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Abstract

Equivalence between rejective core and set of dividend equilibria allocations is studied in finite economy and double infinity economy frameworks in presence of indivisibilities of commodities while also allowing the presence of satiated agents. It is further shown that in the finite economy and the double infinity economy, the core of every renegotiation core, the rejective core of every replica economy and the set of dividend equilibria are identical. Hence, core equivalence is demonstrated in both frameworks.

Keywords: Core equivalence, Indivisible commodities, Satiation, Dividend equilibrium, Rejective core, Renegotiation core.

1 Introduction

We study the core equivalence problem while incorporating indivisibilities and satiation simultaneously into the general equilibrium framework in two different settings, namely, the classical model with finite economy; and the overlapping generations double infinity economy. The seminal work on core equivalence in finite economy is due to Debreu and Scarf, 1963 wherein they show that as economy is replicated an arbitrarily large number times, the core converges to the set of Walrasian equilibria. Corresponding work in a large economy is due to Aumann, 1964, who shows the core equivalence in a large economy. Hildenbrand, 1974 provides another exposition of the large economy problem using Lyapunov's convexity theorem

The question of incorporating indivisibilities in a large economy to model the situations wherein the agents can trade some of the commodities only in integer quantities is due to Mas-Colell, 1977. Core equivalence in this framework is due to Ali Khan and Yamazaki, 1981. Both of these studies limit themselves to a case wherein there is only one divisible commodity in the market. This framework is generalized by Hammond, Kaneko, and Wooders, 1989, who study the widespread externalities in a large economy while having the presence of both divisible and indivisible commodities with no

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restriction on number of divisible or indivisible commodities except that there be at least one divisible commodity. Core equivalence in a large economy in the scenario where all commodities are indivisible is due to Inoue, 2006. Inoue, 2014 shows a similar result under some assumptions for a finite economy. However, all these works do not dispense with non-satiation of all agents.

Dreze and Muller, 1980 conceptualize the notion of dividend equilibrium, while Mas-Colell, 1992 defines the concept of equilibrium with a slack. These concepts are quite similar and deal with satiated agents. A. Konovalov, 1998, Alexander Konovalov, 2005 and Miyazaki and Takekuma, 2010 form a body of works that are quite significant inasmuch as equivalence between rejective core and dividend equilibrium is concerned. Alexander Konovalov, 2005 and Miyazaki and Takekuma, 2010 deal with a large economy, while A. Konovalov, 1998 deals with a large economy with finite number of types. They all show equivalence between rejective core and dividend equilibrium. Murakami and Urai, 2017 show a relation between core of every renegotiation economy and rejective core and prove a limit core theorem between dividend equilibrium and core of every renegotiation economy in the context of a finite economy. This class of works does not dispense with the notion of divisibility of commodities.

Samuelson, 1958 was the pioneer in welfare analysis in overlapping generations economy, and shows that Walrasian equilibrium may not be Pareto optimal. Balasko and Shell, 1980 define the notion of weak Pareto optimality to characterize the core and show that competitive equilibrium is weakly Pareto optimal in overlapping generations economy. Aliprantis and Burkinshaw, 1990 and Chae and Esteban, 1993 study the core equivalence in the framework of overlapping generations. Urai and Murakami, 2016 study a double infinity economy model which has an infinity of traders and commodities, thereby encompassing the double infinity characteristic of overlapping generations economy. They show the equivalence between finite core of every renegotiation economy and set of monetary equilibria in a double infinity economy.

While the standard framework of general equilibrium rests upon assumption of non-satiation and perfectly divisible commodities, studies on core equivalence have relaxed either non-satiation or indivisibilities; to the best of our knowledge, no work in literature relaxes both of these assumptions simultaneously. Our contribution is to relax these assumptions simultaneously while showing the core equivalence result in both of these frameworks, viz. finite economy and double infinity economy. Further, the varied nature of these frameworks endows our results with a wide range of applicability.

The paper is organized as follows: in Section 2, we consider a finite deterministic economy with indivisibilities, and after defining the basic notation and concepts, we proceed to show the relation between the set of dividend equilibria allocations, the rejective core of the replica economy and the core of every renegotiation economy. We then show the main result of the section, which is that as the economy is replicated arbitrarily large number of times, the set of dividend equilibria allocations, the rejective core of the replica economy and the core of every renegotiation economy converge.

In Section 3, we consider a double infinity economy with indivisibilities. We firstly mention as to how the double infinity economy shows the same characteristics as a traditional over-lapping generations economy. We then appropriately modify the assumptions, notations and the basic concepts for the double infinity economy, and proceed to show the analogous relation between the set of dividend equilibria allocations, the finite rejective core and the finite core of a renegotiation economy. In doing so, we use the construction of the rejecting coalition from Section 2. We then go on to prove the main result of the section, which is that as the economy is replicated arbitrarily large number of times, the set of dividend equilibria allocations, the rejective core of the replica economy and the core of every renegotiation economy converge. We also discuss the role of replications, as well as the role of taking finite cores in bringing about the core equivalence. We then conclude the paper.

2 Finite Economy

In this section, we consider an economy $\mathcal{E}$ with a finite number of agents and finitely many commodities. Let I denote the set of agents and K denote the set of commodities. We take $K = N \cup D$, where N represents the set of indivisible or non-divisible commodities which the agents are constrained to consume only in integer amounts and D represents the set of divisible commodities. Throughout, we assume that $D \neq \emptyset$. Thus, the commodity space of the economy is represented as $X := \mathbb{R}^D \times \mathbb{Z}^N$ if $N \neq \emptyset$; and $X := \mathbb{R}^D$, otherwise. The *consumption set* of each agent is assumed to be X_+. Further, we denote by X_{++} the set of strictly positive elements of X.

We assume the following insofar as preferences and endowments are concerned:

A.1 The preference structure of agents: $(\succsim_i, e_i)$, $\succsim_i$ is complete and continuous on X_+ and e_i is the *initial endowment* of agent i. It is assumed that for all $i \in I$, $\succ_i (x_i)$ (set of all bundles strictly preferred to x_i by agent i) is an open set in the subspace topology which X_+ inherits from X.

A.2 The convexity of preferences is modified in the following manner: $\forall\ y_i, z_i\ \in \succ_i (x_i) \subseteq X_+$, the set $\Lambda(y_i, z_i) = \{\lambda.y_i + (1-\lambda).z_i : \lambda \in [0,1]\} \cap X_+ \subset \succ_i (x_i)$.

A.3 We allow for the possibility of satiation, i.e. there may exist some $i \in I$ and some $x_i \in X_+$ such that $\succ_i (x_i) = \emptyset$.

A.4 We also assume that there is at-least one agent b whose preferences satisfy monotonicity. In other words, $\{x_b\} + X_{++} \subseteq \succ_b (x_b)$.

A.5 We also assume that there is at-least one agent b' whose preferences satisfy strict monotonicity with respect to divisible commodities in the interior of the consumption space of divisible commodities. In other words, if $\mathrm{Pr}_{\mathbb{R}^D_+}[x_{b'}] >> 0$, then $\{x_{b'}\} + X_+ \cap ((\mathbb{R}^D_+ \setminus \{0\}) \times \mathbb{Z}^N_+) \subseteq \succ_{b'} (x_{b'})$.

If $N \neq \emptyset$, we impose following additional assumptions:

A.6 For the agent b', for any $x \in X_+$ and and any $y \in X_+$, such that $\mathrm{Pr}_{\mathbb{R}^D_+}[x] >> 0$ and $\mathrm{Pr}_{\mathbb{R}^D_+}[y] \in \mathrm{Bd}\ \mathbb{R}^D_+$, $x \succ_{b'} y$.

A.7 There is an agent c whose preferences satisfy overriding desirability of divisible commodities, in other words, for all $x_c \in X_+$, $\exists\, y_c \in X_+$, such that $\mathrm{Pr}_{\mathbb{Z}_+^N}[y_c] = 0$ and $y_c \succ_c x_c$[1].

A.8 Every agent is endowed with a strictly positive amount of every commodity and the endowment vector is at least as good as any bundle containing indivisible bundles only. In other words, $e_i \in X_{++}$ for all $i \in I$ and $e_i \succ_i (0, a)$ for all $a \in \mathbb{Z}_+^N$.

The above assumptions are well founded in literature in these works: Ali Khan and Yamazaki, 1981, Mas-Colell, 1977 and Hammond et al., 1989.

2.1 Core and Equilibrium Concepts

We now proceed to explain the notations and subsequently define the basic concepts for this section.

Definition 2.1. *A* **coalition** *in $\mathcal{E}$ is a non-empty subset of I and an* **allocation** *of $\mathcal{E}$ is just a n-tuple bundle of commodity vectors. Furthermore, an allocation $x = (x_i)_{i\in I}$ is said to be* feasible *if $\sum_{i\in I} x_i = \sum_{i\in I} e_i$. A coalition S is said to* **block** *an allocation x with an allocation y if the following hold:*

1. $\sum_{i\in S} y_i = \sum_{i\in S} e_i$*; and*
2. $y_i \succ_i x_i$ *for all* $i \in S$.

The **core** *of $\mathcal{E}$ is defined to be the set of all feasible allocations that cannot be blocked by any coalition.*

The following definition is due to Alexander Konovalov, 2005 in an economy with a non-atomic measure space of agents. The following definition is just an adaptation of the one given by Alexander Konovalov, 2005 in a framework with finitely many agents, refer to Murakami and Urai, 2017.

