-empty core as a setting where all-included cooperation is feasible, in the sense that there are possible cost reductions that make all agents better off (or, at least, not worse off). The totally balanced property suggests that this all-included cooperation is consistent. We identify various allocation mechanisms that may arise in the core of PE-games. In particular, we discuss a family of cost allocations with weighted pairwise reduction which is always a subset of the core of PE-games. This is a broad family of core allocations which includes the Shapley value, which is obtained when all the weights work out to a half. We provide a highly intuitive, simple expression for the Shapley value, which matches the Nucleolus in our model. To select one of these core-allocations in the second stage, we take into account the incentives that it generates in the efforts made by agents, and consequently in the aggregate cost of a coalition. We show that the Shapley value can induce inefficient effort strategies in equilibrium in the non-cooperative model. However, it is always possible to find core allocations with weighted pairwise reductions that create appropriate incentives for agents to make optimal efforts that minimize aggregate costs, i.e. core allocations that generate an efficient level of effort in equilibrium.

This paper contributes to the literature by presenting a doubly robust cost sharing mechanism. That mechanism not only has good properties regarding the cooperative game in the second stage but also creates appropriate incentives in the non-cooperative game in the first stage.

Cooperative game theory has developed a substantial mathematical framework to identify and provide suitable cost sharing allocations (see, e.g., Fiestras-Janeiro et al. 2011; Guajardo and Rönnqvist 2016, for a survey). Multiple solutions have been proposed from a wide range of approaches (see, e.g., Moulin 1987; Slikker and Van den Nouweland 2012; Lozano et al. 2013). The Shapley Value (Shapley 1953) is considered one of the most outstanding of them, and suitable solution concept (see, e.g., Moretti and Patrone 2008; Serrano 2009 for a survey). As an allocation rule it has very good properties, such as efficiency, proportionality, and individual and coalitional rationality. However, it has the disadvantage of posing computational difficulties, which increase as the number of players increases. Nonetheless, there is a large body of literature in which the Shapley value is proposed as a simple, easy-to-apply solution in different economic scenarios (see, e.g., Littlechild and Owen 1973; Bilbao et al. 2008; Li and Zhang 2009; Kimms and Kozeletskyi 2016; Le et al. 2018; Meca et al. 2019). These papers give simplified solutions for different classes of games. They take the cost structure as given and do not consider the system externalities that arise when agents make efforts to give and receive cost reductions. Our paper here incorporates both the non-cooperative aspects of making efficient efforts by modeling decisions related to pairwise cost reductions and the cooperative nature of giving and receiving cost reductions in pairwise effort networks.

We refer to action by agents as "effort", as in the principal-agent literature. In this setting, the concept of "effort" is widely used in analyzing different kinds of problem. One of the first was the moral hazard problem: See for example Holmstrom (1982). Other examples are Holmstrom (1999) and Dewatripont et al. (1999), who identify conditions under which more information can induce an agent to make less effort. The approach in our model is quite different, in that we do not consider any kind of principal. As far as we know, our model is novel in that it analyzes the incentive for agents to make efforts in a game with two stages: A non-cooperative stage where agents choose how much effort to make and a cooperative second stage. As mentioned, we show how the solution of the cooperative stage determines the incentives that agents have to make an effort in the first stage, and consequently the efficiency of the final outcome.

Bernstein et al. (2015) also use a bi-form model to analyze the role of process improvement in a decentralized assembly system in which an assembler lays in components from several suppliers. The assembler faces a deterministic demand and suppliers incur variable inventory costs and fixed setup production costs. In the first stage of the game suppliers make investments in process improvement activities to reduce their fixed production costs. Upon establishing a relationship with the suppliers, the assembler sets up a knowledge sharing network which is modeled as a cooperative game between suppliers in which all suppliers achieve reductions in their fixed costs. They compare two classes of allocation mechanism: Altruistic allocation enables non-efficient suppliers to keep the full benefits of the cost reductions achieved due to learning from the efficient supplier. The Tute allocation mechanism benefits a supplier by transferring the incremental benefit generated by the inclusion of an efficient supplier in the network. They find that the system-optimal level of cost reduction is achieved under the Tute allocation rule. Our bi-form game is novel in terms of incentive for efforts by agents and is also richer in results: We provide a procedure for finding the unique efficient effort equilibrium in cooperation with pairwise cost reduction.

2 Model

We consider a model with a finite set of agents $N = \{1, 2, ..n\}$, where each agent has a good (for example resources, knowledge, or infrastructure) and has to perform a certain activity. The total cost of an agent's activity can be reduced if it cooperates with another agent, which means that the two agents share their goods. These cost reductions obtained by sharing goods in pairs depend on the effort made previously by each agent, i.e. $e_i = (e_{i1}, ..., e_{i(i-1)}, e_{i(i+1)}, ...e_{in}) \in [0, 1]^{n-1}$ for each $i \in N$, where $e_{ij} \in [0, 1]$ stands for the level of effort by agent i. Thus, for any pair of cooperating agents

$i, j \in N$ and a given effort e_{ij}, agent i reduces the total cost of agent j by an amount $r_{ji}(e_{ij}) \in \mathbb{R}_+$, and vice versa. These particular reductions between agents $i, j \in N$ are independent of cooperation with other agents. Furthermore, these efforts have an additional associated cost $c_i(e_i) \in \mathbb{R}$ for any $i \in N$. We refer to this kind of model as a pairwise inter-organizational model.

The sequence of events is as follows:

1. In the first stage, each agent decides its effort profile and bears the additional associated costs.
2. In the second stage, given the effort made in the first stage, agents cooperate and reduce their respective total costs in pairs.

Thus, these agents participate in bilateral interactions with multiple independent partners whose cost reductions are coalitionally independent, i.e., the cost reduction given by each agent to another agent remains constant in any possible coalition. This means that the total reduction in cost for each agent in a coalition $S \in N$ is the sum of the pairwise cost reductions given to that agent by the rest of the members of the coalition, i.e. for agent i, it is $\sum_{j \in S \setminus \{i\}} r_{ij}(e_{ji})$. Hence, in the first stage, agents can make efforts, and in the second stage, by the pairwise sharing, they give and receive cost reductions according to these previous efforts.

We assume perfect information regarding to the agents' costs and cost reductions depending on efforts. As mentioned above, agent i incurs a cost $c_i(e_i)$ by making the effort profile e_i, and receives from agent j a cost reduction $r_{ij}(e_{ji})$. We assume that $c_i(.) : [0,1]^{n-1} \to \mathbb{R}_+$ is a scalar field of class $C^2([0,1]^{n-1})$.[1] Moreover, for all $e_{ij} \in [0,1]$ with $j \in N \setminus \{i\}$, where it is assumed that $\frac{\partial c_i(e_i)}{\partial e_{ij}} > 0$, $\frac{\partial c_i^2(e_i)}{\partial^2 e_{ij}} > 0$, and $\frac{\partial c_i^2(e_i)}{\partial e_{ij} \partial e_{ih}} = 0$ for all $h \neq i, j$, which implies that the marginal cost $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ is independent of effort that i exerts with agents other than j.[2] We also assume that, for all $j \in N \setminus \{i\}$, function $r_{ij}(.) : [0,1] \to \mathbb{R}_+$ is class C^2, increasing and concave [3] at $[0,1]$.

Given an effort profile $e = (e_1, ..., e_i, ..., e_n) \in [0,1]^{n(n-1)}$, we denote by $\psi_i(e)$ the final cost allocated to agent i, which results from the allocation of the aggregate total cost achieved through cooperation in the second stage. We assume that in the first stage each agent chooses the effort level that minimizes its final cost. That effort is made in anticipation of the result of the cooperative cost game that follows in the second stage. Therefore, we first analyze the second stage (see Section 3), where we focus on different allocations of the aggregate total cost through cooperation with pairwise cost reduction. Analyzing the second stage first enables the first stage to be analyzed in Section 4, where we calculate the efficient effort strategies in equilibrium[4].

[1] A scalar field is said to be class C^2 at $[0,1]^{n-1}$ if its 2-partial derivatives exist at all points of $[0,1]^{n-1}$ and are continuous.

[2] This last assumption implies that the Hessian matrix is a diagonal matrix.

[3] $\partial r_{ji}(e_{ij})/\delta e_{ij} > 0$ (increasing) and $\partial^2 r_{ji}(e_{ij})/\delta^2 e_{ij} < 0$ (concave).

[4] An effort strategy profile is said to be in equilibrium when each agent has nothing to gain by changing only their own effort strategy given the strategies of all the other agents.

3 Cooperation with Pairwise Cost Reduction

This section presents the analysis of cooperation with pairwise cost reduction in the second stage. Agents make their efforts in pairwise sharing in the first stage, and initiate cooperation with efforts $e = (e_1, ..., e_i, ..., e_n)$. We model the pairwise effort game (henceforth PE-game) as a multiple-agent cooperative game where each agent i incurs an initial cost of $c_i(e_i)$. All agents in a pairwise effort group (coalition) give cost reductions to and receive such reductions from other agents. As a result, all agents in the coalition reduce their initial costs to levels that depend on the efforts made by the others. Any agent outside the pairwise effort network does not benefit from this pairwise cost reduction opportunity. Although we introduce all the game-theoretic concepts used in this paper, readers are referred to González-Díaz et al. (2010) for more details on cooperative and non-cooperative games.

We refer to the pairwise effort network as a PE-network and denote it by the t-uple $(N, e, \{c_i(e_i), \{r_{ji}(e_{ij})\}_{j \in N\setminus\{i\}}\}_{i \in N})$. We associate a TU cost game with each PE-network. That TU cost game is 3-upla (N, e, c), where N is the finite set of players, $e \in [0, 1]^{n(n-1)}$ the effort profile, and $c : 2^N \to \mathbb{R}$ is the so-called characteristic function of the game, which assigns to each subset $S \subset N$ the cost $c(S)$ that is incurred if agents in S cooperate. By convention, $c(\emptyset) = 0$. The cost of agent i in coalition $S \subset N$ is given by $c^S(i) := c_i(e_i) - \sum_{j \in S\setminus\{i\}} r_{ij}(e_{ji})$. This cost can be interpreted as the reduced cost of agent i after participating in the PE-network together with the agents in S. Note that the larger the PE-network an agent participates in, the greater the cost reduction it achieves, i.e. for all $i \in S \subset T \subset N$, $c^T(\{i\}) \leq c^S(\{i\})$. Thus, in a PE-network, the cost function is given by $c(S) := \sum_{i \in S} c^S(\{i\}) = \sum_{i \in S}[c_i(e_i) - \sum_{j \in S\setminus\{i\}} r_{ij}(e_{ji})]$.

The class of PE-games has some similarities with the class of linear cost games introduced in Meca and Sosic (2014). They define the concept of cost-coalitional vectors as a collection of certain a priori information, available in the cooperative model, represented by the costs of the agents in all possible coalitions to which they could belong. There, the cost of a coalition was also the sum of the costs of all agents in that coalition. However, the PE-games considered here are significantly different from their linear cost games. Linear cost games focus on the role played by benefactors (giving) and beneficiaries (receiving) as two groups of disjointed agents, but PE-games consider that all the agents could be dual benefactors, in the sense that they could play the roles of benefactors and/or beneficiaries at the same time. In addition, PE-games are based on a bilateral cooperation between agents that enables both to reduce their costs but it is coalitionally independent.

We now consider a PE-network $(N, e, \{c_i(e_i), \{r_{ij}(e_{ij})\}_{j \in N\setminus\{i\}}\}_{i \in N})$ and consider whether if it is profitable for the agents in N to form the grand coalition to obtain a significant reduction in costs. Here, we prove that the answer to this question is yes, because the associated PE-game (N, e, c) is concave, in the sense that for all $i \in N$ and all $S, T \subseteq N$ such that $S \subseteq T \subset N$ with $i \in S$, then $c(S) - c(S \setminus \{i\}) \geq c(T) - c(T \setminus \{i\})$. This concavity property provides us with additional information about the game: the marginal contribution of an agent diminishes as a coalition grows. This is well-known and is called the "snowball effect".

The first result in this section shows that PE-games are always concave. To prove this, the class of unanimity games must be described. Shapley (1953) proves that the family of unanimity games $\{(N, u_T), T \subseteq N\}$ form a basis of the vector space of all games with sets of players N, where (N, u_T) is defined, for each, $S \subseteq N$ as follows:

$$u_T(S) = \begin{cases} 1, & T \subseteq S \\ 0, & otherwise \end{cases}$$

Hence, for each cost game (N, c) there are unique real coefficients $(\alpha_T)_{T \subseteq N}$ such that $c = \sum_{T \subseteq N} \alpha_T u_T$. Many different classes of games, including airport games (Littlechild and Owen, 1973) and sequencing games (Curiel et al., 1989), can be characterized through constrains on these coefficients.

Proposition 1 *Every PE-game is concave.*

Proof. Let $(N, e, \{c_i(e_i), \{r_{ji}(e_{ij})\}_{j \in N \setminus \{i\}}\}_{i \in N})$ be a PE-network and (N, e, c) the associated PE-game. First, we prove that this game can be rewritten as a weighted sum of unanimity games $u_{\{i\}}$ and $u_{\{i,j\}}$ for all $i, j \in N$ as follows:

$$c = \sum_{i \in N} c_i(e_i) u_{\{i\}} - \sum_{i,j \in N; i \neq j} r_{ij}(e_{ji}) u_{\{i,j\}}. \tag{1}$$

Indeed, for all $S \subseteq N$,

$$\begin{aligned} c(S) &= \sum_{i \in N} c_i(e_i) u_{\{i\}}(S) - \sum_{i,j \in N; i \neq j} r_{ij}(e_{ji}) u_{\{i,j\}}(S) = \\ &= \sum_{i \in S} c_i(e_i) - \sum_{i,j \in S; i \neq j} r_{ij}(e_{ji}) = \\ &= \sum_{i \in S} c_i(e_i) - \sum_{i \in S} \sum_{j \in S \setminus \{i\}} r_{ij}(e_{ji}). \end{aligned}$$

It is easily shown that the additive game $\sum_{i \in N} c_i(e_i) u_{\{i\}}$ is concave and that $u_{\{i,j\}}$ is convex. Thus, the game $-\sum_{i,j \in N; i \neq j} r_{ij}(e_{ji}) u_{\{i,j\}}$ is concave because of $r_{ij}(e_{ji}) > 0$ for all $i, j \in N$. Finally, the concavity of (N, e, c) follows from the fact that game c is the sum of two concave games. ■

This means that the grand coalition can obtain a significant reduction in costs. In that case, the reduced total cost is given by $c(N) = \sum_{i \in N} c_i(e_i) - R(N)$, where $R(N) = \sum_{i \in N} \sum_{j \in N \setminus \{i\}} r_{ij}(e_{ji})$ is the total reduction produced by bilateral reductions between all agents in the network, which turns out to be the total cost savings for all agents.

An allocation rule for PE-games is a map ψ which assigns a vector $\psi(e) \in R^n$ to every (N, e, c), satisfying efficiency, that is, $\sum_{i \in N} \psi_i(e) = c(N)$. Each component $\psi_i(e)$ indicates the cost allocated to

$i \in N$, so an allocation rule for PE-games is a procedure for allocating the reduced total to all the agents in N when they cooperate. It is a well-known result in cooperative game theory that concave games are totally balanced: The core of a concave game is non-empty, and since any subgame of a concave game is concave, the core of any subgame is also non-empty. That means that coalitionally stable allocation rules can always be found for PE-games. We interpret a non-empty core for PE-games as indicating a setting where all included cooperation is feasible, in the sense that there are possible cost reductions that make all agents better off (or, at least, not worse off). The totally balanced property suggests that this all-included cooperation is consistent, i.e. for every group of agents whole-group cooperation is also feasible.

