Sources of Economic Growth in Models with Non-Renewable Resources

Sriket, Hongsilp and Suen, Richard M. H.

12 February 2021

Sources of Economic Growth in Models with Non-Renewable Resources

Hongsilp Sriket* Richard M. H. Suen†

This Version: 12th February, 2021.

Abstract

This paper re-examines the conditions under which endogenous economic growth can emerge in neoclassical models with non-renewable resources. Unlike most of the existing studies which focus exclusively on Cobb-Douglas production function, our analysis is based on a general specification. We formally prove that endogenous growth can emerge only under the "knife-edge" condition of a unitary elasticity of substitution between labour input and resource input. If this elasticity is not equal to one (as suggested by empirical evidence), then long-term economic growth is entirely driven by an exogenous technological factor. We also explore the implications of this on resource taxation.

Keywords: Non-Renewable Resources; Endogenous Growth; Knife-Edge Condition; Elasticity of Substitution.

JEL classification: O13, O41, Q32.

*Department of Economics, Kasetsart University, 50 Ngam Wong Wan Rd, Lat Yao Chatuchak, Bangkok 10900, Thailand. Email: *feco.scr@nontri.ku.ac.th*

†Corresponding Author: School of Business, Department of Economics, Finance and Acconting, University of Leicester, Leicester LE1 7RH, United Kingdom. Phone: +44 116 252 2880. Email: *mhs15@le.ac.uk*

1 Introduction

Economists have long been concerned with natural resource scarcity and its implications on long-term economic growth. In a seminal paper, Stiglitz (1974) examines these issues using the now-familiar neoclassical growth model with infinitely-lived consumers. It is shown that long-term growth in per-capita output is sustainable even when natural resources are limited in quantity but essential for production.[1] Most importantly for the present study, Stiglitz's model is one of endogenous growth. This means the long-term economic growth rate in this model is not *a priori* determined by some exogenous technological factors, but rather it is derived within the model and can potentially be influenced by the choices of the consumers, firms, as well as the government. In a more recent study, Agnani, Gutiérrez and Iza (2005, henceforth AGI) show that the endogenous growth result can also be obtained in a similar neoclassical framework but with overlapping generations of finitely-lived consumers. These findings have far-reaching implications for both resource economics and economic growth theory, as they suggest that practices and policies in natural resource management can influence the long-term growth prospect of an economy. One such policy is resource taxation.[2] Existing studies typically focus on the effects of resource tax on a mining firm's exploration and extraction decisions.[3] Very few have examined the impact of such tax on the wider economy and economic growth.[4] For most resource-producing countries, resource taxation is a significant part of the economy. For instance, Bornhorst *et al.* (2009) report that for a sample of 30 resource-producing countries (with various degrees of economic development), resource taxation on average accounts for 49.1% of total government revenues and 16.2% of GDP over the period 1992-2005. The sheer scale of this type of taxation warrants a thorough understanding about the factors that will determine its impact on the wider economy.

The aim of the present study is twofold. First, to re-examine the conditions under which endogenous economic growth can emerge in neoclassical models with non-renewable resources.[5]

[1] More specifically, perpetual growth in per-capita output is possible in the presence of (resource-augmenting) technological improvements and a high degree of substitutability between capital input and resource input. This result is also mentioned in Jones and Manuelli (1997, p.91).

[2] Similar to Boadway and Keen (2010), we use the terms "resource taxation" and "resource tax" broadly to include also other types of revenues that governments collect from the extraction and utilisation of natural resources (such as royalties and equity sharing arrangements).

[3] See, for instance, Gaudet and Lasserre (2015) for a review of the theoretical literature.

[4] An exception is Groth and Schou (2007), which examine the growth effects of capital income tax and resouce tax in a model with infinitely-lived consumers. Obviously, there is a large literature that examines how pollution tax or carbon tax can curb the negative externalities (pollution) generated by resource-related economic activities. We do not consider this type of externality in the present study.

[5] More specifically, we focus on a decentralised, competitive economy that features one single production sector without any externalities as in Stiglitz (1974) and AGI. Other studies in resource economics have shown that endogenous economic growth can emerge in the presence of production externalities [e.g., Groth and Schou (2007)] and R&D activities [see, for instance, Barbier (1999), Scholz and Ziemes (1999), and Grimaud and Rougé (2003)].

Second, to examine the implications of these conditions on the effects of resource taxation. The present study is motivated by the following observations: Both Stiglitz (1974) and AGI have relied on a specific form of production function, which is a Cobb-Douglas production function with three productive inputs (physical capital, labour and natural resources).[6] This is equivalent to assuming that the elasticity of substitution between any two of these inputs is always equal to one. This assumption, however, is at odds with many empirical findings.[7] While the estimates produced by the empirical literature may vary across datasets and estimation methods, the general consensus is that the Cobb-Douglas specification is not consistent with the data. This raises the question of whether the endogenous growth result in Stiglitz (1974) and AGI will remain valid without the Cobb-Douglas assumption. The present study not only provides an answer to this question, we also go one step further and show that this has important implications on the effects of resource taxation.

Our analysis is conducted within the same theoretical framework as in AGI, except for two changes: First, the Cobb-Douglas production function is replaced by some more general specifications that are in line with empirical evidence. Second, a constant flat tax on resource input is introduced.[8] In our benchmark model, we begin with a general class of production functions that exhibit constant returns to scale in all inputs and in which capital input is functionally separable from labour input and resource input.[9] Similar to AGI, we focus on characterising balanced growth equilibria, i.e., competitive equilibria in which (i) all major economic variables are growing at some constant rate and (ii) factor income shares are all strictly positive and constant over time. We show that two types of balanced growth equilibria are possible, depending on the elasticity of substitution between effective labour input and effective resource input.[10] On the one hand, if this elasticity is constant and equal to one (henceforth referred to as the unitary elasticity assumption), then the long-term economic growth rate is endogenously determined as in AGI. This result holds regardless of the elasticity of substitution between capital input and the

These other sources of endogenous growth are not the subject of the present study.

[6]A Cobb-Douglas production function is one that is multiplicatively separable in all inputs and has constant elasticities. This specification is commonly used in resource economics. See, for instance, Solow (1974), Mitra (1983), Barbier (1999) and Groth and Schou (2002) among others.

[7]See, for instance, Kemfert (1998), Kemfert and Welsch (2000), van der Werf (2008), Henningsen, Henningsen and van der Werf (2018).

[8]In resource economics, this is often referred to as an *ad valorem* severence tax. As explained in Groth and Schou (2007, p.83), this type of tax is closely related to the royalties collected by the government from resource extraction.

[9]The terminology and definition of "functional separability" are taken from Leontief (1947) and Blackorby and Russell (1976). Further details are provided in Section 2.

[10]Effective labour input is defined as (raw) labour input multiplied by a labour-augmenting technological factor. Similarly, effective resource input is defined as (raw) resource input multiplied by a resource-augmenting technological factor.

other two inputs. Thus, this can be viewed as a partial generalisation of the endogenous growth result in AGI. But, on the other hand, if the elasticity of substitution between effective labour input and effective resource input is bounded above or below by one, then long-term economic growth is solely driven by an exogenous labour-augmenting technological factor as in the standard neoclassical growth model. Taken together, our benchmark results underscore the pivotal role of the unitary elasticity assumption in generating endogenous economic growth.[11]

The intuition behind these results is as follows: As is well-known in the economic growth literature, perpetual growth in per-capita output requires certain factor (either exogenous or endogenous) that can counteract the diminishing marginal return of physical capital.[12] Such a factor is dubbed as the "engine of growth." In our benchmark model, if the unitary elasticity assumption is satisfied, then total factor productivity (TFP) and resource input jointly serve as the engine of growth. While the growth rate of TFP is taken as exogenously given, the depletion rate of resource input (which is determined by the utilisation rate of natural resources) is endogenously determined. This opens up a door through which other factors (such as consumers' preferences and government policies) can affect the utilisation of natural resources and, in turn, the engine of growth. But if the elasticity at issue is not equal to one, then balanced growth equilibria are possible only if effective resource input and effective labour input are growing at the same rate. This imposes a restriction on the utilisation rate of natural resources. In particular, this rate is now pinned down by the exogenous growth rate of labour input and technological factors. As a result, the engine of growth is solely determined by exogenous factors.

The present study is also related a growing literature which show that, in most (if not all) of the existing economic growth models, balanced growth equilibria are possible only under some "knife-edge" conditions.[13] These existing studies are primarily concerned about balanced growth equilibria in general, without distinguishing between exogenous and endogenous growth. This distinction, however, is the subject of our analysis. In particular, our results suggest that even if the conditions for balanced growth equilibria are met, endogenous growth will require yet another "knife-edge" condition.

Despite the simplicity of our benchmark model, it is able to produce a rich set of predictions

[11] Our benchmark results are robust to several changes in the benchmark model. For instance, in Section 4.2 we show that the exogenous growth result will prevail under several other specifications of the production function. In Appendix C, we show that our benchmark results can be easily extended to an environment with infinitely-lived consumers as in Stiglitz (1974). This suggests that the "knife-edge" condition of unitary elasticity of substitution also plays a crucial role in Stiglitz's results.

[12] See Jones and Manuelli (1997, Section 2) for more elaboration on this point.

[13] See, for instance, Groth and Schou (2002), Growiec (2007), Bugajewski and Maćkowiak (2015) and the references therein.

regarding the effects of resource taxation. The elasticities of substitution among the three inputs again play a critical role in this matter. To sharpen our results, we adopt a two-stage constant-elasticity-of-substitution (CES) production function in this part of the analysis. In the benchmark specification, effective labour input and effective resource input are placed in the inner CES function. The elasticity of substitution is, for now, denoted by σ_G. Hence, the endogenous growth result will emerge if σ_G is equal to one. The elasticity of substitution of the outer CES function is denoted by σ_F. Our first major finding is that if σ_G is one and σ_F is no less than one, then a unique balanced growth equilibrium exists under some additional conditions and resource tax is growth-enhancing. This scenario can be viewed as a direct extension of the analysis in AGI. The intuition behind a growth-enhancing resource tax can be explained as follows: When the resource tax rate goes up, resource input will become more costly and this will defer the utilisation of natural resources. As a result, a larger stock of resources is available for future use. By the complementarity between capital input and resource input in the production function, this will raise the marginal product of capital (and the rate of return from investment), and in turn promote capital accumulation and long-term economic growth. But if σ_G is one and σ_F is strictly less than one, then multiple balanced growth equilibria may emerge and resource tax is either growth-enhancing or growth-prohibiting depending on the equilibrium in question. Our second major finding is that if σ_G is *not* equal to one, then any changes in resource tax will only affect the level of per-capita variables but not their growth rate. In particular, an increase in resource tax will promote (or depress) capital formation and output if σ_G is strictly greater (or less) than one.

Whether the elasticity of substitution between labour input and resource input is equal to one is ultimately an empirical question. A number of existing studies have estimated the elasticity of substitution between physical capital, labour and commercial energy consumption.[14] The last one is used as a proxy for resource input. These studies usually report a less-than-unity elasticity of substitution between labour and energy [Kemfert (1998), Kemfert and Welsch (2000) and van der Werf (2008)]. When combining with these estimates, our benchmark model suggests that (i) introducing resource input into an otherwise standard neoclassical growth model will not change its fundamental nature (i.e., an exogenous growth model), and (ii) a higher tax rate on resource input will have a negative impact on capital formation and aggregate output. These predictions are in stark contrast to those produced under the unitary elasticity assumption.

[14] See van der Werf (2008) and Henningsen *et al.* (2018) for literature review and discussions on different estimation strategies.

The rest of the paper is organised as follows: Section 2 describes the setup of the benchmark model. Section 3 presents the main results concerning the balanced growth equilibria of the model. Section 4 provides some discussions and robustness checks on our baseline results. Section 5 concludes.

2 The Benchmark Model

2.1 Consumers

Our benchmark model is built upon the two-period overlapping-generation model in AGI, but with a more general specification of production function and a flat tax on resource input. Unless otherwise stated, we will adopt the same notations as in AGI to facilitate comparison between the two work.

Time is discrete and is indexed by $t \in \{0, 1, 2, ...\}$. In each time period, a new generation of identical consumers is born. The size of generation t is given by $N_t = (1+n)^t$, where $n \geq 0$ is the population growth rate. Each consumer lives two periods, which we will refer to as the young age and the old age. All young consumers have one unit of time which is supplied inelastically to work. The market wage rate at time t is denoted by w_t. All consumers are retired when old. There are two types of commodities in this economy: a composite good which can be used for consumption and capital accumulation, and non-renewable natural resources which are primarily used as input of production. All prices are expressed in units of the composite good.

Consider a consumer who is born at time $t \geq 0$. Let $c_{1,t}$ and $c_{2,t+1}$ denote his young-age and old-age consumption, respectively. The consumer's lifetime utility is given by

$$U(c_{1,t}, c_{2,t+1}) \equiv \ln c_{1,t} + \frac{1}{1+\theta} \ln c_{2,t+1}, \tag{1}$$

where $\theta > 0$ is the rate of time preference. The consumer can accumulate wealth by investing in physical capital and natural resources. Let s_t and m_t denote, respectively, the consumer's holdings of physical capital and natural resources. The rate of return from physical capital is denoted by r_{t+1}, and the spot price of natural resources at time t is p_t.

Taking $\{w_t, r_{t+1}, p_t, p_{t+1}\}$ as given, the consumer's problem is to choose a consumption profile $\{c_{1,t}, c_{2,t+1}\}$ and an investment portfolio $\{s_t, m_t\}$ so as to maximise his lifetime utility in (1), subject to the budget constraints:

$$c_{1,t} + s_t + p_t m_t = w_t, \qquad \text{and} \qquad c_{2,t+1} = (1 + r_{t+1})\, s_t + p_{t+1} m_t. \tag{2}$$

The first-order conditions of this problem can be expressed as

$$c_{2,t+1} = \left(\frac{1 + r_{t+1}}{1 + \theta}\right) c_{1,t}, \tag{3}$$

$$\frac{p_{t+1}}{p_t} = 1 + r_{t+1}. \tag{4}$$

Equation (3) is the familiar Euler equation of consumption, which determines the growth rate of individual consumption between young and old ages. Equation (4) is the Hotelling rule, which is essentially a no-arbitrage condition. It states that in order for the consumer to invest in both types of assets, the capital gain from natural resources must be equal to the gross return from physical capital. Using (2)-(4), we can derive the optimal level of consumption,

$$c_{1,t} = \left(\frac{1 + \theta}{2 + \theta}\right) w_t \qquad \text{and} \qquad c_{2,t+1} = \left(\frac{1 + r_{t+1}}{2 + \theta}\right) w_t, \tag{5}$$

and the optimal level of investment in physical capital,

$$s_t = \frac{w_t}{2 + \theta} - p_t m_t. \tag{6}$$

2.2 Production

On the supply side of the economy, there is a large number of identical firms that produce the composite good. In every period $t \geq 0$, each firm hires labour (N_t), rents physical capital (K_t) and purchases extracts of natural resources (X_t) from the competitive factor markets, and produces output (Y_t) according to the production technology

$$Y_t = F\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right). \tag{7}$$

In the above expression, Q_t is a resource-augmenting technological factor and A_t is a labour-augmenting technological factor. Both are assumed to grow at some constant exogenous rate, so that $Q_t = (1 + q)^t$ and $A_t = (1 + a)^t$, with $q > 0$ and $a \geq 0$, for all $t \geq 0$.

The production function in (7) is a composition of two functions, $F(\cdot)$ and $G(\cdot)$. Intuitively, one can interpret this as a two-stage production process: In the first stage, effective units of labour and natural resources are combined using an aggregator function $G(\cdot)$. The resultant is then combined with physical capital using another aggregator function $F(\cdot)$ to produce the final output. To use the terminology of Leontief (1947) and Blackorby and Russell (1976, p.286), the subset of inputs $\{Q_t X_t, A_t N_t\}$ is *functionally separable* from K_t. There is more than one way to define functional separability with three inputs. Another possibility is to assume that $\{K_t, Q_t X_t\}$ is functionally separable from $A_t N_t$. A third possibility is to assume that $\{K_t, A_t N_t\}$ is functionally separable from $Q_t X_t$. We will tackle these alternative specifications in Section 4.2.

The main properties of $F(\cdot)$ and $G(\cdot)$ are summarised in Assumptions A1 and A2. Recall that an input is deemed *essential for production* if output cannot be produced without this input [Dasgupta and Heal (1974) and Solow (1974, p.34)]. Throughout this paper, we will use $F_i(\cdot)$ to denote the partial derivative of $F(\cdot)$ with respect to its ith argument, $i \in \{1, 2\}$. The partial derivatives of $G(\cdot)$ are similarly represented.

