Disentangle the Florentine Families Network by the Pre-Kernel

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Abstract

For different model settings we conduct power analyses on the Florentine families network of the 15th century while referring to the most popular power indices like the Shapley-Shubik or Banzhaf value as well as to the pre-nucleolus and pre-kernel. In order to assess their capacity to identify the main protagonists that correspond with the chronicles, we inspect of how the power distributions are spread around the mean. Distributions that are clustered to close around the mean cannot identify outstanding positions. In this respect, they failed to provide a scenario that corresponds with the annals. As it turns out, the pre-kernel solution – as a solution concept designed for studying bargaining situations – retrieves the most accurate image for the examined network structures. Last but not least, we discovered two new non-homogeneous weighted majority games with a disconnected pre-kernel.

Keywords: Transferable Utility Game, (Non-)Homogeneous Game, Disconnected Pre-Kernel, Convex Analysis, Fenchel-Moreau Conjugation, Pre-Nucleolus, Shapley-Shubik Index, Banzhaf Value, Deegan-Packel Index, Johnston Index, Public Good Index.

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1 Introduction

During the last years one observed a growing interest across disciplines into the Florentine families network. Representatively, we just want to mention the work of Bozzo et al. (2015) in Computer Science, of Ostoic (2018) in Physics, of Krause and Caimo (2019) in Statistics, as well as Fronzetti Colladon and Naldi (2020) in Engineering. Each of these mentioned treatises uses

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a different approach to study for the late medieval and early Renaissance the mesh of relations among the leading families of the Republic of Florence. In particular, Bozzo et al. (2015) used a seasoned concept in Graph Theory, namely of graph regularizability, to make the connection to a vulnerability measure to quantify the tendency of a set of actors of the network to be the potential victims of some group of executioners. This measure is contrasted with a symmetric power measure that assesses the capacity of a set of actors to play the executioners. As they established, both measure concepts are supermodular functions which allow to determine the solution by suitable polynomial algorithms. These supermodular functions define suitable coalitional games on the node sets of the network that can be studied by the Shapley value, as the authors demonstrated on some small examples. Unfortunately, the authors give no answer of how to apply the Shapley value on the Florentine families network.

Although network analysis is at the heart of game theory, it is quite astonishing that the Florentine families network of the 15th century with business and marriage ties did not find over the recent decades any attention. This is all the more surprising under the consideration that game theory provides a vast box of instruments to investigate such a network structure. Rather to test these solution concepts on real outcomes as, for instance, in the tradition of studying the sharing rules applied in the Talmud (see Aumann and Maschler (1985)), some preferred to get bogged down in wrong logic (cf. Meinhardt (2016a,b, 2017a,b, 2019)). It is the remarkable merit of Holler and Rupp (2020) to attract attention of game theory on this topic. These authors depict by the public good index an approach of how to investigate the marriage and business relations of those 16 families as they were selected by Breiger and Pattison (1986). Guided by a special focus on the House of Medici, they presented rankings on this data set.

In the course of our investigation we resume their approach, but extends it twofold. On the one hand, we conduct a power analysis while referring to the most popular power indices apart of the public good index, that is, the Banzhaf value, Shapley-Shubik, Deegan-Packel or Johnston index, as well as to the pre-nucleolus and pre-kernel. Note that for the class of monotonic directed simple games the latter two solution concepts coincide with the nucleolus and kernel, respectively (cf. Peleg et al. (1994, Cor. 2.7)). In accordance with the fact that we are referring to the associated procedure of determining the pre-nucleolus or pre-kernel, we prefer to use this prefix to mark the distinction On the other hand, we do not only take account of a symmetric weighting of the families, but also of an asymmetric power weighting while considering the net-wealth situation as well as the cumulated number of marriage and business ties across the 116 families from the Padgett's data set. In particular, including the cumulated ties into the game models provide a more accurate scenario of identifying the major protagonists from the annals.

Even under this more accurate scenario, the power indices failed in general to provide an accurate picture that corresponds with the chronicles. This is mainly due that a power index distribution is clustered to close around the mean, which is caused by the fact that they base their rule of distributive justice (axioms) to close on the principle of equality. In such a situation all actors are similar strong and none of them exert potential power over others to influence an outcome of a political decision making process in her/his favor. Under this consideration, it is not justified to classify an actor as outstanding in accordance with the historical accounts. Almost the same is the situation for the pre-nucleolus, though it is mainly based on bargaining considerations, coincides with the Talmudic rule, and is part of the pre-kernel.

Contrasted to these results, the pre-kernel presents the most accurate image. This may surprise many. However, what consider those as a conceptual defect of this solution concept, – namely, that it is in general a set-solution and not single-valued – turns from a disadvantage into an advantage. Since, a set-valued pre-kernel offers a wide range of possible outcomes on which the bilateral claims are balanced among each pair of players. Offering a wide range of political settlements on which parties could agree upon in order to equilibrate their interests. Implying that the pre-kernel solution ought to be best adapted for a milieu where the political institutions – as in the case of the Florentine Republic – were oriented toward equalizing the interests among the parties. May be some consider an exposed part of the pre-kernel as unapologetic or ruthless, but it reflects a snippet of the actual negotiating power of parties. And in this range the most powerful are able to enforce their interests, whereas weaker parties may not point to a claim that goes beyond their outside option, since there are no allies who support their claims. Nevertheless, there may exist also ranges within a pre-kernel solution, where the principle of equality is of higher prominence, putting therefore a higher weight to those allocations that distribute the power closer to the mean. Thus, the pre-kernel covers even a wide variety of fairness considerations, which makes it to an attractive tool to settle divergent interests. As a byproduct of this analysis, we detected two new non-homogeneous weighted majority games with a disconnected pre-kernel. There are only a few known games with a disconnected pre-kernel. To the best of our knowledge, all of these examples originated from the work of Kopelowitz (1967) and Stearns (1968).

This treatise is organized as follows: Section 2 introduces the definitions of the solution concepts and of some game properties. In contrast, Section 3 provides the concepts of the indirect function and presents a dual pre-kernel characterization in terms of solution sets. A historical gloss is presented in Section 4 to classify the results within a historical context. The reader who is pressed for time may skip this section and postpone its reading. Nevertheless, we recommend to be systematic in order not to lose the background of the subject. Section 5 introduces the first network setting while focusing on the marriage ties of the leading Florentine families to conduct a power analysis. We take account of a symmetric as well as an asymmetric setting to control a number of votes within the network. Contrasted to the foregoing section, Section 6 changes the setting while considering an à priori union between two leading families. The power analysis is performed for a symmetric as well as an asymmetric weighting of nodes. The investigation concludes in Section 7 with a study of the business ties among the families by discussion symmetric and asymmetric weighting configurations. We terminate this tract by some final remarks in Section 8 and an Appendix in Section 9 provides the summary of the results in table form.

2 Preliminaries

A n-person cooperative game with side-payments is defined by an ordered pair $\langle N, v\rangle$. The set $N := \{1, 2, \ldots, n\}$ represents the player set and v is the characteristic function with $v : 2^N \rightarrow \mathbb{R}$, and the convention that $v(\emptyset) := 0$. Elements of N are denoted as players. A subset S of the player set N is called a coalition. The real number $v(S) \in \mathbb{R}$ is called the value or worth of a coalition $S \in 2^N$. However, the cardinality of the player set N is given by $n := |N|$, and that for a coalition S by $s := |S|$. We assume throughout that $v(N) = 1$ and $n \geq 2$ is valid. Formally, we identify a cooperative game by the vector $v := (v(S))_{S \subseteq N} \in \mathscr{G}^n = \mathbb{R}^{2^{|N|}}$, if no confusion can arise, whereas in case of ambiguity, we identify a game by $\langle N, v\rangle$.

A TU game $\langle N, v\rangle$ is called to be simple whenever it satisfies

- $v(S) \in \{0, 1\}$ for all $S \subset N$ and $v(N) = 1$.
- $v(S) \leq v(T)$ for all $S \subset T \subseteq N$.

Notice, that a coalition S is called winning if $v(S) = 1$, otherwise losing. The set of all winning coalitions is denoted as $\mathscr{W}$. In this respect, we call a player k as critical, whenever the winning coalition $S \subseteq N$ s.t. $k \in S$ will be turned into a losing coalition whenever player k is removed from the coalition, i.e., if $v(S) = 1$ then $v(S\backslash\{k\}) = 0$. Obviously, all sub-coalitions $T \subseteq S\backslash\{k\}$ are losing. A swing of player k is a pair of coalitions of the form $(S, S\backslash\{k\})$ such that $k \in S \in \mathscr{W}$ and $S\backslash\{k\} \notin \mathscr{W}$ is satisfied. We denote the set of all swings for player $k \in N$ by $\Omega_k(N, v)$. Moreover, a coalition is minimal winning if S is winning and no proper sub-coalition T of S is winning, whereas the set of all minimal winning coalitions is defined by $M_w(N, v) := \{S \subseteq N \,|\, v(S) = 1 \text{ and } v(T) = 0 \text{ for } T \subset S\}$. We realize that a minimal winning coalition is a winning coalition where all players are critical. In the literature, simple games are also called voting games. In this respect, a particular set of players is of importance, the so-called set of veto-players, which is defined by $J^v := \{k \in N \,|\, v(N\backslash\{k\}) = 0\}$. A member of J^v is a called a veto-player. Moreover, a player $k \in N$ is said to be a null-player, if for all $S \subseteq N$, it holds $v(S \cup \{k\}) = v(S)$.

A possible payoff allocation of the value $v(S)$ for all $S \subseteq N$ is described by the projection of a feasible vector $\mathbf{x} \in \mathbb{R}^n$ on its $|S|$-coordinates such that $x(S) \leq v(S)$ for all $S \subseteq N$, where we identify the $|S|$-coordinates of the vector $\mathbf{x}$ with the corresponding measure on S, such that $x(S) := \sum_{k \in S} x_k$. The set of vectors $\mathbf{x} \in \mathbb{R}^n$ which satisfies the efficiency principle $v(N) = x(N)$ is called the **pre-imputation set** and it is defined by

$$\mathscr{I}^*(v) := \{\mathbf{x} \in \mathbb{R}^n \,|\, x(N) = v(N)\}, \tag{2.1}$$

where an element $\mathbf{x} \in \mathscr{I}^*(v)$ is called a pre-imputation. The set of pre-imputations which satisfies in addition the **individual rationality property** $x_k \geq v(\{k\})$ for all $k \in N$ is called the **imputation set** $\mathscr{I}(N, v)$.

2.1 Power Indices

For introducing the **Banzhaf-Coleman power index** $\eta(N, v) \in \mathbb{R}^N$, let (N, v) be a simple game to finally define it by

$$\eta_k(N, v) := \frac{1}{2^n - 1} \sum_{S \subseteq N\backslash\{i\}} v(S \cup \{i\}) - v(S) = \frac{|\Omega_k(N, v)|}{2^n - 1}\}, \tag{2.2}$$

for all $k \in N$. Hence, the Banzhaf-Coleman power index averages over all proper coalitions the total counts of swing sets $\Omega_k(N, v)$ for each player $k \in N$. This can be interpreted as indicating the probability that a critical player k turns a losing coalition into a winning one independent of the order of players. Dubey and Shapley (1979, p. 102) simply called this expression the swing probability. In this sense, it is an à priori measure of power to influence a voting outcome to serve one's interests. Note in this respect that the Banzhaf index assigns the total power of a swing coalition to the critical players, whereas the non-critical players are attributed with no power. Hence, it is an à priori measure of power to bias a voting outcome in favor of a player k.

A first characterization of the Banzhaf index was provided by Dubey and Shapley (1979, p. 104) through the null-player, anonymity, transfer and total power property. Alternatively, Feltkamp (1995) characterized it by the null-player, anonymity, and additivity property, whereas additivity can be replaced by strong monotonicity (cf. Peters (2008, p. 292)). However, it does not satisfies the desirable efficiency property in accordance with the fact that not all permutations are considered rather than just the proper power set instead. In order to include this property in the Banzhaf index, it must be normalized, which is given by

$$\phi_k^b(N,v) := \frac{\eta_k(N,v)}{\sum_{j\in N}\eta_j(N,v)}, \tag{2.3}$$

for all $k \in N$. By the normalization it loses additivity, nevertheless it can be alternatively characterized by null-player, anonymity, strong monotonicity, and 2-efficiency (cf. Peters (2008, pp. 291-294)).

The other popular power index that contrast with the probability concept of Banzhaf while intrinsically assuming that the n-players, who are in the process of making a vote, are aligned in descent order of their enthusiasm w.r.t. the political proposal, i.e., the most passionate supporter of the proposal coming first and the most rebellious opponent coming last, is called Shapley-Shubik index. The unique player, who is turning a coalition to winning strength by joining, is called the pivotal player of the ordering. This is the player that must be convinced to incline in favor or disfavor of the proposal. If it is assumed that all permutations of players are equiprobable, then the attribution of power to a specific player is the probability that she/he is pivotal, since each permutation produces exactly one pivot. (cf. Dubey and Shapley (1979, p. 103)).

Definition 2.1 (Shapley and Shubik (1954))**.** *Let (N,v) be a simple game. The **Shapley-Shubik power index** assigns each (N,v) with its Shapley value. The k-th coordinate of the vector is denoted as the Shapley-Shubik power index of player k, and it is given by*

$$\phi_k^{SSI}(N,v) := \sum_{S\in\Omega_k(N,v)} \frac{(s-1)!\,(n-s)!}{n!}.$$

Hence, the Shapley-Shubik power index averages over all permutation of players the total counts of swing sets $\Omega_k(N,v)$ for each player $k \in N$. This number is the probability that a pivotal player k turns a losing coalition into a winning one depending on the order of players.

Let us now discuss a special subclass of simple games, where a council of n members has to pass a bill. To capture a wide range of possible voting games, one is not restricted to the general election rule one man one vote. The number of votes needs not to be tied to the number of players, like for a stockholders' meeting. Hence, to be as general as possible we assume $w_i \in \mathbb{R}_+$. Moreover, the total number of votes are specified by $w(N) \in \mathbb{R}_{++}$. For passing the bill at least $\mathfrak{q}$ votes are needed s.t. $0 < \mathfrak{q} \leq w(N)$ holds. A simple game $\langle N,v \rangle$ is referred to a weighted majority game, if there exists a quota (threshold or quorum) $\mathfrak{q} > 0$ and weights $w_k \geq 0$ for all $k \in N$ such that for all $S \subseteq N$ it holds either $v(S) = 1$ if $w(S) \geq \mathfrak{q}$ or $v(S) = 0$ otherwise. Then the simple game $\langle N,v \rangle$ is representable by real measures $[\mathfrak{q}; w_1,\ldots,w_n]$, which satisfies $v = v_{\mathbf{w}}^{\mathfrak{q}}$. We rewrite these measures as a pair $(\mathfrak{q},\mathbf{w}) \in (\mathbb{R}\times\mathbb{R}^n)$ to denote a representation of the simple game $\langle N,v\rangle$. The representation $(\mathfrak{q},\mathbf{w})$ of $\langle N,v \rangle$ is a homogeneous representation of $\langle N,v \rangle$ whenever the subsequent condition is valid

$$S \in M_w(N,v) \Longrightarrow w(S) = \mathfrak{q}. \tag{2.4}$$

Note that if a weighted majority game has a homogeneous representation, then it is homogeneous (cf. Peleg and Sudhölter (2007)). The term homogeneous game was introduced by von Neumann and Morgenstern (1944) to study the main simple solution (the so-called v.N.-M.-solution).

In the economic literature – i.e., by the articles of Barry (1980a,b); Holler (1982); Holler and Packel (1983) – the Shapley-Shubik, Banzhaf-Coleman or the Deegan-Packel power index were criticized on the ground that they consider a coalition value as a private good, i.e., the allocated spoil of a winning coalition is rivalrous and excludable in consumption, which is due that it equally distributes the value among the members of a winning coalition (Deegan-Packel index, see Deegan and Packel (1978)) or to assign it to the pivotal/critical player (Shapley-Shubik/Banzhaf-Coleman index). Note that a player is considered as critical whenever her/his removal from a winning coalition turns the coalition into a losing coalition. In this context to incorporate even the public good aspect, i.e., no exclusion and no deterrence of consumption, an alternative power index was proposed to only focus on the minimal winning coalitions instead, since they form decisive groups that determine the outcome (cf. Holler and Packel (1983, pp. 23-24)). All members of such a coalition are attributed to the same extent with the spoil, no one can be excluded from the benefit. This gives the interpretation of a public good consumption. Under this perception it was assumed that each coalition has equal probability to form with the consequence that different minimal winning coalitions bring forward different public goods as a possible collective outcome (cf. Holler (2018, pp. 33-34)). This alternative power index that is based on the idea to use the minimal winning coalitions to express the voting power of each individual player was proposed by Holler (1982); Holler and Packel (1983) – and was also denoted as a public good index (PGI). Sometimes this voting power index is denoted in the literature as the Holler or Holler-Packel index, we follow this convention in the sequel for the simple reason that we do not consider this index as a public good index. However, before we shall make our point why it is not adequate to consider this index as a public good index, we want to introduce its definition before we are going to discuss some major deficiencies. To this end, the power for each player $k \in N$ w.r.t. the set of minimal winning coalitions is either specified through

$$p_k(N, v) := \sum_{\substack{S \in M_w(N,v) \\ S \ni k}} v(S), \tag{2.5}$$

or $p_k(N, v) := 0$, i.e., if there does not exist any $S \in M_w(N, v)$ with $k \in S$. This formula is simply a count of the minimal winning coalitions to which player k belongs, which is a non-negative integer, called the public value. All members of a minimal winning coalition S are profiting to the same extent from the attribution of power, no one is excluded, likewise such as the consumption of a public good. To get a normalization, one sums up the counts across all players to use it as a normalizer (divisor) of this count. Then we define for each player $k \in N$ the **Holler power index** by

$$\phi_k^{HI}(N, v) := \frac{p_k(N, v)}{\sum_{j \in N} p_j(N, v)}. \tag{2.6}$$

Involving voting weights, a power distribution w.r.t. a solution concept satisfies *local monotonicity*, whenever it holds $w_i \geq w_j$, then $x_i \geq x_j$ is given for all player pairs $(i, j) \in (N \times N)$ s.t. $i \neq j$. That is to say, whenever player i controls a larger share of votes than player j, player i must be more powerful than player j. To see that the Holler index does not satisfy local monotonicity in general, we provide the following vote distribution $[35, 20, 15, 15, 15]$ with the quorum of $\mathfrak{q}_2 = 51$ to construct a counter-example. Then the Holler index provides the power distribution $(4, 2, 3, 3, 3)/15$ that violates local monotonicity for player 2 (cf. Holler and Packel (1983)).

In order to avoid that the Holler index is violating local monotonicity, all winning coalitions must be considered, which are induced by the set of minimal winning coalitions. Modifying the Holler index in this respect, we first define for each player $k \in N$ either

$$p_k^m(N,v) := \sum_{\substack{T \subseteq S \in \mathscr{W} \\ M_w(N,v) \ni T \ni k}} v(S), \tag{2.7}$$

or $p_k^m(N,v) := 0$, i.e., if there does not exist any $T \in M_w(N,v)$ with $k \in T$. The former case is simply a count of the winning coalitions whenever player k belongs to at least one minimal winning coalition, in the latter case the count is set to zero. This assures that the null-player property is satisfied. All members of a winning coalition S are profiting to the same extent from the attribution of power, no one is excluded. To get a normalization, one sums up the counts across all players to use it as a normalizer of this count. Then we define for each player $k \in N$ the modified Holler power index by

$$\phi_k^{MHI}(N,v) := \frac{p_k^m(N,v)}{\sum_{j \in N} p_j^m(N,v)}, \tag{2.8}$$

Having modified the Holler index, local monotonicity is fulfilled by $(12, 9, 8, 8, 8)/45$. Similar, the Banzhaf-Coleman as well as the Shapley-Shubik power index satisfying local monotonicity (cf. Felsenthal (2016)).

To close the discussion w.r.t. the Holler index, we now want to explain why we do not consider those as a public good index. This is caused that we interpret a transferable utility game as a stylized bargaining or voting procedure where subjects are trying to find an outcome on the basis of an agreed upon set of principles of distributive arbitration or voting rules (axioms). The resultant outcome can then be classified as fair, since it is referring to a solid foundation of upright standards that even apply in voting situations with unequal partners. For a weighted majority game this share means the largest possible power to bias an approving policy into her/his favor that respects their choice of rules of distributive justice (fairness). A choice that can be made, for instance, on a Rawlsian veil of ignorance.

In the line of the Holler index, subjects have agreed upon on a set of fairness rules that comprises the null-player, anonymity, efficiency and the mergeability principle to impose their power on a policy (cf. Holler and Packel (1983, pp. 116-117)). Likewise, subjects, who prefer the Deegan-Packel power index over any other power index, have a different view of fairness, and base their rule of distributive justice on null-player, symmetry, efficiency, and minimal monotonicity (cf. Lorenzo-Freire et al. (2007, p. 438)). Whereas subjects, who made under a Rawlsian type of decision a choice on the principles of null-player, symmetry, efficiency and critical mergeability will apply the Johnston power index (cf. Lorenzo-Freire et al. (2007, p. 439)) to reflect their norms of fairness.

