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Abstract

Default risk models have been widely employed to assess the ability of households and sovereigns to insure themselves against shocks. Grid search has often been used to solve these models because the complexity of the problem prevents the use of faster but less general methods. In this paper, we propose an extension of the endogenous grid method for default risk models, which is faster and more accurate than grid search. In particular, we find that our solution method leads to a more accurate bond price function, thus making substantial differences in the model's main predictions. When applied to Arellano's (2008) model, our approach predicts a standard deviation of the interest rate spread one-third lower and defaults 3 to 5 times less frequently than does the conventional approach. On top of that, our method is efficient. It is approximately 4 to 7 times faster than grid search when applied to a canonical model of Arellano (2008) and 19 to 27 times faster than grid search when applied to the richer model of Nakajima and Ríos-Rull (2014). Finally, we show that our method is applicable to a broad class of default risk models by characterizing sufficient conditions.

JEL classification: C63.

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1 Introduction

In many financial markets, default has been widely observed. World capital markets have often experienced sovereign defaults on a large scale, and consumer bankruptcy protection is an important form of social insurance in many countries. Given this, researchers have employed default risk models to better understand the sources of default, the implications of default-related policies, and the ability of households and countries to insure consumption in richer incomplete market settings.[1] As richer data sets have recently become accessible, there have been more opportunities to observe a diversity of behaviors in consumer loan markets. In response, default risk models have become more complex over time. However, despite the increased computational burden, we have relied on the stability of grid search methods to solve default risk models at the cost of efficiency and accuracy because there are few robust and efficient solution methods for these models.[2]

In this paper, we propose a solution method for default risk models that has several notable advantages. First, our method leads to a more accurate bond price function and thus substantially alters the model's main predictions. When applied to Arellano's (2008) model, our method causes significant differences in stationary distributions as well as simulation results. Under our method, the mean debt-to-income ratio is about 70% lower, defaults are 3 to 5 times less frequent, and standard deviations of the interest rate spread are 3 times lower than those under grid search. Second, our method is faster, and this computational gain increases in richer models. It is approximately 4 to 7 times faster than grid search when applied to a canonical model of Arellano (2008); it is approximately 19 to 27 times faster than grid search when applied to the richer model of Nakajima and Ríos-Rull (2014). Finally, our method is applicable to a broader class of default risk models: both finite and infinite-horizon models, models with other discrete choices and multiple types of defaults.

Our method is an extension of the endogenous grid method (EGM), which was originally developed by Carroll (2006). The EGM is much faster than standard solution methods by avoiding using a forward-looking non-linear solver in finding decision rules.[3] Although this prototype EGM requires many restrictions to be applied, the EGM has progressed to solve a broader class of dy-

[1] For example, there have been studies for the episodes of sovereign default (Aguiar and Gopinath, 2006; Arellano, 2008; Yue, 2010), the implications of consumer bankruptcy reforms (Chatterjee et al., 2007; Livshits et al., 2007; Athreya, 2008; Athreya et al., 2009; Livshits et al., 2010; Chatterjee and Gordon, 2012; Nakajima, 2017), and the interactions between household default and business cycles (Nakajima and Ríos-Rull, 2014; Gordon, 2015).

[2] Arellano, Maliar, Maliar and Tsyrennikov (2016) developed an envelope condition method (ECM) for default risk models. However, as mentioned by the authors, the ECM does not guarantee the convergence of value functions if their default rule is endogenous.

[3] For example, in the neoclassical growth model, forward-looking solution methods search for a next period optimal level of asset (a'), given a current level of asset (a). These forward-looking methods compare values of choosing different a' to find the optimal a', therefore compute values multiple times. This step is computationally costly. On the other hand, the EGM backwardly solves a given a' using the first-order conditions (FOCs).

namic problems.[4] However, these extended EGMs cannot be used to solve default risk models because of the properties of these models that make the basic method inapplicable. First, in default risk models, value functions are non-concave and not everywhere differentiable because defaulting is a discrete choice that introduces kinks in the value functions. The general problems of non-concavity have been addressed ((Fella, 2014; Iskhakov et al., 2017; Druedahl and Jørgensen, 2017)), but none of these papers are immediately applicable to default risk models, as they do not address another issue: the feasible set of asset holdings is ex-ante unknown and varies across individual states.[5] This set must be found before solving a model with default to use the solution methods mentioned above.[6]

Our method comprehensively handles the computational issues above. First, we address the issue of non-concavity and non-differentiability by employing Fella's (2014) algorithm. But, as mentioned previously, Fella's (2014) EGM cannot be directly applied to default risk models because it works only when the feasible set for asset holdings is known, as is the case with an exogenous borrowing constraint or a collateral constraint.

To address this issue, we introduce a numerical procedure to identify the feasible set for the solution, according to theoretical findings in Clausen and Strub (2020). In Arellano's (2008) model with i.i.d. endowment shocks, they showed that the feasible set is characterized by one cut-off rule for assets conditional on endowment shock (and vice versa); above the cut-off, the derivative of the loan price schedule in assets is locally well defined at optimal choices and the first-order condition (FOC) is well established as a necessary condition for the global solutions; and in every optimal debt contract, the discounted value of debt, the product of the price and the quantity of debt, increase in the quantity of debt.

To use these theoretical findings with persistent shocks, we assume a finite Markov chain and characterize one cut-off rule conditional on each Markov state, as suggested in Clausen and Strub (2020).[7] This modification enables us to apply their theorems for i.i.d. shocks to the case with persistent shocks. For each state, we search for the minimum level of debt, above which the discounted value of debt increases in the amount owed; take this minimum as the risky borrowing limit, which serves as the lower bound of the feasible set of debt holdings. Likewise, conditional

[4]For example, the EGM has been extended to solve models with endogenous labor supply (Barillas and Fernández-Villaverde, 2007), discrete choices (Fella, 2014; Iskhakov, Jørgensen, Rust and Schjerning, 2017), and multiple dimensional choices (Hintermaier and Koeniger, 2010; Druedahl and Jørgensen, 2017).

[5]In contrast, when an option to default is unavailable, this issue does not appear because the feasible set of the solution is irrelevant to its equilibrium. It is predetermined through an exogenous borrowing constraint or a collateral constraint.

[6]Villemot (2012) is an exception using the EGM to solve a default risk model in Arellano (2008) by introducing a heuristic algorithm that updates the lower bound of the feasible set. However, this algorithm is not guaranteed to find the correct feasible set.

[7]The theorems in Clausen and Strub (2020) might not be applicable if one assumes a continuous and persistent income process. However, given a finite Markov chain, as we do in computation, the theorems are applicable.

on the current endowment and threshold, we compute the default probability and the loan price schedule, using a continuous probability density function. This approach, combined with Fella's (2014) algorithm, allows us to make the best of the computational benefits of the EGM in solving default risk models.

We illustrate the detailed procedures of our method with a canonical model of Arellano (2008).[8] Our EGM has noticeable computational benefits in efficiency. Our method converges approximately 4 to 7 times faster than grid search. These computational benefits increase when we apply our method to solve the richer model of Nakajima and Ríos-Rull (2014). We show that our method is approximately 19 to 27 times faster than grid search.

We also find that our threshold-based computation for the default probability makes differences in the loan price schedule's shape, thereby improving solution accuracy. The loan price schedule in Arellano (2008) is step-shaped because the default probability is computed based upon a discrete Markov chain. In contrast, our threshold-based approach leads the loan price schedule to be smooth because the default probability is computed in a continuous probability density function based on the cut-off of default in endowment shocks. These improvements in accuracy bring about substantial differences in the simulation results. For example, the standard deviation of the interest rate spread (the difference between the yields of government bonds and the risk-free rate) in the threshold-based method is approximately one-third of that in the discrete method with the step-shaped price schedule. The default is 3 to 5 times more frequent in the discrete method than in our threshold-based method. We find that this threshold-based method is the main driving force behind the improvements in accuracy. Grid search is also as accurate as the EGM when implemented with this threshold-based method.

In addition, as in White (2015), we characterize sufficient conditions for our method to be applicable to a model. The sufficient conditions imply that our EGM can be applied to a broad class of default risk models: both finite and infinite-horizon models, along with other discrete choices (e.g., housing, durable goods, health insurance, and retirement) and multiple types of defaults (e.g., Chapter 7 vs. Chapter 13 in consumer bankruptcy). We also discuss the possibility of the application of our method to default models with long-term debt (Chatterjee and Eyigungor, 2012; Hatchondo et al., 2016; Bocola and Dovis, 2019).

The organization of this paper is as follows. Section 2 describes the model of Arellano (2008), to which we apply our method. In Section 3, the detailed procedures of our algorithm are demonstrated, and Section 4 reports the results. In Section 5, we provide and discuss sufficient conditions for the application of our method. Finally, Section 6 concludes this paper.

[8] This model is well suited as a pedagogical example because it contains all the necessary components in a relatively simple model.

2 Model

In this section, we lay out Arellano's (2008) model.[9] In the model, the government starts each period with assets a and endowments e^S. The natural logarithm of endowments, S, follows an AR(1) process with a persistence of ρ_S:

$$S^{'} = \rho_S S + \epsilon^{'}, \tag{1}$$

where $\epsilon^{'}$ is i.i.d. and follows $N(0, \sigma_S^2)$. Let $f(S^{'}|S)$ describe the continuous stochastic process for $S^{'}$ conditional on S. Following the literature, we approximate the AR(1) process as a Markov chain $\pi_{S,S'}$. The government has an option to default on its debt $a < 0$. Given the option to default, the government solves the following problem:

$$V(S, a) = \max\{V^c(S, a), V^d(S)\}, \tag{2}$$

where $V(S, a)$ is the value of the government, $V^c(S, a)$ is the value associated with not defaulting, and $V^d(S)$ is the value associated with defaulting.

The value associated with not defaulting is as follows:

$$V^c(S, a) = \max_{\{a^{'} \geq -Z\}} \left\{ u(e^S - q(S, a^{'})a^{'} + a) + \beta \sum_{S'} \pi_{S,S'} V(S^{'}, a^{'}) \right\}, \tag{3}$$

where Z is a lower bound on debt to prevent Ponzi schemes but is otherwise not binding in the equilibrium. $u(\cdot)$ is the utility function that is differentiable, and $q(\cdot, \cdot)$ is the loan price schedule over the natural logarithm of current endowments S and the next period asset $a^{'}$.

The value associated with defaulting is as follows:

$$V^d(S) = u(h(S)) + \beta \sum_{S'} \pi_{S,S'} \left[\theta V(S^{'}, 0) + (1-\theta)V^d(S^{'})\right] \tag{4}$$

$$h(S) = \begin{cases} \lambda & \text{if } e^S > \lambda \\ e^S & \text{if } e^S \leq \lambda, \end{cases} \tag{5}$$

where θ is the probability that the economy will regain access to the international credit markets. The above value function implies that default causes two kinds of penalties. The first type of

[9] We focus on describing the government problem here. For detailed explanations of the model environment and other features, please see Arellano (2008).

penalty responds exclusion costs. This is the opportunity cost of not having access to the credit market in the following period with probability 1-θ. The other type of penalty captures output costs. The government must pay $e^S - \lambda$ when e^S is greater than λ under a bad credit history.

The financial market is competitive with risk-neutral lenders whose expected profit is zero. With these lenders, the loan price schedule, $q(S, a^{'})$, satisfies

$$q(S, a^{'}) = \frac{1 - \delta(S, a^{'})}{1 + r}, \tag{6}$$

where $\delta(S, a^{'})$ is the probability of default associated with S and $a^{'}$, and r is the risk-free interest rate.

To set the loan price, we need to characterize the default probability $\delta(S, a^{'})$. To do so, let us define $D(a)$ as

$$D(a) = \{S : V^c(S, a) < V^d(S)\}. \tag{7}$$

The probability of default with endowments S and assets in the next period $a^{'}$ is

$$\delta(S, a^{'}) = \int_{\{S' \in D(a^{'})\}} f(S^{'}|S) d(S^{'}) = \int_{\{S' : V^c(S', a^{'}) < V^d(S')\}} f(S^{'}|S) d(S^{'}), \tag{8}$$

where $f(\cdot|S)$ is the probability density function of $S^{'}$ conditional on the current state S. If $D(a^{'}) = \emptyset$, the equilibrium default probability becomes zero, and the bond price is equal to that of a risk-free bond. Here, although the discrete Markov chain, $\pi_{S,S'}$ is used to compute the expected value functions, the default probability $\delta(S, a^{'})$ is computed with this continuous probability density function of $f(\cdot|S)$ over $S^{'}$, allowing us to apply the theoretical findings in Clausen and Strub (2020). It is important to note that this is an essential difference to earlier work that used the approximation $\pi_{S,S'}$ for both the expected value functions and the default probability $\delta(S, a^{'})$. More details will be addressed in the next section.

3 Algorithm

Let us establish notation to explain the algorithm. Let n be the number of iterations for the value function and loan price schedule. Let $EV^n(S, a^{'})$ be the expected value function, $\beta \sum_{S'} \pi_{S,S'} V^n(S^{'}, a^{'})$. We will denote $G_{a'} = \{a^{'}_1, \ldots, a^{'}_{N_{a'}}\}$ as the grid for assets, $a^{'}$, in the next period. In addition, we define $D_{a'} EV^n(S, a^{'})$ as the derivative of the expected value function with respect to the next period asset holdings, $a^{'}$. We compute the numerical derivative of the expected value function in the following way:

$$D_{a'} EV^n(S, a'_k) = \begin{cases} \frac{EV^n(S, a'_{k+1}) - EV^n(S, a'_k)}{a'_{k+1} - a'_k}, \text{ for } k < N_{a'} \\ \frac{EV^n(S, a'_{N_{a'}}) - EV^n(S, a'_{N_{a'}-1})}{a'_{N_{a'}} - a'_{N_{a'}-1}}, \text{ for } k = N_{a'}, \end{cases} \tag{9}$$

where $N_{a'}$ is the number of grid for a. The numerical derivative of the discount loan rate with respect to a', $D_{a'} q^n(S, a')$, is computed in the same way.

3.1 Calculating Risky Borrowing Limit

We set up the feasible sets of the solution through the risky borrowing limit (credit limit), which is studied in Arellano (2008) and Clausen and Strub (2020). They show that, for each state S, the size of loan $q(S, a')a'$ increases with a' in every optimal debt contract. If the size of loan $q(S, a')a'$ decreases in a', households can increase their consumption by reducing debts (increasing a'), which implies that it cannot be an optimal debt contract. Arellano (2008) defines the risky borrowing limit to be the lower bound of the set for optimal contract. Using this theoretical finding, we numerically compute the risky borrowing limit for each state S using the following definition:

Definition 3.1.1. For each n and S, $a^n_{rbl}(S)$ is the risky borrowing limit if

$$\forall a' > a^n_{rbl}(S), \; D_{a'} \left(q^n(S, a') \cdot a' \right) = D_{a'} q^n(S, a') \cdot a' + q^n(S, a') > 0. \tag{10}$$

Figure 1 illustrates the risky borrowing limit, $a^n_{rbl}(S)$. To the right (left) of $a' = 0$, households are saving (borrowing). The return on savings is $1+r$, and the risk borrowing limit is $a^n_{rbl}(S)$. Note that this borrowing limit varies with S, which determines S' and thus the probability of default.