Assumption A1 Both $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ and $G : \mathbb{R}^2_+ \to \mathbb{R}_+$ are twice continuously differentiable, strictly increasing, strictly concave and exhibit constant returns to scale (CRTS) in their arguments.

Assumption A2 Each input $I \in \{K, X, N\}$ is either essential for production or its marginal product is unbounded when I is arbitrarily close to zero.

Assumption A1 is a list of conditions that are commonly used in the economic growth literature. These conditions imply that the composite function in (7) is also twice continuously differentiable, strictly increasing, strictly concave and exhibits CRTS in all three inputs. In neoclassical growth models (without natural resources), it is also common to impose two other assumptions on the production function: First, both physical capital and labour are essential for production. Second, the marginal product of these inputs are unbounded as their quantity approach zero. These assumptions, however, are rather restrictive. For instance, within the class of constant-elasticity-of-substitution (CES) production functions, only Cobb-Douglas production functions satisfy both of these assumptions.[15] Our Assumption A2 gets around this problem by requiring only one of these properties to hold. This is sufficient to ensure that in equilibrium all three inputs are used in every time period. The argument goes as follows: As suggested by Solow

[15] The same point has been made by Dasgupta and Heal (1974, p.14) and Solow (1974, p.34) in natural resource economics. Solow (1974) cites this as the main reason for using the Cobb-Douglas production function in his work.

(1974), it is natural and reasonable to focus on equilibria that have a strictly positive amount of final output in every period. If an input is deemed essential for production, then a strictly positive amount must always be used in this kind of equilibria. On the other hand, since both factor markets and goods markets are perfectly competitive, the price of any input must be equated to its marginal product in equilibrium. If the marginal product of an input is unbounded at or around zero, then the marginal benefit of using an infinitesimal amount will certainly outweigh the marginal cost. Hence, it is never optimal to use a zero quantity of this input.

Assumptions A1 and A2 are compatible with the two-stage CES production functions proposed by Sato (1967).[16] This class of functions can be obtained by setting

$$F(K_t, Z_t) = \left[\alpha K_t^{\eta} + (1-\alpha) Z_t^{\eta}\right]^{\frac{1}{\eta}}, \qquad \text{with } \alpha \in (0,1) \text{ and } \eta < 1, \tag{8}$$

$$G(Q_t X_t, A_t N_t) \equiv \left[\varphi (Q_t X_t)^{\psi} + (1-\varphi)(A_t N_t)^{\psi}\right]^{\frac{1}{\psi}}, \qquad \text{with } \varphi \in (0,1) \text{ and } \psi < 1. \tag{9}$$

The production function in AGI corresponds to the special case in which $\eta = \psi = 0$. Under this "double Cobb-Douglas" specification, the two technological factors A_t and Q_t are observationally equivalent to a single Hicks neutral technological factor (or total factor productivity). For this reason, the separate effects of A_t and Q_t are not considered in AGI.

Since the production function exhibits CRTS in all three inputs, we can focus on the profit-maximisation problem faced by a single representative firm. Let R_t be the rental price of physical capital at time t and $\delta \in (0,1)$ be the depreciation rate. Expenditures on natural resource input are subject to a constant flat tax $\mu \geq 0$. Taking $\{R_t, w_t, p_t, \mu\}$ as given, the representative firm solves the following problem:

$$\max_{K_t, X_t, N_t} \left\{F(K_t, G(Q_t X_t, A_t N_t)) - R_t K_t - (1+\mu) p_t X_t - w_t N_t\right\}.$$

The first-order conditions are given by

$$R_t = r_t + \delta = F_1(K_t, G(Q_t X_t, A_t N_t)), \tag{10}$$

$$(1+\mu) p_t = Q_t F_2(K_t, G(Q_t X_t, A_t N_t)) G_1(Q_t X_t, A_t N_t), \tag{11}$$

$$w_t = A_t F_2(K_t, G(Q_t X_t, A_t N_t)) G_2(Q_t X_t, A_t N_t). \tag{12}$$

[16] In Appendix A, we verify that Assumption A2 is satisfied by various forms of nested CES production functions.

Equation (11) states that the representative firm will choose a level of X_t so that its marginal product equals the after-tax price. The tax rate μ thus drives a wedge between the marginal product of X_t and the price received by the owners of natural resources (i.e., the consumers).

2.3 Natural Resources

The economy has a fixed and known stock of non-renewable natural resources which can be costlessly extracted in each time period. The initial size of the stock is denoted by $M_0 > 0$.[17] Let M_t be the stock available at the beginning of time t, and X_t be the quantity extracted and sold in the factor market at time t.[18] Define the extraction rate (or utilisation rate) at time t as $\tau_t \equiv X_t/M_t$. The stock of natural resources then evolves according to

$$M_{t+1} = M_t - X_t = (1 - \tau_t) M_t. \tag{13}$$

2.4 Competitive Equilibrium

All the tax revenues collected from the resource tax are spent on some "unproductive" government purchases.[19] The government's budget is balanced in every period.

Given the initial conditions, $K_0 > 0$ and $M_0 > 0$, and the constant tax rate $\mu \geq 0$, a competitive equilibrium of this economy includes sequences of allocation $\{c_{1,t}, c_{2,t+1}, s_t, m_t\}_{t=0}^{\infty}$, aggregate inputs $\{K_t, N_t, X_t\}_{t=0}^{\infty}$, natural resources $\{M_t\}_{t=0}^{\infty}$ and prices $\{w_t, R_t, p_t, r_{t+1}\}_{t=0}^{\infty}$ such that,

(i) Given prices, $\{c_{1,t}, c_{2,t+1}, s_t, m_t\}$ solves the consumer's problem at any time $t \geq 0$.

(ii) Given prices and the tax rate, $\{K_t, N_t, X_t\}$ solves the representative firm's problem at any time $t \geq 0$.

(iii) The stock of natural resources evolves according to (13).

(iv) All markets clear in every period, which means $K_{t+1} = N_t s_t$ and $M_{t+1} = N_t m_t$ for all $t \geq 0$.

[17] At time 0, the initial stock of physical capital and natural resources are owned by a group of "initial old" consumers. The decision problem of these consumers is trivial and does not play any role in our main results.

[18] This notation is slightly different from the one in AGI. Specifically, these authors define M_t as the stock remaining at the end of time t (after extraction). This difference is immaterial since we both focus on balanced growth paths along which M_t depletes at a constant rate.

[19] These purchases are deemed unproductive because they have no direct impact on the consumers' utility and the production of goods. Our main results remain valid if the tax revenues are redistributed evenly to the young consumers through a lump-sum transfer. The details of this are shown in Section 4.1.

Using (6) and $M_{t+1} = N_t m_t$, we can write the capital market clearing condition as

$$K_{t+1} = \frac{w_t N_t}{2+\theta} - p_t M_{t+1}. \tag{14}$$

This shows that capital accumulation (i.e., $K_{t+1} > 0$) is possible only if $w_t N_t > (2+\theta) p_t M_{t+1} \geq 0$. Using (13) and the definition of τ_t, we can get

$$M_{t+1} = (1-\tau_t) \frac{M_t}{X_t} \cdot X_t = \left(\frac{1-\tau_t}{\tau_t} \right) X_t.$$

Substituting this, (11) and (12) into (14) gives

$$K_{t+1} = F_2 \left(K_t, G \left(Q_t X_t, A_t N_t \right) \right) \left[\frac{1}{2+\theta} A_t N_t G_2 \left(Q_t X_t, A_t N_t \right) - \frac{1}{1+\mu} \left(\frac{1-\tau_t}{\tau_t} \right) Q_t X_t G_1 \left(Q_t X_t, A_t N_t \right) \right]. \tag{15}$$

We will use this version of the capital market clearing condition repeatedly in the proof of our results.

3 Baseline Results

Our baseline results focus on equilibria that display the following additional properties:

(v) Per-worker output (Y_t/N_t) grows at a constant rate $\gamma^* - 1$, for some $\gamma^* > 0$, in every period.[20]

(vi) The rate of return from physical capital is constant over time, i.e., $r_t = r^*$, for some $r^* > -\delta$.

(vii) The utilisation rate of non-renewable resources is strictly positive and constant over time, i.e., $\tau_t = \tau^*$, for some $\tau^* \in (0,1)$.

Conditions (v) and (vi) are consistent with the empirical observations made by Kaldor (1963) and many subsequent studies in the economic growth literature. Condition (vii) is commonly used in economic growth models with natural resources.[21] Note that a constant interest rate is consistent with both "balanced" and "unbalanced" growth paths. Balanced growth paths are competitive equilibria in which all major economic variables grow at some constant rate and all the

[20] The size of population at time t is given by $N_t + N_{t-1} = \left[1 + (1+n)^{-1}\right] N_t$. Hence, every per-capita variable is directly proportional to its per-worker counterpart, and the two will always grow at the same rate.

[21] Stiglitz (1974) and Groth and Shou (2007) are among the studies that consider equilibria with a constant extraction rate. Scholz and Ziemes (1999) and Grimaud and Rougé (2003) are two examples that consider equilibria with a constant growth rate of X_t. These two conditions are equivalent given (13).

factor income shares are strictly positive and constant over time. By "unbalanced" growth, we are referring to equilibria in which the income share of a subset of productive factors is asymptotically zero. To explain this further, we first recall some of the results in Palivos and Karagiannis (2010): Let $\sigma_F(k)$ be the elasticity of substitution of $f(k) \equiv F(k,1)$. If $\lim_{k\to\infty} \sigma_F(k) > 1$, then the marginal product of capital $f'(k)$ will converge to a strictly positive constant as k approaches infinity. At the same time, the production function $f(k)$ will converge to a linear function and all the income will be distributed as capital income. When applied to the current context, this means condition (vi) is satisfied if $\lim_{k\to\infty} \sigma_F(k) > 1$ and K_t grows at a faster rate than $G(Q_t X_t, A_t N_t)$ so that

$$\frac{K_t}{G(Q_t X_t, A_t N_t)} \to \infty, \qquad \text{as } t \to \infty.$$

But this also means that total labour income ($w_t N_t$) will converge to zero, which violates the necessary condition for capital accumulation, i.e., $w_t N_t > (2+\theta) p_t M_{t+1} \geq 0$. Hence, this type of unbalanced growth paths are not sustainable in equilibrium. For this reason, we will focus on balanced growth paths.[22] This type of equilibria can be characterised as follows: First, given the simple linear structure of (13), condition (vii) implies that X_t and M_t must be decreasing at the same constant rate, i.e.,

$$\frac{X_{t+1}}{X_t} = \frac{M_{t+1}}{M_t} = 1 - \tau^*.$$

Second, a constant growth rate of p_t is implied by the Hotelling rule in (4) and a constant r^*. Lastly, under Assumptions A1 and A2, a constant interest rate will imply a constant ratio between K_t and Y_t. This result is formally established in Lemma 1.[23] All proofs are given in Appendix B. The remaining variables, such as wage rate and individual consumption, will also grow at a constant rate. This will be established in our baseline results.

Lemma 1 *Suppose the production function in (7) satisfies Assumptions A1 and A2. Then condition (vi) implies the existence of a positive constant κ^* such that $K_t = \kappa^* Y_t$ for all t. This means Y_t and K_t must be growing at the same rate over time.*

Before proceeding further, we first review the fundamental results in AGI, where government intervention is absent (i.e., $\mu = 0$). According to their Lemma 1 and Proposition 1, if the

[22] Our approach to characterising balanced growth equilibria is different from AGI's approach. Instead of imposing a constant growth rate on all variables at the onset, we show that such an equilibrium can be obtained from conditions (v)-(vii) and the assumptions on $F(\cdot)$ and $G(\cdot)$.

[23] The proof of Lemma 1 is specific for the production function in (7). For the alternative specifications considered in Section 4.2, we need all three conditions (v)-(vii) to obtain a constant capital-output ratio. The details of this are shown in the proof of Theorem 3.

production function is given by

$$Y_t = B_t K_t^{\alpha} N_t^{\beta} X_t^{v},$$

where $\alpha > 0$, $\beta > 0$, $v > 0$, $\alpha + \beta + v = 1$, and B_t is a measure of total factor productivity (TFP) that grows exogenously at a constant positive rate $b > 0$, then a unique balanced growth equilibrium exists in which per-worker output, per-worker capital, individual consumption and wage rate all grow at the same rate. The common growth factor γ^* and the utilisation rate τ^* are jointly determined by

$$\frac{\gamma^* (1+n)}{(1-\tau^*)} = \frac{\alpha (1+n)(2+\theta)\gamma^*}{\beta - (2+\theta) v (1-\tau^*)/\tau^*} + 1 - \delta, \tag{16}$$

$$\gamma^* = (1+b)^{\frac{1}{1-\alpha}} \left(\frac{1-\tau^*}{1+n} \right)^{\frac{v}{1-\alpha}}. \tag{17}$$

Once τ^* and γ^* are known, the value of r^* and κ^* are given by

$$1 + r^* = \frac{\gamma^* (1+n)}{1-\tau^*} \qquad \text{and} \qquad \kappa^* = \frac{\alpha}{r^* + \delta}. \tag{18}$$

In the sequel, we will refer to this as the AGI solution or the endogenous growth solution.

The main implication of the AGI solution is that both τ^* and γ^* are jointly determined by a host of factors, including the TFP growth rate (b), population growth rate (n), depreciation rate (δ), the share of factor incomes in total output (α, β and v), and the consumers' rate of time preference (θ). If we decompose B_t according to $B_t \equiv Q_t^v A_t^{\beta}$ and define $\widehat{k}_t \equiv K_t / (A_t N_t)$ and $\widehat{x}_t \equiv (Q_t X_t) / (A_t N_t)$, then the AGI solution also implies

$$\frac{\widehat{k}_{t+1}}{\widehat{k}_t} = \left(\frac{\widehat{x}_{t+1}}{\widehat{x}_t} \right)^{\frac{v}{1-\alpha}} = \left[\frac{(1+q)(1-\tau^*)}{(1+a)(1+n)} \right]^{\frac{v}{1-\alpha}}. \tag{19}$$

Thus, depending on the solution of (16)-(17), $\widehat{k}_t$ and $\widehat{x}_t$ can be monotonically increasing, monotonically decreasing or constant over time in the unique balanced growth equilibrium.

To highlight the significance of these findings, consider an alternate economy with $v = 0$ in AGI's production function. This means natural resources are no longer needed in the production process and, as a result, $B_t \equiv A_t^{1-\alpha}$.[24] In this case, a constant r_t immediately implies a constant $\widehat{k}_t$. This in turn implies that per-worker capital and per-worker output must be growing at the

[24] It follows immediately that $\tau_t = \tau^* = 0$ for all t. In this alternate economy, natural resources play the same role as the intrinsically worthless asset in the rational bubble model of Tirole (1983).

same rate as A_t, so that $\gamma^* = (1+a)$.[25] This is no more than a restatement of a well-known result: In the standard neoclassical growth model where production function exhibits CRTS in K_t and $A_t N_t$, long-term growth in per-capita variables is entirely driven by the exogenous labour-augmenting technological factor.[26]

When compared to this alternative economy, the AGI solution shows that introducing productive natural resources can transform an otherwise exogenous growth model into one with endogenous growth. If, in addition, the solution of (16)-(17) satisfies $(1+q)(1-\tau^*) > (1+a)(1+n)$, then the per-capita variables will grow at a faster rate than the technological factor A_t, i.e., $\gamma^* > 1+a$. Also note that equation (17) dictates an inverse relationship between γ^* and τ^*. This can be explained as follows: Lowering the utilisation rate of natural resources means that the resource stock will deplete at a slower pace. Thus, a larger stock of natural resources will remain in each time period. By the complementarity between physical capital and resource input in the production function, such a change will raise the marginal product of capital (and hence the rate of return from investment) in all future time periods. This will in turn promote capital accumulation and economic growth. As we will see below, this inverse relationship is specific to the endogenous growth solution and it is also useful in understanding the growth effects of the resource tax.

We now return to the question of whether the AGI solution remains valid under a more general production function. Our baseline results provide an answer to this question based on the composite function in (7). At the core of the analysis is the elasticity of substitution between the two inputs of $G(\cdot)$. This elasticity can be defined using the function $g(\widehat{x}) \equiv G(\widehat{x}, 1)$ for $\widehat{x} \geq 0$. By the CRTS property of $G(\cdot)$, we can write

$$G(QX, AN) = AN \cdot g(\widehat{x}),$$

where $\widehat{x} \equiv QX/(AN)$. Under Assumption A1, $g(\cdot)$ is twice continuously differentiable with $g'(\cdot) > 0$ and $g''(\cdot) < 0$. As shown in Arrow *et al.* (1961) and Palivos and Karagiannis (2010), the elasticity of substitution of $G(\cdot)$ can be expressed as[27]

$$\sigma_G(\widehat{x}) = -\frac{g'(\widehat{x})}{\widehat{x} g(\widehat{x})} \frac{g(\widehat{x}) - \widehat{x} g'(\widehat{x})}{g''(\widehat{x})} > 0, \qquad \text{for all } \widehat{x} > 0. \tag{20}$$

[25] This can also be seen by setting $\tau^* = 0$ and $v = 0$ in equations (17) and (19).