In contrast to Holler (2018, pp. 33-34), our point of view does not allow that each minimal winning coalition bring forward a different public good or collective outcome with equal probability, for instance. Apparently, à priori, each coalition can be assumed to be equiprobable, and from that perspective some authors uses this term as a prefix in connection with coalition, and therefore with the power index to stress the point that the voting is still not concluded (cf. Deegan and Packel (1978); Felsenthal (2016)). Obviously, for an à priori power index it does not matter, which coalition manifests à posteriori. A membership within a coalition simply allows a subject to protect or defend her/his interests in the run-up of the weighted majority voting in the line of the once agreed upon fairness perception. In the same vein, we do not follow the consideration that the assignment of spoil to a pivotal/critical player (Shapley-Shubik/Banzhaf-Coleman index) or of an equal share among members – "sharing a cake" – (Deegan-Packel index) indicates a private good consumption. In the former case (public good case by Holler) the total power is attributed to all members of the minimal winning coalition instead, and in the latter case (private good case by the same author), the power is either attributed to the critical players or it is attributed equally among the subjects, since they are considered as equally strong partners. No consumption of a good, let it be public or private, has materialized at this prior stage. Just different attributions of power has been assigned to an à priori winning or minimal winning coalition to finally measure the potential power of a subject to bias the final outcome that has not occurred yet.

To enforce their interests onto an approving policy, subjects have an interest to be a member on as much priori winning coalitions as possible, and in this respect the formation of a winning coalition, which is is not minimal, is not caused "by luck" in our opinion, in contrast to that what was expressed by Holler (2018, pp. 34). This is firstly due that in the run-up of a vote no formation of an à priori coalitions can be materialized, and secondly, to judge if a formation of a coalition was caused by lucky circumstances can only be scrutinized à posteriori. However, and that is the crucial point, this argument should give a justification why only minimal winning coalitions ought to be considered. Which we can not follow, since we are convinced that no feasible winning coalition should be excluded à priori from such a voting model, otherwise one has amputated a subject from an opportunity to push forward her/his interests. And in this context of a prior voting process, a subject will point to all à priori winning coalitions that support her/his claim to power. All partners can immediately verify this claim to power in accordance with the principle of full rationality and complete information. No uncertainty is involved. It should be obvious by this argumentation that at this stage no vote has been carried out. We are still in the middle of a weighted majority voting. And to estimate the power of the individual subject for that vote, all à priori winning coalitions ought be included. Hence, for the reason to correctly reflect the expected individual power, we took them into account for the modified Holler index. This argument must even hold for à priori minimal winning coalitions when one consider them as an appropriate measure to estimate power. By this consideration we do not incline in favor for an interpretation of a public good index rather than on an à priori measure of power to influence a still approving policy.[1]

This might be a reason why this power index does not become as popular as the Shapley-Shubik or Banzhaf index. Nevertheless, the Holler index as well as the other mentioned power indices are providing for the measurement of power a different perception of fairness through their axiomatic foundation. Rather to consider fairness as an opaque concept that is based on subjective feelings of an individual, it is now possible to base the perception of fairness on objective norms. In the line of these norms, an objective à priori measure of power is obtained. Different rules of norms imply different measures of power. Reflecting the fact that there exists no uniform understanding of power among the people. This make them attractive from our point of view. If there is consensus that the objective norms of distributive justice (axioms) of the Holler index reflecting best the fairness preferences of the subjects involved in a voting process, then there is also consensus that only minimal winning coalitions should count to measure the à priori power of subjects. Though they must also accept its defects, like the violation of local monotonicity, for instance. Hence, each of these indices provide a different measure of power on the basis of their objective norms of fairness.

Finally, let us turn our attention to the two still missing power indices, which we have al-

[1] Based on private conversation with M. J. Holler.

ready mentioned, but which have not been formally introduced. To start with, we introduce the **Deegan-Packel power index** through

$$\phi_k^{DPI}(N,v) := \frac{1}{|M_w(N,v)|} \sum_{\substack{S \in M_w(N,v) \\ S \ni k}} \frac{1}{|S|}, \tag{2.9}$$

for all $k \in N$. Similar to the Holler index, we realize by its formula that only minimal winning coalitions ought to be considered by the assignment of power to an individual player. Each of these coalition is assumed to be equally likely. The spoil of a minimal winning coalition is equally distributed among its members, whereas non-members are attributed with a zero payoff. The real number assigned by this formula to player k is the expected attribution of power in turning a losing coalition into a minimal winning coalition.

Next, specify the set of critical players of coalitions by $\mathfrak{N}(S) := \{k \in S \mid S \in \mathscr{W} \wedge S \backslash \{k\} \notin \mathscr{W}\}$. Thus, it holds $\mathfrak{N}(S) = S$ whenever all players in S are critical. And the collection of all coalitions which contains at least one critical player $G_w(N,v)$ is specified through

$$G_w(N,v) := \Big\{ S \in \mathscr{W} \mid \exists i \in S \,\text{s.t.}\; S\backslash\{i\} \notin \mathscr{W} \Big\},$$

then the **Johnston power index** is defined by

$$\phi_k^{JI}(N,v) := \frac{1}{|G_w(N,v)|} \sum_{\substack{S \in G_w(N,v) \\ S \ni k}} \frac{1}{|\mathfrak{N}(S)|}, \tag{2.10}$$

for all $k \in N$. The formula looks similar to the Deegan-Packel index. However, in contrast to those, it takes the total number of critical players within a coalition into account and attributes the resultant spoil equally among the critical players within this coalition. These payoff assignment are added up for each player over all coalition to which she/he is critical, and this sum is divided by the overall number of swings. The real number assigned by this formula to player k is the expected attribution of power in turning an emerging coalition into a victorious.

2.2 The Kernel and Pre-Kernel

Finally, we want to contrast the power indices with the pre-nucleolus and pre-kernel. Given a vector $\mathbf{x} \in \mathscr{I}^*(v)$, we define the **excess** of coalition S with respect to the pre-imputation $\mathbf{x}$ in the game $\langle N,v \rangle$ by

$$e^v(S,\mathbf{x}) := v(S) - x(S). \tag{2.11}$$

A non-negative (non-positive) excess of S at $\mathbf{x}$ in the game $\langle N,v \rangle$ represents a gain (loss) to the members of the coalition S unless the members of S do not accept the payoff distribution $\mathbf{x}$ by forming their own coalition which guarantees $v(S)$ instead of $x(S)$.

Take a game $v \in \mathscr{G}^n$. For any pair of players $i,j \in N, i \neq j$, the **maximum surplus** of player i over player j with respect to any pre-imputation $\mathbf{x} \in \mathscr{I}^*(v)$ is given by the maximum excess at $\mathbf{x}$ over the set of coalitions containing player i but not player j, thus

$$s_{ij}(\mathbf{x},v) := \max_{S \in \mathscr{G}_{ij}} e^v(S,\mathbf{x}) \qquad \text{where } \mathscr{G}_{ij} := \{S \mid i \in S \text{ and } j \notin S\}. \tag{2.12}$$

The expression $s_{ij}(\mathbf{x}, v)$ describes the maximum amount at the pre-imputation $\mathbf{x}$ that player i can gain without the cooperation of player j. The set of all pre-imputations $\mathbf{x} \in \mathscr{I}^*(v)$ that balances the maximum surpluses for each distinct pair of players $i, j \in N, i \neq j$ is called the **pre-kernel** of the game v, and is defined by

$$\mathscr{PK}(N, v) := \{\mathbf{x} \in \mathscr{I}^*(N, v) \mid s_{ij}(\mathbf{x}, v) = s_{ji}(\mathbf{x}, v) \quad \text{for all } i, j \in N, i \neq j\}. \tag{2.13}$$

The pre-kernel has the advantage of addressing a stylized bargaining process, in which the figure of argumentation is a **pairwise equilibrium procedure** of claims while relying on his best arguments, that is, the coalitions that will best support his claim. The pre-kernel solution characterizes all those imputations in which all pairs of players $i, j \in N, i \neq j$ are in equilibrium with respect to their claims.

Observe that in case that the admissible bargaining range is the imputation set $\mathscr{I}(N, v)$ rather than $\mathscr{I}^*(N, v)$, player j cannot get less than $v(\{j\})$, the amount he can assure by himself without relying on the cooperation of the other players. A player i outweighs player j w.r.t. the proposal $\mathbf{x} \in \mathscr{I}(N, v)$ presented in a bilateral bargaining situation if $x_j > v(\{j\})$ and $s_{ij}(\mathbf{x}, v) > s_{ji}(\mathbf{x}, v)$. The set of imputations $\mathscr{I}(N, v)$ for which no player outweighs another player is called the **kernel**of a game $v \in \mathscr{G}^n$ referred to as $\mathscr{K}(N, v)$. More formally, the kernel of a n-person game is the set of imputations $\mathbf{x} \in \mathscr{I}(N, v)$ satisfying for all $i, j \in N, i \neq j$

$$[s_{ij}(\mathbf{x}, v) - s_{ji}(\mathbf{x}, v)] \cdot [x_j - v(\{j\})] \leq 0 \quad \text{and} \tag{2.14}$$

$$[s_{ji}(\mathbf{x}, v) - s_{ij}(\mathbf{x}, v)] \cdot [x_i - v(\{i\})] \leq 0. \tag{2.15}$$

This solution scheme is related to the pre-kernel $\mathscr{PK}(N, v)$ of a TU game. In addition, the following inclusion $\mathscr{PK}(N, v) \cap \mathscr{I}(N, v) \subset \mathscr{K}(N, v)$ is satisfied. The kernel is non-empty whenever the imputation set in non-empty, and it is a finite union of closed convex polyhedra (cf. Davis and Maschler (1965)). Therefore, we can infer that the pre-kernel is non-empty and it coincides with the kernel for the class of zero-monotonic TU games (cf. Maschler et al. (1972)). Moreover, due to Sudhölter (1996) it is known that the pre-kernel of every homogeneous weighted majority game without winning players – i.e., a player who satisfies $v(\{i\}) = 1$ or to put it differently $w_i \geq \mathfrak{q}$ – and without veto-players is star-shaped. In this context, it was established by Peleg et al. (1994) that a weighted majority game without winning players but with a homogeneous representation has a single-valued pre-kernel set, whenever it has veto-players. Notice that in connection with the null-player, these types form a decomposition of the player set N in equivalence classes.

2.3 The Nucleolus and Pre-Nucleolus

The kernel as well as the pre-kernel solution are a set-valued solution scheme with the consequence that it is difficult to justify why a selected element from one of these sets should be preferred over the other. To overcome this selection problem, the **nucleolus** of a n-person game, denoted as $\nu(v)$, might be the solution concept of choice, since it is contained in the kernel, $\nu(v) \in \mathscr{K}(N, v)$, it is non-empty and single-valued. This solution concept is due to Schmeidler (1969).

In order to define the nucleolus $\nu(N, v)$ of a game $v \in \mathscr{G}^n$, take any $\mathbf{x} \in \mathscr{I}(N, v)$ to define a $2^n - 1$-tuple vector $\theta(\mathbf{x}, v)$ whose components are the excesses $e^v(S, \mathbf{x})$ of the $2^n - 1$ non-empty coalitions $\emptyset \neq S \subseteq N$, arranged in decreasing order, that is,

$$\theta_i(\mathbf{x}, v) := e^v(S_i, \mathbf{x}) \geq e^v(S_j, \mathbf{x}) =: \theta_j(\mathbf{x}, v) \qquad \text{if} \qquad 1 \leq i \leq j \leq 2^n - 1. \tag{2.16}$$

Ordering the so-called complaint or dissatisfaction vectors $\theta(\mathbf{x},v)$ for all $\mathbf{x} \in \mathscr{I}(N,v)$ by the lexicographic order $\leq_L$ on $\mathbb{R}^{2^n-1}$, we shall write

$$\theta(\mathbf{x},v) <_L \theta(\mathbf{z},v) \qquad \text{if } \exists \text{ an integer } 1 \leq k \leq 2^n - 1, \tag{2.17}$$

such that $\theta_i(\mathbf{x},v) = \theta_i(\mathbf{z},v)$ for $1 \leq i < k$ and $\theta_k(\mathbf{x},v) < \theta_k(\mathbf{z},v)$. Furthermore, we write $\theta(\mathbf{x},v) \leq_L \theta(\mathbf{z},v)$ if either $\theta(\mathbf{x},v) <_L \theta(\mathbf{z},v)$ or $\theta(\mathbf{x},v) = \theta(\mathbf{z},v)$. Notice that we omit the game in the term θ whenever the game context is clear. Now, the nucleolus $\mathscr{N}(N,v)$ of a game $v \in \mathscr{G}^n$ over the set $\mathscr{I}(N,v)$ is defined as

$$\mathscr{N}(N,v) = \{\mathbf{x} \in \mathscr{I}(N,v) \mid \theta(\mathbf{x}) \leq_L \theta(\mathbf{z}) \; \forall \, \mathbf{z} \in \mathscr{I}(N,v)\}\,. \tag{2.18}$$

At this set the total complaint $\theta(\mathbf{x})$ is lexicographically minimized over the non-empty compact convex imputation set $\mathscr{I}(N,v)$. Schmeidler (1969) proved that the nucleolus $\mathscr{N}(N,v)$ is non-empty whenever $\mathscr{I}(N,v)$ is non-empty and it consists of a unique point, which is referred to as $\nu(N,v)$.

Notice that in this context a game has a nucleolus whenever the game is essential, that is, a TU game $\langle\, N, v\,\rangle$ is said to be **essential** if its characteristic function $v : 2^N \to \mathbb{R}$ satisfies

$$v(N) \geq \sum_{i \in N} v(\{k\}), \tag{2.19}$$

this class of games is denoted by $\mathscr{E}\mathscr{G}^n$, otherwise, it is inessential. Hence, for an inessential game the imputation set is empty, and the nucleolus does not exist. Moreover, we call a TU game **strictly essential** whenever the inequality of Formula (2.19) holds strictly. This class of games is denoted as $\mathscr{S}\mathscr{E}\mathscr{G}^n$.

Similar, the pre-nucleolus $\mathscr{P}\mathscr{N}(N,v)$ over the pre-imputations set $\mathscr{I}^*(N,v)$ is defined by

$$\mathscr{P}\mathscr{N}(N,v) = \{\mathbf{x} \in \mathscr{I}^*(N,v) \mid \theta(\mathbf{x}) \leq_L \theta(\mathbf{z}) \; \forall \, \mathbf{z} \in \mathscr{I}^*(N,v)\}\,. \tag{2.20}$$

The **pre-nucleolus** of any game $v \in \mathscr{G}^n$ is non-empty as well as unique, and it is referred to as $\nu^*(N,v)$. We summarize this discussion by the following theorem.

Theorem 2.1 (Single-Valuedness of the (Pre-)Nucleolus)**.** *Set either $X := \mathscr{I}^*(N,v)$ or $X := \mathscr{I}(N,v)$, which are both non-empty, compact and convex sets, then the (pre-)nucleolus consists of a single point for all $\langle\, N, v\,\rangle \in \mathscr{G}$ w.r.t. X.*

Imposing on the worth of any proper coalition – namely the set of coalitions excluding the grand coalition N and the empty set – the same cost $\epsilon \in \mathbb{R}$, then we can define the **strong ϵ-core** $\mathscr{C}_\epsilon(N,v)$ through

$$\mathscr{C}_\epsilon(N,v) := \{\mathbf{x} \in \mathscr{I}(N,v) \,|\, x(N) = v(N) \text{ and } x(S) \geq v(S) - \epsilon \; \forall \, \emptyset \neq S \subset N\}\,. \tag{2.21}$$

with $\mathscr{C}_0(N,v) = \mathscr{C}(N,v)$. It should be evident that the strong ϵ-core generalizes the core concept. For $n \geq 2$ we note that $\mathscr{C}_\epsilon(N,v) \neq \emptyset$ if ϵ is large enough and $\mathscr{C}_\epsilon(N,v) = \emptyset$ for small enough ϵ. Furthermore, if $\epsilon_1 < \epsilon_2$ then $\mathscr{C}_{\epsilon_1}(N,v) \subseteq \mathscr{C}_{\epsilon_2}(N,v)$ and $\mathscr{C}_{\epsilon_1}(N,v) \subset \mathscr{C}_{\epsilon_2}(N,v)$ whenever $\mathscr{C}_{\epsilon_2}(N,v) \neq \emptyset$. Taking the intersection of all non-empty strong ϵ-cores determines a set of latent allocations, namely the least core. The location of these latent allocations can be identified by computing a critical number through

$$\epsilon_0(v) := \min_{\mathbf{x} \in \mathscr{I}(N,v)} \max_{S \neq \emptyset, N} e(\vec{x}, S). \tag{2.22}$$

Note that the value $\epsilon_0(v)$ specifies the smallest ϵ for which the strong ϵ-core still exists. This critical value can be even negative. It is positive, whenever the core is empty, i.e., $\mathscr{C}_0(N,v) = \emptyset$. This core concept was introduced into the literature Maschler et al. (1979)) to investigate the geometric properties of the kernel, nucleolus and related solutions. This is due to that we have $\nu(v) \in \mathscr{C}_\epsilon(N,v)$, whenever $\mathscr{C}_\epsilon(N,v) \neq \emptyset$ and $\epsilon \leq 0$.

3 A Dual Pre-Kernel Representation

Theorem 3.1 (Martinez-Legaz (1996))**.** *The indirect function $\pi : \mathbb{R}^n \to \mathbb{R}$ of any n-person TU game is a non-increasing polyhedral convex function such that*

(i) $\partial\pi(\mathbf{x}) \cap \{-1,0\}^n \neq \emptyset \qquad \forall \mathbf{x} \in \mathbb{R}^n$,

(ii) $\{-1,0\}^n \subset \bigcup_{\mathbf{x}\in\mathbb{R}^n} \partial\pi(\mathbf{x})$, *and*

(iii) $\min_{\mathbf{x}\in\mathbb{R}^n} \; \pi(\mathbf{x}) = 0$.

Conversely, if $\pi : \mathbb{R}^n \to \mathbb{R}$ satisfies (i)-(iii) then there exists an unique n-person TU game $\langle N, v\rangle$ having π as its indirect function, its characteristic function is given by

$$v(S) = \min_{\mathbf{x}\in\mathbb{R}^n} \left\{ \pi(\mathbf{x}) + \sum_{k\in S} x_k \right\} \qquad \forall\, S \subseteq N. \tag{3.1}$$

According to the above result, the associated **indirect function** $\pi : \mathbb{R}^n \to \mathbb{R}_+$ is given by:

$$\pi(\mathbf{x}) = \max_{S\subseteq N} \left\{ v(S) - \sum_{k\in S} x_k \right\} \qquad \forall \mathbf{x} \in \mathbb{R}^n, \tag{3.2}$$

whereas $\partial\pi$ is the subdifferential of the function π. Hence, $\partial\pi(\mathbf{x})$ is the set of all subgradients of π at $\mathbf{x}$, which is a closed polyhedral convex set. A characterization of the pre-kernel in terms of the indirect function is due to Meseguer-Artola (1997). Here, we present this representation in its most general form, although we restrict ourselves to the trivial coalition structure $\mathscr{B} = \{N\}$.

The pre-imputation that comprises the possibility of compensation between a pair of players $i,j \in N, i \neq j$, is denoted as $\mathbf{x}^{\,i,j,\delta} = (x_k^{i,j,\delta})_{k\in N} \in \mathscr{I}^0(v)$, with $\delta \geq 0$, which is given by

$$\mathbf{x}^{i,j,\delta}_{N\setminus\{i,j\}} = \mathbf{x}_{N\setminus\{\,i,j\}},\; x_i^{i,j,\delta} = x_i - \delta \quad \text{and} \quad x_j^{i,j,\delta} = x_j + \delta$$

Proposition 3.1 (Meseguer-Artola (1997))**.** *For a TU game with indirect function π, a pre-imputation $\mathbf{x} \in \mathscr{I}^*(v)$ is in the pre-kernel of $\langle N, v\rangle$ for the coalition structure $\mathscr{B} = \{B_1, \ldots, B_l\}$, $\mathbf{x} \in \mathscr{PrK}(v,\mathscr{B})$, if, and only if, for every $k \in \{1,2,\ldots,l\}$, every $i,j \in B_k$, $i < j$, and some $\delta \geq \delta_1(v,\mathbf{x})$, one gets*

$$\pi(\mathbf{x}^{\,i,j,\delta}) = \pi(\mathbf{x}^{\,j,i,\delta}).$$

whereas $\delta_1(\mathbf{x}, v) := \max_{k\in N, S\subset N\setminus\{k\}} \; |v(S\cup\{k\}) - v(S) - x_k|$.