Note that most existing endogenous grid methods do not require a procedure of defining the risky borrowing limit. Models without default, to which the previous endogenous grid methods are applicable, have a borrowing constraint that is known ex-ante, as in the case of an exogenous borrowing constraint and a collateral borrowing constraint. However, in default risk models, the borrowing constraint is unknown ex-ante, and thus it is difficult to use the preexisting endogenous grid methods. We here address this issue by introducing a numerical procedure of searching for the risk borrowing limit according to its theoretical properties. Going forward, when we compute the endogenous grid, we will only use grid points that lie above the risky borrowing limit.[10]

[10]We argue that the risky borrowing limit might be a general feature of default risk models. More details will be addressed in Section 5.4.

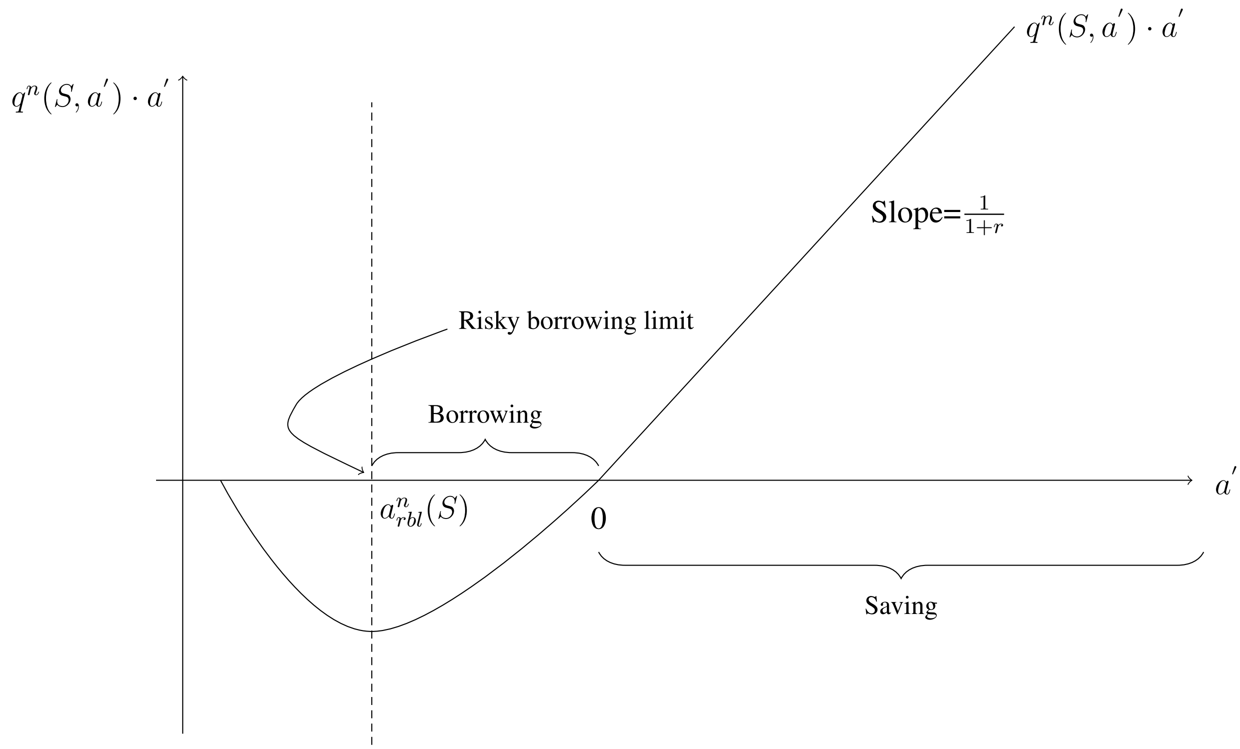


Figure 1: Risky Borrowing Limit

3.2 Identifying the (Non-) Concave Region

Fella (2014) presented an algorithm that divides the space of assets in the next period a' into concave and non-concave regions. In the concave region, the FOC is sufficient and necessary for the global solutions, whereas in the non-concave region, the FOC is only a necessary condition. The algorithm uses information on the curvature of the expected value function. We adjust his algorithm to work in our context. The details are described below.

Following Fella (2014), we use Figure 2 to understand how this algorithm works in our context. The vertical axis represents, for a given S, the values for the derivative of the expected value function, $D_{a'}EV^n(S,\cdot)$, and the marginal utility of consumption, $u'(\cdot)$. The horizontal axis is the value of asset holdings in the next period, a'. Given a level of cash on hand $M(= a+S)$, the marginal utility of present consumption increases with asset holdings in the next period, a'. Let $M''' < M'' < M'$ be three arbitrary levels of cash on hand. Given a', the larger the cash in hand, the more consumption.[11] Since the utility function is concave, marginal utility of present consumption is decreasing in cash on hand. The non-monotonic and discontinuous line is the derivative of the expected value function, $D_{a'}EV^n(S,\cdot)$. The curve is discontinuous at those values of a' for which

[11] From the budget constraint, $c + q(S,a')a' = M$.

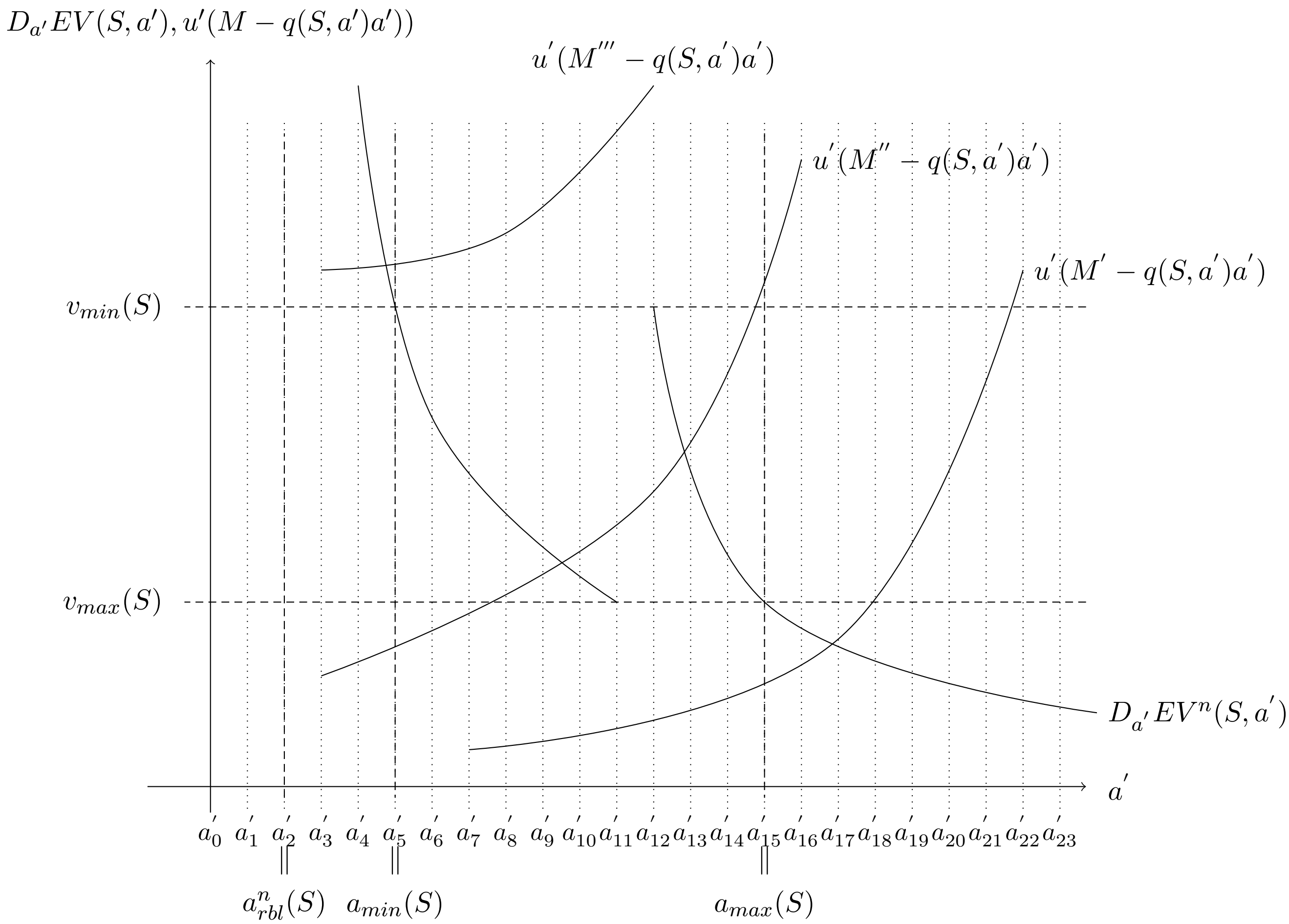


Figure 2: Illustrating the Algorithm

the default probability jumps discretely as $a^{'}$ changes.[12] The risky borrowing limit, $a^n_{rbl}(S)$, is represented at $a^{'}_2$. Let us define $G_{a^n_{rbl}(S)}$ as the set of all grid points for assets above the risky borrowing limit $a^n_{rbl}(S)$. Here, $G_{a^n_{rbl}(S)} = \{a^{'}_2, \ldots, a^{'}_{23}\}$.

This algorithm identifies the concave region by using information related to the FOC.[13] In Figure 2, the intersection points between $u^{'}(\cdot)$ and $D_{a^{'}}EV^n(S,\cdot)$ are where the FOC holds. A non-concavity causes a jump in $D_{a^{'}}EV(S,\cdot)$, which can lead to multiple crossing points (e.g., $u^{'}(M^{''} - q(S,a^{'})a^{'})$ intersects $D_{a^{'}}EV(S,\cdot)$ twice; between $a^{'}_9$ and $a^{'}_{10}$, between $a^{'}_{12}$ and $a^{'}_{13}$). The multiple crossing points mean that the FOC is not a sufficient condition but a necessary condition. In contrast, if there is only one crossing point, then the necessity of the FOC combined with uniqueness means that it is also sufficient for a global solution.[14] As in Fella (2014), we identify the concave region where the two curves are single-crossed by using the following criterion.

$a^{'}_i \in G_{a^n_{rbl}(S)}$ is on the concave region if either

$$\begin{gathered}\forall\ a^{'}_j \in G_{a^n_{rbl}(S)} \text{ with } a^{'}_j < a^{'}_i,\ D_{a^{'}}EV^n(S,a^{'}_i) < D_{a^{'}}EV^n(S,a^{'}_j) \text{ or} \\ \forall\ a^{'}_j \in G_{a^n_{rbl}(S)} \text{ with } a^{'}_j > a^{'}_i,\ D_{a^{'}}EV^n(S,a^{'}_i) > D_{a^{'}}EV^n(S,a^{'}_j).\end{gathered} \tag{11}$$

This condition implies that the derivative of the expected value function, $D_{a^{'}}EV^n(S,a^{'})$, is strictly decreasing on the concave region. In Figure 2, $a^{'}_5$ and $a^{'}_{15}$ are the two thresholds of this condition; we denote them as $a_{min}(S)$ and $a_{max}(S)$, respectively. As a result, in Figure 2, the concave region is $\{a^{'}_2, a^{'}_3, a^{'}_4\} \cup \{a^{'}_{16}, \cdots a^{'}_{23}\}$. The remaining region becomes the non-concave region, $\{a_{min}(S) = a^{'}_5, a^{'}_6, \cdots, a^{'}_{14}, a^{'}_{15} = a_{max}(S)\}$. $v_{max}(S)$ and $v_{min}(S)$ are the corresponding values of $D_{a^{'}}EV(\cdot)$ at $a_{max}(S)$ and $a_{min}(S)$, respectively.

Note that identifying the concave region is equivalent to finding $a_{min}(S)$ and $a_{max}(S)$. To find the thresholds, we take the following steps. First, we check the discontinuous points of the derivative of the expected value function, $D_{a^{'}}EV^n(S,a^{'}_i)$ with respect to $a^{'}_i$. In Figure 2, the discontinuous points arise at $a^{'}_{11}$ and $a^{'}_{12}$. Second, over the discontinuous points, we find the minimum value of $D_{a^{'}}EV^n(S,\cdot)$ and denote it as $v_{max}(S)$. In Figure 2, $v_{max} = D_{a^{'}}EV^n(S,a^{'}_{11})$. Next, we search for a set of $a^{'}_i$ that satisfies $D_{a^{'}}EV^n(S,a^{'}_i) \leq v_{max}(S)$, which is $\{a^{'}_{15}, a^{'}_{16}, \cdots, a^{'}_{23}\}$ in Figure 2. We choose the minimum among this set and define it as $a_{max}(S)$. In Figure 2, $a_{max}(S) = a^{'}_{15}$. v_{min} and a_{min} are computed analogously.

[12] Although the decision on default is the only discrete choice in the model, other types of discrete choices can be addressed along with default options. More details will be discussed in Section 5.

[13] We compute the first-order condition with respect to a' in Equation 3 to get $u'(M - q(S,a')a') = D_{a'}EV(S,a')$.

[14] In Figure 2, this happens at the point where $u^{'}(M^{'''} - q(S,a^{'})a^{'})$ intersects $D_{a^{'}}EV(S,\cdot)$: between $a^{'}_4$ and $a^{'}_5$.

3.3 Computing the Endogenous Grid for the Cash on Hand

For each S and $a_i^{'} \in G_{a_{rbl}^n(S)}$, we compute the endogenously determined cash on hand, $M(S, a_i^{'})$. To retrieve this endogenously determined cash on hand, $M(S, a_i^{'})$, we need to obtain the endogenously determined consumption by using the following FOC:

$$\text{For each } n,\ S \text{ and } a_i^{'} \in G_{a_{rbl}^n(S)}, \quad u^{'}(c(S, a_i^{'})) = \frac{D_{a^{'}} EV^n(S, a_i^{'})}{D_{a^{'}} q^n(S, a_i^{'}) \cdot a_i^{'} + q^n(S, a_i^{'})}, \tag{12}$$

where $c(S, a_i^{'})$ is the endogenously determined consumption. The FOC (12) is locally well defined and easy to compute.[15, 16] Recall that the derivative of the expected value function, $D_{a^{'}} EV^n(S, a_i^{'})$, and the loan price schedules, $D_{a^{'}} q^n(S, a_i^{'})$, are computed using equation (9). Given $D_{a^{'}} EV^n(S, a_i^{'})$ and $D_{a^{'}} q^n(S, a_i^{'})$, $c(S, a_i^{'}) = u^{'-1}\left(\frac{D_{a^{'}} EV^n(S, a_i^{'})}{D_{a^{'}} q^n(S, a_i^{'}) \cdot a_i^{'} + q^n(S, a_i^{'})}\right)$.

Given $c(S, a_i^{'})$, we retrieve the endogenously determined cash on hand $M(S, a_i^{'})$ as follows:

$$M(S, a_i^{'}) = c(S, a_i^{'}) + q^n(S, a_i^{'}) a_i^{'}. \tag{13}$$

For each S and $a_i^{'} \in G_{a_{rbl}^n(S)}$, we save the pairs of $(M(S, a_i^{'}), a^{'})$.