[26] This result holds in both overlapping-generation models and models with infinitely-lived consumers.

[27] The derivation of (20) rests upon two assumptions: (i) the factor markets and goods markets are perfectly competitive and (ii) $G(\cdot)$ exhibits CRTS [see Arrow *et al.* (1961, p.228-229)]. Both assumptions are satisfied in our model.

In particular, $G(\cdot)$ is Cobb-Douglas if and only if $\sigma_G(\cdot)$ is identical to one.

Given that Y_t and K_t are growing at the same rate (Lemma 1), the homogeneity of $F(\cdot)$ implies that $Z_t \equiv G(Q_t X_t, A_t N_t)$ must be growing at the same rate as well, i.e.,

$$\frac{Y_{t+1}}{Y_t} = \frac{K_{t+1}}{K_t} = \frac{Z_{t+1}}{Z_t} = \gamma^* (1+n). \tag{21}$$

If $G(\cdot)$ takes a Cobb-Douglas form as in

$$Z_t = G(Q_t X_t, A_t N_t) = (Q_t X_t)^{1-\phi} (A_t N_t)^{\phi}, \qquad \text{with } \phi \in (0,1), \tag{22}$$

then the growth factor of Z_t is a weighted geometric average of the growth factor of $Q_t X_t$ and $A_t N_t$, i.e.,

$$\frac{Z_{t+1}}{Z_t} = \left(\frac{Q_{t+1}X_{t+1}}{Q_t X_t}\right)^{1-\phi} \left(\frac{A_{t+1}N_{t+1}}{A_t N_t}\right)^{\phi}$$

$$\Rightarrow \gamma^* (1+n) = [(1+q)(1-\tau^*)]^{1-\phi} [(1+a)(1+n)]^{\phi}. \tag{23}$$

Obviously this equation alone is not enough to pin down the two endogenous variables γ^* and τ^*. The extra degree of freedom is what makes the endogenous growth solution possible. In the current model, γ^* and τ^* are jointly determined by equation (23) and the capital market clearing condition in (15). Hence, any factors that appear in these two conditions (which include preference parameters and the resource tax) will affect economic growth. These are the main ideas of our Theorem 1. Note that these results hold even if $F(\cdot)$ does not take the Cobb-Douglas form. Our Theorem 1 thus provides a partial generalisation of the AGI solution. The policy implication of this finding is examined in Proposition 1.

On the other hand, if $\sigma_G(\cdot)$ is never equal to one (which means it is either uniformly bounded above or uniformly bounded below by one), then condition (21) is satisfied only if $\{Z_t, Q_t X_t, A_t N_t\}$ all share the same growth rate, i.e.,

$$\gamma^* (1+n) = (1+q)(1-\tau^*) = (1+a)(1+n). \tag{24}$$

These equations uniquely pin down the value of γ^* and τ^*. In particular, the growth rate of per-worker output is now solely determined by the growth rate of A_t, i.e., $\gamma^* = 1 + a$. Hence, the endogenous growth solution is no longer valid. This also means that the tax rate μ can only affect the *level* of economic variables in a balanced growth equilibrium, but not their growth rate. The exogenous growth solution is presented in Theorem 2.

Theorem 1 *Suppose $F(\cdot)$ satisfies Assumptions A1 and A2 and $G(\cdot)$ takes the Cobb-Douglas form in (22). Define $b \equiv (1+a)^{\phi}(1+q)^{1-\phi} - 1$. Then any equilibrium that satisfies conditions (v)-(vii), if exists, must also satisfy*

$$\gamma^* = (1+b)\left(\frac{1-\tau^*}{1+n}\right)^{1-\phi}, \tag{25}$$

$$(1+r^*)(1-\tau^*) = \gamma^*(1+n), \tag{26}$$

$$\gamma^*(1+n) = \chi^* F_2(1,\chi^*)\left[\frac{\phi}{2+\theta} - \left(\frac{1-\tau^*}{\tau^*}\right)\left(\frac{1-\phi}{1+\mu}\right)\right], \tag{27}$$

$$F_1(1,\chi^*) = r^* + \delta. \tag{28}$$

In addition, wage rate and individual consumption must grow at the same rate as per-worker output.

Theorem 1 describes a balanced growth equilibrium that is similar in spirit to the AGI solution. This equilibrium is characterised by four key variables, namely the growth factor of per-worker output (γ^*), the utilisation rate of natural resources (τ^*), the rate of return from physical capital (r^*) and the ratio between $(\widehat{x}_t)^{1-\phi}$ and $\widehat{k}_t$ (denoted by χ^*). All other variables can be uniquely determined using these four values. Similar to the AGI solution, the utilisation rate τ^* must be greater than a certain threshold $\overline{\tau}(\mu) \in (0,1)$ which depends on μ. To see this, note that both $\gamma^*(1+n)$ and $\chi^* F_2(1,\chi^*)$ are strictly positive, thus it follows from (27) that

$$\frac{\phi}{2+\theta} - \left(\frac{1-\tau^*}{\tau^*}\right)\left(\frac{1-\phi}{1+\mu}\right) > 0$$

$$\Rightarrow \tau^* > \overline{\tau}(\mu) \equiv \frac{(2+\theta)(1-\phi)}{\phi(1+\mu)+(2+\theta)(1-\phi)} \in (0,1). \tag{29}$$

It is obvious from (29) that $\overline{\tau}(\mu)$ is strictly decreasing in μ.

The original AGI result can be recovered as follows: By setting $\mu = 0$ and $F(K_t, Z_t) = K_t^{\alpha} Z_t^{1-\alpha}$, with $\alpha \in (0,1)$, we can get

$$\chi^* = \left(\frac{r^*+\delta}{\alpha}\right)^{\frac{1}{1-\alpha}} \qquad \text{and} \qquad \chi^* F_2(1,\chi^*) = \frac{1-\alpha}{\alpha}(r^*+\delta).$$

Upon substituting these into (27) and setting $\phi = \beta/(1-\alpha)$ and $(1-\phi) = v/(1-\alpha)$, we get

$$\gamma^* (1+n) = \frac{1}{\alpha}(r^* + \delta)\left[\frac{\beta - (2+\theta)v(1-\tau^*)/\tau^*}{2+\theta}\right].$$

This, together with (26), gives us

$$\frac{\alpha(2+\theta)(1+n)\gamma^*}{\beta - (2+\theta)v(1-\tau^*)/\tau^*} = r^* + \delta = \frac{\gamma^*(1+n)}{1-\tau^*} - (1-\delta),$$

which is the same equation that appears in AGI's Lemma 1 part (i). According to their Proposition 1, a unique balanced growth equilibrium exists when both $F(\cdot)$ and $G(\cdot)$ are Cobb-Douglas. We now show that similar results can be obtained if $F(\cdot)$ is a CES function with elasticity of substitution strictly greater than one and $\mu \geq 0$.

Proposition 1 *Suppose $F(\cdot)$ takes the CES form in (8) with elasticity of substitution $\sigma_F \equiv (1-\eta)^{-1} \geq 1$ and $G(\cdot)$ takes the Cobb-Douglas form in (22). Then the economy has at least one balanced growth equilibrium that satisfies (25)-(28). If, in addition,*

$$\left\{(1+b)\left[\frac{1+n}{1-\overline{\tau}(\mu)}\right]^{\phi} - (1-\delta)\right\}^{\eta} > \alpha(1-\eta)^{1-\eta}, \tag{30}$$

where $\overline{\tau}(\mu)$ is the threshold level defined in (29), then a unique balanced growth equilibrium exists.

To better understand the effects of μ, let's consider two economies that are otherwise identical except for the tax rate on resource input, denoted by $\mu_2 > \mu_1 \geq 0$. In both economies, $F(\cdot)$ takes the CES form in (8) with elasticity of substitution $\sigma_F \geq 1$ and $G(\cdot)$ takes the Cobb-Douglas form in (22). Suppose a unique balanced growth equilibrium exists in both economies.[28] Let τ_i^* and γ_i^* denote, respectively, the equilibrium utilisation rate and common growth factor in the economy with tax rate μ_i, for $i \in \{1,2\}$. Then the economy with a higher tax rate will also have a faster growth rate, i.e., $\gamma_2^* > \gamma_1^*$ for any $\mu_2 > \mu_1 \geq 0$. In other words, resource taxation is growth-enhancing. This result is formally stated in Proposition 2. The intuition behind this is straight-forward: Increasing the tax rate μ will raise the cost of resource input and discourage utilisation, i.e., $\tau_2^* < \tau_1^*$ for any $\mu_2 > \mu_1 \geq 0$. A higher growth rate then follows from the inverse relationship between τ^* and γ^* described earlier.

[28] It suffice to assume that condition (30) is satisfied under the higher tax rate, i.e., $\mu_2 > 0$. The details of this are shown in the proof of Proposition 2.

Proposition 2 *Suppose $F(\cdot)$ takes the CES form in (8) with elasticity of substitution $\sigma_F \equiv (1-\eta)^{-1} \geq 1$ and $G(\cdot)$ takes the Cobb-Douglas form in (22). Suppose the condition in (30) is satisfied under $\mu_2 > 0$. Then $\tau_2^* < \tau_1^*$ and $\gamma_2^* > \gamma_1^*$, for any $\mu_2 > \mu_1 \geq 0$.*

Next, we turn to the case when $\sigma_F < 1$ (or equivalently, $\eta < 0$). It turns out to be more difficult to ensure the existence and uniqueness of balanced growth equilibrium in this case. This is because slight changes in σ_F within this range can potentially lead to drastic changes in equilibrium outcomes. The following numerical example is intended to demonstrate this. First, we combine equations (25)-(28) to form a single equation in τ^*, which is

$$\frac{(2+\theta)(1+b)(1+n)^{\phi}(1-\tau^*)^{1-\phi}}{\phi - \left(\frac{1-\tau^*}{\tau^*}\right)(2+\theta)(1-\phi)(1+\mu)^{-1}} = \frac{r(\tau^*)+\delta}{\alpha}\left\{\left[\frac{r(\tau^*)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}} - \alpha\right\}, \tag{31}$$

where $r(\tau^*) \equiv (1+b)(1+n)^{\phi}(1-\tau^*)^{-\phi} - 1$. We then evaluate both sides of this equation over a range of τ using the following parameterisation: Suppose one model period takes 25 years. We set $\theta = 1.775$ so that the annual subjective discount factor is 0.96. We set the annual employment growth rate to 1.6%, which matches the average annual growth rate of U.S. employment over the period 1953-2008. This implies $n = (1.0160)^{25} - 1 = 0.4871$. The annual TFP growth rate is taken to be 1.05%, which is in line with the estimates reported by Feng and Serletis (2008, p.300). The implied value of b is 0.2984 over a 25-year period. We also set $\mu = 0$, $\delta = 1$, $\phi = 0.38$ and $\alpha = 0.24$. Figure 1 plots the left-hand side (LHS) and the right-hand side (RHS) of equation (31) under two different values of σ_F, namely 0.62 and 0.65. Both fall within the range of estimates reported by Henningsen *et al.* (2019, Table 4).[29] As shown in the diagram, equation (31) has no solution when $\sigma_F = 0.62$ $(\eta = -0.613)$, which means there is no equilibrium that satisfies conditions (v)-(vii). But when σ_F is raised to 0.65 $(\eta = -0.538)$, the same equation has at least two solutions, which are $\tau^* = 0.9695$ and $\tau^* = 0.9964$. The possibility of multiple equilibria, however, does not alter the fundamental nature of the AGI solution — in each of these equilibria, the common growth factor γ^* is determined by a host of factors.

When there are more than one balanced growth equilibria, the effects of reource tax may differ across equilibria. For instance, consider the case when $\sigma_F = 0.65$ in the above example. Let (τ_1^*, γ_1^*) and (τ_2^*, γ_2^*) denote the two balanced growth equilibria, with $\tau_1^* < \tau_2^*$. It follows from (25) that $\gamma_1^* > \gamma_2^*$. Note that the resource tax μ only appears on the left-hand side of (31). In

[29] In Henningsen *et al.* (2019, Table 4), the elasticity of substitution between the inputs of $F(\cdot)$ is denoted by $\sigma_{(LE)K}$. In the existing empirical studies, it is conventional to use commerical energy consumption as a proxy for natural resource input.

particular, any increase in μ will lower its value. This will shift the LHS curve in Figure 1 down but leave the RHS curve unaffected. It follows that a small increase in μ will lower the value of τ_1^* and raise the value of γ_1^*, but have the opposite effects on (τ_2^*, γ_2^*).

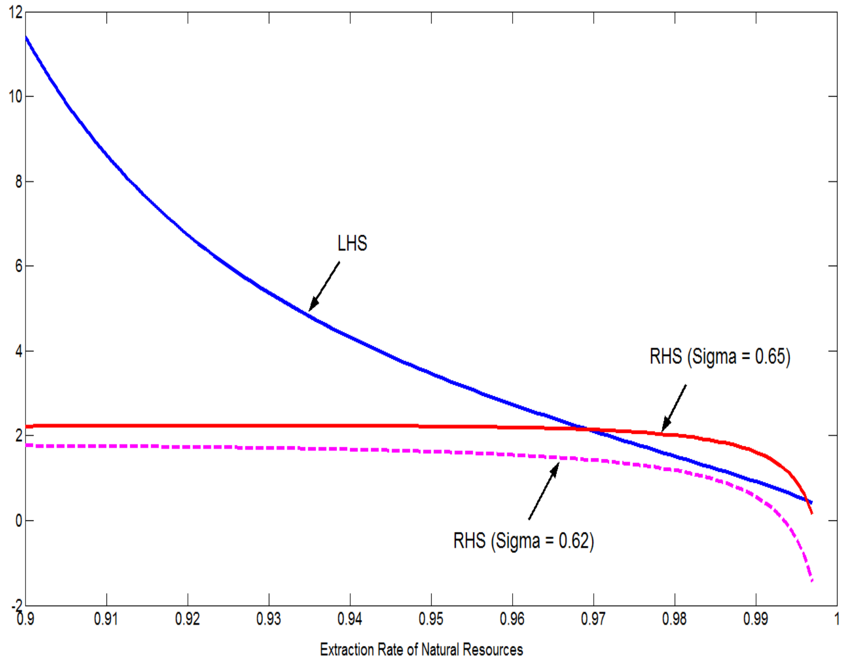

Figure 1 Numerical Example

The rest of this section is devoted to the case when $\sigma_G(\widehat{x}) \neq 1$ for all $\widehat{x} > 0$. The main results are summarised in Theorem 2.

Theorem 2 *Suppose the production function in (7) satisfies Assumptions A1 and A2. Suppose the elasticity of substitution of* $G(\cdot)$ *is never equal to one. Then any equilibrium that satisfies conditions (v)-(vii), if exists, must also satisfy* $\gamma^* = 1 + a$, $r^* = q$, *and*

$$1 - \tau^* = \frac{(1+a)(1+n)}{1+q}. \tag{32}$$

Such an equilibrium will have $\widehat{k}_t = \widehat{k}^*$ *and* $\widehat{x}_t = \widehat{x}^*$ *for all* t, *where* $\widehat{k}^*$ *and* $\widehat{x}^*$ *are determined by*

$$F_1\left(\widehat{k}^*, G(\widehat{x}^*, 1)\right) = q + \delta, \tag{33}$$

$$(1+a)(1+n)\widehat{k}^* = F_2\left(\widehat{k}^*, G(\widehat{x}^*, 1)\right)\left[\frac{G_2(\widehat{x}^*, 1)}{2+\theta} - \left(\frac{1-\tau^*}{\tau^*}\right)\frac{\widehat{x}^* G_1(\widehat{x}^*, 1)}{1+\mu}\right]. \tag{34}$$

In addition, wage rate and individual consumption must grow at the same rate as per-worker output.

Theorem 2 presents a balanced growth equilibrium that is in stark contrast to the AGI solution. Specifically, if $\sigma_G(\cdot)$ is bounded away from one, then either there is no equilibrium that satisfies conditions (v)-(vii) or any such equilibrium will have a common growth rate in per-capita variables that is solely determined by the exogenous growth factor A_t. Thus, there is no room for endogenous growth. It follows that the tax rate μ can only affect the level of major economic variables but not their growth rate. The above theorem also highlights two important differences between the two technological factors A_t and Q_t. First, the growth rate of A_t determines the common growth factor (γ^*), while the growth rate of Q_t determines the rate of return from physical capital (r^*). This follows from the fact that, along any balanced growth path, any changes in Q_t will be absorbed by the resource price p_t. This, together with the Hotelling rule, then implies that $r^* = q$. The second difference is that, holding other factors constant, a higher growth rate of A_t will suppress the utilisation rate τ^* while a higher growth rate of Q_t will promote it. This can be explained as follows: By the complementarity between $Q_t X_t$ and $A_t N_t$ in $G(\cdot)$, a higher growth rate of A_t will raise the marginal product of resource input in all future time periods (when other things are kept constant). This will induce an intertemporal substitution in resource utilisation by shifting the demand from the current period to the future periods. Such a shift will slow down the depletion of the resource stock, which is equivalent to lowering the value of τ^*. A higher growth rate of Q_t will have the opposite effect.