Definition 2.2. *A coalition S is said to* **reject** *an allocation x with a feasible allocation y if there exist S_1 and S_2 such that the following hold:*

1. $S_1 \cap S_2 = \emptyset$, $S_1 \cup S_2 = S$;
2. $\sum_{i\in S} y_i = \sum_{i\in S_1} x_i + \sum_{i\in S_2} e_i$;
3. $y_i \succ_i x_i$ *for all* $i \in S$*; and*
4. $y_i \succsim_i e_i$ *for all* $i \in I \setminus S$.

The **rejective core** *of the economy $\mathcal{E}$ is the set of all such feasible allocations which cannot be rejected by any coalition. We denote by $\mathscr{R}$ the rejective core of the economy $\mathcal{E}$.*

[1] Agents b and c may be same or different.

Note that when S_1 is empty then we can neglect the condition 4 by choosing $y_i = e_i$ for all $i \notin S$. Thus, the above definition of rejection by a coalition is stronger than that of blocking by a coalition, which implies the rejective core is a subset of the core.

Definition 2.3. *A* **dividend equilibrium allocation** *is a feasible allocation $x = (x_i)_{i \in I}$ satisfying the following: there exist*

1. *an element $d \in \mathbb{R}^I_+$ such that $d = (d_i)_{i \in I}$, where $d_i \in \mathbb{R}_+$; and*
2. *an element $p \in \mathbb{R}^K_+$,*

such that x_i is the maximal element (in accordance with $\succsim_i$) of the dividend budget set of agent $i \in I$, where the dividend budget set is given by:

$$\mathbb{B}(p, e_i, d_i) = \{z_i : \ p.z_i \leq p.e_i + d_i\}.$$

The set of all dividend equilibrium allocations of $\mathcal{E}$ is denoted by $\mathscr{D}$.

Definition 2.4. *A* **dividend equilibrium** *is the tuple (x, p, d) where (x, p, d) satisfy the requirements of Definition 2.3*

2.2 Re-negotiation and the equivalence theorem

In above subsection, we mainly focus on the the economy $\mathcal{E}$. We now introduce the concept of re-negotiation in a replica economy and discuss its relation with the notions of rejective core and the set of dividend equilibrium of the original economy.

Let x be any feasible allocation. Then an economy in which initial endowment allocation is x, keeping the preferences and the set of agents unchanged is denoted by $\mathcal{E}(x)$. Thus, we have $\mathcal{E} = \mathcal{E}(\omega)$. An economy in which each agent is replicated n times in the economy $\mathcal{E}(x)$, that is, the n-replica of the economy $\mathcal{E}(x)$, is denoted by $\mathcal{E}^n(x)$. We define the renegotiation replica economy in line with Murakami and Urai, 2017. For integers $m \geq 1$ and $n \geq 0$, the $(m+n)$**-fold re-negotiation replica economy**, denoted by $\mathcal{E}^m(e) \bigoplus \mathcal{E}^n(x)$, is defined in the sense that each agent i is replicated $m+n$ times, with m replicas of agent i having the endowment e_i and n replicas of agent i having the endowment x_i. For an allocation y of the economy $\mathcal{E}(x)$, we denote by y^n an allocation of the economy $\mathcal{E}^n(x)$ in which each replica of agent i consumes the bundle y_i. Analogously, given a common allocation y in economies $\mathcal{E}(e)$ and $\mathcal{E}(x)$, y^{m+n} represents $(m+n)$-fold replica allocation of y in the $(m+n)$-fold re-negotiation replica economy. Finally, we denote by $\mathscr{C}(m,n)$ the core of the $(m+n)$-fold re-negotiation replica economy, and by $\mathscr{R}^n$ the rejective core of the replicated economy $\mathcal{E}^n(e)$.

We now restate the Proposition 1 of Murakami and Urai, 2017 as follows and note that this lemma will hold irrespective of divisibility of commodities.

Proposition 1. *For all integers $m \geq 1$ and $n \geq 0$, let x^{m+n} be a rejective core allocation for replica economy $\mathcal{E}^{m+n}(e)$. Then, x^{m+n} is a core allocation of the economy $\mathcal{E}^m(e) \bigoplus \mathcal{E}^n(x)$. In other words, $\mathscr{R}^{m+n} \subseteq \mathscr{C}(m,n)$.*

Proposition 2. *For each $n \geq 1$, n^{th}-replica of any dividend equilibrium allocation of $\mathcal{E}$ belongs to the rejective core of the economy $\mathcal{E}^n(e)$.*

Proof. Let (x, p, d) be a dividend equilibrium of $\mathcal{E}$. By definition, it is feasible. If x is not a part of the rejective core of $\mathcal{E}^n(e)$, then there exists a finite coalition S of the economy $\mathcal{E}^n(e)$, which rejects x^n with allocation y of $\mathcal{E}^n(e)$. Then, following must go through:[2]

1. there exist $S_1, S_2 \subseteq S$ such that $S_1 \cap S_2 = \emptyset$, $S_1 \cup S_2 = S$;
2. $\sum_{(i,h)\in S} y_i = \sum_{(i,h)\in S_1} x_i + \sum_{(i,h)\in S_2} e_i$;
3. $y_{(i,h)} \succ_i x_{(i,h)}$ for all $(i, h) \in S$; and
4. $y_{(i,h)} \succsim_i e_{(i,h)}$ for all $(i, h) \notin S$.

Condition 3 implies that $p.y_{(i,h)} > p.e_{(i,h)} + d_{(i,h)}$ for all $(i, h) \in S$, where $d_{(i,h)} := d_i$. This yields

$$p. \sum_{(i,h)\in S} y_{(i,h)} > p. \sum_{(i,h)\in S} e_{(i,h)} + \sum_{(i,h)\in S} d_{(i,h)}.$$

Since x_i is affordable for every agent i under the given equilibrium, then $p.x_{(i,h)} \leq p.e_{(i,h)} + d_{(i,h)}$ for all $(i, h) \in S_1$. Summing up across S_1:

$$p. \sum_{(i,h)\in S_1} x_i \leq p. \sum_{(i,h)\in S_1} e_i + \sum_{(i,h)\in S_1} d_{(i,h)}.$$

Coupling this with condition 2, taking an inner product with the vector p and using the fact that $d_i \geq 0$:

$$p. \sum_{(i,h)\in S} y_{(i,h)} \leq p. \sum_{(i,h)\in S} e_{(i,h)} + \sum_{(i,h)\in S} d_{(i,h)}.$$

This is a contradiction, which completes the proof. □

Proposition 3. *Any allocation x of $\mathcal{E}$ whose $(m+n)^{\text{th}}$-replica belongs to the core of the $(m+n)$-re-negotiation economy for all $m \geq 1$ and $n \geq 0$ is a dividend equilibrium allocation of $\mathcal{E}$.*

Proof. Let x be an allocation of $\mathcal{E}$ whose $(m+n)^{\text{th}}$-replica belongs to the core of the $(m+n)$-re-negotiation economy for all $m \geq 1$ and $n \geq 0$. We denote the set of agents in I who are satiated (resp. non-satiated) under allocation x by I^S (resp. I^{NS}). For any agent $i \in I^{NS}$, we define the sets[3]

$$\Gamma_i^1 = \text{Co}(\succ_i (x_i) - x_i) = \text{Co}\left(\left\{z_i^1 \in \mathbb{R}^D \times \mathbb{Z}^N : z_i^1 + x_i \succ_i x_i\right\}\right)$$

$$\Gamma_i^2 = \text{Co}(\succ_i (x_i) - e_i) = \text{Co}\left(\left\{z_i^2 \in \mathbb{R}^D \times \mathbb{Z}^N : z_i^2 + e_i \succ_i x_i\right\}\right)$$

[2]In what follows, the symbol (i, h) will mean h^{th}-replica of consumer i.

[3]For any non-empty set $A \subseteq \mathbb{R}^D \times \mathbb{Z}^N$, the notation $\text{Co}(A)$ stands for the convex hull of A in $\mathbb{R}^D \times \mathbb{R}^N \equiv \mathbb{R}^K$.

Since Γ_i^1 and Γ_i^2 are convex, the convex hull of $\Gamma_i^1 \cup \Gamma_i^2$ is defined as

$$\Gamma_i := \text{Co}(\Gamma_i^1 \cup \Gamma_i^2) = \left\{ z \in \mathbb{R}^K \;\middle|\; \begin{array}{l} z = \beta^i.z_i^1 + (1-\beta^i).z_i^2 \\ 0 \leq \beta^i \leq 1 \\ z_i^1 \in \Gamma_i^1, z_i^2 \in \Gamma_i^2 \end{array} \right\}$$

Lastly, we denote by Γ the convex hull of the finite union of all such Γ_i for non-satiated agents. Thus, $\Gamma := \text{Co}\left(\bigcup_{i \in I^{NS}} \Gamma_i\right)$. A generic element of Γ can be expressed as

$$z = \sum_{i \in I^{NS}} \alpha^i.(\beta^i.z_i^1 + (1-\beta^i).z_i^2)$$

for some $z_i^1 \in \Gamma_i^1, z_i^2 \in \Gamma_i^2$ and $0 \leq \alpha^i \leq 1$ with $\sum\limits_{i \in I^{NS}} \alpha^i = 1$.