3.4 Storing the No-Default Value Function on the Endogenous Grid for Cash on Hand

Given n, for each S and $a_i^{'} \in G_{a_{rbl}^n(S)}$, we compute the value function over the endogenous grid for cash on hand, $M(S, a_i^{'})$, as follows:

$$V^{c,n+1}(S, M(S, a_i^{'})) = u(M(S, a_i^{'}) - q^n(S, a_i^{'}) \cdot a_i^{'}) + EV^n(S, a_i^{'}), \tag{14}$$

It is worth noting two things in this step. First, the value function is computed without any max-operator, which contributes to efficiency. Second, the value functions are defined on the endogenous grid of $M(S, a_i^{'})$, not on its exogenous grid.

3.5 Identifying the Global Solution over the Endogenous Grid for Cash on Hand

Given a level of cash on hand, the corresponding $a^{'}$ may not be a global solution as illustrated in Figure 2. In this step, we identify a set of the global solutions and save the corresponding pairs of

[15] Clausen and Strub (2020) proved the local differentiability of the expected value function and the loan price schedules and showed the existence of the FOC (12). The proof in Clausen and Strub (2020) was for the case of i.i.d. shocks on earnings, yet as they mentioned, the inclusion of AR-1 shocks does not make a huge difference in the proof.

[16] For each $a_i^{'} \in G_{a^{'}}$ with $a_i^{'} > a_{rbl}(S)$, the derivative of the size of the loan, $D_{a^{'}} q^n(S, a_i^{'}) a_i^{'} + q^n(S, a_i^{'})$, is always positive by the definition of the risky borrowing limit, $a_{rbl}^n(S)$. We assume that the utility function $u(\cdot)$ is differentiable with respect to c. The derivative of the expected value function and price function can be obtained numerically using equation (9).

$(M(S,a_i^{'}),a_i^{'})$. Given n and S, $a_i^{'} \in G_{a^n_{rbl}(S)}$ is either on the concave region or on the non-concave region. When $a_i^{'}$ is on the concave region, as $a_3^{'}$ in Figure 2, the pair of $(M(S,a_i^{'}),a_i^{'})$ implies a global solution because the FOC (12) is a sufficient and necessary condition. We save all of the pairs $(M(S,a_i^{'}),a_i^{'})$ on the concave region.

When $a_i^{'}$ is on the non-concave region (e.g., $a_i^{'} = a_9^{'}$) in Figure 2, the pair of $(M(S,a_i^{'}),a_i^{'})$ does not guarantee a global maximum because the FOC (12), while necessary, is not sufficient. As in Fella (2014), for each S and $a_i^{'}$ on the non-concave region, we verify whether this $a_i^{'}$ is the global solution by solving the following problem:

$$a_g^{'} = \underset{\{a_k^{'} \in \{a_{min}(S),\cdots,a_{max}(S)\}\}}{\operatorname{argmax}} \Big[u(M(S,a_i^{'}) - q^n(S,a_k^{'}) \cdot a_k^{'}) + EV^n(S,a_k^{'})\Big], \tag{15}$$

where $\{a_{min}(S),\cdots,a_{max}(S)\}$ is the non-concave region. If $a_i^{'} = a_g^{'}$, this implies that the pair of $(M(S,a_i^{'}),a_i^{'})$ corresponds to a global solution, thus we save this pair. If $a_i^{'} \neq a_g^{'}$, we discard this pair. This step does not add much computational intensity, since it only searches over the non-concave region.

3.6 Computing the Endogenous Grid for the Current Assets Related to the Global Solutions

For the saved pairs of $(M(S,a_i^{'}),a_i^{'})$ for the global solutions, we store the corresponding pairs of $(a(S,a_i^{'}),a_i^{'})$. For each saved $a_i^{'}$, we compute the endogenous grid for the current assets, $a(S,a_i^{'})$, as follows:

$$a(S,a_i^{'}) = M(S,a_i^{'}) - e^S. \tag{16}$$

Note that when a set of $a_i^{'}$ corresponds to the global solutions, $a(S,a_i^{'})$ monotonically increases with $a_i^{'}$, thereby allowing for a one-to-one mapping from a to $a^{'}$. This mapping enables us to use splines to evaluate the policy function over the exogenous grid in the following step.

3.7 Evaluating the Policy Function and the No-Default Value Function on the Exogenous Grid for the Current Assets

Using the one-to-one mapping between $a(S,\cdot)$ and $a^{'}$, we employ a linear interpolation to evaluate the policy function of asset holdings, $a^{'} = g_a(S,\cdot)$, over the exogenous grid for the current asset a_i. Then, we obtain the corresponding value $V^{c,n+1}(S,\cdot)$ over the exogenous grid for the current assets a_i by computing $V^{c,n+1}(S,a_i) = u(a_i + e^S - q^n(S,g_a(S,a_i)) \cdot g_a(S,a_i)) + EV^n(S,g_a(S,a_i))$.

3.8 Computing the Value of Defaulting

So far, we have solved the value of non-defaulting. In default risk models, agents choose whether to default by comparing their non-defaulting value with a defaulting one. In this step, we compute the value of defaulting. The value function with a bad credit history is as follows:

$$V^{d,n+1}(S) = u(y(S)) + \beta \sum_{S'} \pi_{S,S'} \left[\theta V^{c,n+1}(S',0) + (1-\theta)V^{d,n}(S'),\right]. \tag{17}$$

Since the value function of defaulting is not related to any continuous endogenous state, it is not costly to compute it.

3.9 Updating the Value Function and Loan Price Schedules

We update the value function, $V^{n+1}(S,a)$, and the price function, $q^{n+1}(S,a')$, in the following way:

$$V^{n+1}(S,a) = \max\{V^{c,n+1}(S,a), V^{d,n+1}(S)\} \tag{18}$$

$$q^{n+1}(S,a') = \frac{1-\delta(S,a')}{1+r},$$

where

$$\delta(S,a') = \int_{\{S' \in D(a')\}} f(S'|S)d(S') = \int_{\{S' : V^c(S',a') < V^d(S')\}} f(S'|S)d(S')$$

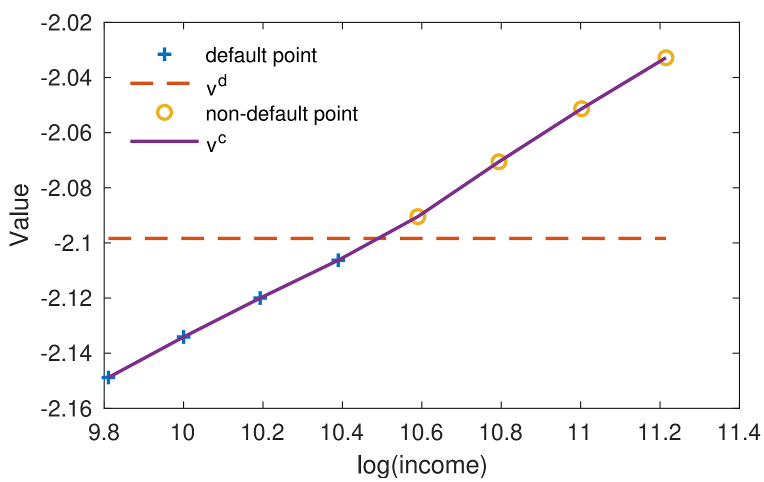


Figure 3: Value of Repaying and Defaulting Given S, a'

As mentioned previously, Clausen and Strub (2020) prove that $D(a')$ is characterized by a unique cutoff rule in endowment shocks when these shocks are i.i.d. To exploit this theoretical finding, we modify the default set $D(a')$ as represented by i.i.d. shocks. Let us define the default set of a' conditional on S, $D(a'|S)$, as

$$D(a'|S) = \{S' : V^c(S'|S, a') < V^d(S'|S)\}. \tag{19}$$

Using $S' = \rho_s S + \epsilon'$, we rearrange $D(a'|S)$ in terms of ϵ' as follows:

$$D(a'|S) = \{\epsilon' : V^c(\rho_s S + \epsilon', a') < V^d(\rho_s S + \epsilon')\}, \tag{20}$$

This equation implies that the default set associated with a' conditional on S can be represented by cut-off rules for i.i.d. shocks, ϵ'. In this case, as shown in Clausen and Strub (2020), for each a' and S, there exists a unique cutoff $\bar{\epsilon}_{S,a'}$ such that $V^c(\rho_s S + \bar{\epsilon}_{S,a'}, a') = V^d(\rho_s S + \bar{\epsilon}_{S,a'})$. The conditional default set can be rearranged in terms of a unique cutoff $\bar{\epsilon}_{S,a'}$ as follows:

$$D(a'|S) = \{\epsilon' \leq \bar{\epsilon}_{S,a'} : \ V^c(\rho_s S + \bar{\epsilon}_{S,a'}, a') = V^d(\rho_s S + \bar{\epsilon}_{S,a'})\}. \tag{21}$$

This unique cutoff, $\bar{\epsilon}_{S,a'}$, can be computed by using information in the value functions. For example, in Figure 3, given a pair of (S, a'), the cutoff $\bar{\epsilon}_{S,a'}$ is $10.5 - \rho_s S$ because v^c and v^d intersect at $S' = 10.5$.

Then, the default probability function associated with S and a' is

$$\delta(S, a') = \int_{-\infty}^{\bar{\epsilon}_{S,a'}} g(\epsilon')d(\epsilon'), \tag{22}$$

where $g(\cdot)$ is the probability density function of $\epsilon' \sim N(0, \sigma_S^2)$. For each S and a', we compute $\bar{\epsilon}_{S,a'}$ by employing an interpolation on the value functions in S'; compute $\delta(S, a')$ directly with the cut-off rule, $\bar{\epsilon}_{S,a'}$, and the probability density function of the normal distribution, $g(\cdot)$.

We compute $EV^{n+1}(S, a)$. If $||EV^{n+1}(S, a) - EV^n(S, a)||_\infty > 10^{-5}$ where $||\cdot||_\infty$ is the sup norm over $\mathbb{S}X\mathbb{A}$, start a new iteration with $n = n + 1$.

3.10 Summary of the Algorithm

To sum up, given an iteration number, n, and the expected value function $EV^n(S, a)$, the algorithm is as follows:

1. For each S, calculate the risky borrowing limit, $a^n_{rbl}(S)$, and save it.
2. Identify the (non-) concave region of asset holdings a' by using the algorithm of Fella (2014).
3. Given $(S, a^n_{rbl(S)})$, compute the endogenously determined cash on hand, $M(S, a_i')$, by solving

the FOC (12). Save these pairs of $(M(S, a_i^{'}), a_i^{'})$.

4. Compute the value function for non-defaulting over the endogenous grid for cash on hand, $(V^{c,n+1}(S, M(S, a_i^{'}))$.
5. Identify the global solution over the endogenous grid for cash on hand.
   - If $a_i^{'}$ is on the concave region, save the pair of $(M(S, a_i^{'}), a_i^{'})$.
   - If $a_i^{'}$ is on the non-concave region, verify whether the candidate $(M(S, a_i^{'}), a_i^{'})$ implies the global solution by solving the value function. If this is the global solution, save the pair of $(M(S, a_i^{'}), a_i^{'})$. Otherwise, discard it.
6. For the saved pairs of $(M(S, a_i^{'}), a_i^{'})$, compute the corresponding endogenous grid for the current assets, $a(S, a_i^{'})$. Save the pairs of $(a(S, a_i^{'}), a_i^{'})$.
7. Using the monotonicity between $a(S, a_i^{'})$ and $a_i^{'}$, compute the policy function of asset holdings and the value function for non-defaulting over the exogenous grid for the current assets.
   - Use a linear interpolation to compute the policy function of asset holdings in the next period, $g_a(S, a_i)$. With $g_a(S, a_i)$, compute $V^{c,n+1}(S, a_i)$.
8. Compute the value function for defaulting, $V^{d,n+1}(S)$.
9. Update the value function, $V^{n+1}(S, a) = \max\{V^{c,n+1}(S, a), V^{d,n+1}(S)\}$, and loan price schedules, $q^{n+1}(S, a^{'})$.
10. Compute $EV^{n+1}(S, a)$.
11. Start a new iteration with $n = n + 1$ if $||EV^{n+1}(S, a) - EV^n(S, a)||_\infty > 10^{-5}$. Otherwise, stop.

4 Results

We first document how computation speed and accuracy can be improved by using our EGM. We compare our EGM results with two versions of grid search results: i) grid search with discrete bond price function (Grid search 1), and ii) grid search with continuous bond price function (Grid search 2). Later we show simulation results and investigate how the method of computing the price function affects quantitative implications of a model.

4.1 Parameterization

Table 1: Parameters

r	1.7%	Risk-free interest rate
σ	2.0	Risk aversion
ρ_S	0.945	Endowment process
σ_S	0.025	Endowment process
β	0.953	Discount factor
θ	0.282	Probability of reentry
λ	$0.969E(S)$	Output cost

We follow Arellano's (2008) choice of parameter values. The utility function is

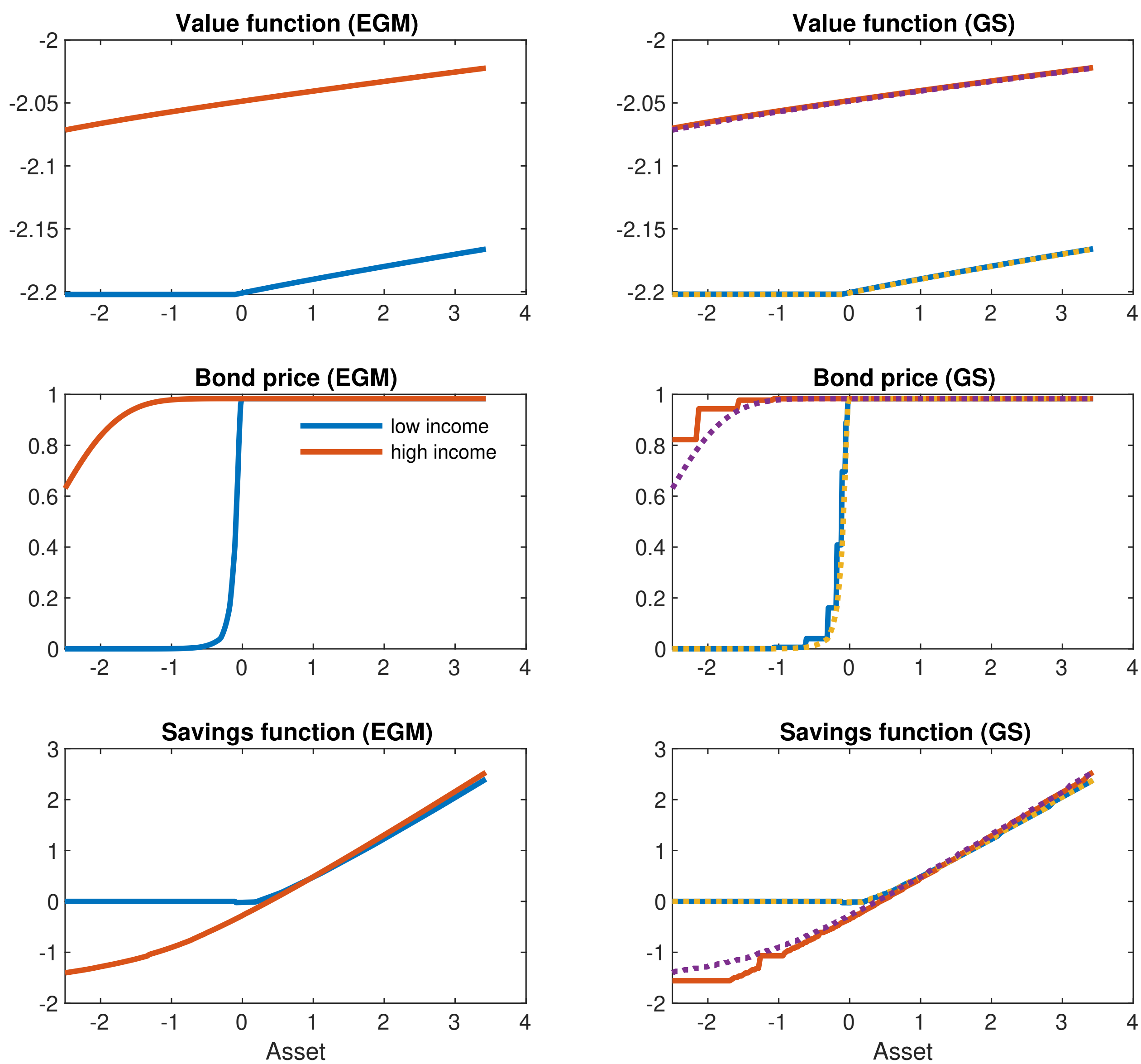
$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}.$$

The Markov chain approximation to the process follows Tauchen (1986). We set the lower and upper bound of a to -2.5 and 3.5. Table 1 shows the values of the chosen parameters.

