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Abstract

Cointegration between Periodically Integrated (PI) processes has been analyzed among other by Birchenhall, Bladen-Hovell, Chui, Osborn, and Smith (1989), Boswijk and Franses (1995), Franses and Paap (2004), Kleibergen and Franses (1999) and del Barrio Castro and Osborn (2008). However, so far there is not a method, published in an academic journal, that allows us to determine the cointegration rank between PI processes. This paper fills the gap, a method to determine the cointegration rank between a set PI Processes based on the idea of pseudo-demodulation is proposed in the context of Seasonal Cointegration by del Barrio Castro, Cubadda and Osborn (2020). Once a pseudo-demodulation time series is obtained the Johansen (1995) procedure could be applied to determine the cointegration rank. A Monte Carlo experiment shows that the proposed approach works satisfactorily for small samples.

Keywords: Reduced Rank Regression,Periodic Cointegration, Periodically Integrated Processes.

JEL codes: C32.

1 Introduction

The two main ways of modeling non-stationary Integration in the seasonal time series are Seasonal Integration (SI) and Periodic Integration (PI), (see Ghysels and Osborn (2001) for details about the main characteristics and differences between PI and SI). PI could be more attractive than SI as the nonstationary behavior is ruled by a common stochastic trend shared by the seasons on the time series, contrary in the case of a SI process, where each season of the time series hs his own stochastic trend (see Osborn (1993) and Ghysels and Osborn (2001) for details). Furthermore, PI shows up as a suitable data generating process for seasonal time series when preferences of economic agents vary along the seasons of the year (see Hansen and Sargent (1993), Gersovitz and McKinnon (1978) and Osborn (1988)).

In terms of long-run relationships (Cointegration) that could be established between seasonal integrated processes, we could also find Seasonal and Periodic Cointegration. For SI processes it is possible to define both Seasonal and Periodic Cointegration, but in the case of PI processes only full Periodic Cointegration could be established (see del Barrio Castro and Osborn (2008) and Ghysels and Osborn (2001) for details). In the case of Seasonal Cointegration, both methods for single equation and reduced rank regression have been proposed to test for the presence of Cointegration and determining the Cointegration rank, (see for example Hylleberg, Engle, Granger and Yoo (1990), Engle, Granger, Hylleberg and Lee (1993), Johansen and Schaumburg (1998), Cubbada (2000) and Ahn and Reinsel (1994)). Periodic Cointegration was proposed by Birchenhall, Bladen-Hovell, Chui, Osborn, and Smith (1989). A single equation method to test for the presence of Periodic Cointegration was proposed by Boswijk and Franses (1995). The authors of the previous paper claim that their method could be applied to both SI and PI processes, but del Barrio Castro and Osborn (2008) show that the asymptotic distribution of the error-correction test for periodic cointegration derived by Boswijk and Franses (1995) does not apply for PI processes. Also del Barrio Castro and Osborn (2008) proposed a residual based cointegration test for Periodic Cointegration between PI processes. But to the best of our knowledge, only the working paper by Kleibergen and Franses (1999) has tried to propose a method to determine the cointegration rank between a set of PI processes, (see also Franses and Paap (2004) for details) the method proposed by Kleibergen and Franses (1999) relies on Periodic Vector Autoregressive (VAR) Models and implies the use of GMM and reduced rank regression techniques. Finally a full dynamic system approach, in which equations are estimated jointly for observations relating to each season, can theoretically be applied (Ghysels and Osborn (2001) pp 171–176) as it was done in the application of Haldrup, Hylleberg, Pons, and Sansó (2007), but the VAR becomes quite over-parameterized, hence this approach is feasible in practice only where data of a relatively high frequency are available.

In this paper, we proposed a simple method to determine the cointegration rank between PI processes inspired in the demodulation method proposed by del Barrio Castro, Osborn and Cubada (2020), that only needs the use of the procedure proposed by Johansen (1995) once the PI processes or time series are "filtered" or "demodulated".

The paper is organized as follows, in the next section, we describe and summarize the main characteristics of PI processes and their consequences in terms of cointegration between PI processes. After that, we present our reduced rank approach to determine the cointegration rank between PI processes, followed by a Monte Carlo Section where it is shown that our approach works well in small samples. Finally, the last section concludes.

It is useful to introduce some notation at this stage. Our analysis is concerned with seasonal processes which have S observations per year; for example, $S = 4$ for quarterly seasonal data. In this paper the vector of seasons representation that indicates a specific observation within the year it is used, and also the double subscript notation and it is important to appreciate that, in this vector notation, $x_{s\tau}$ indicates the s^{th} observation within the τ^{th} year; for example with quarterly data $x_{s\tau}$ is the s^{th} quarter of year τ within the available sample. Assuming that $t = 1$ represents the first period within a cycle, the identity $t = S(\tau - 1) + s$ provides the link between the usual time index and the vector notation.

2 Periodic Integration and Cointegration between Periodically Integrated Processes

In first place, we will focus on the main characteristics of Periodic Integrated processes. One of these characteristics is going to be very important and crucial for the approach proposed in this paper to determine the cointegration rank between PI processes. In second place, we will pay attention to the Cointegration possibilities between PI processes.

2.1 Periodic Integration (PI)

A Periodic autoregressive (PAR) process of order p is a generalization of an autoregressive process where the parameters are allowed to vary with the season of the year, hence we have:

$$\begin{aligned} y_{s\tau} &= \phi_{1s} y_{s-1,\tau} + \phi_{2s} y_{s-2,\tau} + \cdots + \phi_{ps} y_{s-p,\tau} + \varepsilon_{s\tau} \\ s &= 1, 2, \cdots, S \qquad \tau = 1, 2, \cdots, N \end{aligned} \tag{1}$$

where $\varepsilon_{s\tau}$ is the innovation of the process and we assume that $\varepsilon_{s\tau} \sim iid\left(0, \sigma_\varepsilon^2\right)$. In order to understand the concept of Periodic Integration, let focus on the PAR process of order one:

$$y_{s\tau} = \phi_s y_{s-1,\tau} + u_{s\tau}. \tag{2}$$

In (2) we assume that $u_{s\tau}$ is a stationary innovation, this assumption will help us later on to connect (2) with (1)[1]. The condition of Periodic Integration in (2) is $\prod_{s=1}^{S} \phi_s = 1$ and implies that between the seasons of the time series we have $S - 1$ cointegration relationships or equivalently the seasons of the process share a common stochastic trend. This situation is clearly shown in the so called Vector of Seasons representation of a PAR process, where the S seasons of the time series are stacked in a $S \times 1$ vector $Y_\tau = [y_{1\tau}, y_{2\tau}, \ldots, y_{S\tau}]'$ and have:

$$\mathbf{A}_0 Y_\tau = \mathbf{A}_1 Y_{\tau-1} + U_\tau \tag{3}$$

[1] Note that (1) is connected with (2) if we write (1) as:

$$\left(1 - \phi_{1s} L - \phi_{2s} L^2 - \cdots - \phi_{ps} L^p\right) y_{s\tau} = \varepsilon_{s\tau}$$

and factorize the polynomial $\left(1 - \phi_{1s} L - \phi_{2s} L^2 - \cdots - \phi_{ps} L^p\right)$ as:

$$\left(1 - \phi_{1s} L - \phi_{2s} L^2 - \cdots - \phi_{ps} L^p\right) = \left(1 - \phi_s L\right)\left(1 - \psi_{1s} L - \cdots - \psi^*_{p-1,s} L^{p-1}\right)$$

hence (1) is connected with (2) as if $u_{s\tau}$ in (2) is defined as follows:

$$\left(1 - \psi_{1s} L - \cdots - \psi_{p-1,s} L^{p-1}\right) u_{s\tau} = \varepsilon_{s\tau}.$$

where, $U_\tau = [u_{1\tau}, u_{2\tau}, ..., u_{S\tau}]'$ and $\mathbf{A}_0$ and $\mathbf{A}_1$are $S \times S$ matrices with generic elements

$$A_{0(h,j)} = \begin{cases} 1 & h = j,\ j = 1, ..., S \\ -\phi_h & h = j+1,\ j = 1, ..., S-1 \\ 0 & \text{otherwise} \end{cases} \qquad (4)$$
$$A_{1(h,j)} = \begin{cases} \phi_1 & h = 1,\ j = S \\ 0 & \text{otherwise} \end{cases}$$

in which the subscript (h, j) indicates the $(h, j)^{th}$ element of the respective matrix. As in the quarterly case studied by Paap and Franses (1999), successively substituting in (3) yields

$$\begin{aligned} Y_\tau &= [\mathbf{A}_0^{-1}\mathbf{A}_1]^\tau Y_0 + \mathbf{A}_0^{-1}U_\tau + \sum_{j=1}^{\tau-1}[\mathbf{A}_0^{-1}\mathbf{A}_1]^j \mathbf{A}_0^{-1}U_{\tau-j} \\ &= \mathbf{A}_0^{-1}\mathbf{A}_1 Y_0 + \mathbf{A}_0^{-1}U_\tau + \mathbf{A}_0^{-1}\mathbf{A}_1\mathbf{A}_0^{-1}\sum_{j=1}^{\tau-1}U_{\tau-j}. \end{aligned} \qquad (5)$$

Note, that this result follows because matrix $\mathbf{A}_0^{-1}\mathbf{A}_1$ is idempotent. First, note that the matrix $\mathbf{A}_0$ (see chapter 2 pp 45-48 of Pollock (1999)) is an $S \times S$ lower-triangular Toeplitz matrix associated with the polynomial $(1 - \phi_s L)$. Hence the matrix $\mathbf{A}_0^{-1}$ collects the coefficients of the expansion of the inverse polynomial associated with $(1 - \phi_s L)^2$. Based on the form of the matrices $\mathbf{A}_0^{-1}$ and $\mathbf{A}_1$, it is clear that the resulting matrix $\mathbf{A}_0^{-1}\mathbf{A}_1$ is an $S \times S$ matrix with the first $S-1$ columns having elements equal to zero and the last column equal to the column vector $\mathbf{v} = \begin{bmatrix} \phi_1 & \phi_1\phi_2 & \phi_1\phi_2\phi_3 & \cdots & \prod_{s=1}^{S}\phi_s \end{bmatrix}'$. Finally note that the last element of $\mathbf{v}$, that is, $\prod_{s=1}^{S}\phi_s$, is equal to 1, as we have Periodic Integration. Also, as the first $S-1$ columns of $\mathbf{A}_0^{-1}\mathbf{A}_1$ are equal to zero and the lower left element of this matrix is equal to one, implies that $[\mathbf{A}_0^{-1}\mathbf{A}_1]^j = \mathbf{A}_0^{-1}\mathbf{A}_1$ for $j = 2, 3, ...$ Clearly, (5) provides a representation of (2), where the matrix $\mathbf{A}_0^{-1}\mathbf{A}_1\mathbf{A}_0^{-1}$ gives the effect of the accumulated vector of shocks $\sum_{j=1}^{\tau-1} U_{\tau-j}$ (see for example Boswijk and Franses (1996), Paap and Franses (1999) and del Barrio Castro and Osborn (2008)). The matrix $\mathbf{A}_0^{-1}\mathbf{A}_1\mathbf{A}_0^{-1}$ has rank one and hence can be written as

$$\mathbf{A}_0^{-1}\mathbf{A}_1\mathbf{A}_0^{-1} = \mathbf{a}\mathbf{b}' \qquad (6)$$

where, for (6),

$$\begin{aligned} \mathbf{a} &= \begin{bmatrix} 1 & \phi_2 & \phi_2\phi_3 & \cdots & \prod_{s=2}^{S}\phi_s \end{bmatrix}' \\ \mathbf{b} &= \begin{bmatrix} 1 & \phi_1\prod_{s=2}^{S}\phi_s & \phi_1\prod_{s=3}^{S}\phi_s & \cdots & \phi_1 \end{bmatrix}'. \end{aligned} \qquad (7)$$

Therefore, the scalar partial sum $\mathbf{b}'\sum_{j=1}^{\tau-1} U_{\tau-j}$ in (5) is the common stochastic trend shared by the seasons of Y_τ. As we have a common stochastic trend shared by the S seasons of the PI process, we will have $S-1$ cointegration relationships between the seasons of (3), Re-write (3) as:

$$\begin{aligned} Y_\tau &= \mathbf{A}_0^{-1}\mathbf{A}_1 Y_{\tau-1} + \mathbf{A}_0^{-1}U_\tau \\ Y_\tau - Y_{\tau-1} &= \left[\mathbf{A}_0^{-1}\mathbf{A}_1 - I\right] Y_{\tau-1} + \mathbf{A}_0^{-1}U_\tau \end{aligned} \qquad (8)$$

[2] That is:

$$\mathbf{A}_0^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & & 0 \\ \phi_2 & 1 & 0 & 0 & \cdots & 0 \\ \phi_2\phi_3 & \phi_3 & 1 & 0 & \cdots & 0 \\ \phi_2\phi_3\phi_4 & \phi_3\phi_4 & \phi_4 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \prod_{j=2}^{S}\phi_j & \prod_{j=3}^{S}\phi_j & \prod_{j=4}^{S}\phi_j & \prod_{j=5}^{S}\phi_j & \cdots & 1 \end{bmatrix}.$$

and note that matrix $\left[\mathbf{A}_0^{-1}\mathbf{A}_1 - I\right]$ has rank $S-1$ and hence he have $S-1$ cointegration relationships between the S seasons of the PI process. Clearly $\left[\mathbf{A}_0^{-1}\mathbf{A}_1 - I\right] = \alpha\beta'$ where both α and β have dimension $S\times(S-1)$, one possible choice for the columns of β are the last $S-1$ rows of $\mathbf{A}_0$[3]. Finally, It is clear that we have cointegration between the seasons of Y_τ, if we multiply by the left (5) by β' we have:

$$\begin{aligned}\beta' Y_\tau &= \beta'\mathbf{A}_0^{-1}\mathbf{A}_1 Y_0 + \beta'\mathbf{A}_0^{-1}U_\tau + \beta'\mathbf{A}_0^{-1}\mathbf{A}_1\mathbf{A}_0^{-1}\sum_{j=1}^{\tau-1}U_{\tau-j} \\ &= \beta'\mathbf{A}_0^{-1}\mathbf{A}_1 Y_0 + \beta'\mathbf{A}_0^{-1}U_\tau + \beta'\mathbf{a}\mathbf{b}'\sum_{j=1}^{\tau-1}U_{\tau-j}.\end{aligned}$$

As $\beta'\mathbf{a} = 0$ we clearly show that $\beta' Y_\tau \sim I(0)$ and that we have $S-1$ cointegration relationships between the S seasons of $y_{s\tau}$ (or Y_τ). In the following lemma we summarize the stochastic behavior of Y_τ in (3).

