Minimal entropy and uniqueness of price equilibria in a pure exchange economy

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Abstract:

We introduce uncertainty into a pure exchange economy and establish a connection between Shannon's differential entropy and uniqueness of price equilibria. The following conjecture is proposed under the assumption of a uniform probability distribution: entropy is minimal if and only if the price is unique for every economy. We show the validity of this conjecture for an arbitrary number of goods and two consumers and, under certain conditions, for an arbitrary number of consumers and two goods.

Keywords: Entropy, uniqueness of equilibrium, price multiplicity, equilibrium manifold, minimal submanifold.

JEL Classification: D50, D51, D80.

1 Introduction

In a pure exchange economy let us denote by $x = (p, \omega)$ an initial allocation ω and its supporting equilibrium price vector p. Suppose that x is slightly perturbed by exogenous, i.e. shocks, or endogenous factors, e.g. the uncertainty related to the effects of Safra's competitive manipulation [19]. The result of this perturbation is a new allocation and equilibrium price vector, x', belonging to a neighborhood N of x. We can represent this process as a probability model, where the random variable ranges in the set N. Observe that N is not an Euclidean space. It belongs to a space of endowments and prices and consists of points such that aggregate excess demand function is equal to zero. Under standard smooth assumptions and in a fixed total resource setting, N becomes a submanifold with boundary of a manifold called the *equilibrium manifold*, denoted by $E(r)$ which in turn is a smooth submanifold of $S \times \Omega(r)$, where S is the space of prices and $\Omega(r)$ the space of economies (see the seminal work by Balasko [2] and also Section 2).

Thus $E(r)$ can be equipped with a natural measure, namely the Riemannian volume form dM_g associated to the Riemannian metric g induced by the flat metric of its ambient space $S \times \Omega(r)$ (see e.g. [13]). The probability that $x \in E(r)$ belongs to N is

$$Pr(x \in N) = \int_N p(x) dM_g(x),$$

where p is a given probability density on $E(r)$ (the reader is referred to [18] for a geometric approach to probability theory on Riemannian manifolds). Moreover, following Shannon [20] (see also [6]) in this framework we define the *differential entropy of N* as

$$H(N) = -\int_N p(x) log(p(x)) dM_g(x).$$

Obviously when $E(r)$ is an Euclidean flat space then one can take dM_g equals to the Lebesgue measure and $H(N)$ is the differential entropy defined in [20].

Since entropy is a measure of missing information it is natural to investigate under which conditions it is minimized. Therefore we provide he following:

Definition (MEP) *The equilibrium manifold satisfies the minimal entropy property (MEP) if for every economy x belonging to $E(r)$, there exists a neighborhood N of x in $E(r)$ such that $H(N) \leq H(\tilde{N})$, where $\tilde{N}$ is any other submanifold of $S \times \Omega(r)$ containing x which has the same boundary of N and whose volume structure, in the same way as N, is induced by the flat metric of the ambient space $S \times \Omega(r)$.*

It would be interesting (and challenging) to study how the choice of different probability distributions affects the economic properties implied by (MEP). This issue, which deserves further analysis, is beyond the scope of this paper.

On the other hand, it is natural to restrict to the case of uniform distribution, namely when the probability density function is given by

$$p_N = \frac{\chi_n}{V(N)},$$

where $V(N) = \int_N dM_g(x)$ is the volume of N and $\chi_N : E(r) \to \{0, 1\}$ is the characteristic function supported on N. Under this assumption

$$H(N) = -\int_N \frac{1}{V(N)} log(\frac{1}{V(N)}) dM(x) = log(V(N)), \tag{1}$$

and so by the increasing property of the logarithm we deduce that in the case of uniform distribution the (MEP) is equivalent to the following

Definition (MVP) *The equilibrium manifold satisfies the minimal volume property (MVP) if for every economy x belonging to $E(r)$, there exists a neighborhood N of x in $E(r)$ such that $V(N) \leq V(\tilde{N})$, where $\tilde{N}$ is any other submanifold of $S \times \Omega(r)$ containing x which has the same boundary of N.*

Now the (MVP) for $E(r)$ can be translated into the language of differential geometry: the equilibrium manifold $E(r)$ is a *stable minimal submanifold* of $S \times \Omega(r)$, i.e. a local minimum of the volume functional. In particular $E(r)$ is a critical point of the volume functional, namely $E(r)$ is a *minimal submanifold*[1] of $S \times \Omega(r)$.