Meseguer-Artola (1997) was the first who recognized that based on the result of Proposition 3.1 a pre-kernel element can be derived as a solution of an over-determined system of non-linear equations. Every over-determined system can be equivalently expressed as a minimization problem. The set of global minima coalesces with the pre-kernel set. For the trivial coalition structure $\mathscr{B} = \{N\}$ the over-determined system of non-linear equations is given by

$$\begin{cases} f_{ij}(\mathbf{x}) = 0 & \forall i,j \in N, i < j \\ f_0(\mathbf{x}) = 0 \end{cases} \tag{3.3}$$

where, for some $\delta \geq \delta_1(\mathbf{x}, v)$,

$$f_{ij}(\mathbf{x}) := \pi(\mathbf{x}^{\,i,j,\delta}) - \pi(\mathbf{x}^{\,j,i,\delta}) \qquad \forall i,j \in N, i < j, \tag{3.3-a}$$

and

$$f_0(\mathbf{x}) := \sum_{k \in N} x_k - v(N). \tag{3.3-b}$$

$$h(\mathbf{x}) := \sum_{\substack{i,j \in N \\ i<j}} (f_{ij}(\mathbf{x}))^2 + (f_0(\mathbf{x}))^2 \geq 0 \qquad \forall \mathbf{x} \in \mathbb{R}^n. \tag{3.4}$$

For further details see Meinhardt (2013b, Chap. 5 & 6). Then one can establish the subsequent result:

Corollary 3.1 (Meinhardt (2013b))**.** *For a TU game $\langle N, v\rangle$ with indirect function π, it holds that*

$$h(\mathbf{x}) = \sum_{\substack{i,j \in N \\ i<j}} (f_{ij}(\mathbf{x}))^2 + (f_0(\mathbf{x}))^2 = \min_{\mathbf{y} \in \mathscr{I}^0(v)} h(\mathbf{y}) = 0, \tag{3.5}$$

if, and only if, $\mathbf{x} \in \mathscr{P}r\mathscr{K}(v)$*.*

To identify a partition of the domain of function h into payoff equivalence classes we first define the set of **most effective** or **significant coalitions** for each pair of players $i, j \in N, i \neq j$ at the payoff vector $\mathbf{x}$ by

$$\mathscr{C}_{ij}(\mathbf{x}) := \left\{ S \in \mathscr{G}_{ij} \;\middle|\; s_{ij}(\mathbf{x}, v) = e^v(S, \mathbf{x}) \right\}. \tag{3.6}$$

When we gather for all pair of players $i, j \in N, i \neq j$ all these coalitions that support the claim of a specific player over some other players, we have to consider the concept of the collection of most effective or significant coalitions w.r.t. $\mathbf{x}$, which we define as in Maschler et al. (1979, p. 315) by

$$\mathscr{C}(\mathbf{x}) := \bigcup_{\substack{i,j \in N \\ i \neq j}} \mathscr{C}_{ij}(\mathbf{x}). \tag{3.7}$$

Notice that the set $\mathscr{C}_{ij}(\mathbf{x})$ for all $i, j \in N, i \neq j$ does not have cardinality one, which is required to identify a partition on the domain of function h. Now let us choose for each pair $i, j \in N, i \neq j$ a descending ordering on the set of most effective coalitions in accordance with their size, and within such a collection of most effective coalitions having smallest size the lexicographical minimum is singled out, then we obtain the required uniqueness to partition the domain of h. This set is denoted by $\mathscr{S}_{ij}(\mathbf{x})$ for all pairs $i, j \in N, i \neq j$, and gathering all these collections we are able to specify the set of lexicographically smallest most effective coalitions w.r.t. $\mathbf{x}$ through

$$\mathscr{S}(\mathbf{x}) := \left\{ \mathscr{S}_{ij}(\mathbf{x}) \;\middle|\; i, j \in N, i \neq j \right\}. \tag{3.8}$$

This set will be denoted in short as the set of **lexicographically smallest coalitions**. Given the correspondence $\mathscr{S}$ on the payoff space we say that two payoff vectors $\mathbf{x}$ and $\mathbf{y}$ are equivalent w.r.t. the binary relation $\sim$ iff $\mathscr{S}(\mathbf{x}) = \mathscr{S}(\mathbf{y})$. In case that the binary relation $\sim$ is reflexive, symmetric and transitive, then it is an **equivalence relation** and it induces **equivalence classes** $[\vec{\gamma}]$ on $dom\, h$ which we define through $[\vec{\gamma}] := \{\mathbf{x} \in dom\ h\ | \mathbf{x} \sim \vec{\gamma}\}$. Thus, if $\mathbf{x} \sim \vec{\gamma}$, then $[\mathbf{x}] = [\vec{\gamma}]$, and if $\mathbf{x} \nsim \vec{\gamma}$, then $[\mathbf{x}] \cap [\vec{\gamma}] = \emptyset$. This implies that whenever the binary relation $\sim$ induces equivalence classes $[\vec{\gamma}]$ on $dom\, h$, then it partitions the domain $dom\, h$ of the function h. The resulting collection of equivalence classes $[\vec{\gamma}]$ on $dom\, h$ is called the quotient of $dom\, h$ modulo $\sim$, and we denote this collection by $dom\, h/ \sim$. We indicate this set as an equivalence class whenever the context is clear, otherwise we apply the term payoff set or payoff equivalence class.

Proposition 3.2 (Meinhardt (2013b))**.** *The binary relation $\sim$ on the set $dom\, h$ defined by $\mathbf{x} \sim \vec{\gamma} \iff \mathscr{S}(\mathbf{x}) = \mathscr{S}(\vec{\gamma})$ is an equivalence relation, which forms a partition of the set $dom\, h$ by the collection of equivalence classes $\{[\vec{\gamma}_k]\}_{k \in J}$, where J is an arbitrary index set. Furthermore, for all $k \in J$, the induced equivalence class $[\vec{\gamma}_k]$ is a convex set .*

This binary relation induces a partition on the payoff space. Having identified payoff equivalence classes, we can select an arbitrary payoff vector to get a unique quadratic and convex function. To see this, select payoff vector $\mathbf{x}$ from payoff equivalence class $[\vec{\gamma}]$, then we get the set $\mathscr{S}(\mathbf{x})$ from which a rectangular matrix $\mathbf{E}$ can be constructed through $\mathbf{E}_{ij} := (\mathbf{1}_{S_{ji}} - \mathbf{1}_{S_{ij}}) \in \mathbb{R}^n$, $\forall i,j \in N, i < j$, and $\mathbf{E}_0 := -\mathbf{1}_N \in \mathbb{R}^n$. Notice that in this respect the characteristic vector for $\mathbf{x} \in \mathbb{R}^n$ is given by $x_k = 1$ if $k \in S$ and $x_k = 0$ whenever $k \notin S$. Let $q = \binom{n}{2} + 1$; combining these q-column vectors, we can construct matrix $\mathbf{E}$ as an $(n \times q)$-matrix in $\mathbb{R}^{n \times q}$, which is given by

$$\mathbf{E} := [\mathbf{E}_{1,2}, \ldots, \mathbf{E}_{n-1,n}, \mathbf{E}_0] \in \mathbb{R}^{n \times q}. \tag{3.9}$$

A matrix $\mathbf{Q} \in \mathbb{R}^{n^2}$ can now be expressed as $\mathbf{Q} = 2 \cdot \mathbf{E}\ \mathbf{E}^\top$, a column vector $\mathbf{a}$ as $2 \cdot \mathbf{E}\ \vec{\alpha} \in \mathbb{R}^n$. Moreover, defining $\alpha_{ij} := v(S_{ij}) - v(S_{ji}) \in \mathbb{R}\ \forall i,j \in N, i < j$ and $\alpha_0 := v(N)$. Finally, the scalar α is given by $\|\vec{\alpha}\|^2$, whereas $\mathbf{E} \in \mathbb{R}^{n \times q}, \mathbf{E}^\top \in \mathbb{R}^{q \times n}$ and $\vec{\alpha} \in \mathbb{R}^q$. For the details to construct the above set, matrix and vector we refer the reader to Meinhardt (2013b, Chap. 5 & 6).

From vector $\vec{\gamma}$ the set (3.8) is constructed and then matrix $\mathbf{Q}$, column vector $\mathbf{a}$, and scalar α are induced from which a quadratic and convex function can be specified through

$$h_{\vec{\gamma}}(\mathbf{x}) = (1/2) \cdot \langle\, \mathbf{x}, \mathbf{Q}\, \mathbf{x}\, \rangle + \langle\, \mathbf{x}, \mathbf{a}\, \rangle + \alpha \qquad \mathbf{x} \in \mathbb{R}^n. \tag{3.10}$$

In view of Proposition 6.2.2 Meinhardt (2013b) function h as defined by (3.4) is composed of a finite family of quadratic and convex functions of type (3.10). For the details, we again refer the interested reader to Meinhardt (2013b, Chap. 5 & 6). In accordance with Theorem 7.3.1 by Meinhardt (2013b, p. 137) a dual representation of the pre-kernel is obtained as a finite union of convex and restricted solution sets $M(h_{\vec{\gamma}_k}, \overline{[\vec{\gamma}_k]})$ of a quadratic and convex function of type $h_{\vec{\gamma}_k}$, that is,

$$\mathscr{P}r\mathscr{K}(v) = \bigcup_{k \in \mathscr{J}'} M(h_{\vec{\gamma}_k}, \overline{[\vec{\gamma}_k]}). \tag{3.11}$$

where $\mathscr{J}'$ is a finite index set such that $\mathscr{J}' := \{k \in \mathscr{J}\,|\, g(\vec{\gamma}_k) = 0\}$. In addition, $g(\vec{\gamma}_k)$ is the minimum value of a minimization problem under constraints of function $h_{\vec{\gamma}_k}$ over the closed convex payoff set $\overline{[\vec{\gamma}_k]}$. For the index set it is claimed that this minimum value is equal to zero on the closed payoff set $\overline{[\vec{\gamma}_k]}$. The solution sets $M(h_{\vec{\gamma}_k}, \overline{[\vec{\gamma}_k]})$ are convex. Taking the finite union of convex sets may give us a non-convex set. Hence, the pre-kernel set is generically a non-convex set for games with more than 4 players. By the characterization of (3.11) we observe that it can be even disconnected. Some examples of a disconnected pre-kernel were discussed by Kopelowitz (1967) and Stearns (1968). An example form this source was recently reconsidered in Meinhardt (2013b, Sec. 8.5). According to our information, this is the sole example of a disconnected pre-kernel investigated in the literature. However, we have found some further evidence that a disconnected pre-kernel occurs more frequently as this example may suggest. This evidence is supported by the findings of Subsection 5.2 and 6.2. Thus, it might be a rare event having a null measure, but it can be materialized even though the conditions under which such an event can be observed are still unclear. Caused by the fact that the pre-kernel is still not fully understood.

For the class of convex games and three person games we have $|\,\mathscr{J}'\,| = 1$, which implies that the pre-kernel must be a singleton. In this respect, Meinhardt (2020b) has established that whenever a default game has a unique pre-kernel satisfying the non-empty interior condition for a payoff set, then on a restricted subset of the game space constituted by the default game and a set of related games this point is the sole pre-kernel element. The pre-kernel correspondence is single-valued and constant on this subset.

To this end we consider a mapping that sends a point $\vec{\gamma}$ to a point $\vec{\gamma}_\circ \in M(h_{\vec{\gamma}})$ through

$$\Gamma(\vec{\gamma}) := -\Big(\mathbf{Q}^\dagger\, \mathbf{a}\Big)(\vec{\gamma}) = -\Big(\mathbf{Q}^\dagger_{\vec{\gamma}}\, \mathbf{a}_{\vec{\gamma}}\Big) = \vec{\gamma}_\circ \in M(h_{\vec{\gamma}}) \qquad \forall \vec{\gamma} \in \mathbb{R}^n, \tag{3.12}$$

where $\mathbf{Q}_\gamma$ and $\mathbf{a}_\gamma$ are the matrix and the column vector induced by vector $\vec{\gamma}$, respectively. Notice that matrix $\mathbf{Q}^\dagger_{\vec{\gamma}}$ is the pseudo-inverse of matrix $\mathbf{Q}_\gamma$. In addition, the set $M(h_{\vec{\gamma}})$ is the solution set of function $h_{\vec{\gamma}}$. Under a regime of orthogonal projection this mapping induces a cycle free method to evaluate a pre-kernel point for any class of TU games. We restate here Algorithm 8.1.1 of Meinhardt (2013b) in a more succinctly written form by

Algorithm 3.1: Procedure to seek for a Pre-Kernel Element

Data: Arbitrary TU Game $\langle N, v\rangle$, and a payoff vector $\vec{\gamma}_0 \in \mathbb{R}^n$.
Result: A payoff vector s.t. $\vec{\gamma}_{k+1} \in \mathscr{P}r\mathscr{K}(v)$.
begin
0 $k \longleftarrow 0$, $\mathscr{S}(\vec{\gamma}_{-1}) \longleftarrow \emptyset$
1 Select an arbitrary starting point $\vec{\gamma}_0$
if $\vec{\gamma}_0 \notin \mathscr{P}r\mathscr{K}(v)$ **then** Continue
else Stop
2 Determine $\mathscr{S}(\vec{\gamma}_0)$
if $\mathscr{S}(\vec{\gamma}_0) \neq \mathscr{S}(\vec{\gamma}_{-1})$ **then** Continue
else Stop
repeat
3 **if** $\mathscr{S}(\vec{\gamma}_k) \neq \emptyset$ **then** Continue
else Stop
4 Compute $\mathbf{E}_k$ and $\vec{\alpha}_k$ from $\mathscr{S}(\vec{\gamma}_k)$ and v
5 Determine $\mathbf{Q}_k$ and $\mathbf{a}_k$ from $\mathbf{E}_k$ and $\vec{\alpha}_k$
6 Calculate by Formula (3.12) $\mathbf{x}$
7 $k \longleftarrow k+1$
8 $\vec{\gamma}_{k+1} \longleftarrow \mathbf{x}$
9 Determine $\mathscr{S}(\vec{\gamma}_{k+1})$
until $\mathscr{S}(\vec{\gamma}_{k+1}) = \mathscr{S}(\vec{\gamma}_k)$
end

Meinhardt (2013b, Theorem 8.1.2) establishes that this iterative procedure converges towards a pre-kernel point. In view of Meinhardt (2013b, Theorem 9.1.2) we even know that at most $\binom{n}{2} - 1$-iteration steps are sufficient to successfully terminate the search process. However, we have some empirical evidence that generically at most $n+1$-iteration steps are needed to determine an element from the pre-kernel set (cf. Meinhardt (2013b, Appendix A)).[2]

4 A historical Gloss

The House of Medici was a Italian dynasty originated from the Mugello region of Tuscany. Its founder was Giovanni di Bicci de' Medici (1360-1429), who was not from the middle class as commonly believed, the preserved tax records show that during this time period the Medici were by far the richest family in Florence. Giovanni di Bicci also founded in 1397 the bank of the same name, which operated on an international level with branches across Europe till 1494. The rise of the Medici Bank was the result of the collapse of the Bardi, Peruzzi and Acciaiuolli companies caused by excessive loans to Edward III, King of England, with the consequence that the disappearance of these companies deprived the Pope from the facilities to finance the Vatican spending. This place was taken by the Medici Bank, and made it in the 15th-century to one of the largest financial institution in Europe providing the necessary economical resources that allow the House of Medici to seize the political power in Florence (cf. de Roover (1966, pp. 1-5)). Their political influence was strong enough to bring forth four Popes of the Catholic Church and two queens of the Kingdom of France. One of the most prominent political actor of this dynasty was Cosimo de' Medici, called the Elder, and son of Giovanni di Bicci. Even though Padgett and Ansell (1993, p. 1264) stressed the point that "Cosimo did not set out a grand design to take over the state", he was, nevertheless, able to build up a network of supporters around him during the Milan and Lucca wars (1424-33). This network builds up the necessary political capacity to overcome even the disastrous war against the Republic of Lucca in 1433, for which he was made responsible by the faction of the Albizzi and Strozzi families. He was put in jail, however he managed to be exiled to Venice with his brother Lorenzo though the faction of his opponents demanded his execution. Twelve months later the Medici network, operating from exile, imposed a democratic vote to overturn Cosimo's and Lorenzo's banishment. An overturn that had its origin in a failed seize of the city hall and the government by the Albizzi clan in September 1433 as the Medici could only operated from exile. Nevertheless, their network of supporters was strong enough to prevent this subversion. Returned back to Florence, the Medici clan and their supporters taking revenge to permanently banish or punish their enemies, in particular the Albizzi and Guadagni family (cf. Padgett and Ansell (1993)). This was the origin of the prominence of the House of Medici and the decline of the old Florentine oligarchs. Resulting finally into the end of the Florentine Republic by 1569 and made the Medici hereditary rulers of the grand duchy of Florence till their extinction in 1737 and with the accession of Francis Stephen, Duke of Lorraine and husband of Maria Theresa of Austria.

The Strozzi, Albizzi, Peruzzi and the Guadagni families were members of the old oligarchs that ruled the Republic of Florence before the rise of the House of Medici begun. In particular, the Strozzi and Albizzi clans were the great rivals of the Medici. Like the Medici, the House of Strozzi acquired their wealth from banking, had similar political ambitions, and was – in accordance with the 1427 catasto[3] – the most wealthy family of the Florentine elite, albeit their company was overshadowed by the Medici Bank (cf. de Roover (1966, p. 3)). They were unfaltering opponents

[2]Algorithm 3.1 is implemented in our MATLAB toolbox MatTuGames 2020a. The documentation of the toolbox is given by Meinhardt (2013a) and ships with the toolbox.

[3]A register to record the financial assets and real estates of taxpayers to estimate their tax liabilities.

of the Medici hegemony, and played a more crucial role during the 1527 insurrection. In contrast, the House of Albizzi was the leading family and the actual ruler in Florence in the reaction that followed the Ciompi Revolt[4]. Since that time the Albizzi were the enemies of the Medici caused by Salvestro de' Medici (1331–1388), a second cousin of Giovanni di Bicci de' Medici, who showed sympathy with the insurrection of the Ciompi as he was Gonfaloniere di Giustizia[5]. Thereafter the Medici were considered by the oligarchs as the party of the people, and they were regarded with suspicion. Till the prominence of the House of Medici, the Albizzi pursued an aggressive foreign policy to open a passage to sea for the wool and cloth trade while expanding the territories by force and purchase. On the one hand it was the impetus of an increase in prosperity that manifests in an immense jump in wealth of the republic (cf. Hibbert (1974, pp. 30-33)), but on the other hand it was the cause of the disastrous issue of the war against Lucca initiate under the aegis of Rinaldo degli Albizzi at 1429. Not to forget the House of Peruzzi, whose members were also bankers and made their fortune with investments into the textile business. Their company was the second largest of the Florentine banks operating across Europe and the Levant till the company failed in 1343 due to unsecured loans to Edward III of England, who owed them "the value of a realm" (cf. de Roover (1966, p. 3)). The Medici Bank never attained the size of the Peruzzi company (cf. ibid. p. 3). Last but not least, the Guadagni clan, neighbors of the Albizzi with family links to them, and bankers as the others, albeit they did not attain whose financial status (cf. Breiger and Pattison (1986, Table 6)). Nevertheless, they produced Bernardo Guadagni (1367–1434), a powerful adversary of Cosimo de' Medici, who manged during his second term as Gonfaloniere di Giustizia to imprison and to exile him to Venice.

The economic upswing of Florence attracted artisans, laborers, and craftsmen from surrounding regions to settle in this buzzing city. New arrivals who spread over the whole city, and who had no former and embedded relations to the old ruling families. These new men sought ties to the Florentine clientage system[6] that were offered to them by the Medicean parvenus by an extraordinarily centralized star network system with very few relations among the Medici supporters, but with direct ties to the Medici clan itself. Moreover, these partisans were mainly connected to the oligarchic elite through the Medici family (cf. Padgett and Ansell (1993, p.1278)). The Medicean followed according to Padgett and Ansell (1993, p.1280) the simple maxim: to control partisans politically, segregate their social relations with them and isolate them from all others. This socially heterogeneous, but centralized mesh of relations allowed the Medici a direct control over their network for a cohesive common action to hold down their rivals. In this context, Padgett and Ansell (1993, pp. 1262-63) reported that Cosimo used the network defensively, acting only "when need arose", that is, he reacted on aggressive moves of his rivals but never made the first move by himself while taking leadership due to his anxious, passive and indecisive character.

Contrasted to the Medici, the oligarchic network was skewed toward patricians and their partisan loyalties that arose over a prolonged period of time through neighborhood residence at the ward or quarter level. This elite was densely interconnected through marriage and insisted on their status as equal among equals with the consequence that each family claimed leadership (cf. Padgett and Ansell (1993, p.1279)). This socially homogeneous mesh of relations was inimical

[4] A rebellion for political participation in the Florentine government (1378-82) of non-organized laborers who did not belong to any guild.

[5] Head of the government of the Republic of Florence.

[6] A clientage refers to a hierarchical relationship between a patron and clients with mutual obligations. The patron or protector granted benefits to the client, in return the client owed allegiance, for instance, support to the patron running for a political office. The origin of clientelism can be traced back to the ancient Rome.

for a coherent, cohesive, and purposeful action. By these considerations, Padgett and Ansell (1993) considered the Medicean network superior over the oligarchic and as the source of their political prominence.

