Competition in Signaling

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Abstract

I study a multi-sender signaling game between an uninformed decision maker and two senders with common private information and opposed interests. Senders can misreport information at a cost that is tied to the size of the misrepresentation. The main results concern the amount of information that is transmitted in equilibrium and the language used by senders to convey such information. Fully revealing and pure strategy equilibria exist but are not plausible. I identify sufficient conditions under which equilibria always exist, are plausible, and essentially unique, and deliver a complete characterization of such equilibria. As an application, I study the informative value of different judicial procedures.

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Contents

1 Introduction

How and how much information is revealed when two equally informed senders with conflicting interests provide advice to a decision maker? When senders are well informed and misreporting is prohibitively expensive, the decision maker can "rely on the information of the interested parties" to always make the right choice.[1] However, there are many situations where information is not fully verifiable and it is possible to misreport it at a reasonable cost.[2] Intuition would suggest that, in these cases, the decision maker might obtain conflicting advice and make wrong choices as a result of being poorly informed.

On the applied front, this type of interaction is at the core of a large number of applications: during electoral campaigns, candidates competing for consensus provide voters with different accounts of the same facts; newspapers with opposed political leanings deliver conflicting and inaccurate news; prosecutors and defendants may tamper with evidence to persuade a jury; co-workers competing for a promotion may exaggerate their own contribution to a team project; advocacy groups use *amicus curiae* briefs to influence court cases, and methods used in lobbying against public health include "industry-funded research that confuses the evidence and keeps the public in doubt" (Chan, 2013).

I address the above questions with a costly signaling game between an uninformed decision maker and two senders with common information and conflicting goals. The two senders observe the realization of a random variable—the state—and then simultaneously or privately deliver a report to the decision maker. These reports are literal statements about the realized state. Senders can misreport such information, but to do so they incur "misreporting costs" that are increasing with the magnitude of misrepresentation. By contrast, reporting truthfully is costless. After observing the reports, the decision maker must select one of two alternatives, and each player obtains a payoff from the selected alternative that depends on the state. Every player finds the relative value of the two alternatives to be increasing with the state.[3]

Throughout the paper, I restrict attention to equilibria where the decision maker's posterior beliefs satisfy a first-order stochastic dominance condition with respect to the senders' reports. Under this restriction, reports claiming that the state takes strictly higher values cannot signal to the decision maker that the relative value of the two alternatives is strictly lower. This condition is natural given the type of strategic interaction considered here, where senders have opposed goals, reports are literal, and misreporting is costly. It imposes some sort of monotonicity over the senders' reporting strategies, and thus it

[1] See, e.g., P. Milgrom and Roberts (1986b).

[2] Misreporting information is a costly activity due to, e.g., the time and effort that is required to misrepresent information, or because misreporting generates an expected loss in reputation, credibility, and future influence. Misreporting is more difficult, and thus more costly, when information is harder.

[3] The state is a valence or vertical differentiation parameter, and can be thought of representing the relative quality of the two alternatives. Examples are leadership or competence in politicians, durability or product quality of commercial goods, and fit with the state of the word of policies.

is akin to restrictions that are widely used in many economic applications, such as in auction theory and in models of communication with lying costs.

The main results of this paper concern the amount of information that can be plausibly transmitted in equilibrium and the "language" used by senders to deliver such information. I first show that misreporting occurs in every equilibrium. Yet, there are "receiver-efficient" equilibria where the decision maker obtains enough information to always select her preferred alternative *as if* fully informed. In spite of senders' misreporting behavior, the decision maker might even end up obtaining more information than what she needs. All these equilibria, while important for this analysis, turn out to be unreasonable.

I show that all receiver-efficient and fully revealing equilibria rely on an ad-hoc choice of beliefs that have implausible discontinuities to discourage deviations. I identify two well-known refinements that eliminate such equilibria: unprejudiced beliefs (Bagwell & Ramey, 1991) and ε-robsutness (Battaglini, 2002).[4] A similar fate is met by pure strategy equilibria, as I show that they are all receiver-efficient and thus unreasonable. This result motivates the search for mixed strategy equilibria that are robust to such refinements.

The analysis of equilibria in mixed strategies is, however, a daunting task: a notorious problem of signaling games is that they typically yield a wealth of equilibria, and here this issue is exacerbated by the presence of multiple senders and of rich state and signal spaces. Canonical refinements based on the notion of strategic stability (Kohlberg & Mertens, 1986) are of little help, as they are developed for settings with a single sender. I thus proceed by drawing on the implausibility of receiver-efficient and pure strategy equilibria to introduce reasonable restrictions on the decision maker's posterior beliefs.

More specifically, I focus the subsequent analysis on equilibria that satisfy two additional conditions on the posterior beliefs of the decision maker: the first one is a strong form of first-order stochastic dominance which requires that conflicting reports claiming a strictly higher state must signal that the relative value of the two alternatives is, in expectation, strictly higher; the second is a dominance condition under which the decision maker excludes the possibility that senders may deliver reports that are equilibrium dominated.[5] I refer to equilibria satisfying these two conditions as "direct equilibria," as they feature reports which are direct signals of the realized state.

I provide a complete characterization of direct equilibria, and show that they possess desirable properties: they always exists, they are essentially unique, and they survive the refinement criteria that break down fully revealing, receiver-efficient, and pure strategy equilibria. The two conditions imposed by direct equilibria, even though relatively natural and mild, are therefore sufficient to ensure robustness and uniqueness while preserving existence.

[4] See Section 4 for a formal definition of unprejudiced beliefs and ε-robustness. I show that these two refinements are tightly connected: equilibria that are ε-robust must have unprejudiced beliefs (Lemma 3). This result suggests a novel rationale for the use of ε-robustness in multi-sender communication games.

[5] See Definition 4 in Section 5 for a complete and formal statement of these two conditions.

In direct equilibria, the transmission of information is qualitatively different than in comparable models of strategic communication. There is neither "babbling" nor full revelation, in contrast with predictions advanced by related models of cheap talk and verifiable disclosure, respectively. By contrast, "revelation" is a probabilistic phenomenon in the sense that the decision maker fully learns almost every state with some positive probability. Full revelation is more likely to occur in extreme states, while it is relatively unlikely in intermediate states. There are extreme states in which both senders always truthfully reveal the state to the decision maker even though they have opposed goals.

Senders' equilibrium behavior is mixed, as they always report the truth with some positive probability, and they misreport otherwise. Therefore, in (almost) every state the two senders may deliver exactly the same truthful report even though they have conflicting interests. They might also end up delivering different reports that however imply the same recommended action to the decision maker. Whenever one of these two events takes place, the decision maker fully learns the realized state. In the former case full revelation occurs without wasteful signaling expenditures, while the latter case requires a sender to engage in costly misreporting. This is in contrast with previous results in multi-sender signaling games, where full revelation is either always inefficient (Emons & Fluet, 2009) or it is always efficient (Bagwell & Ramey, 1991).[6]

Conditional on misreporting, senders deliver reports in a convex set, and no particular misrepresentation in such set is delivered with strictly positive probability. The misreporting behavior of each sender is directly determined by the feature of its opponent, such as the opponent's costs structure and payoff function, and it is determined only indirectly by its own features. Upon observing two conflicting reports recommending different actions, the decision maker understands that "the truth is somewhere in between" and that at least one of the two senders is misreporting. The decision maker cross-validates reports and allocates the burden of proof across senders by accounting for their characteristics.

The setting studied in the main part of the paper allows for a large number of asymmetries. I also analyze the specific case where senders have a similar payoff and cost structure, and where the distribution of the state is such that no sender is ex-ante advantaged in any way. In this "symmetric environment," I provide a closed-form solution to direct equilibria and show that they naturally display symmetric strategies. The decision maker equally allocates the burden of proof among senders by following the recommendation of the sender delivering the most extreme report.[7] The senders' misreporting behavior depends on the shape of the common cost function: with convex costs, senders are more likely to deliver large misrepresentations of the state rather than

[6] Signaling games with a single sender typically have inefficient separating equailibria. See for example Spence (1973), P. Milgrom and Roberts (1982, 1986a), Kartik (2009), Kartik, Ottaviani, and Squintani (2007).

[7] This result is reminiscent of equilibria in the all-pay auction with complete information, where the prize is assigned to the player submitting the highest bid (Baye, Kovenock, & De Vries, 1996).

small lies, while the opposite is true for concave misreporting costs.

As a brief application, I use insights from the analysis of direct equilibria to study the informational value of different judicial systems. Shin (1998) shows that, when information is fully verifiable, the adversarial judicial procedure is always superior to the inquisitorial procedure. However, Shin (1998) also conjectures that such sharp result may crucially depend on the assumption of verifiability. I show that, when information is not fully verifiable, then the inquisitorial procedure may indeed be superior than the adversarial procedure, thus proving the above conjecture to be correct.

The remainder of this article is organized as follows. In Section 2, I discuss the related literature. Section 3 introduces the model, which I solve in Section 4 and 5. In Section 6, I provide an example and an application. Finally, Section 7 concludes. Formal proofs are relegated to Appendix A.

2 Related Literature

This paper contributes to different strands of literature. First, it relates to models of strategic communication with multiple senders. This line of work shows several channels through which full information revelation can be obtained (Battaglini, 2002; Krishna & Morgan, 2001; P. Milgrom & Roberts, 1986b). Papers in this literature typically assume that misreporting is either costless (cheap talk) or impossible (verifiable disclosure). By contrast, in this article misreporting is possible at a cost that depends on the magnitude of misrepresentation. Under this modelling specification, I show that fully revealing equilibria exist but are not plausible.

Therefore, this paper relates to models of strategic communication with misreporting costs (Chen, 2011; Chen, Kartik, & Sobel, 2008; Kartik, 2009; Kartik et al., 2007; Ottaviani & Squintani, 2006). All these papers are concerned with the single-sender case, while I consider a multi-sender setting. An exception is Dziuda and Salas (2018), where they study a communication game with endogenous lying costs and consider a case with two senders.

The introduction of misreporting costs makes this a costly signaling model. Therefore, this paper contributes to the literature of multi-sender signaling with perfectly correlated types, but it differs from this line of work in a number of ways. First, in my model the messages or signals of senders have the only role of transmitting information, and thus do not directly affect how players value each alternative. This is not the case, e.g., in related models of limit entry (Bagwell & Ramey, 1991; Schultz, 1996), price competition (Bester & Demuth, 2015; Fluet & Garella, 2002; Hertzendorf & Overgaard, 2001; Yehezkel, 2008), and public good provision (Schultz, 1996).[8] Second, I model a setting where the signals of

[8]For example, in these models firms may signal quality through prices, which affect market demand and thus profits. Some of these papers also study signaling by both pricing and advertising together.

senders are fully observable.[9] By contrast, in the entry deterrence models of Harrington (1987) and Orzach and Tauman (1996), incumbent firms simultaneously select their own pre-entry output, but the entrant can observe only the resulting market price.

A key feature of the model analyzed in this paper is that both senders pay their own signaling costs independently of the decision maker's choice. This *all-pay* feature is missing in related multi-sender signaling models of electoral competition (Banks, 1990; Callander & Wilkie, 2007), where only the elected candidate incurs the signaling cost.[10]

The type of strategic interaction and competition that is analyzed in this article is reminiscent of and closely related to all-pay contest models, where contestants compete for a prize by simultaneously delivering costly scores or bids (Baye et al., 1996; Siegel, 2009). In these papers, the mapping from signals or scores to outcomes is exogenously determined by a contest success function. For example, Skaperdas and Vaidya (2012) study persuasion by contending parties as an all-pay contest. The paper studied here differs from this literature in that the decision maker is a strategic actor whose choice is endogenously determined as a part of an equilibrium. Similarly, Gul and Pesendorfer (2012) study political contests were two parties with opposing interests provide costly payoff-relevant signals to a strategic voter. However, in their model only one party incurs a cost at each moment, and parties cannot distort information.

Finally, this paper is also connected to work studying adversarial procedures (Dewatripont & Tirole, 1999; Shin, 1998). Differently than this line of work, I consider a model where information is not fully verifiable. In this regard, Emons and Fluet (2009) constitute an exception. However, they consider a setting with a continuum of types, signals, and receiver's actions, which yields only fully revealing equilibria.

3 The Model

Set-up and timeline. There are three players: two informed senders (1 and 2) and one uninformed decision maker (dm). Let $\theta \in \Theta \subseteq \mathbb{R}$ be the underlying state, distributed according to the full support probability density function f. After observing the realized state θ, each of the two senders simultaneously or privately deliver to the decision maker a report $r_j \in R_j$, where r_j is a report by sender j and R_j is the report space of sender j. The decision maker, after observing the pair of reports (r_1, r_2) but not the state θ, selects an alternative $a \in \{\oplus, \ominus\}$.

Payoffs. Player $i \in \{1, 2, dm\}$ obtains a payoff of $u_i(a, \theta)$ if the decision maker selects

[9] Signals are not fully observable if, e.g., they are aggregated into a single score and the receiver can observe only such score, but cannot observe each individual signal.

[10] These papers also differs from my model in that they consider settings where senders do not have common information. Similarly, Mailath (1989) and Daughety and Reinganum (2007) study price signaling and Honryo (2018) studies risk shifts in settings with imperfectly correlated types. My model should be seen as complementary to this line of work.

alternative a in state θ. I normalize $u_i(\ominus, \theta) = 0$ for all $\theta \in \Theta$ and denote by $u_i(\theta) \equiv u_i(\oplus, \theta)$, where $u_i(\theta)$ is weakly increasing in θ. The decision maker's expected utility from selecting $\oplus$ given the senders' reports is $U_{dm}(r_1, r_2)$. Thus, the state θ is an element of vertical differentiation or valence component over which players share a common preference, and it is interpreted as the relative quality of alternative $\oplus$ with respect to alternative $\ominus$. I may refer to the state θ also as the senders' "type."

Misreporting costs. Sender j bears a cost $k_j C_j(r_j, \theta)$ for delivering report r_j when the state is θ. The cost function $C_j(r_j, \theta) \geq 0$ is continuous and such that, for every $\theta \in \Theta$ and $j \in \{1, 2\}$, we have that $C_j(\theta, \theta) = 0$ and

$$\text{if } \ r_j \gtrless \theta, \ \text{ then } \ \frac{dC_j(r_j, \theta)}{dr_j} \gtrless 0 \gtrless \frac{dC_j(r_j, \theta)}{d\theta}.$$

The scalar $k_j > 0$ is a finite parameter measuring the intensity of misreporting costs. Therefore, misreporting is increasingly costly with the magnitude of misrepresentation, while truthful reporting is always costless. Sender j's total utility is

$$w_j(r_j, \theta, a) = \mathbb{1}\{a = \oplus\} u_j(\theta) - k_j C_j(r_j, \theta),$$

where $\mathbb{1}\{\cdot\}$ is the indicator function. It follows that, conditional on the decision maker's eventual choice, both senders prefer to deliver reports that are closer to the truth.

Definitions and assumptions. I assume that the state space and the report spaces are the same, i.e., $R_1 = R_2 = \Theta$. Thus, a generic report r has the literal or exogenous meaning "The state is $\theta = r$." I say that sender j reports truthfully when $r_j = \theta$, and misreports otherwise. I sometimes use $-j$ to denote the sender other than sender j.

I define the "threshold" τ_i as the state in which player i is indifferent between the two alternatives. Formally, $\tau_i := \{\theta \in \Theta | u_i(\theta) = 0\}$. I assume that utilities $u_i(\theta)$ are such that τ_i exists and is unique[11] for every $i \in \{1, 2, dm\}$. The threshold τ_i tells us that player i prefers $\oplus$ over $\ominus$ when the state θ is greater than τ_i. Throughout the paper, I consider the case where senders have opposing biases, i.e., $\tau_1 < \tau_{dm} < \tau_2$. To make the problem non-trivial I let $\tau_{dm} \in \Theta$, and I normalize $\tau_{dm} = 0$. Therefore, the decision maker prefers to select the positive alternative $\oplus$ when the state θ takes positive values, and prefers to select the negative alternative $\ominus$ when the state is negative. I assume that when the decision maker is indifferent between the two alternatives at given beliefs, she selects $\oplus$.

I define the "reach" of sender j in state θ as the report which associated misreporting costs offset j's gains from having its own preferred alternative eventually selected. Formally,

[11] These assumptions are for notational convenience. The model can accommodate for senders that always strictly prefer one alternative over the other and for utility functions such that $u_i(\theta) \neq 0$ for every $\theta \in \Theta$, including step utility functions.

the "upper reach" $\bar{r}_j(\theta) \geq \theta$ of sender j in state θ is defined as

$$\bar{r}_j(\theta) := \max\left\{r \in \mathbb{R} \mid (-1)^{\mathbb{1}\{\theta<\tau_j\}} u_j(\theta) = k_j C_j(r, \theta)\right\}. \tag{1}$$

Similarly, the "lower reach" $\underline{r}_j(\theta) \leq \theta$ of sender j in state θ is defined as

$$\underline{r}_j(\theta) := \min\left\{r \in \mathbb{R} \mid (-1)^{\mathbb{1}\{\theta<\tau_j\}} u_j(\theta) = k_j C_j(r, \theta)\right\}. \tag{2}$$

I will sometimes use the "inverse reaches" $\bar{r}_1^{-1}(r_1)$ and $\underline{r}_2^{-1}(r_2)$, where $\bar{r}_j^{-1}(\cdot)$ and $\underline{r}_j^{-1}(\cdot)$ map from R_j to Θ and are defined as the inverse functions of $\bar{r}_j(\theta)$ and $\underline{r}_j(\theta)$, respectively.

I assume that the state and report spaces are large enough, that is,

$$\Theta \supseteq \hat{R} := [\underline{r}_2(0), \bar{r}_1(0)].$$

This assumption ensures that the information senders can transmit is not artificially bounded by restrictions in the reports that they can deliver.