Claim 3.1. $\Gamma \cap \mathbb{R}^K_{--} = \emptyset$.[4]

It follows from Claim 3.1 and the separating hyper-plane theorem, there is a vector $p \in \mathbb{R}^K \setminus \{0\}$ such that for all $i \in I^{NS}$: $p.z_i^1 \geq 0$ if $z_i^1 \in \Gamma_i^1$ and $p.z_i^2 \geq 0$ if $z_i^2 \in \Gamma_i^2$. In particular, for all agents $i \in I^{NS}$ and $y_i \succ_i x_i$, we have $p.y_i \geq p.x_i$ and $p.y_i \geq p.e_i$.

Presence of agent b in the economy ensures that prices are non-negative. This is seen as follows: suppose price of a commodity is negative, and let that commodity be k. Consider the bundle $z_M = x_b + (1, \cdots, M, \cdots, 1)$, with an integer M on component corresponding to k. There will exist a large enough integer M such that $z_M \succ_b x_b$ and $p.z_M < p.x_b$, which is a contradiction.

Further, presence of agent c ensures that at-least one divisible good has a positive price. This can be seen as follows. First, note that the strict positivity of e_c implies $p.e_c > 0$. Let e^D be a vector with value 1 on components corresponding to the divisible goods and 0 on every other component. By overriding desirability of divisible commodities for agent c, there exists some $\lambda_c > 0$ such that $\lambda_c.e^D \succ_c x_c$. But $p.\lambda_c.e^D = 0 < p.e_c$. This is a contradiction with the separating hyper-plane argument following Claim 3.1.

Furthermore, presence of agent b' ensures that prices of all divisible commodities are positive. This is seen as follows:

Case 1: Let $N = \emptyset$. Since $p.x_{b'} \geq p.e_{b'} > 0$, there is some commodity $l \in D$ such that $p^l > 0$ and $x_{b'}^l > 0$.

Case 2: Let $N \neq \emptyset$. Then by assumption A.6 and the fact that $x_{b'} \succeq e_{b'}$, we have $x_{b'}^m > 0$ for all $m \in D$.

Thus, irrespective of the fact that whether there is a non-divisible commodity, there is always a divisible commodity l with positive price and $x_{b'}^l > 0$. Let $k \in D$ be a divisible commodity with 0 price. Then, by continuity of preferences, there is some $\lambda > 0$ such that $x_{b'} + e^k - \lambda.e^l \succ_{b'} x_{b'}$. However, $p.(x_{b'} + e^k - \lambda.e^l) < p.x_{b'}$. This is a contradiction with the separating hyper-plane argument following Claim 3.1.

Now, define $d_i = \max\{0, p.x_i - p.e_i\}$ and $d = (d_i)_{i \in I}$. Then, by construction, $x_i \in \mathbb{B}_i(p, e_i, d_i)$. We now show the individual rationality under the price and dividend system (p, d). For each $i \in I$, affordability of x_i follows from construction. It only remains

[4] Proof of this claim is given in Appendix.

to show that x_i maximizes the preference of agent i in the budget set for any non-satiated agent.

Suppose that $y_i \succ_i x_i$ for some $i \in I^{NS}$ and $p.y_i = p.e_i$.[5] Define z_i^λ by setting $\Pr_{\mathbb{R}^N}[z_i^\lambda] = \Pr_{\mathbb{R}^N}[y_i]$ and $\Pr_{\mathbb{R}^D}[z_i^\lambda] = \lambda \Pr_{\mathbb{R}^D}[y_i]$. Continuity of preferences guarantees the existence of some $\lambda \in (0,1)$ close enough to 1 such that $z_i^\lambda \succ_i x_i$. Since $p.y_i > 0$, $0 < p.z_i^\lambda < p.y_i$. Strict inequality follows from the fact that price of every divisible commodity is positive. This implies that $p.z_i < p.e_i = p.y_i$, which is a contradiction to the result obtained in Claim 3.1. Similarly, suppose that $y_i \succ_i x_i$ for some $i \in I_s$ and $p.y_i = p.x_i$. Proceeding similarly, a contradiction is again obtained. These two contradictions together imply that $p.y_i > \max\{p.e_i, p.x_i\}$ or $p.y_i > p.e_i + d_i$. □

Theorem 2.1. *For any feasible allocation x of the economy $\mathcal{E}$, following statements are equivalent:*

1. $x^{m+n} \in \mathscr{R}^{m+n}$ *for all* $m \geq 1$ *and* $n \geq 0$.
2. $x^{m+n} \in \mathscr{C}(m,n)$ *for all* $m \geq 1$ *and* $n \geq 0$.
3. $x \in \mathscr{D}$

Proof. Follows directly from Propositions 1, 2, and 3. □

3 Double Infinity Economy

The model for double infinity economies is based upon Urai and Murakami, 2016, in which there are infinitely many traders and commodities. This model uses overlapping commodities, and considers that set of agents in each time period to be disjoint from the set of agents living in some other time period. This model preserves the double infinity characteristics of a traditional overlapping generations economy in the sense that while there are a infinite number of traders, they do not interact all at once in the same market.

Under the assumption on strict temporal separability of preferences, this model can be considered to be a traditional overlapping generations model. However, this assumption is too strong, and hence, it is appropriate to refer to this model as a double infinity model rather than an overlapping generations model. We follow this terminology throughout the paper.

The main result of this section is to establish the core equivalence in a double infinity economy framework, thereby extending the result of Theorem 2.1. This model considers a sequence of markets which are linked by commodities, rather than agents, and hence is a significant departure from the traditional finite economy model. However, as this section shows, the core equivalence in this framework is an extension of the core equivalence in a traditional finite economy framework. This extension draws upon the construction used in Section 2 for creation of a rejecting coalition. This in

[5] A.8, combined with individual rationality of agent i implies that $x_i \succsim_i e_i$. This, in turn implies that if $y_i \succ_i x_i$, then $\Pr_{\mathbb{R}^D}[y_i] \geq 0$ and $\Pr_{\mathbb{R}^D}[y_i] \neq 0$.

turn, leads to quite analogous results as in Section 2. Thus, we show that insofar as core equivalence is concerned, even a radically different market structure obeys the same result as a traditional finite economy model.

The pure exchange economy, denoted by $\mathcal{E}$, comprises of the following:

1. *Agents*

A.1.1 The set of agents present in the market during time period t is finite and is represented by I_t.

A.1.2 The set of all the agents in the economy is represented by $\{I_t\}_{t\geq 1}$. Each agent lives for only one period. This in turn implies that $\{I_t\}_{t\geq 1}$ is a collection of pairwise disjoint sets.

A.1.3 We also assume that if $I_t = \emptyset$, then $I_s = \emptyset$ for all $s > t$.

2. *Commodities*

A.2.1 The set of commodities available in time period t is a finite set and is denoted by K_t.

A.2.2 Each K_t can be expressed as $K_t = N_t \cup D_t$, where N_t represents the set of indivisible or non-divisible commodities which the agents are constrained to consume only in integer amounts and D_t represents the set of divisible commodities. Throughout, we assume that $D_t \neq \emptyset$ for all t.

A.2.3 The *consumption set* for an agent living in time period t is $X_{t+} := \mathbb{R}_+^{D_t} \times \mathbb{Z}_+^{N_t}$, if $N_t \neq \emptyset$, and $X_{t+} := \mathbb{R}_+^{D_t}$, otherwise.

A.2.4 The commodities overlap in the sense that if $K_t = \{k(t), k(t)+1, \cdots, k(t)+l(t)\}$, then $k(t) < k(t+1) \leq k(t)+l(t)$. For any time period t, the set of commodities which were available to the economy throughout the history is $K(t) = \bigcup_{s=1}^{t} K_s$.

A.2.5 Every agent $i \in I_t$ has a positive *initial endowment* of every commodity that is available in generation t, that is, $e_i >> 0$.

3. *Preferences*

A.3.1 The preferences of the agents are complete and continuous. Continuity of preferences implies that for all $i \in I_t$, $\succ_i (x_i)$ (set of all bundles strictly preferred to x_i by agent i) is an open set in the subspace topology which X_{t+} inherits from $\mathbb{R}^{D_t} \times \mathbb{R}^{N_t}$ if $N_t \neq \emptyset$, and $\mathbb{R}^{D_t}$, otherwise.

A.3.2 The convexity of preferences is modified in the following manner: for all t, for all $i \in I_t$, if $y_i, z_i \in \succ_i (x_i) \subseteq X_{t+}$, then the set $\Lambda(y_i, z_i) := \{\lambda.y_i + (1-\lambda).z_i | \lambda \in [0,1]\} \cap X_{t+} \subset \succ_i (x_i)$.

A.3.3 We allow for the possibility of satiation, i.e. for any t, there may exist some $i \in I_t$ and some $x_i \in X_{t+}$ such that $\succ_i (x_i) = \emptyset$.

A.3.4 There exists at-least one agent $b \in I_t$ for every time period t having strictly monotone preferences with respect to divisible commodities in the interior of the consumption space of divisible commodities. In other words, if $\Pr_{\mathbb{R}_+^{D_t}}[x_b] >> 0$, then $\{x_b\} + (X_{t+} \setminus \{0\}) \subseteq \succ_b (x_b)$.