Lemma 1 *For $Y_\tau = [y_{1\tau}, y_{2\tau}, y_{3\tau}, \ldots, y_{S\tau}]'$ with $y_{s\tau}$ $s = 1, 2, \cdots, S$ defined in (2-3) and with $(1 - \psi_{1s}L - \cdots - \psi_{p-1,s}L^{p-1})\, u_{s\tau} = \varepsilon_{s\tau}$ and $\varepsilon_{s\tau} \sim iid\,(0,\sigma^2)$ then*

$$\begin{aligned}\frac{1}{\sqrt{T}}Y^{-}_{\lfloor Tr\rfloor} &\Rightarrow \sigma\mathbf{A}_0^{-1}\mathbf{A}_1\mathbf{A}_0^{-1}\Psi(1)^{-1}W(r) = \sigma\mathbf{a}\mathbf{b}'\Psi(1)^{-1}W(r) \\ &= \sigma\mathbf{a}w(r)\end{aligned} \tag{9}$$

where $\mathbf{a}$ *and* $\mathbf{b}$ *are defined in (7), $W(r)$ is a $S\times 1$ multivariant Brownian Vector and $w(r)$ is a scalar Brownian motion defined in the appendix. Finally the definition of matrix $\Psi(1)$ could be also found in the appendix.*

In the following subsection we pay attention the cointegration possibilities between PI processes.

2.2 Cointegration between PI processes

Ghysels and Osborn (2001) and del Barrio Castro and Osborn (2008) analyze the cointegration possibilities between PI processes, and show that between PI processes the only cointegration possibilities are full Periodic Cointegration or full Non-Periodic Cointegration.

Periodic Cointegration was introduced by Birchenhall, Bladen-Hovell, Chui, Osborn and Smith (1989) and implies that the long-run relationships are considered season by season, hence we have different cointegration vectors for each season. Periodic Cointegration could be established for both Seasonal Integrated (SI) processes and Periodically Integrated (PI) processes. Boswijk and Franses (1995) distinguished between full and partial Periodic Cointegration, the latter (partial) applies when stationary linear combinations between seasonal non-stationary time series could be established only for some seasons $s = 1, 2, \ldots, S$. And full Periodic Integration implies that the stationary linear combinations exit for all the seasons. Finally, full Non-Periodic Cointegration implies that the same cointegration vectors are shared for all the seasons.

In this paper, we follow the definition of Periodic Cointegration proposed in del Barrio Castro and Osborn (2008) (see definition 1 in section 2.2). Let consider the $m\times 1$ vector process $Y_{s\tau}^{(m)} = \left[\begin{array}{cccc} y_{s\tau}^1 & y_{s\tau}^2 & \ldots & y_{s\tau}^2 \end{array}\right]'$ where each of the elements $y_{s\tau}^k$ are PI processes, such that:

$$y_{s\tau}^k = \phi_s^k y_{s-1,\tau}^k + u_{s\tau}^k \quad \prod_{s=1}^{S}\phi_s^k = 1, \quad s = 1, 2, \ldots, S, \quad k = 1, 2, \ldots, S. \tag{10}$$

[3]Note that we have $S-1$ cointegration relationships between the seasons of (2) of the form $y_{s\tau} - \phi_s y_{s-1,\tau}$, that are clearly identified with the last $S-1$ rows of matrix $\mathbf{A}_0$. that is:

$$\beta' = \begin{bmatrix} -\phi_2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\phi_3 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\phi_S & 1 \end{bmatrix}.$$

Note also, that equivalently we could also use its normalized version

$$\beta^{*\prime} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -\phi_1 \\ 0 & 1 & 0 & \cdots & 0 & -\phi_1\phi_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\prod_{i=1}^{S-1}\phi_i \end{bmatrix}.$$

where each $u_{s\tau}^{(k)}$ is a stationary periodic autoregressive process:

$$\left(1-\psi_{1s}^{k}L-\cdots-\psi_{p-1,s}^{k}L^{p-1}\right)u_{s\tau}^{k}=\varepsilon_{s\tau}^{k}.$$

Finally the vector $E_{s\tau}^{(m)}=\left[\begin{array}{cccc}\varepsilon_{s\tau}^{1} & \varepsilon_{s\tau}^{2} & \ldots & \varepsilon_{s\tau}^{m}\end{array}\right]'$ is a white noise vector with positive definite variance-covariance matrix $E\left[E_{s\tau}^{(m)}E_{s\tau}^{(m)\prime}\right]=\Sigma$. Definition 1 in del Barrio Castro and Osborn (2008), established periodic cointegration for a $m\times 1$ vector $Y_{s\tau}^{(m)}$ of periodic processes satisfying (10) if there exist $m\times r$ matrices β_s of rank r such that the linear combinations $\beta_s'Y_{s\tau}^{(m)}$ are (periodically) stationary for each season $s=1,2,\ldots,S$. Boswijk and Franses (1995) define Partial Periodic Cointegration when stationary linear combinations $\beta_s'Y_{s\tau}^{(m)}$ exists for only some seasons, and Full Periodic Cointegration when the linear combinations exist for all the seasons. Full Non-Periodic Cointegration is a particular case of Full Periodic Cointegration where the same $m\times r$ matrix β allow us to obtain stationary linear combinations for all the seasons.

In Lemma 1 in del Barrio Castro and Osborn (2008) it is shown that between PI processes such as the ones collected in vector $Y_{s\tau}^{(m)}$ above with elements defined in (10), the only cointegration possibilities are Full Periodic Cointegration and Full Non-Periodic Cointegration. The intuition behind this result is that as shown in Lemma 1 of the previous subsection the S seasons of a PI process are driven by the same stochastic common trend, hence if we have cointegration between one of the seasons of PI processes recursive substitution implies that cointegration hold for the rest of seasons, with cointegration vector that will change for each season unless all the PI processes share the same coefficients associated with the PI condition $\prod_{s=1}^{S}\phi_s^k=1$ that is $\phi_s^k=\phi_s$ for $k=1,2,\ldots,m$ and $s=1,2,\ldots,S$. And precisely in this latter case when all the PI processes share the same coefficients $\phi_s^k=\phi_s$ associated to the PI condition we have Full Non-Periodic Cointegration.

For simplicity and to allow us to pay attention to the main facts on the problem we will focus from now onwards on the case of 3 PI processes, that is $m=3$[4]. Between 3 PI processes we could have the following situations: (a) no cointegration, (b) one common stochastic trend shared by the 3 PI processes and (c) two common stochastic trends shared by the 3 PI processes.

2.3 The 3 PI processes case.

Let focus on vector $Y_{s\tau}^{(3)}=\left[\begin{array}{ccc}y_{s\tau}^{1} & y_{s\tau}^{2} & y_{s\tau}^{3}\end{array}\right]'$ with the elements $y_{s\tau}^{k}$ $k=1,2$ and 3, been Periodically Integrated (PI). In order to understand the cointegration possibilities between these Three-variant PI systems, we use the following 3-variant vector of seasons $Y_{\tau}^{(3)}=\left[\begin{array}{ccc}y_{1\tau}^{1},y_{2\tau}^{1},\ldots,y_{S\tau}^{1} & y_{1\tau}^{2},y_{2\tau}^{2},\ldots,y_{S\tau}^{2} & y_{1\tau}^{3},y_{2\tau}^{3},\ldots,y_{S\tau}^{3}\end{array}\right]'$. For the scenarios (a) no cointegration, (b) one common stochastic trend shared by the 3 PI processes and (c) two common stochastic trends shared by the 3 PI processes, hence we have the following VAR(1):

$$\begin{aligned}
\mathbf{A}_0^{(3)}Y_{\tau}^{(3)} &= \mathbf{A}_1^{(3)}Y_{\tau-1}^{(3)}+U_{\tau}^{(3)} \qquad (11)\\
Y_{\tau}^{(3)} &= \left[\begin{array}{ccc}y_{1\tau}^{1},y_{2\tau}^{1},\ldots,y_{S\tau}^{1} & y_{1\tau}^{2},y_{2\tau}^{2},\ldots,y_{S\tau}^{2} & y_{1\tau}^{3},y_{2\tau}^{3},\ldots,y_{S\tau}^{3}\end{array}\right]'\\
&= \left[\begin{array}{c}Y_{\tau}^{1}\\ Y_{\tau}^{2}\\ Y_{\tau}^{3}\end{array}\right]\\
U_{\tau}^{(3)} &= \left[\begin{array}{ccc}u_{1\tau}^{1},u_{2\tau}^{1},\ldots,u_{S\tau}^{1} & u_{1\tau}^{2},u_{2\tau}^{2},\ldots,u_{S\tau}^{2} & u_{1\tau}^{3},u_{2\tau}^{3},\ldots,u_{S\tau}^{3}\end{array}\right]'\\
&= \left[\begin{array}{c}U_{\tau}^{1}\\ U_{\tau}^{2}\\ U_{\tau}^{3}\end{array}\right]. \qquad (12)
\end{aligned}$$

The matrices $\mathbf{A}_0^{(3)}$ and $\mathbf{A}_1^{(3)}$ are square matrices of dimension $(3S)\times(3S)$ and will have a different form in the 3 scenarios.

2.3.1 No Cointegration

In scenario (a) of no cointegration between the 3 PI processes. The matrices $\mathbf{A}_0^{(3)}$ and $\mathbf{A}_1^{(3)}$ in (11) will be as follows:

[4] Note also that in our Monte-Carlo section we are going to focus on the 3 PI processes case.

$$\begin{aligned}\mathbf{A}_0^{(3)} &= diag\left[\mathbf{A}_0^1, \mathbf{A}_0^2, \mathbf{A}_0^4\right] \\ \mathbf{A}_1^{(3)} &= diag\left[\mathbf{A}_1^1, \mathbf{A}_1^2, \mathbf{A}_1^3\right],\end{aligned} \tag{13}$$

hence $\mathbf{A}_0^{(3)}$ and $\mathbf{A}_1^{(3)}$ are block diagonal matrices with the following $S \times S$ submatrices:

$$\mathbf{A}_0^j = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -\phi_2^j & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\phi_3^j & 1 & 0 & \cdots & 0 \\ 0 & 0 & -\phi_4^j & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\phi_S^j & 1 \end{bmatrix} \quad j = 1, 2, 3 \tag{14}$$

$$\mathbf{A}_1^j = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \phi_1^j \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad j = 1, 2, 3. \tag{15}$$

In model (11) we could have by recursive substitution:

$$Y_\tau^{(3)} = \left[\left(\mathbf{A}_0^{(3)}\right)^{-1} \mathbf{A}_1^{(3)}\right]^\tau Y_0^{(3)} + \left(\mathbf{A}_0^{(3)}\right)^{-1} U_\tau^{(3)} + \sum_{j=1}^{\tau-1} \left[\left(\mathbf{A}_0^{(3)}\right)^{-1} \mathbf{A}_1^{(3)}\right]^j \left(\mathbf{A}_0^{(3)}\right)^{-1} U_{\tau-j}^{(3)}, \tag{16}$$

first note that the inverse matrix $\left(\mathbf{A}_0^{(3)}\right)^{-1}$ will be also block diagonal, such that:

$$\begin{aligned}\left(\mathbf{A}_0^{(3)}\right)^{-1} &= diag\left[\left(\mathbf{A}_0^1\right)^{-1}, \left(\mathbf{A}_0^2\right)^{-1}, \left(\mathbf{A}_0^3\right)^{-1}\right] \\ & with: \\ \left(\mathbf{A}_0^j\right)^{-1} &= \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \phi_2^j & 1 & 0 & 0 & \cdots & 0 \\ \phi_2^j\phi_3^j & \phi_3^j & 1 & 0 & \cdots & 0 \\ \phi_2^j\phi_3^j\phi_4^j & \phi_3^j\phi_4^j & \phi_4^j & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \prod_{k=2}^S \phi_k^j & \prod_{k=3}^S \phi_k^j & \prod_{k=4}^S \phi_k^j & \prod_{k=5}^S \phi_k^j & \cdots & 1 \end{bmatrix} \quad j = 1, 2, 3.\end{aligned} \tag{17}$$