Strategies. A pure strategy for sender j is a function $\rho_j : \Theta \to R_j$ such that $\rho_j(\theta)$ is the report delivered by sender j in state θ. A mixed strategy for sender j is a mixed probability measure $\phi_j : \Theta \to \Delta(R_j)$, where $\phi_j(r_j, \theta)$ is the mixed probability density that $\phi_j(\theta)$ assigns to a report $r_j \in R_j$. I denote by $S_j(\theta)$ the support of sender j's strategy in state θ. Section 5 introduces additional notation that is required to study equilibria in mixed strategies.

I say that a pair of reports (r_1, r_2) is off-path if, given the senders' strategies, (r_1, r_2) will never be observed by the decision maker. Otherwise, I say that the pair (r_1, r_2) is on-path. A posterior beliefs function for the decision maker is a mapping $p : R_1 \times R_2 \to \Delta(\Theta)$ which, given any pair of reports (r_1, r_2), generates posterior beliefs $p(\theta|r_1, r_2)$ with CDF $P(\theta|r_1, r_2)$. Given a pair of reports (r_1, r_2) and posterior beliefs $p(\theta|r_1, r_2)$, the decision maker selects an alternative in the sequentially rational set $\beta(r_1, r_2)$, where

$$\beta(r_1, r_2) = \underset{a \in \{\oplus, \ominus\}}{\arg\max}\, \mathbb{E}_p\left[u_{dm}(a, \theta)|r_1, r_2\right].$$

As mentioned before, if $p(\theta|r_1, r_2)$ is such that $U_{dm}(r_1, r_2) = 0$, then $\beta(r_1, r_2) = \oplus$.

Solution concept. The solution concept is perfect Bayesian equilibrium (PBE).[12] Throughout the paper, I restrict attention to equilibria where beliefs p satisfy the following first-order stochastic dominance condition: for every $r_j \geq r_j'$ and $j \in \{1, 2\}$,

$$U_{dm}(r_1, r_2) \geq U_{dm}(r_1', r_2'). \tag{FOSD}$$

Condition (FOSD) says that a higher report cannot signal to the decision maker a lower

[12]For a textbook definition of perfect Bayesian equilibrium, see Fudenberg and Tirole (1991).

expected utility from selecting alternative $\oplus$.[13] A focus on these equilibria is natural given that the value of $\oplus$ is increasing in the state, reports are literal, and misreporting is costly.

Since in equilibrium the decision maker has correct beliefs, imposing conditions on p has consequences over the senders' equilibrium reporting behavior. An immediate implication of (FOSD) is that senders play strategies that satisfy some sort of monotonicity condition: in every equilibrium, a sender that prefers alternative $\oplus$ over $\odot$ is never going to deliver a report that is strictly lower than the actual realized value of θ. The next lemma formalizes this result.

Lemma 1. *In every perfect Bayesian equilibrium satisfying* (FOSD)*,* $\min S_j(\theta) \geq \theta$ *for* $\theta \geq \tau_j$ *and* $\max S_j(\theta) \leq \theta$ *otherwise,* $j \in \{1, 2\}$.

Lemma 1 shows how (FOSD) is akin to assumptions that are widely used in many economic applications, such as the monotone bidding strategies in auction theory (e.g., Wilson (1977)), the monotone likelihood ratio property in signal distributions (e.g., P. R. Milgrom (1981)), and the message monotonicity in related communication games (e.g., Kartik (2009)). To study mixed strategy equilibria, I will use a stronger version of (FOSD) coupled with an additional condition that draws on a dominance argument. Section 5 introduces these conditions together with additional notation that is required to describe mixed strategies. Hereafter, I refer to perfect Bayesian equilibria that satisfy (FOSD) simply as "equilibria."

3.1 Benchmark

Before solving for the equilibria of the model, I briefly consider a number of benchmark cases that are useful to interpret the results in the next sections.

Full information. Under full information about the state θ, the decision maker selects $\oplus$ when $\theta \geq 0$ and selects $\odot$ otherwise. Both senders would always report truthfully. The ex-ante full information welfare obtained by the decision maker in this scenario is

$$W_{fi} = \int_0^{\max \Theta} f(\theta) u_{dm}(\theta) d\theta. \tag{3}$$

Perfect alignment. Sender j is perfectly aligned with the decision maker when $\tau_j = \tau_{dm}$. There is an equilibrium where the perfectly aligned sender j always reports truthfully and the decision maker blindly trusts j's reports. The other sender, even if not perfectly aligned, can do no better than reporting truthfully as well. In this case, the decision maker gets her full information welfare W_{fi}, and no misreporting takes place.

[13] Posterior beliefs $p(\theta|r_1, r_2)$ first-order stochastically dominate $p(\theta|r_1', r_2')$ for $r_j \geq r_j'$, $j \in \{1, 2\}$, if and only if $\int u(\theta)p(\theta|r_1, r_2)d\theta \geq \int u(\theta)p(\theta|r_1', r_2')d\theta$ for every weakly increasing utility function $u(\theta)$. Thus, condition (FOSD) is weaker than that as it needs to apply only to $u(\theta) \equiv u_{dm}(\theta)$.

Verifiable information. Consider the case where information about the state is fully verifiable, that is, $k_j = \infty$, $j \in \{1, 2\}$. Senders cannot profitably withhold information, but even if they could we would obtain an equilibrium where in every state at least one of the two senders discloses truthfully (P. Milgrom & Roberts, 1986b).[14] As before, the decision maker gets its full information welfare W_{fi}.

Cheap talk. Suppose now that $k_1 = k_2 = 0$. A babbling equilibrium exists, where the decision maker adjudicates according to her prior f only, while senders deliver uninformative messages. There is no equilibrium where the decision maker obtains enough information to always select her preferred alternative.[15] In an informative equilibrium, the decision maker can only learn that the state is between the senders' thresholds τ_j. Therefore, when misreporting is "cheap," the decision maker obtains an ex-ante welfare that is strictly lower than W_{fi}.

4 Receiver-efficient and Pure Strategy Equilibria

The goal of this section is that of studying the existence and the plausibility of equilibria where the decision maker always obtains the information she needs to select her preferred alternative. This class of equilibria is important because it is believed that competition in "the marketplace of ideas" may result in the truth becoming known (Gentzkow & Shapiro, 2008). Competing forces may indeed yield full information revelation in cheap talk settings (Battaglini, 2002) as well as in models of verifiable disclosure (P. Milgrom & Roberts, 1986b).

In this setting, the combination of a rich state space together with a binary action space implies that, to select her favorite alternative, the decision maker does not need to know precisely what is the realized state θ. All the decision maker needs to know is, in fact, only whether the state is positive or negative. For the purpose of this section, a focus on fully revealing equilibria would therefore be too restrictive. The following definition gives a weaker notion of revelation that will provide useful for the analysis that follows.

Definition 1. *A "fully revealing equilibrium" (FRE) is an equilibrium where for every $\theta' \in \Theta$, $r_j \in S_j(\theta')$, and $j \in \{1, 2\}$, $P(\theta|r_1, r_2) = 1$ if and only if $\theta \geq \theta'$. A "receiver-efficient equilibrium" (REE) is an equilibrium where for every $\theta \in \Theta$, $r_j \in S_j(\theta)$, and $j \in \{1, 2\}$, $\beta(r_1, r_2) = \oplus$ if $\theta \geq 0$, and $\beta(r_1, r_2) = \odot$ otherwise.*

[14] If withholding is not possible or prohibitively expensive, then this result holds even when only one of the two senders has verifiable information, i.e., $0 \leq k_j < k_i = \infty$ for $i \neq j$: in equilibrium, the decision maker pays attention only to sender i and disregards every report delivered by sender j, which cannot do better than reporting truthfully as well.

[15] Battaglini (2002) shows conditions under which there is full revelation of the state in cheap talk games. With a binary action space the decision maker cannot take extreme actions that punish both senders, and thus there cannot be equilibria where the state is fully revealed.

A fully revealing equilibrium is also receiver-efficient, but a receiver-efficient equilibrium is not necessarily fully revealing. If competing forces could discipline senders into always report truthfully their private information about the state, then full revelation would naturally occur. However, the following observation points out that, in the game considered here and described in Section 3, misreporting occurs in every equilibrium.

Observation 1. *Misreporting occurs in every equilibrium.*

To see why, suppose by way of contradiction that there exists an equilibrium[16] where misreporting never occurs, that is, where $\rho_1(\theta) = \rho_2(\theta) = \theta$ for every $\theta \in \Theta$. Consider such a truthful equilibrium and a state $\theta = \epsilon > 0$, where ϵ is small enough. To discourage deviations, off-path beliefs must be such that $\beta(\epsilon, -\epsilon) = \oplus$. However, there always exists an $\epsilon > 0$ such that, when the state is $\theta = -\epsilon$, sender 1 can profitably deviate from the prescribed truthful strategy by reporting $r_1 = \epsilon$, as $u_1(-\epsilon) > k_1 C_1(\epsilon, -\epsilon)$. This contradicts the existence of equilibria where misreporting never occurs.

The question is: if senders misreport in every equilibrium, do receiver-efficient equilibria exist at all? Figure 1 provides a positive graphical answer by showing reporting strategies that not only constitute a receiver-efficent equilibrium, but are also fully revealing.[17] To verify that Figure 1 depicts an equilibrium, consider the following strategies: sender 1 delivers $\rho_1(\theta) = \bar{r}_1(0)$ for every $\theta \in [0, \bar{r}_1(0)]$, where for simplicity we assume that $\bar{r}_1(0) < \tau_2$. Otherwise, sender 1 reports truthfully. By contrast, sender 2 always report truthfully, i.e., $\rho_2(\theta) = \theta$ for all $\theta \in \Theta$. Given any on-path pair of reports, posterior beliefs are such that $P(\theta|r_1, r_2) = 0$ for every $\theta < r_2$ and $P(\theta|r_1, r_2) = 1$ otherwise, which is consistent with sender 2 playing a separating strategy. Off-path beliefs are such that such that $U_{dm}(r_1, r_2) < 0$ if $r_1 < \bar{r}_1(0)$, and $P(\theta|r_1, r_2) = 1$ if and only if $\theta \geq r_1 \geq \bar{r}_1(0)$. By definition of reach, sender 1 would never find it profitable to deliver a report $r_1 \geq \bar{r}_1(0)$ when $\theta < 0$. Sender 2 cannot deviate from its truthful strategy by delivering a negative report when the state is positive: since $\rho_1(\theta) \geq \bar{r}_1(0)$ for every $\theta \geq 0$, such a deviation would induce $\beta(\cdot) = \oplus$. No sender has a profitable individual deviation from the prescribed equilibrium strategies. Therefore, there exist equilibria where senders always fully reveal the state to the decision maker, even though full revelation involves misreporting.

Incidentally, Figure 1 also proves the existence of equilibria in pure strategies. Intuitively, when two competing senders with opposed interests play pure strategies, the decision maker can "undo" their reports to recover the underlying truth. This argument may suggest that all pure strategy equilibria are receiver-efficient. The next lemma shows that such intuition is correct and, in addition, that all receiver-efficient equilibria are in pure strategies.

[16] Observation 1 applies to every perfect Bayesian equilibria, and not only to those satisfying (FOSD).

[17] In a single-sender setting with unbounded state space, Kartik et al. (2007) study a fully revealing equilibrium where misreporting occurs in every state. There, the reporting strategy is fully separating.

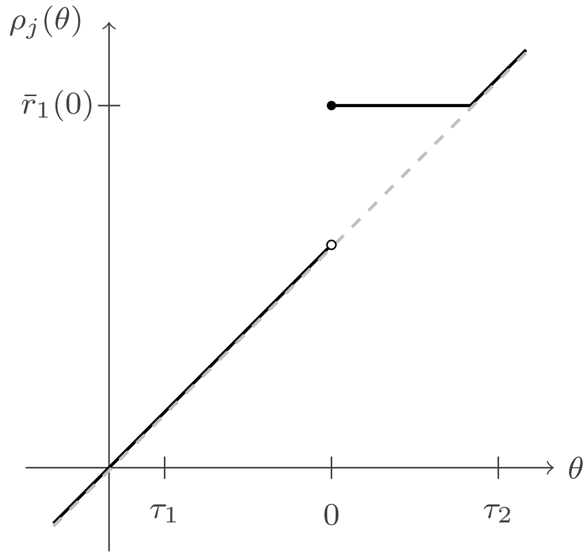

Figure 1: Senders' strategies in a receiver-efficient and fully revealing equilibrium. The reporting rules of sender 1 and 2 are in black and dashed gray, respectively.

Lemma 2. *An equilibrium is receiver-efficient if and only if it is in pure strategies.*

The receiver-efficient and fully revealing equilibrium strategies discussed above are, however, problematic. To see what the problem is, consider again the strategies pictured in Figure 1 and a state $\theta' \in (0, \bar{r}_1(0))$. Suppose that in state θ' sender 1 deviates from the prescribed equilibrium by reporting the truth instead of $\rho_1(\theta') = \bar{r}_1(0)$, whereas sender 2 sticks to its separating reporting rule. Notice that, in the equilibrium under consideration, sender 1 never delivers $r_1 = \theta'$. Upon observing the off-path pair of reports (θ', θ'), beliefs p induce an expected payoff of $U_{dm}(\theta', \theta') < 0$ and lead to $\beta(\theta', \theta') = \circleddash$. These off-path beliefs require the decision maker to conjecture that the state is likely to be negative. However, this means that the decision maker must entertain the possibility that (i) both senders performed at the same time a deviation from the prescribed equilibrium strategies, and that (ii) sender 2 has delivered a strictly dominated report.

In addition, the receiver-efficient equilibrium in Figure 1 is sustained by beliefs that are discontinuous: for every on-path pair of reports $(\bar{r}_1(0), r_2)$ such that $r_2 \in (0, \bar{r}_1(0))$, beliefs are such that $U_{dm}(\bar{r}_1(0), r_2) = u_{dm}(r_2) > 0$; by contrast, $U_{dm}(\bar{r}_1(0) - \epsilon, r_2) < 0$ for every arbitrarily small $\epsilon > 0$. This discontinuity is crucial to discourage deviations, but it does not seem plausible especially when considering its problematic implications discussed above. In the remaining part of this section, I put receiver-efficient equilibria under the scrutiny of two well-known tests for games with multiple senders: unprejudiced beliefs (Bagwell & Ramey, 1991) and ε-robustness (Battaglini, 2002).

Unprejudiced beliefs. Consider again a deviation from the equilibrium depicted in Figure 1 where both senders report truthfully in some state $\theta' \in (0, \bar{r}_1(0))$. If, whenever possible, the decision maker conjectures deviations as individual and thus as originating from one sender only, then she should infer that sender 1 has performed the deviation:

sender 1 never delivers $r_1 = \theta'$ on the equilibrium path, whereas sender 2 truthfully reports $r_2 = \theta'$ only when the state is indeed θ'. Since sender 2 is following its separating strategy, the decision maker should infer that the state is $\theta' > 0$. According to this line of reasoning, off-path beliefs must be such that $P(\theta|\theta', \theta') = 1$ if and only if $\theta \geq \theta'$, and thus $\beta(\theta', \theta') = \oplus$. Therefore, such a deviation becomes profitable for sender 1 because it saves on misreporting costs without affecting the outcome.

Bagwell and Ramey (1991) introduce the concept of "unprejudiced beliefs," which formalize the idea that the decision maker should exclude the possibility that multiple senders are deviating at the same time whenever it is possible that only a single sender is deviating. Vida and Honryo (2019) show that, in generic multi-sender signaling games, strategic stability (Kohlberg & Mertens, 1986) implies unprejudiced beliefs. Apart for its relationship with the notion of strategic stability, unprejudiced beliefs are intuitive, easily applicable, and consistent with the notion of Nash equilibrium, and therefore constitute a sensible way to refine equilibria in multi-sender signaling games where other criteria fail to do so. The following definition formalizes unprejudiced beliefs.[18]

Definition 2 (Vida & Honryo, 2019)**.** *Given senders' strategies ρ_j, beliefs p are unprejudiced if, for every pair of reports (r_1, r_2) such that $\rho_j(\theta') = r_j$ for some $\theta' \in \Theta$ and $j \in \{1, 2\}$, we have that $p(\theta''|r_1, r_2) > 0$ only if there is a sender $i \in \{1, 2\}$ such that $\rho_i(\theta'') = r_i$.*

We have seen how the above "informational free-riding" argument breaks down the receiver-efficient equilibrium depicted in Figure 1. A natural question is whether such argument applies only in that particular case or if instead it prunes out other equilibria. The next proposition tells us that in fact there is no receiver-efficient equilibrium that supports unprejudiced beliefs.

Proposition 1. *There are no receiver-efficient equilibria with unprejudiced beliefs.*

ε-robustness. In the model described in Section 3, senders are perfectly informed and the receiver can perfectly observe the senders' reports. There is no "noise" or perturbation in what senders report or in what the decision maker observes. This modelling strategy allows me to isolate the effects of strategic interactions and inference from the effects of statistical information aggregation. However, this procedure may give us excessive freedom to pick ad-hoc beliefs that would not survive the presence of even arbitrarily small perturbations in the transmission of information.

