Behavioral economics. Forbidden zones. New method and models

Harin, Alexander

Modern University for the Humanities

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Alexander Harin
aaharin@gmail.com
Modern University for the Humanities

A forbidden zone theorem, hypothesis, and applied mathematical method and model are introduced in the present article. The method and model are based on the forbidden zones and hypothesis. The model is uniformly and successfully applied for different domains. The ultimate goal of the research is to solve some generic problems of behavioral economics.

Keywords: Expectation Variation Boundary Utility PACS: 60A86 62C86

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1. Introduction. Motivations and sources

1.1. Preliminaries. Main contributions. Organization of the article

Random variables whose values lie within a finite interval and whose variances are non-zero are analyzed in the present article. Such r.v.s can represent diverse types of data and information including measurement data. A theorem is proven that establishes the existence of certain non-zero boundary bounds (or forbidden zones) on the expectations of these r.v.s.

The theorem provides mathematical support for the analysis (see, e.g., [26]) of well-known generic problems (see, e.g., [30] of behavioral economics, and mainly for a behavioral idea (hypothesis) of presupposed biases, and applied mathematical method (approach) and models.

Two main contributions of the article can be preliminary noted.

1) A necessity of corrections for situations that satisfy the theorem.

2) A mathematical model that is uniformly true for different domains.

The article is organized as follows.

Section 1 presents its motivations and sources.

Section 2 presents the forbidden zone theorem.

Section 3 presents practical examples of the forbidden zones.

Section 4 presents the hypothesis.

Section 5 presents the mathematical method.

Section 6 presents the mathematical models.

Section 7 presents a particular consequence of a special model and practical numerical examples of its application for different domains.

Section 8 presents general consequences of the theorem and method.

Section 9 presents conclusions.

The Appendix presents lemmas for the theorem.

1.2. Moments, functions, utility, noise, biases. Review of the literature

Diverse bounds on moments and functions of r.v.s are considered in a wealth of works, see, e.g., [15], [19], [33], [36]. The works [40], [45], and, especially, [8] consider the closest mathematical situations to that analyzed here. Additionally, the discrete part of the proof in the Appendix can be considered as another variant of the proof in [8]. The continuous and mixed parts of the proof in the Appendix can be considered as its developments.

Mathematical aspects of utility are considered in, e.g., [7], [9], [16], [37]. Works [1] and [48] provide one of the two starting points for the theorem.

Noise and its influence are the subject of a wealth of works.

Channel capacity and noise are considered in a lot of works, see, e.g., [14], [44], [46], [55]. Channel capacity is in a sense similar to the allowed zone that is complementary to the considered forbidden zones.

Some qualitative influences of noise are analyzed as well. For example, stabilization and synchronization by noise are considered in a number of works, see, e.g., [4], [6], [13], [21]. Noise as a possible cause of some periodic behavior is considered in, e.g., [23], [42].

So the cited articles and also, in a sense, this one show that noise can exert not only a quantitative but also some qualitative influence.

Diverse types of biases are considered in a wealth of works. For example:

behavioral biases are considered in, e.g., [32], [56];

psychological biases are considered in, e.g., [38], [50];

home biases are considered in, e.g., [17], [18], [22];

anchoring biases are considered in, e.g., [25], [52];

confirmatory biases are considered in, e.g., [3], [10];

optimism biases are considered in, e.g., [11], [53];

present biases are considered in, e.g., [34], [54].

The nearest to the items of this article are:

hypothetical biases (see, e.g., [35], [39]), [49],

and pull-to-center biases (and close items such as newsvendor problem), see, e.g., [2], [5], [24], [57].

1.3. Practical need for such considerations

A man as an individual actor is a key subject of economics and some other sciences. There are a number of problems concerned with the mathematical description of the behavior of an individual. Examples of these are the underweighting of high and the overweighting of low probabilities, the Allais paradox, risk aversion, loss aversion, equity premium puzzle, fourfold pattern of risk preferences, etc.

1.3.1. Choices between uncertain and sure games

One of the problems of this mathematical description is a comparison of choices between uncertain and sure games.

The essence of the above examples of the problems consists in biases of choices of people (subjects) for the uncertain and sure games in comparison with the predictions of the theory of probability. These problems are generic and well-known. They are the most important in behavioral economics in utility and prospect theories and also in psychology, decision theory, and the social sciences. They are pointed out in a wealth of works.

For example, we see in [30] page 222:
"A long series of modern challenges to utility theory, starting with the paradoxes of Allais (1953) ..., have demonstrated inconsistency in preferences"

For example, we see in [31], page 265:
"PROBLEM 1: Choose between
A: 2,500 with probability .33, // 2,400 with probability .66, // 0 with probability .01;
B: 2,400 with certainty.
N = 72 [18] [82]"
My note. This is the clear inconsistency: 18% for "A," that is less than 82% for "B" (for 72 trials) in opposition with the expectations 2,500×.33+2,400×.66 = 2,409, that is more than 2.400.

For example, we see in [47], page 974:
"... a choice between two lotteries *R'* (for "riskier") and *S'* (for "safer"). *R'* gave a 0.2 chance of winning £10.00 and a 0.75 chance of winning £7.00 (with the residual 0.05 chance of winning nothing); *S'* gave £7.00 for sure."
My note. R' = £10.00×0.2+£7.00×0.75 = £7.25 and S' = £7.00. Here the expectations are R' = £7.25, that is more than S' = £7.00, but the results were 13 choices for R' that is less than 27 choices for S'.

1.3.2. Behavior of subjects in different domains

An additional and, probably, a lot harder problem is, moreover, the radically different behavior of people (subjects) in different domains.

Thaler wrote in 2016 in [51], pages 1581–1582 (the boldfaces are my own): "Kahneman and Tversky's research documented numerous choices that **violate any sensible definition of rational**. ... subjects were **risk averse in the domain of gains** but **risk seeking in the domain of losses**."

My note: at high probabilities.

For example, the data in [31], page 268 Table 1 can be represented as:

Problem 3: (4,000 at 0.80) > (3,000 at 1.00) leads to choices [20%] < [80%].