Also, note that product $\left(\mathbf{A}_0^{(3)}\right)^{-1} \mathbf{A}_1^{(3)}$ is also block diagonal, with the following form:

$$\begin{aligned}\left(\mathbf{A}_0^{(3)}\right)^{-1} \mathbf{A}_1^{(3)} &= diag\left[\left(\mathbf{A}_0^1\right)^{-1} \mathbf{A}_1^1, \left(\mathbf{A}_0^2\right)^{-1} \mathbf{A}_1^2, \left(\mathbf{A}_0^3\right)^{-1} \mathbf{A}_1^3\right] \\ & with: \\ \left(\mathbf{A}_0^j\right)^{-1} \mathbf{A}_1^j &= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \phi_1^j \\ 0 & 0 & 0 & \cdots & 0 & \phi_1^j\phi_2^j \\ 0 & 0 & 0 & \cdots & 0 & \phi_1^j\phi_2^j\phi_3^j \\ 0 & 0 & 0 & \cdots & 0 & \phi_1^j\phi_2^j\phi_3^j\phi_4^j \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \prod_{k=1}^S \phi_k^j \end{bmatrix} \quad j = 1, 2, 3.\end{aligned}$$

Clearly, as we have PI processes the lower right element of the sub-matrices $\left(\mathbf{A}_0^j\right)^{-1}\mathbf{A}_1^j$ are equal to $\prod_{k=1}^{S}\phi_k^j = 1$. Hence, it is easy to check that matrix $\left(\mathbf{A}_0^{(3)}\right)^{-1}\mathbf{A}_1^{(3)}$ is idempotent. Then it is possible to write for (16):

$$Y_\tau^{(3)} = \left(\mathbf{A}_0^{(3)}\right)^{-1}\mathbf{A}_1^{(3)}Y_0^{(3)} + \left(\mathbf{A}_0^{(3)}\right)^{-1}U_\tau^{(3)} + \left(\mathbf{A}_0^{(3)}\right)^{-1}\mathbf{A}_1^{(3)}\left(\mathbf{A}_0^{(3)}\right)^{-1}\sum_{j=1}^{\tau-1}U_{\tau-j}^{(3)} \quad (18)$$

$$\left(\mathbf{A}_0^{(3)}\right)^{-1}\mathbf{A}_1^{(3)}\left(\mathbf{A}_0^{(3)}\right)^{-1} = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1' & 0_{S\times S} & 0_{S\times S} \\ 0_{S\times S} & \mathbf{a}_2\mathbf{b}_2' & 0_{S\times S} \\ 0_{S\times S} & 0_{S\times S} & \mathbf{a}_3\mathbf{b}_3' \end{bmatrix}$$

$$\mathbf{a}_j = \begin{bmatrix} 1 & \phi_2^j & \phi_2^j\phi_3^j & \cdots & \prod_{s=2}^{S}\phi_s^j \end{bmatrix}'$$

$$\mathbf{b}_j = \begin{bmatrix} 1 & \phi_1^j\prod_{s=2}^{S}\phi_s^j & \phi_1^j\prod_{s=3}^{S}\phi_s^j & \cdots & \phi_1^j \end{bmatrix}'.$$

Note that, from (18), each of the 3 PI processes collected in the vector $Y_\tau^{(3)}$ has his own stochastic trend, that is $\mathbf{b}_j'\sum_{k=1}^{\tau-1}U_{\tau-k}^j$ for $k = 1, 2$ and 3. And also we have cointegration between the seasons of each PI process. The stochastic behavior is summarized in the following lemma.

Lemma 2 *For* $Y_\tau^{(3)} = \left[Y_\tau^{1\prime}, Y_\tau^{2\prime}, Y_\tau^{3}\right]'$ *defined in (11-18) and with* $\left(1 - \psi_{1s}^jL - \cdots - \psi_{p-1,s}^jL^{p-1}\right)u_{s\tau}^j = \varepsilon_{s\tau}^j$ *for* $j = 1, 2$ *and* 3 *and* $E_{s\tau}^{(3)} = \begin{bmatrix} \varepsilon_{s\tau}^1 & \varepsilon_{s\tau}^2 & \varepsilon_{s\tau}^3 \end{bmatrix}'$ *is a white noise vector with positive definite variance-covariance matrix* $E\left[E_{s\tau}^{(3)}E_{s\tau}^{(3)\prime}\right] = \Sigma$ *then*

$$\begin{aligned} \frac{1}{\sqrt{T}}Y_{\lfloor Tr\rfloor}^{(3)} &\Rightarrow \left(\mathbf{A}_0^{(3)}\right)^{-1}\mathbf{A}_1^{(3)}\left(\mathbf{A}_0^{(3)}\right)^{-1}\Psi^{(3)}(1)^{-1}\left[\mathbf{P}\otimes\mathbf{I}_S\right]W^{(3)}(r) \\ &= \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1' & 0_{S\times S} & 0_{S\times S} \\ 0_{S\times S} & \mathbf{a}_2\mathbf{b}_2' & 0_{S\times S} \\ 0_{S\times S} & 0_{S\times S} & \mathbf{a}_3\mathbf{b}_3' \end{bmatrix}\Psi^{(3)}(1)^{-1}\left[\mathbf{P}\otimes\mathbf{I}_S\right]W^{(3)}(r) \\ &= \begin{bmatrix} \omega_1\mathbf{a}_1w_1(r) & 0_{S\times S} & 0_{S\times S} \\ 0_{S\times S} & \omega_2\mathbf{a}_2w_2(r) & 0_{S\times S} \\ 0_{S\times S} & 0_{S\times S} & \omega_3\mathbf{a}_3w_3(r) \end{bmatrix} \end{aligned} \quad (19)$$

where $\mathbf{a}_j$ *and* $\mathbf{b}_j$ *for* $j = 1, 2$ *and* 3 *are defined in (18),* $W^{(3)}(r)$ *is a* $(3S)\times 1$ *multivariant Brownian Vector and* $w_j(r)$ *for* $j = 1, 2$ *and* 3 *are scalar Brownian motions defined in the appendix. Finally, the definition of matrix* $\Psi^{(3)}(1)$ *and the scalar terms* ω_j *for* $j = 1, 2$ *and* 3 *could also be found in the appendix and* $\mathbf{P}$ *is a* 3×3 *matrix such that* $\Sigma = \mathbf{PP}'$.

Lemma 2 above is a particular case of Lemma 3 in del Barrio Castro and Osborn (2008) as in Lemma 2, we only have three PI processes but on the other hand Lemma 2 is defined for a general number of seasons and the results in del Barrio Castro and Osborn (2008) are for quarterly data. Clearly, in Lemma 2 we show that between the S seasons of each PI process we have $S-1$ cointegration relationships and a common stochastic trend for each PI process, that is, reflected in each scalar Brownian motion $w_j(r)$ with $j = 1, 2$ and 3.

2.3.2 One Common stochastic trend shared between the three PI processes

In the case of cointegration between PI processes we know from Lemma 1 in del Barrio Castro and Osborn (2008) that we should have Full Periodic Cointegration or Full Non-Periodic Cointegration, the latter situation is restricted to the case where all the PI processes share the same vaule for the coefficients associated to the periodic integration restriction. In the three PI system, a common stochastic trend implies the existence of two periodic cointegration relationships. Let's consider the following situation[5]:

$$\begin{aligned} y^1_{s\tau} &= \alpha_s y^3_{s\tau} + u^1_{s\tau} \\ y^2_{s\tau} &= \beta_s y^3_{s\tau} + u^2_{s\tau} \\ y^3_{s\tau} &= \phi^3_s y^3_{s-1,\tau} + u^3_{s\tau} \\ s &= 1,2,\ldots,S. \end{aligned} \tag{20}$$

with α_s and β_s such that:

$$\begin{aligned} \alpha_S = \alpha = \alpha\frac{\prod_{i=0}^{S-1}\phi^3_{S-i}}{\prod_{i=0}^{S-1}\phi^1_{S-i}} \qquad & \beta_S = \beta = \beta\frac{\prod_{i=0}^{S-1}\phi^3_{S-i}}{\prod_{i=0}^{S-1}\phi^2_{S-i}} \\ \alpha_{S-1} = \alpha\frac{\phi^3_S}{\phi^1_S} \qquad & \beta_{S-1} = \beta\frac{\phi^3_S}{\phi^2_S} \\ \alpha_{S-2} = \alpha\frac{\phi^3_S\phi^3_{S-1}}{\phi^1_S\phi^1_{S-1}} \qquad & \beta_{S-2} = \beta\frac{\phi^3_S\phi^3_{S-1}}{\phi^2_S\phi^2_{S-1}} \\ \alpha_{S-3} = \alpha\frac{\phi^3_S\phi^3_{S-1}\phi^3_{S-2}}{\phi^1_S\phi^1_{S-1}\phi^1_{S-2}} \qquad & \beta_{S-2} = \beta\frac{\phi^3_S\phi^3_{S-1}\phi^3_{S-2}}{\phi^2_S\phi^2_{S-1}\phi^2_{S-2}} \\ \vdots \qquad & \\ \alpha_1 = \alpha\frac{\prod_{i=0}^{S-2}\phi^3_{S-i}}{\prod_{i=0}^{S-2}\phi^1_{S-i}} \qquad & \beta_1 = \beta\frac{\prod_{i=0}^{S-2}\phi^3_{S-i}}{\prod_{i=0}^{S-2}\phi^2_{S-i}}. \end{aligned} \tag{21}$$

The system (20)-(21) admits a Vector of Seasons representation like (11) with matrices $\mathbf{A}_0^{(3)}$ and $\mathbf{A}_1^{(3)}$ in (11) that will be as follows:

$$\begin{aligned} \mathbf{A}_0^{(3)} &= \begin{bmatrix} I_S & 0_{S\times S} & \mathbf{A}_0^{(y_1)} \\ 0_{S\times S} & I_S & \mathbf{A}_0^{(y_2)} \\ 0_{S\times S} & 0_{S\times S} & \mathbf{A}_0^{(y_3)} \end{bmatrix} \\ \mathbf{A}_1^{(3)} &= \begin{bmatrix} 0_{S\times S} & 0_{S\times S} & 0_{S\times S} \\ 0_{S\times S} & 0_{S\times S} & 0_{S\times S} \\ 0_{S\times S} & 0_{S\times S} & \mathbf{A}_1^{(y_3)} \end{bmatrix}, \end{aligned} \tag{22}$$

with the $S \times S$ submatrices $\mathbf{A}_0^{(y_3)}$ and $\mathbf{A}_1^{(y_3)}$, defined equivalently as (14) and (15) that is:

$$\mathbf{A}_0^{(y_3)} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -\phi^3_2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\phi^3_3 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -\phi^3_4 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\phi^3_S & 1 \end{bmatrix} \qquad \mathbf{A}_1^{(y_3)} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \phi^3_1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{23}$$

[5] Trought the paper, we are going to use the normalization collected for example Lütkepohl (2005) pp 250.

Finally both the $S \times S$ submatrices $\mathbf{A}_0^{(y1)}$ and $\mathbf{A}_0^{(y_2)}$ are diagonal matrices of the form:

$$\begin{aligned}
\mathbf{A}_0^{(y1)} &= diag\left[-\alpha_1, -\alpha_2, -\alpha_3, \ldots, -\alpha_S\right] \\
&= diag\left[-\alpha\frac{\prod_{i=0}^{S-2}\phi_{S-i}^3}{\prod_{i=0}^{S-2}\phi_{S-i}^1}, -\alpha\frac{\prod_{i=0}^{S-3}\phi_{S-i}^3}{\prod_{i=0}^{S-3}\phi_{S-i}^1}, -\alpha\frac{\prod_{i=0}^{S-4}\phi_{S-i}^3}{\prod_{i=0}^{S-4}\phi_{S-i}^1}, \ldots, -\alpha\right] \qquad (24)\\
\mathbf{A}_0^{(y_2)} &= diag\left[-\beta_1, -\beta_2, -\beta_3, \ldots, -\beta_S\right] \\
&= diag\left[-\beta\frac{\prod_{i=0}^{S-2}\phi_{S-i}^3}{\prod_{i=0}^{S-2}\phi_{S-i}^2}, -\beta\frac{\prod_{i=0}^{S-3}\phi_{S-i}^3}{\prod_{i=0}^{S-3}\phi_{S-i}^2}, -\beta\frac{\prod_{i=0}^{S-4}\phi_{S-i}^3}{\prod_{i=0}^{S-4}\phi_{S-i}^2}, \ldots, -\beta\right].
\end{aligned}$$