Observe now that according to Theorem 2.1 below, if for every economy there is uniqueness of equilibrium, the equilibrium manifold is "flat" (and hence minimal): i.e., (global) uniqueness implies (MVP). Here we explore the reverse of this implication: if there is price multiplicity, can (MVP) holds true? In other words, does (MVP) implies uniqueness? This is not a trivial issue: in fact the equilibrium manifold can almost arbitrarily be twisted for an appropriate preference profile[2]. Hence one could expect to find an utility profile which gives rise to multiplicity and minimality. Actually, we believe that exactly the opposite is true. Indeed we address the following:

Conjecture: *Under the assumption of uniform distribution the equilibrium manifold satisfies (MEP) if and only if the equilibrium price is unique.*

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[1]Throughout this paper we will content ourselves with this definition since in the proof of our main results we are not using the differential geometric machinery of the theory of minimal submanifolds. The interested reader is referred to [21] for more details and material on minimal submanifolds. The simplest examples of minimal submanifolds arise when $n = 1$: in this case they are simply geodesics of the ambient space. In higher dimensions every totally geodesic submanifold is a minimal submanifold (cf. also [11] for some properties of geodesics and totally geodesic submanifolds of the equilibrium manifold). Nevertheless, there exist a lot of interesting minimal submanifolds (see [21] or Section 3 below).

[2]Even if the equilibrium manifold $E(r)$ is unknotted in its ambient space [7].

In other words, we believe that an utility profile which minimizes entropy (and hence volume) with uniform distribution is incompatible with price multiplicity.

In this paper we show the validity of this conjecture in the case of an arbitrary number of goods and two consumers (Theorem 3.1) and in the case of an arbitrary number of consumers and two goods (Theorem 4.1) under the additional assumption that the normal vector field of $E(r)$ is constant outside a compact subset of the ambient space. The proof of Theorem 3.1 strongly relies on geometric and economic properties: the classification of ruled minimal submanifolds of the Euclidean space, the bundle structure of the equilibrium manifold and the positiveness of prices. On the other hand, the proof of Theorem 4.1 combines deep geometric results relating the topology of a minimal submanifold of the Euclidean space with the fact that $E(r)$ is globally diffeomorphic to an Euclidean space.

It is worth noticing that the choice of a metric depends on the analysis. In [12] the metric on the equilibrium manifold was chosen to deal with asymptotic properties related to economies with an arbitrarily large number of equilibria. In [13] the metric used was a tool to explore geometric properties which are intrinsic, i.e. they do not depend on the ambient space. But the purpose, and the approach, of the present work is entirely different and this affects the choice of the metric used.

We believe that this information-theoretic and geometric approach can be further extended in different directions. Following the seminal contribution by [22] (see also [4, 5] and [15]), an entropy-based metric could be developed in order to compute geodesics representing redistributive policies. Another direction of research (see [19] and [9]) is to analyze the extrinsic uncertainty in $N \subset E(r)$ caused by coalitional manipulation of endowments. This approach could provide new insights into the understanding of coalition formation and sunspot equilibria. Finally, due to the economic relevance of the shape of the equilibrium manifold, it can be worth investigating the connection between its shape and the primitives of the model, an issue still largely unexplored. This local-global view can hopefully lead to new perspectives on issues like uniqueness and stability (see [10, 16] for a survey).

This paper is organized as follows. Section 2 recalls notations, definitions and the existing results which will be used to prove our main results. Section 3 and Section 4 prove our main results, Theorem 3.1 and Theorem 4.1.