4.2 Computing Time and Accuracy

The model is solved on a grid of 200, 500, 1000, and 2000 for the continuous state variable a and a grid of 21 for S. The grid points for S are linearly spaced and the grid points for a are log-spaced around zero, as default decisions are made and measures are located near zero. The programs were written in Fortran 95, and all computations were carried out on a single core of an Intel i7-4770 processor.

We find that the way of computing the bond price function matters for the results. As we explained in Section 3.9, we identified the exact size of income shocks that lead to defaults to compute default probability. As a result, $q(S, a')$ becomes a smooth function, unlike the step-shaped schedule in Arellano (2008). However, this way of computing the bond price function is not restricted to EGM, and we report the results from the two versions of grid search: i) grid search with discrete bond price function (Grid search 1); and ii) grid search with continuous bond price function (Grid search 2). Figure 4 shows the results of our computations. Our solutions using grid search with discrete bond price function resemble the solution of Arellano (2008). When we compute the probability of default more accurately, bond price functions look smooth with EGM and Grid search 2. As a result, the savings functions are also smooth.



Note: In GS, solid lines are grid search 1 and dashed lines are grid search 2.

Figure 4: Value Function, Bond Price Schedule, and Savings Function

4.2.1 Computing Time

Table 2: Computing Time (second)

# of Grid Points	200			500			1000			2000		
Method	EGM	GS1	GS2	EGM	GS1	GS2	EGM	GS1	GS2	EGM	GS1	GS2
Time to converge	2.2	11.0	9.3	10.9	68.0	59.0	43.8	268.5	224.8	187.6	1087.1	901.3
Time per iteration	0.01	0.06	0.05	0.06	0.36	0.31	0.23	1.41	1.18	0.98	5.69	4.72

Table 5 shows that the EGM is approximately 4 to 7 times faster than the grid search method across all grid settings. Some readers might be concerned that identifying the exact size of income shock to compute the bond price schedule may slow down the computation. However, Grid search 2 does not take longer to execute than Grid search 1.

Since the model we solve is simple, one might think that the efficiency gain is small. Further, considering the additional time to implement the EGM algorithm, it might not be worth using the EGM on a simple model to reduce computing time. However, if a model entails more complicated features and hence takes longer to solve, the EGM can reduce computing time significantly. For example, Nakajima and Ríos-Rull (2014) use a more complex income process and aggregate uncertainty. They use a method in Krusell and Smith (1998) to approximate the aggregate states of the model, and thus solving it requires a long simulation (outer loop) after computing value functions and decision rules (inner loop). Also, finding equilibrium requires several iterations of inner loops and outer loops. We use our method to solve Nakajima and Ríos-Rull (2014) and find the EGM is from 18.5 to 27.3 times faster than the grid search method in the inner loops. In the outer loops, the EGM is approximately 7.5 times faster than the grid search method. The details about our implementation and numerical results can be found in the Appendix.

4.2.2 Accuracy

Bellman equation error We compute Bellman equation errors instead of computing Euler equation errors to measure accuracy. Recall the following notation: S is the state vector other than assets a. Then, the Bellman equation

$$V(S,a) = u(c(S,a)) + E_{S'}\left[V(S^{'}, a^{'}(S,a))\right] \tag{23}$$

should hold exactly for the true decision rules. Because the decision rules are numerically computed in our exercises, the Bellman equation (23) does not hold exactly with the numerically calculated decision rules. We define c^* as the solution for

$$u(c^*(S,a)) = V(S,a) - E_{S'}\left[V(S^{'}, \bar{a}^{'}(S,a))\right], \tag{24}$$

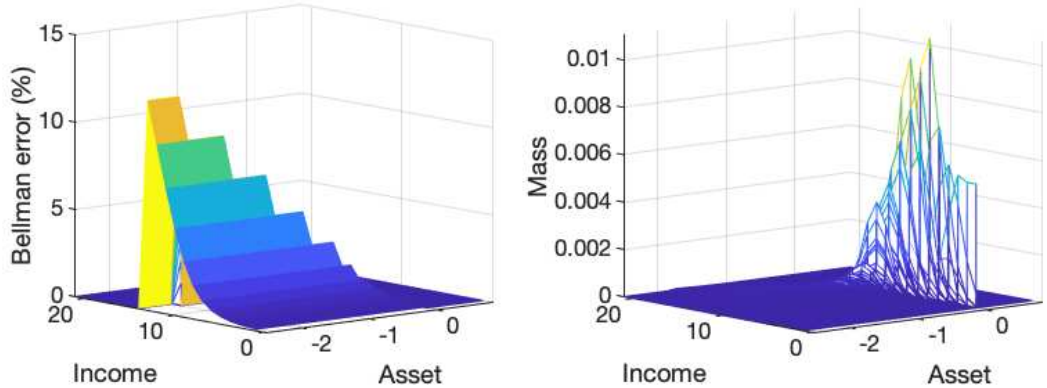


Figure 5: Bellman Error and Stationary Distribution for the EGM

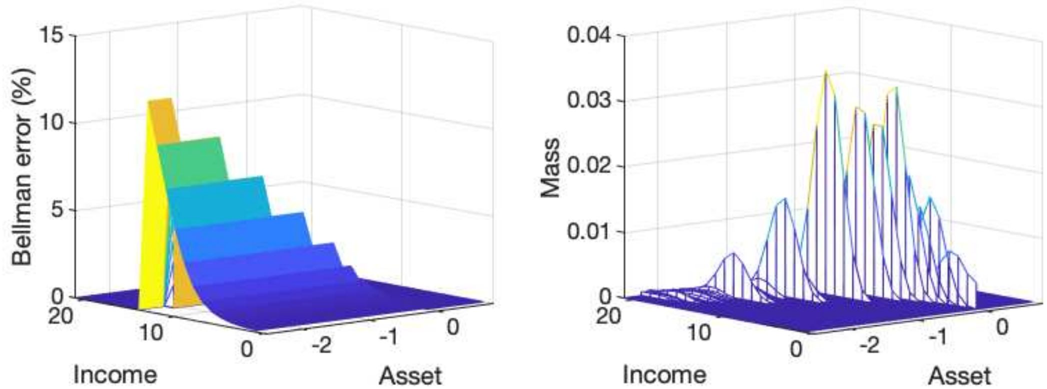


Figure 6: Bellman Error and Stationary Distribution for Grid Search 1

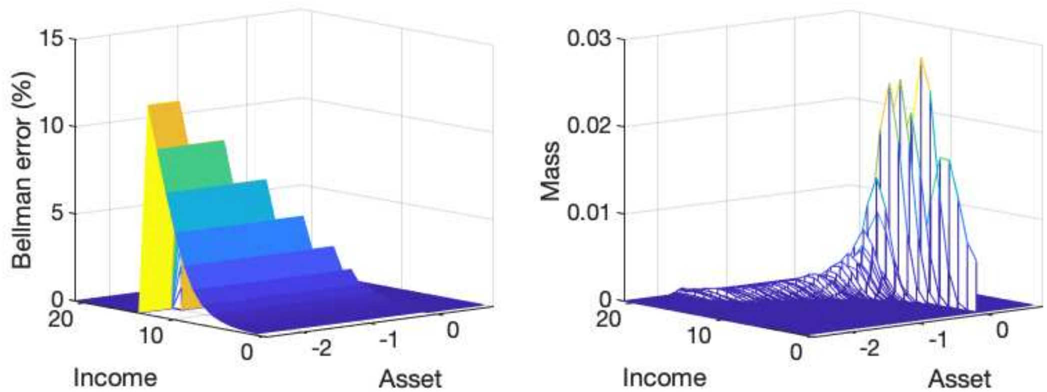


Figure 7: Bellman Error and Stationary Distribution for Grid Search 2

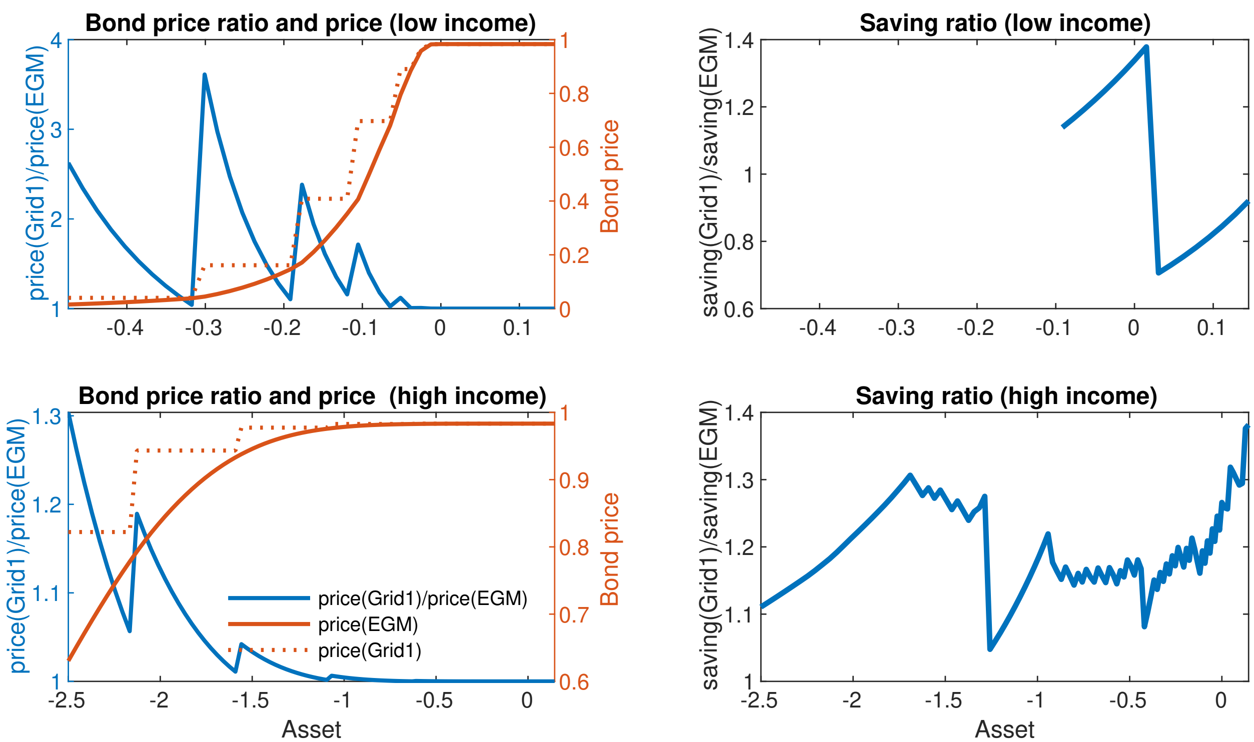
where bars indicate the numerically calculated decision rules. We define the Bellman equation error as

$$BE(S,a) = \left| 1 - \frac{c^*(S,a)}{\bar{c}(S,a)} \right|. \tag{25}$$

Following the literature, we report both the maximum and the average of Bellman equation errors. We compute the average errors as the weighted average of the Bellman equation errors over the stationary distribution and the maximum error as the maximum of the Bellman equation errors lying only on the stationary distribution.[17]

Table 3: Bellman Equation Error (%)

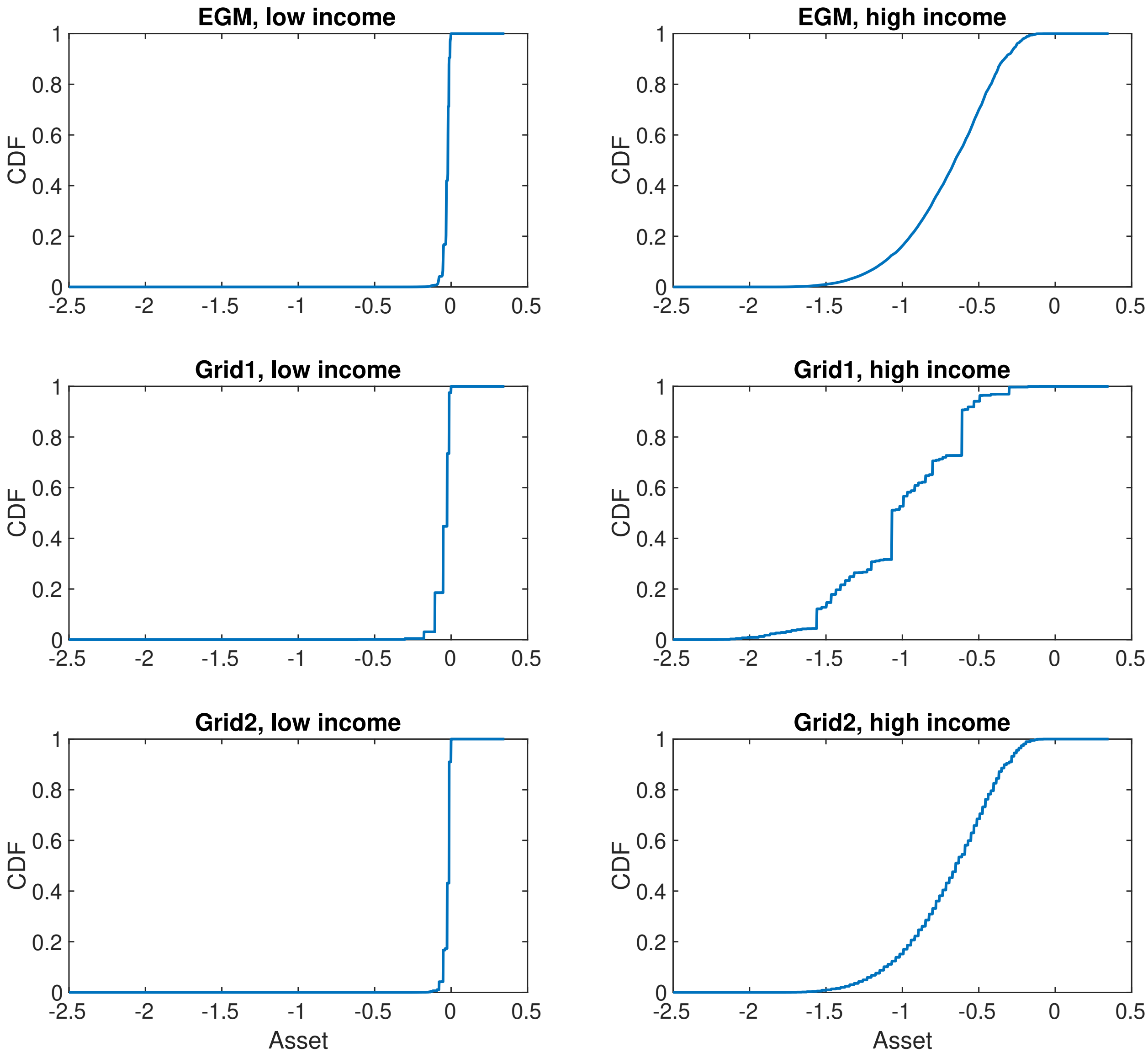
# of Grid Points	200			500			1000			2000		
Method	EGM	GS1	GS2	EGM	GS1	GS2	EGM	GS1	GS2	EGM	GS1	GS2
Mean	0.013	0.028	0.013	0.013	0.027	0.013	0.013	0.029	0.013	0.013	0.029	0.013
Max	9.58	12.05	9.58	9.58	12.06	9.58	9.58	12.06	9.58	9.58	12.06	9.58



Note: Savings ratio (low income) does not have values below $a = -0.1$ because those are default points.