A highly natural allocation rule for PE-games is $\varphi_i(e) = c^N(\{i\}) = c_i(e_i) - R_i(N)$, for all $i \in N$, with $R_i(N) = \sum_{j \in N \setminus \{i\}} r_{ij}(e_{ji})$ being the total reduction received by agent $i \in N$ from the rest of the agents $j \in N \setminus \{i\}$. It has good properties at least with respect to computability and coalitional stability in the sense of the core. This means that, for every PE-game (N, c), $\varphi(e)$ should satisfy the requirement that $\sum_{i \in S} \varphi_i(e) \leq c(S)$ for every $S \subseteq N$. Notice that, for every $S \subseteq N$, $\sum_{i \in S} \varphi_i(e) = \sum_{i \in S} c^N(i) \leq \sum_{i \in S} c^S(i) = c(S)$. Nevertheless, the agents could argue that this allocation does not compensate them for their dual role of giving and receiving. Note that the allocation only considers their role as receivers.

PE-games are concave, so cooperative game theory provides allocation rules for them with good properties, at least with respect to coalitional stability and acceptability of items. We highlight the Shapley value (see Shapley 1953), which assigns a unique allocation (among the agents) of a total surplus generated by the grand coalition. It measures how important each agent is to the overall cooperation, and what cost can it reasonably expect. It is a "fair" allocation in the sense that it is the only distribution with certain desirable properties listed below. The Shapley value of a concave game is the center of gravity of its core (see Shapley 1971). In general, this allocation becomes harder to compute when the number of agents increases. Despite everything, we present a simple expression here for the Shapley value of PE-games that takes into account all bilateral relations between agents and compensates them for their dual role of giving and receiving.

Given a general cost game (N, c), we denote by $\phi(c)$ the Shapley value, where for each agent $i \in N$, the corresponding cost allocation is

$$\phi_i(c) = \sum_{i \in T \subseteq N} \frac{(n-t)!(t-1)!}{n!} \left[(c(T) - c(T \setminus \{i\}) \right], with \mid T \mid = t. \tag{2}$$

The Shapley value has many desirable properties, and it is also the only allocation rule that satisfies a certain subset of those properties (see Moulin, 2004). For example, it is the only allocation rule that satisfies the four properties of Efficiency, Equal treatment of equals, Linearity and Null player (Shapley, 1953). Next, we describe all of these properties of the Shapley value, which are useful in demonstrating our next result.

(EFF) *Efficiency.* The sum of the Shapley values of all agents equals the value of the grand coalition, so that all the gain is allocated to the agents:

$$\sum_{i\in N} \phi_i(c) = c(N). \tag{3}$$

(ETE) *Equal treatment of equals.* If i and j are two agents who are equivalent in the sense that $c(S \cup \{i\}) = c(S \cup \{j\})$ for every coalition S of N which contains neither i nor j, then $\phi_i(c) = \phi_j(c)$.

(LIN) *Linearity.* If two cost games c and c^* are combined, then the cost allocation should correspond to the costs derived from c and the costs derived from c^*:

$$\phi_i(c + c^*) = \phi_i(c) + \phi_i(c^*), \forall i \in N. \tag{4}$$

Also, for any real number a,

$$\phi_i(ac) = a\phi_i(c), \forall i \in N. \tag{5}$$

(NUP) *Null Player.* The Shapley value $\phi_i(c)$ of a null player i in a game c is zero. A player i is null in c if $c(S \cup \{i\}) = c(S)$ for all coalitions S that do not contain i.

Given a PE-game (N, e, c), we denote by $\phi(e)$ the Shapley value of the cost game. The following Theorem shows that the Shapley value provides an acceptable allocation for PE-games. Indeed, it reduces the individual cost of an agent by half the total reduction that it obtains from the others ($R_i(N)$) plus a half of the total reduction that it provides to the rest of the agents, which is $G_i(N) = \sum_{j\in N\setminus\{i\}} r_{ji}(e_{ij})$.

Theorem 1 *Let (N, e, c) be a PE-game. For each agent $k \in N, \phi_k(e) = c_k(e_k) - \frac{1}{2}[R_k(N) + G_k(N)]$.*

Proof. Consider the PE-game (N, e, c) rewritten as a weighted sum of unanimity games given by (1). That is,

$$c = \sum_{i\in N} c_i(e_i)u_{\{i\}} - \sum_{i,j\in N; i\neq j} r_{ij}(e_{ji})u_{\{i,j\}}.$$

Take an agent $k \in N$. By the (LIN) property of the Shapley value, $\phi_k(e)$, it follows that

$$\begin{aligned}\phi_k(e) &= \phi_k\left(\sum_{i\in N} c_i(e_i)u_{\{i\}}\right) - \phi_k\left(\sum_{i,j\in N;i\neq j} r_{ij}(e_{ji})\left(u_{\{i,j\}}\right)\right) \\ &= \sum_{i\in N} c_i(e_i)\phi_k\left(u_{\{i\}}\right) - \sum_{i\in N}\sum_{j\in N\setminus\{i\}} r_{ij}(e_{ji})\phi_k\left(u_{\{i,j\}}\right).\end{aligned} \tag{6}$$

In addition, it is known from the (NUP) property that

$$\phi_k\left(u_{\{i\}}\right) = \begin{cases} 1, & i = k \\ 0, & otherwise \end{cases} \tag{7}$$

and from (ETE) and (NUP), that

$$\phi_k\left(u_{\{i,j\}}\right) = \begin{cases} 1/2, & i = k, j = k, i \neq j \\ 0, & otherwise \end{cases} \tag{8}$$

Consequently, by substituting the values (7) and (8) in equation (6), the following is obtained:

$$\begin{aligned} \phi_k(e) &= c_k(e_k) - \sum_{j \in N\setminus\{k\}} r_{kj}(e_{jk})\phi_k\left(u_{\{k,j\}}\right) - \sum_{j \in N\setminus\{k\}} r_{jk}(e_{kj})\phi_k\left(u_{\{j,k\}}\right) \\ &= c_k(e_k) - \frac{1}{2}\sum_{j \in N\setminus\{k\}} [r_{kj}(e_{jk}) + r_{jk}(e_{kj})]. \end{aligned}$$

Finally, it can be concluded that, for each agent $k \in N$,

$$\phi_k(e) = c_k(e_k) - \frac{1}{2}[R_k(N) + G_k(N)].$$

■

From Theorem 1 it can be derived that the Shapley value, $\phi(e)$, considers the dual role of giving and receiving of all agents, and the final effect on those agents depends on which role is stronger. As mentioned above, if an allocation does not compensate them for their dual role of giving and receiving, and it only considers their role as receivers, as the individual cost in the grand coalition, $\varphi(e)$, does, the cooperation cannot be acceptable to those dual agents. This non-acceptability can be avoided by using the Shapley value, which also matches with the Nucleolus (Schmeiler 1989) for PE-games.

The nucleolus selects the allocation in which the coalition with the smallest excess (the worst treated) has the highest possible excess. The nucleolus maximizes the "welfare" of the worst treated coalitions. Denote by $\nu(e) \in R^n$ the *Nucleolus* of the PE-game (N, e, c), associated with a PE-network $(N, e, \{c_i(e_i), \{r_{ij}(e_{ij})\}_{j \in N\setminus\{i\}}\}_{i \in N})$. First, we define the excess of coalition S in (N, e, c) with respect to allocation x as $d(S, x) = c(S) - \sum_{i \in S} x_i$. This number can be considered as an index of the "welfare" of coalition S at x: The greater $d(S, x)$, the better coalition S is at x. Let $d^*(x)$ be the vector of the 2^n excesses arranged in (weakly) increasing order, i.e., $d_i^*(x) \leq d_j^*(x)$ for all $i < j$. Second, we define the lexicographical order $\succ_l$. For any $x, y \in R^n$, $x \succ_l y$ if and only if there is an index k such that for any $i < k$, $x_i = y_i$ and $x_k > y_k$. The nucleolus of the PE-game (N, e, c) is the set

$$\nu(e) = \{x \in X : d^*(x) \succ_l d^*(y) \text{ for all } y \in X\}$$

with $X = \{x \in R^n : \sum_{i \in N} x_i = c(N),\ x_i \geq c(i) \text{ for all } i \in N\}$.

The Proposition 2 proves that for PE-games the Shapley value matches the Nucleolus. This is a very good property that few cost games satisfy.

Proposition 2 *Let (N, e, c) be a PE-game. For each agent $k \in N$, $\nu_k(e) = \phi_k(e)$.*

Proof. To prove that for PE-games the Shapley value coincides with the Nucleolus, it is first necessary to describe the class of PS-games introduced by Kar et al (2009).

Denote by $M_i c(T)$ the marginal contribution of player $i \in S$, that is, $M_i c(T) = c(T) - c(T \setminus \{i\})$, for all $i \in T \subseteq N$. A cost game (N, c) satisfies the PS property if for all $i \in N$, there exists $k_i \in \mathbb{R}$ such that $M_i c(T \cup \{i\}) + M_i c(N \setminus T) = k_i$, for all $i \in N$ and all $T \subseteq N \setminus \{i\}$. Kar et al (2009) show that for PS games, the Shapley value coincides with the Nucleolus; that is, $\phi_i(c) = \nu_i(c) = \frac{k_i}{2}$, for all $i \in N$.

Therefore, it only remains to show that (N, e, c) is a PS-game with $k_i = [c_i(e_i) - R_i(N)] + [c_i(e_i) - G_i(N)]$, for all $i \in N$.

First, we prove that $M_i c(T) = c_i(e_i) - \sum_{j \in T \setminus \{i\}} [r_{ji}(e_{ij}) + r_{ij}(e_{ji})]$ for all $i \in T \subseteq N$. Indeed, take a coalition $T \subseteq N$ and an agent $i \in T$. Then, by definition of the PE-game,

$$c(T) = \textstyle\sum_{h \in T} c_h^T = c_i^T + \sum_{h \in T \setminus \{i\}} c_h^T.$$

Then, substituting the expressions of c_i^T and c_h^T it follows that

$$c(T) = \left(c_i(e_i) - \textstyle\sum_{j \in T \setminus \{i\}} r_{ij}(e_{ji})\right) + \sum_{h \in T \setminus \{i\}} \left(c_h(e_h) - \sum_{j \in T \setminus \{h\}} r_{hj}(e_{jh})\right) =$$

$$c_i(e_i) - \textstyle\sum_{j \in T \setminus \{i\}} r_{ij}(e_{ji}) + \sum_{h \in T \setminus \{i\}} \left(c_h(e_h) - \left(\sum_{j \in T \setminus \{h,i\}} r_{hj}(e_{jh})\right) - r_{hi}(e_{ih})\right) =$$

$$c_i(e_i) - \textstyle\sum_{j \in T \setminus \{i\}} r_{ij}(e_{ji}) + \sum_{h \in T \setminus \{i\}} \left(c_h(e_h) - \left(\sum_{j \in T \setminus \{h,i\}} r_{hj}(e_{jh})\right)\right) - \sum_{h \in T \setminus \{i\}} r_{hi}(e_{ih}) =$$

$$c_i(e_i) - \textstyle\sum_{j \in T \setminus \{i\}} r_{ij}(e_{ji}) + c(T \setminus \{i\}) - \sum_{h \in T \setminus \{i\}} r_{hi} e_{ih}).$$

Hence,

$$M_i c(T) := c(T) - c(T \setminus \{i\}) = c_i(e_i) - \textstyle\sum_{j \in T \setminus \{i\}} \left(r_{ij}(e_{ji}) + r_{ji}(e_{ij})\right).$$

Second, we show that $M_i c(T \cup \{i\}) + M_i c(N \setminus T) = [c_i(e_i) - R_i(N)] + [c_i(e_i) - G_i(N)]$ for all $i \in N$ and $T \subseteq N \setminus \{i\}$. Indeed, take a coalition $T \subseteq N$ and an agent $i \in T$. It is shown that $M_i c(T \cup \{i\}) = c_i(e_i) - \sum_{j \in T} \left(r_{ji}(e_{ij}) + r_{ij}(e_{ji})\right)$, and $M_i c(N \setminus T) = c_i(e_i) - \sum_{j \in N \setminus (T \cup \{i\})} \left(r_{ji}(e_{ij}) + r_{ij}(e_{ji})\right)$. Therefore,

$$M_i c(T \cup \{i\}) + M_i c(N \setminus T) = 2c_i(e_i) - \textstyle\sum_{j \in N \setminus \{i\}} \left(r_{ji}(e_{ij}) + r_{ij}(e_{ji})\right) =$$

$$\left[c_i(e_i) - \textstyle\sum_{j \in N \setminus \{i\}} r_{ij}(e_{ji})\right] + \left[c_i(e_i) - \sum_{j \in N \setminus \{i\}} r_{ji}(e_{ij})\right].$$

Hence, $M_i c(T \cup \{i\}) + M_i c(N \setminus T) = [c_i(e_i) - R_i(N)] + [c_i(e_i) - G_i(N)] = k_i)$, and so (N, e, c) is a PS game. ■

Therefore, given a profile of efforts, the Shapley value is a very suitable way of allocating the reduced cost due to cooperation. Note that, the cost reduction as a result of cooperation between any pair of agents $i, j \in N$ is $r_{ij}(e_{ji}) + r_{ji}(e_{ij})$, and the Shapley value assigns one half of this amount to i and the other half to j. This seems a reasonable way to split this aggregated cost reduction. However, if agents knew before choosing their levels of efforts that the cost reductions resulting from their efforts were going to be allocated according to the Shapley value, the incentives created could generate inefficiencies. Some agents could find it optimal to exert too little effort and in some situations this could be inefficient.

For example, consider a PE-network in which one agent has the ability to produce a substantial reduction in costs for other agents with a low effort cost and the rest of the agents have almost no ability to reduce costs for others even with a high effort cost. If the Shapley value is used as the allocation rule for this game, agents may not have incentives to make any level of effort. Note that in the first step agents have to decide how much effort to make. However, if the Shapley value is modified to give a greater portion of the pairwise cost reduction to this especially productive agent, it might make more effort and thus produce a greater reduction in cost for other agents. This change in the Shapley value generates new allocation rules, which can reduce the cost of the grand coalition regarding the Shapley allocation. The following example with three agents illustrates these ideas.

Example 1 ***(A 3-firm case).*** *Consider a pairwise inter-organizational situation with three firms, i.e.* $N = \{1, 2, 3\}$*. For any effort profile* $e \in [0, 1]^6$*, the PE-network is given by the following initial costs,*

$c_1(e_{12}, e_{13}) = 100 + 100e_{12} + 4e_{12}^2 + 100e_{13} + 4e_{13}^2$
$c_2(e_{21}, e_{23}) = 100 + 100e_{21} + 4e_{21}^2 + 100e_{23} + 4e_{23}^2$
$c_3(e_{31}, e_{32}) = 100 + 100e_{31} + 4e_{31}^2 + 100e_{32} + 4e_{32}^2$

and the following pairwise reduced costs, all of them in thousands of Euros,

$r_{i1}(e_{1i}) = 2 + 200e_{1i} - 3e_{1i}^2$ *with* $i = 2, 3$
$r_{i2}(e_{2i}) = 2 + 3e_{2i} - e_{2i}^2$ *with* $i = 1, 3$
$r_{i3}(e_{3i}) = 2 + 3e_{2i} - e_{3i}^2$ *with* $i = 1, 2$

If the allocation rule in the second stage is the Shapley value, the firms choose their levels of effort according to this cost allocation function. It is straight forward to show that in this case the unique effort equilibrium e^**, is one in which the three firms make no effort, i.e.* $e_{ij}^* = 0$ *for* $i, j \in N$*.*[5] *Thus, the Shapley value distributes the cost of the grand coalition* $c^*(N) = 288$ *equally, i.e. for each firm* $i = 1, 2, 3$, $\phi_i(e^*) = c_i(e_i^*) - \frac{1}{2}\sum_{j \in N\setminus\{i\}}[r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*)] = 100 - \frac{1}{2}((2+2) + (2+2)) = 96$.