This short historical gloss of the principal actors in the Florentine Republic let us focus on the ruling families as the subject of our analysis. The catasto of 1427 identified in total 10171 families from a population of about 50000 inhabitants, from which only 200 households were paying more than 25 florins (fiorino d'oro) taxes, only three household were paying more than 100 florins, among those were the Medici (cf. de Roover (1966, pp. 28-31)). Padgett and Ansell (1993) identified 215 Florentine leading families drawn from the catasto, but they focused in their study on just 92 families. Breiger and Pattison (1986, p. 219) used an extended data base of 116 families from the same authors from which they picked out those 16 families from the 1427 catasto (cf. ibid. Table 1 and Table 6) "whose support of, or opposition to, the Medicis has been clearly established". We resume this data set for our purpose (cf. Table 4.1 and 4.2). The data set is wide-enough, but not too large to be able to conduct a power index analysis.[7]

Table 4.1: Florentine Marriage Relations

Family[a] \ Family	01	02	03	04	05	06	07	08	09	10	11	16	12	13	14	15
01 ACCIAIUOL	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
02 ALBIZZI	0	0	0	0	0	1	1	0	1	0	0	0	0	0	0	0
03 BARBADORI	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0
04 BISCHERI	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0
05 CASTELLAN	0	0	1	0	0	0	0	0	0	0	1	0	0	0	1	0
06 GINORI	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
07 GUADAGNI	0	1	0	1	0	0	0	1	0	0	0	0	0	0	0	1
08 LAMBERTES	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
09 MEDICI	1	1	1	0	0	0	0	0	0	0	0	0	1	1	0	1
10 PAZZI	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
11 PERUZZI	0	0	0	1	1	0	0	0	0	0	0	0	0	0	1	0
16 PUCCI	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12 RIDOLFI	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	1
13 SALVIATI	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
14 STROZZI	0	0	0	1	1	0	0	0	0	0	1	0	1	0	0	0
15 TORNABUON	0	0	0	0	0	0	1	0	1	0	0	0	1	0	0	0

[a] cf. Table 1 of Breiger and Pattison (1986, p. 218).

[7] MATLAB R2020b can only handle 40 different actors, whereas the double-precision floating-point format would allow to consider up to 51 actors. Quadruple-precision floating-point format extends this bound up to 111. However, a number that still falls short of the mentioned 215 Florentine families.

Table 4.2: Florentine Business Relations

Family[a] \ Family	01	02	03	04	05	06	07	08	09	10	11	16	12	13	14	15
01 ACCIAIUOL	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
02 ALBIZZI	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
03 BARBADORI	0	0	0	0	1	1	0	0	1	0	1	0	0	0	0	0
04 BISCHERI	0	0	0	0	0	0	1	1	0	0	1	0	0	0	0	0
05 CASTELLAN	0	0	1	0	0	0	0	1	0	0	1	0	0	0	0	0
06 GINORI	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0
07 GUADAGNI	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0
08 LAMBERTES	0	0	0	1	1	0	1	0	0	0	1	0	0	0	0	0
09 MEDICI	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	1
10 PAZZI	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
11 PERUZZI	0	0	1	1	1	0	0	1	0	0	0	0	0	0	0	0
16 PUCCI	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12 RIDOLFI	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13 SALVIATI	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
14 STROZZI	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15 TORNABUON	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0

[a] cf. Table 1 of Breiger and Pattison (1986, p. 218).

Our work was inspired by Holler and Rupp (2020) who investigated for the first time the Florentine marriage and business relations from a view point of cooperative game theory. In particular, their findings – that are based on the public good index – provided some further evidence of an accentuated position of the Medici family, albeit not as outstanding as it was already recognized for decades by sociologists such as Breiger and Pattison (1986); Padgett and Ansell (1993); Padgett (2010) to mention just those studies that found even great attention across disciplines. In our opinion, the attributed power could not significantly explain their rise to power, which is our motivation to resume an extended analysis. Rather than to confine ourselves on the public good index w.r.t. an underlying network structure, we include in our analysis even the power indices from our previous discussion to investigate whether this outstanding position of the House of Medici could be at least confirmed by an alternative measurement of power.

To qualify a solution concept as a standard of fairness, we consider as a desirable feature, – apart of an appealing axiomatic foundation, its stability properties, or a comprehensible and efficient computability – its capability of providing empirical evidence, i.e., its capability of replicating a real outcome as close as possible. A solution that does not replicate known events, reveals that real actors do not act in accordance with its principles of distributive arbitration. Hence, the fairness rules that characterize the solution do not constitute any moral guideline for subjects by their decision making. They may offer nice mathematical properties that are interesting and worthwhile to study, but they lack the crucial ingredient of describing human behavior. We would not go so-far to deny for it every value and classify such a solution as useless, but its significance is rather limited.

5 Florentine Elite Marriages

Normally, one analyzes the power of individuals in a political decision-making process without considering any relationships among each other. In such a framework one ignores the most significant ingredient of political success, namely the formation, control and domination of a network of relationships to influence a voting outcome and institutions to serve one's interests.

To incorporate the underlying mesh of relations into the measurement of power, we need to specify the network, for instance, by an undirected graph in order to obtain a game representation in characteristic function form. Exemplarily to conduct a power index analysis for an underlying network structure, we are focusing on the ruling families of the Republic of Florence during the late medieval and the early Renaissance.

During the mentioned time period not individuals but rather than families were the principal actors. A family was comprised in general by people with the same last name, albeit it has more the character of a clan than of a household. The medieval as well as the Renaissance were not the era of individualism. Relations among households were typically organized by a clan structure. The clan, led by a chieftain like Cosimo de' Medici (1389-1464), decided everything from office-holding up to the marriage of their members (cf. Padgett and Ansell (1993, p. 1267)).

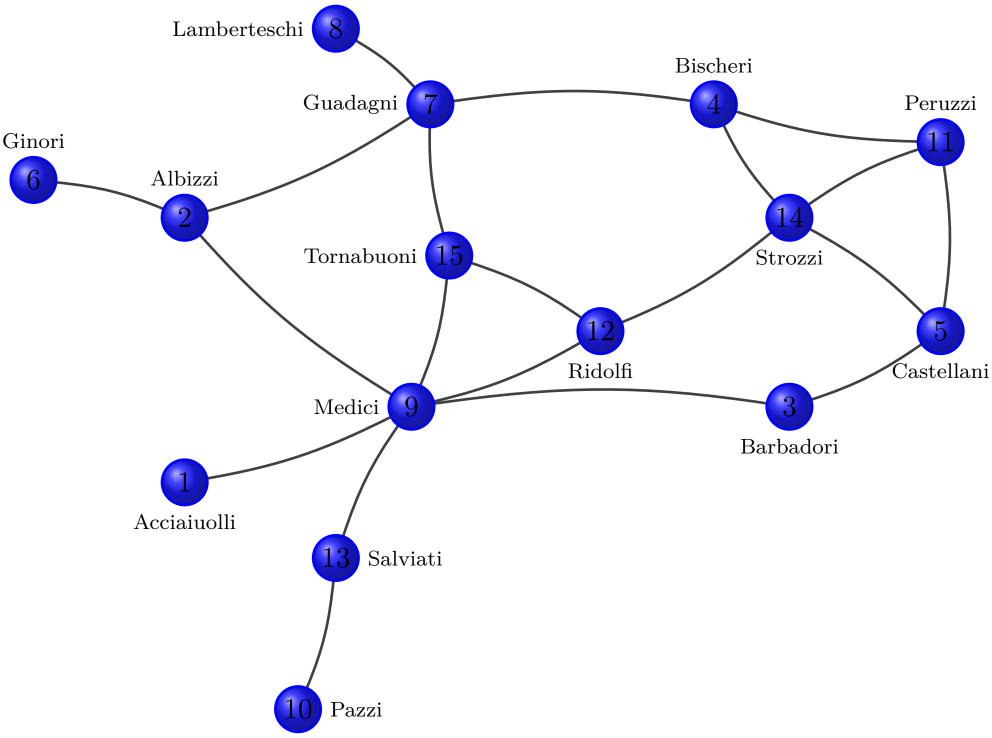

Figure 1: Florentine Elite Marriages 1395-1434[8]

In order to pursue a network power analysis of the leading Florentine families, we need to define a weighted majority graph problem. For doing so, we have to remind that a weighted majority TU game is referring to a weighted majority problem $[\mathfrak{q}; w_1, \ldots, w_n]$ s.t. $v(S) = 1$ whenever $w(S) \geq \mathfrak{q}$ or $v(S) = 0$ otherwise. In addition, to introduce a network game that allows us to analyze the Florentine marriage and business relations by the power index method, we need to modify the weighted graph approach introduced by Deng and Papadimitriou (1994) in some

[8]The used network data set of the Florentine Marriage Relations is provided by the file *florentine_m* that ships with the R software package netrankr of Schoch (2020a) that is based on the work of Breiger and Pattison (1986). Notice that the Pucci family is not part of network graph, since it has no marriage links with other florentine families (cf. ibid. Table 1, p. 218).

respect. Analogously, we set out an undirected graph by $\mathtt{G} := (\mathtt{V}, \mathtt{E})$ that is constituted by a set of vertices (nodes), denoted as $\mathtt{V}$ that is considered as the set of agents; and a set of edges specified by $\mathtt{E} : \mathtt{V} \rightarrow \mathtt{V}$ s.t. a pair $(i, j) \in (\mathtt{V} \times \mathtt{V})$ constitutes the edge to specify the relationship between a pair of agents. Moreover, to define a weighted majority graph problem, we need to introduce a mapping $w : \mathtt{V} \rightarrow \mathbb{R}_{++}$ s.t. the number $w_i > 0$ is the weight of the vertex – gravity of the node –, i.e., the share of votes controlled by player i. We set $\mathbf{w} := (w_i)_{i \in N}$. Contrasted to a weighted graph problem, we do not allow self-loops. A weighted majority graph problem is now specified by the triple $\mathtt{wMP} := (\mathtt{G}, \mathbf{w}, \mathtt{q})$. The corresponding network weighted majority TU game can then be determined by setting $N = \mathtt{V}$, i.e., each player corresponds to a vertex/agent, and the characteristic value of a coalition $S \subseteq N$ is determined by $v(S) = 1$, if S is a connected graph of a weighted majority sub-graph problem $\mathtt{wMP}_S$ s.t. $w(S) \geq \mathtt{q}$, otherwise $v(S) = 0$. We denote the class of network weighted majority games by $\mathscr{G}\mathscr{S}^n_{wMP}$. By the construction of the associated simple game from a weighted majority graph problem, we can then apply a power index analysis based on the characteristic function form. To get the game, we have to transcribe Table 4.1 into a graph, which is given by Figure 1

This graph depicts the marriage relations of the 16 families whose support of, or the opposition to, the House of Medici could be identified through the work of Breiger and Pattison (1986). By inspection of the figure, it is noticeable that only 15 families (nodes) are counted. This is due that in accordance with Table 4.1 no marriage relations to the other 15 families can be identified for the Pucci clan so that they are removed from the network. We referring to this undirected graph as $\mathtt{G}_1$.

To derive from this network structure the associated voting game, we follow – for comparability reasons – the assumptions made by Holler and Rupp (2020, p. 6) that to each player node the same weight of one is assigned, and that just a simple majority rule is applied. Hence, the set of agents has cardinality 15, for all $k \in N$, an agent k has a weight of $w_k = 1$, and the quorum is set to $\mathtt{q} = 8$, so that the weighted majority graph problem based on Table 4.1 is given by

$$\mathtt{wMP}_1 := (\mathtt{G}_1, [1, \underbrace{1, \ldots, 1}_{13 \text{ times}}, 1], 8).$$

These information are enough to derive a voting game. For getting the game, graph $\mathtt{G}_1$ can be characterized by an edge matrix through

$$\mathbf{mE} = \begin{bmatrix} 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 7 & 7 & 9 & 9 & 9 & 10 & 11 & 12 & 12 \\ 9 & 6 & 7 & 9 & 5 & 9 & 7 & 11 & 14 & 11 & 14 & 8 & 15 & 12 & 13 & 15 & 13 & 14 & 14 & 15 \end{bmatrix},$$

which has the form of of a generalized permutation. Having set out the edge matrix to represent graph $\mathtt{G}_1$ as well as the parameter values of $(\mathbf{w}, \mathtt{q})$ the graph problem $\mathtt{wMP}_1$ is well defined and the corresponding simple game can be created, for instance, while using MATLAB.[9] Let us denote this game as $\langle N, v_1 \rangle$.

Since our work was inspired by the work of Holler and Rupp (2020), we start the power analysis with the Holler index – or public good index – for the above graph problem. Applying

[9] A computation method is implemented in our MATLAB toolbox MatTuGames 2020a as well as a set of routines to compute the discussed results. This set of methods can be made available upon request.

Formula (2.6) and its power index distributions is given by

$$\phi^{HI}(N, v_1) = (0.0520, 0.0879, 0.0721, 0.0735, 0.0647, 0.0374, 0.0914, 0.0389, 0.1179, 0.0271, 0.0561, 0.0727, 0.0697, 0.0730, 0.0656),$$

and the array of the associated public values is quantified through

$$p(N, v_1) = (343, 580, 476, 485, 427, 247, 603, 257, 778, 179, 370, 480, 460, 482, 433).$$

The above distribution confirms the results of Holler and Rupp (2020, p. 10), listed by their Table 1. The Medici clan, on coordinate 9, has an accentuated position, followed by their rivals the Guadagni (7) and the Albizzi (2). However, to identify if an actor has an outstanding position, an informal look at the highest value is not sufficient. Rather, we fix such a statement in terms of the squared expected deviation from the mean (variance). The smaller this measure, the closer is the dispersion of the set of power with the consequence that the largest value cannot be designated as outstanding. Contrasted with, an outstanding position is classified to a wide spread from the mean, a large variance. The variance 5.2618 of the power distribution[10] shown by Table 9.11 is too small to identify an actor as outstanding. From this power distribution, we cannot infer on an outstanding position of the Medici network. Rather we would say that the power, expressed by the Holler index, was insufficient to explain their rise to power in Florentine politics. It can also not explain why the Medici network was strong enough to prevent the seize of the city hall and the government by the old oligarchs in September 1433. It is hardly to imagine that this subversion could ever be foiled by the Medicean on the basis of the above power distribution. Furthermore, the power of the Strozzi clan (14), the most wealthy family and great rival of the Medicean, seems to be underestimated, they are ranked on place 5. Contrasted to that, we do not consider the power of the Guadagni, Albizzi and the Strozzi networks as influential enough to prevent them from seizing the power or sending them into exile. This is a disappointed outcome and we turn our attention to some alternative power indices to figure out if at least one of them can confirm the outstanding position of the Medicean recognized by sociologists since decades. If not, we have to ask whether the network representation correctly captures the historical facts or if is it probably too stylized to replicate them.

Rather to count just the minimal winning coalitions to which a critical player can be belong to, we enlarge the assessment basis while incorporating all winning coalitions. By doing so, we compute the power distribution for the modified Holler index through Formula (2.8), which is given by

$$\phi^{MHI}(N, v_1) = (0.0595, 0.0720, 0.0676, 0.0680, 0.0667, 0.0577, 0.0758, 0.0583, 0.0895, 0.0552, 0.0623, 0.0688, 0.0638, 0.0686, 0.0662),$$

and the modified public value is specified by

$$p^m(N, v_1) = (3783, 4575, 4298, 4325, 4239, 3665, 4817, 3704, 5691, 3510, 3962, 4371, 4057, 4360, 4209).$$

Remind that these values count the winning coalitions to which a player k is a member. Inspection of the power distribution reveals to us that the ranking of the most prominent families is also

[10] The power distribution has been multiplied by 100 to calculate the variance.

preserved by the modified Holler index, i.e., the Medicean are placed on position one followed by their rivals the Guadagni (7) and the Albizzi (2). Once more, we judge the attributed power to the Strozzi clan (14) on rank 5 as underestimated. Contrasted to those, the attributed power to all of them have been decreased and the distribution was more balanced. With even less power attributed to the main actors and a smaller variance – just 0.7184 as recorded by Table 9.11 – as compared to the Holler value, this power index characterization is all the more not able to support the outstanding position of the Medici, and the decline of their rivals. This lead us straight away to the Deegan-Packel index (2.9), which is quantified by

$$\begin{aligned}\phi_k^{DPI}(N, v_1) = (&0.0520, 0.0879, 0.0721, 0.0735, 0.0647, 0.0374, 0.0914, 0.0389, 0.1179,\\ &0.0271, 0.0561, 0.0727, 0.0697, 0.0730, 0.0656),\end{aligned}$$

which returns exactly the same distribution as the Holler index. Hence, this index is also unsuitable to support the historical facts. That let us immediately turn to the Johnston power distribution that is quantified via Formula (2.10) through

$$\begin{aligned}\phi^{JI}(N, v_1) = (&0.0231, 0.0881, 0.0615, 0.0627, 0.0547, 0.0181, 0.1238, 0.0198, 0.2744,\\ &0.0116, 0.0331, 0.0676, 0.0470, 0.0646, 0.0498).\end{aligned}$$

Again, we obtain by this power distribution the same ranking of the Medici, Guadagni, and the Albizzi families. Even more clearly, their power attribution has been raised by this index. In particular, the Medici clan is pleased with an over-proportional increment on power, their rank is now further emphasized. Falling, nevertheless, far short of a blocking minority, if one considers this level as a lower bound of political power that ought be at least attained to thwart the political ambitions of the opponents, for instance, the 1433 subversion. Albeit the variance of the power distribution as reported by Table 9.11 with a value of 41.679 is best so-far, we come up once more with the conclusion that these figures can neither support the failure of this subversion nor to mention the rise of the House of Medici. These outcomes let us refocus on the popular power indices, namely the Shapley-Shubik and Banzhaf index. The Shapley-Shubik index defined by Definition 2.1 provides the subsequent power distribution

$$\begin{aligned}\phi^{SSI}(N, v_1) = (&0.0275, 0.0835, 0.0657, 0.0706, 0.0656, 0.0246, 0.1255, 0.0278, 0.1988,\\ &0.0142, 0.0442, 0.0740, 0.0432, 0.0733, 0.0614).\end{aligned}$$

Though the Shapley-Shubik power distribution reflects the ranking of the Medici and their main rivals, and even attributes to all those more power than the Holler index, the outstanding position of the Medicean cannot be supported. As shown by Table 9.11, the value of variance of 21.3984 indicates that the power distribution is still clustered too close around the mean, – i.e., is too balanced – to assume a successful breakthrough of a clan as an all-embracing actor as it was accomplished by the Medici. Thus, these figures cannot neither explain the prominence of the Medici nor the decline of their rivals nor to explain the failure of the 1433 subversion.

Focusing on our power index analysis of the Banzhaf value (2.3), we get

$$\begin{aligned}\phi^{b}(N, v_1) = (&0.0364, 0.0891, 0.0707, 0.0725, 0.0668, 0.0285, 0.1053, 0.0311, 0.1635,\\ &0.0182, 0.0483, 0.0755, 0.0546, 0.0748, 0.0648).\end{aligned}$$

The result reveals that the ranking of the main actors is preserved, but even here the data are too weak to give support for the historical facts. As an interim result, we can record that none of the investigated power indices is able to sufficiently explain at least the outstanding position of the Medicean. Hence, we have to take into consideration that the network structure is too stylized to explain the real world with the consequence that some need arise to adjust our model design. Before doing that, we want just – for the sake of completeness – to present the outcome of pre-nucleolus, which is quantified by

$$\nu^*(N, v_1) = (\frac{100}{3003}, \frac{1}{10}, \frac{267}{4003}, \frac{267}{4003}, \frac{267}{4003}, \frac{100}{3003}, \frac{1}{10}, \frac{100}{3003}, \frac{3}{20}, \frac{100}{3003}, \frac{267}{4003}, \frac{267}{4003}, \frac{1}{20}, \frac{267}{4003}, \frac{267}{4003})$$

Even this distribution cannot provide any evidence for the historical events, though the ranking of the main actors is preserved. A comparison of the results is shown in Table 9.2.

Although it seems that this is the end of the story, and that the solution concepts provided by cooperative game theory are completely unsuitable to disentangle the power mesh of Florentine politics, we have to remind ourselves that the pre-nucleolus possesses a marked position within the bargaining range that is spanned by the pre-kernel whenever it is a set-valued solution. This property, i.e., that it is generic not single-valuedness, is deemed by many scholars as a conceptual defect, which turns in this context into a crucial advantage. Since the political institutions of the Florentine Republic were aimed toward a balancing of interests within the elite, though the system of checks and balances was vestigial and vulnerable with the consequence that it was considered as legitimate to impose a political will by force. Nevertheless, an ideal milieu to apply the pre-kernel as a solution which equalizes the bilateral claims for each pair of players and offering a wide range of political settlements whenever it is not single-valued. Contrasted to the power indices which are all single-valued and mainly offering a too balanced power distribution, that is, putting too much power to the weak players that seems unsuitable to identify the main actors for an epoch where the rule of force was predominant. Thus, we have to figure out if the pre-kernel is single-valued or not.