Since τ^* must be confined between zero and one, it is necessary to impose the restriction $1+q > (1+a)(1+n)$. This means the growth rate of resource-augmenting technological factor must be strictly positive, even when there is no population growth (i.e., $n=0$) and no labour-augmenting technological progress (i.e., $a=0$). Intuitively, this is saying that a minimum degree of resource-augmenting technological progress is necessary in order to compensate for the decline in X_t over time and make perpetual economic growth possible.

To sharpen our understanding of the exogenous growth solution, we focus on the case when $F(\cdot)$ and $G(\cdot)$ take the CES form in (8) and (9). Define an auxiliary notation Θ according to

$$\Theta \equiv \frac{q+\delta}{\alpha(2+\theta)}\left[\left(\frac{q+\delta}{\alpha}\right)^{\frac{\eta}{1-\eta}} - \alpha\right].$$

The first part of Proposition 3 establishes the existence and uniqueness of balanced growth equilibrium under the stated conditions. Since $\{\tau^*, \gamma^*, r^*\}$ are all independent of μ, any changes in this tax rate will only affect the level of $\widehat{x}^*$ and $\widehat{k}^*$. The second part of Proposition 3 states that these effects depend crucially on the value of σ_G. If this elasticity is greater than one [or equivalently, $\psi \in (0,1)$], then a higher tax rate on resource input will raise the value of $\widehat{x}^*$. The effect is opposite if σ_G is less than one (or equivalently, $\psi < 0$). By the homogeneity property of $F_1(\cdot)$ and (33), $\widehat{x}^*$ and $\widehat{k}^*$ will move in the same direction in light of any changes in μ.

Proposition 3 *Suppose* $F(\cdot)$ and $G(\cdot)$ *take the CES form in (8) and (9), respectively. Suppose further that* $\min\{\Theta, 1+q\} > (1+a)(1+n)$. *Then the following results hold.*

(i) *There exists a unique balanced growth equilibrium that satisfies* $\gamma^* = 1+a$, $r^* = q$, *and (32)-(34).*

(ii) *An increase in* μ *will raise the value of* $\widehat{x}^*$ *and* $\widehat{k}^*$ *if* $\sigma_G \equiv (1-\psi)^{-1} > 1$ *and lower their value if* $\sigma_G \equiv (1-\psi)^{-1} < 1$.

Two final remarks are in order. First, Proposition 3 covers the special case in which $F(\cdot)$ and $G(\cdot)$ have the same constant elasticity of substitution, i.e., $\eta = \psi$. In this case, the production function in (7) becomes

$$Y_t = \left[\alpha K_t^{\eta} + (1-\alpha)\varphi(Q_t X_t)^{\eta} + (1-\alpha)(1-\varphi)(A_t N_t)^{\eta}\right]^{\frac{1}{\eta}},$$

which is the familiar Dixit–Stiglitz aggregator function. Second, the main results in Theorem 1 and Theorem 2 can be readily extended to an environment with infinitely-lived consumers. The details are shown in Appendix C.

4 Further Results and Discussions

4.1 Alternative Use of Tax Revenues

Most of the theoretical results in Section 3 will remain valid if all the tax revenues collected from the resource tax are redistributed evenly among the young consumers through a lump-sum transfer.[30] Under this alternative arrangement, a young consumer at time t will face the following

[30]Due to page limitations, we can only highlight the key points here. Further details are available from the authors upon request.

budget constraint:

$$c_{1,t} + s_t + p_t m_t = w_t + \xi_t,$$

where ξ_t is the transfer at time t. The consumer's optimal choices are now given by

$$c_{1,t} = \left(\frac{1+\theta}{2+\theta}\right)(w_t + \xi_t), \qquad c_{2,t+1} = \left(\frac{1+r_{t+1}}{2+\theta}\right)(w_t + \xi_t)$$

$$s_t = \frac{w_t + \xi_t}{2+\theta} - p_t m_t. \tag{35}$$

The government's budget is balanced in every time period, so that

$$\mu p_t X_t = N_t \xi_t, \qquad \text{for all } t \geq 0. \tag{36}$$

The rest of the economy is the same as in the benchmark model.

Since the policy variables μ and ξ_t do not affect the production technology directly, most of the results in Theorem 1 and Theorem 2 will remain valid. Specifically, it remains the case that if the elasticity of substitution of $G(\cdot)$ is constant and equal to one, then the endogenous growth solution will prevail; but if this elasticity is bounded away from one, then γ^* and τ^* are again determined by (24).[31] The only parts that need to be modified are (27) and (34), both of which are derived from the capital-market-clearing condition. In particular, equation (27) in Theorem 1 will now be replaced by

$$\gamma^*(1+n) = \chi^* F_2(1, \chi^*) \left[\frac{\phi}{2+\theta} + \left(\frac{\mu}{2+\theta} - \frac{1-\tau^*}{\tau^*}\right)\frac{1-\phi}{1+\mu}\right]. \tag{37}$$

This equation also implies that τ^* must be greater than the threshold

$$\widetilde{\tau}(\mu) \equiv \frac{(1-\phi)(2+\theta)}{\phi + \mu + (1-\phi)(2+\theta)}, \tag{38}$$

which is strictly decreasing in μ. Similarly, equation (34) in Theorem 2 will be replaced by

$$(1+a)(1+n)\widehat{k}^* = F_2\left(\widehat{k}^*, G(\widehat{x}^*, 1)\right)\left[\frac{G_2(\widehat{x}^*, 1)}{2+\theta} + \left(\frac{\mu}{2+\theta} - \frac{1-\tau^*}{\tau^*}\right)\frac{\widehat{x}^* G_1(\widehat{x}^*, 1)}{1+\mu}\right].$$

The results of Propositions 1-3 also remain valid, except for some minor changes. First, consider the case when $G(\cdot)$ is Cobb-Douglas and $F(\cdot)$ is a CES function with $\sigma_F \equiv (1-\eta)^{-1} \geq 1$.

[31] The proof of these statements are essentially the same as in the proof of Theorem 1 and Theorem 2, hence they are not repeated here.

Then it can be shown that (i) a balanced growth equilibrium that satisfies (25), (26), (28) and (37) always exists; (ii) if the following condition is satisfied, then a unique balanced growth equilibrium exists,

$$\left\{(1+b)\left[\frac{1+n}{1-\widetilde{\tau}(\mu)}\right]^{\phi}-(1-\delta)\right\}^{\eta}>\alpha(1-\eta)^{1-\eta},$$

where $\widetilde{\tau}(\mu)$ is the same threshold defined in (38); and (iii) an increase in μ will lower the value of τ^* but increase the common growth factor γ^*.

Finally, consider the case when both $F(\cdot)$ and $G(\cdot)$ take the CES form as in Proposition 3. In the benchmark model, the value of $\widehat{x}^*$ is uniquely determined by[32]

$$(\widehat{x}^*)^{\psi}=\frac{1-\varphi}{\varphi}\frac{\Theta-(1+a)(1+n)}{(1+a)(1+n)+\left(\frac{1-\tau^*}{\tau^*}\right)\left(\frac{2+\theta}{1+\mu}\right)\Theta}. \tag{39}$$

When the tax revenues are refunded to the consumers, the value of $\widehat{x}^*$ is determined by

$$(\widehat{x}^*)^{\psi}=\frac{1-\varphi}{\varphi}\frac{\Theta-(1+a)(1+n)}{(1+a)(1+n)+\left[\left(\frac{1-\tau^*}{\tau^*}\right)\left(\frac{2+\theta}{1+\mu}\right)-\frac{\mu}{1+\mu}\right]\Theta}. \tag{40}$$

In both settings, the utilisation rate τ^* is determined by (32). Note that the right-hand side of both (39) and (40) are strictly increasing in μ. Thus, an increase in μ will raise (or lower) the value of $\widehat{x}^*$ if $\psi>0$ (or $\psi<0$).

4.2 Alternative Specifications of Production Function

In this subsection, we will consider two alternative specifications of the production function. These are given by

$$Y_t=F(A_tN_t,G(K_t,Q_tX_t)), \tag{41}$$

$$Y_t=F(Q_tX_t,G(K_t,A_tN_t)). \tag{42}$$

To maintain consistency across all three specifications, we use $G(\cdot)$ to represent the "inner" aggregator function and $F(\cdot)$ to represent the "outer" aggregator function in (7), (41) and (42). All three specifications will coincide with AGI's production function if both $G(\cdot)$ and $F(\cdot)$ have the Cobb-Douglas form. Our main interest here is to examine the properties of balanced growth equilibrium when *one* of the aggregator functions in (41) and (42) *does not* take the Cobb-Douglas form. To this end, we consider four different parametric production functions based on (41) and

[32] The derivation of this is shown in the proof of Proposition 3.

(42). In the first two specifications, the inner aggregator function is Cobb-Douglas but the outer one has a CES form, so that

$$Y_t = \left\{ \varphi \left(A_t N_t \right)^{\psi} + (1-\varphi) \left[K_t^{\alpha} \left(Q_t X_t \right)^{1-\alpha} \right]^{\psi} \right\}^{\frac{1}{\psi}}, \tag{43}$$

$$Y_t = \left\{ \varphi \left(Q_t X_t \right)^{\psi} + (1-\varphi) \left[K_t^{\alpha} \left(A_t N_t \right)^{1-\alpha} \right]^{\psi} \right\}^{\frac{1}{\psi}}, \tag{44}$$

with $\alpha \in (0,1)$, $\varphi \in (0,1)$ and $\psi < 1$. In the second group, the inner aggregator function is a CES function and the outer one is Cobb-Douglas, so that

$$Y_t = \left[\varphi K_t^{\psi} + (1-\varphi) \left(Q_t X_t \right)^{\psi} \right]^{\frac{1-\beta}{\psi}} \left(A_t N_t \right)^{\beta} \tag{45}$$

$$Y_t = \left(Q_t X_t \right)^{\upsilon} \left[\varphi K_t^{\psi} + (1-\varphi) \left(A_t N_t \right)^{\psi} \right]^{\frac{1-\upsilon}{\psi}}, \tag{46}$$

with $\beta \in (0,1)$, $\upsilon \in (0,1)$, $\varphi \in (0,1)$ and $\psi < 1$.[33] The rest of the economy is the same as in the benchmark model. The main result of this subsection is summarised in Theorem 3.

Theorem 3 *Suppose the production function takes one of the forms in (43)-(46). Then any balanced growth equilibrium (if exists) must satisfy* $\gamma^* = 1 + a$, $r^* = q$, *and*

$$1 - \tau^* = \frac{(1+a)(1+n)}{1+q}.$$

The main message of Theorem 3 is clear: despite the differences in appearance, all the production functions in (43)-(46) have the same implications for balanced growth equilibrium. Specifically, any balanced growth equilibrium (if exists) must satisfy $\gamma^* = 1 + a$, $r^* = q$, and $(1 - \tau^*) = (1+a)(1+n)/(1+q)$. It follows that the two transformed variables $\widehat{k}_t$ and $\widehat{x}_t$ must be time-invariant, and so there is no room for endogenous growth.

4.3 Discussions

The results in the previous sections suggest that the AGI solution is valid only under the "knife-edge" condition of a unitary elasticity of substitution between labour input and resource input. If we rewrite (22) as

$$G(Q_t X_t, A_t N_t) = \left[A_t \left(Q_t X_t \right)^{\frac{1-\phi}{\phi}} N_t \right]^{\phi},$$

[33] The parameters β and υ have the same economic meaning as in AGI. Specifically, they represent the share of total output distributed as labour income and expenses on natural resource input.

then the expression $\widetilde{X}_t \equiv A_t \left(Q_t X_t\right)^{\frac{1-\phi}{\phi}}$ can be viewed as a labour-augmenting factor and serves as the engine of growth. When viewed through this lens, our results suggest that the AGI solution is valid only when effective resource input is labour-augmenting in the production function, i.e.,

$$Y_t = F\left(K_t, \left(\widetilde{X}_t N_t\right)^{\phi}\right).$$

This result may remind one of the celebrated Uzawa Growth Theorem [Uzawa (1961)]. But there are at least two important differences between the two. First, the Uzawa Growth Theorem and its variants are typically derived from a CRTS production function with only two inputs, namely physical capital and labour [see, for instance, Uzawa (1961), Schlicht (2006), Jones and Scrimgeour (2008) and Grossman *et al.* (2017)]. It is not immediately clear how the Uzawa Growth Theorem can be extended to a general CRTS production function with more than two inputs, such as the one considered here. Second, and more importantly, the Uzawa Growth Theorem states the conditions under which a balanced growth equilibrium can emerge, without explicitly mentioning whether the "engine of growth" is exogenous or endogenous. The distinction between exogenous and endogenous growth, however, is the main focus of our analysis.

5 Conclusions

In this paper, we re-examine the conditions required for endogenous long-term economic growth in neoclassical models with non-renewable resources. Unlike most of the existing studies which focus exclusively on Cobb-Douglas production function, we adopt a general specification and seek general conditions under which endogenous economic growth can emerge. Our benchmark results show that this type of growth is possible only under the "knife-edge" condition of a unitary elasticity of substitution between effective labour input and effective resource input. This condition, however, has found little support in empirical studies. For all other specifications that we have considered, including those that are in line with empirical evidence, the model predicts that long-term economic growth is entirely driven by the exogenous labour-augmenting technological factor. One possible direction of future research is to examine whether unitary elasticity assumption plays a similar role in other endogenous growth models (e.g., those that involve R&D activities). Our model also produces a rich set of predictions regarding the effects of resource taxation. In particular, depending on the elasticities of substution among the three inputs, an increase in resource tax can be either beneficial or adverse to capital accumulation.

When combined with the estimates produced by the empirical literature, our benchmark model suggests that increasing the resource tax will have a negative impact on capital formation and aggregate output. This is in stark contrast to the prediction produced by the endogenous growth solution.

Appendix A: Nested CES Production Functions

In this appendix, we will verify that Assumption A2 is satisfied by all the nested CES production functions considered in Sections 3 and 4. We begin with the specification considered in Section 3, which is

$$F(K_t, Z_t) = \left[\alpha K_t^{\eta} + (1-\alpha) Z_t^{\eta}\right]^{\frac{1}{\eta}}, \qquad \text{with } \alpha \in (0,1) \text{ and } \eta < 1,$$

$$G(Q_t X_t, A_t N_t) \equiv \left[\varphi (Q_t X_t)^{\psi} + (1-\varphi)(A_t N_t)^{\psi}\right]^{\frac{1}{\psi}}, \qquad \text{with } \varphi \in (0,1) \text{ and } \psi < 1.$$

First, consider capital input. If $\eta \leq 0$, then

$$\lim_{K_t \to 0} F(K_t, G(Q_t X_t, A_t N_t)) = 0,$$

for any $Q_t X_t > 0$ and $A_t N_t > 0$, regardless of the value of ψ. Thus, physical capital is essential for production when $\eta \leq 0$. If $\eta \in (0,1)$, then

$$\lim_{K_t \to 0} F_1(K_t, G(Q_t X_t, A_t N_t)) = \infty,$$

regardless of the value of ψ. Next, consider the inputs of $G(\cdot)$. When $\psi \leq 0$, we have

$$\lim_{X_t \to 0} G(Q_t X_t, A_t N_t) = \lim_{N_t \to 0} G(Q_t X_t, A_t N_t) = 0,$$

$$\lim_{X_t \to 0} G_1(Q_t X_t, A_t N_t) = \varphi^{\frac{1}{\psi}} \quad \text{and} \quad \lim_{N_t \to 0} G_2(Q_t X_t, A_t N_t) = (1-\varphi)^{\frac{1}{\psi}}.$$

There are now two subcases to consider: If $\psi \leq 0$ and $\eta \leq 0$, then both natural resources and labour are essential for production, i.e.,

$$\lim_{X_t \to 0} F(K_t, G(Q_t X_t, A_t N_t)) = \lim_{N_t \to 0} F(K_t, G(Q_t X_t, A_t N_t)) = 0.$$

If $\psi \leq 0$ and $\eta \in (0,1)$, then we can show that

$$\begin{aligned} \lim_{X_t \to 0} \frac{\partial Y_t}{\partial X_t} &= (1-\alpha)\left\{\alpha \lim_{X_t \to 0}\left[\frac{G(Q_t X_t, A_t N_t)}{K_t}\right]^{-\eta} + 1 - \alpha\right\}^{\frac{1}{\eta}-1} \cdot \lim_{X_t \to 0} G_1(Q_t X_t, A_t N_t) \cdot Q_t \\ &= (1-\alpha) \cdot \infty \cdot \varphi^{\frac{1}{\psi}} Q_t = \infty. \end{aligned}$$

Likewise,

$$\begin{aligned}\lim_{N_t\to 0}\frac{\partial Y_t}{\partial N_t} &= (1-\alpha)\left\{\alpha\lim_{N_t\to 0}\left[\frac{G(Q_tX_t,A_tN_t)}{K_t}\right]^{-\eta}+1-\alpha\right\}^{\frac{1}{\eta}-1}\cdot\lim_{N_t\to 0}G_2(Q_tX_t,A_tN_t)\cdot A_t\\ &= (1-\alpha)\cdot\infty\cdot(1-\varphi)^{\frac{1}{\psi}}A_t=\infty.\end{aligned}$$

If $\psi\in(0,1)$, then we have

$$\lim_{X_t\to 0}G(Q_tX_t,A_tN_t)=(1-\varphi)^{\frac{1}{\psi}}(A_tN_t)\qquad\text{and}\qquad\lim_{N_t\to 0}G(Q_tX_t,A_tN_t)=\varphi^{\frac{1}{\psi}}(Q_tX_t),$$

$$\lim_{X_t\to 0}G_1(Q_tX_t,A_tN_t)=\lim_{N_t\to 0}G_2(Q_tX_t,A_tN_t)=\infty.$$

Using these we can obtain

$$\lim_{X_t\to 0}\frac{\partial Y_t}{\partial X_t}=F_2\left(K_t,(1-\varphi)^{\frac{1}{\psi}}A_tN_t\right)\left[\lim_{X_t\to 0}G_1(Q_tX_t,A_tN_t)\right]=\infty,$$

$$\lim_{N_t\to 0}\frac{\partial Y_t}{\partial N_t}=F_2\left(K_t,\varphi^{\frac{1}{\psi}}Q_tX_t\right)\left[\lim_{N_t\to 0}G_2(Q_tX_t,A_tN_t)\right]=\infty.$$

Note that these results hold regardless of the value of η.