Figure 8: Bond Price Ratio and Savings Ratio

[17] Although we vary the number of grid points for a when solving the value functions, we fix the number of grid points for a to 2,000 when computing the stationary distributions.



Note: Density is normalized to one at each income level.

Figure 9: Distribution

Figure 5 to 7 show the Bellman equation errors and stationary distributions. In all figures, there is less mass where errors are large. The differences are not noticeable in the figure, but Table 6 shows the differences. EGM and GS2 produce equally accurate outcomes and more accurate outcomes than GS1. Both average and maximum Bellman equation errors in the EGM and GS2 are lower than those in the grid search methods across all grid settings. As we mentioned above, the way of computing $q(S, a')$ affects accuracy significantly.

While Bellman equation errors look similar in Figure 5 to 7, stationary distributions do not. This is also because of the differences in the way we compute the bond price function. As we describe in Section 3.9, we compute the exact size of income shock that leads to a default and infer the default probability from the cumulative distribution function (CDF) of the income shock. Since it removes the upward bias in the bond price, EGM and GS2 price functions are lower than those of GS1.[18] Figure 8 compares bond price computed using GS1 and bond price with EGM and shows the ratio of savings function of the two methods.[19] As is clear from the discussion so far, the bond price ratios are always above one. Since borrowing is more expensive under the EGM solution, borrowing is smaller. As a result, the aggregate borrowing is smaller when we solve the model with EGM, and we see fewer masses with large debts. Figure 9 shows the differences in distributions more closely. They plot cumulative density over assets at a given income level. As mentioned, there are fewer borrowings with the EGM solution. Also, thanks to the smooth savings function, we can see that masses are more smoothly distributed over grids in EGM distributions. These differences not only contribute to achieving lower Bellman errors but also may affect the implications of the model. We will discuss this issue further with the simulation results.

Table 4: Simulation Results

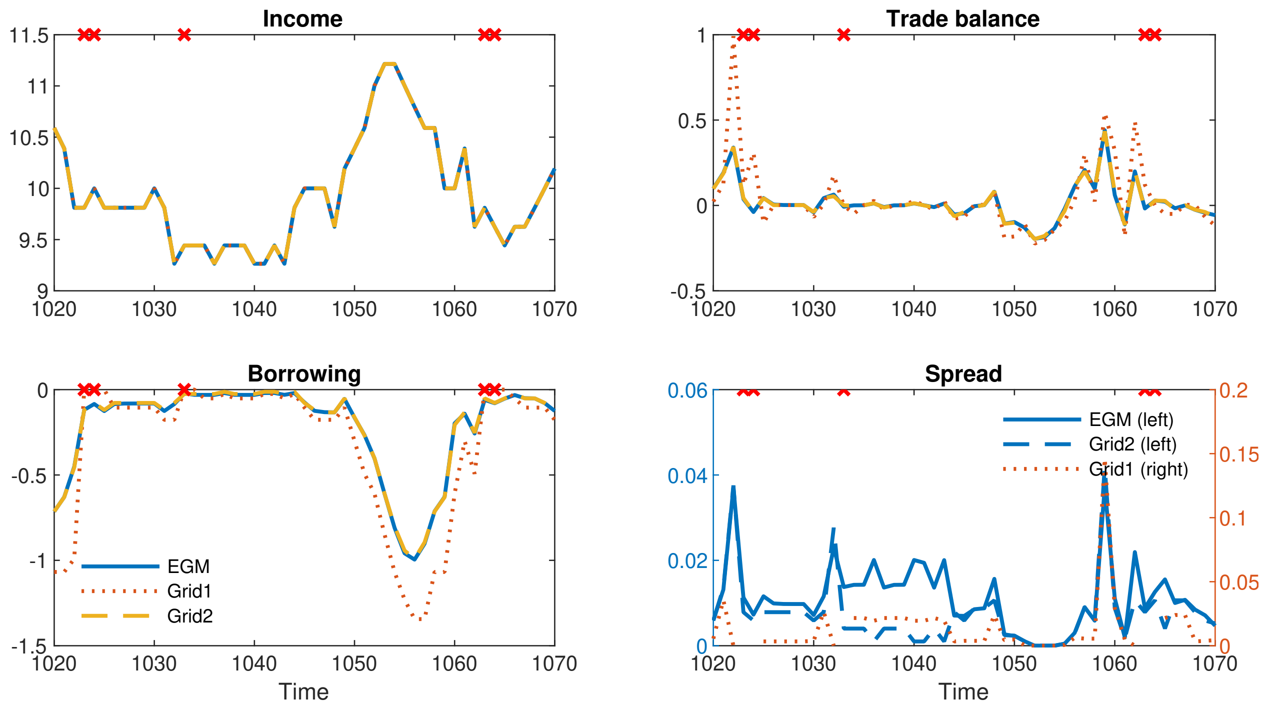
Computational Method	EGM				GS1				GS2			
# of Grid Points	200	500	1000	2000	200	500	1000	2000	200	500	1000	2000
$\sigma(TB/S)(\%)$	0.95	0.94	0.94	0.97	1.38	1.38	1.40	1.40	0.99	0.95	0.97	0.97
$\sigma(r_s)(\%)$	2.06	0.61	0.53	0.48	1.67	1.65	1.73	1.72	0.58	0.54	0.50	0.49
$\rho(r_s, S)$	-0.49	-0.61	-0.53	-0.41	-0.35	-0.41	-0.45	-0.45	-0.20	-0.32	-0.26	-0.25
$\rho(r_s, TB/S)$	0.37	0.59	0.66	0.71	0.40	0.40	0.40	0.40	0.57	0.62	0.67	0.69
$mean(r_s)(\%)$	2.13	0.81	0.69	0.62	1.07	1.08	1.18	1.18	0.54	0.60	0.57	0.56
$mean(a'/S)$	-3.15	-3.10	-3.10	-3.22	-5.48	-5.50	-5.72	-5.73	-3.27	-3.13	-3.21	-3.21
# of default per 500,000q	1665	1508	1337	1004	4879	5035	5509	5508	946	1206	1003	1003

TB: trade balance (output - consumption), S: income, r_s: $\frac{1}{q(S,a')} -$ risk-free rate, σ: standard deviation, ρ: correlation

Simulation results Table 7 reports statistics obtained in the simulations. TB is a trade balance (output minus consumption), and r_s is the interest rate spread (margin of extra yield over the risk-free rate). Standard deviations are denoted by σ and are reported in percentage terms; correlations are denoted by ρ. Statistics were computed following Arellano (2008). Samples were selected using the following criteria: i) a default is declared immediately after the end of the sample, ii) the sample length is 74 quarters, and iii) the last exclusion period was observed at least two periods before the beginning of the sample. We select 1,000 such samples and report the average value of statistics computed across samples of 74 periods.

[18] For example, in Figure 3, the default probability would be the sum of transition probability for the first four grid points if we do not find the exact cutoff point. We account the area from the fourth point to the point where v^c and v^d meets: therefore, the default probability is always higher when using this scheme.

[19] Bond price ratio between EGM and GS2 is close to 1.



Note: Red cross marks indicate exclusion period due to default of Grid1.

Figure 10: Simulated Series

Simulation results reconfirm the importance of accurate bond price function. Overall, the results from EGM and GS2 are similar, and they are different from GS1 results.[20] Mean debt-to-income ratio is approximately 70% higher in GS1, as the upward bias in bond price leads to more borrowings. Average interest spreads are higher in GS1 as loan sizes are larger; spread volatilities are three to four times higher as well. Trade balances are approximately 50% more volatile, and defaults are 3 to 5 times more frequent in GS1. Importantly, these gaps do not become narrow as we increase the number of grid points.[21] Figure 10 illustrates simulated income, trade balance, borrowing, and interest rate spreads.[22]

The inaccuracy of GS1 results is consistent with the findings in Hatchondo et al. (2010). In this paper, we show that grid search can be accurate as long as the bond price function is accurately computed and the EGM can achieve efficiency as well as accuracy.

[20]While the results from EGM and GS2 are similar, the spreads are more counter-cyclical and mean spreads are higher in EGM. This is because we often observe very small amount of borrowing (around 1% of endowment) and high spreads from EGM simulation while no borrowing and zero spreads from GS2 in corresponding periods.

[21]When we increase the number of endowment grids, GS1 results are closer to GS2, as it leads to more accurate loan price schedules.

[22]As can be seen in Figure 10, we use discretized order Markov shocks when simulating the model. Some might wonder how much the cyclical properties are different if we use the original continuous shock. The properties are largely similar; see Appendix B for details.

5 Generalization of the EGM

In Section 2 and 3, we chose Arellano's (2008) model to demonstrate our EGM because it is applicable to this model. However, the EGM does not work in all models. For example, the EGM may not be useful when asymmetric information exists between lenders and borrowers or when the risky borrowing limit does not exist. This section formalizes the method in a theoretical framework and provides its sufficient conditions in order to understand the class of models to which our EGM is applicable. We modify theorems in White (2015), in accordance with the circumstances of default risk models.

5.1 Formalization of the EGM

Consider a set of agents facing a dynamic maximization problem with a number of decision variables, each of which can be either continuous or discrete.[23] Time t begins at period 0 and ends after period $T \in \mathbb{N}$.[24] We denote $s_t \in S_t \subset \mathbb{R}^p$ as the exogenous state where $p \in \mathbb{N}$. Let $a_t \in A_t \subset \mathbb{R}$ be the continuous state. Let us denote $d_t \in D_t = \{1, \cdots, N_d\}$ as the endogenous default state. State variables (s_t, a_t, d_t) correspond to (S, a, d) in Section 3. We define $b_t \in B_t = \{1, \cdots, N_b\}$ as the endogenous discrete state other than the default state, which does not exist in Arellano's (2008) model. Note that changes in the current endogenous state variables (a_t, d_t, b_t) affect $(a_{t+1}, d_{t+1}, b_{t+1})$, but not s_{t+1}.

Agents make a choice of y_t from the closed and convex feasible set $\Gamma_t(s_t, a_t, b_t, d_t)$, where the constraint correspondence, $\Gamma_t(\cdot)$, represents the budget set for the current state: $\Gamma_t : S_t \times A_t \times B_t \times D_t \longrightarrow Y_t \subset \mathbb{R}$. We denote the feasible set $Y_t = \cup_{\{s_t,a_t,b_t,d_t\}\in S_t\times A_t\times D_t\times B_t}\Gamma_t(s_t, a_t, b_t, d_t) \subset \mathbb{R}$. In Section 3, the choice y_t corresponds to consumption c, and the constraint correspondence $\Gamma_t(\cdot)$ corresponds to the budget constraints, $\Gamma(S, d = 0, a) = \{c : 0 < c \leq S + a - q(S, -Z)(-Z)\}$ and $\Gamma(S, d = 1, a) = \{c : 0 < c \leq h(S)\}$. Agents obtain a flow of utility from their choices and states through the utility function $U_t : S_t \times B_t \times D_t \times Y_t \longrightarrow \mathbb{R}$. U_t is continuous, strictly monotonic, strictly concave, and twice differentiable to y_t on its interior domain. In addition, the discount factor β_t is between 0 and 1.

Once agents determine y_t, their state changes from (s_t, a_t, b_t, d_t) to the interim period state, $(s_{t+1}, a_{t+1}, b_{t+1}, d_t) \in S_{t+1} \times A_{t+1} \times B_{t+1} \times D_t$ according to the transition function $\Delta_t(\cdot)$. Note that this interim timing indicates the time before making decision on default in period $t+1$ but after deciding y_t, and thereby s_t, a_t, and b_t change to s_{t+1}, a_{t+1}, and b_{t+1}. The transition $\Delta_t(\cdot)$ depends on random shock $\epsilon_{t+1} \in E_{t+1} \subset \mathbb{R}^l$, drawn from the CDF of $F_{t+1}(\epsilon_{t+1})$. ϵ_{t+1} corresponds to ϵ' in

[23] For the sake of easy exposition, we describe the problem in terms analagous to Section 3. However, while we refer to the discrete decision as "defaulting", our algorithm can be applied in the context of other problems with discrete choices such as housing.

[24] As in White (2015), it can be applied to infinite horizon models, with the time subscripts skipped.

Arellano's (2008) model. Agents do not know the exact value of ϵ_{t+1} when making the decision on y_t, but know the distribution of $F_{t+1}(\epsilon_{t+1})$. The transition function Δ_t is formally defined as

$$\begin{aligned}\Delta_t \colon S_t \times A_t \times B_t \times D_t \times Y_t \times E_{t+1} &\longrightarrow S_{t+1} \times A_{t+1} \times B_{t+1} \times D_t \\ (s_t, a_t, b_t, d_t, y_t, \epsilon_{t+1}) &\longmapsto (s_{t+1}, a_{t+1}, b_{t+1}, d_t).\end{aligned} \tag{26}$$

Since d_t does not change in the interim period, we can express the transition function Δ_t conditional on d_t. Further, we regard the future endogenous state b_{t+1} as given state $\bar{b}_{t+1}$ to use the algorithm of Fella (2014) afterward. We define the transition function conditional on $(d_t \times \bar{b}_{t+1}) \in D_t \times B_{t+1}$, $\Delta_{d_t,\bar{b}_{t+1}}$, as follows:

$$\begin{aligned}\Delta_{d_t,\bar{b}_{t+1}} \colon S_t \times A_t \times B_t \times Y_t \times E_{t+1} &\longrightarrow S_{t+1} \times A_{t+1} \\ (s_t, a_t, b_t, y_t, \epsilon_{t+1}) &\longmapsto (s_{t+1}, a_{t+1}).\end{aligned} \tag{27}$$

Recall that y_t affects the transition from a_t to a_{t+1}, but not that from s_t to s_{t+1}. Thus, we can decompose the conditional transition function $\Delta_{d_t,\bar{b}_{t+1}}$ into the portion $\Delta^S_{d_t,\bar{b}_{t+1}}$ independent of y_t and the portion $\Delta^A_{d_t,\bar{b}_{t+1}}$ dependent on y_t. This decomposition implies that $\partial\Delta^S_{d_t,\bar{b}_{t+1}}/\partial y_t = 0|_{p\times 1}$, $\partial\Delta^S_{d_t,\bar{b}_{t+1}}/\partial a_t = 0|_{p\times 1}$, $\partial\Delta^S_{d_t,\bar{b}_{t+1}}/\partial b_t = 0|_{p\times N_b}$, and $\partial\Delta^A_{d_t,\bar{b}_{t+1}}/\partial y_t \neq 0$. Since $\Delta^S_{d_t,\bar{b}_{t+1}}$ is irrelevant to (a_t, b_t, y_t), we represent $\Delta^S_{d_t,\bar{b}_{t+1}}$ as a function of (s_t, ϵ_{t+1}). For example, in Section 2, $\Delta^S_{d_t,\bar{b}_{t+1}}$ corresponds to the first-order Markov chain $\pi_{S,S'}$ which was independent of consumption c. $\Delta^A_{d_t,\bar{b}_{t+1}}$ corresponds to the transition of the current assets a to the next period assets a' that depends not only on $\pi_{S,S'}$ but also on consumption c.