We could have also recursive substitution as in (16), note that, it is possible to check that the inverse of matrix $\mathbf{A}_0^{(3)}$ in (22) will be as follows:

$$\left(\mathbf{A}_0^{(3)}\right)^{-1} = \begin{bmatrix} I_S & 0_{S\times S} & -\mathbf{A}_0^{(y_1)}\left(\mathbf{A}_0^{(y_3)}\right)^{-1} \\ 0_{S\times S} & I_S & -\mathbf{A}_0^{(y_2)}\left(\mathbf{A}_0^{(y_3)}\right)^{-1} \\ 0_{S\times S} & 0_{S\times S} & \left(\mathbf{A}_0^{(y_3)}\right)^{-1} \end{bmatrix}, \qquad (25)$$

note the inverse of sub-matrix $\mathbf{A}_0^{(y_3)}$, that is, $\left(\mathbf{A}_0^{(y_3)}\right)^{-1}$ is a lower triangular matrix as in (17), that is:

$$\left(\mathbf{A}_0^{(y_3)}\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \phi_2^3 & 1 & 0 & 0 & \cdots & 0 \\ \phi_2^3\phi_3^3 & \phi_3^3 & 1 & 0 & \cdots & 0 \\ \phi_2^3\phi_3^3\phi_4^3 & \phi_3^3\phi_4^3 & \phi_4^3 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \prod_{k=2}^{S}\phi_k^3 & \prod_{k=3}^{S}\phi_k^3 & \prod_{k=4}^{S}\phi_k^3 & \prod_{k=5}^{S}\phi_k^3 & \cdots & 1 \end{bmatrix}. \qquad (26)$$

Based on the form of $\left(\mathbf{A}_0^{(3)}\right)^{-1}$ and $\mathbf{A}_1^{(3)}$ in (25-22-26-24-**??**) it is possible to see that the product $\left(\mathbf{A}_0^{(3)}\right)^{-1} \times \mathbf{A}_1^{(3)}$ has the following expression:

$$\left(\mathbf{A}_0^{(3)}\right)^{-1} \times \mathbf{A}_1^{(3)} = \begin{bmatrix} 0_{S\times S} & 0_{S\times S} & 0_{S\times(S-1)}\mathbf{v}_1 \\ 0_{S\times S} & 0_{S\times S} & 0_{S\times(S-1)}\mathbf{v}_2 \\ 0_{S\times S} & 0_{S\times S} & 0_{S\times(S-1)}\mathbf{v}_3 \end{bmatrix}, \qquad (27)$$

hence all the elements of the $(3S)\times(3S)$ matrix $\left(\mathbf{A}_0^{(3)}\right)^{-1} \times \mathbf{A}_1^{(3)}$ are equal to zero, except for its last column. This last column is the concatenation of the $S \times 1$ vectors $\mathbf{v}_j$ $j = 1, 2$ and 3. Where the vectors are defined as follows:

$$
\begin{aligned}
\mathbf{v}_1 &= \left[\begin{array}{ccccc} \alpha_1\phi_1^3 & \alpha_2\phi_1^3\phi_2^3 & \alpha_3\phi_1^3\phi_2^3\phi_3^3 & \cdots & \alpha_S\prod_{s=1}^{S}\phi_s^3 \end{array} \right]' \\
&= \left[\begin{array}{ccccc} \alpha_1\phi_1^3 & \alpha_2\phi_1^3\phi_2^3 & \alpha_3\phi_1^3\phi_2^3\phi_3^3 & \cdots & \alpha_S \end{array} \right]' \\
\mathbf{v}_2 &= \left[\begin{array}{ccccc} \beta_1\phi_1^3 & \beta_2\phi_1^3\phi_2^3 & \beta_3\phi_1^3\phi_2^3\phi_3^3 & \cdots & \beta_S\prod_{s=1}^{S}\phi_s^3 \end{array} \right]' \\
&= \left[\begin{array}{ccccc} \beta_1\phi_1^3 & \beta_2\phi_1^3\phi_2^3 & \beta_3\phi_1^3\phi_2^3\phi_3^3 & \cdots & \beta_S \end{array} \right]' \\
\mathbf{v}_3 &= \left[\begin{array}{ccccc} \phi_1^3 & \phi_1^3\phi_2^3 & \phi_1^3\phi_2^3\phi_3^3 & \cdots & \prod_{s=1}^{S}\phi_s^3 \end{array} \right]' \\
&= \left[\begin{array}{ccccc} \phi_1^3 & \phi_1^3\phi_2^3 & \phi_1^3\phi_2^3\phi_3^3 & \cdots & 1 \end{array} \right]'.
\end{aligned} \tag{28}
$$

Note that the lower left element of matrix is $\left(\mathbf{A}_0^{(3)}\right)^{-1} \times \mathbf{A}_1^{(3)}$ is equal to one. And due to its form, it is clear that $\left(\mathbf{A}_0^{(3)}\right)^{-1} \times \mathbf{A}_1^{(3)}$ is idempotent. Hence in this case we also have:

$$
Y_\tau^{(3)} = \left(\mathbf{A}_0^{(3)}\right)^{-1} \mathbf{A}_1^{(3)} Y_0^{(3)} + \left(\mathbf{A}_0^{(3)}\right)^{-1} U_\tau^{(3)} + \left(\mathbf{A}_0^{(3)}\right)^{-1} \mathbf{A}_1^{(3)} \left(\mathbf{A}_0^{(3)}\right)^{-1} \sum_{j=1}^{\tau-1} U_{\tau-j}^{(3)} \tag{29}
$$

$$
\begin{aligned}
\left(\mathbf{A}_0^{(3)}\right)^{-1} \mathbf{A}_1^{(3)} \left(\mathbf{A}_0^{(3)}\right)^{-1} &= \left[\begin{array}{ccc} 0_{S\times S} & 0_{S\times S} & \mathbf{v}_1\mathbf{u}_3' \\ 0_{S\times S} & 0_{S\times S} & \mathbf{v}_2\mathbf{u}_3' \\ 0_{S\times S} & 0_{S\times S} & \mathbf{v}_3\mathbf{u}_3' \end{array} \right] \\
\mathbf{u}_3' &= \left[\begin{array}{cccccc} \prod_{k=2}^{S}\phi_k^3 & \prod_{k=3}^{S}\phi_k^3 & \prod_{k=4}^{S}\phi_k^3 & \prod_{k=5}^{S}\phi_k^3 & \cdots & 1 \end{array} \right]
\end{aligned}
$$

Note that $\mathbf{u}_3'$ is the last row of matrix $\left(\mathbf{A}_0^{(y_3)}\right)^{-1}$ (26). And it is also possible to write:

$$
\begin{aligned}
\left(\mathbf{A}_0^{(3)}\right)^{-1} \mathbf{A}_1^{(3)} \left(\mathbf{A}_0^{(3)}\right)^{-1} &= \left[\begin{array}{ccc} 0_{S\times S} & 0_{S\times S} & \alpha\left(\mathbf{a}_1\mathbf{b}_3'\right) \\ 0_{S\times S} & 0_{S\times S} & \beta\left(\mathbf{a}_2\mathbf{b}_3'\right) \\ 0_{S\times S} & 0_{S\times S} & \mathbf{a}_3\mathbf{b}_3' \end{array} \right] \\
\mathbf{a}_j &= \left[\begin{array}{ccccc} 1 & \phi_2^j & \phi_2^j\phi_3^j & \cdots & \prod_{s=2}^{S}\phi_s^j \end{array} \right]' \qquad j = 1, 2, 3 \\
\mathbf{b}_3' &= \left[\begin{array}{cccccc} 1 & \phi_1^3\prod_{k=3}^{S}\phi_k^3 & \phi_1^3\prod_{k=4}^{S}\phi_k^3 & \phi_1^3\prod_{k=5}^{S}\phi_k^3 & \cdots & \phi_1^3 \end{array} \right].
\end{aligned} \tag{30}
$$

Hence clearly the three PI processes share the same stochastic trend $\mathbf{b}_3' \sum_{k=1}^{\tau-1} U_{\tau-k}^3$. As in the previous subsection, the following lemma summarizes the stochastic behavior of the vector of seasons.

Lemma 3 *For* $Y_\tau^{(3)} = \left[Y_\tau^{1\prime}, Y_\tau^{2\prime}, Y_\tau^3\right]'$ *defined in (11-29-30) and with* $\left(1 - \psi_{1s}^j L - \cdots - \psi_{p-1,s}^j L^{p-1}\right) u_{s\tau}^j = \varepsilon_{s\tau}^j$ *for* $j = 1, 2$ *and* 3 *and* $E_{s\tau}^{(3)} = \left[\begin{array}{ccc} \varepsilon_{s\tau}^1 & \varepsilon_{s\tau}^2 & \varepsilon_{s\tau}^3 \end{array} \right]'$ *is a white noise vector with positive definite variance-covariance matrix* $E\left[E_{s\tau}^{(3)} E_{s\tau}^{(3)\prime}\right] = \Sigma$ *then*

$$
\begin{aligned}
\frac{1}{\sqrt{T}} Y_{\lfloor Tr \rfloor}^{(3)} &\Rightarrow \left(\mathbf{A}_0^{(3)}\right)^{-1} \mathbf{A}_1^{(3)} \left(\mathbf{A}_0^{(3)}\right)^{-1} \Psi^{(3)}(1)^{-1} \left[\mathbf{P} \otimes \mathbf{I}_S\right] W^{(3)}(r) \\
&= \left[\begin{array}{ccc} 0_{S\times S} & 0_{S\times S} & \alpha\left(\mathbf{a}_1\mathbf{b}_3'\right) \\ 0_{S\times S} & 0_{S\times S} & \beta\left(\mathbf{a}_2\mathbf{b}_3'\right) \\ 0_{S\times S} & 0_{S\times S} & \mathbf{a}_3\mathbf{b}_3' \end{array} \right] \Psi^{(3)}(1)^{-1} \left[\mathbf{P} \otimes \mathbf{I}_S\right] W^{(3)}(r) \\
&= \left[\begin{array}{ccc} 0_{S\times S} & 0_{S\times S} & \alpha\omega_3\mathbf{a}_1 w_3(r) \\ 0_{S\times S} & 0_{S\times S} & \beta\omega_3\mathbf{a}_2 w_3(r) \\ 0_{S\times S} & 0_{S\times S} & \omega_3\mathbf{a}_3 w_3(r) \end{array} \right]
\end{aligned} \tag{31}
$$

where $\mathbf{a}_j$ *for* $j = 1, 2$ *and* 3 *and* $\mathbf{b}_3$ *are defined in (30),* $W^{(3)}(r)$ *is a* $(3S) \times 1$ *multivariant Brownian Vector and* $w_3(r)$ *is a scalar Brownian motions defined in the appendix. Finally, the definition of matrix* $\Psi^{(3)}(1)$ *could be also found in the appendix and* $\mathbf{P}$ *is a* 3×3 *matrix as in the previous lemma.*

Clearly, Lemma 3 above shows that the common stochastic trend shared by the Seasons of the three PI processes is identified with the scalar Brownian Motion $w_3(r)$, hence we have cointegration within the Seasons of each PI process and also between the Seasons of all the PI processes.

2.3.3 Two Common stochastic trend shared between the three PI processes.

In the three PI system, two common stochastic trends imply the existence of one periodic cointegration relationship. Let consider the following situation:

$$
\begin{aligned}
y_{s\tau}^1 &= \beta_{1,s} y_{s\tau}^2 + \beta_{2,s} y_{s\tau}^3 + u_{s\tau}^1 \\
y_{s\tau}^2 &= \phi_s^3 y_{s-1,\tau}^2 + u_{s\tau}^2 \\
y_{s\tau}^3 &= \phi_s^3 y_{s-1,\tau}^3 + u_{s\tau}^3 \\
s &= 1, 2, \ldots, S.
\end{aligned}
\tag{32}
$$

with $\beta_{1,s}$ and $\beta_{2,s}$ are such that:

$$
\begin{aligned}
\beta_{1,S} = \beta_1 &= \beta_1 \frac{\prod_{i=0}^{S-1} \phi_{S-i}^2}{\prod_{i=0}^{S-1} \phi_{S-i}^1} & \beta_{2,S} = \beta_2 &= \beta_2 \frac{\prod_{i=0}^{S-1} \phi_{S-i}^3}{\prod_{i=0}^{S-1} \phi_{S-i}^1} \\
\beta_{1,S-1} &= \beta_1 \frac{\phi_S^2}{\phi_S^1} & \beta_{2,S-1} &= \beta_2 \frac{\phi_S^3}{\phi_S^1} \\
\beta_{1,S-2} &= \beta_1 \frac{\phi_S^2 \phi_{S-1}^2}{\phi_S^1 \phi_{S-1}^1} & \beta_{2,S-2} &= \beta_2 \frac{\phi_S^3 \phi_{S-1}^3}{\phi_S^1 \phi_{S-1}^1} \\
\beta_{1,S-3} &= \beta_1 \frac{\phi_S^2 \phi_{S-1}^2 \phi_{S-2}^2}{\phi_S^1 \phi_{S-1}^1 \phi_{S-2}^1} & \beta_{2,S-2} &= \beta_2 \frac{\phi_S^3 \phi_{S-1}^3 \phi_{S-2}^3}{\phi_S^1 \phi_{S-1}^1 \phi_{S-2}^1} \\
&\vdots \\
\beta_{1,1} &= \beta_1 \frac{\prod_{i=0}^{S-2} \phi_{S-i}^2}{\prod_{i=0}^{S-2} \phi_{S-i}^1} & \beta_{2,1} &= \beta_2 \frac{\prod_{i=0}^{S-2} \phi_{S-i}^3}{\prod_{i=0}^{S-2} \phi_{S-i}^1}.
\end{aligned}
\tag{33}
$$