Next, we turn to the production function in (43). There are now only two possible cases: If $\psi\leq 0$, then all three inputs are essential for production. If $\psi\in(0,1)$, then we can obtain

$$\lim_{N_t\to 0}\frac{\partial Y_t}{\partial N_t}=\varphi A_t\left\{\varphi+(1-\varphi)\lim_{N_t\to 0}\left[\frac{A_tN_t}{K_t^{\alpha}(Q_tX_t)^{1-\alpha}}\right]^{-\psi}\right\}^{\frac{1}{\psi}-1}=\infty,$$

$$\lim_{K_t\to 0}\frac{\partial Y_t}{\partial K_t}=\alpha(1-\varphi)\left\{\varphi\lim_{N_t\to 0}\left[\frac{K_t^{\alpha}(Q_tX_t)^{1-\alpha}}{A_tN_t}\right]^{-\psi}+1-\varphi\right\}^{\frac{1}{\psi}-1}\left[\lim_{K_t\to 0}\left(\frac{K_t}{Q_tX_t}\right)^{\alpha-1}\right]=\infty,$$

$$\begin{aligned}\lim_{X_t\to 0}\frac{\partial Y_t}{\partial X_t} &= (1-\alpha)(1-\varphi)\left\{\varphi\lim_{X_t\to 0}\left[\frac{K_t^{\alpha}(Q_tX_t)^{1-\alpha}}{A_tN_t}\right]^{-\psi}+1-\varphi\right\}^{\frac{1}{\psi}-1}\left[\lim_{X_t\to 0}\left(\frac{K_t}{Q_tX_t}\right)^{\alpha}\right]Q_t\\ &= \infty.\end{aligned}$$

Note that the production functions in (43) and (44) are essentially identical, except that A_tN_t and Q_tX_t have switched place. Thus, using the same line of argument we can show that (44) satisfies Assumption A2.

We now consider the production function in (45). The first thing to note is that labour input is essential for production regardless of the value of ψ. If $\psi \leq 0$, then both physical capital and natural resources are essential for production. What remains is to consider the marginal product of these inputs when $\psi \in (0, 1)$. Straightforward differentiation gives

$$\frac{\partial Y_t}{\partial K_t} = (1-\beta)\,\varphi \left(\frac{A_t N_t}{Q_t X_t}\right)^{\beta} \left[\varphi + (1-\varphi)\left(\frac{K_t}{Q_t X_t}\right)^{-\psi}\right]^{\frac{1}{\psi}-1} \left[\varphi\left(\frac{K_t}{Q_t X_t}\right)^{\psi} + 1 - \varphi\right]^{-\frac{\beta}{\psi}},$$

$$\frac{\partial Y_t}{\partial X_t} = (1-\beta)\,(1-\varphi) \left(\frac{A_t N_t}{K_t}\right)^{\beta} \left[\varphi\left(\frac{Q_t X_t}{K_t}\right)^{-\psi} + (1-\varphi)\right]^{\frac{1}{\psi}-1} \left[\varphi + (1-\varphi)\left(\frac{Q_t X_t}{K_t}\right)^{\psi}\right]^{-\frac{\beta}{\psi}}.$$

Using these and the following properties,

$$\lim_{K_t \to 0} \left[\varphi + (1-\varphi)\left(\frac{K_t}{Q_t X_t}\right)^{-\psi}\right]^{\frac{1}{\psi}-1} = \lim_{X_t \to 0} \left[\varphi\left(\frac{Q_t X_t}{K_t}\right)^{-\psi} + (1-\varphi)\right]^{\frac{1}{\psi}-1} = \infty,$$

$$\lim_{K_t \to 0} \left[\varphi\left(\frac{K_t}{Q_t X_t}\right)^{\psi} + 1 - \varphi\right]^{-\frac{\beta}{\psi}} = (1-\varphi)^{-\frac{\beta}{\psi}},$$

$$\lim_{X_t \to 0} \left[\varphi + (1-\varphi)\left(\frac{Q_t X_t}{K_t}\right)^{\psi}\right]^{-\frac{\beta}{\psi}} = \varphi^{-\frac{\beta}{\psi}},$$

we can get

$$\lim_{K_t \to 0} \frac{\partial Y_t}{\partial K_t} = \lim_{X_t \to 0} \frac{\partial Y_t}{\partial X_t} = \infty.$$

Since (45) and (46) are symmetric, the same line of argument can be used to show the desired properties for (46).

Appendix B: Proofs

Proof of Lemma 1

Suppose there exists a real number $r^* > -\delta$ such that

$$F_1 \left[K_t, G\left(Q_t X_t, A_t N_t\right) \right] = F_1 \left[1, \frac{G\left(Q_t X_t, A_t N_t\right)}{K_t} \right] = r^* + \delta > 0.$$

The first equality follows from the homogeneity property of $F_1 (\cdot)$. Since $F_1 (1, \cdot)$ is continuous and strictly decreasing under Assumption A1, this implies the existence of a non-negative real number χ^* such that

$$\frac{G\left(Q_t X_t, A_t N_t\right)}{K_t} = \chi^* \geq 0.$$

Note that there are two possible cases: In this first one, $\chi^* = 0$ which can happen if $\lim_{k \to \infty} \sigma_F (k) > 1$. A formal proof of this can be found in Palivos and Karagiannis (2010). Under this scenario, K_t is persistently growing at a higher rate than $G\left(Q_t X_t, A_t N_t\right)$. But as we have explained in the main text, this scenario cannot be supported as an equilibrium. In this second case, we have $\chi^* > 0$. By the homogeneity property of $F(\cdot)$, we can write

$$Y_t = F\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right) = K_t F\left(1, \chi^*\right), \qquad \text{for all } t,$$

$$\Rightarrow K_t = \left[F\left(1, \chi^*\right)\right]^{-1} Y_t.$$

The desired results follow by setting $\kappa^* \equiv \left[F\left(1, \chi^*\right)\right]^{-1} > 0$.

This completes the proof of Lemma 1.

Proof of Theorem 1

The proof is divided into a number of steps:

Step 1 This part of the proof uses the same line of argument as in Schlicht (2006) and Jones and Scrimgeour (2008). First, condition (v) implies that aggregate output Y_t grows at a constant rate $\widehat{\gamma} \equiv \gamma^* (1 + n)$ in every period, i.e., $Y_{t+1} = \widehat{\gamma} Y_t$, for all t. Rearranging terms and applying the CRTS property of $F(\cdot)$ gives

$$\begin{aligned} Y_t &= F\left(\widehat{\gamma}^{-1} K_{t+1}, \widehat{\gamma}^{-1} G\left(Q_{t+1} X_{t+1}, A_{t+1} N_{t+1}\right)\right) \\ &= F\left(K_t, \widehat{\gamma}^{-1} G\left(Q_{t+1} X_{t+1}, A_{t+1} N_{t+1}\right)\right). \end{aligned}$$

The second line uses the fact that K_t and Y_t must grow at the same rate, as per Lemma 1. For any given $K_t > 0$, $F(K_t, Z_t)$ is strictly increasing in Z_t. Hence, the following equality must hold in any equilibrium that satisfies condition (v),

$$G(Q_t X_t, A_t N_t) = \widehat{\gamma}^{-1} G(Q_{t+1} X_{t+1}, A_{t+1} N_{t+1}). \tag{47}$$

Note that (47) holds regardless of whether $G(\cdot)$ is Cobb-Douglas.

Suppose now $G(\cdot)$ is given by

$$G(Q_t X_t, A_t N_t) = (Q_t X_t)^{1-\phi} (A_t N_t)^{\phi}, \qquad \text{for some } \phi \in (0,1).$$

Using this, together with $A_{t+1} = (1+a) A_t$, $Q_{t+1} = (1+q) Q_t$, $X_{t+1} = (1-\tau^*) X_t$ and $N_{t+1} = (1+n) N_t$, we can rewrite (47) as

$$(Q_t X_t)^{1-\phi} (A_t N_t)^{\phi} = \widehat{\gamma}^{-1} [(1+q)(1-\tau^*)] [(1+a)(1+n)]^{\phi} (Q_t X_t)^{1-\phi} (A_t N_t)^{\phi}. \tag{48}$$

Since $(Q_t X_t)^{1-\phi} (A_t N_t)^{\phi} > 0$, (48) is valid if and only if

$$[(1+q)(1-\tau^*)]^{1-\phi} [(1+a)(1+n)]^{\phi} = \widehat{\gamma} \equiv \gamma^* (1+n)$$

$$\Rightarrow \gamma^* = (1+a)^{\phi} \left[\frac{(1+q)(1-\tau^*)}{1+n} \right]^{1-\phi}.$$

This is equation (25) in the theorem.

Step 2 Next, we will show that given condition (vi), the ratio $p_t X_t / Y_t$ must be time-invariant and strictly positive. This can then be used to derive equation (26). Suppose $r_t = r^* > -\delta$. Then by (10), we have

$$F_1\left(1, \frac{G(Q_t X_t, A_t N_t)}{K_t}\right) = F_1\left(1, \frac{\widehat{x}_t^{1-\phi}}{\widehat{k}_t}\right) = r^* + \delta > 0.$$

Since $F_1(1, \cdot)$ is strictly decreasing, it follows that the ratio between $\widehat{x}_t^{1-\phi}$ and $\widehat{k}_t$ must be constant in any equilibrium that satisfies condition (vi). Hence, we can write

$$\frac{G(Q_t X_t, A_t N_t)}{K_t} = \frac{\widehat{x}_t^{1-\phi}}{\widehat{k}_t} = \chi^* > 0. \tag{49}$$

By the homogeneity properties of $F(\cdot)$ and $F_2(\cdot)$, we can write

$$F_2\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right) = F_2\left(1, \chi^*\right),$$

$$F\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right) = K_t F\left(1, \chi^*\right).$$

Using these and (11), we can get

$$\begin{aligned} \frac{p_t X_t}{Y_t} &= \frac{1}{1+\mu} \frac{Q_t X_t}{K_t} \frac{F_2\left(1, \chi^*\right) G_1\left(Q_t X_t, A_t N_t\right)}{F\left(1, \chi^*\right)} \\ &= \frac{1}{1+\mu} \frac{F_2\left(1, \chi^*\right)}{F\left(1, \chi^*\right)} \frac{G\left(Q_t X_t, A_t N_t\right)}{K_t} \frac{Q_t X_t G_1\left(Q_t X_t, A_t N_t\right)}{G\left(Q_t X_t, A_t N_t\right)} \\ &= \left(\frac{1-\phi}{1+\mu}\right) \frac{\chi^* F_2\left(1, \chi^*\right)}{F\left(1, \chi^*\right)}. \end{aligned}$$

The last equality follows from the Cobb-Douglas specification of $G(\cdot)$. Hence, $p_t X_t / Y_t$ must be strictly positive and time-invariant. This in turn implies

$$\frac{p_{t+1}}{p_t} \frac{X_{t+1}}{X_t} = \left(1+r^*\right)\left(1-\tau^*\right) = \frac{Y_{t+1}}{Y_t} = \gamma^*(1+n).$$

Step 3 We now derive equation (27), which is based on the capital market clearing condition in (15). As shown in Step 2, we can rewrite $F_2\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right)$ as $F_2\left(1, \chi^*\right)$. Substituting this and $\tau_t = \tau^*$ into (15) gives

$$K_{t+1} = F_2\left(1, \chi^*\right)\left[\frac{1}{2+\theta} A_t N_t G_2\left(Q_t X_t, A_t N_t\right) - \frac{1}{1+\mu}\left(\frac{1-\tau^*}{\tau^*}\right) Q_t X_t G_1\left(Q_t X_t, A_t N_t\right)\right].$$

Using the Cobb-Douglas specification for $G(\cdot)$, we can simplify this to become

$$K_{t+1} = F_2\left(1, \chi^*\right)\left[\frac{\phi}{2+\theta} - \left(\frac{1-\tau^*}{\tau^*}\right)\left(\frac{1-\phi}{1+\mu}\right)\right] G\left(Q_t X_t, A_t N_t\right).$$

Dividing both sides by K_t and using (49) gives

$$\frac{K_{t+1}}{K_t} = \gamma^*(1+n) = \chi^* F_2\left(1, \chi^*\right)\left[\frac{\phi}{2+\theta} - \left(\frac{1-\tau^*}{\tau^*}\right)\left(\frac{1-\phi}{1+\mu}\right)\right].$$

Step 4 Equation (5) implies that both $c_{1,t}$ and $c_{2,t}$ will grow at the same rate as w_t when r_t is time-invariant. Using the definition of χ^* and (12), we can express the equilibrium wage rate as

$$w_t = \phi A_t \widehat{k}_t \chi^* F_2(1, \chi^*) = \phi \chi^* F_2(1, \chi^*) \frac{K_t}{N_t}.$$

Hence, w_t will grow at the same rate as per-worker capital and per-worker output. This completes the proof of Theorem 1.