Note that, for example, in the relationship between firm 1 *and* 2*, the pairwise cost reduction is* $r_{12}(e_{21}) + r_{21}(e_{12})$*, and the Shapley value gives* $\frac{1}{2}$ *of this amount to firm* 1 *and the other* $\frac{1}{2}$ *to firm* 2*. However, if the proportion that firm* 1 *obtains is increased, e.g. from* $\frac{1}{2}$ *to* $\frac{3}{4}$*, and the part for firm* 2 *is thus reduced to* $\frac{1}{4}$*, the incentive of firm* 1 *to make an effort can be increased. The same*

[5] Theorem 3, in Section 4, shows the efforts of equilibrium in the non-cooperative stage of the game in the general case.

can be done between firms 1 and 3 so that the incentive of firm 1 to make an effort for firm 3 is also increased. These changes in the Shapley value lead to a new allocation rule which we denote by $A(e) = (A_1(e), A_2(e), A_3(e))$ *for any effort profile* $e \in [0,1]^6$. *With this new allocation rule, the equilibrium efforts are zero for firms 2 and 3, and one for by firm 1. That is,* $e^{**}_{1j} = 1$, *for* $j = 2, 3$, $e^{**}_{2j} = 0$, *for* $j = 1, 3$, *and* $e^{**}_{3j} = 0$, *for* $j = 1, 2$. *In this case, the grand coalition cost* $c^{**}(N) = 152$ *is allocated equally between firms 2 and 3, and the rest to firm 1. That is,* $A_i(e^{**}) = 100 - \frac{1}{4}[(2+100-3)+2] - \frac{1}{2}(2+2) = 72,75$ *for* $i = 2, 3$, *and* $A_1(e^{**}) = 100+100+4+100+4-\frac{3}{4}[(2+(2+200-3))+(2+(2+200-3))] = 6,5$.

Hence, the new allocation rule $A(e^{**})$ *greatly reduces the grand coalition cost (by 136.000 Euros) as well as the costs of each firm; i.e. a reduction of 89.500 Euros for firm 1 and 23.250 Euros for firms 2 and 3. In relative terms, with the value of Shapley each company pays 33.33% of the total cost. However, with the modified Shapley value agent 1 only pays 4.4% of the total cost, while agents 2 and 3 pay 47.8% each. Therefore, the modified Shapley value generates a more efficient outcome in the sense that it creates more appropriate incentives for firms.*

To reach efficient effort strategies in equilibrium (henceforth EEE) in the first stage, we consider a new family of allocation rules, for PE-games (second stage), based on the Shapley value. This family consists of the rules $A(e) \in R^n$, where for all $i \in N$,

$$A_i(e) = c_i(e_i) - \textstyle\sum_{j \in N \setminus \{i\}} \alpha_{ij}[r_{ij}(e_{ji}) + r_{ji}(e_{ij})],$$

with $\alpha_{ij} \in [0,1]$, for all $j \in N \setminus \{i\}$, such that $\alpha_{ij} = 1 - \alpha_{ji}$. The Shapley value is a particular case of this family of rules in which $\alpha_{ij} = \frac{1}{2}$. This family of cost allocation for PE-games is referred to as *cost allocation with weighted pairwise reduction*.

The Theorem below shows that the family of cost allocations with weighted pairwise reduction is always a subset of the core of PE-games. This property identifies a wide subset of the large core of PE-games, which includes the Shapley value and the Nucleolus.

Theorem 2 *Let* (N, e, c) *be a PE-game. For every family of weights* $\alpha_{ij} \in [0,1]$, $i, j \in N, i \neq j$, *such that* $\alpha_{ij} + \alpha_{ji} = 1$, $A(e)$ *belongs to the core of* (N, e, c).

Proof. Consider the PE-game (N, e, c) associated with the PE-network $(N, e, \{c_i(e_i), \{r_{ij}(e_{ij})\}_{j \in N \setminus \{i\}}\}_{i \in N})$. Take a family of weights $\alpha_{ij} \in [0,1], i, j \in N, i \neq j$ such that $\alpha_{ij} + \alpha_{ji} = 1$, and $A(e)$ the corresponding cost allocation with weighted pairwise reduction with $A_i(e) = c_i(e_i) - \sum_{j \in N \setminus \{i\}} \alpha_{ij}[r_{ij}(e_{ji}) + r_{ji}(e_{ij})]$, for all $i \in N$. To prove that $A(e) \in C(N, e, c)$ it must be checked that (1) $\sum_{i \in N} A_i(e) = c(N)$, (2) $\sum_{i \in S} A_i(e) \leq c(S)$, for all $S \subset N$.

We start by checking (1). Notice that $\sum_{i \in N} A_i(e) = c(N)$ is equivalent to

$\sum_{i \in N} \sum_{j \in N \setminus \{i\}} \alpha_{ij}[r_{ij}(e_{ji}) + r_{ji}(e_{ij})] = \sum_{i \in N} \sum_{j \in N \setminus \{i\}} r_{ij}(e_{ji})$.

Indeed,

$\sum_{i\in N}\sum_{j\in N\setminus\{i\}}\alpha_{ij}[r_{ij}(e_{ji})+r_{ji}(e_{ij})] = \sum_{i\in N}\sum_{j\in N\setminus\{i\}}(\alpha_{ij}+\alpha_{ji})r_{ij}(e_{ji}) = \sum_{i\in N}\sum_{j\in N\setminus\{i\}}r_{ij}(e_{ji})$,

where the last equality is due to $\alpha_{ij}+\alpha_{ji}=1$ for all $i,j\in N$.

Next we check (2). Take $S\subset N$. Notice now that $\sum_{i\in S}A_i(e)\leq c(S)$ is equivalent to $\sum_{i\in S}\sum_{j\in N\setminus\{i\}}\alpha_{ij}[r_{ij}(e_{ji})+r_{ji}(e_{ij})]-\sum_{i\in S}\sum_{j\in S\setminus\{i\}}r_{ij}(e_{ji})\geq 0$.

Indeed, an argument similar to that used in (1) leads to

$\sum_{i\in S}\sum_{j\in N\setminus\{i\}}\alpha_{ij}[r_{ij}(e_{ji})+r_{ji}(e_{ij})]-\sum_{i\in S}\sum_{j\in S\setminus\{i\}}r_{ij}(e_{ji}) =$

$\sum_{i\in S}\sum_{j\in S\setminus\{i\}}\alpha_{ij}[r_{ij}(e_{ji})+r_{ji}(e_{ij})]+\sum_{i\in S}\sum_{j\in N\setminus S\cup\{i\}}\alpha_{ij}[r_{ij}(e_{ji})+r_{ji}(e_{ij})]-\sum_{i\in S}\sum_{j\in S\setminus\{i\}}r_{ij}(e_{ji}) =$

$\sum_{i\in S}\sum_{j\in S\setminus\{i\}}r_{ij}(e_{ji})+\sum_{i\in S}\sum_{j\in N\setminus S\cup\{i\}}\alpha_{ij}[r_{ij}(e_{ji})+r_{ji}(e_{ij})]-\sum_{i\in S}\sum_{j\in S\setminus\{i\}}r_{ij}(e_{ji}) =$

$\sum_{i\in S}\sum_{j\in N\setminus S\cup\{i\}}\alpha_{ij}[r_{ij}(e_{ji})+r_{ji}(e_{ij})]\geq 0$. ■

Now we are ready to carry out a complete analysis of the EEE for cooperation in pairwise cost reduction.

4 Analysis of Efficient Effort strategies in Equilibrium

This section analyzes the non-cooperative effort game that arises in the first stage. Agents decide how much pairwise effort to make to reduce the costs of other agents, and do so in anticipation of the allocation of the total cost reduction of the PE-game resulting from the second stage. Our goal is to demonstrate that there are always core allocations in the cooperative game of the second stage that induce an efficient effort equilibrium level in the first stage. We consider that an effort profile $e^1\in E$ is more efficient than a profile $e^2\in E$ if the aggregate cost generates in the second stage by e^1 is lower than that generated by e^2.

We therefore first study the non-cooperative effort game that arises under the cost allocation $A(e)\in R^n$ defined in the previous section. This game is defined by $(N,\{E_i\}_{i\in N},\{A_i\}_{i\in N})$, where for every agent $i\in N$, $E_i:=[0,1]^{(n-1)}$ is the players' i strategy set, and for all effort profiles $e\in E:=\prod_{i\in N}E_i$, the cost function is:

$$A_i(e)=c_i(e_i)-\sum_{j\in N\setminus\{i\}}\alpha_{ij}[r_{ij}(e_{ji})+r_{ji}(e_{ij})], \tag{9}$$

with $\alpha_{ij}\in[0,1]$. The interaction between agents i and j generates an aggregate cost reduction which is $r_{ij}(e_{ji})+r_{ji}(e_{ij})$. The parameter α_{ij} measures the proportions in which this reduction is shared between agents i and j, i.e. α_{ij} is the proportion for agent i and $\alpha_{ji}=1-\alpha_{ij}$ for agent j. In this game, we use the following definition of equilibrium.

Definition 1 *The effort profile $e^*=(e_1^*,...,e_n^*)\in E$ is an equilibrium for the game $(N,\{E_i\}_{i\in N},\{A_i\}_{i\in N})$,* *if and only if e_i^* is the optimal effort for agent $i \in N$ given the strategies of all the other agents $j \in N\backslash\{i\}$.*

First, note that the optimal effort for agent $i \in N$ given the strategies of all the other agents $j \in N\backslash\{i\}$ is the effort e_i that minimizes $A_i(e_i, e_{-i})$. To simplify notation and analysis, we consider that for all $i \in N$ and $j \in N\backslash\{i\}$, $c_i'(e_{ij}) := \frac{\partial c_i(e_i)}{\partial e_{ij}}$, $c_i''(e_{ij}) := \frac{\partial^2 c_i(e_i)}{\partial e_{ij}^2}$, $r_{ji}'(e_{ij}) := \frac{\partial r_{ji}(e_{ij})}{\partial e_{ij}}$ and $r_{ji}''(e_{ij}) := \frac{\partial^2 r_{ji}(e_{ij})}{\partial e_{ij}^2}$. Note that $\frac{\partial A_i(e)}{\partial e_{ij}} = c_i'(e_{ij}) - \alpha_{ij} r_{ji}'(e_{ij})$ and $\frac{\partial^2 A_i(e)}{\partial e_{ij}^2} = c_i''(e_{ij}) - \alpha_{ij} r_{ji}''(e_{ij}) > 0$ because, because, as assumed above, $c_i''(e_{ij}) > 0$ and $r_{ji}''(e_{ij}) < 0$. Thus, the function A_i is convex in the effort e_{ij} that agent i exerts for any $j \in N\backslash\{i\}$. This means that for agent i there is a unique optimal level of effort $\hat{e}_{ij}$ for each $j \in N\backslash\{i\}$. That optimal level $\hat{e}_{ij}$ depends on the parameter α_{ij}, on the marginal cost of agent i in regard to effort $\hat{e}_{ij}$, i.e. $c_i'(e_{ij})$, and on the marginal cost-reduction for agent j in regard to effort $\hat{e}_{ij}$, i.e. $r_{ji}'(e_{ij})$. Consequently, although the cost function of agent i depends on other agents' efforts (e_{ji} for all $j \in N\backslash\{i\}$), the optimal effort does not.

Before analyzing the EEE of the non-cooperative effort game, we define thresholds of alpha parameters that will enable them to be reached.

Definition 2 *Given an effort game $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$, we define the following lower and upper thresholds for each pair of agents i and j,*

$\underline{\alpha}_{ij} := \frac{c_i'(0)}{r_{ji}'(0)}$, $\bar{\alpha}_{ij} := \frac{c_i'(1)}{r_{ji}'(1)}$, $\underline{\alpha}_{ji} := \frac{c_j'(0)}{r_{ij}'(0)}$, *and* $\bar{\alpha}_{ji} := \frac{c_j'(1)}{r_{ij}'(1)}$.

It is clear that $0 < \underline{\alpha}_{ij} < \bar{\alpha}_{ij}$ because c_i' is an increasing function and r_{ji}' decreasing. Analogously, $0 < \underline{\alpha}_{ji} < \bar{\alpha}_{ji}$.

The first Theorem in this section characterizes all possible types of effort equilibrium according to the value of the parameter α_{ij}, for all $i, j \in N, i \neq j$. Before proving this Theorem, we consider two previous Lemmas that will be very useful for latter results. The first one characterizes the optimal effort level for agent $i \in N$ in the first stage non-cooperative game.

Lemma 1 *Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ be the effort game, with $\hat{e}_{ij}$ being the optimal level of effort that agent i exerts to reduce the costs of agent j. Thus,*

1. *$\hat{e}_{ij} = 0$ if and only if $\alpha_{ij} \leq \underline{\alpha}_{ij}$*
2. *There is a unique $\hat{e}_{ij} \in (0,1)$ that holds $c_i'(\hat{e}_{ij}) - \alpha_{ij} r_{ji}'(\hat{e}_{ij}) = 0$ if and only if $\underline{\alpha}_{ij} < \alpha_{ij} < \bar{\alpha}_{ij}$.*
3. *$\hat{e}_{ij} = 1$ if and only $\alpha_{ij} \geq \bar{\alpha}_{ij}$.*

Proof. First, remember that the cost function $A_i(e)$ is convex for all $i \in N$. To obtain the optimal effort, we analyze the derivative of this function with respect to e_{ij}, for any $j \in N\backslash\{i\}$. It must be noted that, for all $e_{ij} \in [0,1]$, $\frac{\partial A_i(e)}{\partial e_{ij}} > 0 \iff c_i'(e_{ij}) > \alpha_{ij} r_{ji}'(e_{ij})$, which is a necessary and sufficient condition for $\hat{e}_{ij} = 0$ to be the optimal effort. [6]

We begin by proving point 1. Note that $\underline{\alpha}_{ij} = \frac{c_i'(0)}{r_{ji}'(0)} < \frac{c_i'(e_{ij})}{r_{ji}'(e_{ij})}$ because $c_i' > 0$, $r_{ji}' > 0$, $c_i'' > 0$, and $r_{ji}'' < 0$. Thus, $c_i'(e_{ij})$ is a positive and increasing function, and $r_{ji}'(e_{ij})$ a positive and decreasing function, so for any $e_{ij} > 0$, $c_i'(0) < c_i'(e_{ij})$ and $r_{ji}'(0) > r_{ji}'(e_{ij})$. Therefore, $\alpha_{ij} \leq \underline{\alpha}_{ij} \iff c_i'(e_{ij}) > \alpha_{ij} r_{ji}'(e_{ij})$ for all $e_{ij} > 0 \iff \hat{e}_{ij} = 0$.

The demonstration in point 3 is similar to that of point 1. The above arguments are the same and only the signs of the inequalities change.