The system (32)-(33) admits a Vector of Seasons representation like (11) where the matrices $\mathbf{A}_0^{(3)}$ and $\mathbf{A}_1^{(3)}$ in (11) will be as follows:

$$
\begin{aligned}
\mathbf{A}_0^{(3)} &= \begin{bmatrix} I_S & \mathbf{A}_0^{(y_1 y_2)} & \mathbf{A}_0^{(y_1 y_3)} \\ 0_{S\times S} & \mathbf{A}_0^{(y_2)} & 0_{S\times S} \\ 0_{S\times S} & 0_{S\times S} & \mathbf{A}_0^{(y_3)} \end{bmatrix} \\
\mathbf{A}_1^{(3)} &= \begin{bmatrix} 0_{S\times S} & 0_{S\times S} & 0_{S\times S} \\ 0_{S\times S} & \mathbf{A}_1^{(y_2)} & 0_{S\times S} \\ 0_{S\times S} & 0_{S\times S} & \mathbf{A}_1^{(y_3)} \end{bmatrix},
\end{aligned}
\tag{34}
$$

where the $S \times S$ submatrices $\mathbf{A}_0^{(y_3)}$ and $\mathbf{A}_1^{(y_3)}$ are defined as in (23) and $\mathbf{A}_0^{(y_2)}$ and $\mathbf{A}_1^{(y_2)}$ are defined equivalently, that is:

$$\mathbf{A}_0^{(y_2)} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -\phi_2^2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\phi_3^2 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -\phi_4^2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\phi_S^2 & 1 \end{bmatrix} \qquad \mathbf{A}_1^{(y_2)} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \phi_1^2 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{35}$$

Finally the $S \times S$ sub-matrices $\mathbf{A}_0^{(y_1 y_2)}$ and $\mathbf{A}_0^{(y_1 y_3)}$ are diagonal matrix, defined as follows:

$$\begin{aligned} \mathbf{A}_0^{(y_1 y_2)} &= diag\,[-\beta_{11}, -\beta_{12}, -\beta_{13}, \ldots, -\beta_{1S}] \\ &= diag\left[-\beta_1 \frac{\prod_{i=0}^{S-2} \phi_{S-i}^2}{\prod_{i=0}^{S-2} \phi_{S-i}^1}, -\beta_1 \frac{\prod_{i=0}^{S-3} \phi_{S-i}^2}{\prod_{i=0}^{S-3} \phi_{S-i}^1}, -\beta_1 \frac{\prod_{i=0}^{S-4} \phi_{S-i}^2}{\prod_{i=0}^{S-4} \phi_{S-i}^1}, \ldots, -\beta_1\right] \end{aligned} \tag{36}$$

$$\begin{aligned} \mathbf{A}_0^{(y_1 y_3)} &= diag\,[-\beta_{21}, -\beta_{22}, -\beta_{23}, \ldots, -\beta_{2S}] \\ &= diag\left[-\beta_2 \frac{\prod_{i=0}^{S-2} \phi_{S-i}^3}{\prod_{i=0}^{S-2} \phi_{S-i}^1}, -\beta_2 \frac{\prod_{i=0}^{S-3} \phi_{S-i}^3}{\prod_{i=0}^{S-3} \phi_{S-i}^1}, -\beta_2 \frac{\prod_{i=0}^{S-4} \phi_{S-i}^3}{\prod_{i=0}^{S-4} \phi_{S-i}^1}, \ldots, -\beta_2\right]. \end{aligned}$$

In this case, it is also possible to use recursive substitution as in (16), note that, it is possible to check that the inverse of matrix $\mathbf{A}_0^{(3)}$ in (34) will be as follows:

$$\left(\mathbf{A}_0^{(3)}\right)^{-1} = \begin{bmatrix} I_S & -\mathbf{A}_0^{(y_1 y_2)} \left(\mathbf{A}_0^{(y_2)}\right)^{-1} & -\mathbf{A}_0^{(y_1 y_3)} \left(\mathbf{A}_0^{(y_3)}\right)^{-1} \\ 0_{S\times S} & \left(\mathbf{A}_0^{(y2)}\right)^{-1} & 0_{S\times S} \\ 0_{S\times S} & 0_{S\times S} & \left(\mathbf{A}_0^{(y_3)}\right)^{-1} \end{bmatrix}. \tag{37}$$

Also as mentioned previously for $\left(\mathbf{A}_0^{(y_i)}\right)^{-1}$ with $i = 2$ and 3 we have:

$$\left(\mathbf{A}_0^{(y_i)}\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ \phi_2^i & 1 & 0 & 0 & \cdots & 0 \\ \phi_2^i \phi_3^i & \phi_3^i & 1 & 0 & \cdots & 0 \\ \phi_2^i \phi_3^i \phi_4^i & \phi_3^i \phi_4^i & \phi_4^i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \prod_{k=2}^{S} \phi_k^i & \prod_{k=3}^{S} \phi_k^i & \prod_{k=4}^{S} \phi_k^i & \prod_{k=5}^{S} \phi_k^i & \cdots & 1 \end{bmatrix} \qquad i = 2, 3. \tag{38}$$

The resulting matrix $\left(\mathbf{A}_0^{(3)}\right)^{-1} \times \mathbf{A}_1^{(3)}$:

$$\left(\mathbf{A}_0^{(3)}\right)^{-1} \times \mathbf{A}_1^{(3)} = \begin{bmatrix} 0_{S\times S} & 0_{S\times(S-1)}\mathbf{w}_{12} & 0_{S\times(S-1)}\mathbf{w}_{13} \\ 0_{S\times S} & 0_{S\times(S-1)}\mathbf{w}_2 & 0_{S\times S} \\ 0_{S\times S} & 0_{S\times S} & 0_{S\times(S-1)}\mathbf{w}_3 \end{bmatrix}, \tag{39}$$

hence all the elements of $\left(\mathbf{A}_0^{(3)}\right)^{-1}\times\mathbf{A}_1^{(3)}$ are equal to zero except for the elements of the $S\times 1$ vectors $\mathbf{w}_{12}$, $\mathbf{w}_{13}$, $\mathbf{w}_2$ and $\mathbf{w}_3$, that are defined as follows:

$$
\begin{aligned}
\mathbf{w}_{12} &= \left[\begin{array}{ccccc} \beta_{11}\phi_1^2 & \beta_{12}\phi_1^2\phi_2^2 & \beta_{13}\phi_1^2\phi_2^2\phi_3^2 & \cdots & \beta_{1S}\prod_{s=1}^{S}\phi_s^2 \end{array}\right]' \\
&= \left[\begin{array}{ccccc} \beta_{11}\phi_1^2 & \beta_{12}\phi_1^2\phi_2^2 & \beta_{13}\phi_1^2\phi_2^2\phi_3^2 & \cdots & \beta_{1S} \end{array}\right]' \\
\mathbf{w}_{13} &= \left[\begin{array}{ccccc} \beta_{21}\phi_1^3 & \beta_{22}\phi_1^3\phi_2^3 & \beta_{23}\phi_1^3\phi_2^3\phi_3^3 & \cdots & \beta_{2S}\prod_{s=1}^{S}\phi_s^3 \end{array}\right]' \\
&= \left[\begin{array}{ccccc} \beta_{21}\phi_1^3 & \beta_{22}\phi_1^3\phi_2^3 & \beta_{23}\phi_1^3\phi_2^3\phi_3^3 & \cdots & \beta_{2S} \end{array}\right]' \\
\mathbf{w}_j &= \left[\begin{array}{ccccc} \phi_1^j & \phi_1^j\phi_2^j & \phi_1^j\phi_2^j\phi_3^j & \cdots & \prod_{s=1}^{S}\phi_s^j \end{array}\right]' \\
&= \left[\begin{array}{ccccc} \phi_1^j & \phi_1^j\phi_2^j & \phi_1^j\phi_2^j\phi_3^j & \cdots & 1 \end{array}\right]' \qquad j=2,3.
\end{aligned}
\tag{40}
$$

As in the previous section, due to the form of matrix $\left(\mathbf{A}_0^{(3)}\right)^{-1}\times\mathbf{A}_1^{(3)}$ and noting than the last element of $\mathbf{w}_2$ and $\mathbf{w}_3$ are equal to one, it is possible to see that matrix $\left(\mathbf{A}_0^{(3)}\right)^{-1}\times\mathbf{A}_1^{(3)}$ is going to be idempotent. Hence we are also able to write for (11) with $\mathbf{A}_0^{(3)}$ and $\mathbf{A}_1^{(3)}$ defined in (34):

$$
Y_\tau^{(3)} = \left(\mathbf{A}_0^{(3)}\right)^{-1}\mathbf{A}_1^{(3)}Y_0^{(3)} + \left(\mathbf{A}_0^{(3)}\right)^{-1}U_\tau^{(3)} + \left(\mathbf{A}_0^{(3)}\right)^{-1}\mathbf{A}_1^{(3)}\left(\mathbf{A}_0^{(3)}\right)^{-1}\sum_{j=1}^{\tau-1}U_{\tau-j}^{(3)} \tag{41}
$$

$$
\begin{aligned}
\left(\mathbf{A}_0^{(3)}\right)^{-1}\mathbf{A}_1^{(3)}\left(\mathbf{A}_0^{(3)}\right)^{-1} &= \left[\begin{array}{ccc} 0_{S\times S} & \mathbf{w}_{12}\mathbf{u}_2' & \mathbf{w}_{12}\mathbf{u}_3' \\ 0_{S\times S} & \mathbf{w}_2\mathbf{u}_2' & 0_{S\times S} \\ 0_{S\times S} & 0_{S\times S} & \mathbf{w}_3\mathbf{u}_3' \end{array}\right] \\
\mathbf{u}_i' &= \left[\begin{array}{cccccc} \prod_{k=2}^{S}\phi_k^i & \prod_{k=3}^{S}\phi_k^i & \prod_{k=4}^{S}\phi_k^i & \prod_{k=5}^{S}\phi_k^i & \cdots & 1 \end{array}\right] \qquad i=2,3.
\end{aligned}
$$

It is possible to see that we could rewrite $\left(\mathbf{A}_0^{(3)}\right)^{-1}\mathbf{A}_1^{(3)}\left(\mathbf{A}_0^{(3)}\right)^{-1}$ in (41) as:

$$
\begin{aligned}
\left(\mathbf{A}_0^{(3)}\right)^{-1}\mathbf{A}_1^{(3)}\left(\mathbf{A}_0^{(3)}\right)^{-1} &= \left[\begin{array}{ccc} 0_{S\times S} & \beta_1\left(\mathbf{a}_1\mathbf{b}_2'\right) & \beta_2\left(\mathbf{a}_1\mathbf{b}_3'\right) \\ 0_{S\times S} & \mathbf{a}_2\mathbf{b}_2' & 0_{S\times S} \\ 0_{S\times S} & 0_{S\times S} & \mathbf{a}_3\mathbf{b}_3' \end{array}\right] \\
\mathbf{a}_j &= \left[\begin{array}{cccccc} 1 & \phi_2^j & \phi_2^j\phi_3^j & \cdots & \prod_{s=2}^{S}\phi_s^j \end{array}\right]' \qquad j=1,2,3 \\
\mathbf{b}_k' &= \left[\begin{array}{cccccc} 1 & \phi_1^k\prod_{k=3}^{S}\phi_k^k & \phi_1^k\prod_{k=4}^{S}\phi_k^k & \phi_1^k\prod_{k=5}^{S}\phi_k^k & \cdots & \phi_1^k \end{array}\right] \qquad j=2,3.
\end{aligned}
\tag{42}
$$

Hence in this three PI processes system we have two common stochastic trends, that is $\mathbf{b}_2'\sum_{k=1}^{\tau-1}U_{\tau-k}^2$ and $\mathbf{b}_3'\sum_{k=1}^{\tau-1}U_{\tau-k}^3$. As in the previous subsection the following lemma summarize the stochastic behavior of the vector of seasons.

Lemma 4 *For* $Y_\tau^{(3)} = \left[Y_\tau^{1\prime}, Y_\tau^{2\prime}, Y_\tau^3\right]'$ *defined in (11-41-42) and with* $\left(1-\psi_{1s}^jL-\cdots-\psi_{p-1,s}^jL^{p-1}\right)u_{s\tau}^j = \varepsilon_{s\tau}^j$ *for* $j=1,2$ *and* 3 *and* $E_{s\tau}^{(3)} = \left[\begin{array}{ccc}\varepsilon_{s\tau}^1 & \varepsilon_{s\tau}^2 & \varepsilon_{s\tau}^3\end{array}\right]'$ *is a white noise vector with positive definite variance-*

covariance matrix $E\left[E_{s\tau}^{(3)}E_{s\tau}^{(3)\prime}\right]=\Sigma$ *then*

$$\frac{1}{\sqrt{T}}Y_{\lfloor Tr\rfloor}^{(3)}\Rightarrow\left(\mathbf{A}_0^{(3)}\right)^{-1}\mathbf{A}_1^{(3)}\left(\mathbf{A}_0^{(3)}\right)^{-1}\Psi^{(3)}(1)^{-1}\left[\mathbf{P}\otimes\mathbf{I}_S\right]W^{(3)}(r) \quad (43)$$
$$=\begin{bmatrix}0_{S\times S} & \beta_1(\mathbf{a}_1\mathbf{b}_2') & \beta_2(\mathbf{a}_1\mathbf{b}_3')\\ 0_{S\times S} & \mathbf{a}_2\mathbf{b}_2' & 0_{S\times S}\\ 0_{S\times S} & 0_{S\times S} & \mathbf{a}_3\mathbf{b}_3'\end{bmatrix}\Psi^{(3)}(1)^{-1}\left[\mathbf{P}\otimes\mathbf{I}_S\right]W^{(3)}(r)$$
$$=\begin{bmatrix}0_{S\times S} & \beta_1\omega_3\mathbf{a}_1w_2(r) & \beta_2\omega_3\mathbf{a}_1w_3(r)\\ 0_{S\times S} & \mathbf{a}_2\omega_2w_2(r) & 0_{S\times S}\\ 0_{S\times S} & 0_{S\times S} & \omega_3\mathbf{a}_3w_3(r)\end{bmatrix}$$

where $\mathbf{a}_j$ *for* $j=1,2$ *and* 3 *and* $\mathbf{b}_2$ *and* $\mathbf{b}_3$ *are defined in (42),* $W^{(3)}(r)$ *is a* $(3S)\times 1$ *multivariant Brownian Vector and* $w_2(r)$ *and* $w_3(r)$ *are scalar Brownian motions defined in the appendix. Finally, the definition of matrix* $\Psi^{(3)}(1)$ *could be also found in the appendix and* $\mathbf{P}$ *is a* 3×3 *matrix as in the two previous lemmas.*

Note that, clearly the three PI processes system of Lemma 4, is driven by two common stochastic trends, that are clearly identified with the two scalar Brownian motions $w_2(r)$ and $w_3(r)$.