2   Definitions

The economic setup is represented by a pure exchange smooth economy with L goods and M consumers under the standard smooth assumptions (see [2, Chapter

2]). The set of normalized prices is defined by

$$S = \{p = (p_1, \ldots p_L) \in \mathbb{R}^L \mid p_l > 0, l = 1, \ldots, L, \ p_L = 1\}$$

and the set $\Omega = (\mathbb{R}^L)^M$ denotes the space of endowments $\omega = (\omega_1, \ldots, \omega_M)$, $\omega_i \in \mathbb{R}^L$. The *equilibrium manifold* E is the set of the pairs $(p, \omega) \in S \times \Omega$, which satisfy the equality:

$$\sum_{i=1}^{M} f_i(p, p \cdot \omega_i) = \sum_{i=1}^{M} \omega_i, \tag{2}$$

where $f_i(p, w_i)$ is consumer's i demand.

By [2, Lemma 3.2.1], E is a (closed) smooth submanifold of $S \times \Omega$, globally diffeomorphic to $S \times \mathbb{R}^M \times \mathbb{R}^{(L-1)(M-1)} = \mathbb{R}^{LM}$, i.e. $\phi_{|E} \cong \mathbb{R}^{LM}$, where the smooth mapping

$$\phi : S \times \Omega \to S \times \mathbb{R}^M \times \mathbb{R}^{(L-1)(M-1)}$$

is defined by

$$(p, \omega_1 \ldots, \omega_M) \mapsto (p, p \cdot \omega_1, \ldots, p \cdot \omega_M, \bar{\omega}_1, \ldots, \bar{\omega}_{M-1}),$$

where $\bar{\omega}_i$ denotes the first $L - 1$ components of ω_i, for $i = 1, \ldots, M - 1$.

We also introduce the following two subsets of E:

- the set of *no-trade equilibria* $T = \{(p, \omega) \in E \mid f_i(p, p \cdot \omega_i) = \omega_i, \ i = 1, \ldots, M\}$;

- *the fiber* associated with $(p, w_1, \ldots, w_M) \in S \times \mathbb{R}^M$, which is defined as the set of pairs $(p, \omega) \in S \times \Omega$ such that:

    - $p \cdot \omega_i = w_i$ for $i = 1, \ldots, M$;
    - $\sum_i \omega_i = \sum_i f_i(p, w_i)$.

By defining the two smooth maps

$$f : S \times \mathbb{R}^M \to S \times \mathbb{R}^{LM},$$

where $f(p, w_1, \ldots, w_M) = (p, f_1(p, w_1), \ldots, f_M(p, w_M))$, and

$$\phi_{Fiber} : E \to S \times \mathbb{R}^M,$$

where $\phi_{Fiber}(p, \omega_1, \ldots, \omega_M) = (p, p \cdot \omega_1, \ldots, p \cdot \omega_M)$, since $f(S \times \mathbb{R}^M) = T \subset E$ and $\phi_{Fiber} \circ f$ is the identity mapping, by applying [2, Lemma 3.2.1], Balasko shows [2, Proposition 3.3.2] that T is a smooth submanifold of E diffeomorphic to $S \times \mathbb{R}^M$.

By construction, every fiber associated with $(p, w_1, \ldots, w_M)$ is a subset of E which is the inverse image of $(p, w_1, \ldots, w_M)$ via the mapping ϕ_{Fiber}. It is intuitively clear that while holding $(p, w_1, \ldots, w_M)$ fixed and letting ω varying along the fiber, there are not any nonlinearities which may arise from the aggregate demand. In fact the fiber is a linear submanifold of E of dimension $(L-1)(M-1)$ [2, Proposition 3.4.2].