To end the proof, we prove point 2. First, we show that there is a unique $\hat{e}_{ij} \in (0,1)$ such that $c_i'(\hat{e}_{ij}) = \alpha_{ij} r_{ji}'(\hat{e}_{ij})$, which is the unique optimal effort because $\left.\frac{\partial A_i(e)}{\partial e_{ij}}\right|_{e_i = \hat{e}_{ij}} = 0$ and $A_i(e)$ is a convex function. In addition, $c_i'(e_{ij})$ is a positive and increasing function and $r_{ji}'(e_{ij})$ a positive and decreasing function, in $e_{ij} \in [0,1]$. This implies that equation $\frac{\partial A_i(e)}{\partial e_{ij}} = c_i'(e_{ij}) - \alpha_{ij} r_{ji}'(e_{ij}) = 0$ has a unique root, which belongs to $(0,1)$ if and only if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$. Note that if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$ then $c_i'(0) < \alpha_{ij} r_{ji}'(0)$ and $c_i'(1) > \alpha_{ij} r_{ji}'(1)$, and so there is a unique point $\hat{e}_{ij}$ where $c_i'(\hat{e}_{ij}) = \alpha_{ij} r_{ji}'(\hat{e}_{ij})$. ■

We now need to write down the previous Lemma 1 for agent j but depending on α_{ij}, the proof of which is straightforward.

Lemma 2 *Let $\underline{\alpha}_{ji} = \frac{c_j'(e_{ji}=0)}{r_{ij}'(e_{ji}=0)}$ and $\bar{\alpha}_{ji} = \frac{c_j'(e_{ji}=1)}{r_{ij}'(e_{ji}=1)}$, where $0 < \underline{\alpha}_{ji} < \bar{\alpha}_{ji}$, and $\hat{e}_{ji}$ the optimal level of effort that j exerts to reduce the costs of player i, then:*

1. *$\hat{e}_{ji} = 0 \iff \alpha_{ji} \leq \underline{\alpha}_{ji} \iff \alpha_{ij} \geq 1 - \underline{\alpha}_{ji}$.*
2. *There is a unique $\hat{e}_{ij}$ such that $0 < \hat{e}_{ji} < 1 \iff \underline{\alpha}_{ji} < \alpha_{ji} < \bar{\alpha}_{ji} \iff 1 - \bar{\alpha}_{ji} < \alpha_{ij} < 1 - \underline{\alpha}_{ji}$.*
3. *$\hat{e}_{ji} = 1 \iff \alpha_{ji} \geq \bar{\alpha}_{ji} \iff \alpha_{ij} \leq 1 - \bar{\alpha}_{ji}$.*

To simplify the Theorem notation, $(a,b)_-$ stands for $Min\{a,b\}$ and $(a,b)^+$ for $Max\{a,b\}$.

Theorem 3 *Consider the effort game $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$. Let e_{ij}^* and e_{ji}^* be the pairwise efforts in any unique equilibrium (e_{ij}^*, e_{ji}^*). Thus,*

1. *$e_{ij}^* = e_{ji}^* = 0$ if and only if $\alpha_{ij} \in (1 - \underline{\alpha}_{ji}, \underline{\alpha}_{ij})$.*
2. *$e_{ij}^* = 0$ and there is a unique $e_{ji}^* \in (0,1)$ if and only if $\alpha_{ij} \in \left(1 - \bar{\alpha}_{ji}, (\underline{\alpha}_{ij}, 1 - \underline{\alpha}_{ji})_-\right)$.*
3. *$e_{ij}^* = 0$ and $e_{ji}^* = 1$ if and only if $\alpha_{ij} \in \left(0, (\underline{\alpha}_{ij}, 1 - \bar{\alpha}_{ji})_-\right)$.*
4. *$e_{ij}^* = 1$ and $e_{ji}^* = 0$ if and only if $\alpha_{ij} \in \left((\bar{\alpha}_{ij}, 1 - \underline{\alpha}_{ji})^+, 1\right)$.*

[6] This occurs because $A_i(e)$ is an increasing function in e_{ij} and the minimum value is obtained for $\hat{e}_{ij} = 0$, which is the optimal effort for agent i.

5. $e^*_{ij} = 1$ *and there is a unique* $e^*_{ji} \in (0,1)$ *if and only if* $\alpha_{ij} \in \left((\bar{\alpha}_{ij}, 1-\bar{\alpha}_{ji})^+, 1-\underline{\alpha}_{ji}\right)$.

6. $e^*_{ij} = 1$ *and* $e^*_{ji} = 1$ *if and only if* $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1-\bar{\alpha}_{ji})$.

7. *There is a unique* $e^*_{ij} \in (0,1)$ *and* $e^*_{ji} = 0$ *if and only if* $\alpha_{ij} \in \left((\underline{\alpha}_{ij}, 1-\underline{\alpha}_{ji})^+, \bar{\alpha}_{ij}\right)$.

8. *There is a unique* $e^*_{ij} \in (0,1)$ *and* $e^*_{ji} = 1$ *if and only if* $\alpha_{ij} \in (\underline{\alpha}_{ij}, (\bar{\alpha}_{ij}, 1-\bar{\alpha}_{ji})_-)$.

9. *There is a unique* $e^*_{ij} \in (0,1)$ *and* $e^*_{ji} \in (0,1)$ *if and only if* $\alpha_{ij} \in \left((\underline{\alpha}_{ij}, 1-\bar{\alpha}_{ji})^+, (\bar{\alpha}_{ij}, 1-\underline{\alpha}_{ji})_-\right)$.

In points 7.(2.), 8.(5.), and 9.(9.), e^*_{ij} (e^*_{ji}) *is the unique real solution of equation* $c'_i(e_{ij}) - \alpha_{ij} r'_{ji}(e_{ij}) = 0$ $(c'_j(e_{ji}) - \alpha_{ji} r'_{ij}(e_{ji}) = 0)$.

Proof. We begin with point 1. Note that, by Lemma 1 $\hat{e}_{ij} = 0 \iff \alpha_{ij} \leq \underline{\alpha}_{ij}$, and by Lemma 2, $\hat{e}_{ji} = 0 \iff \alpha_{ji} \leq \underline{\alpha}_{ji} \iff \alpha_{ij} \geq 1-\underline{\alpha}_{ji}$. Therefore, $e^*_{ij} = e^*_{ji} = 0$ if and only if $\alpha_{ij} \in (1-\underline{\alpha}_{ji}, \underline{\alpha}_{ij})$.

For point 2. note that, by Lemma 1 $\hat{e}_{ij} = 0 \iff \alpha_{ij} \leq \underline{\alpha}_{ij}$, and by Lemma 2, there is a unique $\hat{e}_{ij}$ such that $0 < \hat{e}_{ji} < 1 \iff \underline{\alpha}_{ji} < \alpha_{ji} < \bar{\alpha}_{ji} \iff 1-\bar{\alpha}_{ji} < \alpha_{ij} < 1-\underline{\alpha}_{ji}$. Therefore, $e^*_{ij} = 0$ and there is a unique $e^*_{ji} \in (0,1)$ if and only if $\alpha_{ij} \in (1-\bar{\alpha}_{ji}, Min\{\underline{\alpha}_{ij}, 1-\underline{\alpha}_{ji}\})$.

Similarly, demonstrations of the remaining cases can be obtained straightforwardly from Lemmas 1 and 2. Since there are only three possible types of optimal effort for any agent i and j, as described in Lemmas 1 and 2, there are only nine possible cases depending on the values of α_{ij} regarding to $\underline{\alpha}_{ij}$, $\bar{\alpha}_{ij}$, $\underline{\alpha}_{ji}$, and $\bar{\alpha}_{ji}$. ■

The next corollary shows how the pairwise equilibrium efforts e^*_{ij} depend on α_{ij}, for all $i,j \in N, i \neq j$. As expected, as the proportion of aggregate cost reduction obtained by an agent increases, the effort that agent exerts also increases (or at least stays the same).

Corollary 1 *Let* $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ *be the effort game and* (e^*_{ij}, e^*_{ji}) *the pairwise efforts equilibrium. Thus,*

- $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} > 0$, *if* $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$*;* $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} = 0$*, otherwise.*
- $\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} < 0$, *if* $\alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$*;* $\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} = 0$*, otherwise.*

Proof. By the implicit function theorem, $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} = -\frac{\frac{\partial(c'_i(e^*_{ij}) - \alpha_{ij} r'_{ji}(e^*_{ij}))}{\partial \alpha_{ij}}}{\frac{\partial(c'_i(e^*_{ij}) - \alpha_{ij} r'_{ji}(e^*_{ij}))}{\partial e^*_{ij}}} = \frac{r'_{ji}(e^*_{ij})}{c''_i(e^*_{ij}) - \alpha_{ij} r''_{ji}(e^*_{ij})} > 0$, because $r'_{ji}(e^*_{ij}) > 0$, $c''_i(e^*_{ij}) > 0$, and $r''_{ji}(e^*_{ij}) < 0$. Thus, for any $\alpha_{ij} \leq \underline{\alpha}_{ij}$, Lemma 1 implies that $e^*_{ij} = 0$, thus, $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} = 0$. However, if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$, then $e^*_{ij} \in (0,1)$ and $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} > 0$. Finally, if $\alpha_{ij} \geq \bar{\alpha}_{ij}$, then $e^*_{ij} = 1$ and $\frac{\partial e^*_{ij}}{\partial \alpha_{ij}} = 0$. Analogously, if $\alpha_{ji} \leq \underline{\alpha}_{ji} \iff \alpha_{ij} \geq 1-\underline{\alpha}_{ji}$, then $e^*_{ji} = 0$ and $\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} = 0$, if $\alpha_{ji} \in (\underline{\alpha}_{ji}, \bar{\alpha}_{ji}) \iff \alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$, then $e^*_{ji} \in (0,1)$ and $\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} < 0$. Finally, if $\alpha_{ji} \geq \bar{\alpha}_{ji} \iff \alpha_{ij} \leq 1-\bar{\alpha}_{ji}$, then $e^*_{ij} = 1$ and $\frac{\partial e^*_{ji}}{\partial \alpha_{ij}} = 0$. ■

The results above are really useful when the goal is to incentivize agents $i,j \in N$ to make more pairwise effort e_{ij} by means of the parameter α_{ij}. However, we wish to go further, specifically to achieve efficiency within the family of cost allocations with weighted pairwise reduction. In other words we wish to find the α_{ij}, for all $i, j \in N$ that minimizes the aggregate cost function $\sum_{i\in N} A_i(e^*)$ in equilibrium, where both $A_i(e)$ and the effort equilibrium e^* depends on α_{ij}.

4.1 Efficient Effort Equilibrium

The search for alpha parameters which will lead to the EEE can be simplified by taking into account the bilateral and independent interactions of agents. Note first that any pair of agents have a particular α_{ij}, and second that the optimal effort that any agent $i \in N$ makes in regard to any agent $j \in N\backslash\{i\}$ is independent of the optimal effort that agent i exerts in regard to any other agent $h \in N\backslash\{i, j\}$. Thus, minimizing $\sum_{i\in N} A_i(e^*)$ in terms of α_{ij} is equivalent to minimizing the function $L^*(\alpha_{ij}) = A_i(e^*)+A_j(e^*)$, since each particular α_{ij} only appears in $A_i(e^*)$ and $A_j(e^*)$. Fortunately, the problem can be further simplified: Note that, $A_i(e^*)$ and $A_j(e^*)$ are the sums of different terms, but α_{ij} only appears in those terms related to the interaction between i and j (see (9)). These terms are $c_i(e_i^*) - \alpha_{ij}(r_{ij}(e_{ji}^*) + r_{ji}(e_{ij}^*))$ from $A_i(e^*)$, and $c_j(e_j^*) - (1 - \alpha_{ij})(r_{ji}(e_{ij}^*) + r_{ij}(e_{ji}^*))$ from $A_j(e^*)$. Thus, we a new function $A_i^*(\alpha_{ij}) := c_i(e_i^*)-\alpha_{ij}(r_{ij}(e_{ji}^*)+r_{ji}(e_{ij}^*))$ can be considered, and analogously $A_j^*(1-\alpha_{ij})$. Note that $\frac{\partial^x(A_i(e^*))}{\partial\alpha_{ij}^x} = \frac{\partial^x(A_i^*(\alpha_{ij}))}{\partial\alpha_{ij}^x}$ and $\frac{\partial^x(A_j(e^*))}{\partial\alpha_{ij}^x} = \frac{\partial^x(A_j^*(1-\alpha_{ij}))}{\partial\alpha_{ij}^x}$ for $x = 1, 2, ...$ Therefore, for each pair i and j, it is possible to define the function $L_{ij}^*(\alpha_{ij}) := A_i^*(\alpha_{ij}) + A_j^*(1 - \alpha_{ij})$, where $\frac{\partial^x(L^*(\alpha_{ij}))}{\partial\alpha_{ij}^x} = \frac{\partial^x(L_{ij}^*(\alpha_{ij}))}{\partial\alpha_{ij}^x}$ for $x = 1, 2, ...$ So minimizing $L^*(\alpha_{ij})$ is equivalent to minimizing $L_{ij}^*(\alpha_{ij})$.

We now focus on finding the α_{ij} that minimizes function $L_{ij}^*(\alpha_{ij})$, and provide a procedure for finding the unique EEE. First, to solve the above optimization problem it is necessary to know the function $L_{ij}^*(\alpha_{ij})$ very well. The following Proposition shows that, according to the value of the effort equilibrium, the cost function $L_{ij}^*(\alpha_{ij})$ is a continuous piecewise function with four types of piece. This result characterizes all of those pieces, showing the shape of $L_{ij}^*(\alpha_{ij})$ and the optimal α_{ij} in each type of piece. Second, the main theorem in this section characterizes the optimal α_{ij}^*, for all $i, j \in N$ with $i \neq j$, which incentivizes an efficient effort equilibrium, which is also provided.

To demonstrate the following results, three technical lemmas that can be found in the Appendix are needed. Lemmas 3, 4, and 5 characterize the derivative $\frac{\partial(A_i^*(\alpha_{ij}))}{\partial\alpha_{ij}}$, $\frac{\partial(L_{ij}^*(\alpha_{ij}))}{\partial\alpha_{ij}}$, and $\frac{\partial^2(L_{ij}^*(\alpha_{ij}))}{\partial\alpha_{ij}^2}$ respectively.

Proposition 3 *Consider the effort game $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ and e^* as the effort equilibrium. Let $\alpha_{ij} \in [a, b]$ be a piece of $L_{ij}^*(\alpha_{ij})$ with $0 \leq a < b \leq 1$, $L_{ij}^*(\alpha_{ij})$ can have only four types of piece:*

Constant: *(e_{ij}^*, e_{ji}^*) is equal to either $(0, 0)$, $(1, 0)$, $(0, 1)$ or $(1, 1)$. Thus $\frac{\partial(L_{ij}^*(\alpha_{ij}))}{\partial\alpha_{ij}} = 0$ and $L_{ij}^*(\alpha_{ij})$ is always constant. Therefore, any $\alpha_{ij} \in [a, b]$ minimizes $L_{ij}^*(\alpha_{ij})$.*

Increasing: *e_{ij}^* is equal to either 0 or 1, and $0 < e_{ji}^* < 1$. Thus $\frac{\partial(L_{ij}^*(\alpha_{ij}))}{\partial\alpha_{ij}} = -\alpha_{ij}\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\frac{\partial e_{ji}^*}{\partial\alpha_{ij}} > 0$ and $L_{ij}^*(\alpha_{ij})$ is always increasing. Therefore, $\alpha_{ij} = a$ minimizes $L_{ij}^*(\alpha_{ij})$.*

Decreasing: *$0 < e_{ij}^* < 1$, and e_{ji}^* is equal to either 0 or 1. Thus $\frac{\partial(L_{ij}^*(\alpha_{ij}))}{\partial\alpha_{ij}} = -(1-\alpha_{ij})\frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\frac{\partial e_{ij}^*}{\partial\alpha_{ij}} <$*

0 *and* $L^*_{ij}(\alpha_{ij})$ *is always decreasing. Therefore,* $\alpha_{ij} = b$ *minimizes* $L^*_{ij}(\alpha_{ij})$.