I follow Battaglini (2002) and define an ε-perturbed game as the game described in Section 3 in which the decision maker perfectly observes the report of sender j with probability $1 - \varepsilon_j$ and with probability ε_j observes a random report $\tilde{r}_j$, where $\tilde{r}_j$ is a

[18] Definition 2 is weaker than the definition originally introduced by Bagwell and Ramey (1991), and therefore it is useful to test for equilibria that do not support unprejudiced beliefs.

random variable with continuous distribution G_j, density g_j, and support in Θ. This may correspond to a situation where with some probability the decision maker misreads reports; or, alternatively, where with some probability senders commit mistakes in delivering their reports.[19] As before, senders incur misreporting costs that depend only on the realized state θ and on their "intended" report r_j, but not on the wrongly observed or delivered $\tilde{r}_j$. The introduction of noise makes any pair of reports to be possible on the equilibrium path. The decision maker's posterior beliefs depend on $\varepsilon = (\varepsilon_1, \varepsilon_2)$, $G = (G_1, G_2)$, and on the senders' reporting strategies $\rho_j(\theta)$.

Definition 3 (Battaglini (2002))**.** *An equilibrium is ε-robust if there exists a pair of distributions $G = (G_1, G_2)$ and a sequence $\varepsilon^n = (\varepsilon_1^n, \varepsilon_2^n)$ converging to zero such that the off-path beliefs of the equilibrium are the limit as $\varepsilon^n \to 0^+$ of the beliefs that the equilibrium strategies would induce in an ε-perturbed game.*

Intuitively, as the noise ε fades away, the event in which the decision maker misreads both reports becomes negligible. At the limit as $\varepsilon \to 0^+$, the decision maker infers that she is correctly observing at least one of the two reports. Therefore, upon observing an off-path pair of reports, beliefs in an ε-robust equilibrium are as if the decision maker conjectures—whenever possible—that one sender is following its prescribed reporting strategy while the other is not. This conclusion is reminiscent of unprejudiced beliefs, and suggests that there might be a tight connection between these two refinement criteria. The next lemma confirms the existence of such a relationship.

Lemma 3. *If a perfect Bayesian equilibrium is ε-robust, then it has unprejudiced beliefs.*

A straight forward implication of Lemma 3 and Proposition 1 is that no receiver-efficient or fully revealing equilibrium is ε-robust. By Lemma 2, we obtain that also pure strategy equilibria do not have unprejudiced beliefs and are not ε-robust. These results suggest that mixed strategy equilibria are qualitatively important, whereas in related work pure strategies have a prominent role.[20] The next section is dedicated to finding equilibria that are robust in the sense that are ε-robust, and supported by unprejudiced beliefs.

[19] Battaglini (2002) introduces noise in what senders know, while here I perturb the reports observed by the decision maker. This type of perturbation is qualitatively equivalent to that used by Battaglini (2002).

[20] For example, Kartik et al. (2007) and Kartik (2009) focus on pure strategy only, and in Chen (2011) there are no (monotone) mixed strategies. Most work on multi-sender signaling (see Section 2) also study only pure strategy equilibria. Results in Section 4 also suggest that the similarity between this setting and contest theory goes beyond the type of strategic interaction between senders, but it extends also to the equilibrium behavior, which, in contests, is typically in mixed strategies (Siegel, 2009).

5 Direct Equilibria

Findings in the previous section show that pure strategy equilibria exist and are receiver-efficient, but are supported by an unreasonable choice of off-path beliefs. Such results motivate the quest for "robust" equilibria which, if exist, must therefore be in mixed strategies. The two main goals of this section are that of providing sufficient conditions under which equilibria are robust and to characterize such robust equilibria.

Since (FOSD) is not enough to rule out unreasonable equilibria, I need to impose a different set of restrictions to study robust mixed strategy equilibria. However, classical refinements for signaling games such as the "intuitive criterion" (Cho & Kreps, 1987) and the "universal divinity" (Banks & Sobel, 1987) have little bite here, as they are developed for single-sender settings. To date, there is no large consensus on how to extend these criteria to multi-sender settings. By contrast, ε-robustness and unprejudiced beliefs proved to be useful in testing separating equilibria of multi-sender signaling games, but cannot be easily applied when looking for non-separating equilibria in mixed strategies.

Therefore, I draw on the implausibility of receiver-efficient equilibria to impose two conditions on how the decision maker interprets the senders' reports. I refer to equilibria satisfying these conditions as "direct equilibria."

Definition 4. *A "direct equilibrium" (DE) is a perfect Bayesian equilibrium where posterior beliefs p satisfy the following conditions:*

i) condition (FOSD) holds, and for every pair of reports (r_1, r_2) such that $\underline{r}_2(0) < r_2 \leq 0 \leq r_1 < \bar{r}_1(0)$, and for $j \in \{1, 2\}$,

$$\frac{dU_{dm}(r_1, r_2)}{dr_j} > 0; \tag{D}$$

ii) upon observing the pairs of reports $(\bar{r}_1(0), \underline{r}_2(0))$ and $(0, 0)$, beliefs p are such that the decision maker is indifferent between the two alternatives, that is,

$$U_{dm}\left(\bar{r}_1(0), \underline{r}_2(0)\right) = U_{dm}\left(0, 0\right) = 0. \tag{C}$$

The first condition, (D), imposes a "strict" first-order stochastic dominance on posterior beliefs p, but only for pairs of reports consisting of conflicting recommendations. Otherwise, (FOSD) applies. Since (D) implies (FOSD), Lemma 1 applies also to direct equilibria. Intuitively, (D) means that strictly higher conflicting reports inform the decision maker that the expected value of selecting alternative $\oplus$ is strictly higher. As for (FOSD), this condition is natural and consistent with the idea that reports are literal statements about the state and that misreporting is costly.

Condition (C) draws from a simple argument of equilibrium dominance. To see why, consider a report $r_j \in \hat{R}$, and define by $Q_j(r_j)$ the set of states for which delivering report

r_j is potentially profitable for sender j given that beliefs p satisfy (D). By Lemma 1 and the definition of inverse reach, we obtain that $Q_1(r_1) = [\bar{r}_1^{-1}(r_1), r_1] \cap \Theta$ and $Q_2(r_2) = [r_2, \underline{r}_2^{-1}(r_2)] \cap \Theta$. Denote the intersection of these two sets by $Q(r_1, r_2) = Q_1(r_1) \cap Q_2(r_2)$. If $Q(r_1, r_2) \neq \varnothing$, then it would be sensible for the decision maker to exclude the possibility that the realized state lies outside $Q(r_1, r_2)$, i.e., $p(\theta|r_1, r_2) = 0$ for all $\theta \notin Q(r_1, r_2)$. Since $Q(\bar{r}_1(0), \underline{r}_2(0)) = Q(0, 0) = \{0\}$, upon receiving the pairs or reports $(\bar{r}_1(0), \underline{r}_2(0))$ or $(0, 0)$, the decision maker should understand that the realized state[21] is for sure $\theta = 0$. Otherwise, the decision maker would have to believe that at least one of the two senders has delivered a report that is equilibrium dominated. Condition (C) is even less stringent than this argument suggests, as it does not require beliefs to be degenerate at 0, and does not impose conditions over pairs of reports[22] other than $(\bar{r}_1(0), \underline{r}_2(0))$ and $(0, 0)$.

As an immediate application of direct equilibria, reconsider the fully revealing and voter-efficient equilibrium in Figure 1 previously discussed in Section 4. To prevent a deviation by sender 1, beliefs p are such that $U_{dm}(\theta', \theta') < 0$ for any $\theta' \in (0, \bar{r}_1(0))$, and thus $\beta(\theta', \theta') = \ominus$. That cannot be a direct equilibrium: by (C) we have that $U_{dm}(0, 0) = 0$, and by (D) it must be that $U_{dm}(\theta', \theta') \geq 0$, leading to $\beta(\theta', \theta') = \oplus$ and thus to a profitable deviation by sender 1. Therefore, conditions (C) and (D) rule out at least some equilibria that, we have seen, are not plausible.

By the end of this section we will see that direct equilibria have a number of remarkable properties: they always exists, they are essentially unique, and there are direct equilibria that are ε-robust and thus with unprejudiced beliefs.

5.1 Notation for Mixed Strategies

Before analyzing direct equilibria, I first introduce further notation. To describe mixed strategies, I use a "mixed" probability distribution $\phi_j(r_j, \theta)$ which, for every state θ, assigns a mixed probability density to report r_j by sender j. This specification allows me to describe the senders' reporting strategies as mixed random variables which distribution can be partly continuous and partly discrete.[23]

Formally, I partition the support $S_j(\theta)$ of each sender in two subsets, $C_j(\theta)$ and $D_j(\theta)$. To represent atoms in $\phi_j(\theta)$, I define a partial probability density function $\alpha_j(\cdot, \theta)$ on $D_j(\theta)$ such that $0 \leq \alpha_j(r_j, \theta) \leq 1$ for all $r_j \in D_j(\theta)$, and $\hat{\alpha}_j(\theta) = \sum_{r_j \in D_j(\theta)} \alpha_j(r_j, \theta)$. By contrast, the continuous part of the distribution $\phi_j(\theta)$ is described by a partial probability density function $\psi_j(\cdot, \theta)$ on $C_j(\theta)$ such that $\int_{r_j \in C_j(\theta)} \psi_j(r_j, \theta) d\theta = 1 - \hat{\alpha}_j(\theta)$. I set $\alpha_j(r', \theta) = 0$ for

[21] From $P(\theta|\bar{r}_1(0), \underline{r}_2(0)) = P(\theta|0, 0) = 1$ iff $\theta \geq 0$ we get $U_{dm}(\bar{r}_1(0), \underline{r}_2(0)) = U_{dm}(0, 0) = u_{dm}(0) = 0$.

[22] As we shall see, it turns out that in every direct equilibrium the pair $(\bar{r}_1(0), \underline{r}_2(0))$ is on-path only for $\theta = 0$, and thus it fully reveals that the state is indeed zero. By contrast, no sender ever delivers $r_j = 0$, and thus the pair of reports $(0, 0)$ is not only off-path, but it must constitute a double deviation.

[23] Mixed type distributions that have both a continuous and a discrete component to their probability distributions are widely used to model zero-inflated data such as queuing times. For example, the "rectified gaussian" is a mixed discrete-continuous distribution.

all $r' \notin D_j(\theta)$ and $\psi_j(r'', \theta) = 0$ for all $r'' \notin C_j(\theta)$.

As we shall see (Lemma 7 and Proposition 4), in every direct equilibrium $D_j(\theta) = \{\theta\}$ for all $\theta \in \Theta$ and $j \in \{1, 2\}$. Therefore, I hereafter simplify notation by setting $\alpha_j(\theta) \equiv \alpha_j(\theta, \theta) = \hat{\alpha}_j(\theta)$. The score $\alpha_j(\theta)$ thus represents the probability that sender j reports truthfully in state $\theta \in \Theta$. The partial density probabilities[24] $\alpha_j(\theta)$ and $\psi_j(\cdot, \theta)$ determine the "generalized" density function $\phi_j(\theta)$ through the well defined mixed distribution

$$\phi_j(x, \theta) = \delta(x - \theta)\alpha_j(\theta) + \psi_j(x, \theta),$$

where $\delta(\cdot)$ is the Dirac delta "generalized" function.[25]

A mixed strategy for sender j is a mixed probability measure $\phi_j(\theta) : \Theta \to \Delta(R_j)$ with support $S_j(\theta)$. I indicate with $\phi_j(r_j, \theta)$ the mixed probability assigned by $\phi_j(\theta)$ to a report r_j in state θ that satisfies

$$\int_{r_j \in S_j(\theta)} \phi_j(r_j, \theta) dr_j = \alpha_j(\theta) + \int_{r_j \in C_j(\theta)} \psi_j(r_j, \theta) dr_j = 1.$$

I denote by $\Phi_j(r_j, \theta)$ and $\Psi_j(r_j, \theta)$ the CDFs of ϕ_j and ψ_j, respectively. Sender j's expected utility from delivering r_j when the state is θ in a direct equilibrium ω is $W_j^\omega(r_j, \theta)$.

5.2 Solving for Direct Equilibria

In the remaining parts of this section, I characterize direct equilibria and show their properties. All proofs and a number of intermediate results are relegated to Appendix A.2.

Given a pair of reports (r_1, r_2) the decision maker forms posterior beliefs $p(\theta|r_1, r_2)$, which determine whether she rationally selects $\oplus$ or $\odot$. Consider a direct equilibrium and a pair of reports (r_1, r_2) such that $r_2 < 0$ and $U_{dm}(r_1, r_2) < 0$, and suppose that there exists a report $r_1' \in R_1$ such that $U_{dm}(r_1', r_2) > 0$. By conditions (C) and (D), it must be[26] that there exists a report $r_1'' \in (r_1, r_1')$ such that $U_{dm}(r_1'', r_2) = 0$. In this case, r_1'' "swings" the decision maker's choice as $\beta(r, r_2) = \oplus$ for all $r \geq r_1''$ and $\beta(r, r_2) = \odot$ otherwise, and I say that r_1'' is the "swing report" of r_2. The notion of swing report is key for the analysis of direct equilibria, and the following definition formalizes this concept.

Definition 5. *Given a report r, I define the "swing report" $s(r)$ as*

$$s(r) = \begin{cases} \{r_2 \in R_2 \mid U_{dm}(r, r_2) = 0\} & \textit{if } r \geq 0 \\ \{r_1 \in R_1 \mid U_{dm}(r_1, r) = 0\} & \textit{otherwise.} \end{cases}$$

[24] Under this specification, even the "mass" $\alpha_j(\cdot)$ is a partial probability "density."

[25] The Dirac delta $\delta(x)$ is a generalized function such that $\delta(x) = 0$ for all $x \neq 0$, $\delta(0) = \infty$ and $\int_{-\epsilon}^{\epsilon} \delta(x) dx = 1$ for any $\epsilon > 0$.

[26] By (C) we have $U_{dm}(0, 0) = 0$, and by (D) we have $U_{dm}(0, r_2) < 0$ and $r_1' > r_1$. Since the differentiability of U_{dm} for conflicting reports implies its continuity, and since $U_{dm}(r_1', r_2) > 0$, it follows from the intermediate value theorem that there must be a $r_1'' \in (0, r_1')$ such that $U_{dm}(r_1'', r_2) = 0$.

If $s(r) = \varnothing$, then I set $s(r) = -\infty$ for $r \geq 0$, and $s(r) = \infty$ otherwise.

With some abuse of language, I hereafter say that sender j "swings" the report of its opponent $-j$ whenever the pair of reports (r_1, r_2) induce the selection of sender j's preferred alternative. When there is a conflict of interests between senders, that is for some $\theta \in (\tau_1, \tau_2)$, sender 1 swings the report of sender 2 whenever $r_1 \geq s(r_2)$. Similarly, sender 2 swings the report of sender 1 when $r_2 < s(r_1)$.

In a direct equilibrium, the swing report $s(r)$ has a number of intuitive properties: first, condition (D) ensures that the swing report, if it exists, is unique; second, condition (C) pins down the swing report for $s(\bar{r}_1(0)) = \underline{r}_2(0)$, $s(\underline{r}_2(0)) = \bar{r}_1(0)$, and $s(0) = 0$. From the interaction of conditions (C) and (D), it follows that every report $r \in \hat{R} = [\underline{r}_2(0), \bar{r}_1(0)]$ has a unique swing report $s(r) \in \hat{R}$ such that if $r > 0$ then $s(r) < 0$. Moreover, for all $r \in \hat{R}$, the swing report of a swing report is the report itself, i.e., $s(s(r)) = r$, and strictly higher reports have strictly lower swing reports. Importantly, $s(r)$ is endogenously determined in equilibrium through the posterior beliefs p. The following lemma formalizes these equilibrium features of the swing report function.

Lemma 4. *In a direct equilibrium, every report $r \in \hat{R}$ has a swing report $s(r) \in \hat{R}$ such that (i) if $r \gtreqless 0$ then $s(r) \lesseqgtr 0$ and $s(0) = 0$; (ii) $s(s(r)) = r$; (iii) for every $r \in \hat{R}$, $\frac{ds(r)}{dr} < 0$; (iv) $s(\bar{r}_1(0)) = \underline{r}_2(0)$.*

Therefore, $s(r)$ is effectively a strictly decreasing function of r in the set $[\underline{r}_2(0), \bar{r}_1(0)]$, and in such domain I refer to $s(r)$ as the "swing report function." When the state takes extreme values, a sender may not be able to profitably swing the report of its opponent even when the opponent reports truthfully. This happens when $s(\theta)$ is beyond a sender's reach. In such cases, we should expect both senders to always report truthfully, and thus to deliver congruent reports that reveal the state. It is therefore useful to define cutoffs in the state space that help determine when truthful reporting always occurs in direct equilibria.

Definition 6. *The "truthful cutoffs" are defined as*

$$\theta_1 := \{\theta \in \Theta \mid s(\theta) = \bar{r}_1(\theta)\},$$

$$\theta_2 := \{\theta \in \Theta \mid s(\theta) = \underline{r}_2(\theta)\}.$$

The truthful cutoffs are also determined in equilibrium, as they depend on $s(r)$. Recall that by condition (C) we have that $s(\underline{r}_2(0)) = \bar{r}_1(0)$ and $s(0) = 0$. Since $\bar{r}_1(\theta)$ is increasing in θ and $\bar{r}_1(\tau_1) = \tau_1 < 0 < s(\tau_1)$, it follows that $0 > \theta_1 > \max\{\tau_1, \underline{r}_2(0)\}$. Similarly, we obtain that $0 < \theta_2 < \min\{\tau_2, \bar{r}_1(0)\}$. Therefore, in any direct equilibrium the set of states that lie within the truthful cutoffs (θ_1, θ_2) is also a strict subset of $[\tau_1, \tau_2]$ and of $[\underline{r}_2(0), \bar{r}_1(0)]$. This equilibrium feature of the truthful cutoffs is convenient because it implies that for every state $\theta \in (\theta_1, \theta_2)$ there is always a conflict of interest between senders, and that the swing report function $s(\theta)$ exists and is well defined in such a set.[27]

In a direct equilibrium, we should expect both senders to always report truthfully—and thus to play pure strategies—whenever the state lies outside the truthful cutoffs. By contrast, when the state takes values within the truthful cutoffs, we might expect senders to play mixed strategies and to engage in some misreporting activity. As the following lemma shows, these two conjectures turn out to be correct in every direct equilibrium.