Since every fiber contains only one no-trade equilibrium [2, Proposition 3.4.3], the equilibrium manifold E can be thought as a disjoint union of fibers parametrized by the no-trade equilibria T via the mapping $\phi_{|E} : E \to S \times \mathbb{R}^M \times \mathbb{R}^{(L-1)(M-1)}$: for a fixed $(p, w_1, \ldots, w_M) \in S \times \mathbb{R}^M$, each fiber is parametrized by $\bar{\omega}_1, \ldots, \bar{\omega}_{M-1}$. By letting $(p, w_1, \ldots, w_M)$ varying in $S \times \mathbb{R}^M$, we obtain the *bundle structure* of the equilibrium manifold.

If total resources are fixed, the equilibrium manifold is defined as

$$E(r) = \{(p, \omega) \in S \times \Omega(r) \mid \sum_{i=1}^{M} f_i(p, p \cdot \omega_i) = r\}, \tag{3}$$

where $r \in \mathbb{R}^L$ is the vector that represents the total resources of the economy and $\Omega(r) = \{\omega \in \mathbb{R}^{LM} \mid \sum_{i=1}^{M} \omega_i = r\}$.
Let

$$B(r) = \{(p, w_1, \ldots, w_M) \in S \times \mathbb{R}^M \mid \sum_{i=1}^{M} f_i(p, w_i) = r\} \tag{4}$$

be the set of *price-income equilibria* (see [2, Definition 5.1.1]). $B(r)$ is a submanifold of $S \times \mathbb{R}^M$ diffeomorphic to $\mathbb{R}^{M-1}$ [2, Corollary 5.2.4] through the map $\theta : S \times \mathbb{R}^M \to \mathbb{R}^L \times \mathbb{R}^{M-1}$, defined by

$$(p, w) \mapsto (\sum_i f_i(p, w_i), u_1(f_1(p, w_1)), \ldots, u_{M-1}(f_{M-1}(p, w_{M-1})). \tag{5}$$

The equilibrium manifold $E(r)$ is a submanifold of $S \times \Omega(r)$ diffeomorphic to $\mathbb{R}^{L(M-1)}$ [2, Corollary 5.2.5]

$$\phi(E(r)) = B(r) \times \mathbb{R}^{(L-1)(M-1)}. \tag{6}$$

Moreover we can define and $T(r) = T \cap S \times \Omega(r)$. By construction, even in a fixed total resource setting, the equilibrium manifold preserves its bundle structure property and, hence, $E(r)$ can be written as the disjoint union

$$E(r) = \sqcup_{x \in T(r)} F_x, \tag{7}$$

where F_x is an $(L-1)(M-1)$-affine subspace of $\mathbb{R}^{L(M-1)}$.

6

Let $t = (t_1, \ldots, t_{l-1})$, $\bar{\omega}_j = (\omega_1^1, \ldots, \omega_1^{l-1})$ and $p(t) = (p_1(t), \ldots, p_{l-1}(t))$. Following [2] and [13], we can parametrize $B(r)$ via the map:

$$\varphi : \mathbb{R}^{M-1} \to B(r), \; t \to (p(t), w_1(t) \ldots, w_{m-1}(t)) \tag{8}$$

and $E(r)$ via the map:

$$\Phi : \mathbb{R}^{L(M-1)} \to E(r), \tag{9}$$

$(t, \omega_1^1, \ldots, \omega_{M-1}^1, \ldots, \omega_1^1, \ldots, \omega_{M-1}^{L-1}) \mapsto (p(t), \bar{\omega}_1, w_1(t) - p(t) \cdot \bar{\omega}_1, \ldots, w_{M-1}(t) - p(t) \cdot \bar{\omega}_{M-1})$

We end this section with the following result due to Balasko, deeply related to the issue raised in this paper.

Theorem 2.1 [2, p. 188 Theorem 7.3.9 part (2)] *If for every $\omega \in \Omega(r)$ there is uniqueness of equilibrium, the equilibrium correspondence is constant: The equilibrium price vector p associated with ω does not depend on ω.*

Remark 2.2 Hence (global) uniqueness implies (MEP) for $E(r)$ under the assumption of a uniform distribution. This theorem will be used to prove the "only if" part of Theorem 3.1 and Theorem 4.1.