Depending on cost function shape: $0 < e^*_{ij} < 1$ *and* $0 < e^*_{ji} < 1$. *Thus,*

$$\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = -\alpha_{ij}\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} - (1-\alpha_{ij})\frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}.$$

In this case, there is always a unique $\breve{\alpha}^{[a,b]}_{ij} \in [a,b]$ *that minimizes* $L^*_{ij}(\alpha_{ij})$*, which is:*

$$\breve{\alpha}^{[a,b]}_{ij} = \begin{cases} a & \text{if } \frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} > 0 \text{ for all } \alpha_{ij} \in [a,b] \\ b & \text{if } \frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} < 0 \text{ for all } \alpha_{ij} \in [a,b] \\ \text{Solution of } \frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = 0 & \text{otherwise} \end{cases}$$

Proof. First note that Theorem 3 determines the nine possible types of effort equilibrium, all of which are considered in the four types of piece of Proposition 3. The four types of equilibrium associated with the constant piece in Proposition 3 are characterized in points 1, 3, 4, and 6 of Theorem 3. For the increasing case of Proposition 3, the equilibria correspond to points 2 and 5 of Theorem 3, and for the decreasing case points 7 and 8. Finally, in the case depending on cost function shape the point 9 of Theorem 3 characterizes the last equilibrium.

Proposition 3 is straightforward when comparing Theorem 3 and Lemma 4. Additionally, the case depending on cost function shape needs Lemma 5, so in this piece, $\frac{\partial^2(L^*_{ij}(\alpha_{ij}))}{\partial\alpha^2_{ij}} > 0$. In this last case, it is also straightforward to show that $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}}$ is continuous, so there is always a unique $\breve{\alpha}_{ij}$ that minimizes $L^*_{ij}(\alpha_{ij})$ in such pieces. ■

It is now possible to characterize the optimal α_{ij} that minimizes the pairwise cost $L^*_{ij}(\alpha_{ij})$ of any pair of agents $i, j \in N$. For any effort game considered here, there are only six possible distributions of the lower and upper thresholds of the alpha parameter.[7] These cases are[8]:

$$\begin{array}{ll} \textbf{Case A} & \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1-\bar{\alpha}_{ji} < 1-\underline{\alpha}_{ji} \\ \textbf{Case B} & \underline{\alpha}_{ij} < 1-\bar{\alpha}_{ji} < \bar{\alpha}_{ij} < 1-\underline{\alpha}_{ji} \\ \textbf{Case C} & \underline{\alpha}_{ij} < 1-\bar{\alpha}_{ji} < 1-\underline{\alpha}_{ji} < \bar{\alpha}_{ij} \\ \textbf{Case D} & 1-\bar{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1-\underline{\alpha}_{ji} \\ \textbf{Case E} & 1-\bar{\alpha}_{ji} < \underline{\alpha}_{ij} < 1-\underline{\alpha}_{ji} < \bar{\alpha}_{ij} \\ \textbf{Case F} & 1-\bar{\alpha}_{ji} < 1-\underline{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} \end{array} \tag{10}$$

This means that the continuous piecewise cost function $L^*_{ij}(\alpha_{ij})$ has only five pieces, and each piece must belong to one of the four types described in Proposition 3.

The last Theorem characterizes the optimal α^*_{ij} in all six cases [see (10)]. Thus, given an effort game, Theorem 4 provides the α^*_{ij} that incentivizes an efficient effort equilibrium, and that equilibrium is also provided.

[7] Note that $\underline{\alpha}_{ji} < \bar{\alpha}_{ji}$ and $\underline{\alpha}_{ij} < \bar{\alpha}_{ij}$.

[8] For expositional purposes, we omit the particular possible knife-edge cases. For example in Case A the only tie that might occur is when $\bar{\alpha}_{ij} = 1-\bar{\alpha}_{ji}$. In this case there are only four pieces. The piece missing, i.e. $(\hat{\alpha}_{ij}, 1-\hat{\alpha}_{ji})$, is the piece where the optimal α_{ij} was according to Theorem 4. In that case, the optimal will be the endpoints of this interval, $\{\hat{\alpha}_{ij}, 1-\hat{\alpha}_{ji}\}$, which must be equal. If the missing interval does not contain the optimal, the latter will be as described in Theorem 4.

Theorem 4 *Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ be an effort game and $\Lambda(\alpha) = min\{\alpha, 1\}$, with α^*_{ij} to be the solution of $\underset{\alpha_{ij}\in[0,1]}{min} L^*_{ij}(\alpha_{ij})$. Thus,*

Case A *α^*_{ij} is equal to any $\alpha_{ij} \in [\bar{\alpha}_{ij}, 1 - \bar{\alpha}_{ji}]$, with $e^*_{ij} = 1 = e^*_{ji}$.*

Case B *$\alpha^*_{ij} = \check{\alpha}_{ij}^{[1-\bar{\alpha}_{ji}, \bar{\alpha}_{ij}]}$, where e^*_{ij} and e^*_{ji} are, respectively, the unique solution of $c'_i(e_{ij}) - \alpha^*_{ij} r'_{ji}(e_{ij}) = 0$ and $c'_j(e_{ji}) - \alpha^*_{ji} r'_{ij}(e_{ji}) = 0$.*

Case C *$\alpha^C_{ij} = \arg min\{L^*_{ij}(\check{\alpha}_{ij}^{[1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}]}), L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}$, where*

- **C.1** *if $\alpha^C_{ij} = \check{\alpha}_{ij}^{[1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}]}$, then $\alpha^*_{ij} = \alpha^C_{ij}$ and e^*_{ij} and e^*_{ji} are, respectively, the unique solution of $c'_i(e_{ij}) - \alpha^*_{ij} r'_{ji}(e_{ij}) = 0$ and $c'_j(e_{ji}) - \alpha^*_{ji} r'_{ij}(e_{ji}) = 0$;*
- **C.2** *if $\alpha^C_{ij} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} > 1$, then $\alpha^*_{ij} = \alpha^C_{ij}$ and e^*_{ij} is the solution of $c'_i(e_{ij}) - \alpha^*_{ij} r'_{ji}(e_{ij}) = 0$ and $e^*_{ji} = 0$;*
- **C.3** *if $\alpha^C_{ij} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} < 1$, then α^*_{ij} is equal to any $\alpha_{ij} \in [\bar{\alpha}_{ij}, 1]$ and $e^*_{ij} = 1$, $e^*_{ji} = 0$.*

Case D *$\alpha^D_{ij} = \arg min\{L^*_{ij}(\Lambda(1 - \bar{\alpha}_{ji})), L^*_{ij}(\check{\alpha}_{ij}^{[\underline{\alpha}_{ij}, \bar{\alpha}_{ij}]})\}$, where*

- **D.1** *if $\alpha^D_{ij} = \Lambda(1 - \bar{\alpha}_{ji})$ and $1 - \bar{\alpha}_{ji} > 0$, then α^*_{ij} is equal to any $\alpha_{ij} \in [0, 1 - \bar{\alpha}_{ji}]$ and $e^*_{ij} = 0$, $e^*_{ji} = 1$;*
- **D.2** *if $\alpha^D_{ij} = \Lambda(1 - \bar{\alpha}_{ji})$ and $1 - \bar{\alpha}_{ji} < 0$, then $\alpha^*_{ij} = \alpha^D_{ij}$, $e^*_{ij} = 0$, and e^*_{ji} is the unique solution of $c'_j(e_{ji}) - \alpha_{ji} r'_{ij}(e_{ji}) = 0$;*
- **D.3** *if $\alpha^D_{ij} = \check{\alpha}_{ij}^{[\underline{\alpha}_{ij}, \bar{\alpha}_{ij}]}$, then $\alpha^*_{ij} = \alpha^D_{ij}$, and e^*_{ij}, and e^*_{ji} are, respectively, the unique solution of $c'_i(e_{ij}) - \alpha_{ij} r'_{ji}(e_{ij}) = 0$ and $c'_j(e_{ji}) - \alpha_{ji} r'_{ij}(e_{ji}) = 0$.*

Case E *$\alpha^E_{ij} = \arg min\{L^*_{ij}(\Lambda(1 - \bar{\alpha}_{ji})), \check{\alpha}_{ij}^{[\underline{\alpha}_{ij}, 1-\underline{\alpha}_{ji}]}, L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}$, where*

- **E.1** *if $\alpha^E_{ij} = \Lambda(1 - \bar{\alpha}_{ji})$ and $1 - \bar{\alpha}_{ji} > 0$, then α^*_{ij} is equal to any $\alpha_{ij} \in [0, 1 - \bar{\alpha}_{ji}]$, $e^*_{ij} = 0$, and $e^*_{ji} = 1$;*
- **E.2** *if $\alpha^E_{ij} = \Lambda(1 - \bar{\alpha}_{ji})$ and $1 - \bar{\alpha}_{ji} < 0$, then $\alpha^*_{ij} = \alpha^E_{ij}$, $e^*_{ij} = 0$, and e^*_{ji} is the solution of $c'_j(e_{ji}) - \alpha_{ji} r'_{ij}(e_{ji}) = 0$;*
- **E.3** *if $\alpha^E_{ij} = \check{\alpha}_{ij}^{[\underline{\alpha}_{ij}, \bar{\alpha}_{ij}]}$, then $\alpha^*_{ij} = \alpha^E_{ij}$, and e^*_{ij} and e^*_{ji} are, respectively, the solution of $c'_i(e_{ij}) - \alpha^*_{ij} r'_{ji}(e_{ij}) = 0$ and $c'_j(e_{ji}) - \alpha^*_{ji} r'_{ij}(e_{ji}) = 0$;*
- **E.4** *if $\alpha^E_{ij} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} > 1$, then $\alpha^*_{ij} = \alpha^E_{ij}$, e^*_{ij} is the solution of $c'_i(e_{ij}) - \alpha^*_{ij} r'_{ji}(e_{ij}) = 0$ and $e^*_{ji} = 0$;*
- **E.5** *if $\alpha^E_{ij} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} < 1$, then α^*_{ij} is equal to any $\alpha_{ij} \in [\bar{\alpha}_{ij}, 1]$, $e^*_{ij} = 1$, and $e^*_{ji} = 0$.*

Case F *$\alpha^F_{ij} = \arg min\{L^*_{ij}(\Lambda(1 - \bar{\alpha}_{ji})), L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}$, where*

- **F.1** *if $\alpha^F_{ij} = \Lambda(1 - \bar{\alpha}_{ji})$ and $1 - \bar{\alpha}_{ji} > 0$, then α^*_{ij} is equal to any $\alpha_{ij} \in [0, 1 - \bar{\alpha}_{ji}]$, $e^*_{ij} = 0$, and $e^*_{ji} = 1$;*

F.2 *if* $\alpha_{ij}^F = \Lambda(1-\bar{\alpha}_{ji})$, $1-\bar{\alpha}_{ji} < 0$ *and* $1-\underline{\alpha}_{ji} > 0$, *then* $\alpha_{ij}^* = \alpha_{ij}^F$, $e_{ij}^* = 0$, *and* e_{ji}^* *is the unique solution of* $c_j'(e_{ji}) - \alpha_{ji}^* r_{ij}'(e_{ji}) = 0$;

F.3 *if* $\alpha_{ij}^F = \Lambda(\bar{\alpha}_{ij})$ *and* $\bar{\alpha}_{ij} < 1$, *then* α_{ij}^* *is equal to any* $\alpha_{ij} \in [\bar{\alpha}_{ij}, 1]$, $e_{ij}^* = 1$, *and* $e_{ji}^* = 0$;

F.4 *if* $\alpha_{ij}^F = \Lambda(\bar{\alpha}_{ij})$, $\bar{\alpha}_{ij} > 1$ *and* $\underline{\alpha}_{ij} < 1$, *then* $\alpha_{ij}^* = \alpha_{ij}^F$, e_{ij}^* *is the unique solution of* $c_i'(e_{ij}) - \alpha_{ij}^* r_{ji}'(e_{ij}) = 0$ *and* $e_{ji}^* = 0$;

F.5 *if* $\alpha_{ij}^F = \Lambda(\bar{\alpha}_{ij})$, $1-\underline{\alpha}_{ji} < 0$ *and* $\underline{\alpha}_{ij} > 1$, *then* α_{ij}^* *is equal to any* $\alpha_{ij} \in [0,1]$, *and* $e_{ij}^* = 0 = e_{ji}^*$.

Proof. As $L_{ij}^*(\alpha_{ij})$ is a continuous piecewise function, we analyze the five pieces that define it in each case. First, by using Theorem 3, we show the value of the efforts in the unique equilibrium in each piece. This enables to the type of the piece to be determined according to Proposition 3, thus giving the value of α_{ij} that minimizes $L_{ij}^*(\alpha_{ij})$ in each piece. Comparing the pieces gives the α_{ij}^* that minimizes the aggregate cost for each of the six cases. This value need not be unique. Note, in addition, that $\underline{\alpha}_{ij}$, $\bar{\alpha}_{ij}$, $\bar{\alpha}_{ji}$ and $\underline{\alpha}_{ji}$ are always greater than zero, but any of them may be greater than one, which implies that some pieces of certain cases may not exist. We prove the theorem case by case:

1. **Case A** $(\underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1-\bar{\alpha}_{ji} < 1-\underline{\alpha}_{ji})$

   Note that those thresholds are always greater than zero, so $0 < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1-\bar{\alpha}_{ji} < 1-\underline{\alpha}_{ji} < 1$.

   If $\alpha_{ij} \in \left(0, \underline{\alpha}_{ij}\right)$, then $e_{ij}^* = 0$, $e_{ji}^* = 1$, and $L_{ij}^*(\alpha_{ij})$ is constant in this interval.

   If $\alpha_{ij} \in \left(\underline{\alpha}_{ij}, \bar{\alpha}_{ij}\right)$, then $0 < e_{ij}^* < 1$, $e_{ji}^* = 1$, and $L_{ij}^*(\alpha_{ij})$ is decreasing, which implies that $\alpha_{ij} = 1-\bar{\alpha}_{ji}$ minimizes $L_{ij}^*(\alpha_{ij})$.

   If $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1-\bar{\alpha}_{ji})$, then $e_{ij}^* = 1$, and $e_{ji}^* = 1$, and $L_{ij}^*(\alpha_{ij})$ is constant in this interval.

   If $\alpha_{ij} \in \left(1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}\right)$, then $e_{ij}^* = 1$, $0 < e_{ji}^* < 1$, and $L_{ij}^*(\alpha_{ij})$ is increasing, which implies that $1-\bar{\alpha}_{ji}$ minimizes $L_{ij}^*(\alpha_{ij})$.

   If $\alpha_{ij} \in \left(1-\underline{\alpha}_{ji}, 1\right)$, then $e_{ij}^* = 1$, $e_{ji}^* = 0$, and $L_{ij}^*(\alpha_{ij})$ is constant in this interval.

   Therefore, α_{ij}^* is equal to any $\alpha_{ij} \in [\bar{\alpha}_{ij}, 1-\bar{\alpha}_{ji}]$ so by Theorem 3, $e_{ij}^* = 1 = e_{ji}^* = 1$.

2. **Case B** $(\underline{\alpha}_{ij} < 1-\bar{\alpha}_{ji} < \bar{\alpha}_{ij} < 1-\underline{\alpha}_{ji})$

   Analogously, $0 < \underline{\alpha}_{ij} < 1-\bar{\alpha}_{ji} < \bar{\alpha}_{ij} < 1-\underline{\alpha}_{ji} < 1$.