If for any t, $N_t \neq \emptyset$, we impose following additional assumptions:

A.3.5 Preferences of agent b having strict monotonicity satisfy overriding desirability of divisible commodities, in other words, for all $x_b \in X_{t+}$, there exists $y_b \in X_{t+}$, such that $\Pr_{\mathbb{Z}_+^{N_t}}[y_b] = 0$ and $y_b \succ_b x_b$.[6][7]

A.3.6 For the monotone agent b, for any $x \in X_{t+}$ and and any $y \in X_{t+}$, such that $\Pr_{\mathbb{R}_+^{D_t}}[x] >> 0$ and $\Pr_{\mathbb{R}_+^{D_t}}[y] \in \mathrm{Bd}\, \mathbb{R}_+^{D_t}$, $x \succ_b y$.[8]

A.3.7 For every agent i, $e_i \succ_i (0, x)$ for all $x \in \mathbb{Z}_+^{N_t}$.

Thus, the economy $\mathcal{E}$ can be summarized as

$$\mathcal{E} = \langle \bigcup_{t\geq 1} I_t,\ \prod_{t\geq 1} K_t, \{X_{t+},\ \succsim_i,\ e_i\}_{i\in \bigcup\limits_{t\geq 1} I_t} \rangle$$

The replication structure is taken to be same as in previous structure, as is the definition of renegotiation replica economy. We follow the same replication structure and notation as in Section 2 to maintain homogeneity.

We can extend each of x_i's so that x_i's can be written as elements of $(\mathbb{R}_+^{D_1} \times \mathbb{Z}_+^{N_1}) \times \prod_{t>1} (\mathbb{R}_+^{D_t \setminus D(t-1)} \times \mathbb{Z}_+^{N_t \setminus N(t-1)})$ [9] under the restriction that only those components of x_i's which correspond to the commodities available in the time period when the agent lives are allowed to take non zero values. An allocation x is defined as $x = (x_i)_{i\in \bigcup\limits_{t\geq 1} I_t}$.

Note: If X and Y are two homeomorphic topological spaces, then we write $X \approx Y$
We note that[10] :

$$(\mathbb{R}_+^{D_1} \times \mathbb{Z}_+^{N_1}) \times \prod_{t>1} (\mathbb{R}_+^{D_t \setminus D(t-1)} \times \mathbb{Z}_+^{N_t \setminus N(t-1)}) \subseteq (\mathbb{R}_+^{D_1} \times \mathbb{R}_+^{N_1}) \times \prod_{t>1} (\mathbb{R}_+^{D_t \setminus D(t-1)} \times \mathbb{R}_+^{N_t \setminus N(t-1)}) \approx \mathbb{R}_+^{\infty}$$

Here, $\mathbb{R}_+^{\infty}$ is the space of all non-negative real-valued sequences. Thus, each of the x_i's can themselves be written as elements of $\mathbb{R}_+^{\infty}$. This abuse of notation allows us to

[6] A.3.5, A.3.6 and A.3.7 deal with overriding desirability of commodities. A.3.5 and A.3.6 are imposed only on the monotone agent. Coupled with A.3.4, they are tantamount to saying that preferences of the monotone agent b are neoclassical insofar as divisible commodities are concerned and strictly monotone as far as indivisible commodities go.

[7] In this section, we have combined agents b, b' and c of Section 2 into one agent b. This "combining" makes no material difference in the proof and has been done solely for ease of notation.

[8] This is the only additional assumption on preferences made vis-a-vis Section 2.

[9] If any of the $N_t \setminus N(t-1) = \emptyset$, we just drop $\mathbb{Z}_+^{N_t \setminus N(t-1)}$ and proceed.

[10] Here too, the same caveat "if any of the $N_t \setminus N(t-1) = \emptyset$, we just drop $\mathbb{Z}_+^{N_t \setminus N(t-1)}$ and proceed" applies.

define the feasibility of an allocation. An allocation is said to be feasible if

$$\sum_{t\in\mathbb{N}}\sum_{i\in I_t} x_i = \sum_{t\in\mathbb{N}}\sum_{i\in I_t} e_i$$

3.1 Core and Equilibrium Concepts

A **coalition** is defined as a non-empty subset of all agents, that is, $S \subseteq \bigcup_{t\geq 1} I_t$ and $S \neq \emptyset$.

Definition 3.1. *A coalition S is said to* **block** *an allocation x with an allocation y if the following hold:*

1. $\sum\limits_{t\in\mathbb{N}}\sum\limits_{i\in S\cap I_t} y_i = \sum\limits_{t\in\mathbb{N}}\sum\limits_{i\in S\cap I_t} e_i$*; and*
2. $y_i \succ_i x_i$ *for all* $i \in S$.

The **core** *of $\mathcal{E}$ is defined to be the set of all feasible allocations of $\mathcal{E}$ that cannot be blocked by any coalition.*

Definition 3.2. *A coalition S is said to* **reject** *an allocation x with a feasible allocation y if $\exists\, S_1 \subseteq S$ and $\exists\, S_2 \subseteq S$ such that the following hold:*

1. $S_1 \cap S_2 = \emptyset$, $S_1 \cup S_2 = S$;
2. $\sum\limits_{t\in\mathbb{N}}\sum\limits_{i\in S\cap I_t} y_i = \sum\limits_{t\in\mathbb{N}}\sum\limits_{i\in S_1\cap I_t} x_i + \sum\limits_{t\in\mathbb{N}}\sum\limits_{i\in S_2\cap I_t} e_i$;
3. $y_i \succ_i x_i$ *for all* $i \in S$*; and*
4. $y_i \succsim_i e_i$ *for all* $i \in \bigcup\limits_{t\geq 1} I_t \setminus S$.

Definition 3.3. *The* **finite rejective core** *of $\mathcal{E}$ is the set of all such feasible allocations of $\mathcal{E}$ which cannot be rejected by any* **finite** *coalition of $\mathcal{E}$. We denote by $\mathscr{F}$ the finite rejective core of the economy $\mathcal{E}$. We denote by $\mathscr{F}^n$ the finite rejective core of the replicated economy $\mathcal{E}^n(e)$.*

Definition 3.4. *The* **finite core** *of the re-negotiation economy $\mathcal{E}^m(e)\bigoplus\mathcal{E}^n(x)$ is the set of all feasible allocations of $\mathcal{E}^m(e)\bigoplus\mathcal{E}^n(x)$ which cannot be blocked by any* **finite** *coalition of the same economy. It is denoted by $\mathscr{L}(m,n)$.*

Definition 3.5. *A* **dividend equilibrium allocation** *is a feasible allocation $x = (x_i)_{i\in\bigcup\limits_{t\geq 1} I_t}$ if there exists*

1. *an element $d \in \mathbb{R}_+^\infty$ such that $d = (d_i)_{i\in\bigcup\limits_{t\geq 1} I_t}$, where $d_i \in \mathbb{R}_+$; and*

2. *an element $p \in \mathbb{R}^{K(1)} \times \prod\limits_{t>1} \mathbb{R}^{K(t)\setminus K(t-1)}$.*

such that x_i is a maximal element of the dividend budget set of agent $i \in I_t$:

$$\mathbb{B}(p, e_i, d_i) = \{z_i \in X_{t+} | \Pr_{\mathbb{R}^{K_t}}[p].z_i \leq \Pr_{\mathbb{R}^{K_t}}[p].e_i + d_i\}$$

The set of all dividend equilibria allocations is denoted by $\mathscr{D}$[11]

Definition 3.6. *A* **dividend equilibrium** *is the tuple (x, p, d) where (x, p, d) satisfy the requirements of the Definition 3.5*

Note: Urai and Murakami, 2016 consider a monetary equilibrium which involves the transfer of money through generations. Since we consider the transfer of dividends through the generations, our definition of dividend equilibrium is quite analogous to theirs. Therefore, we stick with the terminology of dividend equilibrium rather than monetary equilibrium to emphasize this fact.

Note: It is worth noting that definition of equilibrium concept in the double infinity economy is a straightforward adaptation of the concept of equilibrium in the finite economy. However, things do not remain as straightforward insofar as definition of renegotiation core or rejective core is concerned. In these definitions, we emphasise that blocking (rejection) be done by a finite coalition. These definitions are adaptations of definition of renegotiation core used by Urai and Murakami, 2016.

3.2 Renegotiation and the Equivalence Theorem

The renegotiation economy has been defined analogously as in Section 2. We now restate the Proposition 1 as follows and note that this proposition will hold irrespective of divisibility of commodities.

Proposition 4. *Let x^{m+n} denote a finite rejective core allocation for replica economy $\mathcal{E}^{m+n}(e)$ for some $m \geq 1$ and $n \geq 0$. Then, x^{m+n} is a finite renegotiation core allocation for the economy $\mathcal{E}^m(e) \bigoplus \mathcal{E}^n(x)$. In other words, $\mathscr{F}^{m+n} \subseteq \mathscr{L}(m, n)$.*

Proof. Suppose not and let S be a finite coalition in $\mathcal{E}^m(e) \bigoplus \mathcal{E}^n(x)$ for some m, n which blocks x^{m+n} with some y. Now, $S = S_1 \cup S_2$, where $S_1 \subseteq \bigcup_{t \geq 1} I_t^m$ and $S_2 \subseteq \bigcup_{t \geq 1} I_t^n$, where I_t^m denotes the n replicas of agents in I_t with an endowment e_i and I_t^n denotes the n replicas of agents in I_t who have an endowment x_i.