3 The case $M = 2$

In this section we prove the following:

Theorem 3.1 *Let $M = 2$ and assume a uniform distribution. Then $E(r)$ satisfies the (MEP) if and only if the price is unique.*

Before proving the theorem we need some definitions.

- a submanifold $\mathcal{M}^n \subset \mathbb{R}^{n+k}$ is said to be ruled if $\mathcal{M}^n$ is foliated by affine subspaces of dimension $n - 1$ in $\mathbb{R}^{n+k}$.

- a *generalized helicoid* is the ruled submanifold $\mathcal{M}^n(a_1, \ldots, a_k, b) \subset \mathbb{R}^{n+k}$, $k \leq n$, admitting the following parametrization:

$(s, t_1, \ldots, t_{n-1}) \mapsto (t_1 \cos(a_1 s), t_1 \sin(a_1 s), \ldots, t_k \cos(a_k s), t_k \sin(a_k s), t_{k+1}, \ldots, t_{n-1}, bs)),$

where a_j, $j = 1, \ldots, k$, and b are real numbers (we are not escluding that one of these coefficients could vanish and the generalized helicoid becomes an affine subspace).

The key ingredient in the proof of Theorem 3.1 is the following classification result on ruled minimal submanifolds of the Euclidean space. We refer the reader to [8, Section 1] and references therein (in particular [14] for a proof).

Theorem 3.2 ([14]) *A minimal ruled submanifold $\mathcal{M}^n \subset \mathbb{R}^{n+k}$ is, up to rigid motions[3] of $\mathbb{R}^{n+k}$, a generalized helicoid.*

We need also the following simple but fundamental fact:

Lemma 3.3 *Let $\mathcal{M}^n(a_1, \ldots, a_k, b) \subset \mathbb{R}^{n+k}$ be a generalized helicoid such that $b \cdot \prod_{i=1}^{k} a_i \neq 0$. Then $\mathcal{M}^n$ intersects any affine hyperplane of $\mathbb{R}^{n+k}$.*

Proof: In cartesian coordinates $x_1, y_1, \ldots, x_k, y_k, x_{k+1}, \ldots, x_{n-1}, x_n$ an hyperplane of $\mathbb{R}^{n+k}$ has equation:

$$\alpha_1 x_1 + \beta_1 y_1 + \cdots + \alpha_k x_k + \beta_k y_k + \alpha_{k+1} x_{k+1} + \cdots + \alpha_{n-1} x_{n-1} + \alpha_n x_n = \delta,$$

where $\alpha_i, \beta_i, i = 1, \ldots k$, $\alpha_j, j = k+1, \ldots n$ and δ are real numbers such that

$$\sum_{i=1}^{k} (\alpha_i^2 + \beta_i^2) + \sum_{j=1}^{n} \alpha_j^2 \neq 0.$$

On the other hand the following equation represents the condition to be satisfied for a point of the hyperplane to belong to the generalized helicoid:

$$\sum_{i=1}^{k} t_i(\alpha_i \cos(a_i s) + \beta_i \sin(a_i s)) + \sum_{j=k+1}^{n-1} \alpha_j t_j + \alpha_n b s = \delta.$$

Since one can always find a pair (s_0, t_0) satisfying the previous equation, the lemma is proved. $\square$

Proof of Theorem 3.1: Since (MEP) is equivalent to (MVP), $E(r)$ is a minimal submanifold of $S \times \Omega(r)$. Since $M = 2$, by the bundle structure property (see (7) above) $E(r)$ is a ruled submanifold in $\mathbb{R}^{2L-1}$. By Theorem 3.2, $E(r)$ is (up to rigid motions) a generalized helicoid. If some a_i or b are equal to zero then $E(r)$ is an hyperplane and, by Theorem 2.1, the price is unique. Otherwise if $b \cdot \prod_i a_i \neq 0$, by combining Lemma 3.3 with the fact that $E(r)$ is contained in the open set of $\mathbb{R}^{2L-1}$ consisting of those points with $p > 0$ (p being the price) one deduces that $E(r)$ is an affine hyperplane and so the price is unique. The "only if" part follows by Theorem 2.1 (see Remark 2.2). $\square$

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[3] A rigid motion of the Euclidean space $\mathbb{R}^l$ is an isometry of $\mathbb{R}^l$ given by the composition of an orthogonal $l \times l$ matrix and a translation by some vector $v \in \mathbb{R}^l$.