For pursuing the latter piste, we have to study whether the imposed weighted majority game has a homogeneous representation without any winning or veto-player to conclude that the pre-kernel is a set-valued solution and span an enlarged settlement range. Clarifying that the game has neither a winning nor a veto-player is left to the reader. In contrast, determining whether the network weighted majority game has a homogeneous representation is a little bit more cumbersome as for the case of the generic weighted majority game. Therefore, we endow the reader with some assistance to answer this issue. Analogously, it requires to single out the set of minimal winning coalitions, but now from the underlying sub-graph problem. Having them single out, one has again to figure out that the cumulative vote equals the quorum for each minimal winning coalition (cf. Formula (2.4)). Notice that a weighted majority game having a homogeneous representation is said to be homogeneous, which can be affirmed in this case.

Apart from the homogeneity, we can figure out that the game possesses neither winning nor veto-players. Due to Sudhölter (1996), we can draw the conclusion that the pre-kernel is a star-shaped set-solution spanning some bargaining range. A non-empty subset $\mathcal{X}$ of the real vector space $\mathbb{R}^n$ is denoted as a star whenever it contains an element $\mathbf{x}_0$ such that for every other element $\mathbf{x} \in \mathcal{X}$ the line segment constituted by the endpoints of $\mathbf{x}_0$ and $\mathbf{x}$ is contained in $\mathcal{X}$. A vector $\mathbf{x}_0$ which satisfies this property is called a center solution of $\mathcal{X}$. Obviously, a center solution needs not to be unique. Moreover, for determining a pre-kernel element it is enough to consider the minimal winning coalitions, since the maximum surplus of a player i over a player j is attained by a minimal winning coalition, for all pairs $i, j \in N, i \neq j$. Hence, balancedness of a pre-imputation occurs on the set of minimal coalitions (cf. Peleg (1966); Sudhölter (1996)). Obviously, the pre-nucleolus as part of the pre-kernel is determined by the set of minimal coalitions too. However, identifying the set of minimal winning coalitions is much more harder than to determine the maximum surpluses of players on the power set. Having identified this solution structure of the pre-kernel, it is save to search for a second pre-kernel element. To find an additional pre-kernel element, one can apply the method described by Algorithm 3.1. Having found a second point one can apply the procedure discussed in Meinhardt (2014) to establish that this pre-kernel element – in connection with the pre-nucleolus – is an endpoint of the line segment that forms the pre-kernel solution of that game, hence the pre-kernel is given by the convex hull

$$\begin{aligned}\mathscr{PK}(v_1) = conv\Big\{&(0,0,0,0,0,0,1,0,1,0,0,0,0,0,0)/2,\\ &(\frac{100}{3003},\frac{1}{10},\frac{267}{4003},\frac{267}{4003},\frac{267}{4003},\frac{100}{3003},\frac{1}{10},\frac{100}{3003},\frac{3}{20},\frac{100}{3003},\frac{267}{4003},\frac{267}{4003},\frac{1}{20},\frac{267}{4003},\frac{267}{4003})\Big\},\end{aligned}$$

and having two center solutions, namely the endpoints. Moreover, the variance of the second pre-kernel point is about 309.52. The largest value recognized so-far and indicating outstanding positions.

Inspection of the found pre-kernel endpoint reveals to us two outstanding protagonists, namely the Guadagni at coordinate seven and the Medici at coordinate nine. Finally, we get a result that fits to one event of the historical accounts. The clash of Bernardo Guadagni with Cosimo de' Medici. Recall that Bernardo Guadagni was the adversary who managed to arrest, imprison and exile Cosimo de' Medici that resulted in a political retaliation as the Medici managed to overturn their banishment while returning back to Florence for taking revenge. The figures suggest that both parties were of equal strength so that the final outcome cannot be predicted, but a clash among these parties seem conceivable and supportable by this power distribution. Albeit these figures seem to be convincing on a first glance, we have nevertheless to contrast them with the historical events. It is well known that Bernardo became head of the government (Gonfaloniere) by the support of Rinaldo degli Albizzi, who settled his debts in order to make him eligible for office. And it was the pressure of Rinaldo that Cosimo was summoned and arrested by Bernardo in September of 1433. Despite his liability against Rinaldo, the impecunious Bernardo was bribed by Cosimo to retreat from office on health grounds that allowed him to escape from the death sentence (cf. Hibbert (1974, pp. 49-51)). In this sense the data can only be interpreted as a concealed operation, initiated by Rinaldo degli Albizzi behind the scenes.

For some, our first investigation of the Florentine elite marriage network by means of cooperative game theory has established a surprising outcome. None of the presented power indices could give sufficiently large support for an outstanding position of the Medici dynasty or of one of their rival in Florentine politics. Solely by the second pre-kernel element, two outstanding protagonists can be identified, and from which a link to a historical event can be drawn. A historical outcome is never the result of a clear and unique chain of events. A wide variety of factors have determined the issue, not a sole one. And it was for contemporaries not predictable at all of how events evolve over time and what would be the final outcome. Apparently, they were aware of their opportunities set, and how they can use it in their interest. But they could never be certain about the next move of their opponents. An event, negligible to contemporaries, could incline the weighing pan into one's favor or disfavor. To capture a broader range of possible outcomes, single-valued solution concepts are not flexible and sensitive enough, a flexibility that can be provided by a set-solution whenever it is not too large to slip off in arbitrariness or tautologies. The pre-kernel as a set-solution that balances the maximum surpluses of each pair of players, i.e., their maximum bilateral claims, and which may define a wider bargaining range in which a possible outcome may materialize, can exactly accomplish this kind of task. Hence, we should not just regard it as a tool for the investigation of the bargaining set with nice mathematical properties (cf. Peleg (1966)), even though it was already realized by the same author that it represents an extreme form of negotiation (cf. ibid. p.1), rather it ought to be considered as a useful tool to investigate market behavior (cf. Ostmann and Meinhardt (2007, 2008); Meinhardt (2018)) or even of historical events as in the case of Florentine politics to significantly explain the accession to power of the Medici or the failure of their rivals. To what extent the pre-kernel can perform in this direction is the subject matter to analysis in the remaining part of our study.

5.1 Some Modifications in the Model Setting

A crucial feature of the implemented network structure of Holler and Rupp (2020) was the underlying assumption that each player node has equal weight implying that the weighted majority graph problem was coded symmetrically. To take account of asymmetric power to control a network of clansmen, we referring to the net-wealth as it was reported by Table 6 of Breiger and Pattison (1986, p. 239)). During the Middle Ages and Renaissance people lived in hierarchical societies that are based in most Italian cities on clientage systems. Financial means were used as a vehicle to build up and to control a network of supporters in order to implement a collective common action in the political decision-making process. It is indispensable that the accession to power of the House of Medici had been taken place without the acquired fortune from their banking activities. By these considerations, we use the net-wealth values as a proxy for financing a clientage system and to exercise control on it. Reported net-wealth of the 1427 catasto are denominated on thousand florins, where we use the rounded values as they are provided by the R software package netrankr of Schoch (2020a). From the data panel, the entry of the Pucci family is once again removed. Hence, the weight vector of net-wealth is represented by the subsequent array of positive integers

$$w_{1w} = (10, 36, 55, 44, 20, 32, 8, 42, 103, 48, 49, 27, 10, 146, 48),$$

to which we refer to as w_{1w}. This ranking of net-wealth is basically supported by the source of de Roover (1966, p. 28), who identified from the catasto of 1427 Giovanni di Bicci de' Medici on the third place rather on second, surpassed by the Panciatichi brothers and Palla Strozzi. Thirty years later, the catasto of 1457 records that the Medici were by far the richest family (cf. ibid p.28). Nevertheless, we have to be cautious in interpreting these reported net-wealth data at face value. As was it emphasized by Hibbert (1974, p. 60), Cosimo de' Medici was a prudent men who kept special accounts while exaggerating bad debts with the intention to declare a much lower taxable fortune than it really was. This implies that the wealth situation of the Medici is undervalued, and their power may be underestimated too. However, creative accounting can also be expected for the other main actors in Florentine politics, albeit the extent is unclear and whether the effect is statistically offset. More crucially we judge that the Medici – as bankers operating on an international level with branches spread across Europe – were able to dissemble their fortune before the eyes of their adversary, as it did Cosimo de' Medici in autumn of 1433 shortly before he was arrested while transferring huge amount of sums of money from his bank in Florence to his branches in Rome and Naples to keep them safe from confiscation by the Albizzi (cf. ibid. p.49). Due to this smart hedging of his fortune, all attempts of Rinaldo degli Albizzi failed to bankrupt him during his exile of 1433-1434 (cf. ibid. p.54). A practice he picked up later to secure his power while using the Florentine taxation system to break his enemies (cf. ibid. p.61). From this point of view, the net-wealth figure reported by the catasto for the Medici is certainly undervalued.

Despite these doubts and in the absence of any better and more sound data set we still make use of them. Pursuing our analysis in this direction, we assume a simple majority rule through the quorum to $\mathtt{q}_{1w} = 340$. We denote the associated simple game by $\langle N, v_{1w} \rangle$. Then the adjusted weighted majority graph problem is specified by $\mathtt{wMP}_{1w} =: (\mathtt{G}_1, w_{1w}, \mathtt{q}_{1w})$. For getting the reference solution, the Holler index, associated to this new underlying graph problem is provided by

$$\begin{aligned}\phi^{HI}(N, v_{1w}) = (&0.0335, 0.0871, 0.0780, 0.0835, 0.0664, 0.0390, 0.0945, 0.0524, 0.1115,\\ &0.0378, 0.0640, 0.0731, 0.0615, 0.0500, 0.0676),\end{aligned}$$

and the distribution of the public values are given by

$$p(N, v_{1w}) = (55, 143, 128, 137, 109, 64, 155, 86, 183, 62, 105, 120, 101, 82, 111).$$

Comparing the power distribution of the Holler index with the initial graph problem $\mathtt{wMP}_1$ reveals that the main actors getting less power though their ranking is preserved, and the variance is diminished from 5.2618 to 4.9081. Though the Peruzzi (11) gained on power, whereas the Strozzi (14) as the most wealthy family in accordance with the catasto is still losing power and amplifying their unimportance, which is astonishing while falling short of their actual influence. Once again the result does not mimic the historical facts, and the Holler index seems not be a suitable for that purpose. This lets us immediately turn to the Johnston index that performed best from all other power indices but not good enough during the foregoing analysis to explain the historical events. To this end, computing the Johnston index gives

$$\begin{aligned}\phi^{JI}(N, v_{1w}) = (&0.0033, 0.0380, 0.0689, 0.0609, 0.0588, 0.0082, 0.0637, 0.0129, 0.2304,\\ &0.0069, 0.0291, 0.1165, 0.0113, 0.2516, 0.0395).\end{aligned}$$

This power distribution is still clustered too close around the mean with a variance of 59.6757 (cf. Table 9.11) so that none of the figures can be designated as outstanding, and does not provide any support to the historical events either. Neither it places the Medici on the first rank nor it preserves the ranking of the other families. Remarkable is the fact that another great rival of the Medici assumes their place as the most accentuated actor, that is the Strozzi clan. This is surprising from the point of view of the previous investigation, there all indices have not attributed to them the strength of an equivalent rival of the Medici family, rather one had to classify them as a negligible political actor without any means to influence an event in their favor. Astonished, one has to recognize that the Strozzi – in accordance with this power distribution – are stronger than the Medici. Probably strong enough to prevent their rise to power. Knowing that the records are speaking a totally different language, we have to assess this power attribution as excessive. Another distinctive feature is the third rank of the Ridolfi, though this family produced Lorenzo Ridolfi (1362-1443) and Breiger and Pattison (1986) made them an object of investigation of the block analysis, their political influence seems exaggerated too (cf. Table 9.2). Obviously contradicting the annals in respect thereof. Observing that the above distribution has completely reversed the trend, we dismiss it. Letting us focus on the pre-nucleolus instead, which does not change the trend either as we notify through

$$\nu^*(N, v_{1w}) = (\frac{47}{5000}, \frac{140}{1649}, \frac{359}{3807}, \frac{283}{5000}, \frac{89}{4709}, \frac{47}{5000}, \frac{283}{5000}, \frac{89}{4709}, \frac{709}{3579}, \frac{89}{4709}, \frac{151}{2000}, \frac{283}{5000}, \frac{89}{4709}, \frac{83}{400}, \frac{151}{2000}).$$

This distribution attributes the highest power to the Strozzi (14), followed by the Medici (9), Barbadori (3), Albizzi (2), and Peruzzi (11), however, the accentuated position of the Ridolfi was with rank 9 not confirmed. Hence, it mainly preserved the previous ranking on the first two position, though it adjusts their power downward. With a variance of 38.8377 (cf. Table 9.11) an outstanding position of the Medici or any other major protagonist in the Florentine politics cannot be affirmed. In the same vein, the Shapley-Shubik and the Banzhaf value failed to explain the historical incidents (see Table 9.3). This lets us remind that the pre-nucleolus as part of the pre-kernel reflects only a marked position within the settlement range spanned by a set-valued pre-kernel solution. For a homogeneous weighted majority game without winning or veto-players the set-valuedness of the pre-kernel is guaranteed. Apparently, this representation is not homogeneous implying that in advance no conclusion about the shape can be drawn, nevertheless, the pre-kernel is constituted by the line segment given by

$$\mathscr{PK}(v_{1w}) = conv\Big\{(0,0,1,0,0,0,1,0,1,0,0,0,0,1,0)/4, \\ (\frac{47}{5000}, \frac{140}{1649}, \frac{359}{3807}, \frac{283}{5000}, \frac{89}{4709}, \frac{47}{5000}, \frac{283}{5000}, \frac{89}{4709}, \frac{709}{3579}, \frac{89}{4709}, \frac{151}{2000}, \frac{283}{5000}, \frac{89}{4709}, \frac{83}{400}, \frac{151}{2000})\Big\},$$

with two center solutions, namely the endpoints. Moreover, the variance of the second endpoint is about 130.95, and it reveals that the set of actors can be reduced to four main actors. All of them can be classified due to the variance as outstanding. These protagonists are the Medici, the Strozzi, and the Guadagni as one may expect by the historical incidents, but not, as one might expect the Albizzi clan, the enemies of the Medici, as the fourth main actor in Florentine politics, rather a new major actor appears with the Barbadori. This result is surprising, since this clan was neither recognized as a principal actor in the chronicles nor can we assign to them an outstanding position within the network structure of the Graph 1. Analogously to the historical accounts – in particular, Hibbert (1974); Padgett and Ansell (1993) assigned them to the Albizzi faction – they occupied just a peripheral position there, albeit the 1427 catasto ranked them on position three after the Strozzi and Medici, but outstripping the Albizzi and Guadagni (cf. wealth vector w_{1w} on Page 25). Though Niccolò Barbadori was one of those seventy names who felt the revenge of the Medici party while being banished (cf. Hibbert (1974, p. 58)). Or do we have determined a novel main actor whose role in the Florentine power struggle was not really recognized by historians? We cannot answer this question, since it goes beyond our analysis. Nevertheless, this issue is interesting enough to resume this thread in Subsection 7 again to exclude that it was an unique event.

5.2 Including the Combined Ties into the Network Structure

Although the pre-kernel solution confirmed three main protagonists who played a principal role in the Florentine republic, the result seems disputable on the ground that it even assigns an outstanding position to the Barbadori clan that could not be affirmed by the annals. From this point of view, we have to contest our approach, and we consider some need to redesign the model in order to grasp better the underlying mesh of power between the leading families. For modifying the game model in this direction, we introduce in the marriage network of the above 16 families, the number of combined business and marriage ties to whom each family is connected across all 116 families as it was reported by Table 6 of Breiger and Pattison (1986, p. 239)). Thus, we impose on the network structure with its 16 families an additional layer to take account in a stylized form the more rich and comprehensive family relations of the Florentine elite in the hope to increase the gravity of the main families in order to entangle their power network.

The combined number of business and marriage ties are quantified by the subsequent array

$$w_{1t} = (2, 3, 14, 9, 18, 9, 14, 14, 54, 7, 32, 4, 5, 29, 7).$$

Noteworthy is the fact that the Albizzi dynasty, as the actual ruler of the republic of Florence, counts just three family ties. Contrasted to the 54 ties of the Medici clan as best integrated actor in the Florentine mesh of power. Applying again a majority voting rule, the quorum $\mathfrak{q}_{1t}$ is set to 111. The associated simple game is denoted by $\langle N, v_{1t}\rangle$. By far the Johnston index performed best from all power indices, we provide its power distribution by

$$\begin{aligned}\phi^{JI}(N, v_{1t}) = (&0.0015, 0.0268, 0.0880, 0.0522, 0.0866, 0.0060, 0.0801, 0.0118, 0.3405,\\ &0.0031, 0.0582, 0.0804, 0.0076, 0.1297, 0.0275).\end{aligned}$$

Inspection of the power distribution reveals that the Medici (9) are positioned on the first rank followed by the Strozzi (14), Barbadori (3), Castellani (5), Ridolfi (12), Guadagni (7), and Peruzzi (11). According to these figures the Albizzi and Guadagni do not play a principal role in the power struggle. This is not surprising, since their network ties with 3 respective 14 combined counts seem too weak of considering them as principal actors. A distinctive feature is that the Ridolfi with just 4 ties are surpassing from rank 13 to 5. In comparison to the previous analysis of the Johnston index, they lose two places, since they lose some power. In relation to the other main actors, their power seem to excessive. These both counterfactual results contradicting the historical facts. Nevertheless, the variance of the power distribution in accordance with the Johnston index is close to 72.8866, it is the highest value w.r.t. the investigated sample of power indices letting us to identify a quasi outstanding position of the Medicean and Strozzi.

In the next step, we want again study the pre-kernel solution, as a solution concept that performed by far best in replicating the relative strength of actors in the Republic of Florence. For doing so, we have to remember that for getting a pre-kernel element we need to apply the Algorithm 3.1. The presentation of the pre-nucleolus is skipped her, since it performs worse than the Johnston index. The solution can be retrieved from Table 9.4. For convenience sake, we just mention the ranking of the first five actors and the variance. These are the Medici, Strozzi, Guadagni, Castellani, and Barbadori with a variance of 49.05 assigning no outstanding positions. Although the pre-nucleolus can be eliminated on this ground, we turn our attention to the pre-kernel solution to find an element that may mirror the reported power configuration of the annals. To follow this direction, we need to figure out whether the game has a homogeneous representation or not. If the game is homogeneous without any winning or veto-player it is star-shaped, if not, its shape cannot be anticipated in advance. To make things much more complicated in the latter case, we cannot even expect that the pre-kernel is connected whenever it is set-valued. Some examples of weighted majority games with a disconnected pre-kernel are given by Kopelowitz (1967) and Stearns (1968). In Meinhardt (2013b, Section 8.5) a selected example of Kopelowitz was extensively discussed. Here, we are going to present a new example of a non-homogeneous weighted majority game having a disconnected pre-kernel. Before we present the disconnected pre-kernel of the game, we focus by an initial step on a pre-kernel element that can be support by the recordings, which is given by

$$(0, 0, \frac{1}{12}, 0, \frac{1}{12}, 0, \frac{1}{6}, 0, \frac{1}{3}, 0, \frac{1}{6}, 0, 0, \frac{1}{6}, 0).$$

We observe that it attributes for the Medici the highest power, and places again the Strozzi and Guadagni on position two. The variance of the vector is about 101.19, indicating outstanding positions to the Medici, Peruzzi, Strozzi and the Guadagni, for the last three figures of equal strength. Striking is the fact that for the first time the Peruzzi are emphasized with an outstanding position in Florentine politics in accordance with that power distribution. Followed by the Barbadori and Castellani, both are attributed by Padgett and Ansell (1993, p. 1289) to the Peruzzi faction. Except for the Albizzi, – which is not surprising due to their negligible counts of combined ties – this distribution identifies the major protagonists of the events dated for the period 1433-1434. Providing some empirical evidence that the principles of distributive arbitration in accordance with the pre-kernel may be a basis of a political decision making process.