Problem 3': (-4,000 at 0.80) < (-3,000 at 1.00) leads to choices [92%] > [8%].

My note. These data lead to the undoubted deduction of the clear inconsistency between the behavior of subjects in the domains of gains and losses.

This article is motivated in large measure by the need for mathematical support for the performed analysis (see, e.g., [26]) of the influence of the scatter and noisiness of data. This analysis is mathematically supported here. It has explained the above problems, at least partially or qualitatively.

1.4. Two ways. Variance, expectation, and forbidden zones

Many efforts have been made to explain the above generic problems.

One of the possible ways to explain them has been widely discussed, e.g., in [12], [29], [43]. It consists in paying proper attention to imprecision, noise, incompleteness, and other reasons that can cause spread of data.

Another possible way is to consider the vicinities of the boundaries of the probability scale, e.g., at $p \to 1$. So [1] and [48] emphasized a fundamental question: whether Prelec's function (see [41]) is equal to 1 at $p = 1$.

In any case, one may suppose that a synthesis of these two possible ways can be of some interest. This idea of a synthesis turned out to be useful indeed. It has successfully explained, at least partially, the underweighting of high and the overweighting of low probabilities, risk aversion, and some other problems (see, e.g., [26]). There are also works providing experimental support of this synthesis (see, e.g., [27], [47]).

Here it is proved that bounds on the variances and ranges of random variables lead to bounds (or forbidden zones) for their expectations near the boundaries of the ranges (intervals). The role of noise, as a possible cause of these zones, and their possible influence on the results of measurements near the boundaries of the intervals are considered in a preliminary way also.

Keeping in mind the above bounds in, e.g., [15], [19], [33], [40] for functions, various functions of the expectations of r.v.s can be also investigated.

2. Theorem

2.1. Preliminaries

Let us consider a set $\{X_i\}$, $i = 1, \dots, n$, of random variables X_i whose values lie within an interval $[a, b]$. For the sake of simplicity, X_i, μ_i, σ_i^2 and similar symbols will often be written without the subscript "i."

If there is at least one discrete value of X, then let us denote the discrete value(s) of X by $\{x_k\}$, $k = 1, \dots, K$, where $K \geq 1$, and the probability mass function (PMF) by $p_X(x_k)$. If there are none, then let us ignore all the expressions involving discrete value(s).

If there are continuous values of X, then let us denote them by x and the probability density function (PDF) by $f_X(x)$. If there are none, then let us ignore all the expressions involving continuous values.

Under the normalizing condition

$$\sum_{k=1}^{K} p_X(x_k) + \int_{-\infty}^{+\infty} f_X(x)dx = \sum_{x_k \in [a,b]} p_X(x_k) + \int_{a}^{b} f_X(x)dx = 1, \tag{1}$$

let us consider the expectation and variance of X, and their relations.

In connection with the terms "bound" and "forbidden zone," the abbreviation "r_μ" (arising from the first letter "r" of the term "restriction") will be used here, due to its consonance with the usage in previous works.

Non-trivial forbidden zones of non-zero width will sometimes be referred to as non-zero forbidden zones.

2.2. Maximality of the variance

A proof is given in [8] that, for the variance σ^2 of a discrete random variable with range $[a, b]$ and expectation μ,

$$\sigma^2 \leq (\mu - a)(b - \mu). \tag{2}$$

An alternate proof is given in the Appendix that this inequality holds also for any real-valued random variable X_i as in above subsection 2.1.

2.3. Existence theorem

Theorem 1. Consider a set $\{X_i\}$, $i = 1, \ldots, n$, of random variables X_i whose values lie within an interval $[a, b]$. If $0 < (b\text{-}a) < \infty$ and there exists a forbidden zone (or lower bound) of a non-zero width $\sigma^2_{\min}$ for the variances σ_i^2 of X_i, such that for all i,

$$\sigma^2{}_i \geq \sigma^2{}_{\min} > 0\,, \tag{3}$$

then certain forbidden zones (or boundary bounds, or restrictions) of a non-zero width r_μ exist for the expectations μ_i of each X_i such that

$$a < (a + r_\mu\ ) \leq \mu_i \leq (b - r_\mu\ ) < b\,. \tag{4}$$

Proof. Inequalities (2) and (3) lead to $0 < \sigma^2{}_{\min} \leq \sigma_i^{\ 2} \leq (\mu_i - a)(b - \mu_i)$. At, e.g., the boundary a, this leads to $\sigma^2{}_{\min} \leq \sigma^2{}_{\mathrm{i}} \leq (\mu_i\text{-}a)(b\text{-}a)$ and

$$\mu_i \geq a + \frac{\sigma^2{}_i}{b-a} \geq a + \frac{\sigma^2{}_{\min}}{b-a}\,.$$

At the boundary b, the considerations are similar and give

$$\mu_i \leq b - \frac{\sigma^2{}_i}{b-a} \leq b - \frac{\sigma^2{}_{\min}}{b-a}\,.$$

Defining the bounds (restrictions) r_μ on the expectation μ_i as

$$r_\mu \equiv \frac{\sigma^2{}_{\min}}{b-a} \leq \frac{\sigma^2{}_i}{b-a}\,, \tag{5}$$

we obtain the inequalities $a + r_\mu \leq \mu_i \leq b - r_\mu$.

Due to $0 < (b\text{-}a) < \infty$ and $\sigma^2{}_{\min} > 0$, the bounds r_μ are non-zero and this leads to the inequalities

$$a < \left(a + \frac{\sigma^2{}_{\min}}{b-a} \right) \leq \mu_i \leq \left(b - \frac{\sigma^2{}_{\min}}{b-a} \right) < b \tag{6}$$

those are equivalent to (4). □

2.4. Comments to the theorem

We see that the particular bounds for the expectation of some a particular r. v. are determined by its variance. If the variance is non-zero, then these bounds are non-zero also. If the minimal variance $\sigma^2{}_{\min}$ for the set of random variables $\{X_i\}$ is non-zero, then the common bounds for the set of all X_i are non-zero as well. These bounds cannot be less than r_μ in (5).