Now, consider the following allocation z such that

1. $\forall\, i \in S_1 \cup S_2$, $z_i = y_i$
2. $\forall\, i \in I_t^m \setminus S_1$, $z_i = e_i$
3. $\forall\, i \in I_t^n \setminus S_2$, $z_i = x_i$

The coalition S then rejects allocation x^{m+n} with allocation z, which is a contradiction. □

[11]Henceforth, we shall write $\Pr_{\mathbb{R}^{K_t}}[p].z_i = p.z_i$.

Proposition 5. *For each $n \geq 1$, n^{th}-replica of any dividend equilibrium allocation of $\mathcal{E}$ belongs to the rejective core of the economy $\mathcal{E}^n(e)$.*

Proof. The proof mimics the proof of Proposition 2 and has been omitted to avoid repetition. □

Note: The use of finite cores in this framework arises from the fact that if infinite coalitions are allowed, then, $\sum_{i \in S} p.x_i$ may become infinite, which will not result in the desired contraction showing that $\mathscr{D} \subseteq \mathscr{F}$.

Proposition 6. *Any allocation x of $\mathcal{E}$ whose $(m+n)^{\text{th}}$-replica belongs to the core of the $(m+n)$-re-negotiation economy for all $m \geq 1$ and $n \geq 0$ is a dividend equilibrium allocation of $\mathcal{E}$.*

Proof. Let x be a feasible allocation of $\mathcal{E}$ such that for any $m \geq 1$ and $n \geq 0$, $x^{m+n} \in \mathscr{L}(m,n)$ and let t be any given time period. For any time period t, we denote the satiated agents by I_t^S, $I_t^S \subseteq I_t$.

Now, for any agents $i \in I_s \setminus I_s^S$, $1 \leq s \leq t$, we define the sets[12]:

$$\Gamma_i^1(s) := \mathrm{Co}(\succ_i(x_i) - x_i) = Co(\{z_i^1 \in \mathbb{R}^{D_s} \times \mathbb{Z}^{N_s} | z_i^1 + x_i \succ_i x_i\})$$

$$\Gamma_i^2(s) := \mathrm{Co}(\succ_i(x_i) - e_i) = Co(\{z_i^2 \in \mathbb{R}^{D_s} \times \mathbb{Z}^{N_s} | z_i^2 + e_i \succ_i x_i\})$$

We define the convex hull of $\Gamma_i^1(s) \cup \Gamma_i^2(s)$, $\Gamma_i(s) := \mathrm{Co}(\Gamma_i^1(s) \cup \Gamma_i^2(s))$.

$$\Gamma_i(s) = \left\{ z \in \mathbb{R}^{K_s} \;\middle|\; \begin{array}{l} z = \beta^i.z_i^1 + (1-\beta^i).z_i^2 \\ 0 \leq \beta^i \leq 1 \\ z_i^1 \in \Gamma_i^1(s), z_i^2 \in \Gamma_i^2(s) \end{array} \right\}$$

[12] We allow the following abuse of notation:

1. While defining the vectors z_i^1 and z_i^2, we implicitly consider that for an agent $i \in I_s$, $x_i \in X_{t+}$
2. We denote $\mathbb{R}^{D_1} \times \mathbb{R}^{N_1} \times \prod_{s>1}^{t} (\mathbb{R}^{D_s \setminus D(s-1)} \times \mathbb{R}^{N_s \setminus N(s-1)})$ by Y_t. While constructing the convex hulls $\Gamma_1(t)$, $\Gamma_2(t)$ and $\Gamma(t)$, we extend the vectors z_i^1, z_i^2 and x_i in such a way that if $i \in I_s, 1 \leq s \leq t$, then:
   - $z_i^1 \in Y_t \approx \mathbb{R}^{K(t)}$
   - $z_i^2 \in Y_t \approx \mathbb{R}^{K(t)}$
   - $x_i \in \mathbb{R}_+^{D_1} \times \mathbb{Z}_+^{N_1} \times \prod_{s>1}^{t} (\mathbb{R}_+^{D_s \setminus D(s-1)} \times \mathbb{Z}_+^{N_s \setminus N(s-1)}) \subseteq Y_t \approx \mathbb{R}^{K(t)}$

   by setting all components of z_i^1, z_i^2 and x_i corresponding to $K(t) \setminus K_s$ to be identically 0. In other words, $\mathrm{Pr}_{\mathbb{R}^{K(t) \setminus K_s}}[z_i^1] \equiv 0$, $\mathrm{Pr}_{\mathbb{R}^{K(t) \setminus K_s}}[z_i^2] \equiv 0$ and $\mathrm{Pr}_{\mathbb{R}^{K(t) \setminus K_s}}[x_i] \equiv 0$.

We take the convex hull of the finite union of all such $\Gamma_i(s)$ for $1 \leq s \leq t$. $\Gamma(t) := \mathrm{Co}(\bigcup_{1\leq s\leq t} \bigcup_{i\in(I_s\setminus I_s^S)} \Gamma_i(s))$.

$$\Gamma(t) = \left\{ z \in \mathbb{R}^{K(t)} \;\middle|\; \begin{array}{l} z = \sum_{i\in\bigcup_{1\leq s\leq t} I_s\setminus I_s^S} \alpha^i.(\beta^i.z_i^1 + (1-\beta^i).z_i^2) \\ 0 \leq \alpha^i \leq 1 \\ \sum_{i\in\bigcup_{1\leq s\leq t} I_s\setminus I_s^S} \alpha^i = 1 \\ z_i^1 \in \Gamma_i^1(s), z_i^2 \in \Gamma_i^2(s) \end{array} \right\}$$

Claim 6.1. $\Gamma(t) \cap \mathbb{R}_{--}^{K(t)} = \emptyset \; \forall \, t$.[13]

Since $\Gamma(t) \cap \mathbb{R}_{--}^{K(t)} = \emptyset \; \forall \, t$; therefore, for any time period t there is a vector $p(t) \neq 0$, $p(t) \in \mathbb{R}_+^{K(t)}$ such that if $i \in I_s \setminus I_s^S$ for any $s = 1, \cdots, t$, then $p(t).z_i^1 \geq 0$ if $z_i^1 \in \Gamma_i^1(s)$ and $p(t).z_i^2 \geq 0$ if $z_i^2 \in \Gamma_i^2(s)$[14]. In particular, for any $s = 1, \cdots, t$, for all agents $i \in I_s \setminus I_s^S$ and $\forall \, y_i \succ_i x_i$, we have $p(t).y_i \geq p(t).x_i$ and $p(t).y_i \geq p(t).e_i$.

We note that $\mathbb{R}_+^{K(t)}$ is homeomorphic to $\mathbb{R}_+^{K_1} \times \prod_{s>1}^{t} \mathbb{R}_+^{K_s\setminus K(s-1)}$ under the identity function and use this fact to write $p(t) \in \mathbb{R}_+^{K_1} \times \prod_{s>1}^{t} \mathbb{R}_+^{K_s\setminus K(s-1)}$. Let $\Omega(t)$ be the set of all such $p(t)$. It is trivially seen that $\Omega(t)$ is closed and $\Omega(t+1) \subseteq \Omega(t) \times \mathbb{R}_+^{K_{t+1}\setminus K(t)}$.

Claim 6.2. *Let* $p(t) \in \Omega(t)$ *for any time period* t*. Then* $\mathrm{Pr}_{\mathbb{R}^{D_t}}[p(t)] >> 0$.

Proof. **Case 1**: Let $N_t = \emptyset$. Since $p(t).x_b \geq p(t).e_b > 0$, there is some commodity $l \in D_t$ such that $p(t)^l > 0$ and $x_b^l > 0$.

Case 2: Let $N_t \neq \emptyset$. Then by assumption A.3.5 and the fact that $e_i >> 0 \; \forall \, i \in I_t$, there is a divisible good $l \in D_1$ such that $p(t)^l > 0$[15]. Moreover, by assumption A.3.6 and the fact that $x_b \succsim_b e_b$, we have $x_b^k > 0$ for all $k \in D_t$.

Thus, in either case, there is always a commodity $l \in D_t$ such that $p(t)^l > 0$ and $x_b^l > 0$. Suppose price of a divisible commodity $k \in D_t$ is 0. Then, by continuity of preferences, there is some $\lambda^l \in (0, x_b^l)$ such that $x_b + e^k - \lambda^l.e^l \succ_b x_b$ and $p(t).x_b > p(t).(x_b + e^k - \lambda^l.e^l)$. This is a contradiction to the separating hyperplane argument following Claim 6.1. □

[13]Proof of this claim is a minor modification of Claim 3.1. This can be seen as follows: for any $i \in I_s \setminus I_s^S$, z_i^1 or z_i^2 will have value of 0 on components corresponding to all those commodities which do not exist during time period s. Since $\mathbb{Q}^{K(t)}$ is dense in $\mathbb{R}^{K(t)}$, imposition of this restriction makes no major difference.