Finally, in the following section, we present our proposal to determine the cointegration rank with reduced rank regression techniques in systems of PI processes. As we will see the methodology is based on the demodulation approach proposed by del Barrio Castro, Cubadda and Osborn (2020).

3 Econometric Methodology

As mentioned previously our proposal is based on the demodulation approach used in del Barrio Castro, Cubadda and Osborn (2020). In the previous section, it is clearly shown that for a particular PI process we have $S-1$ cointegration relationships between the seasons of the PI process or equivalently there is a common stochastic trend shared by the seasons of the PI process. Clearly, in Lemmas 1 to 4, it is evident that the stochastic common trend could be identified with scalar Brownian motions, that drives the long run behavior of the seasons of each PI process in the systems of the previous section. For example, in Lemma 2 we have 3 common stochastic trends identified with the scalar Brownian motions $w_1(r)$, $w_2(r)$ and $w_3(r)$. These stochastic trends are adjusted to each season of the PI process by the elements of the $S\times 1$ vectors $\mathbf{a}_j$ for $j=1,2$ and 3. Note that the elements of the vector are the coefficients associated with the PI restriction, that is $\prod_{s=1}^{S}\phi_s^j=1$ for $j=1,2$ and 3. Note also, that in Lemmas 2, 3 and 4, the stochastic trend associated with scalar Brownian motions $w_1(r)$, $w_2(r)$ and $w_3(r)$ in Lemma 2, $w_3(r)$ in Lemma 3 and $w_2(r)$ and $w_3(r)$ in Lemma 3, are transmitted to each season of the three PI processes thought the elements of the vectors $\mathbf{a}_j$ for $j=1,2$ and 3.

Hence our approach is based on the simple idea of demodulating each time series by multiplying each season by the reciprocal (or inverse) of the corresponding element of vector $\mathbf{a}_j$. That is, assume for a PI time series $y_{s\tau}^j$ where the condition of PI $\prod_{s=1}^{S}\phi_s^j=1$ holds, and we have the $S\times 1$ vector $\mathbf{a}_j=\begin{bmatrix}\mathbf{a}_j^1 & \mathbf{a}_j^2 & \mathbf{a}_j^3 & \cdots & \mathbf{a}_j^S\end{bmatrix}'=\begin{bmatrix}1 & \phi_2^j & \phi_2^j\phi_3^j & \cdots & \prod_{s=2}^{S}\phi_s^j\end{bmatrix}'$. Our demodulation approach consists in multiply each observation of the PI time series $y_{s\tau}^j$ by the corresponding reciprocal of $\mathbf{a}_j^s$, that is $\left(\mathbf{a}_j^s\right)^{-1}$, and work with the new time series $\tilde{y}_{s\tau}^j=\left(\mathbf{a}_j^s\right)^{-1}y_{s\tau}^j$. Clearly, our approach implies knowledge of the coefficients associated with the PI restriction $\prod_{s=1}^{S}\phi_s^j=1$. This limitation could be easily solved with a test of Periodic Integration, like the Likelihood Ratio test of Periodic Integration proposed by Boswijk and Franses (1996) and the multivariant proposal of Franses (1994)[6]. In this paper, we will use the Boswijk and Franses (1996) test instead of the proposal by Franses (1994), as the latter has overparametrization problems (for quarterly data you need to run de Johansen procedure with four time series, that is, each quarter is treated as a

[6]Although a non-parametric tests to the the null of Periodic integration where proposed by del Barrio Castro and Osborn (2011,2012), these tests are not valid for our approach as the test do not need an estimation of the coefficients associated with the Periodic Integration restriction.

different time series). If we want to determine the cointegration rank between PI processes, a previous and necessary condition is to test (or be sure) that all the analyzed time series behave like PI processes. Hence, we could take advantage of this initial step and use it to obtain information about the values of the parameters associated with the PI condition (that is $\prod_{s=1}^{S} \phi_s^j = 1$).

To summarize our approach consists in the following steps:

- Test for Periodic integration using the Boswijk and Franses (1996) Likelihood Ratio test. And retain the values of the fitted coefficients associated to vector $\mathbf{a}_j = \begin{bmatrix} \mathbf{a}_j^1 & \mathbf{a}_j^2 & \mathbf{a}_j^3 & \cdots & \mathbf{a}_j^S \end{bmatrix}' = \begin{bmatrix} 1 & \phi_2^j & \phi_2^j \phi_3^j & \cdots & \prod_{s=2}^{S} \phi_s^j \end{bmatrix}'$.
- Obtain the $\tilde{y}_{s\tau}^j = \left(\mathbf{a}_j^s\right)^{-1} y_{s\tau}^j$ based on the estimation of the elements of $\mathbf{a}_j$ in the previous step.
- Finally include the demodulated time series $\tilde{y}_{s\tau}^j$ in the usual Johansen procedure. And determine the cointegration rank.

Note that we could use the usual critical values of the Johansen procedure. Also, it is important to note, that our approach has a clear advantage over the Boswijk and Franses (1995) and del Barrio Castro and Osborn (2008) as these methods do not allow us to determine the cointegration rank between a set of PI time series. Finally, we do not need to use a Periodic VAR framework and GMM jointly with reduced rank regression techniques as in Kleibergen and Franses (1999).

In the next section, we present results of a Monte Carlo experiment that clearly show that our approach works fine.

4 Monte Carlo

For our Monte Carlo experiment, we use a three variables approach like in subsections 2.3.1, 2.3.2 and 2.3.3. We explore the three situations presented in subsections 2.3.1 (Lemma 2), 2.3.2 (Lemma 3) and 2.3.3 (Lemma 4). That is, a situation with no cointegration between three PI processes, one common stochastic trend shared by three PI processes (that is, two periodic cointegration relationship between three PI processes) and finally two common stochastic trends shared by three PI processes (that is, one periodic cointegration relationship between three PI processes).

4.1 No Cointegration

We consider the three PI processes with no cointegration like in subsection 2.3.1, that is:

$$
\begin{aligned}
y_{s\tau}^1 &= \phi_s^1 y_{s-1,\tau}^1 + u_{s\tau}^1 \\
y_{s\tau}^2 &= \phi_s^2 y_{s-1\tau}^2 + u_{s\tau}^2 \\
y_{s\tau}^3 &= \phi_s^3 y_{s-1,\tau}^3 + u_{s\tau}^3 \\
s &= 1, 2, 3 \ and \ 4 \\
\tau &= 1, 2, \ldots, N
\end{aligned}
\tag{44}
$$

with the following combinations of the parameters:

Table 1

	ϕ_1^1	ϕ_2^1	ϕ_3^1	ϕ_1^2	ϕ_2^2	ϕ_3^2	ϕ_1^3	ϕ_2^3	ϕ_3^3
i	1.05	1.1	0.9	1.05	0.9	1.1	0.9	1.05	1.1
ii	1.2	0.8	1	1.2	1	0.8	1	1.2	0.8
iii	0.8	0.5	0.6	0.8	0.6	0.5	0.6	0.8	0.5
iv	0.5	1.2	1	0.5	1	1.2	1	0.5	1.2

Note that in table 1 we only provide the value of the first three parameters for each process, the non reported parameter, that is, ϕ_4^j for $j = 1, 2$ and 3, will be such that the PI condition holds, hence we will have $\phi_4^j = 1/\left(\phi_1^j \phi_2^j \phi_3^j\right)$. Also for the innovations $u_{s\tau}^j$ we consider the following three possibilities:

$$\begin{aligned} &(1) \quad u_{s\tau}^j = \varepsilon_{s\tau}^j \\ &(2) \quad u_{s\tau}^j = \varepsilon_{s\tau}^j - \theta \varepsilon_{s-1,\tau}^j \quad \theta = \{0.5, 0.8\} \\ &(3) \quad u_{s\tau}^j = \phi u_{s-1,\tau}^j + \varepsilon_{s\tau}^j \quad \phi = \{0.5, 0.8\} \\ &\qquad j = 1, 2, 3. \end{aligned} \tag{45}$$

where $E_{s\tau}^{(3)} = \begin{bmatrix} \varepsilon_{s\tau}^1 & \varepsilon_{s\tau}^2 & \varepsilon_{s\tau}^3 \end{bmatrix}'$ is a white noise vector with positive definite variance-covariance matrix $E\left[E_{s\tau}^{(3)} E_{s\tau}^{(3)\prime}\right] = \Sigma$. With the following three possibilities for Σ:

$$\begin{aligned} &(a) \quad \Sigma_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &(b) \quad \Sigma_2 = \begin{bmatrix} 1 & 0.8 & 0.3 \\ 0.8 & 1 & 0.5 \\ 0.3 & 0.5 & 1 \end{bmatrix} \\ &(b) \quad \Sigma_3 = \begin{bmatrix} 1 & 0.8 & 0.95 \\ 0.8 & 1 & 0.8 \\ 0.95 & 0.8 & 1 \end{bmatrix}. \end{aligned} \tag{46}$$

We consider quarterly data, that is $S = 4$, and we consider the following possibilities about the total number of year $N = 50$, 100, 200 and 500. Finally, all the results are obtained with 10.000 replications.

The results are collected in tables 2.A, 2.B and 2.C. Table 2.A collects the results obtained for white noise innovation and AR(1) innovation with $\phi = 0.5$. Table 2.B collects the results obtained for AR(1) innovation with $\phi = 0.8$ and a MA(1) innovation with $\theta = 0.5$. Finally, table 2.C collects the results obtained with a MA(1) innovation with $\theta = 0.8$. The columns labelled with *i*, *ii*, *iii* and *iv* refer to the values of the coefficients ϕ_1^j, ϕ_2^j and ϕ_3^j for $j = 1, 2, 3$ collected in Table 1. Finally, the labels Σ_1, Σ_2 and Σ_3 refer to three options of the variance covariance matrix $E\left[E_{s\tau}^{(3)} E_{s\tau}^{(3)\prime}\right]$ (46) used in the Monte-Carlo experiments.

As mentioned in the econometric strategy section, we first apply to all the time series the LR test by Boswijk and Franses (1996) and retain the fitted values of $\hat{\phi}_1^j$, $\hat{\phi}_2^j$, $\hat{\phi}_3^j$ and $\hat{\phi}_4^j$for $j = 1, 2, 3$ under the restriction $\prod_{s=1}^{4} \hat{\phi}_s^j = 1$. For case (1) in (45) to compute the LR test we fit a restricted and unrestricted PAR(1) and also the order pf the VAR in the Johansen procedure is set to 1. For case (2) in (45) the order of the PAR and VAR is 5 in the case of $\theta = 0.5$, and for $\theta = 0.8$ the PAR is of order 5 and for the VAR we obtain results for order 5 and 10. Finally, for case (2) in (45) both the PAR and VAR the order is 2. In the two remaining sections, the order of the PAR and VAR models fitted will be as defined here.

Clearly, the results of table 2.A, show that with white noise innovation the Johansen method applied to the "demodulated" time series works fine in detecting that we do not have cointegration between the three PI processes. And the results are very similar in the three scenarios about the Variance-Covariance Matrix of the $E\left[E_{s\tau}^{(3)} E_{s\tau}^{(3)\prime}\right] = \Sigma$. In the lower part of table 2.A we have the result with innovation following an AR(1) with $\phi = 0.5$, in this case we observe that the performance of the tests is very similar to the case of the white noise innovation, overall the Johansen procedure correctly detects the absence of cointegration between the three PI processes. The only difference observed if we compare the two panels of table 2.A is that in the case of $\phi = 0.5$, there is a slight over-size of the Johansen test for $r_0 = 0$ compared with the white noise innovation. Finally, the over-size situation is solved as the sample increases. In Table 2.B we have the results for an AR(1) innovation with $\phi = 0.8$ in the upper panel of the table and with a MA(1) innovation with $\theta = 0.5$ in the lower panel. Overall the performance of the Johansen procedure does a good job in detecting the cointegration rank. We observe an over-size for $\phi = 0.8$ bigger than with $\phi = 0.5$ and in some cases this situation is not totally solved as the sample size increased. In the case of $\theta = 0.5$ the observed over-size is even smaller than with $\phi = 0.5$. Table 2.c collects the results for a MA(1) innovation with $\theta = 0.8$, with an augmentation of 5 lags in the upper panel and 10 lags in the lower panel. Clearly, the use of 5 lags is not enough and in order to have a similar performance to the case of the MA(1) innovation with $\theta = 0.5$ we need to use 10 lags. Overall we could say that the Johansen procedure applied to the pseudo-demodulated time series does a good job detecting the absence of cointegration between the three PI processes.