The boundary bounds (restrictions) r_μ can be considered as some forbidden zones of the width r_μ for the expectations of the random variables X_i near the boundaries of the interval $[a, b]$. Consequently the allowed zone for the expectations of X_i is located in the center of the interval. The allowed zone is compressed by the forbidden zones (in comparison with the entire interval), and the expectations are biased from the boundaries to the middle of the interval (in comparison with the case of zero forbidden zones). This is similar to the pull-to-center biases, see, e.g., [2], [24], [57].

The importance of this simple theorem lies in its particular and general consequences that are used and/or considered in next sections.

3. Practical examples of the occurrence of the forbidden zones

3.1 Practical examples of the forbidden zones. Boat and waves

Consider a calm or mirror-like sea and a small rigid boat or any other small rigid floating body at rest in the sea. Suppose that this boat or body rests right against (or is constantly touching) a rigid moorage wall. As long as the sea is calm, the expectations of their sides can touch the wall.

Suppose there is a heavy sea. Consider a small rigid boat or any other small rigid floating body which oscillates on the waves in the heavy sea. Suppose that this boat or body oscillates on the waves near this rigid moorage wall.

When the boat is oscillated by sea waves, then its side oscillates also (both up–down and left–right) and it can touch the wall only in the (nearest) extremity of the oscillations. Hence the expectation of the side cannot touch the wall. Hence the expectation of the side is biased away from the wall.

So, one can say that, in the presence of waves, a forbidden zone exists between the expectation of the side and the wall.

This forbidden zone biases the expectation away from the wall. The width of the forbidden zone is roughly one-half of the amplitude of the oscillations.

3.2. Practical examples of the forbidden zones. Washing machine, drill

Consider a washing machine or drill (or any other rigid body) that can vibrate when it works. Suppose its edgeless rigid side (or some rigid limiter of the movement of its side) is located near a rigid surface or wall.

If the machine or drill is at rest, then the expectation of this side can be located right against (be constantly touching) the wall.

If the machine or drill vibrates, then the expectation of this side is biased and kept away from the wall due to the vibrations.

So, in the presence of the vibrations, a forbidden zone exists between the expectation of the side of the rigid body and the rigid wall. This is evidently true for any rigid body near any rigid surface or wall.

3.3. Vibration suppression. Sure games

Vibrations or oscillations can be suppressed by means of some efforts of some forces. Such efforts can be, e.g., physical in the case of physical vibrations.

A vibrating rigid body can be pressed by some means. In this case the corresponding forbidden zone can be suppressed either partially or even totally, depending on the parameters of the suppression.

This suppression can correspond to the case of sure games (and outcomes) in behavioral economics, decision theory, the social sciences, etc.

In behavioral sciences, the term "sure" presumes usually that some efforts are applied to guarantee the sure games in comparison with the uncertain ones. Due to these guaranteeing efforts, the widths of the forbidden zones and, hence, the biases for the sure games can be less than the widths and biases for the corresponding uncertain games. In the limiting case, when the efforts are sufficiently hard, there are no forbidden zones for the sure games.

So, sure games are guaranteed by some efforts. Due to these efforts, the forbidden zones and biases for the sure games can be suppressed and reduced.

4. Hypothesis of presupposed biases

4.1. Preliminary remark

First of all, the above hard, complex, and old problems evidently cannot be solved by any single article. Such a solution needs a lot of elaborated works of a sufficient number of high-powered research teams. Hence and especially keeping in mind the above statement "… numerous choices … violate any sensible definition of rational" of a Nobel Laureate in this field in [51], I may formulate a *principle of gradualism* for the present research and article. This principle can sound like "*stage by stage and step by step*."

Hence the applied mathematical method (or approach) that will be proposed in the present article should be only a preliminary stage for subsequent verifications, changes, modifications and refinements by a sufficient number of independent research teams. So for such a preliminary stage, some good step can be even the above theorem with its consequences, and a collection of some suppositions and mathematical relations.

4.2. Behavioral hypothesis

The practical examples of the previous section evidently illustrate possible forbidden zones of the theorem. Similar examples are widely disseminated in real life. Due to this dissemination, subjects (people) can keep in mind the feasibility of such forbidden zones and the biases of the expectations caused by the zones. This can influence the behavior and choices of the subjects.

In consequence of this consideration, I propose a statement that can be named as a behavioral hypothesis:

"People, as economic subjects, behave and decide (at least to a considerable degree) as if there were some presupposed (hypothetical) biases of the expectations for games."

Note. This hypothesis can be supported by the reason that such biases may be proposed and tested even from a purely formal point of view.

This hypothesis can be found in hidden forms in the literature or derived from it (see, e.g., [26], [31], [51], etc.) in this particular field, and in an explicit form in neighboring fields (see, e.g., [35], [39]). Nevertheless one should state it in an explicit form and emphasize it.

5. Mathematical method of biases of expectations (MMBE)

5.1. Propositions

Two main propositions can be suggested for a mathematical method of solution of the above problems. The first one is the above hypothesis. Shortly it is:

Proposition 1. *Presupposed (Hypothetical) biases of the expectations.*

Or in details: "Subjects behave and decide (at least to a considerable degree) as if there were some presupposed biases of the expectations for games."

Due to this proposition, the method (approach) can be called an Applied Mathematical Method of Biases of Expectations, or AMMBE, or shortly MMBE. The MMBE is to explain not only the objective situations but also and mainly the subjective behavior and choices of subjects.

The second main proposition is:

Proposition 2. *Explanation by the forbidden zones of the theorem.*

That is these biases (real biases or subjective reactions and choices of the subjects) can be explained (at least to a considerable degree) with the help of the forbidden zones of the theorem.

5.2. Notation

One can introduce following denotations.