Consider the renegotiation economy $\mathcal{E}^{R_t.L_t}(e) \bigoplus \mathcal{E}^{R_t.L_t+1}(x)$,where R_t and L_t are constructed analogous to Section 2. The distribution of consumption bundles across replicas will also be analogously defined. Further, for all $s > t$, for any agent $i \in I_s$, $R_t.L_t + 1$ replicas of i consume x_i and $R_t.L_t$ replicas consume e_i. This construction satisfies all the conditions required to prove the claim. Thus, all the arguments of Claim 3.1 go through. We omit the proof to avoid repetition.

[14]The inner product is taken in the sense of footnote for Definition 3.5.

[15]If not, then by A.3.5, $\exists \, y_b \succ_b x_b$ such that $p(t).y_b = 0$. However, $p.e_i > 0$ for every agent $i \in I_t$.

We define the following sets:

$$\Phi(1) := \Omega(1) \cap \{x \in \mathbb{R}^{K(1)} | \|x\| = 1\}$$

$$\Phi(t) := \{p(t) \in \Omega(t) | \Pr_{\mathbb{R}^{K(1)}}[p(t)] \in \Phi(1)\}$$

It is trivially seen that $\Phi(t)$ is closed for each t.

Claim 6.3. $\Phi(t) \subseteq \mathbb{R}^{K(1)} \times \prod_{s>1}^{s\leq t} \mathbb{R}^{K(s)\setminus K(s-1)}$ *is compact. Further,* $\forall\, t$, $\exists\, \delta(t) > 0$ *such that* $\Phi(t) \subseteq \Phi(t-1) \times [0, \delta(t)]^{K(t)\setminus K(t-1)}$.

Proof. Since $\Phi(t)$ is closed, it suffices to prove that $\Phi(t)$ is bounded.

Now, to show that $\Phi(t)$ is bounded, we argue as follows. We first show that in time period 2, prices of all commodities are bounded above. Then we repeat this process of constructing the upper bounds on prices in all time periods till t by analogous arguments.

For time period 2 and consider a commodity k in the same period. There are three possibilities.

1. $k \in D_2$ and agent b (the agent with strictly monotone preferences) consumes k, that is, $x_b^k > 0$.

2. $k \in N_2$ and agent b (the agent with strictly monotone preferences) consumes k, that is, $x_b^k > 0$.[16]

3. $k \in K_2$ and agent b (the agent with strictly monotone preferences) does not consumes k, that is, $x_b^k = 0$. Define $\Psi(2) := \{l \in K_2\, | x_b^l = 0\}$.[17]

Case 1: This case deals with imposition of an upper bound on prices of divisible commodities. By the individual rationality of the agent b and A.3.6, we have $D_2 \cap \Psi(2) = \emptyset$. Let $k \in D_2 \setminus K_1$. Let $s \in K_1 \cap K_2$. Consider the unit vectors $e^s \in \mathbb{R}^{K_2}$ and $e^k \in \mathbb{R}^{K_2}$ which have the value 1 on components corresponding to s^{th} and k^{th} commodities and 0 on every other component. Then, $\exists\ \lambda_b^k \in (0, x_b^k)$ such that $x_b + e^s - \lambda_b^k.e^k \succ_b x_b$. Since $p(2)^s < 1$, therefore $\lambda_b^k.p(2)^k < 1$, or $p(2)^k < \frac{1}{\lambda_b^k} = \delta_1^k(2)$. Define $\delta_1(2) := \max_{k\in D_2\setminus K_1} \delta_1^k(2)$. It is then seen by construction that $p(2)^k < \delta_1(2)$ for every commodity $k \in D_2 \setminus K_1$.

Case 2: This case deals with imposition of an upper bound on prices of indivisible commodities which are consumed by the agent b in time-period 2. Let e^{D_2} represent the vector with value 1 on components corresponding to all divisible commodities in period 2. A.3.5 and A.3.6 together imply that $\exists\ \mu > 0$ such that $x_b + \mu.e^{D_2} - e^k \succ_b x_b$. Then, $p(2).(x_b + \mu.e^{D_2} - e^k) \geq p(2).(x_b)$. This implies that $p(2)^k \leq \mu \sum_{s\in D_2} p(2)^s$, which in turn implies that:[18]

[16] If $N_2 = \emptyset$, we skip this case.
[17] If $\Psi(2) = \emptyset$, we skip this case.
[18] For any finite set S, denote by $\#S$ its cardinality.

$$p(2)^k < \mu\left(\#[K_1 \cap D_2] + \delta_1(2).\#[D_2 \setminus K_1]\right) = \delta_2^k(2)$$

Analogous to Case 1, we define $\delta_2(2) := \max_{k \in (N_2 \setminus K_1) \setminus \Psi(2)} \delta_2^k(2)$. By construction, it is seen that $p(2)^k < \delta_2(2)$ for every commodity $k \in (N_2 \setminus K_1) \setminus \Psi(2)$.

Case 3: This case deals with imposition of an upper bound on prices of all indivisible commodities which are not consumed by the agent b in time-period 2. It is seen from continuity of preferences that $p(2).x_b \geq p(2).e_b$. Non-negativity of prices implies that:

$$p(2)^k.e_b^k \leq \sum_{l \in \Psi(2)} p(2)^l.e_b^l \leq \sum_{s \in K_2 \setminus \Psi(2)} p(2)^s(x_b^s - e_b^s)$$

Define $\mu^s := |x_b^s - e_b^s|$ for all commodities $s \in K_2 \setminus \Psi(2)$.We note that μ^s's are bounded above because of feasibility of a rejective core allocation. Then:

$$p(2)^k < \frac{\sum_{s \in K_2 \setminus \Psi(2)} \max\{1, \delta_1(2), \delta_2(2)\}.\mu^s}{e_b^k} =: \delta_3^k(2).$$

As in previous cases, we analogously define $\delta_3(2) := \max_{k \in \Psi(2)} \delta_3^k(2)$. We now define $\delta(2) := \max\{\delta_1(2), \delta_2(2), \delta_3(2)\}$. Thus, it is clear that for any $q(2) \in \Omega(2)$, if $\mathrm{Pr}_{\mathbb{R}^{K(1)}}[q(2)] \in \Phi(1)$, then $\mathrm{Pr}_{\mathbb{R}^{K_2 \setminus K(1)}}[q(2)] \in [0, \delta(2)]^{K_2 \setminus K(1)}$. This, in turn implies that

$$\Phi(2) \subseteq \Phi(1) \times [0, \delta(2)]^{K_2 \setminus K(1)} \subseteq [0,1]^{K_1} \times [0, \delta(2)]^{K_2 \setminus K(1)}.$$

Thus, $\Phi(2)$ is non-empty, bounded and closed, and hence, non-empty compact subset of $\mathbb{R}^{K_1} \times \mathbb{R}^{K_2 \setminus K(1)}$.

We repeat this process for every time period $s \leq t$ to obtain $\delta(s)$[19]. By construction, $\mathrm{Pr}_{\mathbb{R}^{K_s \setminus K(s-1)}}[\Phi(s)] \subseteq [0, \delta(s)]^{K_s \setminus K(s-1)}$. This argument, coupled with the fact that $\Omega(t) \subseteq \Omega(t-1) \times \mathbb{R}_+^{K_t \setminus K(t-1)}$ completes the proof of the second part of the Claim. □

Now, for any time period t, we define the set $\mathscr{P}_t := \Phi(t) \times \prod_{s>t} [0, \delta(s)]^{K_s \setminus K(s-1)}$ and note that $\mathscr{P}_t \subseteq \mathbb{R}^{K_1} \times \prod_{s>1} \mathbb{R}^{K_s \setminus K(s-1)}$, where $\mathbb{R}^{K_1} \times \prod_{s>1} \mathbb{R}^{K_s \setminus K(s-1)}$ is endowed with product topology and is a Hausdorff space. By Claim 6.3 and Tychonoff Theorem, $\mathscr{P}_t$ is compact in product topology for each t and $\mathscr{P}_{t+1} \subseteq \mathscr{P}_t$. By the nested intersection property of compact sets, $\bigcap_{t \geq 1} \mathscr{P}_t \neq \emptyset$.

Now, let $p \in \bigcap_{t \geq 1} \mathscr{P}_t$. Define $d_i = \max\{0, p.x_i - p.e_i\}$ and $d = (d_i)_{i \in \bigcup_{t \geq 1} I_t}$. Then, by construction, $x_i \in \mathbb{B}_i(p, e_i, d_i)$. We now show the individual rationality under the price system p. Affordability of x_i follows from construction and hence, individual rationality follows trivially for the satiated agents.