Lemma 5. *In a direct equilibrium, $S_j(\theta) = \{\theta\}$ for all $\theta \notin (\theta_1, \theta_2)$, and $|S_j(\theta)| > 1$ for every $\theta \in (\theta_1, \theta_2)$, $j \in \{1, 2\}$.*

This result, together with the previous observation that $(\theta_1, \theta_2) \subset [\tau_1, \tau_2]$, shows an interesting characteristic of direct equilibria: in relatively extreme states, both senders always deliver matching and truthful reports even though they have opposing interests. Since senders' reports coincide, it follows from Lemma 1 that in these cases the decision maker learns the underlying state. Therefore, full information revelation always occurs in extreme states that lie outside the truthful cutoffs.

Given the results outlined above, from now on I focus on players' behavior when the state takes values within the truthful cutoffs. I proceed by first studying the reporting strategies when senders misreport their private information. Conditional on misreporting in state θ, sender j's strategy $\phi_j(\theta)$ has support in the set $S_j(\theta) \setminus \{\theta\}$. To describe and study equilibrium supports and strategies, it is useful to understand if such set is convex or not. The next lemma tells us that $S_j(\theta) \setminus \{\theta\}$ is always convex.

Lemma 6. *In a direct equilibrium, $S_j(\theta) \setminus \{\theta\}$ is convex for all $\theta \in (\theta_1, \theta_2)$ and $j \in \{1, 2\}$.*

The intuition behind Lemma 6 is the following: in equilibrium, the presence of a "hole" in the set $S_j(\theta) \setminus \{\theta\}$ must imply that j's opponent never wastes resources to swing reports that are in such a hole, as they are never delivered. However, this means that in $S_j(\theta) \setminus \{\theta\}$ there are two different reports that yield approximately the same probability of inducing the selection of j's preferred alternative but have different costs. This cannot be possible in an equilibrium, and therefore the set $S_j(\theta) \setminus \{\theta\}$ must be convex.

While senders may misrepresent the same state in a number of ways, the above argument also suggests that, conditional on misreporting, there is no report that they deliver with strictly positive probability. To see why, suppose by way of contradiction that sender j misreports some state θ by delivering $r_j \in S_j(\theta) \setminus \{\theta\}$ with some strictly positive "mass" probability $\alpha_j(r_j, \theta) > 0$. That is, j's strategy $\phi_j(\theta)$ has an atom in r_j. It follows that sender $-j$'s expected payoff is discontinuous around $r_{-j} = s(r_j)$, and therefore $s(r_j)$ cannot be in the interior[28] of $S_{-j}(\theta)$. If $s(r_j) \notin S_{-j}(\theta)$, then j can profitably "move" the

[27] These results are formalized by Lemma A.1 in Appendix A.2.

[28] Recall that every report in the equilibrium support must yield the same expected payoff.

atom to some cheaper report that ensures the selection of its own favorite alternative. If instead $s(r_j)$ is on the boundary of $S_{-j}(\theta)$, then one of the two senders would have a profitable deviation: either there are reports outside $S_{-j}(\theta)$ that yield an higher expected payoff than reports inside the support, or there is some report that dominates r_j. In both cases, we obtain a contradiction with j's strategy being part of an equilibrium. The following lemma formalizes the idea that the equilibrium reporting strategies are non-atomic whenever senders misreport their private information.[29]

Lemma 7. *In a direct equilibrium, strategies $\phi_j(\theta)$ have no atoms in $S_j(\theta) \setminus \{\theta\}$ for every $\theta \in (\theta_1, \theta_2)$ and $j \in \{1, 2\}$.*

5.2.1 Strategies, Supports, and Beliefs

I am now ready to state the main results of this section. Lemmata 6 and 7 tell us that, conditional on misreporting, senders play an atomless reporting strategy with support in a convex set. By using the method of payoff-equation, I obtain the partial probability densities $\psi_j(r_j, \theta)$. The next proposition establishes senders' misreporting behavior.

Proposition 2. *In a direct equilibrium, for every $\theta \in (\theta_1, \theta_2)$ and $i, j \in \{1, 2\}$ with $i \neq j$, sender j delivers report $r_j \in S_j(\theta) \setminus \{\theta\}$ according to*

$$\psi_j(r_j, \theta) = \frac{k_i}{-u_i(\theta)} \frac{dC_i(s(r_j), \theta)}{dr_j}.$$

Each sender's misreporting behavior depends directly on its opponent's utility and costs, while it may only depend indirectly on its own characteristics through the swing report function $s(r)$. Whether a sender is more likely to deliver small lies or large misrepresentations, depends on the shape of its opponent's misreporting costs function together with the shape of the swing report function, where the latter is determined in equilibrium. In Section 6.1 I discuss more in detail the senders' misreporting behavior for the particular case where senders have symmetric features.

Since the sets $S_j(\theta) \setminus \{\theta\}$ are convex and the strategies $\phi_j(\theta)$ are atomless on $S_j(\theta) \setminus \{\theta\}$, I can integrate the partial probability densities ψ_j to pin down the senders' equilibrium supports. This procedure allows me to prove the the next proposition.

Proposition 3. *In a direct equilibrium, for every state $\theta \in (\theta_1, \theta_2)$, supports $S_j(\theta)$ are*

$$S_1(\theta) = \{\theta\} \cup [\max\{s(\theta), \theta\}, \min\{\bar{r}_1(\theta), s(\underline{r}_2(\theta))\}],$$

$$S_2(\theta) = \{\theta\} \cup [\max\{\underline{r}_2(\theta), s(\bar{r}_1(\theta))\}, \min\{s(\theta), \theta\}].$$

[29] The intuition of results provided in this section omits a number of additional steps that are necessary to prove Lemmata 6 and 7. See Lemmata A.2 to A.7 in Appendix A.2.

So far, I focused the analysis on senders' misreporting behavior. However, the above proposition shows that "the truth" is always part of equilibrium supports. Having fully characterized the senders' misreporting strategies $\psi_j(\cdot,\theta)$ and supports $S_j(\theta)$, I can now proceed to establish senders' truthful reporting behavior.

Proposition 4. *In a direct equilibrium, for every state $\theta \in (\theta_1, \theta_2)$, strategies $\phi_j(\theta)$ have an atom at $r_j = \theta$ of size $\alpha_j(\theta)$, where*

$$\alpha_1(\theta) = \begin{cases} \frac{k_2}{-u_2(\theta)} C_2\left(s(\theta), \theta\right) & \text{if } \theta \in [0, \theta_2) \\ 1 - \frac{k_2}{-u_2(\theta)} C_2\left(s(\bar{r}_1(\theta)), \theta\right) & \text{if } \theta \in (\theta_1, 0], \end{cases}$$

$$\alpha_2(\theta) = \begin{cases} 1 - \frac{k_1}{u_1(\theta)} C_1\left(s(\underline{r}_2(\theta)), \theta\right) & \text{if } \theta \in [0, \theta_2) \\ \frac{k_1}{u_1(\theta)} C_1\left(s(\theta), \theta\right) & \text{if } \theta \in (\theta_1, 0]. \end{cases}$$

Both senders report truthfully with strictly positive probability in almost every state. The only exception is $\theta = 0$, where the truth is never reported as $\alpha_1(0) = \alpha_2(0) = 0$. With probability $\alpha_1(\theta)\alpha_2(\theta)$ both senders deliver the truth, and by Lemma 1 we obtain that whenever this event occurs the decision maker fully learns the realized state. Moreover, by Proposition 3 we get that the decision maker may learn the realized state even when only one of the two senders reports truthfully: if the realized state is positive, then full revelation occurs whenever sender 2 reports truthfully; if the state is negative, then full revelation occurs when sender 1 reports truthfully. In these cases, senders deliver different reports which nevertheless recommend the decision maker to select same alternative.

The probability that full revelation takes place and the probability of observing congruent reports are both increasing as the realized state is further away from zero.[30] Therefore, in direct equilibria we obtain that the revelation of the state and the congruence of reports are phenomena that are more likely to occur in extreme states than in intermediate or central states. To see this, note that

$$\frac{d\alpha_1(\theta)}{d\theta} = \begin{cases} \frac{k_2}{u_2(\theta)^2}\frac{du_2(\theta)}{d\theta} C_2\left(s(\theta), \theta\right) + \frac{k_2}{-u_2(\theta)}\frac{dC_2(s(\theta),\theta)}{d\theta} > 0 & \text{if } \theta \in [0, \theta_2) \\ -\frac{k_2}{u_2(\theta)^2}\frac{du_2(\theta)}{d\theta} C_2\left(s(\bar{r}_1(\theta)), \theta\right) - \frac{k_2}{-u_2(\theta)}\frac{dC_2(s(\bar{r}_1(\theta)),\theta)}{d\theta} < 0 & \text{if } \theta \in (\theta_1, 0), \end{cases}$$

$$\frac{d\alpha_2(\theta)}{d\theta} = \begin{cases} \frac{k_1}{u_1(\theta)^2}\frac{du_1(\theta)}{d\theta} C_1\left(s(\underline{r}_2(\theta)), \theta\right) - \frac{k_1}{u_1(\theta)}\frac{dC_1(s(\underline{r}_2(\theta)),\theta)}{d\theta} > 0 & \text{if } \theta \in [0, \theta_2) \\ -\frac{k_1}{u_1(\theta)^2}\frac{du_1(\theta)}{d\theta} C_1\left(s(\theta), \theta\right) + \frac{k_1}{u_1(\theta)}\frac{dC_1(s(\theta),\theta)}{d\theta} < 0 & \text{if } \theta \in (\theta_1, 0). \end{cases}$$

Figure 2 depicts both the probability that senders deliver the same report and the probability that the decision maker fully learns the realized state.

After obtaining the senders' equilibrium supports and strategies, I can now proceed to study the decision maker's posterior beliefs. It is key for this analysis to understand how

[30] This is because the decision maker's threshold τ_{dm} is normalized to zero.

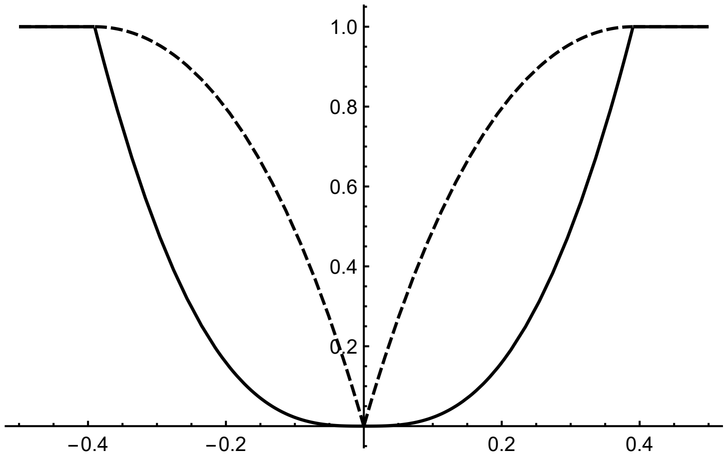

Figure 2: The probability that the decision maker fully learns the state (dashed black line) and the probability that senders deliver matching reports (full black line) as functions of the realized state in a direct equilibrium of a symmetric environment with linear utilities and quadratic loss misreporting costs.

beliefs p determine the decision maker's choice given any pair of reports. To this end, it is sufficient to examine how posterior beliefs shape the swing report function $s(r)$. By Lemma 4, we have that $s(r) \in \hat{R}$ for every $r \in \hat{R}$, with $s(r) < 0$ if $r > 0$, $s(r) > 0$ if $r < 0$, and $s(0) = 0$. Given the supports and the strategies as in Propositions 2, 3, and 4, we obtain that every pair of reports (r_1, r_2) such that $\underline{r}_2(0) \leq r_2 < 0 < r_1 \leq \bar{r}_1(0)$ is on-path. By Definition 5 and Lemma 4 we have that, for a pair of reports $(r_1, r_2 = s(r_1))$,

$$U_{dm}(r_1, s(r_1)) = U_{dm}(s(r_2), r_2) = \int_\Theta u_{dm}(\theta) p(\theta | r_1, s(r_1)) d\theta = 0.$$

Therefore, I can use $p(r_1, s(r_1)|\theta) = \phi_1(r_1, \theta) \cdot \phi_2(s(r_1), \theta)$ and previous results to show how posterior beliefs p pin down the swing report function $s(r)$ in a direct equilibrium. The next proposition shows how the swing report depends on the model's parameters.

Proposition 5. *In a direct equilibrium, the swing report function $s(r_i)$ is implicitly defined for $i, j \in \{1, 2\}$, $i \neq j$, and $r_i \in \hat{R}$, as*

$$s(r_i) = \left\{ r_j \in R_j \;\middle|\; \int_{\max\{r_2, \bar{r}_1^{-1}(r_1)\}}^{\min\{r_1, \underline{r}_2^{-1}(r_2)\}} f(\theta) \frac{u_{dm}(\theta)}{u_1(\theta) u_2(\theta)} \frac{dC_j(r_j, \theta)}{dr_j} \frac{dC_i(r_i, \theta)}{dr_i} d\theta = 0 \right\}. \tag{4}$$

5.2.2 Uniqueness, Robustness, and Existence

Propositions 2 to 5 complete the characterization of direct equilibria. However, there are three potential issues that must be addressed: first, there may be multiple direct equilibria which yield different solutions; second, direct equilibria may not be robust to the refinements introduced and discussed in Section 4, and thus they may be unreasonable; third, direct equilibria might not exists at all. I conclude this section by showing that direct equilibria are essentially unique, are robust, and always exist.

The issue of multiplicity is cleared out by the observation that equation (4), which implicitly determines the swing report function $s(r)$, depends only on the primitives of the model. In particular, the swing report function depends on the prior beliefs, the players' utilities, and the senders' costs only. Given these primitives, the swing report function is the same in every direct equilibrium, and therefore also the senders' reporting strategies and supports are the same across all direct equilibria. Conditions (C) and (D) are thus sufficient to ensure that all equilibria are essentially unique in the sense that they are all strategy and outcome equivalent.

Corollary 1. *Direct equilibria are essentially unique.*

In Section 4, I find that all pure strategy and all receiver-efficient equilibria are not plausible for two different reasons: they feature informational free-riding opportunities that generate individual profitable deviations, and they are not robust to the presence of even arbitrarily small noise in communication. Robustness to informational free-riding opportunities and to noise require equilibria to support unprejudiced beliefs (Bagwell & Ramey, 1991) and to be ε-robust (Battaglini, 2002), respectively. I also show that these two different criteria are tightly connected, as ε-robust equilibria have unprejudiced beliefs. The question is: can direct equilibria support unprejudiced beliefs and be ε-robust?

To study whether there exists direct equilibria with unprejudiced beliefs I apply the following definition, which is adapted from Bagwell and Ramey (1991) to accommodate for non-degenerate mixed strategies.[31]

Definition 7. *Given senders' strategies ϕ_j, beliefs p are unprejudiced if, for every off-path pair of reports (r_1, r_2) such that $\phi_j(r_j, \theta') > 0$ for some $j \in \{1, 2\}$ and $\theta' \in \Theta$, we have that $p(\theta''|r_1, r_2) > 0$ if and only if there is a sender $i \in \{1, 2\}$ such that $\phi_i(r_i, \theta'') > 0$.*

The next corollary confirms that there exists direct equilibria supported by unprejudiced beliefs (as in both Definition 2 and 7) that are also ε-robust.[32]

Corollary 2. *There are direct equilibria with unprejudiced beliefs that are also ε-robust.*

Even well behaved signaling games may have no equilibria (Manelli, 1996). However, given beliefs p, the equilibrium reporting strategies and supports in Proposition 2 to 4 are by construction such that no sender has individual profitable deviations. Moreover, given such strategies, the decision maker choice is sequentially rational. Therefore, as long as the assumptions established in Section 3 are satisfied, a direct equilibrium always exists.

Corollary 3. *A direct equilibrium always exists.*

[31] Definition 2, which is introduced by Vida and Honryo (2019) and is used in Section 4, is a weaker version of Definition 7. Lemma 3 applies to unprejudiced beliefs as in both definitions.

[32] Since ε-robustness implies unprejudiced beliefs, it would be sufficient to show that there exist direct equilibria that are ε-robust. Corollary 2 simply remarks that the two refinements are different.

6 An Example and Application

6.1 Example: Symmetric Environments

As follows, I provide an example where senders have similar features and the state is symmetrically distributed. This environment is an important benchmark because it deals with situations where no sender is ex-ante advantaged. In addition, it gives us a closed-form solution for senders' equilibrium strategies and supports. The following definition formalizes what I mean by a "symmetric environment."

Definition 8. *In a symmetric environment,*

i) the state is symmetrically distributed around zero, i.e., $f(\theta) = f(-\theta)$ for all $\theta \in \Theta$;

ii) $k_j C_j(r, \theta) = kC(r, \theta)$ for $j \in \{1, 2\}$, where $k > 0$ and $C(\cdot)$ satisfies $C(\theta + x, \theta) = C(\theta - x, \theta)$ for every $\theta \in \Theta$ and $x \in \mathbb{R}$;

iii) payoffs satisfy[33] *$u_{dm}(\theta) = -u_{dm}(-\theta)$ and $u_1(\theta) = -u_2(-\theta)$ for all $\theta \in \Theta$.*

Conditions i) to iii) are in addition to the assumptions in Section 3.

In symmetric environments the two senders differ only because they have opposed interests. In these cases, there is no particular reason why the decision maker should give more importance to the report of one sender than the other. Intuition would suggest that, in a symmetric environment, the decision maker should equally assign the "burden of proof" among senders. The next corollary confirms that this intuition is indeed correct in a direct equilibrium.