Denote the real expectations for the games by

$$\mu_{sure} \quad \text{and} \quad \mu_{uncert} \equiv \mu_{uncertain}.$$

Denote the presupposed biases (of the expectations) that are required to obtain the data corresponding to the choices of the subjects by

$$\Delta_{ch-\mu.uncert} \equiv \Delta_{choice-\mu.uncertain} \quad \text{and} \quad \Delta_{ch-\mu.sure} \equiv \Delta_{choice-\mu.sure}.$$

That is the resulting expectations (i.e., expectations including these biases) for the observed choices of the subjects can be written as $\mu_{uncert} + \Delta_{ch-\mu.uncert}$ for the uncertain games and as $\mu_{sure} + \Delta_{ch-\mu.sure}$ for the sure ones.

5.3. General relations of the MMBE

Let us consider some essential features of the examined situations and, using the above notations, develop some relations.

1. Condition for the MMBE. Due to the first main proposition, the method of biases of expectations can be useful only if these biases for the choices for the uncertain (see the third relation below) games are non-zero

$$|\Delta_{ch-\mu.uncert}| > 0 \quad \text{or} \quad \operatorname{sgn}(\Delta_{ch-\mu.uncert}) \neq 0. \qquad (7)$$

2. Forbidden zones as, at least, one of the origins of biases. The presupposed bias $\Delta_{ch\text{-}\mu.uncert}$ may be introduced and considered purely formally. The question is not only whether $\Delta_{ch\text{-}\mu.uncert}$ can explain the problems. Due to the above second proposition, $\Delta_{ch\text{-}\mu.uncert}$ itself should be explained by the forbidden zones of the theorem, at least partially.

First of all, the theorem should be applicable. Therefore inequalities (3), that is $\sigma^2 \geq \sigma^2{}_{\min} > 0$, of the non-zero minimal variance are required to be true.

Further, let us denote the bias caused by the forbidden zone of the theorem by

$\Delta_{theorem}$. The sign of the presupposed bias should coincide with that for the bias caused by the theorem

$$\text{sgn}\,(\Delta_{ch-\mu.uncert}) = \text{sgn}\,(\Delta_{theorem})\,.$$

Then the conditions for the explanation can be written as $\Delta_{ch\text{-}\mu.uncert} \approx \Delta_{theorem}$ in the case when the forbidden zones are the main source of the biases. If these zones are only one of the essential sources of the biases, then these conditions can be represented as $\Delta_{ch\text{-}\mu.uncert} = O(\Delta_{theorem})$.

So the relations of the explanation by the theorem are

$$\sigma^2 \geq \sigma^2{}_{\min} > 0 \quad \text{and also} \quad \Delta_{ch-\mu.uncert} \approx \Delta_{theorem}$$

$$\text{or at least} \;\; \Delta_{ch-\mu.uncert} = O(\Delta_{theorem})\,. \qquad 8)$$

3. Biases for sure games. The above considerations about noise suppression and sure games emphasize the condition that the sure games are guaranteed by some guaranteeing efforts. Due to these efforts, the biases for the sure games can be suppressed and reduced. They can be moreover equal to zero.

Therefore I assume that the presupposed biases of the data for the sure games are equal to zero or, more generally, are strictly less than the presupposed biases for the corresponding uncertain games.

So, the relation for the sure and uncertain games is

$$|\,\Delta_{ch-\mu.uncert}\,| > |\,\Delta_{ch-\mu.sure}\,|\,. \qquad (9)$$

5.4. Restrictions. One of the main questions

There are two causes of restrictions on the method and models.

First. Evidently, if $\sigma_{\min} \rightarrow 0$ then, due to (5), $r_\mu/\sigma_{\min} \rightarrow 0$ as well.

Second. The preliminary estimate [28] shows that the real relative biases are sometimes comparable with the upper bound of the relative biases that can be derived from biases (5) guaranteed by the theorem.

Due to these two reasons, and also from general and formal points of view, one may suppose: “In general cases, along with the non-zero minimal variance, other sources of the biases cannot be excluded so far.” Hence, general models can be considered at present as only preliminary ones. So, one of the main questions is to determine whether the forbidden zones can lead to sufficiently high values for the biases (for both low and high minimal variances). So, one of the main questions of future research is to analyze the possible widths of the forbidden zones for various types of distributions.

5.5. First stage. Qualitative problems, models and explanations

Due to the above principle of gradualism, the first stage of the approach (method) can be constituted by qualitative models. That is one can both deal with qualitative problems and give qualitative explanations.

The statements of this first stage can be formulated as follows:

Qualitative problems. Only qualitative problems will be considered.

Qualitative analysis. Only a qualitative analysis will be performed.

Qualitative explanation. Only qualitative explanations of the existing problems will be given. No predictions will be made in during this first stage.

Choices of subjects. The method and models will explain mainly the subjective behavior and choices of subjects.

6. Qualitative mathematical models. Novelty

6.1. Need for qualitative models

First of all, is there a real need for qualitative models?

Suppose you are considering a confused situation where you know the exact magnitude of some effect, which can be either positive or negative, but you cannot predict its sign. Evidently the goal is, first of all, to understand and explain the origins of the effect and predict its sign, and only then to calculate its exact magnitude.

The literature analysis states that this problem of the determination of the signs was posed not later than in 1979 (see, e.g., [31] page 268 ”The reflection effect”), but is still unsolved (see, e.g., [51] pages 1581–1582 “violate any sensible definition of rational. ... subjects were risk averse in the domain of gains but risk seeking in the domain of losses”). So the theory takes into account the observed signs of the biases but does not explain them, and there is a need for such an explanation.

6.2. Elements of a general qualitative model

First let us define what problems can be named here as qualitative.

Definition 1. A *qualitative problem* is defined for the purposes of the present article and research as the problem such that the sign of the difference between the resulting expectations for the choices of the subjects (people) for the uncertain and sure games is distinct from the sign of the difference between the real expectations for these games.

This type of problems is chosen as the example that is usual in experiments (see, e.g., [31], [47], [51]). It can make manifest a qualitative representation of the problems and can be a background for further research. Such problems will be the item of the first stage of the MMBE.