Proof of Proposition 1

Using (25) and (26), we can get

$$\gamma^* (1+n) = (1+b)(1+n)^{\phi} (1-\tau^*)^{1-\phi},$$

$$r^* = (1+b)(1+n)^{\phi} (1-\tau^*)^{-\phi} - 1 \equiv r(\tau^*).$$

The CES function in (8) implies

$$F_1(1, \chi^*) = \alpha \left[\alpha + (1-\alpha)(\chi^*)^{\eta}\right]^{\frac{1-\eta}{\eta}}, \tag{50}$$

$$F_2(1, \chi^*) = (1-\alpha)(\chi^*)^{\eta-1} \left[\alpha + (1-\alpha)(\chi^*)^{\eta}\right]^{\frac{1-\eta}{\eta}}. \tag{51}$$

Combining (28) and (50) gives

$$(1-\alpha)(\chi^*)^{\eta} = \left[\frac{r(\tau^*)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}} - \alpha. \tag{52}$$

Substituting (52) into (51) gives

$$\begin{aligned} \chi^* F_2(1, \chi^*) &= (1-\alpha)(\chi^*)^{\eta} \left[\alpha + (1-\alpha)(\chi^*)^{\eta}\right]^{\frac{1-\eta}{\eta}} \\ &= \frac{r(\tau^*)+\delta}{\alpha} \left\{ \left[\frac{r(\tau^*)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}} - \alpha \right\}. \end{aligned}$$

Using these expressions, we can rewrite (27) as

$$\frac{(2+\theta)(1+b)(1+n)^{\phi}(1-\tau^*)^{1-\phi}}{\phi - \left(\frac{1-\tau^*}{\tau^*}\right)(2+\theta)(1-\phi)(1+\mu)^{-1}} = \frac{r(\tau^*)+\delta}{\alpha} \left\{ \left[\frac{r(\tau^*)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}} - \alpha \right\}.$$

A unique balanced growth equilibrium exists if there is a unique solution for this equation. Fix $\mu \geq 0$ and define two auxiliary functions $\Lambda(\cdot)$ and $\Gamma(\cdot)$ according to

$$\Lambda(\tau;\mu) \equiv \frac{(2+\theta)(1+b)(1+n)^{\phi}(1-\tau)^{1-\phi}}{\phi - \left(\frac{1-\tau}{\tau}\right)(2+\theta)(1-\phi)(1+\mu)^{-1}}, \tag{53}$$

$$\Gamma(\tau) \equiv \frac{r(\tau)+\delta}{\alpha}\left\{\left[\frac{r(\tau)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}} - \alpha\right\}. \tag{54}$$

The following properties of $\Lambda(\cdot)$ can be easily verified: $\Lambda(1;\mu) = 0$; $\Lambda(\tau;\mu) \to \infty$ as τ approaches $\overline{\tau}(\mu)$ from the right, where $\overline{\tau}(\mu) \in (0,1)$ is the threshold value defined in (29); $\Lambda(\tau;\mu) < 0$ for all $\tau < \overline{\tau}(\mu)$; and $\Lambda(\tau;\mu)$ is strictly decreasing in τ over the range $(\overline{\tau}(\mu), 1]$. Similarly, one can show that $\Gamma[\overline{\tau}(\mu)] < \infty$ and $\Gamma(\tau) \to \infty$ as $\tau \to 1$ if $\eta \in (0,1)$. Since both $\Lambda(\cdot;\mu)$ and $\Gamma(\cdot)$ are continuous functions in τ over the range between $\overline{\tau}(\mu)$ and one, these properties ensure the existence of at least one value $\tau^* \in (\overline{\tau}(\mu), 1)$ such that $\Lambda(\tau^*;\mu) = \Gamma(\tau^*)$.

If, in addition, $\Gamma(\cdot)$ is strictly increasing between $\overline{\tau}(\mu)$ and one, then a unique solution exists. Straightforward differentiation gives

$$\Gamma'(\tau) = \frac{1}{\alpha}\left\{\frac{1}{1-\eta}\left[\frac{r(\tau)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}} - \alpha\right\}(1+b)(1+n)^{\phi}\phi(1-\tau)^{-(1+\phi)}.$$

Hence, $\Gamma'(\tau) \gtrless 0$ if and only if

$$\left[\frac{r(\tau)+\delta}{\alpha}\right]^{\frac{\eta}{1-\eta}} \gtrless \alpha(1-\eta) \Leftrightarrow r(\tau)+\delta \gtrless \alpha^{\frac{1}{\eta}}(1-\eta)^{\frac{1}{\eta}-1}.$$

Since $r(\tau)$ is a strictly increasing function, it follows that $\Gamma(\cdot)$ is strictly increasing between $\overline{\tau}(\mu)$ and one if and only if

$$r[\overline{\tau}(\mu)] + \delta > \alpha^{\frac{1}{\eta}}(1-\eta)^{\frac{1}{\eta}-1}.$$

This condition can be rewritten as (30). A graphical illustration of the existence and uniqueness result is shown in Figure A1. This completes the proof of Proposition 1.

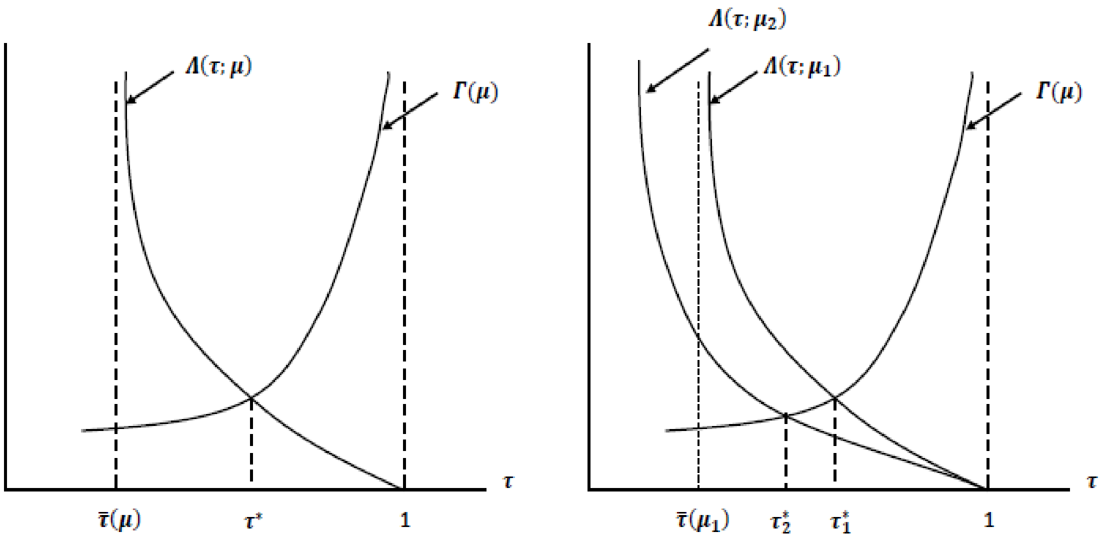

Figure A1: Existence and Uniqueness of Balanced Growth Equilibrium.

Proof of Proposition 2

Suppose condition (30) is satisfied under $\mu_2 > 0$, i.e.,

$$\left\{(1+b)\left[\frac{1+n}{1-\overline{\tau}(\mu_2)}\right]^{\phi}-(1-\delta)\right\}^{\eta}>\alpha(1-\eta)^{1-\eta}.$$

As shown in the proof of Proposition 1, this condition is sufficient to ensure the existence of a uniqueness balanced growth equilibrium in the economy with tax rate μ_2. Rewrite the above condition as

$$r[\overline{\tau}(\mu_2)]+\delta>\alpha^{\frac{1}{\eta}}(1-\eta)^{\frac{1}{\eta}-1},$$

where $r(\tau)\equiv(1+b)(1+n)^{\phi}(1-\tau^*)^{-\phi}-1$. Since $r(\cdot)$ is a strictly increasing function and $\overline{\tau}(\cdot)$ is strictly decreasing, it follows that

$$r[\overline{\tau}(\mu_1)]+\delta>r[\overline{\tau}(\mu_2)]+\delta>\alpha^{\frac{1}{\eta}}(1-\eta)^{\frac{1}{\eta}-1},$$

for any $\mu_2 > \mu_1 \geq 0$. Hence, (30) is also satisfied under μ_1, which ensures the existence of a unique balanced growth equilibrium in the economy with μ_1.

To establish the comparative statics result, first recall the auxiliary function $\Lambda(\tau;\mu)$ defined in (53). It is straightforward to verify that

$$\Lambda(\tau;\mu_2) < \Lambda(\tau;\mu_1),$$

over the range $\overline{\tau}(\mu_1) \leq \tau < 1$. In addition, $\Lambda(1;\mu_2) = \Lambda(1;\mu_1) = 0$ and $\Lambda(\tau;\mu_1) \to \infty$ as $\tau \to \overline{\tau}(\mu_1)$. These conditions ensure that $\tau_2^* < \tau_1^*$ [see Figure A1]. Finally using (25), we can write

$$\gamma_2^* = (1+b)\left(\frac{1-\tau_2^*}{1+n}\right)^{1-\phi} > (1+b)\left(\frac{1-\tau_1^*}{1+n}\right)^{1-\phi} = \gamma_1^*.$$

This concludes the proof of Proposition 2.

Proof of Theorem 2

Step 1 First, we will show that $\gamma^* = 1+a$ if the elasticity of substitution of $G(\cdot)$ is never equal to one. Recall that equation (47) in the proof of Theorem 1 is valid even if $G(\cdot)$ is not Cobb-Douglas. Define $\widehat{x}_t \equiv Q_t X_t/(A_t N_t)$. Then by the CRTS property of $G(\cdot)$, equation (47) can be equivalently stated as

$$G(Q_t X_t, A_t N_t) = G\left[\frac{(1+q)(1-\tau^*)}{\widehat{\gamma}} Q_t X_t, \frac{(1+a)(1+n)}{\widehat{\gamma}} A_t N_t,\right]. \tag{55}$$

Define the following notations

$$\varsigma \equiv \frac{(1+a)(1+n)}{\widehat{\gamma}} \qquad \text{and} \qquad \varpi \equiv \frac{(1+q)(1-\tau^*)}{\widehat{\gamma}}.$$

Dividing both sides of (55) by $\varsigma A_t N_t$ and using $g(\widehat{x}) \equiv G(\widehat{x}, 1)$ give

$$g(\widehat{x}_t) = \varsigma g\left(\frac{\varpi}{\varsigma}\widehat{x}_t\right), \qquad \text{for all } \widehat{x}_t > 0. \tag{56}$$

Equation (56) is trivially satisfied if $\varsigma = \varpi = 1$, which immediately implies

$$\gamma^* = 1+a \qquad \text{and} \qquad 1-\tau^* = \frac{(1+a)(1+n)}{1+q}.$$

We now show that if $\sigma_G(\cdot) \neq 1$, then equation (56) holds if and only if $\varsigma = \varpi = 1$.

We first establish an intermediate result: For any $\widehat{x} > 0$,

$$\frac{d}{d\widehat{x}}\left[\frac{\widehat{x}g'(\widehat{x})}{g(\widehat{x})}\right] \gtrless 0 \qquad \text{if and only if} \qquad \sigma_G(\widehat{x}) \gtrless 1.$$

To start, straightforward differentiation gives

$$\frac{d}{d\widehat{x}}\left[\frac{\widehat{x}g'(\widehat{x})}{g(\widehat{x})}\right]=\frac{g'(\widehat{x})}{g(\widehat{x})}-\frac{\widehat{x}\left[g'(\widehat{x})\right]^2}{\left[g(\widehat{x})\right]^2}+\frac{\widehat{x}g''(\widehat{x})}{g(\widehat{x})}. \tag{57}$$

Next, using the expression in (20), $\sigma_G(\widehat{x}) \gtrless 1$ if and only if

$$\frac{g'(\widehat{x})\left[g(\widehat{x})-\widehat{x}g'(\widehat{x})\right]}{g(\widehat{x})} \gtrless -\widehat{x}g''(\widehat{x})$$

$$\Leftrightarrow \frac{g'(\widehat{x})}{g(\widehat{x})}\left[1-\frac{\widehat{x}g'(\widehat{x})}{g(\widehat{x})}\right] \gtrless \frac{-\widehat{x}g''(\widehat{x})}{g(\widehat{x})}$$

$$\Leftrightarrow \frac{g'(\widehat{x})}{g(\widehat{x})}-\frac{\widehat{x}\left[g'(\widehat{x})\right]^2}{\left[g(\widehat{x})\right]^2}-\frac{\widehat{x}g''(\widehat{x})}{g(\widehat{x})}=\frac{d}{d\widehat{x}}\left[\frac{\widehat{x}g'(\widehat{x})}{g(\widehat{x})}\right] \gtrless 0. \tag{58}$$

This intermediate result says that if $\sigma_G(\cdot)$ is never equal to one, then $\widehat{x}g'(\widehat{x})/g(\widehat{x})$ must be either strictly increasing or strictly decreasing for all $\widehat{x}>0$. We will now apply this result on (56).

Since $g(\cdot)$ is continuously differentiable and (56) holds for all $\widehat{x}_t>0$, we can differentiate both sides of (56) with respect to $\widehat{x}_t$ and get

$$g'(\widehat{x}_t)=\varpi g'\left(\frac{\varpi}{\varsigma}\widehat{x}_t\right).$$

Combining this and (56) gives

$$\frac{\widehat{x}_t g'(\widehat{x}_t)}{g(\widehat{x}_t)}=\frac{\frac{\varpi}{\varsigma}\widehat{x}_t g'\left(\frac{\varpi}{\varsigma}\widehat{x}_t\right)}{g\left(\frac{\varpi}{\varsigma}\widehat{x}_t\right)}. \tag{59}$$

As mentioned above, if $\sigma_G(\cdot)$ is never equal to one, then $\widehat{x}g'(\widehat{x})/g(\widehat{x})$ must be either strictly increasing or strictly decreasing for all $\widehat{x}>0$. Hence, the equality in (59) holds if and only if $\varpi=\varsigma$. Using this, we can rewrite (56) as $g'(\widehat{x}_t)=\varpi g'(\widehat{x}_t)$, which implies that $\varpi=1$.

Step 2 The equalities $\varsigma=\varpi=1$ imply that $\widehat{k}_t$ and $\widehat{x}_t$ are time-invariant in any balanced growth equilibrium, i.e., $\widehat{k}_t=\widehat{k}^*$ and $\widehat{x}_t=\widehat{x}^*$. Using these, we can rewrite (10) and (11) as

$$r^*+\delta=F_1\left(\widehat{k}^*,G(\widehat{x}^*,1)\right)$$

$$(1+\mu)p_t=Q_tF_2\left(\widehat{k}^*,G(\widehat{x}^*,1)\right)G_1(\widehat{x}^*,1).$$

Equation (4) can now be used to obtain $r^*=q$. Equation (33) then follows.

Step 3 Dividing both sides of (15) by $A_t N_t$ gives

$$(1+a)(1+n)\widehat{k}_{t+1} = F_2\left(\widehat{k}_t, G(\widehat{x}_t, 1)\right)\left[\frac{1}{2+\theta}G_2(\widehat{x}_t, 1) - \left(\frac{1-\tau^*}{\tau^*}\right)\frac{\widehat{x}_t G_1(\widehat{x}_t, 1)}{1+\mu}\right].$$

Equation (34) can be obtained by setting $\widehat{k}_{t+1} = \widehat{k}_t = \widehat{k}^*$ and $\widehat{x}_t = \widehat{x}^*$.

Step 4 Equation (5) implies that both $c_{1,t}$ and $c_{2,t}$ will grow at the same rate as w_t when r_t is time-invariant. By the homogeneity property of $F_2(\cdot)$ and $G_2(\cdot)$, we can rewrite (12) as

$$w_t = A_t F_2\left(\widehat{k}_t, G(\widehat{x}_t, 1)\right) G_2(\widehat{x}_t, 1).$$

Since $\widehat{k}_t$ and $\widehat{x}_t$ are both constant over time, it follows that w_t will grow at the same rate as A_t. This completes the proof of Theorem 2.

Proof of Proposition 3

Part (i) Fix $\mu \geq 0$. Suppose $F(\cdot)$ takes the CES form in (8), with $\alpha \in (0,1)$ and $\eta < 1$. Then (33) can be rewritten as

$$\alpha\left\{\alpha + (1-\alpha)\left[\frac{G(\widehat{x}^*, 1)}{\widehat{k}^*}\right]^{\eta}\right\}^{\frac{1-\eta}{\eta}} = q + \delta$$

$$\Rightarrow (1-\alpha)\left[\frac{G(\widehat{x}^*, 1)}{\widehat{k}^*}\right]^{\eta} = \left(\frac{q+\delta}{\alpha}\right)^{\frac{\eta}{1-\eta}} - \alpha \qquad (60)$$

Using these, we can write

$$\frac{G(\widehat{x}^*, 1)}{\widehat{k}^*} F_2\left(\widehat{k}^*, G(\widehat{x}^*, 1)\right) = \frac{q+\delta}{\alpha}\left[\left(\frac{q+\delta}{\alpha}\right)^{\frac{\eta}{1-\eta}} - \alpha\right] \equiv (2+\theta)\Theta$$

Similarly, if $G(\cdot)$ takes the CES form in (9), then we can get

$$G_2(\widehat{x}^*, 1) = (1-\varphi)\left[\varphi(\widehat{x}^*)^{\psi} + 1 - \varphi\right]^{\frac{1}{\psi}-1} = \frac{(1-\varphi) G(\widehat{x}^*, 1)}{\varphi(\widehat{x}^*)^{\psi} + 1 - \varphi},$$

$$G_1(\widehat{x}^*, 1) = \frac{\varphi(\widehat{x}^*)^{\psi-1} G(\widehat{x}^*, 1)}{\varphi(\widehat{x}^*)^{\psi} + 1 - \varphi}.$$

Based on these observations, we can rewrite (34) as

$$\begin{aligned}(1+a)(1+n)\left[\varphi(\widehat{x}^*)^{\psi}+1-\varphi\right] &= \frac{G(\widehat{x}^*,1)}{\widehat{k}^*}F_2\left(\widehat{k}^*,G(\widehat{x}^*,1)\right)\left[\frac{1-\varphi}{2+\theta}-\left(\frac{1-\tau^*}{\tau^*}\right)\frac{\varphi}{1+\mu}(\widehat{x}^*)^{\psi}\right] \\ &= \Theta\left[1-\varphi-\left(\frac{1-\tau^*}{\tau^*}\right)\left(\frac{2+\theta}{1+\mu}\right)\varphi(\widehat{x}^*)^{\psi}\right],\end{aligned}$$

which can be simplified to become

$$(\widehat{x}^*)^{\psi}=\frac{1-\varphi}{\varphi}\frac{\Theta-(1+a)(1+n)}{(1+a)(1+n)+\left(\frac{1-\tau^*}{\tau^*}\right)\left(\frac{2+\theta}{1+\mu}\right)\Theta}. \tag{61}$$

The purpose of the additional condition $\min\{\Theta,1+q\}>(1+a)(1+n)$ is twofold: First, it ensures that a unique, strictly positive value of $\widehat{x}^*$ can be obtained from the above equation. Second, it ensures that $\tau^*\in(0,1)$.