As hinted in one previous paragraph, the pre-kernel is disconnected, which is constituted by three convex sets, namely a rhombus, a line segment, and as an isolated point, the pre-nucleolus. Applying the methods described by Meinhardt (2014) the extent of each of these separated convex set-valued objects can be easily specified. The rhombus is the convex hull constituted by four extreme points given through

$$\begin{aligned}
S1 = conv\Big\{&(0, \frac{2}{49}, \frac{3}{49}, \frac{2}{49}, \frac{4}{49}, \frac{1}{49}, \frac{9}{98}, \frac{3}{98}, \frac{55}{196}, \frac{1}{98}, \frac{1}{7}, \frac{1}{49}, \frac{3}{196}, \frac{6}{49}, \frac{2}{49}),\\
&(0, \frac{53}{1193}, \frac{157}{2356}, \frac{53}{1193}, \frac{51}{574}, \frac{52}{2341}, \frac{235}{2351}, \frac{78}{2341}, \frac{99}{379}, \frac{1}{90}, \frac{75}{674}, \frac{52}{2341}, \frac{39}{2341}, \frac{157}{1178}, \frac{53}{1193}),\\
&(0, \frac{3}{59}, \frac{3}{59}, \frac{3}{59}, \frac{5}{59}, \frac{1}{59}, \frac{11}{118}, \frac{3}{118}, \frac{16}{59}, \frac{1}{59}, \frac{8}{59}, \frac{2}{59}, \frac{1}{59}, \frac{7}{59}, \frac{2}{59}),\\
&(0, \frac{48}{913}, \frac{48}{913}, \frac{48}{913}, \frac{80}{913}, \frac{16}{913}, \frac{8}{83}, \frac{24}{913}, \frac{54}{205}, \frac{16}{913}, \frac{26}{211}, \frac{15}{428}, \frac{16}{913}, \frac{33}{269}, \frac{15}{428})\Big\}.
\end{aligned}$$

Investigating of the extreme points reveals that the rank order of the first eight positions is identical, except for the second extreme point, where the Peruzzi and the Strozzi exchange their positions. Thus, the order of the eight largest figures is given as follows: apparently the Medici on the first position, followed then by the Peruzzi, Strozzi, Guadagni, Castellani, Barbadori, Albizzi, and finally the Bischeri. Establishing that the ranking of the five major actors is not preserved from the Johnston index. Recalling for convenience sake its ranking, which is given by the Medici, Strozzi, Barbadori, Castellani, and finally the Ridolfi. Contrasted to the discussed pre-kernel element from above, which orders the main four protagonist accordingly. Though the variances of the pre-kernel set $S1$ are far from reaching the foregoing levels with the consequences that these power distributions cannot give any strong support of the historical accounts. The distributions are clustered to close around the mean. Hence, we discard them on the ground that they do not provide any empirical evidence.

Analogously to the former set, we get the second set while applying the methods of Meinhardt (2014) to determine from an extreme point the whole line segment. Notice that the second endpoint is the known power distribution already determined above.

$$S2 = conv\left\{(0,0,\frac{1}{8},0,\frac{1}{8},0,\frac{1}{4},0,\frac{1}{4},0,0,0,0,\frac{1}{4},0),(0,0,\frac{1}{12},0,\frac{1}{12},0,\frac{1}{6},0,\frac{1}{3},0,\frac{1}{6},0,0,\frac{1}{6},0)\right\},$$

The first endpoint has a higher variance than the second, i,e., 108.63 versus 101.19, identifies the same protagonists, except for the Peruzzi, reducing the power of those and the Medici while redistributing it to the aforementioned actors with the consequence that now the Strozzi and Guadagni are of equal strength with the Medici.

This line segment offering a wide range of possible outcomes on which the bilateral claims are balanced among each pair of players, implying by its axiomatization that no subgroup of players should have an incentive to deviate from an agreement and to play their own game in order to improve their situation. Offering a wide range of political settlements on which the main protagonists could find an equilibrium of interests rather than an outbreak of conflict. However, it is hardly conceivable that the leading Florentine families had an unified perception of fairness making it less likely that they had agreed upon on the set of principles of distributive arbitration (axioms) related to the pre-kernel or any other solution, though the Florentine institutions were designed to obtain a balance of interests among the oligarchs. And even if, a pre-kernel settlement would be contestable on the simple reasoning that it could not be stabilized by a figure of argumentation that is based on the balance of claims, since their understanding of fairness were too different and driven by subject feelings. Elucidating that one cannot expect despite a homogeneous mesh of relations among the oligarchs an unitary, cohesive, and purposeful action. Making it impossible to predict the outcome of the game on the basis of these distributions, an outcome finally determined by factors that cannot be grasped by the game model. Nevertheless, it reveals the main figures from the annals and that no event was so apparent after the fact. Establishing that the political equilibrium was fragile, and a small disturbance may determine the final outcome. In this context, we have to remind that the Guadagni are acting on the behalf of the Albizzi, and these endpoints allowing us of making the links to two extreme events: on the one hand the arrest and exile of the Medici, but on the other hand their revenge and prominence. In the former event, the Peruzzi acting as observer while trying to profit from the event with minimal effort, whereas in the latter case, the power of the Medici revived, which was ignored by their enemies, since one supposed the backing of the Strozzi, Barbadori and the strengthened Peruzzi, but was unable to conduct a joint and coherent action. Though the power of the Medici overshadow their opponents, they seem nevertheless not strong enough to subdue them. However, the chronicles report that by ignorance, passivity and disunity of their main rivals, the Medici were be able to outmaneuver them and to impose on them their will.

To conclude, the pre-kernel is the union of these two sets in connection with the pre-nucleolus, hence $\mathscr{PK}(v_{1t}) = S1 \cup S2 \cup \nu^*(N, v_{1t})$. Moreover, it best identifies the main historical protagonists of the Republic of Florence of all investigated solution concepts.

6 Florentine Elite Marriages Reconsidered

As aforementioned, Rinaldo degli Albizzi was the actual ruler behind the scenes of the Republic of Florence during the time period from 1429 till September 1434. Though Bernardo Guadagni was the elected Gonfaloniere, he acted on behalf of Rinaldo, who settled his debts to become eligible for office. Both families were members of the old oligarchs, lived in the same quarter, had close family ties, and were enemies of the Medici, albeit Bernardo refrained from his enmity by a sufficient large amount of bribe money from Cosimo. Despite this latter limitation, the fact that Bernardo insisted of a reciprocal recognition and on his status equality, one may consider both families as an unity. As a side effect, this would uncover the actual power structures. Even though we may consider them as a single actor that does not mean that we can just add their power. Not every solution concept applied in our network analysis satisfies additivity. Although we assume that both families form an à priori union, we have nevertheless to conduct a completely new investigation from scratch. By the annals, the Albizzi played the more prominent role in Florentine politics, that's why we follow Holler and Rupp (2020) while assigning for simplicity the network links of the Guadagni to the Albizzi, and treating them as a null-player, albeit the gravity of the node, i.e., the number of votes, remains unaltered by their unification as one may grasp by Graph 2. We referring to this undirected graph as $\mathtt{G}_2$.

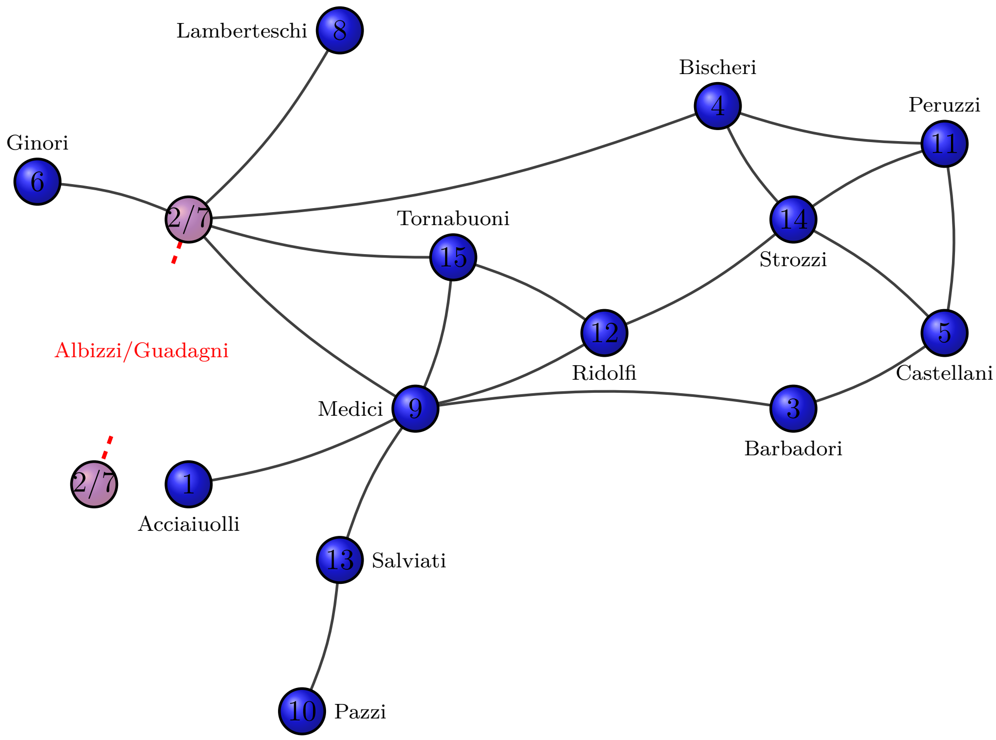

Figure 2: Florentine Elite Marriages 1395-1434: Priori Union Between Albizzi and Guadagni[11]

This undirected graph of Figure 2 represented in form of an edge matrix by

$$\mathbf{mE}_2 = \begin{bmatrix} 1 & 2 & 2 & 3 & 3 & 2 & 4 & 4 & 5 & 5 & 2 & 2 & 9 & 9 & 9 & 10 & 11 & 12 & 12 \\ 9 & 6 & 9 & 5 & 9 & 4 & 11 & 14 & 11 & 14 & 8 & 15 & 12 & 13 & 15 & 13 & 14 & 14 & 15 \end{bmatrix},$$

and the decision rule is set to the quorum $\mathfrak{q}_2 = 8$, so that the weighted majority graph problem

[11]The adjusted network data set with à priori union between the Albizzi and Guadagni families of the Florentine Elite Marriages is based on the file *florentine_m* that ships with the R software package netrankr Schoch (2020a). This network was taken from Figure 2 of Holler and Rupp (2020).

based on Graph 2 is given by

$$\mathtt{wMP}_2 := (\mathtt{G}_2, [1, \underbrace{1, \ldots, 1}_{\text{5 times}}, 0, \underbrace{1, \ldots, 1}_{\text{7 times}}, 1], 8).$$

In the sequel, we referring to the associated simple game as $\langle N, v_2 \rangle$, which is again a 15-person game having in total 32767 non-empty coalitions.

For comparing the power index distribution obtained by Holler and Rupp (2020, p. 11) with our result, which is given by

$$\begin{aligned}\phi^{HI}(N, v_2) = (&0.0581, 0.1114, 0.0745, 0.0788, 0.0673, 0.0532, 0, 0.0532, 0.1199,\\ &0.0347, 0.0622, 0.0722, 0.0794, 0.0728, 0.0622),\end{aligned}$$

and the distribution of the public values are given by

$$p(N, v_2) = (273, 523, 350, 370, 316, 250, 0, 250, 563, 163, 292, 339, 373, 342, 292).$$

Recall that family 7, the Guadagni, are treated as null-player, hence the Holler index attributes to them a power of zero. The above power and public value distributions are identical to the results obtained by Holler and Rupp (2020). Of course, the Medicean are ranked on the first place followed by the Albizzi, actually the Albizzi-Guadagni. However, we treat the Guadagni as null-player so that we just referring to the Albizzi, though we have to keep in mind that they formed an à priori union with the Guadagni. Furthermore, we place the Salviati before the Bischeri, then appears the Barbadori, the Strozzi and the Ridolfi,. Apparently, the Medici have an accentuated position like the Albizzi, but the power of the Strozzi falling short of their actual significance. The variance has increased from 5.2618 to 8.008 (cf. Table 9.11), but this value is still too low to indicate to the Medici or Albizzi an outstanding position. Analogously to page 21, we do not attribute to this power distribution any significant evidence for supporting the annals.

From all indices, the Johnston index performed best in the foregoing examinations of the power structure, for that reason we list its distribution through

$$\begin{aligned}\phi^{JI}(N, v_2) = (&0.0282, 0.1910, 0.0599, 0.0664, 0.0562, 0.0259, 0, 0.0259, 0.2749,\\ &0.0152, 0.0383, 0.0594, 0.0623, 0.0593, 0.0372).\end{aligned}$$

This distribution has the highest variance of all power indices, it is about 51.8961. Apparently, this value is lower than for the Johnston index of game v_{1t} (cf. Table 9.11). Thus, this power distribution as well as any other distribution cannot give any strong support for an outstanding position of any major actor like the Medici or Albizzi. For sake of completeness, we want just mention that in comparison to the Holler index, the Bischeri and Salviati change their positions as well as the Ridolfi and Strozzi. All index solutions in connection with the pre-nucleolus can be retrieved from Table 9.5.

After having investigated the power indices, we turn our attention to the pre-kernel solution to figure out if it could provide some higher evidence to reflect the accounts. For doing so, homogeneity of the game must be checked in a first step, which can be confirmed. Then we must test on the presence of any winning or veto-player, which can be denied. Therefore, the pre-kernel is star-shaped. To determine its exact shape, we apply the procedure of Meinhardt (2014) to establish that it spans a bargaining range constituted by the pre-nucleolus and the point

$(0,1,0,0,0,0,0,0,1,0,0,0,0,0,0)/2$. Hence, the pre-kernel has the same structure as game v_1, it is a convex hull

$$\mathscr{PK}(v_2) = conv\,\{(0,1,0,0,0,0,0,0,1,0,0,0,0,0,0)/2,\\ (2,8,4,4,4,2,0,2,9,2,4,4,3,4,4)/56\},$$

with two center solutions, namely the endpoints, whereas the first endpoint of the pre-kernel segment has a variance of 309.52 providing strong support of an outstanding position of the Medici and Albizzi. Notice that the Albizzi has inherited the power of the Guadagni of game v_1. Both protagonists have equal strength and it is therefore not predictable who can assert oneself against the other. Hence, we may make the link to the clash of Rinaldo degli Albizzi with Cosimo de' Medici. Contrasted to the identified clash of Bernardo Guadagni with Cosimo on page 24, the present power distribution replicates the actual accounts, and do not place Rinaldo behind the scenes.

6.1 Including the Net-Wealth into the Network Structure

Analogously to the foregoing investigation of Section 5 we deviate from the assumption of equal gravity for each node while taking account of asymmetries within the network structure. As an initial step, we refer to the net-wealth data as they were reported by Breiger and Pattison (1986, p. 239)) to overcome the underestimation of the Strozzi clan by the foregoing analysis. Pursuing the study in this direction, the adjusted net-wealth vector is given by

$$w_{2w} = (10,44,55,44,20,32,0,42,103,48,49,27,10,146,48),$$

to which we refer to as w_{2w}. Finally, we impose a simple majority voting rule given by the subsequent quorum $\mathtt{q}_{2w} = 340$, whereas the undirected graph $\mathtt{G}_2$ remains unaltered. Hence, the modified weighted majority graph problem based on Graph 2 is specified by

$$\mathtt{wMP}_{2w} := (\mathtt{G}_2, \mathbf{w}_{2w}, \mathtt{q}_{2w}).$$

A summary of the results is grasped in Table 9.6. Here, we restrict ourselves to present the main ingredients of the results. Firstly, the public good index ranks the Medici before the Albizzi, but failed to support the position of the Strozzi. Moreover, the variance is with 9.7811 rather weak to recognize for any main protagonist of the chronicles an outstanding position, that's why we dismiss this index on that ground. Again the Johnston index performs best. The variance of its power distribution is highest of all power indices pointing out a figure of 58.9851, though in our opinion, not high enough to recognize for any major actor any outstanding position. Nevertheless, it identifies the Strozzi, Medici and Albizzi as a leading actors, in this order. The Strozzi clearly outstrips the Albizzi, and they are even stronger than the Medici. In comparison with game v_{1t} they lose some power, but not as much as the Medici (cf. Table 9.3). Making them relatively more powerful, and making it much more likelier to prevent the rise of Medici. Once more, we consider on this ground their power as overestimated.

Concerning the pre-nucleolus, it attributes the same ranking as under the Johnston index, and provides the Medici as well as the Strozzi with almost the same power, but reduces the power of the Strozzi. However, not sufficiently enough to reverse the trend. Therefore, we consider also the power of the Strozzi as overvalued under the pre-nucleolus. This lets us turn to the pre-kernel to identify a bargaining range that fits better with the annals. By doing so, we test on homogeneity, which can be denied. Thus, we may expect a disconnected pre-kernel, but even this can be rejected, though it is a non-convex set given by the union of the three line segments having as its center the pre-nucleolus. To be more precise, these three line segments are

$$\begin{aligned}S1 &= conv\,\{(0,24,18,12,6,3,0,3,40,4,12,6,4,42,12)/186,\\ &\quad (0,2,1,0,0,0,0,0,2,0,1,1,0,2,1)/10\},\\ S2 &= conv\,\{(0,24,18,12,6,3,0,3,40,4,12,6,4,42,12)/186,\\ &\quad (0,2,1,0,1,0,0,0,2,0,0,0,0,2,0)/8\},\\ S3 &= conv\,\{(0,24,18,12,6,3,0,3,40,4,12,6,4,42,12)/186,\\ &\quad (0,1,1,1,0,0,0,0,2,0,0,0,0,2,1)/8\},\end{aligned}$$

and the pre-kernel is formed by $\mathscr{PK}(v_{2w}) = S1 \cup S2 \cup S3$. The distribution of the endpoint of the second line segment has highest variance with 108.63 indicating outstanding positions to the Medici, Albizzi and Strozzi, but contrasted to the power indices with equal strength. One may think that the Strozzi and Albizzi were together strong enough to prevent the rise of the Medici, so why not add them together. First of all the pre-kernel is not an additive solution. Secondly, we have to remind that the oligarchs network was unable to conduct an unitary, cohesive, and purposeful action. And though both families "were patrician to the core" (cf. Padgett and Ansell (1993, p. 1284)), mainly follow the same interests, consider the Medici as their enemies, they nevertheless follow different political agendas. In addition, the Strozzi were extremely rich bankers allowing them to retain their independence, and judge every political action under the angle whether it is bad for business or not. Under this perspective we have to classify that Rinaldo degli Albizzi and Palla Strozzi pulling in the same direction during the subversion of 1433 as Rinaldo tried to assemble more troops, he got the support of Palla, though their effort were offset by other supporters changing their mind (cf. ibid. p.1279). But Palla, a moderate in Florentine politics, committed desertion as Rinaldo claimed death sentence against Cosimo while those got backing from mighty foreign customers – like the Pope – of the Medici Bank with the consequence that Rinaldo had finally to accept Cosimo's banishment instead (cf. Hibbert (1974, p. 52)). From this context, this distribution may make the link of Cosimo's exile.

6.2 Including the Combined Ties into the Network Structure

Pursuant to Subsection 5.2 we include now the combined business and marriage ties to change the gravity of the nodes. Due to the fact that the Albizzi and Guadagni form an à priori union, and the Guadagni are treated as null-player, we attribute their ties to the Albizzi clan while augmenting their counts from 3 to 17, and annihilating their gravity, then the weights vector is specified by

$$w_{2t} = (2,17,14,9,18,9,0,14,54,7,32,4,5,29,7),$$

whereas the quorum remains unaltered with $\mathtt{q}_{2t} = 111$. Apparently the weighted majority graph problem is still based on Graph 2, and is given by

$$\mathtt{wMP}_{2t} := (\mathtt{G}_2, \mathbf{w}_{2t}, \mathtt{q}_{2t}).$$

Table 9.7 contains a synopsis for the results of the power indices and pre-nucleolus. Here, we confine ourselves to the indices that support best the annals. First of all, we referring to the power distribution of the Johnston index with a variance of 80.3623, the best value seen by far for a power index. Identifying as major protagonists the Medici, Albizzi, and Strozzi, followed by the Barbadori. Thus, incorporating the cumulated business and marriage ties into the network structure and attributing the power of the Guadagni to the actual ruler of the Florentine politics bring to the fore the Albizzi while revealing their power, and place them on the second position. Disentangling the real power structure, and revealing who is the actual antagonist of the Medici. Allowing us of pointing to the revenge and prominence of the Medici while attributing them a power of 0.353 that overshadows the power of the Albizzi with 0.126 and those of the Strozzi with 0.1141. Still we do not consider the Medici as strong enough to subdue their rivals and to impose on them their will. Though we have to remember that it was ignorance, passivity and disunity of their main rivals that allow them to seize the power in the Republic of Florence, and incline the weighing pan into their favor. Nevertheless, we judge the variance still not as sufficient enough to give any significant support for making a link to the annals.

Turning to the pre-nucleolus shows that the variance of its distribution is worse than the distribution of the Johnston index, hence we dismiss it (cf. Table 9.11). Even worse is the situation for the Shapley-Shubik index or Banzhaf value. Letting us again immediately turn to the per-kernel as a solution that settles the bilateral claims of the pair of players, providing a wide range of outcomes where actors should be able to manage a balance of interests. In this context, we have to recall that the political institutions of the Florentine Republic were oriented toward equalizing the interests between the oligarchs, since only within a pacified state, bankers and traders found the environment for running a successful and profitable business implying that the pre-kernel solution ought to be best adapted for such a milieu, we have nevertheless to remind ourselves that the late Middle Ages and early Renaissance was an era of the right to private warfare. Thus, the political equilibrium was fragile, and the main protagonist considered it as legitimate to impose on the political decision process the rule of force making it difficult to anticipate the final outcome.