So the inalienable feature of the analyzed qualitative problems is the necessary change of the sign. There can be only three combinations of the signs: the expectation for the uncertain game (or outcome) can be greater than, less than, or equal to that for the sure game. So, the signs of their differences can be correspondingly positive, negative, or zero.

In other words, when the difference between the real expectations is, e.g., positive (that is, $\text{sgn}\ (\mu_{uncert} - \mu_{sure}) > 0$), then, to obtain the observed data, the difference for the choices (resulting expectations) should be non-positive, that is, $\text{sgn}\ (\mu_{uncert} + \Delta_{ch\text{-}\mu.uncert} - \mu_{sure} - \Delta_{ch\text{-}\mu.sure}) \leq 0$. When it is negative, then the difference for the choices should be non-negative. When the difference between the real expectations is equal to zero, then the difference for the choices should be undoubtedly positive or negative.

This feature can be represented by

$$\text{sgn}\,(\mu_{uncert} - \mu_{sure}) \neq \text{sgn}\,(\mu_{uncert} + \Delta_{ch-\mu.uncert} - \mu_{sure} - \Delta_{ch-\mu.sure})\,. \qquad (10)$$

To overcome the real difference between the expectations for the uncertain and sure games, the absolute value of the presupposed bias for the uncertain game should be evidently not less than this real difference. That is

$$|\,\Delta_{ch-\mu.uncert}\,| \geq |\,\mu_{uncert} - \mu_{sure}\,|. \qquad (11)$$

So, relations (10) and (11) constitute an addition to the method. The sum can be named as a preliminary general qualitative mathematical model.

Note. Relation (11) implies, in particular, that if $\mu_{uncert} = \mu_{sure}$, then (11) takes the form of (7) (that is $|\,\Delta_{ch-\mu.uncert}\,| > 0$). For the other problems (11) takes the form $|\,\Delta_{ch-\mu.uncert}\,| > |\,\mu_{uncert} - \mu_{sure}\,|$.

The trial examples of [28] of applications of the general model show that it can qualitatively explain the practical examples cited here. Nevertheless, this preliminary general qualitative mathematical model still needs proofs.

6.3. Special qualitative mathematical model (SQMM)

A preliminary estimate of [28] restricts applications of the general model. One of the main questions for future research is to analyze the possible widths of the forbidden zones for various types of distributions.

Let us consider the qualitative problems under the special condition

$$\mu_{uncert} = \mu_{sure} \tag{12}$$

This special condition and relation assert that the expectations for the uncertain games are exactly equal to the expectations of the corresponding sure games. This is the well-known and important case of real experimental situations. Here (12) (keeping in mind (7)) substitutes (10) and (11).

Such a special situation enables avoiding the constraints of preliminary estimate [28] of the secure upper bound (5) for the bias, and making this special model less formal. The biases can be selected to be much less than the secure upper bound (5), and the suppositions will be simpler.

This Special Practical Qualitative Mathematical Model (SPQMM or shortly SQMM) can be considered as a first step of the first stage of the approach (method) MMBE.

6.4. Novelty

The literature analysis including the above citation from [51], leads to the reliable statement that the forbidden zones, theorem, hypothesis, method, and models introduced here have not been described before and are new.

The responses and comments of journals' editors and reviewers on the articles related to this research confirm this statement.

7. Particular consequence. Practical numerical examples for different domains

The above theorem leads to both particular and general consequences. They will be considered in this and next sections.

7.1. Particular consequence. Mathematical support for the analysis

Some well-known generic problems (see, e.g., [30]) were analyzed in, e.g., [26]. The problems include examples of typical paradoxes of prospect theories such as the underweighting of high and the overweighting of low probabilities, risk aversion, etc. The analysis was performed for the purposes of behavioral economics, psychology, decision theory, and the social sciences.

The analysis explained, at least partially or qualitatively, the analyzed paradoxes. Experimental and analytical works (see, e.g., [47] and [26]) devoted to the experimental methods of behavioral economics support it as well.

The analysis used the idea of the considered forbidden zones of the theorem. The r.v.s considered in the theorem include those used in this analysis.

So the theorem supports analysis [26]. This mathematical support can be considered as a particular consequence of the theorem.

7.2. Practical numerical example. First domain. Gains

Suppose that the parameters of the special practical qualitative mathematical model for the gains are: the presupposed bias for the choices for the uncertain game is equal to \$2, and for the sure game it is equal to \$1.

The typical examples (see, e.g., [31] and [47]) can be simplified to the special qualitative situations similar to that of the preceding section and [26].

Imagine that you face the following pair of concurrent games (a sure game and an uncertain game). Choose between:

A) A sure gain of \$99.

B) A 99% chance to gain \$100 and a 1% chance to gain or lose nothing.

7.2.1. Ideal case

In the ideal case, without taking into account the dispersion of the data, the expectations μ_{sure} and μ_{uncert} are equal to each other: $\mu_{sure} = \$99 \times 100\% = \$99$ and $\mu_{uncert} = \$100 \times 99\% = \$99$.

So, in the ideal case we have

$$\$99 = \$99,$$

that is, the uncertain and sure games are equally preferable.

7.2.2. Forbidden zones

In the real case, one should take into account some dispersion of the data, and hence the minimal non-zero variance (3) caused by this dispersion, and the forbidden zones (4) caused by this variance, at least for the uncertain games.

Let us consider the real case of a non-zero variance of the data, the corresponding forbidden zones, and presupposed biases.

The biases are $\Delta_{ch\text{-}\mu.uncert} = \$2$ and $\Delta_{ch\text{-}\mu.sure} = \$1$. So we have $\mu_{sure} - \Delta_{ch\text{-}\mu.sure}$ $= \$99\times 100\% - \$1 = \$98$ and $\mu_{uncert} - \Delta_{ch\text{-}\mu.uncert} = \$100\times 99\% - \$2 = \$97$. The expectation μ_{uncert} is biased more than μ_{sure} and

$$\$98 > \$97 .$$

We see the clear and evident difference between the resulting expectations (with their biases caused by the forbidden zones of the theorem) and its correspondence with the salient and unequivocal choices of the subjects.