Part (ii) Differentiating both sides of (61) with respect to $\widehat{x}^*$ and μ gives

$$\psi(\widehat{x}^*)^{\psi-1}\frac{d\widehat{x}^*}{d\mu}=\frac{1-\varphi}{\varphi}\frac{\Theta-(1+a)(1+n)}{\left[(1+a)(1+n)+\left(\frac{1-\tau^*}{\tau^*}\right)\left(\frac{2+\theta}{1+\mu}\right)\Theta\right]^2}\left(\frac{1-\tau^*}{\tau^*}\right)\frac{2+\theta}{(1+\mu)^2}\Theta.$$

Since the right-hand side of the above equation is always strictly positive, it follows that

$$\frac{d\widehat{x}^*}{d\mu}\gtreqless 0 \qquad \text{iff} \qquad \psi\gtreqless 0.$$

Using (60), we can get

$$G_1(\widehat{x}^*,1)\frac{d\widehat{x}^*}{d\mu}=\left\{\frac{1}{1-\alpha}\left[\left(\frac{q+\delta}{\alpha}\right)^{\frac{\eta}{1-\eta}}-\alpha\right]\right\}^{\frac{1}{\eta}}\frac{d\widehat{k}^*}{d\mu}.$$

This equation shows that $\widehat{x}^*$ and $\widehat{k}^*$ will move in the same direction whenever there is a change in μ. This completes the proof of Proposition 3.

Proof of Theorem 3

We will consider each of the specifications in (43)-(46) separately. For each specification we will first verify the existence of a positive constant κ^* such that $K_t=\kappa^*Y_t$ for all t under conditions (v)-(vii) in Section 3.

Specification 1 We begin with the production function in (43). Under this specification, the first-order conditions for the representative firm's problem are given by

$$(1-\varphi)\,\alpha Y_t^{1-\psi} K_t^{\alpha\psi-1} (Q_t X_t)^{(1-\alpha)\psi} = r_t + \delta, \tag{62}$$

$$(1-\varphi)(1-\alpha) Y_t^{1-\psi} K_t^{\alpha\psi} (Q_t X_t)^{(1-\alpha)\psi-1} Q_t = (1+\mu)\, p_t, \tag{63}$$

$$\varphi Y_t^{1-\psi} (A_t N_t)^{\psi-1} A_t = w_t. \tag{64}$$

Combining (62) and (63) gives

$$\frac{p_t X_t}{K_t} = \frac{(1-\alpha)(r_t+\delta)}{\alpha(1+\mu)}. \tag{65}$$

Suppose conditions (vi) and (vii) are satisfied, i.e., $r_t = r^* > -\delta$ and $\tau_t = \tau^*$ for all t. Then both p_t and X_t are growing at some constant rate. It follows from (65) that K_t must also be growing at a constant rate. Next, dividing both sides of (14) by K_t gives

$$\frac{K_{t+1}}{K_t} = \frac{1}{2+\theta}\frac{w_t N_t}{K_t} - \frac{1-\tau_t}{\tau_t}\frac{p_t X_t}{K_t}. \tag{66}$$

If conditions (vi) and (vii) are satisfied, then τ_t, $p_t X_t / K_t$ and K_{t+1}/K_t are all constant over time. Hence, $w_t N_t / K_t$ must be constant over time as well. Finally, rewrite the production function in (43) as

$$Y_t^{\psi} = \varphi (A_t N_t)^{\psi} + (1-\varphi)\left[K_t^{\alpha} (Q_t X_t)^{1-\alpha}\right]^{\psi}.$$

Substituting (63) and (64) into this expression gives

$$Y_t^{\psi} = w_t N_t Y_t^{\psi-1} + \frac{1+\mu}{1-\alpha} p_t X_t Y_t^{\psi-1} \Longrightarrow \frac{Y_t}{K_t} = \frac{w_t N_t}{K_t} + \frac{1+\mu}{1-\alpha}\frac{p_t X_t}{K_t}.$$

This shows that Y_t / K_t is constant over time under conditions (vi) and (vii).

Substituting $r_t = r^*$ and $K_t = \kappa^* Y_t$ into (62) gives

$$(1-\varphi)\,\alpha (\kappa^*)^{\psi-1}\left(\frac{K_t}{Q_t X_t}\right)^{(\alpha-1)\psi} = (1-\varphi)\,\alpha(\kappa^*)^{\psi-1}\left(\frac{\widehat{k}_t}{\widehat{x}_t}\right)^{(\alpha-1)\psi} = r^* + \delta.$$

This shows that the ratio between $\widehat{k}_t$ and $\widehat{x}_t$ must be constant over time, or equivalently,

$$\frac{\widehat{x}_{t+1}}{\widehat{x}_t} = \frac{\widehat{k}_{t+1}}{\widehat{k}_t} = \frac{\gamma^*}{1+a} = \frac{(1+q)(1-\tau^*)}{(1+a)(1+n)}.$$

By the same token, we can also rewrite (63) and (64) as

$$\left(1+\mu\right)p_t = \left(1-\varphi\right)\left(1-\alpha\right)\left(\kappa^*\right)^{\psi-1}\left(\frac{\widehat{k}_t}{\widehat{x}_t}\right)^{(\alpha-1)\psi+1} Q_t, \tag{67}$$

$$w_t = \varphi\left(\kappa^*\right)^{\psi-1}\widehat{k}_t^{1-\psi} A_t. \tag{68}$$

Since the ratio between $\widehat{k}_t$ and $\widehat{x}_t$ is constant over time, it follows from (67) that p_t must be growing at the same rate as Q_t. By (4), we can write

$$\frac{p_{t+1}}{p_t} = 1 + r^* = \frac{Q_{t+1}}{Q_t} = 1 + q.$$

The last step is to substitute (67) and (68) into (15). This will give

$$\left(1+a\right)\left(1+n\right)\widehat{k}_{t+1} = \left(\kappa^*\right)^{\psi-1}\left[\frac{\varphi}{2+\theta}\widehat{k}_t^{1-\psi} - \left(\frac{1-\tau^*}{\tau^*}\right)\frac{\left(1-\varphi\right)\left(1-\alpha\right)}{1+\mu}\left(\frac{\widehat{k}_t}{\widehat{x}_t}\right)^{(\alpha-1)\psi}\widehat{k}_t\right]$$

$$\Rightarrow \left(1+a\right)\left(1+n\right)\frac{\widehat{k}_{t+1}}{\widehat{k}_t} = \left(\kappa^*\right)^{\psi-1}\left[\frac{\varphi}{2+\theta}\widehat{k}_t^{-\psi} - \left(\frac{1-\tau^*}{\tau^*}\right)\frac{\left(1-\varphi\right)\left(1-\alpha\right)}{1+\mu}\left(\frac{\widehat{k}_t}{\widehat{x}_t}\right)^{(\alpha-1)\psi}\right].$$

Since both $\widehat{k}_{t+1}/\widehat{k}_t$ and $\widehat{k}_t/\widehat{x}_t$ are constant over time, it follows that the *level* of $\widehat{k}_t$ must be constant over time in any equilibrium that satisfies conditions (v)-(vii). Hence, we have $\gamma^* = 1+a$, $r^* = q$, and $\left(1-\tau^*\right) = \left(1+a\right)\left(1+n\right)/\left(1+q\right)$.

Specification 2 Consider the production function in (44). The first-order conditions for the firm's problem are now given by

$$\left(1-\varphi\right)\alpha Y_t^{1-\psi}K_t^{\alpha\psi-1}\left(A_tN_t\right)^{(1-\alpha)\psi} = r_t + \delta, \tag{69}$$

$$\varphi Y_t^{1-\psi}\left(Q_tX_t\right)^{\psi-1}Q_t = \left(1+\mu\right)p_t, \tag{70}$$

$$\left(1-\varphi\right)\left(1-\alpha\right)Y_t^{1-\psi}K_t^{\alpha\psi}\left(A_tN_t\right)^{\psi(1-\alpha)-1}A_t = w_t. \tag{71}$$

Combining (69) and (71) gives

$$\frac{w_tN_t}{K_t} = \frac{1-\alpha}{\alpha}\left(r_t+\delta\right),$$

which is constant over time under condition (vi). By assumption, both A_t and N_t grow at some exogenous constant rate. Condition (v) implies that Y_t is growing at a constant rate, while condition (vi) states that r_t is time-invariant. Thus, it follows immediately from (69) that K_t must be growing at a constant rate. Equation (66) then implies that $p_t X_t / K_t$.must also be constant over time under conditions (v)-(vii). Finally, rewrite the production function in (44) as

$$Y_t^{\psi} = \varphi \left(Q_t X_t \right)^{\psi} + \left(1 - \varphi \right) \left[K_t^{\alpha} \left(A_t N_t \right)^{1-\alpha} \right]^{\psi}.$$

Substituting (69) and (70) into the above expression and rearranging terms gives

$$\frac{Y_t}{K_t} = \left(1 + \mu \right) \frac{p_t X_t}{K_t} + \left(\frac{1 - \varphi}{\alpha} \right) \left(r_t + \delta \right).$$

Thus, a constant r_t and a constant ratio $p_t X_t / K_t$ will imply a constant capital-output ratio.

Using the two conditions: $K_t = \kappa^* Y_t$ and $r_t = r^*$, we can rewrite the first-order conditions (69)-(71) as

$$\begin{aligned} \left(1 - \varphi \right) \left(\kappa^* \right)^{\psi - 1} \alpha \widehat{k}_t^{(\alpha - 1)\psi} &= r^* + \delta, \\ \varphi \left(\kappa^* \right)^{\psi - 1} \left(\frac{\widehat{k}_t}{\widehat{x}_t} \right)^{1-\psi} Q_t &= \left(1 + \mu \right) p_t, \\ \left(1 - \varphi \right) \left(1 - \alpha \right) \left(\kappa^* \right)^{\psi - 1} \widehat{k}_t^{(\alpha - 1)\psi + 1} A_t &= w_t. \end{aligned} \tag{72}$$

The first one of these equations immediately implies that $\widehat{k}_t$ is constant over time, so that $\gamma^* = 1 + a$. Substituting the last two equations into (15) gives

$$K_{t+1} = A_t N_t \left(\kappa^* \right)^{\psi - 1} \left[\frac{\left(1 - \varphi \right) \left(1 - \alpha \right)}{2 + \theta} \widehat{k}_t^{(\alpha - 1)\psi + 1} - \left(\frac{1 - \tau^*}{\tau^*} \right) \frac{\varphi}{1 + \mu} \left(\frac{\widehat{k}_t}{\widehat{x}_t} \right)^{1-\psi} \widehat{x}_t \right]$$

$$\Rightarrow \left(1 + a \right) \left(1 + n \right) \widehat{k}_{t+1} = \left(\kappa^* \right)^{\psi - 1} \left[\frac{\left(1 - \varphi \right) \left(1 - \alpha \right)}{2 + \theta} \widehat{k}_t^{(\alpha - 1)\psi + 1} - \left(\frac{1 - \tau^*}{\tau^*} \right) \frac{\varphi}{1 + \mu} \widehat{k}_t^{1-\psi} \widehat{x}_t^{\psi} \right].$$

Since $\widehat{k}_t$ is constant over time, the above equation implies that $\widehat{x}_t$ must be constant over time as well. Finally, (72) implies that p_t must be growing at the same rate as Q_t in any equilibrium that satisfies conditions (v)-(vii), so that $r^* = q$.

Specification 3 Next, we consider the production function in (45). The equilibrium factor prices are now characterised by

$$\left(1 - \beta \right) \varphi \left[\varphi K_t^{\psi} + \left(1 - \varphi \right) \left(Q_t X_t \right)^{\psi} \right]^{\frac{1-\beta}{\psi} - 1} \left(A_t N_t \right)^{\beta} K_t^{\psi - 1} = r_t + \delta, \tag{73}$$

$$(1-\beta)(1-\varphi)\left[\varphi K_t^{\psi}+(1-\varphi)(Q_tX_t)^{\psi}\right]^{\frac{1-\beta}{\psi}-1}(A_tN_t)^{\beta}(Q_tX_t)^{\psi-1}Q_t=(1+\mu)p_t, \tag{74}$$

$$\left[\varphi K_t^{\psi}+(1-\varphi)(Q_tX_t)^{\psi}\right]^{\frac{1-\beta}{\psi}}\beta(A_tN_t)^{\beta-1}A_t=w_t. \tag{75}$$

Combining (73) and (74) gives

$$\frac{p_tX_t}{K_t}=\frac{1-\varphi}{\varphi}\frac{(r_t+\delta)}{(1+\mu)}\left(\frac{Q_tX_t}{K_t}\right)^{\psi}. \tag{76}$$

Suppose both r_t and τ_t are constant over time. Then the above expression implies that K_t must be growing at a constant rate over time. From (75), we can get $\beta Y_t = w_tN_t$. Substituting this into (66) gives

$$\frac{K_{t+1}}{K_t}=\frac{\beta}{2+\theta}\frac{Y_t}{K_t}-\frac{1-\tau_t}{\tau_t}\frac{p_tX_t}{K_t}. \tag{77}$$

Finally, rewrite (73) to become

$$(1-\beta)\left(\frac{Y_t}{K_t}\right)\left[\frac{\varphi K_t^{\psi}}{\varphi K_t^{\psi}+(1-\varphi)(Q_tX_t)^{\psi}}\right]=(r_t+\delta)$$

$$\Longrightarrow\frac{Y_t}{K_t}=\frac{(r_t+\delta)}{1+\beta}\left[1+\frac{1-\varphi}{\varphi}\left(\frac{Q_tX_t}{K_t}\right)^{\psi}\right]=\frac{r_t+\delta}{1+\beta}+\frac{1+\mu}{1+\beta}\frac{p_tX_t}{K_t}. \tag{78}$$

The second equality is obtained by using (76). Equations (77) and (78) now form a system of linear equations that can be used to solve for the value of Y_t/K_t and p_tX_t/K_t in terms of K_{t+1}/K_t, τ_t and r_t. Since K_{t+1}/K_t, τ_t and r_t are all time-invariant under conditions (v)-(vii), it follows that Y_t/K_t and p_tX_t/K_t are also time-invariant.

Note that the condition $Y_t=\frac{1}{\kappa^*}K_t$ can be rewritten as

$$\left[\varphi\widehat{k}_t^{\psi}+(1-\varphi)\widehat{x}_t^{\psi}\right]^{\frac{1-\beta}{\psi}}=\frac{1}{\kappa^*}\widehat{k}_t$$

Using this, we can rewrite (73)-(75) as

$$(1-\beta)\varphi(\kappa^*)^{\frac{\psi}{1-\beta}-1}\widehat{k}_t^{-\frac{\beta\psi}{1-\beta}}=r_t+\delta,$$

$$(1-\beta)(1-\varphi)(\kappa^*)^{\frac{\psi}{1-\beta}-1}\widehat{k}_t^{1-\frac{\psi}{1-\beta}}\widehat{x}_t^{\psi-1}Q_t=(1+\mu)p_t,$$

$$\frac{1}{\kappa^*}\beta A_t\widehat{k}_t=w_t.$$

The first of these three equations, together with $r_t = r^*$, implies that $\widehat{k}_t$ must be constant over time. Hence, $\gamma^* = 1 + a$. Substituting the last two equations into (15) gives

$$K_{t+1} = A_t N_t \left[\frac{\beta \widehat{k}_t}{(2+\theta)\kappa^*} - \left(\frac{1-\tau^*}{\tau^*} \right) \frac{(1-\beta)(1-\varphi)}{1+\mu} (\kappa^*)^{\frac{\psi}{1-\beta}-1} \widehat{k}_t^{1-\frac{\psi}{1-\beta}} \widehat{x}_t^{\psi} \right]$$

$$\Rightarrow (1+a)(1+n)\widehat{k}_{t+1} = \frac{\beta \widehat{k}_t}{(2+\theta)\kappa^*} - \left(\frac{1-\tau^*}{\tau^*} \right) \frac{(1-\beta)(1-\varphi)}{1+\mu} (\kappa^*)^{\frac{\psi}{1-\beta}-1} \widehat{k}_t^{1-\frac{\psi}{1-\beta}} \widehat{x}_t^{\psi}.$$

Since $\widehat{k}_t$ is constant over time, the above equation implies that $\widehat{x}_t$ must be constant over time as well. The remaining results follow by the same line argument as in Specification 2.