   If $\alpha_{ij} \in \left(0, \underline{\alpha}_{ij}\right)$, then $e_{ij}^* = 0$, $e_{ji}^* = 1$, and $L_{ij}^*(\alpha_{ij})$ is constant in this interval.

   If $\alpha_{ij} \in \left(\underline{\alpha}_{ij}, 1-\bar{\alpha}_{ji}\right)$, then $0 < e_{ij}^* < 1$, $e_{ji}^* = 1$, and $L_{ij}^*(\alpha_{ij})$ is decreasing, which implies that $\alpha_{ij} = 1-\bar{\alpha}_{ji}$ minimizes $L_{ij}^*(\alpha_{ij})$.

   If $\alpha_{ij} \in (1-\bar{\alpha}_{ji}, \bar{\alpha}_{ij})$, then $0 < e_{ij}^* < 1$, and $0 < e_{ji}^* < 1$, which implies that $\check{\alpha}_{ij}$ minimizes $L_{ij}^*(\alpha_{ij})$.

   If $\alpha_{ij} \in \left(\bar{\alpha}_{ij}, 1-\underline{\alpha}_{ji}\right)$, then $e_{ij}^* = 1$, $0 < e_{ji}^* < 1$, and $L_{ij}^*(\alpha_{ij})$ is increasing, which implies that $\bar{\alpha}_{ij}$ minimizes $L_{ij}^*(\alpha_{ij})$.

If $\alpha_{ij} \in \left(1-\underline{\alpha}_{ji}, 1\right)$, then $e^*_{ij} = 1$, $e^*_{ji} = 0$, and $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

Therefore, $\alpha^*_{ij} = \breve{\alpha}^{[1-\bar{\alpha}_{ji}, \bar{\alpha}_{ij}]}_{ij}$ and, by Theorem 3, e^*_{ij} and e^*_{ji} are, respectively, the solution of $c'_i(e_{ij}) - \alpha_{ij} r'_{ji}(e_{ij}) = 0$ and $c'_j(e_{ji}) - \alpha_{ji} r'_{ij}(e_{ji}) = 0$.

3. **Case C** $(\underline{\alpha}_{ij} < 1-\bar{\alpha}_{ji} < 1-\underline{\alpha}_{ji} < \bar{\alpha}_{ij})$

   It may happen here that either $\bar{\alpha}_{ij} < 1$ or $\bar{\alpha}_{ij} \geq 1$. Thus there are two subcases:

   $0 < \underline{\alpha}_{ij} < 1-\bar{\alpha}_{ji} < 1-\underline{\alpha}_{ji} < \bar{\alpha}_{ij} < 1$

   $0 < \underline{\alpha}_{ij} < 1-\bar{\alpha}_{ji} < 1-\underline{\alpha}_{ji} < 1 < \bar{\alpha}_{ij}$

   Starting with the first subcase,

   if $\alpha_{ij} \in \left(0, \underline{\alpha}_{ij}\right)$, then $e^*_{ij} = 0$, $e^*_{ji} = 1$, and $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

   If $\alpha_{ij} \in \left(\underline{\alpha}_{ij}, 1-\bar{\alpha}_{ji}\right)$, then $0 < e^*_{ij} < 1$, $e^*_{ji} = 1$, and $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\alpha_{ij} = 1-\bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.

   If $\alpha_{ij} \in \left(1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}\right)$, then $0 < e^*_{ij} < 1$, and $0 < e^*_{ji} < 1$, which implies that $\breve{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.

   If $\alpha_{ij} \in \left(1-\underline{\alpha}_{ji}, \bar{\alpha}_{ij}\right)$, then $0 < e^*_{ij} < 1$, $e^*_{ji} = 0$, and $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\bar{\alpha}$ minimizes $L^*_{ij}(\alpha_{ij})$.

   If $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1)$, then $e^*_{ij} = 1$, $e^*_{ji} = 0$, and $L^*_{ij}(\alpha_{ij})$ is constant, in this interval.

   However, in the second subcase $\bar{\alpha}_{ij} > 1$, which implies that the last interval described above does not exist. The rest of the analysis is similar to the first subcase.

   Therefore, $\alpha^C_{ij} = \arg min\{L^*_{ij}(\breve{\alpha}^{\left[1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}\right]}_{ij}), L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}$. Note that, if $\alpha^C_{ij} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} < 1$, then α^*_{ij} is equal to any $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1)$. Otherwise, $\alpha^*_{ij} = \alpha^C_{ij}$.

   Finally, by Theorem 3,

   if $\alpha^C_{ij} = \breve{\alpha}^{\left[1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}\right]}_{ij}$, then e^*_{ij} and e^*_{ji} are, respectively, the solution of $c'_i(e_{ij}) - \alpha_{ij} r'_{ji}(e_{ij}) = 0$ and $c'_j(e_{ji}) - \alpha_{ji} r'_{ij}(e_{ji}) = 0$,

   if $\alpha^C_{ij} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} > 1$, then e^*_{ij} is the solution of $c'_i(e_{ij}) - \alpha_{ij} r'_{ji}(e_{ij}) = 0$ and $e^*_{ji} = 0$,

   if $\alpha^C_{ij} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} < 1$, then $e^*_{ij} = 1$, $e^*_{ji} = 0$.

4. **Case D** $(1-\bar{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1-\underline{\alpha}_{ji})$

   It may happen here that either $1-\bar{\alpha}_{ji} > 0$ or $1-\bar{\alpha}_{ji} \leq 0$. Thus there are two subcases:

   $0 < 1-\bar{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1-\underline{\alpha}_{ji} < 1$

   $1-\bar{\alpha}_{ji} < 0 < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1-\underline{\alpha}_{ji} < 1$

   Starting with the first subcase,

   if $\alpha_{ij} \in (0, 1-\bar{\alpha}_{ji})$, then $e^*_{ij} = 0$, $e^*_{ji} = 1$, and $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

   If $\alpha_{ij} \in \left(1-\bar{\alpha}_{ji}, \underline{\alpha}_{ij}\right)$, then $e^*_{ij} = 0$, $0 < e^*_{ji} < 1$, and $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $\alpha_{ij} = 1-\bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.

   If $\alpha_{ij} \in \left(\underline{\alpha}_{ij}, \bar{\alpha}_{ij}\right)$, then $0 < e^*_{ji} < 1$, and $0 < e^*_{ji} < 1$, which implies that $\breve{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in \left(\bar{\alpha}_{ij}, 1 - \underline{\alpha}_{ji}\right)$, then $e^*_{ij} = 1$, $0 < e^*_{ji} < 1$, and $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $\bar{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1)$, then $e^*_{ij} = 1$, $e^*_{ji} = 0$, and $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

However, if $1 - \bar{\alpha}_{ji} < 0$ the first interval above does not exist. Again, the rest of the analysis is similar to the first subcase.

Therefore, $\alpha^D_{ij} = \arg min\{L^*_{ij}(\Lambda(1 - \bar{\alpha}_{ji})), L^*_{ij}(\check{\alpha}_{ij}^{[\underline{\alpha}_{ij}, \bar{\alpha}_{ij}]})\}$.

Note that if $\alpha^D_{ij} = \Lambda(1 - \bar{\alpha}_{ji})$ and $1 - \bar{\alpha}_{ji} > 0$, then α^*_{ij} is equal to any $\alpha_{ij} \in [0, 1 - \bar{\alpha}_{ji}]$. Otherwise, $\alpha^*_{ij} = \alpha^D_{ij}$.

Finally, by Theorem 3,

if $\alpha^D_{ij} = \Lambda(1 - \bar{\alpha}_{ji})$ and $1 - \bar{\alpha}_{ji} > 0$, then $e^*_{ij} = 0$, $e^*_{ji} = 1$,

if $\alpha^D_{ij} = \Lambda(1 - \bar{\alpha}_{ji})$ and $1 - \bar{\alpha}_{ji} < 0$, then $e^*_{ij} = 0$, and e^*_{ji} is the solution of $c'_j(e_{ji}) - \alpha_{ji} r'_{ij}(e_{ji}) = 0$

if $\alpha^D_{ij} = \check{\alpha}_{ij}^{[\underline{\alpha}_{ij}, \bar{\alpha}_{ij}]}$, then e^*_{ij} and e^*_{ji} are, respectively, the solution of $c'_i(e_{ij}) - \alpha_{ij} r'_{ji}(e_{ij}) = 0$ and $c'_j(e_{ji}) - \alpha_{ji} r'_{ij}(e_{ji}) = 0$.

5. **Case E** $(1 - \bar{\alpha}_{ji} < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij})$

In this case, it may happen that either $1 - \bar{\alpha}_{ji} > 0$ or $1 - \bar{\alpha}_{ji} \leq 0$, and either $\bar{\alpha}_{ij} < 1$ or $\bar{\alpha}_{ij} \geq 1$. Thus there are four subcases:

$0 < 1 - \bar{\alpha}_{ji} < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij} < 1$

$1 - \bar{\alpha}_{ji} < 0 < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < \bar{\alpha}_{ij} < 1$

$0 < 1 - \bar{\alpha}_{ji} < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < 1 < \bar{\alpha}_{ij}$

$1 - \bar{\alpha}_{ji} < 0 < \underline{\alpha}_{ij} < 1 - \underline{\alpha}_{ji} < 1 < \bar{\alpha}_{ij}$

Focusing on the first subcase,

if $\alpha_{ij} \in (0, 1 - \bar{\alpha}_{ji})$, then $e^*_{ij} = 0$, $e^*_{ji} = 1$, and $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

If $\alpha_{ij} \in \left(1 - \bar{\alpha}_{ji}, \underline{\alpha}_{ij}\right)$, then $e^*_{ij} = 0$, $0 < e^*_{ji} < 1$, and $L^*_{ij}(\alpha_{ij})$ is increasing, which implies that $\alpha_{ij} = 1 - \bar{\alpha}_{ji}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in \left(\underline{\alpha}_{ij}, 1 - \underline{\alpha}_{ji}\right)$, then $0 < e^*_{ji} < 1$, and $0 < e^*_{ji} < 1$, which implies that $\check{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in \left(1 - \underline{\alpha}_{ji}, \bar{\alpha}_{ij}\right)$, then $0 < e^*_{ij} < 1$, $e^*_{ji} = 0$, and $L^*_{ij}(\alpha_{ij})$ is decreasing, which implies that $\bar{\alpha}_{ij}$ minimizes $L^*_{ij}(\alpha_{ij})$.

If $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1)$, then $e^*_{ij} = 1$, $e^*_{ji} = 0$, and $L^*_{ij}(\alpha_{ij})$ is constant in this interval.

In the other three subcases, the first and/or last interval may not exist. Once again, the rest of the analysis for those subcases is similar to the first one.

Therefore, $\alpha^E_{ij} = \arg min\{L^*_{ij}(\Lambda(1 - \bar{\alpha}_{ji})), \check{\alpha}_{ij}^{[\underline{\alpha}_{ij}, 1 - \underline{\alpha}_{ji}]}, L^*_{ij}(\Lambda(\bar{\alpha}_{ij}))\}$

Note that if $\alpha^E_{ij} = \Lambda(1 - \bar{\alpha}_{ji})$ and $1 - \bar{\alpha}_{ji} > 0$ then α^*_{ij} is equal to any $\alpha_{ij} \in [0, 1 - \bar{\alpha}_{ji}]$, and if $\alpha^E_{ij} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} < 1$, then α^*_{ij} is equal to any $\alpha_{ij} \in [\bar{\alpha}_{ij}, 1]$. Otherwise $\alpha^*_{ij} = \alpha^E_{ij}$.

Thus, by Theorem 3,

if $\alpha_{ij}^{E} = \Lambda(1-\bar{\alpha}_{ji})$ and $1-\bar{\alpha}_{ji} > 0$, then $e_{ij}^{*} = 0$, $e_{ji}^{*} = 1$,

if $\alpha_{ij}^{E} = \Lambda(1-\bar{\alpha}_{ji})$ and $1-\bar{\alpha}_{ji} < 0$, then $e_{ij}^{*} = 0$, and e_{ji}^{*} is the solution of $c_j'(e_{ji}) - \alpha_{ji} r_{ij}'(e_{ji}) = 0$,

if $\alpha_{ij}^{E} = \check{\alpha}_{ij}^{[\underline{\alpha}_{ij},\bar{\alpha}_{ij}]}$, then e_{ij}^{*} and e_{ji}^{*} are, respectively, the solution of $c_i'(e_{ij}) - \alpha_{ij} r_{ji}'(e_{ij}) = 0$ and $c_j'(e_{ji}) - \alpha_{ji} r_{ij}'(e_{ji}) = 0$,

if $\alpha_{ij}^{E} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} > 1$, then e_{ij}^{*} is the solution of $c_i'(e_{ij}) - \alpha_{ij} r_{ji}'(e_{ij}) = 0$ and $e_{ji}^{*} = 0$,

if $\alpha_{ij}^{E} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} < 1$, then $e_{ij}^{*} = 1$, $e_{ji}^{*} = 0$.

6. **Case F** $(1-\bar{\alpha}_{ji} < 1-\underline{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij})$

This is the most general case and anything could happen with thresholds greater than one. Thus there are nine subcases.

First consider the case $0 < 1-\bar{\alpha}_{ji} < 1-\underline{\alpha}_{ji} < \underline{\alpha}_{ij} < \bar{\alpha}_{ij} < 1$

If $\alpha_{ij} \in (0, 1-\bar{\alpha}_{ji})$, then $e_{ij}^{*} = 0$, $e_{ji}^{*} = 1$, and $L_{ij}^{*}(\alpha_{ij})$ is constant in this interval.

If $\alpha_{ij} \in \left(1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}\right)$, then $e_{ij}^{*} = 0$, $0 < e_{ji}^{*} < 1$, and $L_{ij}^{*}(\alpha_{ij})$ is increasing, which implies that $\alpha_{ij} = 1-\bar{\alpha}_{ji}$ minimizes $L_{ij}^{*}(\alpha_{ij})$.

If $\alpha_{ij} \in \left(1-\underline{\alpha}_{ji}, \underline{\alpha}_{ij}\right)$, then $e_{ij}^{*} = 0$, $e_{ji}^{*} = 0$, and $L_{ij}^{*}(\alpha_{ij})$ is constant in this interval.

If $\alpha_{ij} \in \left(\underline{\alpha}_{ij}, \bar{\alpha}_{ij}\right)$, then $0 < e_{ij}^{*} < 1$, $e_{ji}^{*} = 0$, and $L_{ij}^{*}(\alpha_{ij})$ is decreasing, which implies that $\alpha_{ij} = \bar{\alpha}_{ij}$ minimizes $L_{ij}^{*}(\alpha_{ij})$.

If $\alpha_{ij} \in (\bar{\alpha}_{ij}, 1)$, then $e_{ij}^{*} = 1$, $e_{ji}^{*} = 0$, and $L_{ij}^{*}(\alpha_{ij})$ is constant in this interval.

In any other subcase, the first, second, second to last, and last intervals considered above, may not exist. The rest of the analysis for those subcases is similar to the first one.

Therefore, $\alpha_{ij}^{F} = \arg Min\{L_{ij}^{*}(\Lambda(1-\bar{\alpha}_{ji})), L_{ij}^{*}(\Lambda(\bar{\alpha}_{ij}))\}$. Note that, if $\alpha_{ij}^{F} = \Lambda(1-\bar{\alpha}_{ji})$ and $1-\bar{\alpha}_{ji} > 0$, then α_{ij}^{*} is equal to any $\alpha_{ij} \in [0, 1-\bar{\alpha}_{ji}]$, but if $\alpha_{ij}^{F} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} < 1$, then α_{ij}^{*} is equal to any $\alpha_{ij} \in [\bar{\alpha}_{ij}, 1]$. Additionally, if $1-\bar{\alpha}_{ji} < 0$ and $\bar{\alpha}_{ij} > 1$, then $L_{ij}^{*}(\Lambda(1-\bar{\alpha}_{ji})) = L_{ij}^{*}(\Lambda(\bar{\alpha}_{ij})$, so α_{ij}^{*} is equal to any $\alpha_{ij} \in [0,1]$.