Corollary 4. *In a direct equilibrium of a symmetric environment, $s(r) = -r$ for every $r \in \hat{R}$.*

In a symmetric environment, the decision maker follows the recommendation of the sender that delivers the most extreme report. The burden of proof is equally distributed among senders, as Corollary 4 shows. Moreover, the swing report function is linear even though some fundamentals, e.g., the costs functions, may be non-linear. Remarkably, in symmetric environments direct equilibria naturally display symmetric strategies.[34]

With an explicit solution for the swing report function, we obtain a natural closed-form solution for the senders' equilibrium strategies and supports. In applications this is particularly useful because in similar environments, such as in contests, typically little

[33] By definition of threshold τ_j (see Section 3), this last condition implies that $\tau_2 = -\tau_1$.

[34] Corollary 4 is reminiscent of results in all-pay contests or auctions, where it is shown that with two bidders or contestants, only symmetric solutions exists (Baye et al., 1996). Moreover, in all-pay auctions the bidder with the highest bid (or the greatest effort) always wins. By contrast, here the sender with the most extreme report wins, but it may not be the one paying the highest misreporting costs.

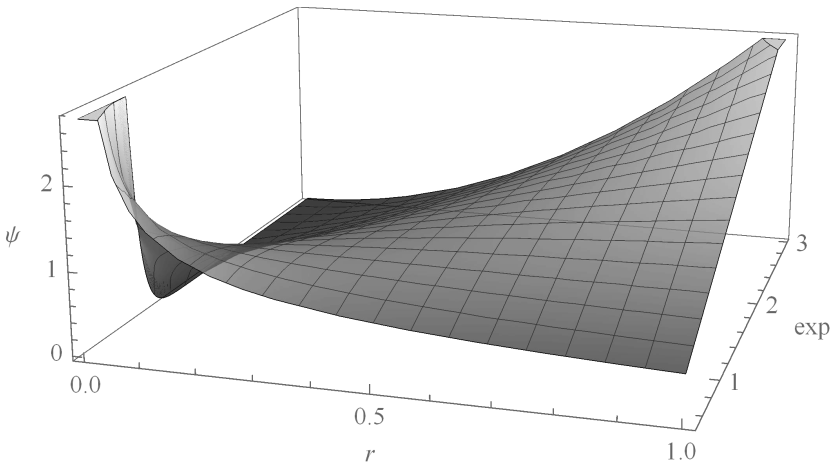

Figure 3: The partial probability density ψ as a function of the misreporting costs' shape and of the extent of misreporting, in a symmetric environment where $C(r, \theta) = |(r - \theta)^{exp}|$ and $\theta = 0$. With square loss costs, $exp = 2$, the density ψ grows linearly as reports get further away from the truth. With absolute value linear costs, $exp = 1$, every misreport in the support has the same partial density. With concave costs, $exp \in (0, 1)$, small misrepresentation are more likely than large lies, and when $exp > 1$ the opposite is true.

is known about mixed strategy equilibria except in some special cases (see Levine and Mattozzi (2019); Siegel (2009)).

I can now use this closed-form solution to examine the determinants and the features of senders' misreporting behavior. I show that the shape of the costs function, in particular its convexity/concavity or second derivative, determines whether senders are more likely to deliver large lies than small misrepresentation or the other way around. From Proposition 2 and Corollary 4 we obtain that, in a symmetric environment, misreporting behavior is described by the following partial density, for $j \in \{1, 2\}$ and $j \neq i$,

$$\psi_j(r_j, \theta) = \frac{k}{-u_i(\theta)} \frac{dC(-r_j, \theta)}{dr_j}.$$

Therefore, if $C(\cdot)$ is strictly convex, we have that $d\psi_1(r_1, \theta)/dr_1 > 0$ for all $\theta \in S_1(\theta) \setminus \{\theta\}$ and $d\psi_2(r_2, \theta)/dr_2 < 0$ for all $\theta \in S_2(\theta) \setminus \{\theta\}$. This means that, conditional on misreporting, senders are more likely to deliver large misrepresentation of the state rather than small lies. By contrast, when senders have concave costs, misreports that are closer to the truth are more likely to be delivered than large lies. The type of senders' interim misreporting behavior is entirely driven by the shape of the cost function C, and not by k or by utilities u_j. Figure 3 shows senders' misreporting behavior for different concavities of the misreporting costs function.

6.2 Application: Judicial Procedures

In a seminal paper, Shin (1998) compares the informative value of adversarial and inquisitorial procedures. Under the adversarial procedure, two parties with opposing interests make their case to an uninformed decision maker. By contrast, the inquisitorial procedure requires the decision maker to adjudicate only based on her own acquired information. The question of which procedure allows the decision maker to take more informed decisions is of interest in a host of applications.

To answer this question, Shin (1998) studies a model of verifiable disclosure where parties can either disclose or withhold information, but they cannot misrepresent evidence because such information is fully verifiable. In the adversarial procedure, the decision maker cannot rely on the information of the interested parties to secure full revelation because the two parties may be uninformed. In the inquisitorial procedure, the decision maker obtains with some probability an informative signal of the underlying evidence. The tension faced by the decision maker is thus that of obtaining two pieces of biased information versus one piece of unbiased information. Within this framework, Shin (1998) finds that the adversarial procedure is always superior to the inquisitorial procedure.

This sharp result raises a natural question: why then systems that are reminiscent of inquisitorial procedures so are often used in practice? On this point, Shin (1998) argues that the assumption of full verifiability might play a key role in determining the superiority of adversarial procedures, and that "potential violations of the verifiability assumption will be an important limiting factor in qualifying our findings in favor of the adversarial procedure" (Shin, 1998, p. 403).

Here, I analyze the validity of this conjecture by using results derived in this paper. The framework introduced in Section 3 allows me to model the adversarial procedure for when information is not fully verifiable and parties can misrepresent evidence. Results derived in Section 4 suggest that under this procedure the decision maker cannot plausibly achieve receiver-efficiency and obtain the full information welfare W_{fi}. Moreover, if we accept that conditions (C) and (D) are sensible modeling assumptions, then results in Section 5 indicate that the ex-ante equilibrium welfare of the decision maker is also strictly lower and bounded away from W_{fi}. To see this, notice that the expected payoff obtained by the decision maker in direct equilibria is bounded above by $\bar{W}_{dm}$, where[35]

$$\bar{W}_{dm} = \underbrace{\int_0^{\max\Theta} f(\theta)u_{dm}(\theta)d\theta}_{=W_{fi}} + \underbrace{\int_{\theta_1}^{0} f(\theta)u_{dm}(\theta)(1-\alpha_1(\theta))\alpha_2(\theta)d\theta}_{<0} < W_{fi}.$$

[35]The upper bound $\bar{W}_{dm}$ is obtained by assuming that the decision maker makes less mistakes than she would in a direct equilibrium: she mistakenly selects $\oplus$ only when $\theta \in (\theta_1, 0)$ and sender 2 reports truthfully while sender 1 misreports. Otherwise, she chooses the correct alternative. Therefore, $\bar{W}_{dm}$ is an upper bound of the ex-ante welfare obtained by the decision maker in direct equilibria.

To model the inquisitorial procedure, I follow Shin (1998) in assuming that the decision maker obtains with probability q a potentially noisy signal σ of the realized state θ. It is straight forward to see that, under the inquisitorial procedure, the decision maker can obtain an expected payoff which, for high q and sufficiently precise σ, is arbitrarily close to W_{fi} and thus higher than $\bar{W}_{dm}$. Therefore, there is always a combination of parameters under which the inquisitorial procedure is superior to the adversarial procedure in that it yields more information to the decision maker. The conjecture of Shin (1998) is thus proved correct for any finite intensity of misreporting costs $k_j > 0$.

It is worth pointing out that, in addition to the verifiability assumption, there are other modeling differences between my setting and Shin (1998): first, I assume that the two parties are always perfectly informed about the realized state, while in Shin (1998) they may be uninformed or observe a noisy signal of the realized state; second, I consider a decision maker that is less informed than the two parties, while in Shin (1998) every player is, on average, equally informed.[36] These two differences give in my setting a relative advantage to the adversarial procedure, and therefore add further force to the potential superiority of inquisitorial procedures.[37]

7 Concluding Remarks

This article studies a multi-sender signaling model with two informed senders and one uninformed decision maker. Senders have perfectly correlated information, which they can misreport at a cost that is tied to the magnitude of misrepresentation. This setting covers a number of applications in economics and politics, including electoral campaigns, contested takeovers, lobbying, informative advertising, and judicial decision making.

I restrict attention to equilibria where the decision maker's posterior beliefs satisfy a first-order stochastic dominance condition. Fully revealing, receiver-efficient, and pure strategy equilibria exist, but they are not robust. I identify two natural restrictions on the decision maker's posterior beliefs under which equilibria always exists, are robust, and are essentially unique. I dub equilibria that satisfy these two conditions as "direct equilibria."

Therefore, this paper provides a tractable and appealing approach to study strategic communication from multiple senders with common information that is neither fully verifiable nor totally "cheap." As an application of direct equilibria, I study the informative value of judicial procedures and show that, when information is not fully verifiable, then inquisitorial systems may be superior than adversarial systems.

[36] In Shin (1998), as we increase the decision maker's ability to gather precise information in the inquisitorial system, we also increase the information possessed in expectation by the contending parties precisely because all players are assumed to be equally informed on average.

[37] Moreover, in my setting "withholding" is not possible or it is prohibitively expensive. In Shin (1998), if parties are perfectly informed but cannot withhold information, then the decision maker could obtain full revelation out of the adversarial procedure, making it always superior than the inquisitorial system.

The transmission of information in direct equilibria takes place in a qualitatively different way with respect to related models of strategic communication. I conclude that the introduction of misreporting costs is not just a technical twist that adds an element of realism; rather, it is an essential component to understand the strategic interaction underlying the setting considered in this paper.[38]

[38] Accounting for misreporting costs also allows to perform comparative statics on such costs that are currently unexplored. For example, it allows to study the effects of "fake news laws" or of technological advancements such as "deepfake videos" which affect senders' misreporting costs. This is left for future research.

A Appendix

Lemma 1. *In every perfect Bayesian equilibrium satisfying* (FOSD)*,* $\min S_j(\theta) \geq \theta$ *for* $\theta \geq \tau_j$ *and* $\max S_j(\theta) \leq \theta$ *otherwise,* $j \in \{1, 2\}$*.*

Proof of Lemma 1. Consider a PBE satisfying (FOSD) and consider a state $\theta \geq \tau_1$. For sender 1, every report $r_1 < \theta$ is dominated by truthful reporting because $C_1(r_1, \theta) > 0 = C_1(\theta, \theta)$ and by (FOSD) we have that $U_{dm}(\theta, r_2) \geq U_{dm}(r_1, r_2)$ for every $r_2 \in R_2$. Therefore, it must be that $r_1 \notin S_1(\theta)$ for all $r_1 < \theta$ and $\theta \geq \tau_1$. A similar argument applies to sender 2 and to states $\theta \leq \tau_j$, $j \in \{1, 2\}$. □

A.1 Receiver-efficient and Pure Strategy Equilibria

Lemma 2. *An equilibrium is receiver-efficient if and only if it is in pure strategies.*

Proof of Lemma 2. Consider a pure strategy equilibrium and suppose that it is not receiver-efficient, e.g., because $\beta(\rho_1(\theta'), \rho_2(\theta')) = \odot$ for some $\theta' \geq 0$. In equilibrium, senders never engage in misreporting to implement their less preferred alternative with certainty, and therefore it must be that $\rho_1(\theta') = \theta'$. Beliefs p must be such that $\beta(r_1, \rho_2(\theta')) = \odot$ for all $r_1 \in (\underline{r}_1(\theta'), \bar{r}_1(\theta'))$, otherwise sender 1 would have a profitable deviation. The pair of reports $(\theta', \rho_2(\theta'))$ can induce $\odot$ only if $(\rho_1(\theta''), \rho_2(\theta'')) = (\theta', \rho_2(\theta'))$ for some $\theta'' < 0$. There is no $\theta \in [\tau_1, 0)$ such that sender 1 would misreport by delivering $r_1 = \theta' \geq 0$ to implement $\odot$, thus it must be that $\theta'' < \tau_1$. Since there is always a $r_1' \in (\underline{r}_1(\theta'), \theta')$ such that $C_1(r_1', \theta'') < C_1(\theta', \theta'')$ and $\beta(r_1', \rho_2(\theta'')) = \odot$, sender 1 has a profitable deviation in state θ'', contradicting that there exists a pure strategy equilibrium that is not receiver-efficient.

Now consider a REE and suppose that it is not in pure strategies, but there is a state $\theta' \in \Theta$ and sender $j \in \{1, 2\}$ such that $S_j(\theta') \supseteq \{r_j', r_j''\}$, with $r_j' \neq r_j''$. Since in a REE we have that $\beta(r_1', r_2') = \beta(r_1'', r_2'')$ for every $r_i', r_i'' \in S_i(\theta)$, $i \in \{1, 2\}$, it must be that $C_j(r_j', \theta') = C_j(r_j'', \theta')$. By Lemma 1, this is possible only if $r_j' = r_j''$, contradicting that there exists a REE that is not in pure strategies. □

Proposition 1. *There are no receiver-efficient equilibria with unprejudiced beliefs.*

Proof of Proposition 1. In a REE, senders play pure strategies (Lemma 2) and the decision maker always selects her preferred alternative *as if* under complete information, that is, $\beta(\rho_1(\theta), \rho_2(\theta)) = \oplus$ for all $\theta \geq 0$ and $\beta(\rho_1(\theta), \rho_2(\theta)) = \odot$ otherwise. Since misreporting is costly, senders report truthfully in states where their least preferred alternative is implemented: $\rho_2(\theta) = \theta$ for all $\theta \in [0, \tau_2]$ and $\rho_1(\theta) = \theta$ for all $\theta \in [\tau_1, 0)$. However, there are no REE where $\rho_j(\theta) = \theta$ for all $\theta \in [\tau_1, \tau_2]$, $j \in \{1, 2\}$: there would always be a state $\theta \in (\tau_1, \tau_2)$ and an off-path pair of reports (r_1, r_2), $r_1 \neq r_2$, such that a sender can profitably deviate from truthful reporting (see also Observation 1). Therefore, in every

REE either sender 1 misreports in some state $\theta \in [0, \tau_2)$, or sender 2 misreports in some $\theta \in (\tau_1, 0]$, or both.

Consider now a REE where $\rho_1(\theta') \neq \theta'$ for some $\theta' \in [0, \tau_2)$. By Lemma 1, we have that $\rho_1(\theta') > \theta'$. To sustain the equilibrium, off-path beliefs p must be such that $\beta(r_1, \theta') = \ominus$ for all $r_1 \in [\theta', \rho_1(\theta'))$ and $\beta(\rho_1(\theta''), r_2) = \oplus$ for all $r_2 \in (\underline{r}_2(\theta''), \theta'']$ and $\theta'' \in [\theta', \tau_2)$. This implies that there must be an open set S of non-negative states such that $\rho_1(\theta''') \geq \rho_1(\theta') > \theta''' = \rho_2(\theta''')$ for all $\theta''' \in S$. It follows that, for every $\theta''' \in S$, the pair of reports (θ''', θ''') is off-path. By Lemma 1, and since $\rho_2(\theta) = \theta$ for all $\theta \in [0, \tau_2]$ and $\rho_1(\theta) = \theta$ for all $\theta \in [\tau_1, 0)$, we have that beliefs p are unprejudiced (Definition 2) only if $p(\theta|\theta''', \theta''') = 0$ for all $\theta < 0$. Therefore, unprejudiced beliefs imply that $\beta(\theta''', \theta''') = \oplus$, and thus sender 1 can profitably deviate by reporting the truth in state $\theta''' \in S$. A similar argument applies for REE where $\rho_2(\theta') \neq \theta'$ for some $\theta' \in (\tau_1, 0]$. Therefore, there are no REE (and, by Lemma 2, no pure strategy equilibria) with unprejudiced beliefs. □

Lemma 3. *If a perfect Bayesian equilibrium is ε-robust, then it has unprejudiced beliefs.*

Proof. Consider the posterior beliefs $p_{G,\varepsilon}$ that the strategies ϕ_j of a PBE (see Section 5 for the notation used to describe mixed strategies) induce in an ε-perturbed game for some distribution G and sequence ε^n,

$$p_{G,\varepsilon}(\theta|r_1, r_2) = f(\theta)\frac{p(r_1, r_2|\theta)}{p(r_1, r_2)}$$
$$= \frac{f(\theta)\left[\varepsilon_1\varepsilon_2 g_1(r_1)g_2(r_2) + \varepsilon_1(1-\varepsilon_2)g_1(r_1)\phi_2(r_2,\theta) + (1-\varepsilon_1)\varepsilon_2 g_2(r_2)\phi_1(r_1,\theta)\right]}{\varepsilon_1\varepsilon_2 g_1(r_1)g_2(r_2) + \varepsilon_1(1-\varepsilon_2)g_1(r_1)\int_\Theta f(\theta)\phi_2(r_2,\theta)d\theta + (1-\varepsilon_1)\varepsilon_2 g_2(r_2)\int_\Theta f(\theta)\phi_1(r_1,\theta)d\theta}.$$

As $\varepsilon^n \to 0^+$ the event in which both reports are wrongly delivered or observed becomes negligible, and thus we have that $p_{G,\varepsilon} \to p_{G,0^+}$, where

$$p_{G,0^+}(\theta|r_1, r_2) = \frac{f(\theta)\left[\varepsilon_1 g_1(r_1)\phi_2(r_2,\theta) + \varepsilon_2 g_2(r_2)\phi_1(r_1,\theta)\right]}{\varepsilon_1 g_1(r_1)\int_\Theta f(\theta)\phi_2(r_2,\theta)d\theta + \varepsilon_2 g_2(r_2)\int_\Theta f(\theta)\phi_1(r_1,\theta)d\theta}. \tag{5}$$

From (5) we obtain that, for any distribution G with full support and any sequence $\varepsilon^n \to 0^+$, $p_{G,0^+}(\theta|r_1, r_2) > 0$ if and only if $\phi_j(r_j, \theta) > 0$ for some $j \in \{1, 2\}$. By Definition 7 (and thus even by Definition 2) we get that the limit beliefs $p_{G,0^+}$ are unprejudiced, and therefore every PBE that is ε-robust has unprejudiced beliefs.[39] □

A.2 Direct Equilibria

Lemma 4. *In a direct equilibrium, every report $r \in \hat{R}$ has a swing report $s(r) \in \hat{R}$ such that (i) if $r \gtrless 0$ then $s(r) \lessgtr 0$ and $s(0) = 0$; (ii) $s(s(r)) = r$; (iii) for every $r \in \hat{R}$,*

[39]Notice that the proof of Lemma 3 readily extends to a n-senders version of the game, for any finite $n \geq 2$. In particular, given a profile of reports $(r_1, \ldots, r_n)$ and a set of senders $N = \{1, \ldots, n\}$, then $p_{G,0^+}(\theta|r_1, \ldots, r_n) > 0$ if and only if $\phi_j(r_j, \theta) > 0$ for $n-1$ senders. This is consistent with the idea behind unprejudiced beliefs that the decision maker conjectures deviations as individual.