We recursively represent the agent's problem. At the beginning of each period, agents solve

$$V_t(s_t, a_t, b_t) = \max_{d_t \in \{0,1,\cdots N_d\}} \{v_t(s_t, a_t, b_t, d_t))\}, \tag{28}$$

where $V_t(s_t, a_t, b_t)$ is the value after default decision and $v_t(s_t, a_t, b_t, d_t)$ is the value before default decision. Before agents decide whether to default or not; for each $(d_t \times \bar{b}_{t+1}) \in D_t \times B_{t+1}$, they solve

$$v_t(s_t,a_t,b_t,d_t;\bar{b}_{t+1}) = \max_{y_t\in\Gamma_t(s_t,a_t,b_t,d_t;\bar{b}_{t+1})} U_t(s_t,b_t,d_t,y_t) + \beta_t E\left[\{V_{t+1}(s_{t+1},a_{t+1},\bar{b}_{t+1})\}\right]$$

$$= \max_{y_t\in\Gamma_t(s_t,a_t,b_t,d_t;\bar{b}_{t+1})} U_t(s_t,b_t,d_t,y_t) + \beta_t \int V_{t+1}(\Delta_{d_t,\bar{b}_{t+1}}(s_t,a_t,b_t,y_t,\epsilon_{t+1}))dF_{t+1}(\epsilon_{t+1}). \tag{29}$$

$$= \max_{y_t\in\Gamma_t(s_t,a_t,b_t,d_t;\bar{b}_{t+1})} U_t(s_t,b_t,d_t,y_t) + \beta_t \int V_{t+1}(\Delta^S_{d_t,\bar{b}_{t+1}}(s_t,\epsilon_{t+1}),\Delta^A_{d_t,\bar{b}_{t+1}}(s_t,a_t,b_t,y_t,\epsilon_{t+1}))dF_{t+1}(\epsilon_{t+1}).$$

The terminal value is defined as:

$$v_t(s_T,a_T,b_T,d_T) = \max_{y_T\in\Gamma_T(s_T,a_T,b_T,d_T)} U_t(s_T,b_T,d_T,y_T). \tag{30}$$

We define the policy function $\Psi_t(s_t,a_t,b_t,d_t;\bar{b}_{t+1})$ as follows:

$$\Psi_t(s_t,a_t,b_t,d_t;\bar{b}_{t+1}) \tag{31}$$

$$= \underset{y_t\in\Gamma_t(s_t,a_t,b_t,d_t;\bar{b}_{t+1})}{\operatorname{argmax}} U_t(s_t,b_t,d_t,y_t) + \beta_t \int V_{t+1}(\Delta^S_{d_t,\bar{b}_{t+1}}(s_t,\epsilon_{t+1}),\Delta^A_{d_t,\bar{b}_{t+1}}(s_t,a_t,b_t,y_t,\epsilon_{t+1}))dF_{t+1}(\epsilon_{t+1}).$$

If a model is included in a subset of the general class of problems described above, then the EGM is applicable when the model satisfies the conditions described in the next subsection.

5.2 Conditions for the EGM

The EGM works only when the FOC exists and is well defined as a necessary condition for the global solutions. This holds when condition C1 is satisfied.

- **C1** : $E[V_{t+1}(s_{t+1},a_{t+1},\bar{b}_{t+1})]$ is locally differentiable with respect to the optimal choice of a_{t+1}. In addition, $a_{t+1} = \Delta^A_{d_t,\bar{b}_{t+1}}(s_t,a_t,b_t,y_t,\epsilon_{t+1})$ is locally differentiable with respect to the optimal choice of y_t.

Researchers can check whether this condition holds in a specific problem by using the "Reverse Calculus" method from Clausen and Strub (2020). They provide useful instruments for checking the local differentiability applicable to models with discrete choices and default options.

Assuming C1 is satisfied, the following FOC will then be a necessary but not sufficient condition for optimal consumption.

$$\frac{\partial U_t(s_t, b_t, d_t, y_t)}{\partial y_t} = \tag{32}$$
$$-\beta_t \int \left[\frac{\partial V_{t+1}(\Delta^S_{d_t,\bar{b}_{t+1}}(s_t, \epsilon_{t+1}), \Delta^A_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t, \epsilon_{t+1}))}{\partial a_{t+1}} \cdot \frac{\partial \Delta^A_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t, \epsilon_{t+1}))}{\partial y_t} \right] dF_{t+1}(\epsilon_{t+1}).$$

Since U_t is differentiable, $\frac{\partial U_t(s_t,b_t,d_t,y_t)}{\partial y_t}$ is well defined. Note that the FOC (32) is not sufficient but necessary for the global solution of a_{t+1} because EV_{t+1} might not be strictly concave due to the default options and other discrete choices.

Let us define $Z_t \subset \mathbb{R}$ as the set of post-decision endogenous state. $z_t \in Z_t$ is the intermediate state after agents have chosen and executed their choices y_t but before the transition shock ϵ_{t+1} arises. For example, in Section 3, the post-decision endogenous state z_t is the size of debt $q(S, a^{'}) \cdot a^{'}$ because it is determined after consumption c is chosen, but before the transition shock $\epsilon^{'}$ is realized. This post-decision endogenous state enables us to decompose the endogenous state transition function $\Delta^A_{d_t,\bar{b}_{t+1}}(\cdot)$ into intra- and inter-period components. This decomposition is important in employing the EGM because it requires the use of the FOC (32) represented by the post-decision endogenous state z_t. Formally, we decompose the endogenous state transition function $\Delta^A_{d_t,\bar{b}_{t+1}}(\cdot)$ as follows:

- **C2** : For each d_t and for each $\bar{b}_{t+1}$, there exist functions $\Xi_{d_t,\bar{b}_{t+1}} : S_t \times A_t \times B_t \times Y_t \longrightarrow \mathbb{R}$ and $\chi_{d_t,\bar{b}_{t+1}} : S_t \times Z_t \times B_t \times E_{t+1} \longrightarrow A_{t+1}$ such that $\Delta^A_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t, \epsilon_{t+1}) = \chi_{d_t,\bar{b}_{t+1}}(s_t, \Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t), b_t, \epsilon_{t+1})$ for all $(s_t, a_t, b_t, y_t) \in W_t \times A_t \times B_t \times Y_t$ when $y_t \in \Gamma_t(s_t, a_t, b_t, d_t; \bar{b}_{t+1})$.

This condition implies that for each d_t and $\bar{b}_{t+1}$, through the intra-period transition function $\Xi_{d_t,\bar{b}_{t+1}}$, the states (s_t, a_t, b_t) and choice y_t generate a post-decision state z_t. In Section 3, recall that $z_t = q(S, a^{'})a^{'}$ and $a_{t+1} = a^{'}$. Our EGM requires z_t to act as a sufficient statistic for the endogenous state a_{t+1}. This implies that there must exist a unique one-to-one mapping between $z_t = q(S, a^{'})a^{'}$ and $a_{t+1} = a^{'}$.

To ensure that this mapping exists, we first need to ensure that $q(S, a^{'})a^{'}$ should increase with $a^{'}$ (so that the mapping is one-to-one). Second, $z_t = q(S, a^{'}) \cdot a^{'}$ needs to be independent of the value of $\epsilon^{'}$, so that the decomposition of $a^{'}$ out of $z_t = q(S, a^{'}) \cdot a^{'}$ is unique.[25] We generalize these conditions as follows:

[25] In Section 3, $q(S, a^{'})$ is not a function of shocks $\epsilon^{'}$ (or $S^{'}$) but a function of the conditional mean of defaulting, $\delta(S, a^{'}) = \sum_{\{S^{'}:V^c(S^{'},a^{'})<V^d(S^{'})\}} \pi_{S,S^{'}}$. Note that the default set $\{S^{'} : V^c(S^{'}, a^{'}) < V^d(S^{'})\}$ can be different across default risk models because it depends on their default rules. What matters is that $q(S, a^{'})$ is affected not by a

- **C3** : For each s_t, d_t and $\bar{b}_{t+1}$, a_{t+1} is independent of ϵ_{t+1}, but dependent on the conditional expectation of the default decision $E_t(D(\epsilon_{t+1}))$ that is formed in the current period t. i.e, $a_{t+1} = \chi_{d_t,\bar{b}_{t+1}}(s_t, z_t, b_t, E_t(D(\epsilon_{t+1})))$.

- **C4** : For each s_t, d_t and $\bar{b}_{t+1}$, $a_{t+1} = \chi_{d_t,\bar{b}_{t+1}}(s_t, z_t, b_t, E_t(D(\epsilon_{t+1})))$ monotonically increases with z_t; therefore, there exists a function $\chi^{-1}_{d_t,\bar{b}_{t+1}}(\cdot)$ where $z_t = \chi^{-1}_{d_t,\bar{b}_{t+1}}(s_t, a_{t+1}, b_t, E_t(D(\epsilon_{t+1})))$ and $\chi^{-1}_{d_t,\bar{b}_{t+1}}(\cdot)$ is increasing in a_{t+1}.

C3 implies that $\chi_{d_t,\bar{b}_{t+1}}(s_t, z_t, b_t, E_t(D(\epsilon_{t+1})))$ is affected not by a future value of ϵ_{t+1} but by $E_t(D(\epsilon_{t+1}))$ that is formed in the current period t. **C4** is a generalized version of the risky borrowing limit. In Section 3, on the region of assets greater than the risky borrowing limit, $z_t = q(S, a^{'})a^{'}$ is monotonic increasing in $a_{t+1} = a^{'}$. Note that $z_t = \chi^{-1}_{d_t,\bar{b}_{t+1}}(s_t, a_{t+1}, b_t, E_t(D(\epsilon_{t+1})))$ is independent of the current continuous state a_t. This feature allows us to predetermine the lower bound of the feasible set for the solution of a_{t+1} at the initial step, thereby insulating the interactions between the step of searching for the lower bound and that of computing the endogenously determined current state a_t in our EGM.

C2, **C3**, and **C4** do not guarantee that the solution to the problem will be contained in the feasibility set implied by Γ_t. To make the post-decision consistent with the feasible set such as budget constraint, we need the following condition:

- **C5** : Let d_t and $\bar{b}_{t+1}$ be given. For all $y_t \in Y_t$ and $(s_t, a_t, b_t) \in S_t \times A_t \times B_t$, $y_t \in \Gamma_t(s_t, a_t, b_t, d_t; \bar{b}_{t+1})$ if and only if $\Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t) \in Z_t$ where $Z_t = \cup_{(s_t,a_t,b_t)\in S_t\times A_t\times B_t}\Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, \Gamma_t(s_t, a_t, b_t, d_t; \bar{b}_{t+1}))$.

C5 means that the post-decision endogenous state z_t is a sufficient statistic for evaluating the feasibility of choice y_t. Additionally, this condition implies that Z_t spans the entire space of possible outcome. In Section 3, this condition implies that given S and a, knowing $z = q(S, a^{'})a^{'}$ is equivalent to knowing c because they satisfy the budget constraint, $c + z = c + q(S, a^{'})a^{'} = S + a$.

Unlike the transitional functions in White (2015), the transitional functions might not be differentiable because of the inclusion of discrete decision variables. We need a condition to ensure the local differentiability of these functions.

- **C6** : For each d_t and $\bar{b}_{t+1}$, (1) $z_t = \Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t)$ is locally differentiable with respect to the optimal choices of y_t, and (2) $a_{t+1} = \chi_{d_t,\bar{b}_{t+1}}(s_t, z_t, b_t, E_t[D(\epsilon_{t+1})])$ is locally differentiable with respect to the optimal post-decision state of z_t.

specific value of $S^{'}$ but by a function of its conditional expectation $\sum_{\{S^{'}:V^c(S^{'},a^{'})<V^d(S^{'})\}} \pi_{S,S^{'}}$ that is formed in the current period.

In Arellano's (2008) model, (1) of $\mathbf{C6}$ implies that $z_t = q(S, a^{'}) \cdot a^{'}$ is differentiable with respect to c. This condition is satisfied for any S and $a^{'}$ as $\frac{\partial q(S,a^{'})\cdot a^{'}}{\partial c} = -1$. (2) of $\mathbf{C6}$ implies that $a^{'}$ is differentiable with respect to $z = q(S, a^{'})a^{'}$. The differentiability of optimal next-period asset holdings $a^{'}$ with regard to z implies that $q(S, a^{'})$ is differentiable with regard to $a^{'}$ because $\frac{\partial a^{'}}{\partial z} = [\frac{\partial z}{\partial a^{'}}]^{-1} = 1/(D_{a^{'}} q(S, a^{'})a^{'} + q(S, a^{'}))$. As with $\mathbf{C1}$, a researcher can check whether their setting has this property using Clausen and Strub's (2020) reverse calculus technique.

With $\mathbf{C1}, \mathbf{C2}, \mathbf{C3}, \mathbf{C4}, \mathbf{C5}$, and $\mathbf{C6}$, for each $(d_t, \bar{b}_{t+1}) \in D_t \times B_{t+1}$, the FOC (32) can be rewritten as follows:

$$\begin{aligned} &\frac{\partial U_t(s_t, b_t, d_t, y_t)}{\partial y_t} \\ &= -\beta_t \int \left[\frac{\partial V_{t+1}(\Delta^S_{d_t,\bar{b}_{t+1}}(s_t, \epsilon_{t+1}), \chi_{d_t,\bar{b}_{t+1}}(s_t, \Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t), b_t, E_t[D(\epsilon_{t+1})]))}{\partial a_{t+1}} \right. \\ &\left. \cdot \frac{\partial \chi_{d_t,\bar{b}_{t+1}}(s_t, \Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t), b_t, E_t[D(\epsilon_{t+1})]))}{\partial z_t} \cdot \frac{\partial \Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t)}{\partial y_t} \right] dF_{t+1}(\epsilon_{t+1}) \\ &= -\beta_t \int \left[\frac{\partial V_{t+1}(\Delta^S_{d_t,\bar{b}_{t+1}}(s_t, \epsilon_{t+1}), a_{t+1})}{\partial a_{t+1}} \right] dF_{t+1}(\epsilon_{t+1}) \cdot \frac{\partial \chi_{d_t,\bar{b}_{t+1}}(s_t, z_t, b_t, E_t[D(\epsilon_{t+1})]))}{\partial z_t} \\ &\cdot \frac{\partial \Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t)}{\partial y_t}, \end{aligned} \tag{33}$$

where $\chi_{d_t,\bar{b}_{t+1}}(s_t, \Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t), b_t, E_t[D(\epsilon_{t+1})]) = a_{t+1}$ and $\Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, y_t) = z_t$. Note that $\frac{\partial a_{t+1}}{\partial z_t} = \frac{\partial \chi_{d_t,\bar{b}_{t+1}}(s_t,z_t,b_t,E_t[D(\epsilon_{t+1})])}{\partial z_t}$ and $\frac{\partial z_t}{\partial y_t} = \frac{\partial \Xi_{d_t,\bar{b}_{t+1}}(s_t,a_t,b_t,y_t)}{\partial y_t}$ can be pulled out of the integral because they are independent of ϵ_{t+1}. Recall that $\mathbf{C4}$ and $\mathbf{C6}$ imply $\frac{\partial a_{t+1}}{\partial z_t} > 0$. Furthermore, we assume the following condition:

- $\mathbf{C7}$: For each $(d_t, \bar{b}_{t+1}) \in D_t \times B_{t+1}$, $\frac{\partial z_t}{\partial y_t} = \frac{\partial \Xi_{d_t,\bar{b}_{t+1}}(s_t,a_t,b_t,y_t)}{\partial y_t} = K \in \mathbb{R} \setminus \{0\}$, and $sgn(\frac{\partial \Xi_{d_t,\bar{b}_{t+1}}(s_t,a_t,b_t,y_t)}{\partial y_t}) = -sgn(\frac{\partial U_t(s_t,a_t,b_t,d_t,y_t)}{\partial y_t})$.