It is worth noting an interesting difference between the pre-kernel element determined for game v_{1t}, and its counterpart for the current game while providing a power distribution with an equal variance of 101.19. Striking is fact that ceteris paribus the Albizzi and Guadagni interchange their positions, as we see through

$$(0, \frac{1}{6}, \frac{1}{12}, 0, \frac{1}{12}, 0, 0, 0, \frac{1}{3}, 0, \frac{1}{6}, 0, 0, \frac{1}{6}, 0).$$

Hence, the Albizzi taking the part of the Guadagni, and the distribution bring to the fore the actual ruler, who had so far been covered up behind the scenes. Enabling us to identify all major protagonists of the events dated for the period 1433-1434, namely, the Medici, Albizzi, Strozzi and the Peruzzi, the last three with equal strength. And due to the variance, all major actors can be attributed outstanding positions.

The game has not a homogeneous representation and possesses a pre-kernel that is disconnected, which is constituted by three line segments. Again applying the procedures described by Meinhardt (2014) the extent of each of these separated line segments can be easily specified through

$$
\begin{aligned}
&S1 = conv\Big\{(0,8,4,2,4,1,0,1,12,0,4,2,0,6,2)/46,\\
&\quad(0,10,4,2,4,1,0,1,14,0,6,4,0,6,2)/54\Big\},\\
&S2 = conv\Big\{(0,8,4,2,4,1,0,1,15,0,8,2,1,6,2)/54,\\
&\quad(0,\frac{79}{494},\frac{79}{988},\frac{78}{1951},\frac{79}{988},\frac{39}{1951},0,\frac{39}{1951},\frac{212}{815},0,\frac{25}{208},\frac{78}{1951},\frac{39}{1951},\frac{77}{642},\frac{78}{1951})\Big\},\\
&S3 = conv\Big\{(0,2,1,0,1,0,0,0,4,0,2,0,0,2,0)/12,\\
&\quad(0,2,1,0,1,0,0,0,2,0,0,0,0,2,0)/8\Big\},
\end{aligned}
$$

and the pre-kernel is formed by $\mathscr{PK}(v_{2t}) = S1 \cup S2 \cup S3$. Line segment $S3$ is offering by its endpoints highest variances, and therefore outstanding positions to the major actors allowing to make the link to two historical events, that is, the arrest and exile of the Medici by the latter endpoint, their revenge and seize of power by the former. Contrasted to Subsection 5.2, this time with the correct sign, the Albizzi onstage.

7 Florentine Elite Business Relations

We continue the power index analysis of the Florentine family network while focusing on those 11 families from the basic population of the 16 leading families that have business relationships among each other.

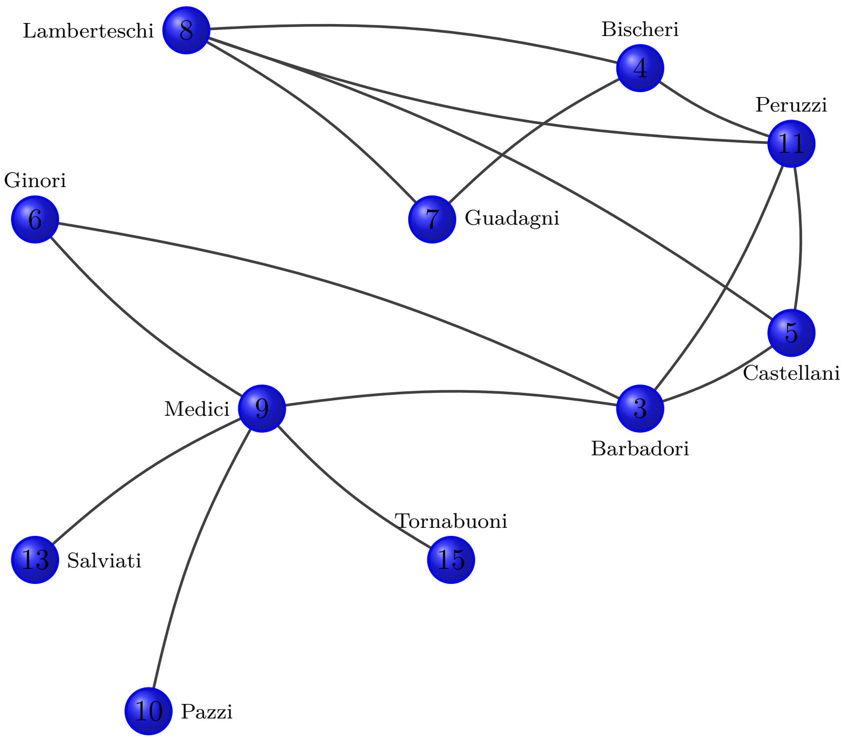

Figure 3: Florentine Elite Business Relations 1395-1434[12]

This removes the Acciaiuolli, Albizzi, Pucci, Ridolfi, and Strozzi from the network structure, since to those no intratrade relations can be assigned (cf. Table 4.2), leading to a reduced form of the weighted majority graph problem based on Figure 3, to which we refer to as $\mathtt{G}_3$.

To get the associated simple game, we follow the assumptions made by Holler and Rupp (2020, p. 12) to assign to each player node an equal weight of one, and just apply a simple majority rule. Hence, the set of agents has cardinality 11, for all $k \in N$, an agent k has a weight of $w_k = 1$, and the quorum is set to $\mathtt{q} = 6$, so that the weighted majority graph problem based on Table 4.2 is given by

$$\mathtt{wMP}_3 := (\mathtt{G}_3, [1, \underbrace{1, \ldots, 1}_{\text{9 times}}, 1], 6).$$

As described above, these information are enough to derive a voting game. Whereas the undirected graph $\mathtt{G}_3$ from Figure 3 is represented in form of an edge matrix through

$$\mathbf{mE}_3 = \begin{bmatrix} 3 & 3 & 3 & 3 & 4 & 4 & 4 & 5 & 5 & 6 & 7 & 8 & 9 & 9 & 9 \\ 5 & 6 & 9 & 11 & 7 & 8 & 11 & 8 & 11 & 9 & 8 & 11 & 10 & 13 & 15 \end{bmatrix},$$

The undirected graph from Figure 3 has in total 15 edges that is captured in the associated edge matrix $\mathbf{mE}_3$, whereas the matrix used by Holler and Rupp (2020, p. 12) has just 14 edges. We referring to the associated simple game as $\langle N, v_3 \rangle$, which has in total 11 non null-players and 4 null-players. It would be sufficient to just rely on the 11-person representation, but computing the associated 15-person game with its 32767 non-empty coalitions is under MATLAB no great deal, which requires some additional seconds of computation time.[13]. Applying the representation used by these authors, we are able to reproduce their results of the public good index and public values. Using the edge matrix $\mathbf{mE}_3$, we get in lieu thereof

$$\begin{aligned}\phi^{HI}(N, v_3) = (&0, 0, 0.1667, 0.0718, 0.0949, 0.0718, 0.0486, 0.0949, 0.1528,\\ &0.0602, 0.1181, 0, 0.0602, 0, 0.0602),\end{aligned}$$

and the distribution of the public values are given by

$$p(N, v_3) = (0, 0, 72, 31, 41, 31, 21, 41, 66, 26, 51, 0, 26, 0, 26).$$

The distribution of the public good index reveals that there are in total 15 players in the game rather than 11. Four of them are receiving nothing indicating that those are treated as null-players. These are the Acciaiuolli (1), Albizzi (2), Ridolfi (12), and Strozzi (14).

Restricting the positioning of the public value distribution to the first five families, places the Barbadori on first with a public value of 77, followed by the Medici (66), Peruzzi (51), finally the Castellani (41) and Lamberteschi (41) with equal strength. Contrasted to the order from Holler and Rupp (2020, p. 13) with the Barbadori on first having a public value of 56, whereas the Medici (51) and Peruzzi (51) doing equally well. Four families have a public good value of 25, three of

[12] The applied network data set of the Florentine Business Relations is part of the file *flo_business* that ships with the R software package networkdata of Schoch (2020b) that is based on Table 1 of Breiger and Pattison (1986), see also Table 4.2.

[13] The evaluation time of the routine from our toolbox MatTuGames 2020a to determine the simple games needed in general not more than 10 seconds on a thin compute node with 40 cores of the Intel Xeon Gold 6230 processor and 90 GB physical memory.

21, and the Guadagni are the last with a value 15. The above power distribution has by far with a variance of 28.5334 highest value ever attributed to the public good index, however, we judge its value still as too weak to assign to any figure an outstanding position.

A synopsis of the results can be retrieved from Table 9.8. In the course, we confine the discussion of the results to the Johnston, Shapley-Shubik index and the pre-nucleolus as a solution that is heavily based on bargaining considerations.

$$\begin{aligned}\phi^{JI}(N, v_3) = (&0, 0, 0.3745, 0.0388, 0.0781, 0.0419, 0.0267, 0.0614, 0.1983,\\ &0.0229, 0.1117, 0, 0.0229, 0, 0.0229).\end{aligned}$$

The power distribution has a variance of 100.59 and attributes to the Barbadori, Medici, and Peruzzi outstanding positions. Remarkable is that the Barbadori are positioned at the first place with a power of 0.3745, they widely overshadow the Medici with 0.1983 as well as the Peruzzi with 0.1117. This is astonishing, since like the Peruzzi and Strozzi, they are just regarded as an ally of the Albizzi. Neither they did play a dominant role during the events of 1433-1434 nor it was recorded that they dominated the course of business. Though their remarkable position seems to be surprising w.r.t. the chronicles, this is not anymore the case by a more thorough inspection of the Graph 3, which reveals that they occupied a very central position within the network contrasted to the Graphs 1 and 2, where they just inhabited at the periphery. As it becomes more clear when we are going to discuss the pre-nucleolus, they hold now an inevitable position in the underlying graph problem $\texttt{wMP}_3$, i.e., all successful relations between the families must run via the Barbadori. Even though the discussed power indices intrinsically balancing the distribution of power among the players, they are also performance-based measures while satisfying in general local monotonicity, an exception is the Holler or public good index. Implying that the seized position by the Barbadori must be compensated by an increase in power as it was registered for the Johnston index. The assignment of power to the Barbadori is much more impressive for the Shapley-Shubik index, as we observe through

$$\begin{aligned}\phi^{SSI}(N, v_3) = (&0.0000, 0.0000, 0.4276, 0.0435, 0.0792, 0.0522, 0.0355, 0.0585, 0.1419,\\ &0.0224, 0.0943, 0.0000, 0.0224, 0.0000, 0.0224).\end{aligned}$$

Moreover, the variance of the power distribution with 115.91 is also much higher than under the Johnston index that intensifies their outstanding position. Apart of the Barbadori, even the Medici as well as the Peruzzi can be assigned with the attribute outstanding. Preserving the ranking of the identified principal protagonists. However, even this result will be outperformed by the pre-nucleolus.

This is due that the Barbadori family occupy a central position within the graph making them to an inevitable player for forming a winning coalition. As one can directly observe by Graph 3, the Barbadori subdivides the graph into two equal sub-graphs having each 6 nodes, where the Barbadori clue them together. Therefore, each winning coalition needs the Barbadori, otherwise they are loosing. Under the figure of argumentation of the pre-kernel, the negotiation power of the Barbadori is so strong that they push each other family against their outside option. An actor i other than the Barbadori cannot claim an amount that goes beyond the outside option, otherwise this can be countered by j while referring to those coalitions that contain j but not i (cf. Meinhardt (2013b, Chap. 3)). Noteworthy is that the player set is complete and directed, in which the Barbadori as a power of veto is the most desirable actor, which must be recompensed by the other actors. By homogeneity and veto-player, the pre-kernel is single-valued and coincides with the pre-nucleolus, and assigning the total spoil to the veto-player at position 3. Thus, it quantifies by

$$\mathscr{PK}(v_3) = \nu^*(N, v_3) = \{(0,0,1,0,0,0,0,0,0,0,0,0,0,0,0)\},$$

with a variance of 666.667. The highest variance ever seen, and attributing solely to the Barbadori an outstanding position. A prominence never indicated in the annals. This lets us infer that the network structure of Graph 3 is too stylized of identifying the actual actors. That these business mesh is thinned out, can be immediately recognized that neither the Ridolfi nor the Strozzi have any commercial links. On a first glance, it is hardly conceivable that the Ridolfi as traders should not have any commercial relations with the other families. More astonishing is the missing of the Strozzi. As the second largest bank behind the Medici Bank, the fact that the Strozzi are according to the catasto richer than the Medici, and in permanent rivalry with them on political and financial supremacy, should not have any business links, does not seem possible.[14] We conclude that these thinned out business relations certainly do not correctly replicate the trade links between the Florentine families. Most of these links are still hidden in the archives and await its discovery, in the worst case they are lost for ever. Just with a sufficient correct image of the trade relations of the Florentine families, their power structure can be revealed. Rather to wait for these discoveries, we impose in the course some additional gravity on the player nodes while considering again the net-wealth data and combined families ties in the hope to get a sharper image of the real power structure.

7.1 Including the Net-Wealth into the Network Structure

To this end let us introduce again the net-wealth values into the network structure. For doing so, the adjusted array of net-wealth values is given by

$$w_{3w} = (0,0,55,44,20,32,8,42,103,48,49,0,10,0,48),$$

to which we refer to as w_{3w}. Finally, we need just to apply a single majority rule, that is given by the subsequent quorum $\mathtt{q}_{3w} = 230$. Apparently, the undirected graph $\mathtt{G}_3$ remains unaltered. Hence, the modified weighted majority graph problem based on Graph 3 is specified by

$$\mathtt{wMP}_{3w} := (\mathtt{G}_3, \mathbf{w}_{3w}, \mathtt{q}_{3w}).$$

To the associated simple game is referred to as $\langle N, v_{3w}\rangle$.

[14]The business links reported by Breiger and Pattison (1986, p. 218) are primarily based on loans. However, the usury ban by the Catholic church made a loan to an illicit transaction, since usury was strictly defined as everything that exceeds the principal, for instance, the face value of a loan. Therefore, even a sufficiently small interest rate on a loan was considered as usury preventing the loaner of salvation and damned him to eternal purgatory. The practical consequence was that a loan was a gratuitous contract ruling out interest rates and offering therefore no profit margin. Nevertheless, it was considered as legitimate to receive compensation letting bankers to rely on the means of exchange by bills. A bill was a request of payment to another place and currency due at a fixed date, though it was exposed to the risk of price fluctuations not only between currencies, but also between gold and silver. The price of the bill can be regarded as the interest rate. In the accounting books, one does not find in general any traces of loans or discount rates rather of exchange transactions (cf. de Roover (1966, pp. 10-14)). This practice makes it quite difficult to discover a business relation that is based on loans. Apparently more difficult than to identify a marriage tie.

A synopsis of the results is grasped in Table 9.9. Here, we restrict ourselves on those solutions that provide highest variances. These are the Johnston index with a value of 138.61 and the pre-nucleolus having a variance of 190.47. For the other solution concepts the power distribution is clustered too close around the mean indicating no outstanding position with the consequence that we dismiss them. For convenience sake, we present the power distribution of the Johnston index that is given by

$$\begin{aligned}\phi^{JI}(N, v_{3w}) = (&0, 0, 0.2927, 0.0124, 0.0221, 0.0646, 0.0027, 0.0216, 0.3986,\\ &0.0603, 0.0599, 0, 0.0048, 0, 0.0603).\end{aligned}$$

It orders now the Medici on first, the Barbadori on second, and the Ginori on third. All of those are marked as outstanding due to the obtained variance. This ranking neither preserves the ranking of the net-wealth distribution nor it reflects in any way the annals, except for the Medici, and with some limitations for the Barbadori. Contrasted to the foregoing study, the Barbadori family lose their central position, they are not anymore an inevitable actor. The Medici do not take over their role either, though they now overshadow the Barbadori. This lets us turn to the pre-kernel and pre-nucleolus. This game has neither a homogeneous representation nor a veto-player, however, applying the method proposed by Meinhardt (2014), we find out that the pre-kernel is single-valued and coincides with the pre-nucleolus. Hence, it is given by

$$\mathscr{PK}(v_{3w}) = \nu^{*}(N, v_{3w}) = \{(0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0)/3\},$$

This distribution identifies the same major protagonists as the Johnston index, which are the Medici, Barbadori, and the Ginori, all with equal strength. Again, we cannot discover any link to the annals with the consequence that we do not consider this distribution as an image of the main historical actors, except for the Medici, and with some limitations for the Barbadori, but not for the Ginori. Their role seems to be overvalued by the weighted majority graph problem $\mathtt{wMP}_{3w}$. Hence, we infer that $\mathtt{wMP}_{3w}$ does not correctly depict the Florentine power structure.

7.2 Including the Combined Ties into the Network Structure

For proceeding our analysis, we need to correct the combined number of business and marriage ties across the 116 families into the network while assigning the null-players with a zero value. The corrected array of the combined number of ties is given by

$$w_{3t} = (0, 0, 14, 9, 18, 9, 14, 14, 54, 7, 32, 0, 5, 0, 7),$$

adjusting then the simple majority rule to set the quorum $\mathtt{q}_{3t}$ to 92. And notifying that the undirected graph $\mathtt{G}_3$ remains unaltered. Hence, the modified weighted majority graph problem based on Graph 3 is specified by $\mathtt{wMP}_{3t} := (\mathtt{G}_3, \mathbf{w}_{3t}, \mathtt{q}_{3t})$. Within this subsection, we referring to the associated simple game as $\langle N, v_{3t} \rangle$.

We present a summary of the results in Table 9.10. Similar to the foregoing analysis we restrict ourselves on those solutions that provide highest variances. These are the Shapley-Shubik index with a value of 154.15 and the pre-nucleolus with a variance of 666.6667. The other solution concepts are dismissed, because their distributions are again clustered too close around the mean that does not assign to any actor an outstanding position.

$$\phi^{SSI}(N, v_{3t}) = (0.4679, 0.0179, 0.0896, 0.0274, 0.0274, 0.0310, 0.1964,\\ 0.0095, 0.1202, 0.0000, 0.0060, 0.0000, 0.0095).$$

This distribution establishes that the Barbadori are on first, the second are the Medici, followed by the Peruzzi. By the variance, to all of them prominent roles are assigned. Analogously to the findings for a pre-kernel element of game v_{2t}, the Peruzzi are again assigned with a dominant role w.r.t. the power distribution, though their power is not as strong as for the mentioned games and solution concept. Remarkable is that the Barbadori considerably overshadow all others actors. This prominence is not covered by the annals, though they again occupy a central position within weighted majority graph problem $\mathtt{wMP}_{3t}$ implying that their power attribution should not be surprising. In particular under the consideration that one can attribute to them veto power making them to an inevitable player. This lets us infer that a pre-kernel element must assign to them the whole spoil. Due to Meinhardt (2014), this assertion can be extended to the whole pre-kernel and pre-nucleolus despite non-homogeneity, hence both solutions coincide. Its single-valuedness is quantified by the point

$$\mathscr{PK}(v_{3t}) = \nu^*(N, v_{3t}) = \{(0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)\},$$

Once again, the Barbadori got a prominent place that is not supported by the chronicles. Concluding that the modified weighted majority graph problem based on Graph 3 is too stylized to get a correct image of the real power distribution. Thus, the imposed additional structure by the cumulated number of business and marriage ties are not sufficient to sharpen the image. We regard the mesh of business relations characterized by Graph 3 as too coarse for identifying the main protagonists, too many business links are missing, for instance those of the Strozzi or even of the Alberti or Rucellai and many others never mentioned in the underlying data set. But, nevertheless, they are crucial actors in the Florentine business network.

8 Concluding Remarks

The network structure provided by the Tables 4.1 and 4.2 are based on a reduced data set of 16 families of a basic population of 215 leading Florentine families. It can therefore only represent an extremely simplified and vestigial image of the real family ties among the oligarchs. From each of these stylized network structures a weighted majority graph problem is derived to embed the mesh of relations within a political decision-making process. This allows us to conduct a power analysis while referring to several solution concepts with the purpose to disentangle the power structure and retrieving the main protagonists from the annals. Three different voting scenarios have been studied, namely a marriage network without an à priori union, a marriage network with an à priori union, and finally a business network. The reference setting for each of these voting scenarios are based on a symmetric control of the share of votes for each family. Later the settings were extended to take account for asymmetries by the consideration of the net-wealth data and the cumulative ties across the families.

As it turns out, the pre-kernel – as a solution concept designed for studying bargaining situations – performed best in retrieving the leading actors from the chronicles. This is mainly caused by its generic set-valuedness with the effect that it is offering therefore a wide range of political settlements to balance the interests among the negotiating parties, which was ideal for a milieu where the political institutions were exactly designed for that purpose. Secondly, by the desirability property, it assigns the highest spoil to the party with strongest negotiation power and does not cluster the power mainly around the mean, as it is the case for the power indices to satisfy the principle of equality. This allows to single out those actors who have the greatest influence to determine the outcome of a political decision making process.

9 Appendix

This section is devoted to summarize the different parameter settings as well as the derived results. In particular, Table 9.1 provides the parameter settings for every weighted majority graph problem from which the associated simple games are obtained, whereas the other tables retrieve the results of all point solutions. The various pre-kernel solutions are not once more listed.