$\frac{ds(r)}{dr} < 0$; *(iv)* $s(\bar{r}_1(0)) = \underline{r}_2(0)$.

Proof. Consider a report r_1 by sender 1 such that $r_1 \in (0, \bar{r}_1(0)]$. By (C) and (D) we obtain that $U_{dm}(r_1, \underline{r}_2(0)) < 0 < U_{dm}(r_1, 0)$, and therefore there must exists a $r_2 \in [\underline{r}_2(0), 0)$ such that $U_{dm}(r_1, r_2) = 0$. Thus, $r_2 = s(r_1)$. A similar argument holds for a report $r_2 \in [\underline{r}_2(0), 0)$. It follows that, for every $r \in \hat{R}$, there exists a $s(r) \in \hat{R}$ such that if $r > 0$ then $s(r) < 0$, and if $r < 0$ then $s(r) > 0$. From (C) and Definition 5 we obtain that $s(0) = 0$ and $s(\bar{r}_1(0)) = \underline{r}_2(0)$. From Definition 5 and point (i) we get that if $r' = s(r)$ then $r = s(r')$, and thus $s(s(r)) = r$. By applying the implicit function theorem and (D) on $s(r)$, we obtain that for every $r \in \hat{R}$, $\frac{ds(r)}{dr} < 0$. □

Lemma A.1. *In a direct equilibrium, truthful cutoffs are such that* $\theta_1 < 0 < \theta_2$ *and* $(\theta_1, \theta_2) \subset [\tau_1, \tau_2] \cap \hat{R}$.

Proof. By Lemma 4 we have that $s(\bar{r}_1(0)) = \underline{r}_2(0) < 0$ and, for every $r \in \hat{R}$, $ds(r)/dr < 0$. Moreover, $d\underline{r}_2(\theta)/d\theta > 0$ and thus $\underline{r}_2(\theta) > \underline{r}_2(0)$ for every $\theta > 0$. Since $s(0) = 0$, there is a state $\theta' \in (0, \bar{r}_1(0))$ such that $s(\theta') = \underline{r}_2(\theta')$. From Definition 5, we obtain that $\theta' = \theta_2 \in (0, \bar{r}_1(0))$. Similarly, we get that $\theta_1 \in (\underline{r}_2(0), 0)$. Since $\bar{r}_1(\tau_1) = \tau_1 < 0$ and $\underline{r}_2(\tau_2) = \tau_2 > 0$, it follows from Definition 6 that $(\theta_1, \theta_2) \subset [\tau_1, \tau_2]$. □

Lemma 5. *In a direct equilibrium,* $S_j(\theta) = \{\theta\}$ *for all* $\theta \notin (\theta_1, \theta_2)$*, and* $|S_j(\theta)| > 1$ *for every* $\theta \in (\theta_1, \theta_2)$*,* $j \in \{1, 2\}$*.*

Proof. I begin by proving first that $S_j(\theta) = \{\theta\}$ for all $\theta \notin (\theta_1, \theta_2)$. Consider a DE and a state $\theta \geq \theta_2$. Since by Lemma 1 we have that $\min S_1(\theta) \geq \theta \geq \theta_2$, it must be that $S_2(\theta) = \{\theta\}$ as $s(r_1) \leq \underline{r}_2(\theta)$ for every $r_1 \in S_1(\theta)$. Since $\beta(\theta, \theta) = \oplus$, sender 1 best replies to $r_2 = \theta$ with $r_1 = \theta$ and thus $S_1(\theta) = \{\theta\}$ as well. A similar argument applies to states $\theta \leq \theta_1$, completing the first part of the proof. Note that when $\theta = \theta_1$, sender 1 is actually indifferent between reporting θ_1 and $\bar{r}_1(\theta_1)$. Since this is a measure zero event which is irrelevant for the analysis that follows, I will consider only the case where $S_1(\theta_1) = \{\theta_1\}$, without any loss of generality.

I turn now to prove that $S_j(\theta)$ contains more than one element for every $\theta \in (\theta_1, \theta_2)$. Suppose by way of contradiction that $S_1(\theta) = \{r_1\}$ for some $\theta \in (\theta_1, \theta_2)$. By Lemma 1, we have that $r_1 \geq \theta$. Consider first the case where $\theta \leq r_1 < 0$. In a DE, sender 2 best replies to $r_1 \in [\theta, 0)$ with $r_2 = \theta = S_2(\theta)$ because, by (C) and (D), we get $\beta(r_1, \theta) = \odot$. However, sender 1 can profitably deviate from the prescribed strategy by delivering $r_1' = s(\theta)$, where $0 < s(\theta) < \bar{r}_1(\theta)$ (Lemmata 4 and A.1), contradicting that $S_1(\theta) = \{r_1\}$. Consider now the case where $r_1 \geq 0$ and $r_1 \geq \theta$. If $s(r_1) \leq \underline{r}_2(\theta)$, then it must be that $S_2(\theta) = \{\theta\}$. By Definition 6 and Lemma 4 we have that $\underline{r}_2(\theta) < 0$ and $r_1 \geq s(\underline{r}_2(\theta)) > 0$. Since $\underline{r}_2(\theta) < \theta$, sender 1 can profitably deviate from the prescribed strategy by reporting either $r_1' = s(\theta) \in (0, r_1)$ if $\theta < 0$, or $r_1' = \theta$ if $\theta \geq 0$, as in both cases we get that $\beta(r_1', \theta) = \oplus$ and $C_1(r_1', \theta) < C_1(r_1, \theta)$. If instead $s(r_1) > \underline{r}_2(\theta)$, then sender 2 must be delivering some

$r_2' \in (\underline{r}_2(\theta), s(r_1))$. Therefore, if $r_1 > \theta$, then sender 1 is strictly better off by reporting θ rather than r_1 because $\beta(\theta, r_2') = \beta(r_1, r_2') = \circleddash$ and $C_1(r_1, \theta) > 0 = C_1(\theta, \theta)$. If instead $r_1 = \theta$, then $\theta \geq 0$ and since $\underline{r}_2(\theta) \geq \underline{r}_2(0)$ we have that $s(r_2') \leq \bar{r}_1(\theta)$ (Lemma 4). In this case, sender 1 can profitably deviate from the prescribed strategy by reporting $r_1' = s(r_2')$. Similar arguments apply to $S_2(\theta) = \{r_2\}$, completing the proof. $\square$

Lemma A.2. *In a direct equilibrium, for every $\theta \in (\theta_1, \theta_2)$ supports $S_j(\theta)$ are such that*

$$\max S_1(\theta) \leq \min \{\bar{r}_1(0), \bar{r}_1(\theta), s(\underline{r}_2(\theta))\},$$

$$\min S_2(\theta) \geq \max \{\underline{r}_2(0), \underline{r}_2(\theta), s(\bar{r}_1(\theta))\}.$$

Proof. Consider a DE and a $\theta \in (\theta_1, \theta_2)$. By definition of reach (equations (1) and (2)) every $r_1 > \bar{r}_1(\theta)$ is strictly dominated by truthful reporting, and thus $\max S_1(\theta) \leq \bar{r}_1(\theta)$. Similarly, we obtain that $\min S_2(\theta) \geq \underline{r}_2(\theta)$ and therefore by (D) and by Definition 5 every $r_1 > s(\underline{r}_2(\theta))$ is dominated by $r_1' = s(\underline{r}_2(\theta))$ and every $r_2 < s(\bar{r}_1(\theta))$ is dominated by $r_2' = s(\bar{r}_1(\theta))$. Thus, $\max S_1(\theta) \leq s(\underline{r}_2(\theta))$ and $\min S_2(\theta) \geq s(\bar{r}_1(\theta))$. For every $\theta \in [0, \theta_2)$ we have $\bar{r}_1(\theta) \geq \bar{r}_1(0)$ and $\underline{r}_2(\theta) \geq \underline{r}_2(0)$, and thus $\min S_2(\theta) \geq \underline{r}_2(0)$. Since $s(\underline{r}_2(0)) = \bar{r}_1(0)$ (Lemma 4), it follows by (D) and by Definition 5 that $s(r_2) \leq \bar{r}_1(0)$ for every $r_2 \in S_2(\theta)$, and thus $\max S_1(\theta) \leq \bar{r}_1(0)$. Similarly, we obtain that $\min S_2(\theta) \geq s(\bar{r}_1(0))$ for every $\theta \in (\theta_1, 0)$. $\square$

Lemma A.3. *In a direct equilibrium, $r_2 \notin S_2(\theta)$ for every $r_2 \in (s(\min S_1(\theta)), \theta)$ and $\theta > 0$, and $r_1 \notin S_1(\theta)$ for every $r_1 \in (\theta, s(\max S_2(\theta)))$ and $\theta < 0$.*

Proof. Consider a $\theta \in (0, \theta_2)$. By Lemmata 1 and 4 we have that $s(\min S_1(\theta)) < 0$, and by Definition 5 we have that $\beta(r_1, r_2) = \oplus$ for every $r_1 \in S_1(\theta)$ and $r_2 \in (s(\min S_1(\theta)), \theta)$. Therefore, for sender 2 every $r_2 \in (s(\min S_1(\theta)), \theta)$ is strictly dominated by truthful reporting, and thus $r_2 \notin S_2(\theta)$. A similar argument applies to sender 1 for $\theta \in (\theta_1, 0)$ and Lemma 5 shows the case $\theta \notin (\theta_1, \theta_2)$, completing the proof. $\square$

Lemma A.4. *In a direct equilibrium, for every $\theta \in (\theta_1, \theta_2)$, reports $r_1 \in (\min S_1(\theta), \max S_1(\theta))$ have $s(r_1) > \underline{r}_2(\theta)$, and reports $r_2 \in (\min S_2(\theta), \max S_2(\theta))$ have $s(r_2) < \bar{r}_1(\theta)$.*

Proof. Suppose not, and consider $r_1' \in (\min S_1(\theta), \max S_1(\theta))$ for some $\theta \in (\theta_1, \theta_2)$ such that $s(r_1') < \underline{r}_2(\theta)$. By Definition 6 we have $\underline{r}_2(\theta) < 0$ and by Lemma 4 we have $s(\underline{r}_2(\theta)) < r_1'$. This is in contradiction with Lemma A.2, which states that $\max S_1(\theta) \leq s(\underline{r}_2(\theta))$. A similar argument holds for reports $r_2 \in (\min S_2(\theta), \max S_2(\theta))$, completing the proof. $\square$

Lemma A.5. *In a direct equilibrium, $\alpha_j(r_j, \theta) = 0$ for all $r_j \in (\min S_j(\theta), \max S_j(\theta))$, $j \in \{1, 2\}$, and $\theta \in (\theta_1, \theta_2)$.*

Proof. Consider a $\theta \in (\theta_1, \theta_2)$ and suppose that there is a DE where sender 1's strategy $\phi_1(\theta)$ has an atom $\alpha_1(r_1', \theta) > 0$ in some report $r_1' \in (\min S_1(\theta), \max S_1(\theta))$. By Lemma A.4

we have that $s(r_1') > \underline{r}_2(\theta)$. The expected payoff of sender 2 is discontinuous around $r_2 = s(r_1')$ and thus it must be that, for some $\epsilon > 0$ small enough, $(s(r_1'), s(r_1') + \epsilon) \cap S_2(\theta) = \varnothing$. Therefore, there exists an ϵ' small enough such that sender 1 can profitably deviate from the prescribed strategy by moving probability from r_1' to some $r_1'' \in (s\left(s(r_1') + \epsilon'\right), r_1')$, where by Lemma 4 $s\left(s(r_1') + \epsilon'\right) < r_1'$, thus contradicting that this is an equilibrium. A similar argument applies to atoms in sender 2's strategy, completing the proof. □

Lemma A.6. *In a direct equilibrium,* $\min S_1(\theta) = \theta$ *for all* $\theta \geq 0$*, and* $\max S_2(\theta) = \theta$ *for all* $\theta \leq 0$.

Proof. Consider a DE and a $\theta \geq 0$. By Lemma 1, it must be that $\min S_1(\theta) \geq \theta$. Suppose by way of contradiction that $\min S_1(\theta) > \theta$. By Lemma 5 it has to be that $\theta < \theta_2$ and by Lemma A.3 we obtain that $S_2(\theta) \cap (s(\min S_1(\theta)), \theta) = \varnothing$. Therefore, unless sender 2's strategy has an atom $\alpha_2(s(\min S_1(\theta)), \theta) > 0$, we have that $\Phi_2(s(\min S_1(\theta)), \theta) = \Phi_2(s(\theta), \theta)$. However, since $\beta(r_1, s(\min S_1(\theta))) = \oplus$ for all $r_1 \in S_1(\theta)$ and $C_2(s(\min S_1(\theta)), \theta) > 0$, it must be that $\alpha_2(s(\min S_1(\theta)), \theta) = 0$ as $s(\min S_1(\theta))$ is strictly dominated by $r_2 = \theta$. Hence, for some $\epsilon > 0$, sender 1 can profitably deviate from the prescribed strategy by moving probability from every $r_1 \in [\min S_1(\theta), \min S_1(\theta) + \epsilon) \cap S_1(\theta)$ to $r_1 = \theta$, contradicting that there can be a DE with $\min S_1(\theta) > \theta$ for a $\theta \geq 0$. A similar argument holds for sender 2 and $\theta \leq 0$, completing the proof. □

Lemma A.7. *In a direct equilibrium,* $|S_j(\theta) \setminus \{\theta\}| > 1$ *for every* $\theta \in (\theta_1, \theta_2)$ *and* $j \in \{1, 2\}$.

Proof. Consider a DE and a state $\theta \in [0, \theta_2)$. By Lemma A.6 we have that $\min S_1(\theta) = \theta$, and by Lemma 5 we have that $|S_1(\theta)| > 1$. Suppose by way of contradiction that $S_1(\theta) \setminus \{\theta\} = \{r_1\}$ for some $r_1 > 0$. Since $C_1(r_1, \theta) > 0$, in equilibrium it must be that r_1 induces $\oplus$ with strictly higher probability than truthful reporting. This implies that there is some $r_2 \in [s(r_1), s(\theta))$ in the support of sender 2's strategy, $r_2 \in S_2(\theta)$. Since reports that are further away from the realized state are more expensive, it must be that $\alpha_2(r_2', \theta) > 0$ for some $r_2' \in [s(r_1), s(\theta))$, and $\phi_2(r_2, \theta) = 0$ for all $r_2 \in [s(r_1), r_2')$. But then sender 1 can profitably deviate from the prescribed strategy by moving probability from r_1 to $s(r_2')$, contradicting that this is an equilibrium.