Finally, by using Theorem 3,

if $\alpha_{ij}^{F} = \Lambda(1-\bar{\alpha}_{ji})$ and $1-\bar{\alpha}_{ji} > 0$, then $e_{ij}^{*} = 0$, $e_{ji}^{*} = 1$,

if $\alpha_{ij}^{F} = \Lambda(1-\bar{\alpha}_{ji})$, $1-\bar{\alpha}_{ji} < 0$ and $1-\underline{\alpha}_{ji} > 0$, then $e_{ij}^{*} = 0$, and e_{ji}^{*} is the unique solution of $c_j'(e_{ji}) - \alpha_{ji} r_{ij}'(e_{ji}) = 0$,

if $\alpha_{ij}^{F} = \Lambda(\bar{\alpha}_{ij})$ and $\bar{\alpha}_{ij} < 1$, then $e_{ij}^{*} = 1$, $e_{ji}^{*} = 0$,

if $\alpha_{ij}^{F} = \Lambda(\bar{\alpha}_{ij})$, $\bar{\alpha}_{ij} > 1$ and $\underline{\alpha}_{ij} < 1$, then e_{ij}^{*} is the unique solution of $c_i'(e_{ij}) - \alpha_{ij} r_{ji}'(e_{ij}) = 0$ and $e_{ji}^{*} = 0$,

if $1-\bar{\alpha}_{ji} < 0$ and $\underline{\alpha}_{ij} > 1$, α_{ij}^{*} has two solutions $\Lambda(1-\bar{\alpha}_{ji})$ and $\Lambda(\bar{\alpha}_{ij})$ because $L_{ij}^{*}(\Lambda(1-\bar{\alpha}_{ji})) = L_{ij}^{*}(\Lambda(\bar{\alpha}_{ij}))$, then $e_{ij}^{*} = 0 = e_{ji}^{*}$.

■

To conclude the paper, we describe a procedure for finding the unique EEE.

EEE Procedure

Given an effort game $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$,

1. we first calculate the lower and upper thresholds of the bilateral interaction between any pair of agents by using Definition 2;
2. we then focus on the list (10) and determine which case (A-F) applies;
3. Theorem 4 provides the optimal α^*_{ij} for all $i, j \in N$, to minimize the centralized (aggregate) cost allocation $\sum_{i\in N} A_i(e^*)$, the efficient effort equilibrium (e^*_{ij}, e^*_{ji}) for every pair of agents, and thus the unique efficient effort equilibrium e^* for the game;
4. at this point the optimal cost allocation that incentivizes agents $i, j \in N$ to make an efficient effort equilibrium e^*_{ij} and e^*_{ji} is known, i.e.

$$A_i(e^*) = c_i(e^*_i) - \sum_{j\in N\setminus\{i\}} \alpha^*_{ij}[r_{ij}(e^*_{ji}) + r_{ji}(e^*_{ij})];$$

The 3-firm case given in Example 1 can be used to illustrate this procedure.

Example 2 *Consider again the PE-network with 3 firms given by Example 1. By Definition 2, the pair of firms* $\{1,2\}$ *has the thresholds* $\underline{\alpha}_{12} = 0.5$, $\bar{\alpha}_{12} = 0.557$, $\underline{\alpha}_{21} = 33.3$, *and* $\bar{\alpha}_{21} = 108$, *which correspond to Case F in the Table 10. By using Theorem 4, it can easily be checked that it is subcase F.3. Thus,* $e^{**}_{12} = 1$, $e^{**}_{21} = 0$, *and* $\alpha^{**}_{12} \in [0.557, 1]$. *As firms* 2 *and* 3 *are identical,* $e^{**}_{13} = 1$, $e^{**}_{31} = 0$, *and* $\alpha^{**}_{13} \in [0.557, 1]$.

Finally, for the pair $\{2,3\}$, $\underline{\alpha}_{23} = 33.3$, $\bar{\alpha}_{23} = 108$, $\underline{\alpha}_{32} = 33.3$, *and* $\bar{\alpha}_{32} = 108$. *This is again Case F, but now subcase F.5. Thus,* $e^{**}_{23} = 0$, $e^{**}_{32} = 0$, *and* $\alpha^{**}_{23} \in [0,1]$.

It can be concluded that $e^{**}_1 = (1,1)$, $e^{**}_2 = e^{**}_3 = (0,0)$, *so the unique EEF is* $e^{**} = ((1,1), (0,0), (0,0))$. *The cost allocation with weighted pairwise reduction for each firm is*

$$A_1(e^{**}) = 308 - 201\left(\alpha^{**}_{12} + \alpha^{**}_{13}\right),$$

$$A_2(e^{**}) = 100 - 201\alpha^{**}_{21} - 4\alpha^{**}_{23},$$

$$A_3(e^{**}) = 100 - 201\alpha^{**}_{31} - 4\alpha^{**}_{32},$$

where $\alpha^{**}_{12} \in [0.557, 1]$, $\alpha^{**}_{13} \in [0.557, 1]$, $\alpha^{**}_{23} \in [0,1]$, $\alpha^{**}_{21} = 1-\alpha^{**}_{12}$, $\alpha^{**}_{31} = 1-\alpha^{**}_{13}$, *and* $\alpha^{**}_{32} = 1-\alpha^{**}_{23}$.

As discussed above, this cost allocation with weights for firm 1 (between 0.5574 *and* 1*) greater than the Shapley value (0.5) encourages that firm to exert the maximum level of effort in regard to each of the other firms (i.e.* $e^{**}_1 = (1,1)$*).*

5 Conclusions and future research

This paper presents a model of cooperation with pairwise cost reduction. The direct impact of pairwise effort on cost reductions is investigated by means of a two-stage bi-form game. First, the agents determine the level of pairwise effort to be made to reduce the cost of their partners. Second, they participate in a bilateral interaction with multiple independent partners where the cost reduction that each agent gives to another agent remains constant in any possible coalition. As a result of cooperation, agents reduce each other's costs. In the non-cooperative game that precedes cooperation, the agents anticipate the cost allocation that will result from the cooperative game by incorporating the effect of the effort made into their cost functions. We show that all-included cooperation is feasible, in the sense that there are possible cost reductions that make all agents better off (or, at least, not worse off), and consistent. We then identify a family of feasible cost allocations with weighted pairwise reduction. One of these cost allocations is selected by taking into account the incentives generated in the efforts that agents make, and consequently in the total cost of coalitions. Surprisingly, we find that the Shapley value, which coincides with the Nucleolus in this model, can induce inefficient effort strategies in equilibrium in the non-cooperative model. However, it is always possible to select a core-allocation with appropriate pairwise weights that can generate an optimal level of efficient effort. We provide a procedure for obtaining the unique EEE in cooperation with pairwise cost reduction.

There are several directions for future research. First, this paper assumes that the individual effort cost function $c_i(e_i)$ is independent of the effort of other agents, and that the marginal cost $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ is independent of the effort that i makes in regard to agents other than j, i.e. $\frac{\partial c_i^2(e_i)}{\partial e_{ij}\partial e_{ih}} = 0$. We make a similar assumption with the cost reduction function $r_{ij}(e_{ji}^*)$. There is some degree of independence between efforts. This is a reasonable assumption in many contexts, but in some settings different assumptions might be needed. For example, there are situations with strategic complementarity in which the efforts of agents reinforce each other. In such cases the function is supermodular. In other cases there is strategic substitutability, so that efforts offset each other and the function is submodular. Focusing on the effort cost function of one agent, if $\frac{\partial c_i^2(e_i)}{\partial e_{ij}\partial e_{ih}} > 0$ then there is complementarity between the efforts, and if $\frac{\partial c_i^2(e_i)}{\partial e_{ij}\partial e_{ih}} < 0$, then there is substitutability. This is a very interesting future extension. It could also be worth considering this complementarity/substitutability not only between the different efforts that one agent makes in regard to other agents but also between the efforts made by different agents. This assumption can be made on both the effort cost functions and the cost reduction function. Obviously, complementarity on the effort cost function has the opposite effect to that on the cost reduction function.

The second direction is related to the assumption of bilateral interaction between agents. This has the advantage of being analytically more tractable and is widely applied in practice (e.g., Fang and Wang 2019; Amin et al. 2020, Park et al. 2010), but overall interaction between agents, dependent on groups, is an important factor that we believe does not affect the success of cooperation. One possible future extension would be to investigate the cooperative model with multiple cost reduction, and also the impact of the efforts made on those cost reductions. Finally, we identify a large family of core-allocations with weighted pairwise reduction which contains the Shapley value and the Nucleolus and always provides a level of efficient effort in equilibrium. This family is very rich in itself, as a set solution concept for our cooperative model. Research into this core-allocation family can be furthered through an in-depth analysis of its structure and its geometric relationship to the core.

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6 Appendix

The first lemma shows how the optimal cost function of agent $i \in N$ depends on α_{ij}. Henceforth, to simplify notation, we consider that for any $i, j \in N$, $\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}$ and $\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}}$ stand for derivatives $\frac{\partial r_{ij}(e_{ji})}{\partial e_{ji}}$ and $\frac{\partial c_i(e_i)}{\partial e_{ij}}$ evaluated in the unique effort equilibrium.

Lemma 3 *Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ be the effort game and e^* the effort equilibrium. Thus,*

1. $$\frac{\partial(A_i(e^*))}{\partial\alpha_{ij}} = \frac{\partial(A^*_i(\alpha_{ij}))}{\partial\alpha_{ij}} = \begin{cases} -r_{ij}(e^*_{ji}) - \alpha_{ij}\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} - r_{ji}(e^*_{ij}), & if \quad \alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}) \\ -r_{ij}(e^*_{ji}) - r_{ji}(e^*_{ij}) < 0, & otherwise \end{cases}$$

2. $$\frac{\partial(A_j(e^*))}{\partial\alpha_{ij}} = \frac{\partial(A^*_j(1-\alpha_{ij}))}{\partial\alpha_{ij}} = \begin{cases} r_{ji}(e^*_{ij}) - (1-\alpha_{ij})\frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} + r_{ij}(e^*_{ji}), & if \quad \alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \\ r_{ji}(e^*_{ij}) + r_{ij}(e^*_{ji}) > 0, & otherwise. \end{cases}$$

Proof. It is known that $A_i(e^*) = c_i(e^*_i) - \sum_{z\in N\setminus\{i\}} \alpha_{iz}(r_{iz}(e^*_{zi}) + r_{zi}(e^*_{iz}))$, and $A^*_i(\alpha_{ij}) = c_i(e^*_i) - \alpha_{ij}(r_{ij}(e^*_{ji}) + r_{ji}(e^*_{ij}))$, thus

$$\frac{\partial(A_i(e^*))}{\partial\alpha_{ij}} = \frac{\partial(A^*_i(\alpha_{ij}))}{\partial\alpha_{ij}} = \frac{\partial c_i(e^*_i)}{\partial e^*_{ij}}\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} - r_{ij}(e^*_{ji}) - \alpha_{ij}\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} - r_{ji}(e^*_{ij}) - \alpha_{ij}\frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\frac{\partial e^*_{ij}}{\partial\alpha_{ij}},$$
$$= \left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \alpha_{ij}\frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} - r_{ij}(e^*_{ji}) - \alpha_{ij}\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} - r_{ji}(e^*_{ij}).$$

The first term of the above expression is always zero, i.e. $\left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \alpha_{ij}\frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right)\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} = 0$. To see this, note that if $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij})$, then $e^*_{ij} \in (0,1)$ by Lemma 1, so $\left(\frac{\partial c_i(e^*_i)}{\partial e^*_{ij}} - \alpha_{ij}\frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\right) = 0$ because it is evaluated in equilibrium. In the other case, where $\alpha_{ij} \leq \underline{\alpha}_{ij}$ or $\alpha_{ij} \geq \bar{\alpha}_{ij}$, $e^*_{ij} = 0$ by Proposition 1, so $\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} = 0$. Therefore, $\frac{\partial(A_i(e^*))}{\partial\alpha_{ij}} = -r_{ij}(e^*_{ji}) - \alpha_{ij}\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} - r_{ji}(e^*_{ij})$.

It is known by assumption that $r_{ij}(e^*_{ji}) \geq 0$, $\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}} > 0$. If $\alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$, then by Proposition 1, $\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} < 0$. However, if $\alpha_{ij} \notin (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$ then, by Proposition 1, $\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} = 0$, so $\frac{\partial(A_i(e^*))}{\partial\alpha_{ij}} = -r_{ij}(e^*_{ji}) - r_{ji}(e^*_{ij})$.

The proof is analogous for $\frac{\partial(A_j(e^*))}{\partial\alpha_{ij}}$. ■

Notice that the effect of α_{ij} on the cost function of agent i could be positive or negative because of two simultaneous effects. First effect: as expected, if α_{ij} increases so does the proportion of cost reduction that agent i can obtain increases, and thus the cost function, $A_i(e^*)$, decreases. This decrease is measured by the term $-r_{ij}(e^*_{ji}) - r_{ji}(e^*_{ij}) < 0$ in the derivative. Second effect: when α_{ij} increases, the effort of agent j decreases in equilibrium, so the cost function of agent i increases. The term $-\alpha_{ij}\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\frac{e^*_{ji}}{\partial\alpha_{ij}} > 0$ measures this second effect. The sum of these two effects determines the sign of the derivative. Therefore, an increase in the proportion of the aggregate cost reduction an agent obtains could increase the costa of that agent if the second effect dominates the first. This is an interesting result: Giving too much to a particular agent could be not only worse for the aggregate cost but also for that particular agent.

The second lemma calculates the derivative of the aggregate cost function $L^*_{ij}(\alpha_{ij}) = A^*_i(\alpha_{ij}) + A^*_j(1-\alpha_{ij})$ in the effort equilibrium for any $i, j \in N$.