$\mathbf{C7}$ implies that given a budget set, a change in y_t must be accompanied by a change in z_t of the opposite sign. Furthermore, this change is a constant proportion such that the tradeoff between y_t and z_t is linear. This condition is a general feature of dynamic problems in economics. For example, in the model of Arellano (2008), $\mathbf{C7}$ implies that $\frac{\partial q(S,a^{'})\cdot a^{'}}{\partial c} = -1 < 0$; and thus, $sgn(\frac{\partial q(S,a^{'})\cdot a^{'}}{\partial c}) = -sgn(u'(c)) < 0$. In the neoclassical growth model, this condition means $\frac{\partial k_{t+1}}{\partial c_t} = -1$ in the budget constraint $c_t + k_{t+1} = (1 + r_t)k_t + w_t l_t$.

With **C1** to **C7**, we can rearrange the previous FOC as follows:

$$\frac{\partial U_t(s_t, b_t, d_t, y_t)}{\partial y_t} \cdot \frac{1}{K} \tag{34}$$
$$= -\left[\frac{\partial \chi^{-1}_{d_t,\bar{b}_{t+1}}(s_t, a_{t+1}, b_t, E_t[D(\epsilon_{t+1})]))}{\partial a_{t+1}}\right]^{-1} \cdot \beta_t \int \left[\frac{\partial V_{t+1}(\Delta^S_{d_t,\bar{b}_{t+1}}(s_t, \epsilon_{t+1}), a_{t+1})}{\partial a_{t+1}}\right] dF_{t+1}(\epsilon_{t+1}),$$

where $\chi^{-1}_{d_t,\bar{b}_{t+1}}(s_t, a_{t+1}, b_t, E_t[D(\epsilon_{t+1})])) = z_t$. Note that $\left[\frac{\partial \chi^{-1}_{d_t,\bar{b}_{t+1}}(s_t,a_{t+1},b_t,E_t[D(\epsilon_{t+1})]))}{\partial a_{t+1}}\right]^{-1}$ is well defined because of **C4** and (2) of **C6**. Now, a_t and z_t disappear in the state vector.

We need a monotonicity of a_{t+1} with respect to a_t to use a spline to approximate the decision rule of a_{t+1} over the exogenous grid. Condition **C8** guarantees this monotonicity.

- **C8** : For each $(d_t, \bar{b}_{t+1}) \in D_t \times B_{t+1}$, at the policy function $\Psi_t(s_t, a_t, b_t, d_t; \bar{b}_{t+1})$, $z_t = \Xi_{d_t,\bar{b}_{t+1}}(s_t, a_t, b_t, \Psi_t(s_t, a_t, b_t, d_t; \bar{b}_{t+1}))$ is non-decreasing in a_t.

In Arellano's (2008) model, **C8** implies that $z_t = q(s, a^{'})a^{'} = q(s, g_a(S, a)) \cdot g_a(S, a)$ is non-decreasing in a. Since **C4** implies that $q(s, a^{'})a^{'}$ is non-decreasing in $a^{'}$, these two conditions indicate that $a^{'} = g_a(S, a)$ is weakly monotonic increasing in a. This monotonicity must be required to use splines.

Let us denote $\hat{g}_{d_t,\bar{b}_{t+1}}(s_t, b_t, y_t)$ and $\hat{V}_{d_t,\bar{b}_{t+1}}(s_t, b_t, a_{t+1})$ as follows:

$$\hat{g}_{d_t,\bar{b}_{t+1}}(s_t, b_t, y_t) = \frac{\partial U_t(s_t, b_t, d_t, y_t)}{\partial y_t} \cdot \frac{1}{K} \tag{35}$$

$$\hat{V}_{d_t,\bar{b}_{t+1}}(s_t, b_t, a_{t+1})$$
$$= -\left[\frac{\partial \chi^{-1}_{d_t,\bar{b}_{t+1}}(s_t, a_{t+1}, b_t, E_t[D(\epsilon_{t+1})]))}{\partial a_{t+1}}\right]^{-1} \cdot \beta_t \int \left[\frac{\partial V_{t+1}(\Delta^S_{d_t,\bar{b}_{t+1}}(s_t, \epsilon_{t+1}), a_{t+1})}{\partial a_{t+1}}\right] dF_{t+1}(\epsilon_{t+1}). \tag{36}$$

In Arellano's (2008) model, $K = \left[\frac{\partial \Xi_{d_t,\bar{b}_{t+1}}(s_t,a_t,b_t,y_t)}{\partial y_t}\right]^{-1} = -1$ and $[\frac{\partial z_t}{\partial a_{t+1}}]^{-1} = \frac{1}{D_{a^{'}} q(S,a^{'})a^{'} + q(S,a^{'})}$. Therefore, $\hat{g}_{d_t,\bar{b}_{t+1}}(s_t, b_t, y_t) = \hat{V}_{d_t,\bar{b}_{t+1}}(s_t, b_t, a_{t+1})$ is consistent with the FOC (12) in Arellano's (2008) model.

Note that for each $(d_t, \bar{b}_{t+1}) \in D_t \times B_{t+1}$, the EGM retrieves the endogenously driven choice variable y_t from the given decision states (s_t, b_t, a_{t+1}). Next, the EGM finds the endogenously driven current state a_t by using the retrieved $y_t(s_t, b_t, a_{t+1})$ with information on the budget set $\Gamma_t(\cdot)$. As a result, for each $(d_t, \bar{b}_{t+1}) \in D_t \times B_{t+1}$ and given (s_t, b_t, a_{t+1}), the EGM solves the following system:

$$\hat{g}_{d_t,\bar{b}_{t+1}}(s_t, b_t, y_t(s_t, b_t, a_{t+1})) = \hat{V}_{d_t,\bar{b}_{t+1}}(s_t, b_t, a_{t+1}), \tag{37}$$

where

$$\hat{g}_{d_t,\bar{b}_{t+1}}(s_t, b_t, y_t(s_t, b_t, a_{t+1})) = \left.\frac{\partial U_t(s_t, b_t, d_t, y_t)}{\partial y_t}\right|_{y_t=y_t(s_t,b_t,a_{t+1})} \cdot \frac{1}{K}$$

$$\hat{V}_{d_t,\bar{b}_{t+1}}(s_t, b_t, a_{t+1})$$

$$= -\left[\frac{\partial \chi^{-1}_{d_t,\bar{b}_{t+1}}(s_t, a_{t+1}, b_t, E_t[D(\epsilon_{t+1})]))}{\partial a_{t+1}}\right]^{-1} \cdot \int \left[\frac{\partial V_{t+1}(\Delta^S_{d_t,\bar{b}_{t+1}}(s_t, \epsilon_{t+1}), a_{t+1})}{\partial a_{t+1}}\right] dF_{t+1}(\epsilon_{t+1}).$$

It is worth noting that there is no max operator, thereby leading to sufficient improvements in computational efficiency.

Let us explain notations for the procedure of the implementation. G_{a_t} is the exogenous grid of a_t and $G_{a_{t+1}}$ is the exogenous grid of a_{t+1}.

5.3 Implementation

The EGM is applicable if a problem satisfies **C1** $-$ **C8**. The method follows a seven-step procedure. Initially, let us begin with $t = T$ and $E[V_{T+1}] = 0$.

1. In each period t, let the expected value function $E_t[V_{t+1}(s_{t+1}, a_{t+1}, \bar{b}_{t+1})]$ and the conditional mean of defaulting $E_t[D(\epsilon_{t+1})]$ be given. For each s_t, b_t, d_t, and $\bar{b}_{t+1}$, characterize an interval of $a_{t+1} \in G_{a_{t+1}}$ satisfying **C4**; find the minimum; and save it as $a^{rbl}_{d_t,\bar{b}_{t+1}}(s_t, b_t, E_t[D(\epsilon_{t+1})])$ (generalized risky borrowing limit). Going forward, take the states $(d_t, \bar{b}_{t+1}) \in D_t \times B_{t+1}$ and $(s_t, b_t) \in S_t \times B_t$ as given to make the notations simple.
2. For each $a_{t+1} \in G_{a_{t+1}}$ with $a_{t+1} > a^{rbl}_{d_t,\bar{b}_{t+1}}(s_t, b_t, E_t[D(\epsilon_{t+1})])$, compute the endogenously driven $y_t(s_t, b_t, a_{t+1})$ by solving $\hat{g}_{d_t,\bar{b}_{t+1}}(s_t, b_t, y_t(s_t, b_t, a_{t+1})) = \hat{V}_{d_t,\bar{b}_{t+1}}(s_t, b_t, a_{t+1})$ (Equation (37)). Then, save the pairs of $(y_t(s_t, b_t, a_{t+1}), b_t, a_{t+1})$.
3. For each $a_{t+1} \in G_{a_{t+1}}$ with $a_{t+1} > a^{rbl}_{d_t,\bar{b}_{t+1}}(s_t, b_t, E_t[D(\epsilon_{t+1})])$, use the algorithm of Fella (2014) to refine the global solution out of the candidates from the previous step. Save the refined pairs of $(y_t(s_t, b_t, a_{t+1}), b_t, a_{t+1})$ for the global solutions.
4. For the refined pairs of $(y_t(s_t, b_t, a_{t+1}), b_t, a_{t+1})$, retrieve the corresponding endogenously driven current state $a_t(s_t, b_t, a_{t+1})$. To do so, first, compute $z_t = \chi^{-1}_{d_t,\bar{b}_{t+1}}(s_t, a_{t+1}, b_t, E_t[D(\epsilon_{t+1})]))$. Then, use z_t, $y_t(s_t, b_t, a_{t+1})$, and the budget set Γ_t to find

$a_t(s_t, b_t, a_{t+1})$ that satisfies $\Gamma_t(s_t, a_t(s_t, b_t, a_{t+1}), b_t, d_t; b_{t+1})$. This search is possible due to $\mathbf{C5}$. Now, $a_t(s_t, b_t, a_{t+1})$ lies in the endogenously determined grid points of a_t. Save the pairs of $(a_t(s_t, b_t, a_{t+1}), b_t, a_{t+1})$. Note that these pairs correspond to the global solutions.

5. For the saved pairs of $(a_t(s_t, b_t, a_{t+1}), b_t, a_{t+1})$, approximate the decision rule $a_{t+1} = g_a(s_t, \cdot, b_t, d_t, b_{t+1})$ over the exogenous grid of a_t, G_{a_t}. We can do this using a spline due to the one-to-one mapping between a_{t+1} and a_t ($\mathbf{C8}$). Then, using the approximated decision rule $a_{t+1} = g_a(s_t, a_t, b_t, d_t, b_{t+1})$, compute the value function $v_t(s_t, a_t, b_t, d_t; \bar{b}_{t+1})$ over G_{a_t}.
6. Solve $\max_{\{(d_t, \bar{b}_{t+1}) \in D_t \times B_{t+1}\}} v_t(s_t, a_t, b_t, d_t; \bar{b}_{t+1})$ and update $V_t(s_t, a_t, b_t)$.
7. Compute $E_{t-1}[V_t(s_t, a_t, b_t)]$ and $E_{t-1}[D(\epsilon_t)]$. Take $t = t - 1$; go back to Step 1 until $t = 0$.

5.4 Discussion

It is worth discussing how restrictive the sufficient conditions are. If these conditions were too restrictive to address many types of default risk models, our method might not be more useful than the conventional approach. Since $\mathbf{C5}$ to $\mathbf{C8}$ are widely shared features in general dynamic problems in macroeconomics, we focus on $\mathbf{C1}$ to $\mathbf{C4}$.

$\mathbf{C1}$ is about the local differentiability of the expected value function $E_t[V_{t+1}]$ with respect to the optimal decision of a_{t+1} and that of the decision rule a_{t+1} with respect to the optimal choice of y_t. It might be hard to claim that all default risk models satisfy $\mathbf{C1}$; however, Clausen and Strub (2020) have shown that this feature is prevalent in many types of dynamic problems. More importantly, we can, at least, certainly check whether this condition holds to a specific problem with default risks by using the "Reverse Calculus" in Clausen and Strub (2020).

$\mathbf{C2}$, $\mathbf{C3}$, and $\mathbf{C4}$ imply that there exists a post-decision state z_t that is a sufficient statistic for the future endogenous state a_{t+1}. According to White (2015), this property of sufficient statistics must be required to use EGMs. To do so, in addition to $\mathbf{C2}$, our EGM requires more conditions than the EGM of White (2015) because in default risk models, the post-decision state z_t is non-linearly related to the future endogenous state a_{t+1} (i.e., $z = q(s, a^{'})a^{'}$). This additional feature brings $\mathbf{C3}$ and $\mathbf{C4}$. $\mathbf{C3}$ implies that the EGM might not work in default risk models with shocks on assets or investment, in which z_t may not uniquely identify a_{t+1} because the realization of shocks ϵ_{t+1} may affect assets a_{t+1} (i.e., Glover and Shorts (2010)). Shocks on defaultable assets, however, are not a commonly used assumption in the literature.

$\mathbf{C4}$ implies that the EGM might not be applicable to default risk models where the generalized risky borrowing limit is not well defined or depends on endogenous states other than $(s_t, z_t, b_t, E_t(D(\epsilon_{t+1})))$. We argue that the first issue might not be problematic, but the second issue does impose some limitations on our method. The first issue implies that either the borrowing constraint is unbounded, or utility is decreasing in the amount of debt anytime a_{t+1} is negative. An unbounded borrowing limit allows for Ponzi schemes, which most researchers prefer to rule out. For the case of utility decreasing in debt for the whole borrowing region, the generalized borrowing limit is well defined at zero-assets, $a_{t+1} = 0$, if the return on savings $(a_{t+1} > 0)$ is independent of the individual choice of a_{t+1}. For example, in Arellano (2008), $a^{'} = 0$ is the upper bound of the feasible set for the risk-borrowing limit because $\partial[q(S, a^{'}) \cdot a^{'}]/\partial a^{'} = q(S, a^{'}) = 1/(1 + r_f) > 0$ if $a^{'} \geq 0$. This assumption is quite common in the literature.