7.3. Practical numerical example. Second domain. Losses

The case of gains has been explained many times, and in a lot of ways. But a uniform explanation for both gains and losses, without any additional suppositions (as, e.g., in [31]), had not been nevertheless recognized by the author of the present article (see a slightly similar work [20]).

SQMM turns out to be useful for such a uniform explanation.

Let us consider the case of losses under the same suppositions as for the case of gains.

Imagine that you face the following pair of concurrent games (a sure game and an uncertain game). Choose between:

A) A sure loss of -$99.

B) A 99% chance to lose -$100 and a 1% chance to lose or gain nothing.

7.3.1. Ideal case

In the ideal case, $\mu_{sure} = -\$99 \times 100\% = -\$99$ and $\mu_{uncert} = -\$100\times 99\% = -\$99$. So they are exactly equal to each other:

$$-\$99 = -\$99 .$$

Therefore the both choices (games) should be equally preferable.

7.3.2. Forbidden zones

The forbidden zone biases the expectation from the boundary of the interval to its middle (see also, e.g., [2], [5], [57]). Therefore, at high probabilities, the biases are subtracted from the absolute values for both cases, gains and losses. That is, due to the opposite signs of the values for gains and losses, the bias is subtracted for the gains and added for the losses.

Note. This is not a supposition but a rigorous conclusion. Hence the conditions of the SQMM are naturally uniform for more than one domain.

Let us consider the forbidden zones under the same suppositions as for the gains, that is for the same, uniform parameters.

The biases are $\Delta_{ch\text{-}\mu.uncert}$ = \$2 and $\Delta_{ch\text{-}\mu.sure}$ = \$1. So we have $\mu_{sure} + \Delta_{ch\text{-}\mu.sure}$ = -\$99×100% + \$1 = -\$98 and $\mu_{uncert} + \Delta_{ch\text{-}\mu.uncert}$ = -\$100×99% + \$2 = -\$97. The expectation μ_{uncert} is biased more than μ_{sure} and

$$-\$98 < -\$97 .$$

The expectation for the uncertain game is biased more than that for the sure one, as was also the case for the gains, but here the bias increases the preferability of the uncertain loss and it is (due to the obvious difference between the resulting expectations) more preferable than the sure one.

We see the clear difference between the resulting expectations and its correspondence with the salient preferences and choices. So the SQMM provides the explanation for the domain of losses as well. Moreover, this explanation is uniform for the both domains of gains and losses.

8. General consequences

8.1. Necessity of corrections

The expectations of r.v.s cannot lie within the forbidden zones.

Suppose a situation of an uncertain game at very high or very low probabilities. Suppose that without taking into account the theorem, the expectation is calculated to lie within this zone. If the situation satisfies the conditions of the theorem, then this calculation should necessarily be corrected.

So the descriptions of situations of uncertain games should be necessarily corrected when these situations satisfy the conditions of the theorem, at least within the forbidden zones.

8.2. Possible additional tools for various theories and models

The ideas, considerations and results of the present article can be used in various theories and models of behavioral sciences. In particular the relations and formulae (5)-(12) can be used as additional mathematical tools.

For example, relations (7) and (9) can be combined in a relation

$$|\Delta_{ch-\mu.uncert}| > |\Delta_{ch-\mu.sure}| \geq 0, \qquad (13)$$

that is more compact than the sum of (7) and (9). This relation can be especially useful near the boundaries of the probability scale, that is at very high and very low probabilities.

For another example, the existence of the non-zero forbidden zones leads to the necessity of essential revision of the form of the probability weighting curve (or Prelec curve), see, e.g., [41].

8.3. Possible description of the influence of noise

Let us make some preliminary considerations for possible general consequences of the theorem for a mathematical description of noise.

If some type of noise leads to some non-zero minimal variance (3) for the considered set of random variables, then this non-zero minimal variance (and, consequently, this type of noise) leads to the above non-trivial forbidden zones (4) for the expectations of these variables. If some type of noise leads to an increase in the value of this minimal variance, then the width of these forbidden zones increases also.

If this noise leads to a non-zero minimal variance $\sigma^2_{\min} : \sigma^2_i > \sigma^2_{\min} > 0$ for the set $\{\sigma^2_i\}$ of variances of the random variables X_i, then the theorem predicts there will be forbidden zones whose width $r_{\mu.noise}$ is not less than

$$r_{\mu.noise} \geq \frac{\sigma^2_{\min}}{b-a}.$$

So, the proven theorem can be a preliminary step towards a general mathematical description of the possible influence of noise near the boundaries of finite intervals.

Some general questions concerning this item can arise. For example, general determinations of level, strength, power, etc. of noise are needed. They should lead to the general determination of the non-negligible noise.

There are many types of noise. Another thing that is needed is the specification of common widespread types of measurement noise those can lead to a certain non-zero minimal variance of the measurement data in the usual circumstances and environments.

Due to the general character of the above questions and due to the demand for widespread experimental support, there is a need for a variety of research teams to give reliable answers to these questions.

8.4. Biases of measurement data

Let us preliminary consider potential general consequences of the theorem for a general mathematical description of the biases of data.

The forbidden zones (4) can lead evidently to some biases in measurements.

Suppose a set (like the above $\{X_i\}$) of series of measurements whose data all lie within a common finite interval. The set of the data series forms the set of their expectations. If there is some non-zero minimal variance of the data such that the inequality (3) is true for the data of any series, then there exist forbidden zones (4) for the set of the expectations of the series.

The allowed zone for the expectations is compressed by the forbidden zones (in comparison with the entire interval), and the expectations are biased from the boundaries to the middle of the interval (in comparison with the case of zero forbidden zones), see also the pull-to-center bias, e.g., [2], [24], [57].

An analysis of the biases in the expectations of the data needs much more volume than the present article permits. Nevertheless some possible results of such an analysis can be briefly outlined.