It only remains to show that x_i maximizes the preference of agent i in the budget set for any non-satiated agent. Let $i \in I_s \setminus I_s^S$. Suppose that $y_i \succ_i x_i$ for some $i \in I_s \setminus I_s^S$

[19] If for any time period s, if either of N_s or $\Psi(s)$ are empty, we skip the corresponding arguments in Cases 2 and 3.

and $p.y_i = p.e_i$.[20] Define z_i by setting $\Pr_{\mathbb{R}^{N_s}}[z_i] = \Pr_{\mathbb{R}^{N_s}}[y_i]$ and $\Pr_{\mathbb{R}^{D_s}}[z_i] \leq \Pr_{\mathbb{R}^{D_s}}[y_i]$ and $\Pr_{\mathbb{R}^{D_s}}[z_i] \neq \Pr_{\mathbb{R}^{D_s}}[y_i]$ so that $z_i \succ_i x_i$. Continuity of preferences guarantees the existence of such z_i. This implies that $p.z_i < p.y_i = p.e_i$[21], which is a contradiction to the result obtained in Claim 6.1. Similarly, suppose that $y_i \succ_i x_i$ for some $i \in I_s \setminus I_s^S$ and $p.y_i = p.x_i$. Proceeding similarly, a contradiction is again obtained. These two contradictions together imply that $p.y > \max\{p.e_i, p.x_i\}$ or $p.y > p.e_i + d_i$. □

Theorem 3.1. *For any feasible allocation x of the economy $\mathcal{E}$, following statements are equivalent:*

1. $x^{m+n} \in \mathscr{F}^{m+n}$ *for all* $m \geq 1$ *and* $n \geq 0$.
2. $x^{m+n} \in \mathscr{L}(m, n)$ *for all* $m \geq 1$ *and* $n \geq 0$.
3. $x \in \mathscr{D}$

Proof. Follows directly from Propositions 4, 5, and 6. □

4 Concluding Remarks

Theorems 2.1 and 3.1 demonstrate the equivalence between rejective core of replica economy, cores of all renegotiation economies and the dividend equilibria allocations for a finite and double infinity economy respectively. We note that Section 3 directly implies the equivalence for finite number of time periods as well.

Our assumptions regarding the preferences of non-satiated agents are relatively quite weak. Our assumptions on presence of a few agents with preferences satisfying monotonicity is a departure from other literature on indivisible commodities, which impose such assumptions on all agents. Insofar as overriding desirability is concerned, we impose a weak version on all agents, while reserving the stronger version only for few agents. This approach makes our result quite general compared to other literature.

Further, the nature of assumptions in our model allows for presence of agents who may treat certain commodities as *bads*. In addition to markets with satiated agents and having indivisible commodities, this generalization also expands the class of markets that our paper models.

Section 2 of our paper is significant extension of Murakami and Urai, 2017 in that we have incorporated indivisibilities in a finite economy. This necessitated a completely different approach for formulating the proofs in this section.

Section 3 of our paper is bidirectional extension of Urai and Murakami, 2016 in the sense that we have incorporated satiation and indivisibilities in their model. Our approach to the problem at hand is quite different from Urai and Murakami, 2016. In

[20] A.3.7, coupled with individual rationality of agent i implies that if $y_i \succ_i x_i$ then $\Pr_{\mathbb{R}^{D_s}}[y_i] \geq 0$ and $\Pr_{\mathbb{R}^{D_s}}[y_i] \neq 0$.

[21] The first inequality follows from the fact that prices of all divisible commodities are strictly positive in all time periods.

order to account for indivisibilities and satiation, the proofs that we have given in this section are quite novel and differ significantly from Urai and Murakami, 2016.

Future work in this area may deal with dispensing with the strictly monotone agent (b' in Section 2 and b in Section 3). It will be an interesting result to see if the core equivalence still holds in the absence of this agent.

5 Appendix

In this section, we provide a proof of Claim 3.1, which is the heart of the proof of Proposition 3. Before that, we give an information discussion on issues of this proof.

Firstly we discuss the broad idea behind the proof. The proof rests upon contradiction, wherein we aim to construct a coalition which will block the allocation x if the Claim 3.1 happens to be false. Debreu and Scarf, 1963, Murakami and Urai, 2017 and Urai and Murakami, 2016 show that $0 \notin \Gamma$ (in the context of analogous definitions of Γ). However, the presence of indivisibilities in our framework prevent us from directly using the logic of Debreu and Scarf, 1963. This can be illustrated as follows. Suppose there is one divisible and one indivisible commodity and three agents. Further suppose that $\beta^1 = \beta^2 = \beta^3 = 1$. Then :

- $(\beta^1.z_1^1 + (1-\beta^1).z_1^2) = z_1^1 = (-6, 2) \in \Gamma_1$
- $(\beta^2.z_2^1 + (1-\beta^2).z_2^2) = z_2^1 = (-6, 2) \in \Gamma_2$
- $(\beta^3.z_3^1 + (1-\beta^3).z_3^2) = z_3^1 = (3, -1) \in \Gamma_3$

Consider the following α's:

- $\alpha^1 = \frac{1}{3} - \frac{1}{2\pi}$
- $\alpha^2 = \frac{1}{2\pi}$
- $\alpha^3 = \frac{2}{3}$

It is verified that $\sum_{i \in I^{NS}} \alpha^i.(\beta^i.z_i^1 + (1-\beta^i).z_i^2) = 0$. However if we aim to replicate our economy on the basis of denominators of α's, we can never reach a finite number of replications which provides the agents with integer quantity of consumption in indivisible commodity. Similar example can be constructed if z_i^1 or z_i^2 were to have irrational coordinates.

To get over this issue, we use the fact that $\forall\ L \in \mathbb{N}$, $\mathbb{Q}^L$ is a dense subset of $\mathbb{R}^L$. However, this leads to a different issue, which is defining the appropriate renegotiation economy. In other words, the issue is to identify m and n in the economy $\mathcal{E}^n(e) \bigoplus \mathcal{E}^m(x)$. This discussion in mind, we divide our proof in 2 broad steps.

1. In Step 1, we identify the number of replications required so that each replica can have a consumption which ensures the integer amounts of consumption of indivisible commodities. We then identify the appropriate renegotiation economy.

2. In Step 2, we identify the distribution of consumption bundles across replicas of each non-satiated agent. Our construction is such that different replicas of an agent may have different consumption bundles. For a given agent $i \in I^{NS}$, we construct a group of replicas, wherein members of the group (replicas of agent i) have different consumption bundles. We use this replication structure to identify the blocking coalition. We then follow up with a check for the feasibility, thereby establishing the desired contradiction.

Proof of Claim 3.1: Suppose Claim 3.1 does not hold. It follows that, for each $i \in I^{NS}$, there must exist $\alpha^i, \beta^i \in [0,1]$ with $\sum_{i \in I^{NS}} \alpha^i = 1$, $z_i^1 \in \Gamma_i^1 \cap \mathbb{Q}^K$ and $z_i^2 \in \Gamma_i^2 \cap \mathbb{Q}^K$ such that

$$\sum_{i \in I^{NS}} \alpha^i.(\beta^i.z_i^1 + (1-\beta^i).z_i^2) = -z$$

for some $z \in \mathbb{Q}^K_{++}$, where z_i^1 and z_i^2 are of the form:

$$z_i^1 = \sum_{j=1}^{J} \delta_1^{ij}.z_{i1}^j \text{ and } z_i^2 = \sum_{j=1}^{J} \delta_2^{ij}.z_{i2}^j$$

where $z_{i1}^j \in \succ_i (x_i) - x_i$ and $z_{i2}^j \in \succ_i (x_i) - e_i$; and δ_1^{ij}'s and δ_2^{ij}'s are positive rational numbers satisfying[22] $\sum_{j=1}^{J} \delta_1^{ij} = \sum_{j=1}^{J} \delta_2^{ij} = 1$ for all $i \in I^{NS}$. Rest of the proof is basically a construction of an appropriate re-negotiation economy in which x can be blocked by some coalition of this economy. We plan to divide it into three steps.

Step 1. *Finding an appropriate re-negotiation economy:* For any $i \in I^{NS}$, let

$$\alpha^i = \frac{a_i^\alpha}{b_i^\alpha}; \quad \beta^i = \frac{a_i^\beta}{b_i^\beta}; \quad \delta_1^{ij} = \frac{a_{ij}^{\delta_1}}{b_{ij}^{\delta_1}}; \quad \delta_2^{ij} = \frac{a_{ij}^{\delta_2}}{b_{ij}^{\delta_2}},$$

where, for α^i and β^i, the numerators are non-negative and denominators are natural numbers, while for δ_1^{ij} and δ_2^{ij}, both numerator and denominator are natural numbers. We define the sets $\mathfrak{X}_1$ and $\mathfrak{X}_2$ as follows:

$$\mathfrak{X}_1 = \left\{ i \in I^{NS} : \beta^i \neq 0 \right\} \text{ and } \mathfrak{X}_2 = \left\{ i \in I^{NS} : \beta^i \neq 1 \right\}.$$

Therefore, the set $\mathfrak{Z}$, defined by $\mathfrak{Z} := \{ i \in I^{NS} : \alpha^i > 0 \}$, is non-empty and $\mathfrak{Z} \subseteq \mathfrak{X}_1 \cup \mathfrak{X}_2$. Let $\mathfrak{I}_1 := \mathfrak{Z} \cap \mathfrak{X}_1$ and $\mathfrak{I}_2 := \mathfrak{Z} \cap \mathfrak{X}_2$. In what follows, we introduce some notations in order to define an appropriate replicated economy at the same time maintaining integer amount consumption corresponding to any indivisible commodity.

[22] Without loss of generality, we can choose same number of elements in the convex combinations for z_i^1 and z_i^2 for all $i \in I^{NS}$.