Specification 4 Finally, we consider the production function in (46). The first-order conditions for the firm's problem are now given by

$$(1-v)\varphi (Q_t X_t)^v \left[\varphi K_t^{\psi} + (1-\varphi)(A_t N_t)^{\psi} \right]^{\frac{1-v}{\psi}-1} K_t^{\psi-1} = r_t + \delta, \tag{79}$$

$$\nu (Q_t X_t)^{v-1} Q_t \left[\varphi K_t^{\psi} + (1-\varphi)(A_t N_t)^{\psi} \right]^{\frac{1-v}{\psi}} = (1+\mu) p_t, \tag{80}$$

$$(1-v)(1-\varphi)(Q_t X_t)^v \left[\varphi K_t^{\psi} + (1-\varphi)(A_t N_t)^{\psi} \right]^{\frac{1-v}{\psi}-1} (A_t N_t)^{\psi-1} A_t = w_t. \tag{81}$$

To start, using (46) and (80) we can obtain

$$\frac{p_t X_t}{K_t} = \frac{\nu}{1+\mu} \frac{Y_t}{K_t}.$$

Next, combining (79) and (81) gives

$$\frac{w_t N_t}{K_t} = \frac{(1-\varphi)(r_t+\delta)}{\varphi} \left(\frac{A_t N_t}{K_t} \right)^{\psi}. \tag{82}$$

Substituting these into (66) gives

$$\frac{K_{t+1}}{K_t} = \frac{(1-\varphi)(r_t+\delta)}{\varphi(2+\theta)} \left(\frac{A_t N_t}{K_t} \right)^{\psi} - \frac{\nu}{1+\mu} \left(\frac{1-\tau_t}{\tau_t} \right) \frac{Y_t}{K_t}. \tag{83}$$

We then use (79) to derive

$$\frac{Y_t}{K_t} = \left(\frac{r_t+\delta}{1-v} \right) \left[1 + \frac{1-\varphi}{\varphi} \left(\frac{A_t N_t}{K_t} \right)^{\psi} \right]. \tag{84}$$

Equations (83) and (84) form a system of linear equations which can be used to solve for Y_t/K_t and $(A_t N_t/K_t)^{\psi}$ in terms of K_{t+1}/K_t, r_t and τ_t. By conditions (vi) and (vii), both r_t and τ_t

are time-invariant. Thus, what remains is to show that K_{t+1}/K_t is a constant under conditions (v)-(vii). To this end, rewrite (80) as

$$\varphi K_t^{\psi} + (1-\varphi)(A_t N_t)^{\psi} = \left[\left(\frac{1+\mu}{\nu}\right)\frac{p_t}{Q_t}\right]^{\frac{\psi}{1-\nu}} (Q_t X_t)^{\psi}$$

and substitutes the above expression into (79) to get

$$(1-v)\varphi\left[\left(\frac{1+\mu}{\nu}\right)\frac{p_t}{Q_t}\right]^{\frac{1-\nu-\psi}{1-\nu}} (Q_t X_t)^{1-\psi} K_t^{\psi-1} = r_t + \delta.$$

The desired result follows from the fact that r_t is time-invariant and $\{p_t, X_t\}$ are growing at a constant rate under conditions (vi) and (vii). This proves that Y_t/K_t is a constant under conditions (v)-(vii).

Next, we rewrite equations (79) and (80) as

$$(1-v)\varphi\widehat{x}_t^{v}\left(\varphi\widehat{k}_t^{\psi} + 1 - \varphi\right)^{\frac{1-v-\psi}{\psi}}\widehat{k}_t^{\psi-1} = r_t + \delta \tag{85}$$

$$v\frac{Y_t}{X_t} = v\widehat{x}_t^{v-1}\left(\varphi\widehat{k}_t^{\psi} + 1 - \varphi\right)^{\frac{1-v}{\psi}} Q_t = (1+\mu)p_t. \tag{86}$$

The condition $Y_t = \frac{1}{\kappa^*}K_t$ can be rewritten as

$$\widehat{x}_t^{v}\left(\varphi\widehat{k}_t^{\psi} + 1 - \varphi\right)^{\frac{1-v}{\psi}} = \frac{1}{\kappa^*}\widehat{k}_t. \tag{87}$$

Combining (85), (87) and $r_t = r^*$ gives

$$\frac{1}{\kappa^*}\frac{(1-v)\varphi\widehat{k}_t^{\psi}}{\varphi\widehat{k}_t^{\psi} + 1 - \varphi} = r^* + \delta$$

$$\Rightarrow (1-v)\varphi\widehat{k}_t^{\psi} = (r^* + \delta)\kappa^*\left(\varphi\widehat{k}_t^{\psi} + 1 - \varphi\right).$$

This can be used to derive a unique solution for $\widehat{k}_t$ which depends only on r^* and some parameters. Hence, $\gamma^* = 1 + a$. Equation (87) then implies that $\widehat{x}_t$ is also constant over time. Hence, $1 - \tau^* = (1+a)(1+n)/(1+q)$. Finally, given $\widehat{k}_t = \widehat{k}^*$ and $\widehat{x}_t = \widehat{x}^*$, equation (86) implies that p_t and Q_t must be growing at the same rate. Hence, $r^* = q$.

This concludes the proof of Theorem 3.

Appendix C: Infinitely-Lived Consumers

In this appendix, we will show that the "knife-edge" condition of a unitary elasticity of substitution between effective labour input and effective resource input plays the same critical role in generating endogenous economic growth in an environment with infinitely-lived consumers. Specifically, an endogenous growth solution similar to the one in Agnani, Gutiérrez and Iza (2005) can be obtained when the elasticity of substitution of $G(\cdot)$ is identical to one. But if this elasticity is bounded away from one, then the common growth factor γ^* and interest rate r^* are solely determined by the growth rates of the exogenous technological factors (i.e., A_t and Q_t).

Consider an economy that is populated by $H > 0$ identical households. Each household contains a growing number of identical, infinitely-lived consumers. The size of each household at time t is given by $N_t = (1+n)^t$, with $n > 0$. Since all households are identical, we can focus on the choices made by a representative household and normalise H (which is just a scaling factor) to one. The representative household solves the following problem:

$$\max_{\{c_t, K_{t+1}, M_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t N_t \frac{c_t^{1-\sigma}}{1-\sigma}$$

subject to the sequential budget constraint

$$N_t c_t + K_{t+1} + p_t M_{t+1} = w_t N_t + (1+r_t) K_t + p_t M_t,$$

where $\beta \in (0,1)$ is the subjective discount factor; $\sigma > 0$ is the reciprocal of the elasticity of intertemporal substitution (EIS); c_t denotes individual consumption at time t; K_t and M_t are, respectively, the household's holding of physical capital and non-renewable resources; p_t, w_t and r_t are as defined in Section 2.1. The first-order conditions of this problem imply the Euler equation for consumption,

$$\frac{c_{t+1}}{c_t} = \left[\beta(1+r_{t+1})\right]^{\frac{1}{\sigma}}, \tag{88}$$

and the Hotelling rule,

$$\frac{p_{t+1}}{p_t} = 1 + r_{t+1}.$$

We do not consider the resource tax in this setting (i.e., $\mu = 0$). The rest of the economy is the same as in Sections 2.2 and 2.3. In particular, the first-order conditions for the firm's problem, (8)-(9), and the dynamic equation for natural resources, (11), remain unchanged. In any competitive equilibrium, goods market clear in every period so that

$$N_t c_t + K_{t+1} - (1-\delta)K_t = F\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right), \qquad \text{for all } t \geq 0. \tag{89}$$

This replaces the capital market clearing condition in (15).

When characterising a balanced growth equilibrium, we maintain the three conditions (v)-(vii) listed in Section 3. Note that Lemma 1 is also valid in this environment. First, consider the case when $G(\cdot)$ takes the Cobb-Douglas form, or equivalently, $\sigma_G(\cdot)$ is identical to one. Dividing both sides of (89) gives

$$\frac{N_t c_t}{K_t} + \frac{K_{t+1}}{K_t} - (1-\delta) = \frac{F\left(K_t, G\left(Q_t X_t, A_t N_t\right)\right)}{K_t}.$$

Hence, in any balanced growth equilibrium, aggregate consumption $N_t c_t$ must be growing at the same rate as K_t and Y_t. This, together with the Euler equation in (88) implies

$$\gamma^* = \left[\beta\left(1+r^*\right)\right]^{\frac{1}{\sigma}},$$

where γ^* is again the growth factor of per-capita output in a balanced growth equilibrium. Next, note that the arguments in Step 1 and Step 2 of the proof of Theorem 1 are built upon the properties of the production function and the characterising properties of balanced growth equilibrium. In particular, these arguments do not rely on the consumer side of the economy. Hence, they remain valid in this environment. Consequently, we have

$$\gamma^* = (1+b)\left(\frac{1-\tau^*}{1+n}\right)^{1-\phi},$$

$$\left(1+r^*\right)\left(1-\tau^*\right) = \gamma^*\left(1+n\right),$$

where $1+b \equiv (1+a)^{\phi}(1+q)^{1-\phi}$. Using these three equations, we can derive

$$1+r^* = \beta^{-\frac{\phi}{\varpi}}\left(1+b\right)^{\frac{\sigma}{\varpi}},$$

$$1-\tau^* = \beta^{\frac{1}{\varpi}}\left(1+b\right)^{\frac{1-\sigma}{\varpi}}\left(1+n\right),$$

$$\gamma^* = \beta^{\frac{1-\phi}{\varpi}}\left(1+b\right)^{\frac{\sigma}{\varpi}},$$

where $\varpi \equiv 1 - (1-\sigma)(1-\phi)$. Thus, a unique balanced growth equilibrium exists if

$$\beta^{\frac{1}{\varpi}} (1+b)^{\frac{1-\sigma}{\varpi}} (1+n) \in (0,1),$$

which ensures that $\tau^* \in (0,1)$. Notice that both γ^* and τ^* are endogenously determined by a host of factors as in the AGI solution.

Suppose now $\sigma_G(\cdot)$ is never equal to one. Since the arguments in Step 1 and Step 2 of the proof of Theorem 2 remain valid in this environment, we have $\gamma^* = 1+a$, $r^* = q$, $\widehat{k}_t = \widehat{k}^*$ and $\widehat{x}_t = \widehat{x}^*$. These in turn imply that

$$1 - \tau^* = \frac{(1+a)(1+n)}{1+q}.$$

References

[1] Agnani, B., M. Gutiérrez and A. Iza, (2005) "Growth in Overlapping Generation Economies with Non-Renewable Resources," *Journal of Environmental Economics and Management*, vol. 50, pp.387-407.

[2] Arrow, K.J., H.B. Chenery, B.S. Minhas and R.M. Solow, (1961) "Capita-Labor Substitution and Economic Efficiency," *Review of Economics and Statistics*, vol. 43, pp.225-250.

[3] Barbier, E.B., (1999) "Endogenous Growth and Natural Resource Scarcity," *Environmental and Resource Economics*, vol. 14, pp.51-74.

[4] Blackorby, C., and R.R. Russell, (1976) "Functional Structure and the Allen Partial Elasticities of Substitution: An Application of Duality Theory," *Review of Economic Studies*, vol. 43, pp.285-291.

[5] Boadway, R., and M. Keen, (2010) "Theoretical Perspectives on Resource Tax Design," in *The Taxation of Petroleum and Minerals: Principles, Problems and Practice,* edited by C. McPherson, M. Keen and P. Daniel. Published by Taylor & Francis.

[6] Bornhorst, F., S. Gupta and J. Thornton, (2009) "Natural Resource Endowments and the Domestic Revenue Effort," *European Journal of Political Economy*, vol. 25, pp.439-446.

[7] Bugajewski, D., and P. Maćkowiak, (2015) "On Knife-Edge Conditions with Unbounded Growth," *Journal of Macroeconomics*, vol. 45, pp.274-283.

[8] Dasgupta, P., and G. Heal, (1974) "The Optimal Depletion of Exhaustible Resources," *Review of Economic Studies*, vol. 41, pp. 3-28.

[9] Feng, G., and A. Serletis, (2008) "Productivity Trends in U.S. Manufacturing: Evidence from the NQ and AIM Cost Functions," *Journal of Econometrics*, vol. 142, pp.281-311.

[10] Gaudet, G., and P. Lasserre, (2015) "The Taxation of Nonrenewable Natural Resources," in *Handbook on the Economics of Natural Resources*, ed. R. Halvorsen and D.F. Layton. (Edward Elgar Publishing Ltd).

[11] Grimaud, A., and L. Rougé, (2003) "Non-Renewable Resources and Growth with Vertical Innovations: Optimum, Equilibrium and Economic Policies," *Journal of Environmental Economics and Management*, vol. 45, pp.433-453.

[12] Grossman, G.M., E. Helpman, E. Oberfield and Thomas Sampson, (2017) "Balanced Growth Despite Uzawa," *American Economic Review*, vol. 107, pp.1293-1312.

[13] Groth, C., and P. Schou, (2002) "Can Non-Renewable Resources Alleviate the Knife-Edge Character of Endogenous Growth?" *Oxford Economic Papers*, vol. 54, pp.386-411.

[14] Groth, C., and P. Schou, (2007) "Growth and Non-Renewable Resources: The Different Roles of Capital and Resource Taxes," *Journal of Environmental Economics and Management*, vol. 53, pp.80-98.

[15] Growiec, J., (2007) "Beyond the Linearity Critique: the Knife-Edge Assumption of Steady-State Growth," *Economic Theory*, vol. 31, pp.489-499.

[16] Henningsen, A., G. Henningsen and E. van der Werf, (2019) "Capital-Labour-Energy Substitution in a Nested CES Framework: A Replication and Update of Kemfert (1998)," *Energy Economics*, vol. 82, p.16-25.

[17] Jones, C.I., and D. Scrimgeour, (2008) "A New Proof of Uzawa's Steady-State Growth Theorem," *Review of Economics and Statistics*, vol. 90, pp.180-182.

[18] Jones, L.E., and R.E. Manuelli, (1997) "The Sources of Growth," *Journal of Economic Dynamics and Control*, vol. 21, pp.75-114.

[19] Kaldor, N., (1963) "Capital Accumulation and Economic Growth," in *Proceedings of a Conference Held by the International Economics Association,* ed. F.A. Lutz and D.C. Hague (London: Macmillan Publishers).

[20] Kemfert, C., (1998) "Estimated Substitution Elasticities of a Nested CES Production Function Approach for Germany," *Energy Economics*, vol. 20, pp.249-264.

[21] Kemfert, C., and H. Welsch, (2000) "Energy-Capital-Labor Substitution and the Economic Effects of CO2 Abatement: Evidence for Germany," *Journal of Policy Modeling*, vol. 22, pp.641-660.

[22] Leontief, W., (1947) "Introduction to a Theory of the Internal Structure of Functional Relationships," *Econometrica*, vol. 15, pp.361-373.

[23] Mitra, T., (1983) "Limits on Population Growth under Exhaustible Resource Constraints," *International Economic Review*, vol. 24, pp.155-168.

[24] Palivos, T., and G. Karagiannis, (2010) "The Elasticity of Substitution as an Engine of Growth," *Macroeconomic Dynamics*, vol. 14, pp.617-628.

[25] Sato, K., (1967) "A Two-Level Constant-Elasticity-of-Substitution Production Function," *Review of Economic Studies*, vol. 34, pp.201-218.

[26] Schlicht, E., (2006) "A Variant of Uzawa's Theorem," *Economics Bulletin*, vol. 5, pp.1-5.

[27] Scholz, C.M., and G. Ziemes (1999) "Exhaustible Resources, Monopolistic Competition, and Endogenous Growth," *Environmental and Resource Economics*, vol. 13, pp.169-185.

[28] Solow, R.M., (1974) "Intergenerational Equity and Exhaustible Resources," *Review of Economic Studies*, vol. 41, pp.29-45.

[29] Stiglitz, J., (1974) "Growth with Exhaustible Natural Resources: Efficient and Optimal Growth Paths," *Review of Economic Studies*, vol. 41, pp.123-137.

[30] Tirole, J., (1985) "Asset Bubbles and Overlapping Generations," *Econometrica*, vol. 53, pp.1499-1528.

[31] Uzawa, H., (1961) "Neutral Inventions and the Stability of Growth Equilibrium," *Review of Economic Studies*, vol. 28, pp.117-124.

[32] van der Werf, E., (2008) "Production Functions for Climate Policy Modeling: An Empirical Analysis," *Energy Economics*, vol. 30, pp.2964-2979.