Remark 3.4 In the previous theorem we use the fact that $E(r) \subset S \times \Omega(r)$ is a minimal submanifold We can prove the same result by only assuming that the no-trade equilibria $T(r)$ (which is one dimensional for $M = 2$) is a minimal submanifold of $E(r)$, namely it is a geodesic. Indeed, by using the diffeomorphism between $T(r)$ and $B(r)$, and the parametrization Φ of $E(r)$ (see (9)), $T(r)$ can be parametrized through Φ by letting $v = 0$:

$$\Phi(t, 0) = \gamma(t).$$

Hence, if $T(r)$ is a geodesic in $E(r)$, its acceleration $\gamma''(t)$ is parallel, for every t, to the unit normal vector $N(t)_{|v=0}$ of $E(r)$ or, equivalently, $\gamma''(t) \wedge N(t)_{|v=0} = 0$. We have $\gamma''(t) = \beta''(t) = (\ddot{p}, 0, \ddot{w})$ and, since $v = 0$, $\Phi_t \wedge \Phi_v = \dot{\beta} \wedge \delta = (-\dot{w}, p\dot{p}, \dot{p})$. Hence $\gamma''(t) \wedge N(t)_{|v=0} = \beta'' \wedge (\beta' \wedge \delta) = 0$ if and only if

$$(-p\dot{p}\ddot{w}, p\ddot{p} + \dot{w}\ddot{w}, p\dot{p}\ddot{p}) = (0, 0, 0).$$

This implies that, for every t, $\dot{p}\ddot{p} = 0$, i.e. $(\dot{p}\dot{p})' = 0$, hence p is (constant and) unique.

4 The case $L = 2$

In this section we consider an economy with two goods and an arbitrary number of consumers. In this case the equilibrium manifold is a hypersurface. More precisely, the equilibrium manifold $E(r)$ has dimension $\mathbb{R}^{2M-2}$ and the ambient space has dimension $\mathbb{R}^{2M-1}$. So it makes sense to consider the normal vector field N along $E(r)$, namely for each $x \in E(r)$ we consider a unit vector $N(x)$ parallel to the affine line $T_x X^\perp$ normal to the tangent space $T_x X$ of X at x. The smooth map $N : E(r) \to S^{2M-2}$ which takes x to the point $N(x)$ of the unit sphere $S^{2M-2} \subset \mathbb{R}^{2M-1}$ is called the *Gauss map*. Obviously, if the Gauss map is constant then the price is constant and hence $E(r)$ is an affine hyperplane in $\mathbb{R}^{2M-1}$. In the following theorem, which represents the second main result of the paper, we show that the minimality assumption together with the constancy of the Gauss map outside a compact set imply uniqueness of the equilibrium price.

Theorem 4.1 *Let $L = 2$. Assume that the Gauss map is constant outside a compact subset of $E(r)$. Under the assumption of uniform distribution, $E(r)$ satisfies (MEP) if and only if the price is unique.*

This theorem can be intepreted by saying that if the equilibrium manifold is minimal and there exists a compact subset K of $\mathbb{R}^{2M-1}$ such that $(\mathbb{R}^{2M-1} \setminus K) \cap E(r)$ is the union of open subsets of hyperplanes each parallel to the hyperplane