Consider now the case where $\theta \in (\theta_1, 0)$ and suppose again that $S_1(\theta) \setminus \{\theta\} = \{r_1\}$. By Lemma 5, we have that $|S_j(\theta)| > 1$ for $j \in \{1, 2\}$, and thus $\min S_1(\theta) = \theta$. By Lemmata A.2 and A.3 we have that $r_1 \geq s(\theta) > 0$ and $\max S_2(\theta) = \theta$. If $r_1 = s(\theta)$, then sender 2 can profitably deviate from the prescribed strategy by always reporting $\theta - \epsilon$ for some $\epsilon > 0$ small enough. If instead $r_1 > s(\theta)$, then it must be that $S_2(\theta) \cap [s(r_1), \theta) = \varnothing$ as every $r_2 \in [s(r_1), \theta)$ would be strictly dominated by truthful reporting. Since $|S_2(\theta)| > 1$, there must be some $r_2 < s(r_1)$ such that $r_2 \in S_2(\theta)$. Therefore, sender 1 can profitably deviate by moving probability from r_1 to $s(\theta)$, contradicting that this is an equilibrium. A similar argument applies to $S_2(\theta) \setminus \{\theta\}$, completing the proof. □

Lemma 6. *In a direct equilibrium, $S_j(\theta)\setminus\{\theta\}$ is convex for all $\theta \in (\theta_1, \theta_2)$ and $j \in \{1, 2\}$.*

Proof. Consider a DE and a state $\theta \in (\theta_1, \theta_2)$. By Lemma A.7 we have that $|S_j(\theta)\setminus\{\theta\}| > 1$, $j \in \{1, 2\}$. Suppose by way of contradiction that $S_1(\theta) \setminus \{\theta\}$ is not convex, but instead there are two reports $r_1', r_1'' \in S_1(\theta) \setminus \{\theta\}$ with $r_1' < r_1''$, such that $r_1 \notin S_1(\theta) \setminus \{\theta\}$ for every $r_1 \in (r_1', r_1'')$. By Lemmata 1, 4, and A.3 we have that $r_1' > 0$, $r_1' \geq s(\theta)$, and $s(r_1'') < s(r_1') < 0$. Since $C_1(r_1'', \theta) > C_1(r_1', \theta)$ and $\frac{dC_j(r,\theta)}{dr} > 0$ for every $r > \theta$, it must be that every report $r_1 \geq r_1''$ such that $\phi_1(r_1, \theta) > 0$ induces the implementation of alternative $\oplus$ with a strictly higher probability than every report $r_1''' \leq r_1'$ such that $\phi_1(r_1''', \theta) > 0$. This is possible only if $r_2 \in S_2(\theta)$ for some $r_2 \in [s(r_1''), s(r_1')]$. Since $\Phi_1(r_1, \theta)$ is constant for all $r_1 \in (r_1', r_1'')$, it must be that sender 2's strategy has an atom $\alpha_2(r_2, \theta) > 0$ in some $r_2 \in (s(r_1''), s(r_1')]$, and $\phi_2(r_2', \theta) = 0$ for all $r_2' \in [s(r_1''), s(r_1')]$ such that $r_2' \neq r_2$. However, for some $\epsilon > 0$ small enough, sender 1 can profitably deviate from the prescribed strategy by moving probability from all $r_1 \in [r_1'', r_1'' + \epsilon)$ such that $r_1 \in S_1(\theta)$ to $s(r_2) < r_1''$, contradicting that this is an equilibrium. A similar argument applies to $S_2(\theta) \setminus \{\theta\}$, completing the proof. □

Lemma 7. *In a direct equilibrium, strategies $\phi_j(\theta)$ have no atoms in $S_j(\theta)\setminus\{\theta\}$ for every $\theta \in (\theta_1, \theta_2)$ and $j \in \{1, 2\}$.*

Proof. Lemma 5 shows that $|S_j(\theta)| > 1$ for all $\theta \in (\theta_1, \theta_2)$ and Lemma A.5 shows that $\phi_j(\theta)$ has no atoms in $(\min S_j(\theta), \max S_j(\theta))$. Consider a $\theta \in (\theta_1, \theta_2)$, and suppose that $\phi_1(\theta)$ has an atom in $\max S_1(\theta)$, i.e., $\alpha_1(\max S_1(\theta), \theta) > 0$. By Lemma A.2, we have that $\max S_1(\theta) \leq \min\{s(\underline{r}_2(\theta)), \bar{r}_1(\theta)\}$ and $\min S_2(\theta) \geq \max\{\underline{r}_2(\theta), s(\bar{r}_1(\theta))\}$. If $\min S_2(\theta) > s(\max S_1(\theta))$, then sender 1 can profitably deviate from the prescribed strategy by moving probability from the atom in $\max S_1(\theta)$ to some $r_1 \in [s(\min S_2(\theta)), \max S_1(\theta))$. If $\min S_2(\theta) = s(\max S_1(\theta))$, then, since the probability of implementing $\ominus$ is discontinuous in $r_2 = s(\max S_1(\theta))$, it must be that $r_2' \notin S_2(\theta)$ for all $r_2' \in [s(\max S_1(\theta)), s(\max S_1(\theta)) + \epsilon]$ and some $\epsilon > 0$. Otherwise, sender 2 could profitably deviate by moving probability from some $r_2' \in [s(\max S_1(\theta)), s(\max S_1(\theta)) + \epsilon]$ to some report $r_2'' = s(\max S_2(\theta)) - \epsilon'$ for some $\epsilon' > 0$. However, this would contradict Lemmata A.7 and 6, and thus it would not be possible in a DE.

Suppose now that $\phi_1(\theta)$ has an atom in $\min S_1(\theta)$, i.e., $\alpha_1(\min S_1(\theta), \theta) > 0$. By Lemma A.6, if $\theta \geq 0$ then $\min S_1(\theta) = \theta$, and thus suppose that $\theta \in (\theta_1, 0)$ and that $\min S_1(\theta) > \theta$ when $\theta < 0$. By Lemmata 4, A.3, and A.6 we have that $\min S_1(\theta) \geq s(\theta) > 0$. If $\min S_1(\theta) = s(\theta)$, then it must be that $\phi_2(\theta, \theta) = 0$, otherwise sender 2 could profitably deviate from the prescribed strategy by moving probability from θ to $\theta - \epsilon$ for some $\epsilon > 0$ small enough. But then, the atom in $\min S_1(\theta)$ would be strictly dominated by truthful reporting as $C_1(s(\theta), \theta) > 0$ and $\beta(s(\theta), r_2) = \ominus$ for every $r_2 \in S_2(\theta)$, contradicting that this is an equilibrium. Consider now the case where $\min S_1(\theta) > s(\theta)$. By definition, we have that $\Phi_1(r_1, \theta) = 0$ for every $r_1 < \min S_1(\theta)$, and by Lemma 4 we have that

$s(\min S_1(\theta)) < \theta$. Therefore, it must be that $\phi_2(r_2, \theta) = 0$ for every $r_2 \in [s(\min S_1(\theta)), \theta)$. However this implies that, for sender 1, $\min S_1(\theta)$ is dominated by $s(\theta)$, contradicting that this can be an equilibrium. Similar arguments hold for atoms $\alpha_2(r_2, \theta)$ for $r_2 \in S_2(\theta) \setminus \{\theta\}$, completing the proof. □

Proposition 2. *In a direct equilibrium, for every $\theta \in (\theta_1, \theta_2)$ and $i, j \in \{1, 2\}$ with $i \neq j$, sender j delivers report $r_j \in S_j(\theta) \setminus \{\theta\}$ according to*

$$\psi_j(r_j, \theta) = \frac{k_i}{-u_i(\theta)} \frac{dC_i(s(r_j), \theta)}{dr_j}.$$

Proof. Consider a DE and a state $\theta \in (\theta_1, \theta_2)$. Given strategy $\phi_1(\theta)$, sender 2 gets an expected utility of $W_2^\omega(r_2, \theta) = (1 - \Phi_1(s(r_2), \theta))u_2(\theta) - k_2 C_2(r_2, \theta)$ from delivering $r_2 \in S_2(\theta) \setminus \{\theta\}$. By Lemmata 6 and 7 we have that $S_j(\theta) \setminus \{\theta\}$ is convex and atomless. By Lemmata 1, A.1, and A.2, we have that $S_j(\theta) \subset \hat{R}$ for all $\theta \in (\theta_1, \theta_2)$, and thus by Lemma 4 we have that $\frac{ds(r)}{dr} < 0$ for all $r_j \in S_j(\theta)$. Therefore, we can apply the method of payoff-equation: by setting $\frac{dW_2^\omega(r_2,\theta)}{dr_2} = 0$, and since $\phi_j(r_j, \theta) = \psi_j(r_j, \theta)$ for all $r_j \in S_j(\theta) \setminus \{\theta\}$ (Lemma 7), we obtain the partial pdf $\psi_1(s(r_2), \theta) = \frac{k_2}{-u_2(\theta)} \frac{dC_2(r_2,\theta)}{dr_2} \frac{dr_2}{ds(r_2)} = \frac{k_2}{-u_2(\theta)} \frac{dC_2(r_2,\theta)}{ds(r_2)}$. By replacing $r_1 = s(r_2)$ we obtain that $\psi_1(r_1, \theta) = \frac{k_2}{-u_2(\theta)} \frac{dC_2(s(r_1),\theta)}{dr_1}$ for $r_1 \in S_1(\theta) \setminus \{\theta\}$. Similarly, we obtain that for $r_2 \in S_2(\theta) \setminus \{\theta\}$, $\psi_2(r_2, \theta) = \frac{k_1}{-u_1(\theta)} \frac{dC_1(s(r_2),\theta)}{dr_2}$. □

Lemma A.8. *In a direct equilibrium, $S_1(\theta)$ is convex for all $\theta \geq 0$ and $S_2(\theta)$ is convex for all $\theta \leq 0$.*

Proof. Consider a DE and suppose by way of contradiction that $S_1(\theta)$ is not convex for some $\theta \in [0, \theta_2)$. By Lemma A.6 we have that $\min S_1(\theta) = \theta$, and by Lemma 6 we have that $S_1(\theta) \setminus \{\theta\}$ is convex. Therefore, it must be that $\min S_1(\theta) \setminus \{\theta\} > \theta$ and $\phi_1(r_1, \theta) = 0$ for every $r_1 \in (\theta, \min S_1(\theta) \setminus \{\theta\})$. In equilibrium, every $r_1 > \min S_1(\theta) \setminus \{\theta\}$ such that $\phi_1(r_1, \theta) > 0$ must yield the implementation of alternative $\oplus$ with strictly higher probability than truthful reporting, as $C_1(r_1, \theta) > 0$. This is possible only if $\phi_2(r_2, \theta) > 0$ for some $r_2 \in [s(\min S_1(\theta) \setminus \{\theta\}), s(\theta))$. However, for some $\epsilon > 0$ small enough, it must be that $\phi_2(r_2', \theta) = 0$ for every $r_2' \in [s(\min S_1(\theta) \setminus \{\theta\}), s(\theta) - \epsilon)$, as every such a report r_2' is dominated by reporting $s(\theta) - \epsilon$. Therefore, there exists an $\epsilon' > 0$ such that sender 1 can profitably deviate from the prescribed strategy by moving probability from reports in the set $[\min S_1(\theta) \setminus \{\theta\}, \min S_1(\theta) \setminus \{\theta\} + \epsilon')$ to $s(s(\theta) - \epsilon)$, contradicting that this is an equilibrium. Lemma 5 considers the case where $\theta \notin (\theta_1, \theta_2)$, and a similar argument applies to states $\theta \leq 0$ and support $S_2(\theta)$. □

Proposition 3. *In a direct equilibrium, for every state $\theta \in (\theta_1, \theta_2)$, supports $S_j(\theta)$ are*

$$S_1(\theta) = \{\theta\} \cup [\max\{s(\theta), \theta\}, \min\{\bar{r}_1(\theta), s(\underline{r}_2(\theta))\}],$$

$$S_2(\theta) = \{\theta\} \cup [\max\{\underline{r}_2(\theta), s(\bar{r}_1(\theta))\}, \min\{s(\theta), \theta\}].$$

Proof. Consider a direct equilibrium and a state $\theta \in [0, \theta_2)$. Since for every $\theta \geq 0$ we have that $\theta \in S_1(\theta)$ (Lemma A.6) and both sets $S_1(\theta)$ and $S_1(\theta) \setminus \{\theta\}$ are convex (Lemmata A.8 and 6), it follows that $S_1(\theta) = [\theta, \max S_1(\theta)]$.

Lemma 6 shows that also $S_2(\theta) \setminus \{\theta\}$ is convex. Since $\min S_1(\theta) = \theta$, Lemma A.3 says that when $\theta > 0$ we have $\phi_2(r_2, \theta) = 0$ for all $r_2 \in (s(\theta), \theta)$, and thus $\max S_2(\theta) \setminus \{\theta\} \leq s(\theta)$ for all $\theta \in (0, \theta_2)$. Suppose that $\max S_2(\theta) \setminus \{\theta\} < s(\theta)$. In this case, it must be that $\phi_1(r_1, \theta) = 0$ for every $r_1 \in (\theta, s(\max S_2(\theta) \setminus \{\theta\}))$, as for sender 1 every such a report r_1 would be dominated by truthful reporting. This is in contradiction with Lemma A.8, and therefore it must be that $\max S_2(\theta) \setminus \{\theta\} = s(\theta)$ for every $\theta \in (0, \theta_2)$. When $\theta = 0$, we have that $\max S_2(0) = 0$ (Lemma A.6).

Lemma 7 shows that $\phi_2(r_2, \theta)$ is atomless in $S_2(\theta) \setminus \{\theta\}$. Therefore, for a $r_2 \in S_2(\theta) \setminus \{\theta\}$ we have that $\Phi_2(r_2, \theta) = \Psi_2(r_2, \theta)$, and thus by using Proposition 2 we can write

$$\Phi_2(r_2, \theta)|_{r_2 \in S_2(\theta) \setminus \{\theta\}} = \int_{\min S_2(\theta)}^{r_2} \psi_2(r, \theta) dr = \frac{k_1}{u_1(\theta)} \left[C_1(s(\min S_2(\theta)), \theta) - C_1(s(r_2), \theta) \right].$$

The probability that sender 2 misreports information in state $\theta \in (0, \theta_2)$ is thus

$$\Phi_2(s(\theta), \theta) = \frac{k_1}{u_1(\theta)} C_1(s(\min S_2(\theta)), \theta). \tag{6}$$

Since $\min S_2(\theta) \geq \underline{r}_2(\theta)$ (Lemma A.2), it follows from Lemma 4 that, for every $\theta \in (0, \theta_2)$, $s(\min S_2(\theta)) < \bar{r}_1(\theta)$. Lemma A.7 shows that the set $S_2(\theta) \setminus \{\theta\}$ is not a singleton, and since $\max S_2(\theta) \setminus \{\theta\} \leq s(\theta)$ it must be that $\min S_2(\theta) < s(\theta)$. Thus by Lemma 4 we have, for $\theta \in (0, \theta_2)$, that $s(\min S_2(\theta)) \in (\theta, \bar{r}_1(\theta))$. Finally, by definition of upper reach we get that $C_1(\bar{r}_1(\theta), \theta) = u_1(\theta)/k_1$, and $C_1(r_1, \theta) < u_1(\theta)/k_1$ for every $r_1 \in [\theta, \bar{r}_1(\theta))$. Therefore, it follows that $\Phi_2(s(\theta), \theta) \in (0, 1)$ for every $\theta \in (0, \theta_2)$. By using $s(s(r)) = r$ and $s(0) = 0$ (Lemma 4), when $\theta = 0$ we obtain that $\Phi_2(s(0), 0) = 1$ only if $\min S_2(0) = \underline{r}_2(0)$.

The above argument shows that $\theta \in S_2(\theta)$ and that $\phi_2(\theta)$ has an atom in $r_2 = \theta$ of size $\alpha_2(\theta) = 1 - \Phi_2(s(\theta), \theta)$. Lemma 1 implies that every pair of on-path reports (r_1, r_2) such that $r_j \geq 0$, $j \in \{1, 2\}$, must yield $\beta(r_1, r_2) = \oplus$. Therefore, by reporting truthfully when $\theta \geq 0$, sender 2 obtains a payoff of $W_2^\omega(\theta, \theta) = u_2(\theta)$. It must be that $\max S_1(\theta) \leq s(\min S_2(\theta))$, otherwise every report $r_1 > s(\min S_2(\theta))$ would be dominated by $s(\min S_2(\theta))$. Since $\phi_1(\theta)$ has no atom in $s(\min S_2(\theta)) > \theta$ (Lemma 7), by reporting $r_2 = \min S_2(\theta)$ sender 2 (almost) always induces the selection of its preferred alternative $\ominus$, and gets an expected payoff of $W_2^\omega(\min S_2(\theta), \theta) = -k_2 C_2(\min S_2(\theta), \theta)$.