Table 1: *Integers defined for proof*

Integer	Definition	Integer	Definition
L_i^1	$\prod_{j=1}^{J} b_{ij}^{\delta_1}$	L_i^2	$\prod_{j=1}^{J} b_{ij}^{\delta_2}$
M_i^1	$a_i^\alpha.a_i^\beta.\prod_{m\neq i} b_m^\alpha.b_m^\beta$	M_i^2	$a_i^\alpha.(b_i^\beta - a_i^\beta).\prod_{m\neq i} b_m^\alpha.b_m^\beta$
L	$(\prod_{i\in\mathfrak{I}_1} L_i^1).(\prod_{i\in\mathfrak{I}_2} L_i^2)$	R	$\prod_{i\in I\setminus I^S} b_i^\alpha.b_i^\beta$

We consider the renegotiation economy $\mathcal{E}^{RL+1}(x)\bigoplus\mathcal{E}^{RL}(e)$. For any agent i, we index the replica of agent as (i,h), where $h = 1,\cdots,2RL+1$. For any replica of the agent i taken from $\mathcal{E}^{RL+1}(x)$, the index h runs from 1 to $RL+1$, and for any replica of the agent i taken from $\mathcal{E}^{RL}(e)$, the index h runs from $RL+2$ to $2RL+1$.

Step 2. *Constructing a blocking coalition:* Using the fact that $\delta_1^{ij} = \frac{a_{ij}^{\delta_1}}{b_{ij}^{\delta_1}}$ and $\delta_2^{ij} = \frac{a_{ij}^{\delta_2}}{b_{ij}^{\delta_2}}$, it can be easily verified that

$$L_i^1.z_i^1 = \sum_{j=1}^{J}\left[a_{ij}^{\delta_1}.\prod_{m\neq j} b_{im}^{\delta_1}.z_{i1}^j\right] \text{ and } L_i^2.z_i^2 = \sum_{j=1}^{J}\left[a_{ij}^{\delta_2}.\prod_{m\neq j} b_{im}^{\delta_2}.z_{i2}^j\right].$$

First, consider the replicas in $\mathcal{E}^{RL+1}(x)$.

1. If $i\in\mathfrak{I}_1$, for each $j = 1\cdots J$, $M_i^1.\frac{L}{L_i^1}.a_{ij}^{\delta_1}.\prod_{m\neq j} b_{im}^{\delta_1}$ replicas of i have the consumption bundle $y_{(i,h)} = x_i + z_{i1}^j$, while their endowment is x_i. We call the coalition of all such replicas for each $i\in\mathfrak{I}_1$ by S_1^1 (Table 2). Using our notation for replicas of agents, we say that for any agent $i\in\mathfrak{I}_1$, the replicas $(i,2),\cdots,(i,M_i^1.L+1)\in S_1^1$.

2. Consider one set of replicas of all agents in I^{NS} from $\mathcal{E}^{RL+1}(x)$ and denote it by $S_1^2 := \{(i,1) : i\in I^{NS}\}$. We now assign each agent $(i,1)$ the consumption bundle $y_{(i,1)} = x_i + w_i$, where $w_i\in X_+\setminus\{0\}$ satisfying $(x_i+w_i)\succ_i x_i$ and

$$\sum_{i\in S_1^2} w_i = -L.\left[\sum_{i\in\mathfrak{I}_1} M_i^1.z_i^1 + \sum_{i\in\mathfrak{I}_2} M_i^2.z_i^2\right].$$

   Due to the local non-satation at x_i for all $i\in I^{NS}$ and the monotonicity of agent b, such a collection of w_i's indeed exists.

Now consider the replicas in $\mathcal{E}^{RL}(e)$. If $i\in\mathfrak{I}_2$, for each $j = 1\cdots J$, $M_i^2.\frac{L}{L_i^2}.a_{ij}^{\delta_2}.\prod_{m\neq j} b_{im}^{\delta_2}$ replicas of i have the consumption bundle $y_{(i,h)} = e_i + z_{i2}^j$, while their endowment is e_i. Let S_2 be a coalition containing all these replicas for each $i\in\mathcal{I}_2$ (Table 3). Using our notation for replicas of agents, we say that for any agent $i\in\mathfrak{I}_2$, the replicas $(i,R.L+2),\cdots,(i,R.L+1+M_i^2.L)\in S_2$.

Table 2: *Consumption distribution of agent* $i \in \mathfrak{I}_1$

Value of j	**Replicas Required**	**Consumption Bundle**
$j = 1$	$M_i^1 \cdot \frac{L}{L_i^1} . a_{i1}^{\delta_1} \cdot \prod_{m \neq 1} b_{im}^{\delta_1}$	$x_i + z_{i1}^1$
$j = 2$	$M_i^1 \cdot \frac{L}{L_i^1} . a_{i2}^{\delta_1} \cdot \prod_{m \neq 2} b_{im}^{\delta_1}$	$x_i + z_{i1}^2$
$\vdots$	$\vdots$	$\vdots$
$j = j$	$M_i^1 \cdot \frac{L}{L_i^1} . a_{ij}^{\delta_1} \cdot \prod_{m \neq j} b_{im}^{\delta_1}$	$x_i + z_{i1}^j$
$\vdots$	$\vdots$	$\vdots$
$j = J$	$M_i^1 \cdot \frac{L}{L_i^1} . a_{iJ}^{\delta_1} \cdot \prod_{m \neq J} b_{im}^{\delta_1}$	$x_i + z_{i1}^J$

Table 3: *Consumption distribution of agent* $i \in \mathfrak{I}_2$

Value of j	**Replicas Required**	**Consumption Bundle**
$j = 1$	$M_i^2 \cdot \frac{L}{L_i^2} . a_{i1}^{\delta_2} \cdot \prod_{m \neq 1} b_{im}^{\delta_2}$	$e_i + z_{i2}^1$
$j = 2$	$M_i^2 \cdot \frac{L}{L_i^2} . a_{i2}^{\delta_2} \cdot \prod_{m \neq 2} b_{im}^{\delta_2}$	$e_i + z_{i2}^2$
$\vdots$	$\vdots$	$\vdots$
$j = j$	$M_i^2 \cdot \frac{L}{L_i^2} . a_{ij}^{\delta_2} \cdot \prod_{m \neq j} b_{im}^{\delta_2}$	$e_i + z_{i2}^j$
$\vdots$	$\vdots$	$\vdots$
$j = J$	$M_i^2 \cdot \frac{L}{L_i^2} . a_{iJ}^{\delta_2} \cdot \prod_{m \neq J} b_{im}^{\delta_2}$	$e_i + z_{i2}^J$

We are now ready to define the coalition S as $S = S_1^1 \cup S_1^2 \cup S_2$.

Step 3. *Effectiveness of a blocking coalition:* Let $i \in \mathfrak{I}_1$. Then, the aggregate of commodities consumed by replicas of i in S_1^1 is given by:

$$\sum_{(i,h) \in S_1^1} y_{(i,h)} = \sum_{j=1}^{J} M_i^1 \cdot \frac{L}{L_i^1} . a_{ij}^{\delta_1} \cdot \prod_{m \neq j} b_{im}^{\delta_1} . (x_i + z_{i1}^j)$$

For agents in S_1^2, the aggregate of commodities consumed is:

$$\sum_{(i,1) \in S_1^2} y_{(i,1)} = \sum_{(i,1) \in S_1^2} x_i - L. \left(\sum_{i \in \mathfrak{I}_1} M_i^1 . z_i^1 + \sum_{i \in \mathfrak{I}_2} M_i^2 . z_i^2 \right).$$

Analogously, for $i \in \mathfrak{I}_2$, the aggregate of commodities consumed by replicas of i in S_2 is given by:

$$\sum_{(i,h) \in S_2} y_{(i,h)} = \sum_{j=1}^{J} M_i^2 \cdot \frac{L}{L_i^2} . a_{ij}^{\delta_2} \cdot \prod_{m \neq j} b_{im}^{\delta_2} . (e_i + z_{i2}^j).$$

Summing up across all the replicas present in S_1^1 and S_2 and simplifying the notation by substituting the terms involving z_{i1}^j's and z_{i2}^j's in terms of z_i^1 and z_i^2, the aggregate of commodities consumed by members of $S_1^1 \cup S_2$ is:

$$\sum_{i \in \mathfrak{I}_1 \cup \mathfrak{I}_2} \sum_{(i,h) \in S} y_{(i,h)} = L.\left(\sum_{i \in \mathfrak{I}_1} M_i^1.x_i + \sum_{i \in \mathfrak{I}_2} M_i^2.e_i\right) + L.\left(\sum_{i \in \mathfrak{I}_1} M_i^1.z_i^1 + \sum_{i \in \mathfrak{I}_2} M_i^2.z_i^2\right).$$

Adding S_1^2 to complete the coalition S, we get the aggregate of commodities consumed by members of S to be:

$$\sum_{(i,1) \in S_1^2} x_i + L.\left(\sum_{i \in \mathfrak{I}_1} M_i^1.x_i + \sum_{i \in \mathfrak{I}_2} M_i^2.e_i\right).$$

which is that same as the aggregate of endowments of members of S. This completes the proof of effectiveness of the coalition S.

Combining Steps 1-3, we conclude that the coalition S blocks x^{2RL+1} in the renegotiation economy $\mathcal{E}^{RL+1}(x) \bigoplus \mathcal{E}^{RL}(e)$, which establishes the contradiction to x being to the core of any $(m+n)$-re-negotiation economy, for all $m \geq 1$ and $n \geq 0$. $\square$

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