Note that, however, when the generalized risky borrowing limit is a function of endogenous states other than $(s_t, z_t, b_t, E_t(D(\epsilon_{t+1})))$, it is uncertain whether our method works. For example, when there exists asymmetric information between lenders and borrowers, the price function might depend on the distribution over agents (i.e., Athreya et al. (2012)). In this type of models, we cannot make sure whether the generalized risky borrowing limit can be predetermined with the states, $(s_t, z_t, b_t, E_t(D(\epsilon_{t+1})))$. In this case, the EGM might not be applicable. To address this issue, one might need to include the additional endogenous states in the generalized risky borrowing limit. This inclusion, however, might dampen the efficiency gain of our method.

One might wonder whether our method is applicable to the sovereign default models that emphasize the role of coordination failures such as Cole and Kehoe (2000). In these models, it is common to assume that the timing of the events in the debt market is as follows:

1. Shocks are realized
2. Government chooses a'
3. Lenders choose price for bonds
4. Government decides whether to default.

A government has the option to default after observing the outcome of the current period's bond auction, which depends on lenders' belief on its default probability. We believe our methods can solve the decision problems as long as the government is aware of lenders' belief as well as their bond price schedule. If a government knows that lenders believe that it will repay and hence offer a relevant price schedule, we can find the risky borrowing limit and can compute the rest of the steps. On the other hand, if lenders believe that the government will default and hence offer a 100% discount rate on the bond, the risky borrowing limit becomes zero. Therefore, when our threshold approach for the default probability is employed in the model, **C1** would be satisfied. **C2-C4** would be satisfied because the structure of lending and borrowing is the same as that in Arelleno when the lender's belief indicates a repayment.

It is worth mentioning default models with long-term debt, as more papers feature long-term debt.[26] We believe that our method can solve the baseline environment as in Chatterjee and Eyigungor (2012). It is required to show that the price function is locally differentiable at optimal

[26]For example, Chatterjee and Eyigungor (2012), Hatchondo et al. (2016), Bocola and Dovis (2019).

choices (**C1** and (2) of **C6**). Given that Chatterjee and Eyigungor (2012) show that there exists an equilibrium price function that is increasing in a' (**C4**), it is possible to find a one-to-one mapping between post-decision state $(q(s, a')a')$ and future endogenous state (a') (**C2**). Since $a^{'}$ is determined in the current period in these types of models, too (**C3**), our EGM might work with long-term debt models.[27]

Also, our method can be used for life-cycle consumer bankruptcy models (i.e., Athreya (2008); Athreya et al. (2009); Livshits et al. (2007, 2010); Gordon (2015)) since the EGM solves the problem in a backward direction. These life-cycle models cannot be solved with the envelope condition method of Arellano et al. (2016) because it is a forward-solving algorithm. Additionally, the EGM can address multiple options to default (i.e., Chapter 7 and Chapter 13 in Consumer bankruptcy) with other discrete choices (i.e., housing, durable goods, health insurance, and retirement) because these models satisfy **C1** to **C8**. This versatility might be useful in investigating the interaction between default and other types of policies related to these discrete choices. Jang (2020), for example, used the EGM to solve a life-cycle model that examines the role of consumer bankruptcy in designing optimal health insurance policies.

Overall, the sufficient conditions imply that our EGM can cover a broad class of default risk models.

6 Conclusion

We presented an extension of the endogenous grid method for default risk models. This method combines Fella's (2014) endogenous grid method by introducing a numerical step to search for the risky borrowing limit, which is the lower bound of the feasible set for the solution of asset holdings. By using the algorithm of Fella (2014) and our step for the risky borrowing limit, we identified the region of solution sets to which Carroll's (2006) endogenous grid method is applicable. Compared to the conventional grid search method, the method brings substantial improvements in computational efficiency and accuracy. We further showed that our EGM is applicable to a broad class of default risk models by providing sufficient conditions for the application. We hope that this method opens up possibilities for researchers to investigate topics with default options that have previously been left unexplored due to computational complexity.

[27] Chatterjee and Eyigungor (2012) mention that the continuous transitory income shock m is crucial for computation. We think that our method can achieve convergence without m as long as the persistent component of income is continuous. We guess that m is necessary in Chatterjee and Eyigungor (2012) because they assume that the persistent component of income is discrete. Here, even though we solve the model on the discretized grids, we compute the cutoff of the income shock given S and S' to get continuous default probability over the asset. This is consistent with Chatterjee and Eyigungor (2012)'s insight; continuous shock is important on computational stability in a model with a discrete choice.

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A Additional Results

We apply our algorithm to solve Nakajima and Ríos-Rull (2014), which is computationally heavier than Arellano (2008). We compare the computing time and accuracy of our EGM with those of the grid search method as well. For details of the model and parameterization, see Nakajima and Ríos-Rull (2014). As in Nakajima and Ríos-Rull (2014), we use Krusell and Smith's (1998) method to handle the aggregate uncertainty. Note that this method approximates aggregate states using a few moments, and agents expect next-period states using parameterized functional forms of those moments. The method achieves high accuracy, but it requires a long simulation to update forecasting rules and may take many trials to find a proper functional form.

A.1 Specification of Krusell and Smith's (1998) Method

Nakajima and Ríos-Rull (2014) approximated $(z, K; m)$ with (z, K, O), where z is total factor productivity, K is aggregate capital, m is household distribution, and O is average individual labor productivity. They use forecasting rules for K', L, r, and O', where L is aggregate labor and r is risk-free rate. Here, we abstract from the counter-cyclical earnings risk and approximate aggregate states $(z, K; m)$ to (z, K). Additionally, instead of forecasting L, which is necessary to calculate the wage w, we forecast the wage directly. We specify the forecasting functions for K', r, and w as the following log-linear forms:

$$log\, K' = \phi_{k1}(z, K) + \phi_{k2}(z, K) \cdot log\, K$$

$$log\, r = \phi_{r1}(z, K) + \phi_{r2}(z, K) \cdot log\, K$$

$$log\, w = \phi_{w1}(z, K) + \phi_{w2}(z, K) \cdot log\, K$$

A.2 Computing Time and Accuracy

We vary the size of the grid for assets across computational exercises. In all computational exercises, we keep the number of the grid points for the other variables as follows. The size of the grid for the permanent labor productivity shock is 2, that for the persistent shock is 15, and that for the transitory shock is 3. The number of the grid for the TFP shock is 3, and that for K is 5. Because we use Krusell and Smith's (1998) method, we must go through the inner and outer loops several times until the forecasting rules converge. We compute the average CPU time per iteration in the inner loop and outer loop, respectively. We simulated the model for 2,000 periods with Krusell and Smith's (1998) method, and all computations were carried out on on a single core of an Intel i7-4770 processor. The programs were written in Fortran 95.

Table 5: Computing Time

# of GRD. PTS. for INR. - OTR.	200-500		300-500		400-500		500-600	
Computational Method	EGM	GS	EGM	GS	EGM	GS	EGM	GS
AVG CPU Time in INR. per ITER.*	0.68	12.54	1.25	29.27	1.99	54.39	2.99	79.65
AVG CPU Time in OTR. per ITER.*	29.49	173.49	27.05	185.62	24.48	182.79	38.66	286.97

.$*$: Unit = minute.
'# of GRD. PTS. for INR. - OTR.' refers to the number of grid points for assets in the inner loops and in the outer loops, respectively. Note that we re-solve the decision rules on the finer grids in the outer loops.

Table 5 indicates that the EGM is faster than the grid search method both in the inner loop and in the outer loop. In the inner loops, the EGM is from 18.5 to 27.3 times faster than the grid search method. In the outer loop, the EGM is approximately 7.5 times faster than the grid search method. The gap differs across the size of the asset grid, but the EGM is much more efficient than the grid search method across all grid settings.

To measure accuracy, we use three criteria in the literature. First, we compute Bellman equation errors (BE error), which is defined the same way in Section 4. Second, we take Den Haan's forecasting test (DH error), described in Algan et al. (2014). It is the difference between expected capital K'_e by the forecasting rules and realized capital K'_r from the simulations: $|logK'_r - logK'_e|$. Finally, we report the R^2 of the forecasting rules in the simulation step.

Figure 11 shows that with the EGM, the price dynamics in the simulation are very close to those generated by the forecasting rules. Since they are very close to one another, it is hard to observe blue lines in the dynamics of the risk-free interest rate and wage. Figure 12 shows that, with the grid search method, there are differences between the simulated dynamics of these prices and those generated by the forecasting rules. DH error measures those differences. Overall, DH errors from the EGM are smaller than those from the grid search method.

Table 6: Computational Accuracy

# of GRD. PTS. for INR. - OTR.	200-500		300-500		400-500		500-600	
Computational Method	EGM	GS	EGM	GS	EGM	GS	EGM	GS
Average of BE error*	0.11%	0.36%	0.06%	0.16%	0.03%	0.09%	0.02%	0.06%
Max of BE error*	10.77%	15.53%	11.34%	11.76%	11.57%	17.49%	11.71%	15.46%
R^2 of $K^{'}$ function	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
R^2 of r function	0.9987	0.9911	0.9971	0.9882	0.9975	0.9963	0.9977	0.9940
R^2 of w function	0.9999	0.9997	0.9999	0.9997	0.9997	0.9997	0.9997	0.9996
Mean of DH error	0.004%	0.01%	0.005%	0.012%	0.009%	0.01%	0.007%	0.01%
Max of DH error	0.029%	0.06%	0.038%	0.081%	0.047%	0.073%	0.04%	0.05%

.$*$: The Bellman equation errors are computed in stationary equilibrium. The number of grid points refers to the number of points for asset grid.

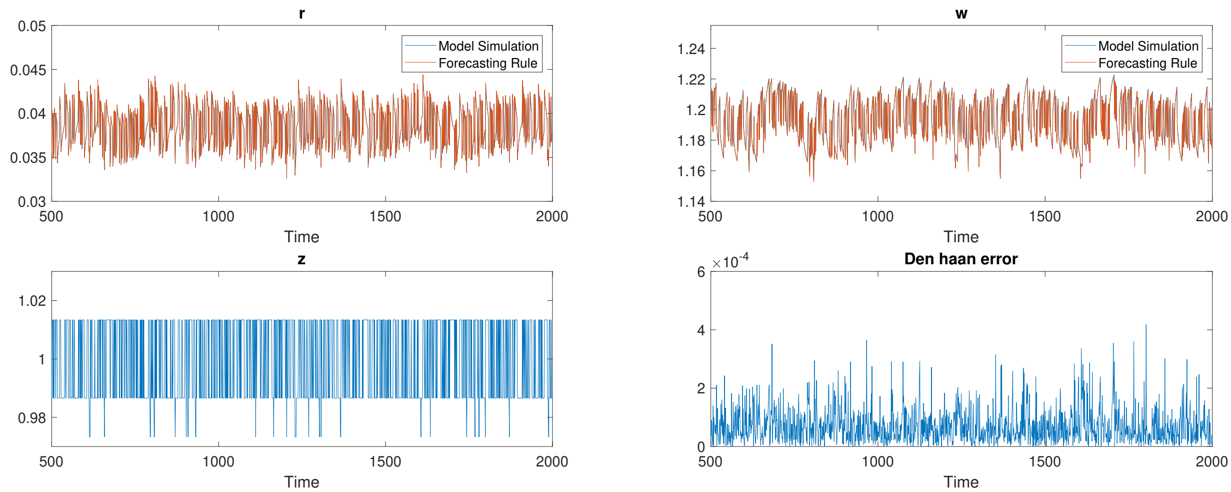

Figure 11: Simulation Results for the EGM with the 500-600 Grid

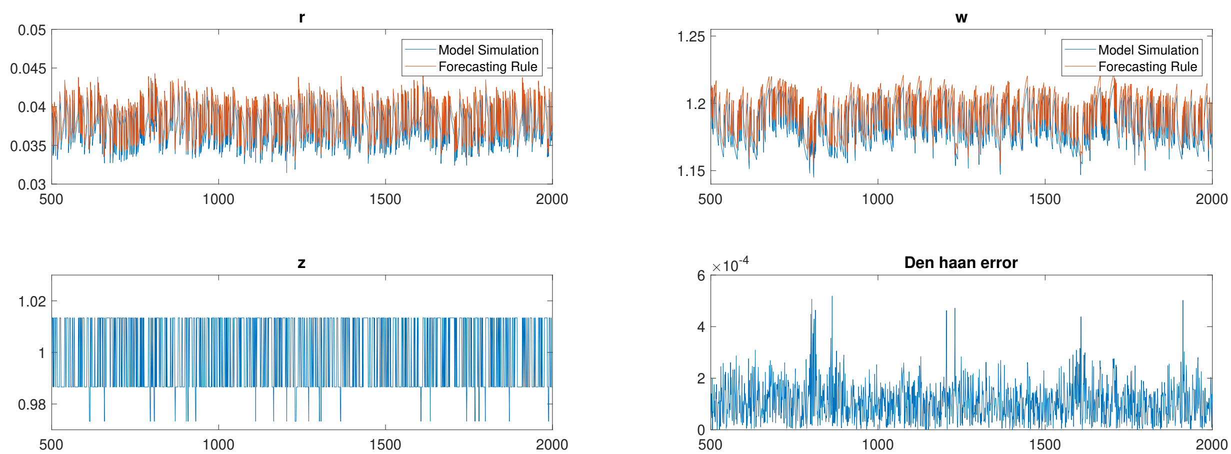

Figure 12: Simulation Results for the Grid Search Method with the 500-600 Grid

Table 6 shows that the EGM produces more accurate outcomes than the grid search method. Regarding the Bellman equation errors, the average Bellman equation errors in the EGM are approximately three times lower than those in the grid search methods. Although the gaps in the maximum Bellman errors are smaller than those in the average Bellman error, the EGM generates smaller values of the maximum Bellman errors than the grid search method. These smaller gaps appear because our EGM also uses the grid search method for the borrowing region. Additionally, the EGM produces higher R^2s of the forecasting functions than the grid search method. Lastly, the average of Den Haan errors and maximum of Den Haan errors from the EGM are lower than those from the grid search method across all grid settings.

B Simulation Results

Table 7: Simulation Results

Computational Method	EGM				EGM (continuous income shock)			
# of Grid Points	200	500	1000	2000	200	500	1000	2000
$\sigma(TB/S)(\%)$	0.95	0.94	0.94	0.97	0.93	0.91	0.91	0.91
$\sigma(r_s)(\%)$	2.06	0.61	0.53	0.48	2.13	0.74	0.65	0.62
$\rho(r_s, S)$	-0.49	-0.61	-0.53	-0.41	-0.50	-0.42	-0.33	-0.29
$\rho(r_s, TB/S)$	0.37	0.59	0.66	0.71	0.42	0.58	0.63	0.64
$mean(r_s)(\%)$	1.13	0.81	0.69	0.62	1.33	0.96	0.86	0.83
$mean(a'/S)$	-3.15	-3.10	-3.10	-3.22	-3.13	-3.08	-3.08	-3.11
# of default per 500,000q	1665	1508	1337	1004	1552	1544	1427	1287

TB: trade balance (output - consumption), S: income, r_s: $\frac{1}{q(S,a')}-$ risk-free rate, σ: standard deviation, ρ: correlation