These biases can possess the following features:

1) They have opposite signs near the opposite boundaries.
2) Their moduli are decreased from the boundaries to the middle.
3) They are directed from the boundaries to the middle (of the interval).

When the minimal variance of the data is equal to zero (that is when (3) is not true), then the expectations of the data of measurements can touch the boundaries of the interval. When the above (non-trivial) forbidden zones exist and are not taken into the consideration, then the predicted results are located closer to the boundaries than in the real case. Hence the predicted results are biased in the comparison with the real ones.

We will now look at a particular example of these biases. If the minimal variance (3) of the data is non-zero, that is if $\sigma^2 > \sigma^2_{\min} > 0$ is true, then the theorem predicts (5), i.e. near the boundaries of intervals the biases are

$$|\Delta_{bias}| \geq \frac{\sigma^2{}_{\min}}{b-a}.$$

So, the theorem, and its consequences and applications can be considered as a preliminary step to the general mathematical description of the biases of measurement data near the boundaries of finite intervals.

9. Conclusions

Theorem 1 is presented in this article. The theorem proves the existence of the forbidden zones for the expectations of the random variables. This proof can be applied to various types of data. The theorem leads to the three main results (contributions) of the present article.

One can summarize these three main new results as follows.

1) The necessity of corrections of descriptions, at least within the forbidden zones (i.e., at very high and very low probabilities), for any situation that satisfies the conditions of the theorem.

2) Estimate (6)

$$a < \left(a + \frac{\sigma^2{}_{\min}}{b-a}\right) \leq \mu_i \leq \left(b - \frac{\sigma^2{}_{\min}}{b-a}\right) < b$$

for the forbidden zones and their widths.

3) The special qualitative mathematical model (SQMM) that is uniformly true for more than one domain.

The relations of the SQMM can be compiled as follows:

Relations (7) for the non-zero biases

$$|\Delta_{ch-\mu.uncert}| > 0 \quad \text{or} \quad \operatorname{sgn}(\Delta_{ch-\mu.uncert}) \neq 0.$$

Relations (8) for the theorem and choices

$$\sigma^2 \geq \sigma^2{}_{\min} > 0 \quad \text{and}$$

$$\Delta_{ch-\mu.uncert} \approx \Delta_{theorem} \quad \text{or at least} \quad \Delta_{ch-\mu.uncert} = O(\Delta_{theorem}).$$

Relation (9) for the choices for the sure and uncertain games

$$|\Delta_{ch-\mu.uncert}| > |\Delta_{ch-\mu.sure}|.$$

And relation (12) for the special qualitative problems

$$\mu_{uncert} = \mu_{sure}.$$

The SQMM is a qualitative model both for practical estimation of special qualitative situations and for determination of the turning points in the conditions of quantitative experiments and situations.

At least one of the main goals for future research is to analyze the possible widths of the forbidden zones for various types of distributions.

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A. Appendix. Lemmas

Let us prove three lemmas for theorem (1). Namely let us prove that the maximal variance of any discrete or continuous or real valued random variable whose values lie within a finite interval is not more than the variance of the discrete random variable whose probability mass function has only two non-zero values, which are located at the boundaries of the interval.

A1. Lemma 1: Discrete part

Lemma 1. If the values of a random variable X lie within an interval $[a, b]$: $0 < (b\text{-}a) < \infty$, (1) holds, and the variance of X can be represented as

$$E[X-\mu]^2 = \sum_{x_k \in [a,b]} (x_k - \mu)^2 p_X(x_k) + \int_a^b (x-\mu)^2 f_X(x)dx, \qquad (14)$$

then

$$\sum_{x_k \in [a,b]} (x_k - \mu)^2 p_X(x_k) \le \sum_{x_k \in [a,b]} \left[(\mu - a)^2 \frac{b - x_k}{b-a} + (b-\mu)^2 \frac{x_k - a}{b-a} \right] p_X(x_k).$$

Proof. Consider the difference between these transformed and initial expressions for the discrete part of the variance for the cases $x_k \ge \mu$ and $x_k \le \mu$.

Case $x_k \ge \mu$.

If $a \le \mu \le x_k \le b$, then

$$\left[(a-\mu)^2 \frac{b-x_k}{b-a} + (b-\mu)^2 \frac{x_k - a}{b-a} \right] - (x_k - \mu)^2 \ge$$

$$\ge (b-\mu)^2 \left[\frac{x_k - a}{b-a} - \left(\frac{x_k - \mu}{b-\mu} \right)^2 \right]$$

and

$$0 \le \frac{x_k - \mu}{b-\mu} \le 1, \quad \text{and} \quad \left(\frac{x_k - \mu}{b - \mu} \right)^2 \le \frac{x_k - \mu}{b-\mu}.$$

Then

$$\frac{x_k - a}{b-a} - \left(\frac{x_k - \mu}{b-\mu} \right)^2 \ge \frac{x_k - a}{b-a} - \frac{x_k - \mu}{b-\mu} \equiv \frac{(x_k - \mu) + (\mu - a)}{(b-\mu) + (\mu - a)} - \frac{x_k - \mu}{b-\mu}$$

and we have

$$(b-\mu)^2\left[\frac{x_k-a}{b-a}-\left(\frac{x_k-\mu}{b-\mu}\right)^2\right]\geq 0\,.$$

So in the case $x_k \geq \mu$, the difference between the transformed and initial expressions for the discrete part of the variance is non-negative.

Case $x_k \leq \mu$.

If $a \leq x_k \leq \mu \leq b$, then, analogously to the above case,

$$\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]-(x_k-\mu)^2 \geq$$

$$\geq (\mu-a)^2\left[\frac{b-x_k}{b-a}-\left(\frac{\mu-x_k}{\mu-a}\right)^2\right]$$

and

$$(\mu-a)^2\left[\frac{b-x_k}{b-a}-\left(\frac{\mu-x_k}{\mu-a}\right)^2\right]\geq 0\,.$$

So in the case when $x_k \leq \mu$, the difference between the transformed and initial expressions for the discrete part of the variance is non-negative also.

Maximality.