Lemma 4 *Let $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ be the effort game, and e^* the effort equilibrium. Thus,*

$$\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = -\alpha_{ij}\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}}I_j - (1-\alpha_{ij})\frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}I_i,$$

$$\textit{where } I_i = \begin{cases} 1 & \textit{if} \quad \alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \\ 0 & \textit{otherwise} \end{cases} \quad \textit{and } I_j = \begin{cases} 1 & \textit{if} \quad \alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}) \\ 0 & \textit{otherwise} \end{cases}.$$

Proof. It is known by Lemma 3 that $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = -r_{ij}(e^*_{ji}) - \alpha_{ij}\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} - r_{ji}(e^*_{ij}) + r_{ji}(e^*_{ij}) - (1-\alpha_{ij})\frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} + r_{ij}(e^*_{ji})$. Simplifying for the different subsets of α_{ij}, the following emerges:

$$\frac{\partial(L^*_{ij}(\alpha_{ij})}{\partial\alpha_{ij}} = -\alpha_{ij}\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} - (1-\alpha_{ij})\frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} \text{ if } \alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cap (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}),$$

$$\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = 0 \text{ if } \alpha_{ij} \notin (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cup (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}),$$

$$\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = -\alpha_{ij}\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} > 0 \text{ if } \alpha_{ij} \in (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji}) \cap \big((0, \underline{\alpha}_{ij}) \cup (\bar{\alpha}_{ij}, 1)\big)$$

$$\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = -(1-\alpha_{ij})\frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} < 0 \text{ if } \alpha_{ij} \in \big((0, 1-\bar{\alpha}_{ji}) \cup \big(1-\underline{\alpha}_{ji}, 1\big)\big) \cap \big(\underline{\alpha}_{ij} \cup \bar{\alpha}_{ij}\big)\big). \ ■$$

The derivative is a piecewise function and there are intervals where its sign is independent of the particular form of the functions of the game. For those cases, it is straightforward to find the optimal α_{ij} that minimizes the function $L^*_{ij}(\alpha_{ij}) = A^*_i(\alpha_{ij}) + A^*_j(1-\alpha_{ij})$. In those intervals, the derivative is either positive, negative or zero throughout the interval. These cases are respectively $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = -\alpha_{ij}\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} > 0$, $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = -(1-\alpha_{ij})\frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\frac{\partial e^*_{ij}}{\partial\alpha_{ij}} < 0$, and $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = 0$. However, there is an interval where the sign of the derivative depends on the particular form of functions of the game. In this particular case $\frac{\partial(L^*_{ij}(\alpha_{ij}))}{\partial\alpha_{ij}} = -\alpha_{ij}\frac{\partial r_{ij}(e^*_{ji})}{\partial e^*_{ji}}\frac{\partial e^*_{ji}}{\partial\alpha_{ij}} - (1-\alpha_{ij})\frac{\partial r_{ji}(e^*_{ij})}{\partial e^*_{ij}}\frac{\partial e^*_{ij}}{\partial\alpha_{ij}}$. This occurs when $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cap (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$, which implies that in equilibrium simultaneously $0 < e^*_{ij} < 1$ and $0 < e^*_{ji} < 1$. Therefore, in this case only, the derivative may be equal to zero for some α_{ij} within this interval. In that case, the second derivative is needed to solve the optimization problem.

The third Lemma shows that the aggregate cost function $L^*_{ij}(\alpha_{ij})$ is convex in α_{ij}. Two additional assumptions about third derivatives need to be introduced.

Lemma 5 *Let* $(N, \{E_i\}_{i\in N}, \{A_i\}_{i\in N})$ *be the effort game,* e^* *the effort equilibrium, and* $\frac{\partial^3 c_i(e_i^*)}{\partial e_{ij}^{*3}} > 0$ *and* $\frac{\partial^3 r_{ji}(e_{ij}^*)}{\partial e_{ij}^{*3}} < 0$*, for any* $i, j \in N$*. Thus* $\frac{\partial^2 L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}^{*2}} > 0$ *for all* $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cap (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$*.*

Proof. Take $\alpha_{ij} \in (\underline{\alpha}_{ij}, \bar{\alpha}_{ij}) \cap (1-\bar{\alpha}_{ji}, 1-\underline{\alpha}_{ji})$. Thus,

$$\frac{\partial^2 (L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}^2} = \frac{\partial^2 (A_i^*(\alpha_{ij}) + A_j^*(1-\alpha_{ij}))}{\partial \alpha_{ij}^2}$$

$$= \frac{\partial\left(-\alpha_{ij}\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} - (1-\alpha_{ij})\frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\frac{\partial e_{ij}^*}{\partial \alpha_{ij}}\right)}{\partial \alpha_{ij}}$$

$$= -\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} - \alpha_{ij}\left(\frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^*\partial \alpha_{ij}}\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} + \frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\frac{\partial^2 e_{ji}^*}{\partial \alpha_{ij}^2}\right)$$

$$+\frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\frac{\partial e_{ij}^*}{\partial \alpha_{ij}} - (1-\alpha_{ij})\left(\frac{\partial^2 r_{ji}(e_{ij}^*)}{\partial e_{ij}^*\partial \alpha_{ij}}\frac{\partial e_{ij}^*}{\partial \alpha_{ij}} + \frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\frac{\partial^2 e_{ij}^*}{\partial \alpha_{ij}^2}\right)$$

$$= -\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} - \alpha_{ij}\left(\frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}\frac{\partial e_{ij}^*}{\partial \alpha_{ij}}\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} + \frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\frac{\partial^2 e_{ji}^*}{\partial \alpha_{ij}^2}\right)$$

$$+\frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\frac{\partial e_{ij}^*}{\partial \alpha_{ij}} - (1-\alpha_{ij})\left(\frac{\partial^2 r_{ji}(e_{ij}^*)}{\partial e_{ij}^{*2}}\frac{\partial e_{ij}^*}{\partial \alpha_{ij}}\frac{\partial e_{ij}^*}{\partial \alpha_{ij}} + \frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\frac{\partial^2 e_{ij}^*}{\partial \alpha_{ij}^2}\right)$$

$$= \frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\frac{\partial e_{ij}^*}{\partial \alpha_{ij}} - \frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} - \alpha_{ij}\left(\frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}\left(\frac{\partial e_{ji}^*}{\partial \alpha_{ij}}\right)^2 + \frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*}\frac{\partial^2 e_{ji}^*}{\partial \alpha_{ij}^2}\right)$$

$$-(1-\alpha_{ij})\left(\frac{\partial^2 r_{ji}(e_{ij}^*)}{\partial e_{ij}^{*2}}\left(\frac{\partial e_{ij}^*}{\partial \alpha_{ij}}\right)^2 + \frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*}\frac{\partial^2 e_{ij}^*}{\partial \alpha_{ij}^2}\right).$$

Now we prove that $\frac{\partial^2 e_{ji}^*}{\partial \alpha_{ij}^2} < 0$ and $\frac{\partial^2 e_{ij}^*}{\partial \alpha_{ij}^2} < 0$, so $\frac{\partial^2 (L^*_{ij}(\alpha_{ij}))}{\partial \alpha_{ij}^2} > 0$.

We first prove that $\frac{\partial^2 e_{ji}^*}{\partial \alpha_{ij}^2} < 0$. It is known that

$$\frac{\partial A_j(e^*)}{\partial e_{ji}} = \frac{\partial c_j(e_j^*)}{\partial e_{ji}^*} - (1-\alpha_{ij})\frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*} = 0$$

We now derive the second term regarding α_{ij}.

$$\frac{\partial^2 c_j(e_j^*)}{\partial e_{ji}^{*2}}\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} + \frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*} - (1-\alpha_{ij})\frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} = 0$$

We now do the same to α_{ij}.

$$\left(\frac{\partial^3 c_j(e_j^*)}{\partial e_{ji}^{*3}}\left(\frac{\partial e_{ji}^*}{\partial \alpha_{ij}}\right)^2 + \frac{\partial^2 c_j(e_j^*)}{\partial e_{ji}^{*2}}\frac{\partial^2 e_{ji}^*}{\partial \alpha_{ji}^2}\right) + \frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}\frac{\partial e_{ji}^*}{\partial \alpha_{ij}}$$

$$-(1-\alpha_{ij})\left(\frac{\partial^3 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*3}}\left(\frac{\partial e_{ji}^*}{\partial \alpha_{ij}}\right)^2 + \frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}\frac{\partial^2 e_{ji}^*}{\partial \alpha_{ji}^2}\right) = 0$$

$$\left(\frac{\partial^2 c_j(e_j^*)}{\partial e_{ji}^{*2}} - (1-\alpha_{ij})\frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}\right)\frac{\partial^2 e_{ji}^*}{\partial \alpha_{ji}^2} + \frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}\frac{\partial e_{ji}^*}{\partial \alpha_{ij}}$$

$$+\left(\frac{\partial^3 c_j(e_j^*)}{\partial e_{ji}^{*3}} - (1-\alpha_{ij})\frac{\partial^3 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*3}}\right)\left(\frac{\partial e_{ji}^*}{\partial \alpha_{ij}}\right)^2 = 0$$

$$\frac{\partial^2 e_{ji}^*}{\partial \alpha_{ij}^2} = \frac{-\frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}} \frac{\partial e_{ji}^*}{\partial \alpha_{ij}} - \left(\frac{\partial^3 c_j(e_j^*)}{\partial e_{ji}^{*3}} - (1-\alpha_{ij}) \frac{\partial^3 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*3}} \right) \left(\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} \right)^2}{\frac{\partial^2 c_j(e_j^*)}{\partial e_{ji}^{*2}} - (1-\alpha_{ij}) \frac{\partial^2 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*2}}}$$

Clearly, this expression is lower than zero if $\frac{\partial^3 c_j(e_j^*)}{\partial e_{ji}^{*3}} > 0$ and $\frac{\partial^3 r_{ij}(e_{ji}^*)}{\partial e_{ji}^{*3}} < 0$; note that $\frac{\partial e_{ji}^*}{\partial \alpha_{ij}} < 0$ by Proposition 1.

Analogously, we obtain

$$\frac{\partial^2 e_{ij}^*}{\partial \alpha_{ij}^2} = \frac{\frac{\partial^2 r_{ji}(e_{ij}^*)}{\partial e_{ij}^{*2}} \frac{\partial e_{ij}^*}{\partial \alpha_{ij}} - \left(\frac{\partial^3 c_i(e_i^*)}{\partial e_{ij}^{*3}} - \alpha_{ij} \frac{\partial^3 r_{ji}(e_{ij}^*)}{\partial e_{ij}^{*3}} \right) \left(\frac{\partial e_{ij}^*}{\partial \alpha_{ij}} \right)^2}{\frac{\partial^2 c_i(e_i^*)}{\partial e_{ij}^{*2}} - \alpha_{ij} \frac{\partial^2 r_{ji}(e_{ij}^*)}{\partial e_{ij}^{*2}}} < 0. \quad \blacksquare$$

Lemma 5 enables us to state that in any interval where the piecewise derivative function takes the value $\frac{\partial(L_{ij}^*(\alpha_{ij}))}{\partial \alpha_{ij}} = -\alpha_{ij} \frac{\partial r_{ij}(e_{ji}^*)}{\partial e_{ji}^*} \frac{\partial e_{ji}^*}{\partial \alpha_{ij}} - (1-\alpha_{ij}) \frac{\partial r_{ji}(e_{ij}^*)}{\partial e_{ij}^*} \frac{\partial e_{ij}^*}{\partial \alpha_{ij}}$, the function $L_{ij}^*(\alpha_{ij})$ is convex (see also Lemma 4).

References

[1] Amin, W., Huang, Q., Afzal, M., Khan, A. A., Umer, K., and Ahmed, S. A. (2020). A converging non-cooperative & cooperative game theory approach for stabilizing peer-to-peer electricity trading. Electric Power Systems Research, 183, 106278.

[2] Bernstein F., Kok A.G., and Meca A. (2015). Cooperation in assembly systems: The role of knowledge sharing networks. European Journal of Operational Research, 240, 160-171.

[3] Bilbao, J. M., Fernández, J. R., Jiménez, N., and López, J. J. (2008). The Shapley value for bicooperative games. Annals of Operations Research, 158(1), 99-115.

[4] Brandenburger, A., and Stuart, H. (2007). Biform games. Management science, 53(4), 537-549.

[5] Cachon, G. P., and Netessine, S. (2006). Game theory in supply chain analysis. In Models, methods, and applications for innovative decision making (pp. 200-233). INFORMS.

[6] Curiel I., Pederzoli G., and Tijs S. (1989). Sequencing games, European Journal of Operational Research, 40, 344-351.

[7] Dewatripont, M., Jewitt, I. and and Tirole, J. (1999). The economics of career concerns, Part I: Comparing information structures. Review of Economic Studies 66(1), 183-198.

[8] Edhan, O. (2014). Value Solutions in Cooperative Games, Journal of Economic Literature 52(1), 211-249.

[9] Fiestras-Janeiro, M. G., García-Jurado, I., and Mosquera, M. A. (2011). Cooperative games and cost allocation problems. Top, 19(1), 1-22.

[10] González-Díaz, J. , Fiestras-Janeiro, M. G., García-Jurado (2010). An introductory course on mathematical game theory. Graduate Studies in Mathematics, vol 115. American Mathematical Society.

[11] Guajardo, M., and Rönnqvist, M. (2016). A review on cost allocation methods in collaborative transportation. International transactions in operational research, 23(3), 371-392.

[12] Fang, D., and Wang, J., (2019). Horizontal capacity sharing between asymmetric competitors. Omega, in press, Article 102109.

[13] Holmstrom, B. (1982). Moral hazard in teams. Bell Journal of Economics 13(2), 324-40.

[14] Holmstrom, B. (1999). Managerial incentive problems: A dynamic perspective. Review of Economic Studies 66(1), 169-182.

[15] Kar, A., Mitra, M., and Mutuswami, S. (2009). On the coincidence of the prenucleolus with the Shapley value. Mathematical Social Sciences, 57(1):16–25.

[16] Kimms, A.,and Kozeletskyi, I. (2016). Shapley value-based cost allocation in the cooperative traveling salesman problem under rolling horizon planning. EURO Journal on Transportation and Logistics, 5(4), 371-392.

[17] Le, P. H., Nguyen, T. D., and Bektaş, T. (2018). Efficient computation of the Shapley value for large-scale linear production games. Annals of Operations Research, 1-21.

[18] Li, S., and Zhang, Q. (2009). A simplified expression of the Shapley function for fuzzy game. European Journal of Operational Research, 196(1), 234-245.

[19] Littlechild S.C. and Owen G., 1973, A simple expression for the Shapley value in a special case, Management Science, 20, 370-372.

[20] Lozano, S., Moreno, P., Adenso-Díaz, B., and Algaba, E. (2013). Cooperative game theory approach to allocating benefits of horizontal cooperation. European Journal of Operational Research, 229(2), 444-452.

[21] Meca, A.,and Sošić, G. (2014). Benefactors and Beneficiaries: The Effects of Giving and Receiving on Cost-Coalitional Problems. Production and Operations Management, 23(9), 1549-1560.

[22] Meca, A., García-Martínez, J. A., and Mayor-Serra, A. J. (2019). The Shapley Value of Corporation Tax Games with Dual Benefactors. Handbook of the Shapley Value, 313.

[23] Moretti, S., and Patrone, F. (2008). Transversality of the Shapley value. Top, 16(1), 1.

[24] Moulin, H. (1987). Equal or proportional division of a surplus, and other methods. International Journal of Game Theory, 16(3), 161-186.

[25] Moulin, H. (2004). Fair Division and Collective Welfare. Cambridge, Massachusetts: MIT Press.

[26] Nash, J. F. (1950). Equilibrium points in n-person games. Proceedings of the national academy of sciences, 36(1), 48-49.

[27] Nash, J. (1951). Non-cooperative games. Annals of mathematics, 286-295.

[28] Park, J., and Van der Schaar, M. (2010). A game theoretic analysis of incentives in content production and sharing over peer-to-peer networks. IEEE Journal of Selected Topics in Signal Processing, 4(4), 704-717.

[29] Schmeidler, D. (1969). The nucleolus of a characteristic function game. SIAM Journal on applied mathematics, 17(6), 1163-1170.

[30] Serrano, R. (2009), Cooperative Games: Core and Shapley Value, in R. Meyers (ed.), Encyclopedia of Complexity and Systems Science, Springer, New York

[31] Shapley, L. S. (1953). A value for n-person games. Contributions to the Theory of Games, 2(28), 307-317.

[32] Shapley, L. S. (1971). Cores of convex games. International journal of game theory, 1(1), 11-26.

[33] Slikker, M., and Van den Nouweland, A. (2012). Social and economic networks in cooperative game theory (Vol. 27). Springer Science & Business Media.

[34] Thomson, W. (2010). Fair Allocation Rules. Chapter 21 in the Handbook of Social Choice and Welfare, vol 2, 393-492.

[35] Todeva, E., and Knoke, D. (2005). Strategic Alliances & Models of Collaboration. Management Decision, 43(1).