$p = const$, then $E(r)$ is indeed an hyperplane. As a consequence, the usual one-dimension representation of the equilibrium manifold cannot be minimal (see figure below).
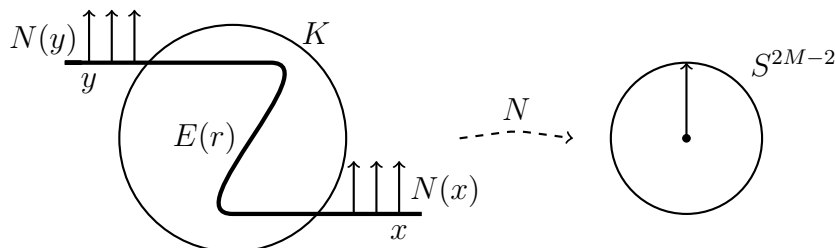
The proof of Theorem 4.1 relies on the following theorem obtained in turn by suitably combining some deep results obtained by Anderson in [1].

Theorem 4.2 *Let $\mathcal{M}^n \subset \mathbb{R}^{n+1}$, $n > 2$, be a minimal hypersurface such that the following conditions are satisfied:*

1. *$\mathcal{M}^n$ has one end;*

2. *$\mathcal{M}^n$ is a C^1-diffeomorphic to a compact manifold $\bar{\mathcal{M}}^n$ punctured at a finite number of points $\{p_i\}$.*

3. *the Gauss map $N : \mathcal{M}^n \to S^n$ extends to a C^1-map of $\bar{\mathcal{M}}^n$.*

Then $\mathcal{M}^n$ is an affine n-plane.

Remark 4.3 The number of ends of a smooth manifold is a topological invariant which, roughly speaking, measures the number of connected components "at infinity". The reader is referred to [1] for a rigorous definition. What we are going to use in the proof of Theorem 4.1 is that for $n > 1$, the Euclidean space $\mathbb{R}^n$ has only one end. This is because $\mathbb{R}^n \setminus K$ has only one unbounded component for any compact set K.

Proof of Theorem 4.1: Since (MEP) is equivalent to (MVP), $E(r)$ is a minimal submanifold of $S \times \Omega(r)$. If $M = 2$ we can apply Theorem 3.1. We can then assume $M > 2$ and so $\dim E(r) = 2M - 2 > 2$. Hence, in order to prove the "if" part it is enough to verify that the three conditions of Theorem 4.2 are satisfied for $E(r) \subset \mathbb{R}^{2M-1}$. Condition 1 follows by the previous remark, since $E(r)$ is globally diffeomorphic to $\mathbb{R}^{2M-2}$. Notice that the unit sphere S^{2M-2} is the Alexandroff compactification of $E(r) \cong \mathbb{R}^{2M-2}$, namely it can be obtained by adding one point, called ∞, to $E(r)$. In other words $E(r)$ is diffeomorphic to the sphere

S^{2M-2} punctured to ∞ and so also condition 2 holds true. The assumption that the Gauss map $N : E(r) \to S^{2M-2}$ is constant outside a compact set K means that $N(x) = N_0$, where N_0 is a fixed vector in S^{2M-2}, for all $x \in E(r) \backslash K$. Therefore, one can extend N to a C^∞-map $\hat{N} : S^{2M-2} \to S^{2M-2}$ by simply defining $\hat{N}(\infty) = N_0$, and so also condition 3 is satisfied. The "only if" part follows by Theorem 2.1 (see Remark 2.2). $\qquad\square$

Remark 4.4 Given a submanifold $\mathcal{M}^n$ of a Riemannian manifold $\mathcal{N}^{n+k}$, one can express the minimality condition in terms of the vanishing of the mean curvature H. If $k = 1$, namely when $\mathcal{M}^n$ is an hypersurface, the minimality condition, namely $H = 0$, is equivalent to the vanishing of the trace of the differential of the Gauss map (see e.g. [3]). Thus, for $L = 2$ one could try to show that minimality of $E(r)$ implies uniqueness of the equiilbrium price without imposing the extra condition on the constancy of the Gauss map outside a compact set (as in Theorem 4.1) by computing the Gauss map through the parametrization (9) above and imposing that the vanishing of the trace of its differential. This gives rise to a complicated PDE equation, which the authors were not able to handle even when $M = 3$.

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