So the difference

$$\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p_X(x_k)-(x_k-\mu)^2\,p_X(x_k)$$

is non-negative for any x_k such that $a \leq x_k \leq b$.

Let us estimate the difference between the transformed and initial expressions for the discrete part of the variance

$$\sum_{x_k\in[a,b]}\left[(a-\mu)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}-(x_k-\mu)^2\right]p_X(x_k)\,.$$

Every member of the sum is non-negative. Hence the total sum is non-negative as well. Lemma 1 has been proved. □

So, the variance of any discrete random variable whose values lie within a finite interval is not more than the variance of the discrete random variable (with the same expectation) which has only two values at the two boundary points of the interval. And the discrete part of the variance of X is not more than the variance for the PMF (with the same norm and expectation as for this discrete part) which has only two values, located at a and b.

A2. Lemma 2: Continuous part

Lemma 2. If the values of an r.v. X lie within $[a,b]$: $0 < (b-a) < \infty$, condition (1) holds, and the variance of X can be represented as (14), then

$$\int_a^b (x-\mu)^2 f_X(x)dx \le \int_a^b \left[(\mu-a)^2 \frac{b-x}{b-a} + (b-\mu)^2 \frac{x-a}{b-a} \right] f_X(x)dx .$$

Proof. Let us find the difference between these transformed and initial expressions for the continuous part of the variance. Let us consider separately the cases $x \ge \mu$ and $x \le \mu$.

Case $x \ge \mu$.

If $a \le \mu \le x \le b$, then, analogously to the above cases, for

$$\left[(\mu-a)^2 \frac{b-x}{b-a} + (b-\mu)^2 \frac{x-a}{b-a} \right] - (x-\mu)^2 \ge (b-\mu)^2 \left[\frac{x-a}{b-a} - \left(\frac{x-\mu}{b-\mu} \right)^2 \right]$$

we have

$$(b-\mu)^2 \left[\frac{x-a}{b-a} - \left(\frac{x-\mu}{b-\mu} \right)^2 \right] \ge 0 .$$

So in the case when $x \ge \mu$, the difference between the transformed and initial expressions for the continuous part of the variance is non-negative.

Case $x \le \mu$.

If $a \le x \le \mu \le b$, then considerations that are entirely analogous to the above cases lead to the conclusion

$$\left[(\mu-a)^2 \frac{b-x}{b-a} + (b-\mu)^2 \frac{x-a}{b-a} \right] - (\mu-x)^2 \ge 0 .$$

So in the case when $x \le \mu$ the difference between the transformed and initial expressions for the variance is non-negative as well.

Maximality.

Let us estimate the difference between the transformed and initial expressions of the continuous part of the variance

$$E_{contin.transformed}[X-\mu]^2 - E_{contin.initial}[X-\mu]^2 =$$
$$= \int_a^b \left[(a-\mu)^2 \frac{b-x}{b-a} + (b-\mu)^2 \frac{x-a}{b-a} - (x-\mu)^2 \right] f_X(x)dx .$$

Since the integrand is non-negative for every point in the scope of the limits of integration, the integral is non-negative as well. The difference between the expressions is therefore non-negative.

Lemma 2 has been proved. □

So, the variance of any continuous random variable whose values lie within a finite interval is not more than the variance of the discrete random variable (with the same expectation) which has only two values located at the two boundary points of the interval.

And the continuous part of the variance of X is not more than the corresponding part (with the same norm and expectation as for this continuous part) of the variance of the probability mass function which has only two values located at the boundary points a and b.

A3. Lemma 3: General mixed case

Lemma 3 (General mixed case). If the values of a random variable X lie within an interval $[a,b]: 0<(b-a)<\infty$, normalizing condition (1) holds, and the variance of the variable X can be represented as

$$E[X-\mu]^2 = \sum_{x_k\in[a,b]}(x_k-\mu)^2 p_X(x_k)+\int_a^b (x-\mu)^2 f_X(x)dx,$$

then the following inequality is true

$$\sum_{x_k\in[a,b]}(x_k-\mu)^2 p_X(x_k)+\int_a^b (x-\mu)^2 f_X(x)dx \le$$
$$\le \sum_{x_k\in[a,b]}\left[(\mu-a)^2\frac{b-x_k}{b-a}+(b-\mu)^2\frac{x_k-a}{b-a}\right]p_X(x_k)+ .$$
$$+\int_a^b\left[(\mu-a)^2\frac{b-x}{b-a}+(b-\mu)^2\frac{x-a}{b-a}\right]f_X(x)dx$$

Proof. The discrete part of this inequality has been proved (by means of lemma (1)) to be true independently of its continuous part for any combination of norms within the framework of normalizing condition (1).

The continuous part of this inequality has been proved (by means of lemma (2)) to be also true independently of its discrete part for any combination of norms within the framework of normalizing condition (1).

Therefore the sum of these two parts is true as well.

So lemma (3) has been proved. □

So in any case, the variance is maximal for the PMF that has only two values located at the two boundary points of the interval.

The transformations that are considered in lemmas (1) and (2) evidently do not change the expectation of the variable X. The expectation of the PMF for these two boundary points is therefore equal to the expectation of the initial random variable. Any two-point PMF $p_{ab} \equiv p_{ab}(a) + p_{ab}(b)$ is determined by its expectation (and these two points).

So $p_{ab}(a) = (b\text{-}\mu)/(b\text{-}a)$, and $p_{ab}(b) = (\mu\text{-}a)/(b\text{-}a)$, and

$$E_{ab}[X-\mu]^2 = (\mu-a)^2\frac{b-\mu}{b-a} + (b-\mu)^2\frac{\mu-a}{b-a} = (\mu-a)(b-\mu).$$

This expression agrees naturally with the result of [8] for discrete variables. So, Lemma 1 can be treated as another version of this result and Lemma (3) can be treated as its expansion.

So the variance of any random variable whose values lie within a finite interval $[a, b]$ is not more than that in inequality (2), that is,

$$E[X-\mu]^2 \le (\mu-a)(b-\mu).$$