4.2 One Periodic Cointegration Relationship

Compatible with subsection 2.3.3, here we explore the situation with three PI processes, with one Periodic long-run relationship or equivalently, a system of three PI processes ruled by two common stochastic trends, see Lemma 4. Hence we have a situation like in (32)-(33) with $S = 4$, $\beta_1 = \beta_2 = 1$. As in the previous subsection the values for ϕ_1^j, ϕ_2^j and ϕ_3^j for $j = 1, 2, 3$ with $\phi_4^j = 1/\left(\phi_1^j \phi_2^j \phi_3^j\right)$ for $j = 1, 2, 3$ are collected in Table 1. Hence we have:

$$
\begin{aligned}
y_{s\tau}^1 &= \beta_{1,s} y_{s\tau}^2 + \beta_{2,s} y_{s\tau}^3 + u_{s\tau}^1 \\
y_{s\tau}^2 &= \phi_s^3 y_{s-1,\tau}^2 + u_{s\tau}^2 \\
y_{s\tau}^3 &= \phi_s^3 y_{s-1,\tau}^3 + u_{s\tau}^3 \\
s &= 1, 2, \ldots, 4. \\
\beta_{1,4} &= 1 & \beta_{2,4} &= 1 \\
\beta_{1,3} &= \frac{\phi_4^2}{\phi_4^1} & \beta_{2,3} &= \frac{\phi_4^3}{\phi_4^1} \\
\beta_{1,2} &= \frac{\phi_4^2 \phi_4^2}{\phi 4 \phi_4^1} & \beta_{2,S-2} &= \beta_2 \frac{\phi_4^3 \phi_3^3}{\phi_4^1 \phi_3^1} \\
\beta_{1,1} &= \frac{\phi_4^2 \phi_3^2 \phi_2^2}{\phi_4^1 \phi_3^1 \phi_2^1} & \beta_{2,S-2} &= \beta_2 \frac{\phi_4^3 \phi_3^3 \phi_2^3}{\phi_4^1 \phi_3^1 \phi_2^1}.
\end{aligned}
$$

With $u_{s\tau}^1$, $u_{s\tau}^2$ and $u_{s\tau}^3$ with the four possibilities collected in (45) and also with the three cases considered in (46) for $E\left[E_{s\tau}^{(3)} E_{s\tau}^{(3)\prime}\right] = \Sigma$. Finally, we use the same sample sizes and number of replications as in the previous subsection. The results are collected in tables 3.a, 3.b and 3.c with the same organization in terms of the different schemes of serial correlation that in the previous section. Overall we could say that the Johansen procedure does a good job determining that the three PI processes share two common stochastic trends and that we have one periodic cointegration relationship between the three PI processes. In tables 3.a to 3.c we observe the same kind of over-size for $r_0 = 1$ than in tables 2.a to 2.c for $r_0 = 0$.

4.3 Two Periodic Cointegration Relationships

Finally compatible with subsection 2.3.2, here we explore the situation with three PI processes, with two Periodic long-run relationships or equivalently, a system of three PI processes ruled by one common stochastic trend, see Lemma 3. Hence we have a situation like in (20)-(21) with $S = 4$, $\beta = \alpha = 1$, as in the previous two cases the values for ϕ_1^j, ϕ_2^j and ϕ_3^j for $j = 1, 2, 3$ with $\phi_4^j = 1/\left(\phi_1^j \phi_2^j \phi_3^j\right)$ for $j = 1, 2, 3$ are collected in Table 1. Hence we have:

$$
\begin{aligned}
y_{s\tau}^1 &= \alpha_s y_{s\tau}^3 + u_{s\tau}^1 \\
y_{s\tau}^2 &= \beta_s y_{s\tau}^3 + u_{s\tau}^2 \\
y_{s\tau}^3 &= \phi_s^3 y_{s-1,\tau}^3 + u_{s\tau}^3 \\
s &= 1, 2, \ldots, 4. \\
\alpha_4 &= 1 & \beta_4 &= 1 \\
\alpha_3 &= \frac{\phi_4^3}{\phi_4^1} & \beta_3 &= \frac{\phi_4^3}{\phi_4^2} \\
\alpha_2 &= \frac{\phi_4^3 \phi_3^3}{\phi_4^1 \phi_3^1} & \beta_2 &= \frac{\phi_4^3 \phi_3^3}{\phi_4^2 \phi_3^2} \\
\alpha_1 &= \frac{\phi_4^3 \phi_3^3 \phi_2^3}{\phi_4^1 \phi_3^1 \phi_2^1} & \beta_1 &= \frac{\phi_4^3 \phi_3^3 \phi_2^3}{\phi_4^2 \phi_3^2 \phi_2^2}
\end{aligned}
\tag{47}
$$

We consider the same options for the innovations $u_{s\tau}^1$, $u_{s\tau}^2$ and $u_{s\tau}^3$ and the variance covariance matrix$E\left[E_{s\tau}^{(3)} E_{s\tau}^{(3)\prime}\right] = \Sigma$ that in the two previous subsections. Finally, the sample size and number of replications as in the two previous subsections as well. The results are collected in tables 4.a, 4.b and 4.c, following the same structure about the serial correlation that in the two sets of tables of the previous subsection. In general terms we could say that in tables 4.a, 4.b and 4.c the performance of the Johansen procedure with the pseudo-demodulated approach does a good job determining the cointegration rank, clealrly the Johansen procedure detects that there is a common stochastic trend shared by the three PI processes. Hence the procedure correctly detects that we have two periodic cointegration relationship between the three PI processes. Finally, the over-size observed in tables 4.a to 4.c with $r_0 = 2$ is equivalent to the one reported for $r_0 = 1$ in tables 3.a to 3.c.

5 Conclusion

In this paper, we propose a method to determine the cointegration rank between a set of PI processes that could be easily implemented. Our method relies on the use of pseudo-demodulated time series, that could be obtained based on the estimation of the parameters associated with the periodic integration restriction $\prod_{s=1}^{S}\phi_s^j = 1$ that could be obtained from the use of the test of Periodic Integration, like the Likelihood Ratio test of Periodic Integration proposed by Boswijk and Franses (1996). Once we have these pseudo-demodulated time series, they could be introduced in the reduced rank regression procedure of Johansen. In a Monte-Carlo section, we show that our proposal to determine the cointegration rank between a set of Periodically Integrated processes has a satisfactory performance for small samples.

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Table 2.a

rank		Σ_1				Σ_2				Σ_3			
	N	i	ii	iii	iv	i	ii	iii	iv	i	ii	iii	iv
	WN												
$r_0 = 0$	50	0.0667	0.0705	0.0734	0.0639	0.0678	0.0667	0.0715	0.0598	0.0665	0.0674	0.0767	0.0663
$r_0 = 0$	100	0.0621	0.0633	0.0646	0.0636	0.0630	0.0651	0.0684	0.0604	0.0598	0.0627	0.0673	0.0633
$r_0 = 0$	250	0.0526	0.0608	0.0632	0.0617	0.0595	0.0618	0.0625	0.0610	0.0629	0.0607	0.0633	0.0595
$r_0 = 0$	500	0.0586	0.0600	0.0620	0.0590	0.0576	0.0577	0.0576	0.0579	0.0579	0.0566	0.0578	0.0580
$r_0 = 1$	50	0.0047	0.0055	0.0052	0.0055	0.0048	0.0050	0.0050	0.0043	0.0047	0.0053	0.0072	0.0052
$r_0 = 1$	100	0.0041	0.0047	0.0049	0.0051	0.0055	0.0044	0.0047	0.0044	0.0050	0.0041	0.0044	0.0042
$r_0 = 1$	250	0.0043	0.0033	0.0043	0.0042	0.0040	0.0043	0.0037	0.0040	0.0044	0.0035	0.0052	0.0041
$r_0 = 1$	500	0.0047	0.0040	0.0028	0.0033	0.0039	0.0043	0.0035	0.0037	0.0049	0.0041	0.0050	0.0030
$r_0 = 2$	50	0.0011	0.0009	0.0005	0.0007	0.0001	0.0007	0.0005	0.0010	0.0003	0.0007	0.0008	0.0008
$r_0 = 2$	100	0.0005	0.0007	0.0006	0.0002	0.0005	0.0009	0.0009	0.0009	0.0008	0.0005	0.0006	0.0004
$r_0 = 2$	250	0.0007	0.0003	0.0005	0.0004	0.0006	0.0005	0.0003	0.0005	0.0001	0.0002	0.0010	0.0002
$r_0 = 2$	500	0.0005	0.0003	0.0004	0.0004	0.0007	0.0005	0.0004	0.0006	0.0004	0.0008	0.0006	0.0005
	$AR(1)$ $\phi = 0.5$												
$r_0 = 0$	50	0.1156	0.1035	0.0985	0.0921	0.1112	0.0947	0.0955	0.0924	0.1047	0.0968	0.0935	0.0918
$r_0 = 0$	100	0.0736	0.0767	0.0856	0.0640	0.0797	0.0722	0.0931	0.0714	0.0868	0.0759	0.0856	0.0788
$r_0 = 0$	250	0.0620	0.0648	0.0701	0.0623	0.0676	0.0671	0.0749	0.0729	0.0694	0.0634	0.0704	0.0687
$r_0 = 0$	500	0.0594	0.0616	0.0586	0.0574	0.0691	0.0585	0.0653	0.0660	0.0632	0.0653	0.0659	0.0672
$r_0 = 1$	50	0.0086	0.0083	0.0077	0.0089	0.0179	0.0114	0.0071	0.0073	0.0142	0.0117	0.0081	0.0055
$r_0 = 1$	100	0.0067	0.0063	0.0068	0.0056	0.0080	0.0056	0.0067	0.0042	0.0082	0.0069	0.0063	0.0045
$r_0 = 1$	250	0.0048	0.0041	0.0047	0.0045	0.0052	0.0044	0.0048	0.0058	0.0040	0.0055	0.0048	0.0041
$r_0 = 1$	500	0.0039	0.0040	0.0039	0.0037	0.0037	0.0038	0.0062	0.0049	0.0046	0.0032	0.0043	0.0043
$r_0 = 2$	50	0.0015	0.0010	0.0017	0.0010	0.0009	0.0012	0.0012	0.0009	0.0013	0.0013	0.0013	0.0005
$r_0 = 2$	100	0.0008	0.0011	0.0011	0.0005	0.0012	0.0007	0.0009	0.0006	0.0009	0.0008	0.0010	0.0007
$r_0 = 2$	250	0.0007	0.0003	0.0005	0.0005	0.0009	0.0005	0.0007	0.0011	0.0004	0.0008	0.0006	0.0005
$r_0 = 2$	500	0.0006	0.0009	0.0006	0.0005	0.0008	0.0004	0.0007	0.0007	0.0006	0.0004	0.0007	0.0006