In equilibrium each sender must receive the same expected payoff from delivering any report that is in the support of its own strategy. Since by definition of lower reach we obtain $C_2(\underline{r}_2(\theta), \theta) = -u_2(\theta)/k_2$, it follows that $W_2^\omega(\min S_2(\theta), \theta) = u_2(\theta) = W_2^\omega(\theta, \theta)$ only if $\min S_2(\theta) = \underline{r}_2(\theta)$. Therefore, for a $\theta \in [0, \theta_2)$, we have that $S_2(\theta) = [\underline{r}_2(\theta), s(\theta)] \cup \{\theta\}$. It also follows that $\max S_1(\theta) = s(\underline{r}_2(\theta))$: if $\max S_1(\theta) < s(\underline{r}_2(\theta))$, then $\underline{r}_2(\theta) < s(\max S_1(\theta))$

and every $r_2 < s(\max S_1(\theta))$ would be strictly dominated by $s(\max S_1(\theta))$. Thus, $S_1(\theta) = [\theta, s(\underline{r}_2(\theta))]$. Similar arguments apply to the case $\theta \in (\theta_1, 0)$, completing the proof. $\square$

Proposition 4. *In a direct equilibrium, for every state $\theta \in (\theta_1, \theta_2)$, strategies $\phi_j(\theta)$ have an atom at $r_j = \theta$ of size $\alpha_j(\theta)$, where*

$$\alpha_1(\theta) = \begin{cases} \frac{k_2}{-u_2(\theta)} C_2\left(s(\theta), \theta\right) & \text{if } \theta \in [0, \theta_2) \\ 1 - \frac{k_2}{-u_2(\theta)} C_2\left(s(\bar{r}_1(\theta)), \theta\right) & \text{if } \theta \in (\theta_1, 0], \end{cases}$$

$$\alpha_2(\theta) = \begin{cases} 1 - \frac{k_1}{u_1(\theta)} C_1\left(s(\underline{r}_2(\theta)), \theta\right) & \text{if } \theta \in [0, \theta_2) \\ \frac{k_1}{u_1(\theta)} C_1\left(s(\theta), \theta\right) & \text{if } \theta \in (\theta_1, 0]. \end{cases}$$

Proof. Consider a direct equilibrium and a state $\theta \in [0, \theta_2)$. The proof of Proposition 3 shows that $\phi_2(\theta)$ has an atom in $r_2 = \theta$ of size $\alpha_2(\theta) = 1 - \Phi_2(s(\theta), \theta)$. From equation (6) and given $\min S_2(\theta) = \underline{r}_2(\theta)$, we obtain that

$$\alpha_2(\theta) = 1 - \frac{k_1}{u_1(\theta)} C_1(s(\underline{r}_2(\theta)), \theta).$$

By Lemma 7, sender 1's strategy $\phi_1(\theta)$ admits an atom only in $\min S_1(\theta) = \theta$. Therefore, we can use Proposition 2 to write

$$\begin{aligned} \Phi_1(r_1, \theta)|_{r_1 \in S_1(\theta)} &= \alpha_1(\theta) + \int_\theta^{r_1} \psi_1(r, \theta) dr \\ &= \alpha_1(\theta) + \frac{k_2}{-u_2(\theta)} \left[C_2(s(r_1), \theta) - C_2(s(\theta), \theta) \right]. \end{aligned}$$

Since $\max S_1(\theta) = s(\underline{r}_2(\theta))$, it must be that $\Phi_1(s(\underline{r}_2(\theta)), \theta) = 1$. By using $s(s(\underline{r}_2(\theta))) = \underline{r}_2(\theta)$ (Lemma 4) and given that from the definition of lower reach we obtain $C_2(\underline{r}_2(\theta), \theta) = -k_2/u_2(\theta)$, we have

$$\begin{aligned} \Phi_1(s(\underline{r}_2(\theta)), \theta) &= \alpha_1(\theta) + \frac{k_2}{-u_2(\theta)} \left[C_2(s(s(\underline{r}_2(\theta))), \theta) - C_2(s(\theta), \theta) \right] \\ &= \alpha_1(\theta) + 1 - \frac{k_2}{-u_2(\theta)} C_2(s(\theta), \theta) = 1, \end{aligned}$$

from which we obtain that

$$\alpha_1(\theta) = \frac{k_2}{-u_2(\theta)} C_2(s(\theta), \theta).$$

A similar procedure can be used for $\theta \in (\theta_1, 0)$, completing the proof. $\square$

Lemma A.9. *In a direct equilibrium, for every (on-path) pair of reports (r_1, r_2) such that*

$r_2 = s(r_1)$, *posterior beliefs are*

$$p(\theta|r_1, r_2) > 0 \;\; \textit{if and only if} \;\; \theta \in [\max\{r_2, \bar{r}_1^{-1}(r_1)\}, \min\{r_1, \underline{r}_2^{-1}(r_2)\}].$$

Proof. Consider a DE and a pair of reports (r_1, r_2) such that $\bar{r}_1(0) \geq r_1 > 0 > r_2 \geq \underline{r}_2(0)$. Given equilibrium supports in Proposition 3, all such pairs are on-path (e.g., for $\theta = 0$). Upon observing (r_1, r_2), the decision maker forms posterior beliefs $p(\theta|r_1, r_2)$. By Lemma 1, it must be that $p(\theta|r_1, r_2) = 0$ for every $\theta \notin [r_2, r_1]$. By Lemma 5, it must be that $p(\theta|r_1, r_2) = 0$ for every $\theta \notin [\theta_1, \theta_2]$. By Proposition 3 we have that $\min S_2(\theta) \geq \underline{r}_2(\theta)$ and $\max S_1(\theta) \leq \bar{r}_1(\theta)$, and therefore $p(\theta|r_1, r_2) = 0$ for every $\theta \notin \left[\bar{r}_1^{-1}(r_1), \underline{r}_2^{-1}(r_2)\right]$, where from equations (1) and (2) we obtain

$$\bar{r}_1^{-1}(r_1) = \min \left\{\theta \in \Theta | u_1(\theta) = k_1 C_1(r_1, \theta)\right\},$$

$$\underline{r}_2^{-1}(r_2) = \max \left\{\theta \in \Theta | -u_2(\theta) = k_2 C_2(r_2, \theta)\right\}.$$

From Proposition 3 we also have that, for every $\theta \in [0, \theta_2)$, $\max S_1(\theta) = s(\underline{r}_2(\theta)) \leq r_1(\theta)$. Therefore, given the report $r_1 \in (0, \bar{r}_1(0)]$, it must be that $p(\theta|r_1, r_2) = 0$ for all θ such that $s(\underline{r}_2(\theta)) < r_1$. By Lemma 4 and since $d\underline{r}_2(\theta)/d\theta > 0$, there is a state θ' such that $s(\underline{r}_2(\theta')) = r_1$. Denote such state with $t_1(r_1) := \{\theta \in \Theta | s(\underline{r}_2(\theta)) = r_1\}$, where $t_1(r_1) > 0$ and $dt_1(r_1)/dr_1 > 0$. Similarly, denote $t_2(r_2) := \{\theta \in \Theta | s(\bar{r}_1(\theta)) = r_2\}$. Given equilibrium supports, it must be that $p(\theta|r_1, r_2) = 0$ for all $\theta \notin [t_2(r_2), t_1(r_1)]$.

By Lemma 4 and since $s(\underline{r}_2(\theta_2)) = \theta_2$ (Definition 6), we obtain that $t_1(r_1) \leq \theta_2$ for every $r_1 \in [\theta_2, \bar{r}_1(0)]$, and thus $\min\{r_1, t_1(r_1)\} \leq \theta_2$ for all $r_1 \in (0, \bar{r}_1(0)]$. Similarly, we get that $\max\{r_2, t_2(r_2)\} \geq \theta_1$ for all $r_2 \in [\underline{r}_2(0), 0)$. Therefore, we have that $p(\theta|r_1, r_2) = 0$ for every $\theta \notin [\max\{r_2, \bar{r}_1^{-1}(r_1), t_2(r_2)\}, \min\{r_1, \underline{r}_2^{-1}(r_2), t_1(r_1)\}]$, and by Proposition 3 we obtain that $p(\theta|r_1, r_2) \propto f(\theta) \cdot \phi_1(r_1, \theta) \cdot \phi_2(r_2, \theta) > 0$ otherwise.

Consider now the case where $r_2 = s(r_1)$ (or, by Lemma 4, $r_1 = s(r_2)$). By definition, at state $\theta' = t_1(r_1)$ we have $s(\underline{r}_2(\theta')) = r_1$. Thus, we get that $s(r_1) = \underline{r}_2(\theta') = r_2$ and $\underline{r}_2^{-1}(r_2) = \theta' = t_1(r_1)$. Similarly, we obtain that $\bar{r}_1^{-1}(r_1) = t_2(r_2)$. Therefore, for every pair of reports $(r_1, s(r_1))$ we have that $p(\theta|r_1, s(r_1)) > 0$ if and only if $\theta \in \left[\max\left\{r_2, \bar{r}_1^{-1}(r_1)\right\}, \min\left\{r_1, \underline{r}_2^{-1}(r_2)\right\}\right]$. □

Proposition 5. *In a direct equilibrium, the swing report function $s(r_i)$ is implicitly defined for $i, j \in \{1, 2\}$, $i \neq j$, and $r_i \in \hat{R}$, as*

$$s(r_i) = \left\{ r_j \in R_j \;\middle|\; \int_{\max\left\{r_2, \bar{r}_1^{-1}(r_1)\right\}}^{\min\left\{r_1, \underline{r}_2^{-1}(r_2)\right\}} f(\theta) \frac{u_{dm}(\theta)}{u_1(\theta) u_2(\theta)} \frac{dC_j(r_j, \theta)}{dr_j} \frac{dC_i(r_i, \theta)}{dr_i} d\theta = 0 \right\}. \quad (4)$$

Proof. Given the equilibrium reporting strategies $\phi_j(r_j|\theta) = \delta(r_j - \theta)\alpha_j(\theta) + \psi_j(r_j|\theta)$, $j \in \{1, 2\}$ (Propositions 2, 3, and 4), the mixed probability distribution $p(r_1, r_2|\theta) =$

$\phi_1(r_1,\theta)\phi_2(r_2,\theta)$ is

$$p(r_1,r_2|\theta)=\delta(r_1-\theta)\delta(r_2-\theta)\alpha_1(\theta)\alpha_2(\theta)+\delta(r_1-\theta)\alpha_1(\theta)\psi_2(r_2,\theta)\\ +\delta(r_2-\theta)\psi_1(r_1,\theta)\alpha_2(\theta)+\psi_1(r_1,\theta)\psi_2(r_2,\theta).$$

Consider a pair of reports (r_1,r_2) such that $\bar{r}_1(0)\geq r_1>0>r_2\geq \underline{r}_2(0)$ and $r_2=s(r_1)$ (as by Lemma 4 we have that if $r>0$, then $s(r)<0$). Since $\frac{dC_j(r_j,\theta)}{dr_j}\Big|_{r_j=\theta}=0$ for every $\theta\in\Theta$, we obtain that $\psi_j(s(\theta),\theta)=0$ for $i,j\in\{1,2\}$, $i\neq j$, and therefore $p(r_1,s(r_1)|\theta)=\psi_1(r_1,\theta)\psi_2(s(r_1),\theta)$.

The swing report $s(r_1)$ is defined in Definition 5 as the $r_2\in R_2$ such that $U_{dm}(r_1,r_2)=\int_\Theta u_{dm}(\theta)p(\theta|r_1,r_2)d\theta=0$, and by Lemma 4 we know that $s(r_1)\in[\underline{r}_2(0),0)$. By Lemma A.9 we have that $p(\theta|r_1,s(r_1))>0$ if and only if $\theta\in[\max\{r_2,\bar{r}_1^{-1}(r_1)\},\min\{r_1,\underline{r}_2^{-1}(r_2)\}]$, and therefore by using Bayes' rule we can rewrite the condition $U_{dm}(r_1,s(r_1))=0$ as $G_s(r_1,s(r_1))=0$, where

$$G_s(r_1,r_2)=\frac{1}{p(r_1,r_2)}\int_{\max\{r_2,\bar{r}_1^{-1}(r_1)\}}^{\min\{r_1,\underline{r}_2^{-1}(r_2)\}}u_{dm}(\theta)f(\theta)\psi_1(r_1,\theta)\psi_2(r_2,\theta)d\theta.$$

By substituting for the equilibrium strategies $\psi_j(r_j,\theta)$ as described in Proposition 2, we obtain the implicit definition of the swing report given in equation (4). □

Corollary 1. *Direct equilibria are essentially unique.*

Proof. The solution of equation (4) is unique and depends only on the model's primitives $u_{dm}(\theta)$, $f(\theta)$, $u_i(\theta)$, τ_i, k_i, $C_i(r_i,\theta)$, for $i\in\{1,2\}$. Therefore, for every $r\in[\underline{r}_2(0),\bar{r}_1(0)]$, the swing report $s(r)$ is the same across every DE. It follows that the truthful cutoffs θ_1 and θ_2, and the senders' reporting strategies $\phi_j(\theta)$ and supports $S_j(\theta)$, $j\in\{1,2\}$, are also the same in all DE. Thus, all DE are strategy and outcome equivalent. □

Lemma A.10. *There exists direct equilibria with unprejudiced beliefs.*

Proof. Consider a DE and an off-path pair reports (r_1,r_2). By Propositions 2, 3, and 4, and by Lemma 5, we obtain that the only pair of reports such that $\phi_j(r_j,\theta)=0$ for all $\theta\in\Theta$ and $j\in\{1,2\}$ is $(0,0)$. For every other off-path pair of reports, there is always a sender i such that $\phi_i(r_i,\theta)>0$ for some $\theta\in\Theta$. There are three types of off-path pairs of reports that need to be considered: those that violate Lemma 1, such as when $r_1>r_2$; those that violate Proposition 3, such as when $r_1>s(\underline{r}_2(r_2))$; those that violate Lemma 5, such as when $r_1\neq r_2$ for some $(r_1,r_2)\notin(\theta_1,\theta_2)^2$.

For beliefs to be unprejudiced, Definition 7 requires that for every such off-path pair of reports we have that $p(\theta''|r_1,r_2)>0$ if and only if there is a sender $i\in\{1,2\}$ such that $\phi_i(r_i,\theta'')>0$. Since $p(\theta''|r_1,r_2)$ can be arbitrarily small, I can just focus on beliefs that rationalize deviations as originating from only one sender. Hereafter, I will consider some posterior beliefs p' which, given an off-path pair of reports (r_1, r_2), rationalize deviations as originating with certainty from one specific sender i under the constraint that $\phi_j(r_j, \theta) > 0$ for some $\theta \in \Theta$ and $j \in \{1, 2\}$. If there is a DE with such beliefs p', then there exists a DE with beliefs p'' (e.g., a small perturbation of p') that satisfy Definition 7 (and thus also Definition 2).

First, if $0 \leq r_1 < r_2$ (resp. $r_1 < r_2 \leq 0$), then set p' such that the decision maker believes the deviation has been performed by sender 1 (2). Given the equilibrium strategies, it must be that sender 2 (1) is reporting truthfully, and thus p' leads to $\beta(r_1, r_2) = \oplus$ ($\ominus$). Consider now the case $r_1 < 0 < r_2$, and set p' such that the decision maker believes that only sender 1 (or 2) is deviating. Therefore, it must be that sender 2 (1) is reporting truthfully, and thus $\beta(r_1, r_2) = \oplus$ ($\ominus$). Second, consider an off-path pair of reports such that, for an $x \geq 0$, $r_2 > x$ and $r_1 \geq s(\underline{r}_2(x))$ (resp. $r_1 < y \leq 0$ and $r_2 \leq s(\bar{r}_1(y))$). If through p' the decision maker believes that sender 1 (2) is the deviator, then it must be that $\theta = r_2$ ($\theta = r_1$) and therefore $\beta(r_1, r_2) = \oplus$ ($\ominus$). Finally, consider an off-path pair $(r_1, r_2) \notin (\theta_1, \theta_2)^2$ with $r_1 \neq r_2$. If both $r_1, r_2 \geq \theta_2$ (resp. $r_1, r_2 \leq \theta_1$), then, by rationalizing the deviation as originating from one sender, the decision maker would infer that $\theta \geq \theta_2 > 0$ ($\theta \leq \theta_1 < 0$) and select $\beta(r_1, r_2) = \oplus$ ($\ominus$).

Since p' is consistent with conditions (C) and (D), and since given p' no sender has individual profitable deviations from the prescribed equilibrium strategies, it follows that there are DE with unprejudiced beliefs as defined in Definition 2 and 7. □

Corollary 2. *There are direct equilibria with unprejudiced beliefs that are also ε-robust.*

Proof. Consider an ε-perturbed game with sequence ε^n and full support distributions $\hat{G} = (\hat{G}_1, \hat{G}_2)$ such that $\hat{g}_1(r_1) \approx 0$ for all $r_1 < 0$ and $\hat{g}_2(r_2) \approx 0$ for all $r_2 > 0$. This means that it is relatively unlikely that the report of sender 1 (resp. 2) is by mistake observed to be negative (resp. positive). By equation (5), the limit beliefs $\hat{p}_{0^+}$ induced by the strategies of a DE after observing a pair of reports (r_1, r_2) such that $0 \leq r_1 < r_2$ are

$$\hat{p}_{0^+}(\theta | r_1 \geq 0, r_2 > 0) \approx f(\theta) \frac{\delta(r_2 - \theta)\alpha_2(\theta)}{f(r_2)\alpha_2(r_2)}.$$

Therefore, the CDF $\hat{P}_{0^+} = \int p_{0^+}(\theta | r_1, r_2) d\theta$ is such that

$$\hat{P}_{0^+}(\theta | r_1 \geq 0, r_2 > 0) \approx \begin{cases} 0 & \text{if } \theta < r_2 \\ 1 & \text{if } \theta \geq r_2. \end{cases}$$

As $\varepsilon^n \to 0^+$ and for every off-path pair of reports that are both positive, the decision maker is almost sure that the realized state coincides with the report of sender 2. Similarly, we obtain that $\hat{P}_{0^+}(r_1 < 0, r_2 \leq 0) \approx 0$ for all $\theta < r_1$ and ≈ 1 otherwise, and by Lemma 3, we have that $\hat{p}_{0^+}(\theta | r_1 < 0, r_2 > 0) > 0$ only for $\theta \in \{r_1, r_2\}$. Therefore, limit beliefs $\hat{p}_{0^+}$

are arbitrarily close to the posterior beliefs p' in the proof of Lemma A.10, and therefore can support a DE. Since a DE is also a PBE, by Lemma 3 we obtain that there exists direct equilibria with unprejudiced beliefs that are also ε-robust. □

Corollary 3. *A direct equilibrium always exists.*

Proof. Given strategies $\phi_j(r_j, \theta) = \delta(r_j - \theta)\alpha_j(\theta) + \psi_j(r_j, \theta)$ as in Proposition 2 and 4, with support $S_j(\theta)$ as in Proposition 3, posterior beliefs $p(\theta|r_1, r_2)$ are such that the swing report function $s(r)$ is as in Proposition 5. Given $s(r)$, strategies $\phi_j(r_j, \theta)$ are optimal by construction, and thus no sender $j \in \{1, 2\}$ can perform a profitable individual deviation from $\phi_j(r_j, \theta)$. Therefore, for every primitive of the model that satisfies the conditions outlined in Section 3, there must exist a direct equilibrium as defined by Definition 4. □

A.3 Example: Symmetric Environments

Corollary 4. *In a direct equilibrium of a symmetric environment, $s(r) = -r$ for every $r \in \hat{R}$.*

Proof. The proof follows directly from Proposition 5: consider a symmetric environment and suppose that $s(r) = -r$. Given a report $r \in (0, \bar{r}_1(0))$, the interval of integration in (4) has $\max\{-r, \bar{r}_1^{-1}(r)\} = -\min\{r, \underline{r}_2^{-1}(-r)\}$. Since the integrand in (4) is symmetric around zero, we obtain that $G_s(r, -r) = 0$, confirming that indeed $s(r) = -r$. □

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