Table 9.1: Parameter Settings

Values \ Prob.[a] / Parameter	$\mathtt{wMP}_1$	$\mathtt{wMP}_{1w}$	$\mathtt{wMP}_{1t}$	$\mathtt{wMP}_2$	$\mathtt{wMP}_{2w}$	$\mathtt{wMP}_{2t}$	$\mathtt{wMP}_3$	$\mathtt{wMP}_{3w}$	$\mathtt{wMP}_{3t}$
Graph[b]	$\mathtt{G}_1$	$\mathtt{G}_1$	$\mathtt{G}_1$	$\mathtt{G}_2$	$\mathtt{G}_2$	$\mathtt{G}_2$	$\mathtt{G}_3$	$\mathtt{G}_3$	$\mathtt{G}_3$
01 ACCIAIUOL[c]	1	10	2	1	10	2	0	0	0
02 ALBIZZI[c]	1	36	3	1	44	17	0	0	0
03 BARBADORI[c]	1	55	14	1	55	14	1	55	14
04 BISCHERI[c]	1	44	9	1	44	9	1	44	9
05 CASTELLAN[c]	1	20	18	1	20	18	1	20	18
06 GINORI[c]	1	32	9	1	32	9	1	32	9
07 GUADAGNI[c]	1	8	14	0	0	0	1	8	14
08 LAMBERTES[c]	1	42	14	1	42	14	1	42	14
09 MEDICI[c]	1	103	54	1	103	54	1	103	54
10 PAZZI[c]	1	48	7	1	48	7	1	48	7
11 PERUZZI[c]	1	49	32	1	49	32	1	49	32
12 RIDOLFI[c]	1	27	4	1	27	4	0	0	0
13 SALVIATI[c]	1	10	5	1	10	5	1	10	5
14 STROZZI[c]	1	146	29	1	146	29	0	0	0
15 TORNABUON[c]	1	48	7	1	48	7	1	48	7
$\mathtt{q}$[d]	8	340	111	8	340	111	6	230	92
Game[e]	v_1	v_{1w}	v_{1t}	v_2	v_{2w}	v_{2t}	v_3	v_{3w}	v_{3t}

[a] Weighted Majority Graph Problem
[b] Network Graph
[c] Weights
[d] Quorum
[e] Derived Simple Game

Table 9.2: Solution Matrix of Game v_1

Power \ Solution / Family	PGI[a]	MPGI[b]	DPI[c]	JHI[d]	SSI[e]	BZF[f]	PN[g]
01 ACCIAIUOL	0.0520	0.0595	0.0520	0.0231	0.0275	0.0364	0.0333
02 ALBIZZI	0.0879	0.0720	0.0879	0.0881	0.0835	0.0891	0.1000
03 BARBADORI	0.0721	0.0676	0.0721	0.0615	0.0657	0.0707	0.0667
04 BISCHERI	0.0735	0.0680	0.0735	0.0627	0.0706	0.0725	0.0667
05 CASTELLAN	0.0647	0.0667	0.0647	0.0547	0.0656	0.0668	0.0667
06 GINORI	0.0374	0.0577	0.0374	0.0181	0.0246	0.0285	0.0333
07 GUADAGNI	0.0914	0.0758	0.0914	0.1238	0.1255	0.1053	0.1000
08 LAMBERTES	0.0389	0.0583	0.0389	0.0198	0.0278	0.0311	0.0333
09 MEDICI	0.1179	0.0895	0.1179	0.2744	0.1988	0.1635	0.1500
10 PAZZI	0.0271	0.0552	0.0271	0.0116	0.0142	0.0182	0.0333
11 PERUZZI	0.0561	0.0623	0.0561	0.0331	0.0442	0.0483	0.0667
12 RIDOLFI	0.0727	0.0688	0.0727	0.0676	0.0740	0.0755	0.0667
13 SALVIATI	0.0697	0.0638	0.0697	0.0470	0.0432	0.0546	0.0500
14 STROZZI	0.0730	0.0686	0.0730	0.0646	0.0733	0.0748	0.0667
15 TORNABUON	0.0656	0.0662	0.0656	0.0498	0.0614	0.0648	0.0667

[a] Holler/Public Good Index
[b] Modified Holler/Public Good Index
[c] Deegan-Packel Index
[d] Johnston Index
[e] Shapley-Shubik Index
[f] Banzhaf Value
[g] Nucleolus

Table 9.3: Solution Matrix of Game v_{1w}

Power \ Solution / Family	PGI[a]	MPGI[b]	DPI[c]	JHI[d]	SSI[e]	BZF[f]	PN[g]
01 ACCIAIUOL	0.0335	0.0547	0.0312	0.0033	0.0050	0.0059	0.0094
02 ALBIZZI	0.0871	0.0639	0.0856	0.0380	0.0507	0.0525	0.0849
03 BARBADORI	0.0780	0.0694	0.0773	0.0689	0.0841	0.0805	0.0943
04 BISCHERI	0.0835	0.0682	0.0831	0.0609	0.0761	0.0742	0.0566
05 CASTELLAN	0.0664	0.0673	0.0667	0.0588	0.0680	0.0700	0.0189
06 GINORI	0.0390	0.0565	0.0361	0.0082	0.0149	0.0150	0.0094
07 GUADAGNI	0.0945	0.0680	0.0909	0.0637	0.0840	0.0733	0.0566
08 LAMBERTES	0.0524	0.0581	0.0497	0.0129	0.0215	0.0230	0.0189
09 MEDICI	0.1115	0.0890	0.1146	0.2304	0.1891	0.1801	0.1981
10 PAZZI	0.0378	0.0560	0.0351	0.0069	0.0118	0.0125	0.0189
11 PERUZZI	0.0640	0.0623	0.0636	0.0291	0.0411	0.0444	0.0755
12 RIDOLFI	0.0731	0.0751	0.0777	0.1165	0.1035	0.1093	0.0566
13 SALVIATI	0.0615	0.0571	0.0577	0.0113	0.0174	0.0184	0.0189
14 STROZZI	0.0500	0.0898	0.0628	0.2516	0.1774	0.1840	0.2075
15 TORNABUON	0.0676	0.0647	0.0679	0.0395	0.0553	0.0568	0.0755

[a] Holler/Public Good Index
[b] Modified Holler/Public Good Index
[c] Deegan-Packel Index
[d] Johnston Index
[e] Shapley-Shubik Index
[f] Banzhaf Value
[g] Nucleolus

Table 9.4: Solution Matrix of Game v_{1t}

Power \ Solution / Family	PGI[a]	MPGI[b]	DPI[c]	JHI[d]	SSI[e]	BZF[f]	PN[g]
01 ACCIAIUOL	0.0281	0.0548	0.0255	0.0015	0.0023	0.0030	0
02 ALBIZZI	0.0965	0.0615	0.0935	0.0268	0.0372	0.0388	0.0488
03 BARBADORI	0.0763	0.0717	0.0772	0.0880	0.0888	0.0937	0.0732
04 BISCHERI	0.0741	0.0664	0.0747	0.0522	0.0646	0.0652	0.0488
05 CASTELLAN	0.0494	0.0717	0.0522	0.0866	0.0928	0.0937	0.0976
06 GINORI	0.0505	0.0565	0.0463	0.0060	0.0121	0.0119	0.0244
07 GUADAGNI	0.1033	0.0716	0.1032	0.0801	0.1032	0.0931	0.1098
08 LAMBERTES	0.0494	0.0582	0.0466	0.0118	0.0250	0.0210	0.0366
09 MEDICI	0.1201	0.0984	0.1263	0.3405	0.2468	0.2378	0.2683
10 PAZZI	0.0337	0.0555	0.0306	0.0031	0.0052	0.0063	0
11 PERUZZI	0.0224	0.0676	0.0243	0.0582	0.0721	0.0717	0.0732
12 RIDOLFI	0.0864	0.0697	0.0877	0.0804	0.0785	0.0831	0.0244
13 SALVIATI	0.0651	0.0568	0.0618	0.0076	0.0109	0.0136	0
14 STROZZI	0.0718	0.0778	0.0794	0.1297	0.1207	0.1264	0.1463
15 TORNABUON	0.0730	0.0619	0.0708	0.0275	0.0398	0.0408	0.0488

[a] Holler/Public Good Index
[b] Modified Holler/Public Good Index
[c] Deegan-Packel Index
[d] Johnston Index
[e] Shapley-Shubik Index
[f] Banzhaf Value
[g] Nucleolus

Table 9.5: Solution Matrix of Game v_2

Power \ Solution / Family	PGI[a]	MPGI[b]	DPI[c]	JHI[d]	SSI[e]	BZF[f]	PN[g]
01 ACCIAIUOL	0.0581	0.0643	0.0581	0.0282	0.0332	0.0436	0.0357
02 ALBIZZI	0.1114	0.0881	0.1114	0.1910	0.1538	0.1364	0.1429
03 BARBADORI	0.0745	0.0720	0.0745	0.0599	0.0681	0.0735	0.0714
04 BISCHERI	0.0788	0.0731	0.0788	0.0664	0.0736	0.0780	0.0714
05 CASTELLAN	0.0673	0.0713	0.0673	0.0562	0.0772	0.0708	0.0714
06 GINORI	0.0532	0.0634	0.0532	0.0259	0.0339	0.0400	0.0357
07 GUADAGNI	0	0	0	0	0	0	0
08 LAMBERTES	0.0532	0.0634	0.0532	0.0259	0.0339	0.0400	0.0357
09 MEDICI	0.1199	0.0956	0.1199	0.2749	0.2003	0.1653	0.1607
10 PAZZI	0.0347	0.0591	0.0347	0.0152	0.0175	0.0236	0.0357
11 PERUZZI	0.0622	0.0675	0.0622	0.0383	0.0515	0.0560	0.0714
12 RIDOLFI	0.0722	0.0720	0.0722	0.0594	0.0724	0.0738	0.0714
13 SALVIATI	0.0794	0.0705	0.0794	0.0623	0.0558	0.0679	0.0536
14 STROZZI	0.0728	0.0721	0.0728	0.0593	0.0751	0.0741	0.0714
15 TORNABUON	0.0622	0.0677	0.0622	0.0372	0.0536	0.0569	0.0714

[a] Holler/Public Good Index
[b] Modified Holler/Public Good Index
[c] Deegan-Packel Index
[d] Johnston Index
[e] Shapley-Shubik Index
[f] Banzhaf Value
[g] Nucleolus

Table 9.6: Solution Matrix of Game v_{2w}

Power / Solution / Family	PGI[a]	MPGI[b]	DPI[c]	JHI[d]	SSI[e]	BZF[f]	PN[g]
01 ACCIAIUOL	0.0320	0.0584	0.0294	0.0028	0.0039	0.0050	0
02 ALBIZZI	0.1178	0.0812	0.1168	0.1223	0.1308	0.1209	0.1290
03 BARBADORI	0.0822	0.0724	0.0808	0.0624	0.0782	0.0763	0.0968
04 BISCHERI	0.0913	0.0756	0.0925	0.0843	0.0905	0.0928	0.0645
05 CASTELLAN	0.0667	0.0696	0.0660	0.0487	0.0620	0.0621	0.0323
06 GINORI	0.0521	0.0618	0.0485	0.0131	0.0198	0.0224	0.0161
07 GUADAGNI	0	0	0	0	0	0	0
08 LAMBERTES	0.0557	0.0628	0.0532	0.0167	0.0237	0.0275	0.0161
09 MEDICI	0.1251	0.0917	0.1276	0.2119	0.1833	0.1744	0.2151
10 PAZZI	0.0402	0.0598	0.0364	0.0069	0.0115	0.0125	0.0215
11 PERUZZI	0.0648	0.0666	0.0647	0.0312	0.0433	0.0469	0.0645
12 RIDOLFI	0.0813	0.0775	0.0842	0.1032	0.1011	0.1023	0.0323
13 SALVIATI	0.0639	0.0610	0.0587	0.0111	0.0163	0.0182	0.0215
14 STROZZI	0.0612	0.0941	0.0750	0.2501	0.1865	0.1865	0.2258
15 TORNABUON	0.0658	0.0677	0.0660	0.0353	0.0491	0.0523	0.0645

[a] Holler/Public Good Index
[b] Modified Holler/Public Good Index
[c] Deegan-Packel Index
[d] Johnston Index
[e] Shapley-Shubik Index
[f] Banzhaf Value
[g] Nucleolus

Table 9.7: Solution Matrix of Game v_{2t}

Power / Solution / Family	PGI[a]	MPGI[b]	DPI[c]	JHI[d]	SSI[e]	BZF[f]	PN[g]
01 ACCIAIUOL	0.0302	0.0589	0.0272	0.0018	0.0030	0.0035	0
02 ALBIZZI	0.1317	0.0821	0.1330	0.1255	0.1345	0.1289	0.1739
03 BARBADORI	0.0854	0.0755	0.0862	0.0855	0.0891	0.0934	0.0870
04 BISCHERI	0.0872	0.0725	0.0876	0.0669	0.0744	0.0771	0.0435
05 CASTELLAN	0.0552	0.0742	0.0586	0.0739	0.0854	0.0862	0.0870
06 GINORI	0.0587	0.0615	0.0546	0.0096	0.0169	0.0180	0.0217
07 GUADAGNI	0	0	0	0	0	0 0	
08 LAMBERTES	0.0641	0.0635	0.0608	0.0171	0.0317	0.0287	0.0217
09 MEDICI	0.1263	0.1041	0.1345	0.3525	0.2524	0.2482	0.2609
10 PAZZI	0.0320	0.0595	0.0290	0.0034	0.0058	0.0068 0	
11 PERUZZI	0.0302	0.0720	0.0320	0.0589	0.0753	0.0744	0.0870
12 RIDOLFI	0.0854	0.0723	0.0843	0.0689	0.0741	0.0759	0.0435
13 SALVIATI	0.0641	0.0609	0.0602	0.0084	0.0120	0.0146	0
14 STROZZI	0.0765	0.0803	0.0837	0.1141	0.1183	0.1195	0.1304
15 TORNABUON	0.0730	0.0628	0.0683	0.0134	0.0271	0.0248	0.0435

[a] Holler/Public Good Index
[b] Modified Holler/Public Good Index
[c] Deegan-Packel Index
[d] Johnston Index
[e] Shapley-Shubik Index
[f] Banzhaf Value
[g] Nucleolus

Table 9.8: Solution Matrix of Game v_3

Power \ Solution / Family	PGI[a]	MPGI[b]	DPI[c]	JHI[d]	SSI[e]	BZF[f]	PN[g]
01 ACCIAIUOL	0	0	0	0	0	0	0
02 ALBIZZI	0	0	0	0	0	0	0
03 BARBADORI	0.1667	0.1345	0.1667	0.3745	0.4276	0.2581	1.0000
04 BISCHERI	0.0718	0.0818	0.0718	0.0388	0.0435	0.0560	0
05 CASTELLAN	0.0949	0.0926	0.0949	0.0781	0.0792	0.0973	0
06 GINORI	0.0718	0.0841	0.0718	0.0419	0.0522	0.0649	0
07 GUADAGNI	0.0486	0.0780	0.0486	0.0267	0.0355	0.0413	0
08 LAMBERTES	0.0949	0.0880	0.0949	0.0614	0.0585	0.0796	0
09 MEDICI	0.1528	0.1129	0.1528	0.1983	0.1419	0.1755	0
10 PAZZI	0.0602	0.0765	0.0602	0.0229	0.0224	0.0354	0
11 PERUZZI	0.1181	0.0987	0.1181	0.1117	0.0943	0.1209	0
12 RIDOLFI	0	0	0	0	0	0	0
13 SALVIATI	0.0602	0.0765	0.0602	0.0229	0.0224	0.0354	0
14 STROZZI	0	0	0	0	0	0	0
15 TORNABUON	0.0602	0.0765	0.0602	0.0229	0.0224	0.0354	0

[a] Holler/Public Good Index
[b] Modified Holler/Public Good Index
[c] Deegan-Packel Index
[d] Johnston Index
[e] Shapley-Shubik Index
[f] Banzhaf Value
[g] Nucleolus

Table 9.9: Solution Matrix of Game v_{3w}

Power \ Solution / Family	PGI[a]	MPGI[b]	DPI[c]	JHI[d]	SSI[e]	BZF[f]	PN[g]
01 ACCIAIUOL	0	0	0	0	0	0	0
02 ALBIZZI	0	0	0	0	0	0	0
03 BARBADORI	0.2045	0.1255	0.2070	0.2927	0.2575	0.2561	0.3333
04 BISCHERI	0.0455	0.0764	0.0412	0.0124	0.0333	0.0218	0
05 CASTELLAN	0.1023	0.0787	0.0930	0.0221	0.0230	0.0327	0
06 GINORI	0.0795	0.0913	0.0807	0.0646	0.1008	0.0926	0.3333
07 GUADAGNI	0.0114	0.0730	0.0088	0.0027	0.0060	0.0054	0
08 LAMBERTES	0.0909	0.0793	0.0833	0.0216	0.0417	0.0354	0
09 MEDICI	0.1932	0.1369	0.2026	0.3986	0.3171	0.3106	0.3333
10 PAZZI	0.0682	0.0890	0.0737	0.0603	0.0746	0.0817	0
11 PERUZZI	0.0909	0.0873	0.0939	0.0599	0.0667	0.0736	0
12 RIDOLFI	0	0	0	0	0	0	0
13 SALVIATI	0.0455	0.0736	0.0421	0.0048	0.0048	0.0082	0
14 STROZZI	0	0	0	0	0	0	0
15 TORNABUON	0.0682	0.0890	0.0737	0.0603	0.0746	0.0817	0

[a] Holler/Public Good Index
[b] Modified Holler/Public Good Index
[c] Deegan-Packel Index
[d] Johnston Index
[e] Shapley-Shubik Index
[f] Banzhaf Value
[g] Nucleolus

Table 9.10: Solution Matrix of Game v_{3t}

Power \ Solution Family	PGI[a]	MPGI[b]	DPI[c]	JHI[d]	SSI[e]	BZF[f]	PN[g]
01 ACCIAIUOL	0	0	0	0	0	0	0
02 ALBIZZI	0	0	0	0	0	0	0
03 BARBADORI	0.2083	0.1434	0.2200	0.4208	0.4679	0.3395	1.0000
04 BISCHERI	0.0625	0.0756	0.0500	0.0091	0.0179	0.0185	0
05 CASTELLAN	0.1458	0.0965	0.1533	0.0958	0.0869	0.1173	0
06 GINORI	0.1042	0.0782	0.0917	0.0167	0.0274	0.0309	0
07 GUADAGNI	0.0625	0.0782	0.0533	0.0170	0.0274	0.0309	0
08 LAMBERTES	0.0833	0.0795	0.0783	0.0215	0.0310	0.0370	0
09 MEDICI	0.1250	0.1252	0.1500	0.2723	0.1964	0.2531	0
10 PAZZI	0.0417	0.0743	0.0417	0.0076	0.0095	0.0123	0
11 PERUZZI	0.1042	0.1017	0.1033	0.1287	0.1202	0.1420	0
12 RIDOLFI	0	0	0	0	0	0	0
13 SALVIATI	0.0208	0.0730	0.0167	0.0030	0.0060	0.0062	0
14 STROZZI	0	0	0	0	0	0	0
15 TORNABUON	0.0417	0.0743	0.0417	0.0076	0.0095	0.0123	0

[a] Holler/Public Good Index
[b] Modified Holler/Public Good Index
[c] Deegan-Packel Index
[d] Johnston Index
[e] Shapley-Shubik Index
[f] Banzhaf Value
[g] Nucleolus

Table 9.11: Variances of Power Distribution[h]

Variance \ Solution Game	PGI[a]	MPGI[b]	DPI[c]	JHI[d]	SSI[e]	BZF[f]	PN[g]
01 v_1	5.2618	0.7184	5.2618	41.6818	21.3984	12.8957	9.9206
02 v_{1w}	4.9081	1.1983	5.2355	59.6757	31.6523	30.7726	38.8377
03 v_{1t}	7.9173	1.2885	8.7435	72.8866	39.5512	37.4060	49.0496
04 v_2	8.0080	4.2793	8.0080	51.8961	25.9732	16.6757	16.6120
05 v_{2w}	9.8711	4.6124	10.5750	58.9851	37.7435	34.6968	51.8557
06 v_{2t}	12.5372	4.7833	13.8096	80.3623	45.1758	43.7429	55.6756
07 v_3	28.5234	19.7285	28.5234	100.5857	115.9110	52.8755	666.6667
08 v_{3w}	42.4046	20.4917	44.6929	138.6146	92.3996	89.7715	190.4762
09 v_{3t}	38.5665	21.3400	44.3254	151.0825	154.1505	107.5185	666.6667

[a] Holler/Public Good Index
[b] Modified Holler/Public Good Index
[c] Deegan-Packel Index
[d] Johnston Index
[e] Shapley-Shubik Index
[f] Banzhaf Value
[g] Nucleolus
[h] Normalized to 100

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