Table 2.b

rank		Σ_1				Σ_2				Σ_3			
	N	i	ii	iii	iv	i	ii	iii	iv	i	ii	iii	iv
	$AR(1)$ $\phi = 0.8$												
$r_0 = 0$	50	0.1496	0.1485	0.1479	0.1768	0.2166	0.2083	0.1606	0.2461	0.2189	0.2079	0.1586	0.2523
$r_0 = 0$	100	0.0947	0.0919	0.1035	0.1197	0.1653	0.1552	0.1266	0.1906	0.1697	0.1638	0.1203	0.1934
$r_0 = 0$	250	0.0749	0.0782	0.0840	0.0774	0.1386	0.1325	0.1064	0.1451	0.1401	0.1323	0.1062	0.1459
$r_0 = 0$	500	0.0619	0.0652	0.0733	0.0648	0.1301	0.1228	0.0979	0.1307	0.1355	0.1181	0.0994	0.1349
$r_0 = 1$	50	0.0144	0.0131	0.0149	0.0168	0.0191	0.0197	0.0134	0.0236	0.0193	0.0213	0.0135	0.0240
$r_0 = 1$	100	0.0076	0.0089	0.0097	0.0098	0.0111	0.0131	0.0097	0.0155	0.0121	0.0094	0.0090	0.0136
$r_0 = 1$	250	0.0052	0.0059	0.0060	0.0067	0.0091	0.0087	0.0082	0.0085	0.0099	0.0089	0.0083	0.0087
$r_0 = 1$	500	0.0035	0.0046	0.0057	0.0047	0.0076	0.0077	0.0073	0.0090	0.0087	0.0076	0.0072	0.0083
$r_0 = 2$	50	0.0028	0.0015	0.0016	0.0021	0.0024	0.0015	0.0013	0.0022	0.0022	0.0030	0.0013	0.0031
$r_0 = 2$	100	0.0011	0.0012	0.0015	0.0016	0.0010	0.0014	0.0012	0.0016	0.0017	0.0015	0.0009	0.0010
$r_0 = 2$	250	0.0007	0.0006	0.0011	0.0007	0.0006	0.0022	0.0015	0.0007	0.0012	0.0011	0.0010	0.0010
$r_0 = 2$	500	0.0003	0.0004	0.0005	0.0003	0.0010	0.0004	0.0010	0.0009	0.0009	0.0006	0.0009	0.0012
	$MA(1)$ $\theta = 0.5$												
$r_0 = 0$	50	0.0873	0.0860	0.0693	0.0912	0.0914	0.0904	0.0752	0.0883	0.0929	0.0904	0.0787	0.0839
$r_0 = 0$	100	0.0822	0.0730	0.0636	0.0742	0.0797	0.0840	0.0613	0.0723	0.0796	0.0755	0.0650	0.0692
$r_0 = 0$	250	0.0749	0.0763	0.0624	0.0681	0.0689	0.0688	0.0582	0.0684	0.0688	0.0733	0.0657	0.0646
$r_0 = 0$	500	0.0754	0.0716	0.0593	0.0671	0.0737	0.0686	0.0615	0.0672	0.0664	0.0662	0.0652	0.0651
$r_0 = 1$	50	0.0065	0.0061	0.0064	0.0070	0.0071	0.0070	0.0069	0.0062	0.0060	0.0064	0.0062	0.0060
$r_0 = 1$	100	0.0063	0.0051	0.0042	0.0046	0.0062	0.0063	0.0047	0.0043	0.0050	0.0054	0.0045	0.0046
$r_0 = 1$	250	0.0052	0.0057	0.0047	0.0045	0.0047	0.0043	0.0048	0.0039	0.0051	0.0048	0.0048	0.0045
$r_0 = 1$	500	0.0067	0.0055	0.0046	0.0054	0.0046	0.0060	0.0041	0.0052	0.0034	0.0044	0.0042	0.0051
$r_0 = 2$	50	0.0010	0.0008	0.0005	0.0014	0.0008	0.0010	0.0013	0.0007	0.0009	0.0009	0.0004	0.0012
$r_0 = 2$	100	0.0017	0.0005	0.0005	0.0005	0.0014	0.0010	0.0002	0.0004	0.0009	0.0008	0.0004	0.0000
$r_0 = 2$	250	0.0009	0.0008	0.0008	0.0008	0.0007	0.0006	0.0002	0.0006	0.0007	0.0002	0.0004	0.0007
$r_0 = 2$	500	0.0012	0.0012	0.0008	0.0007	0.0004	0.0008	0.0007	0.0006	0.0002	0.0002	0.0004	0.0008

Table 2.c

rank	N	Σ_1 *i*	*ii*	*iii*	*iv*	Σ_2 *i*	*ii*	*iii*	*iv*	Σ_3 *i*	*ii*	*iii*	*iv*
	$MA(1)$ $\theta = 0.8$ order 5												
$r_0 = 0$	50	0.6340	0.5112	0.0728	0.1451	0.4560	0.3757	0.0724	0.1370	0.4749	0.3869	0.0743	0.1378
$r_0 = 0$	100	0.7415	0.6043	0.0632	0.1410	0.5380	0.4388	0.0640	0.1341	0.5590	0.4480	0.0622	0.1274
$r_0 = 0$	250	0.7974	0.6645	0.0622	0.1466	0.5854	0.4754	0.0587	0.1320	0.5976	0.4832	0.0648	0.1324
$r_0 = 0$	500	0.8216	0.6781	0.0640	0.1466	0.5996	0.4804	0.0581	0.1343	0.6093	0.4854	0.0623	0.1303
$r_0 = 1$	50	0.1565	0.0993	0.0056	0.0129	0.0648	0.0437	0.0057	0.0108	0.0660	0.0463	0.0046	0.0082
$r_0 = 1$	100	0.2204	0.1239	0.0041	0.0123	0.0777	0.0545	0.0043	0.0076	0.0830	0.0571	0.0036	0.0094
$r_0 = 1$	250	0.2443	0.1532	0.0058	0.0116	0.0885	0.0580	0.0060	0.0088	0.0971	0.0603	0.0039	0.0096
$r_0 = 1$	500	0.2651	0.1648	0.0047	0.0120	0.0919	0.0543	0.0035	0.0099	0.0912	0.0528	0.0040	0.0085
$r_0 = 2$	50	0.0237	0.0121	0.0012	0.0021	0.0058	0.0037	0.0011	0.0011	0.0060	0.0045	0.0005	0.0010
$r_0 = 2$	100	0.0277	0.0116	0.0010	0.0011	0.0044	0.0048	0.0009	0.0008	0.0077	0.0054	0.0003	0.0014
$r_0 = 2$	250	0.0292	0.0165	0.0009	0.0008	0.0058	0.0036	0.0010	0.0008	0.0058	0.0044	0.0003	0.0007
$r_0 = 2$	500	0.0340	0.0196	0.0006	0.0009	0.0058	0.0030	0.0005	0.0006	0.0061	0.0047	0.0009	0.0008
	$MA(1)$ $\theta = 0.8$ order 10												
$r_0 = 0$	50	0.1508	0.1388	0.1327	0.1099	0.1472	0.1506	0.1828	0.1229	0.1409	0.1454	0.1721	0.1263
$r_0 = 0$	100	0.1415	0.1192	0.0896	0.0840	0.1229	0.1074	0.1005	0.0885	0.1243	0.1114	0.1031	0.0880
$r_0 = 0$	250	0.1455	0.1158	0.0704	0.0691	0.1189	0.1057	0.0751	0.0668	0.1247	0.0992	0.0677	0.0730
$r_0 = 0$	500	0.1511	0.1169	0.0667	0.0649	0.1134	0.0955	0.0623	0.0651	0.1148	0.1031	0.0664	0.0673
$r_0 = 1$	50	0.0127	0.0119	0.0126	0.0104	0.0104	0.0137	0.0180	0.0100	0.0110	0.0139	0.0182	0.0134
$r_0 = 1$	100	0.0120	0.0087	0.0065	0.0063	0.0093	0.0093	0.0070	0.0090	0.0094	0.0081	0.0071	0.0056
$r_0 = 1$	250	0.0120	0.0075	0.0052	0.0052	0.0093	0.0077	0.0067	0.0046	0.0083	0.0066	0.0054	0.0039
$r_0 = 1$	500	0.0110	0.0081	0.0038	0.0059	0.0079	0.0070	0.0055	0.0052	0.0080	0.0064	0.0050	0.0057
$r_0 = 2$	50	0.0013	0.0015	0.0021	0.0008	0.0012	0.0017	0.0023	0.0014	0.0009	0.0020	0.0024	0.0019
$r_0 = 2$	100	0.0014	0.0010	0.0009	0.0011	0.0017	0.0012	0.0012	0.0022	0.0008	0.0012	0.0013	0.0006
$r_0 = 2$	250	0.0010	0.0004	0.0010	0.0005	0.0008	0.0016	0.0007	0.0009	0.0012	0.0007	0.0011	0.0006
$r_0 = 2$	500	0.0009	0.0007	0.0005	0.0009	0.0008	0.0006	0.0007	0.0005	0.0009	0.0008	0.0008	0.0003

Table 3.a

rank		Σ_1				Σ_2				Σ_3			
	N	i	ii	iii	iv	i	ii	iii	iv	i	ii	iii	iv
	WN												
$r_0 = 0$	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	0.0582	0.0571	0.0622	0.0544	0.0608	0.0629	0.0558	0.0550	0.0586	0.0596	0.0584	0.0519
$r_0 = 1$	100	0.0562	0.0533	0.0600	0.0568	0.0577	0.0553	0.0515	0.0559	0.0617	0.0543	0.0515	0.0580
$r_0 = 1$	250	0.0549	0.0515	0.0546	0.0559	0.0564	0.0522	0.0531	0.0555	0.0530	0.0498	0.0524	0.0569
$r_0 = 1$	500	0.0554	0.0545	0.0538	0.0511	0.0532	0.0586	0.0526	0.0540	0.0603	0.0521	0.0560	0.0541
$r_0 = 2$	50	0.0050	0.0040	0.0061	0.0048	0.0037	0.0048	0.0056	0.0049	0.0038	0.0055	0.0037	0.0047
$r_0 = 2$	100	0.0048	0.0044	0.0062	0.0054	0.0046	0.0033	0.0041	0.0036	0.0047	0.0036	0.0045	0.0042
$r_0 = 2$	250	0.0042	0.0035	0.0036	0.0042	0.0051	0.0039	0.0040	0.0045	0.0039	0.0036	0.0033	0.0038
$r_0 = 2$	500	0.0050	0.0055	0.0035	0.0044	0.0028	0.0046	0.0028	0.0046	0.0047	0.0044	0.0032	0.0044
	$AR(1)$ $\phi = 0.5$												
$r_0 = 0$	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	0.0686	0.0629	0.0691	0.0579	0.0672	0.0659	0.0551	0.0616	0.0654	0.0616	0.0512	0.0594
$r_0 = 1$	100	0.0568	0.0535	0.0636	0.0532	0.0583	0.0614	0.0519	0.0547	0.0631	0.0589	0.0540	0.0561
$r_0 = 1$	250	0.0560	0.0517	0.0612	0.0504	0.0572	0.0587	0.0557	0.0546	0.0548	0.0534	0.0549	0.0528
$r_0 = 1$	500	0.0573	0.0541	0.0499	0.0498	0.0537	0.0551	0.0517	0.0487	0.0520	0.0545	0.0501	0.0526
$r_0 = 2$	50	0.0062	0.0038	0.0052	0.0035	0.0065	0.0049	0.0032	0.0040	0.0050	0.0056	0.0042	0.0060
$r_0 = 2$	100	0.0044	0.0051	0.0048	0.0031	0.0042	0.0046	0.0035	0.0055	0.0067	0.0060	0.0049	0.0042
$r_0 = 2$	250	0.0046	0.0042	0.0036	0.0037	0.0052	0.0040	0.0040	0.0044	0.0048	0.0043	0.0037	0.0031
$r_0 = 2$	500	0.0044	0.0044	0.0036	0.0039	0.0041	0.0063	0.0047	0.0031	0.0040	0.0045	0.0038	0.0040

Table 3.b

rank		Σ_1				Σ_2				Σ_3			
	N	i	ii	iii	iv	i	ii	iii	iv	i	ii	iii	iv
	$AR(1)$ $\phi = 0.8$												
$r_0 = 0$	50	0.8762	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	0.0707	0.0738	0.0919	0.0920	0.0768	0.0782	0.0919	0.0703	0.0804	0.0784	0.0878	0.0742
$r_0 = 1$	100	0.0634	0.0627	0.0840	0.0900	0.0659	0.0622	0.0826	0.0689	0.0656	0.0580	0.0799	0.0689
$r_0 = 1$	250	0.0582	0.0565	0.0806	0.0582	0.0612	0.0534	0.0773	0.0583	0.0584	0.0585	0.0781	0.0547
$r_0 = 1$	500	0.0525	0.0564	0.0728	0.0473	0.0525	0.0522	0.0739	0.0455	0.0541	0.0534	0.0771	0.0476
$r_0 = 2$	50	0.0068	0.0066	0.0089	0.0062	0.0073	0.0069	0.0123	0.0053	0.0065	0.0075	0.0085	0.0064
$r_0 = 2$	100	0.0046	0.0054	0.0117	0.0055	0.0047	0.0057	0.0087	0.0044	0.0055	0.0045	0.0110	0.0048
$r_0 = 2$	250	0.0052	0.0040	0.0082	0.0044	0.0049	0.0034	0.0089	0.0046	0.0046	0.0047	0.0082	0.0050
$r_0 = 2$	500	0.0042	0.0040	0.0085	0.0040	0.0040	0.0043	0.0080	0.0039	0.0048	0.0039	0.0077	0.0045
	$MA(1)$ $\theta = 0.5$												
$r_0 = 0$	50	1.0000	1.0000	0.9989	0.9994	0.9747	0.9937	0.9982	0.9767	0.9724	0.9950	0.9989	0.9789
$r_0 = 0$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	0.0690	0.0718	0.0546	0.0650	0.0643	0.0678	0.0632	0.0646	0.0661	0.0696	0.0694	0.0684
$r_0 = 1$	100	0.0683	0.0703	0.0620	0.0645	0.0623	0.0700	0.0589	0.0603	0.0679	0.0685	0.0651	0.0615
$r_0 = 1$	250	0.0706	0.0691	0.0571	0.0604	0.0646	0.0650	0.0633	0.0622	0.0641	0.0638	0.0589	0.0619
$r_0 = 1$	500	0.0657	0.0680	0.0548	0.0626	0.0652	0.0587	0.0604	0.0582	0.0653	0.0672	0.0576	0.0608
$r_0 = 2$	50	0.0060	0.0045	0.0042	0.0051	0.0044	0.0050	0.0054	0.0045	0.0054	0.0065	0.0052	0.0056
$r_0 = 2$	100	0.0063	0.0053	0.0045	0.0061	0.0040	0.0054	0.0050	0.0050	0.0061	0.0053	0.0059	0.0062
$r_0 = 2$	250	0.0044	0.0061	0.0055	0.0054	0.0053	0.0051	0.0056	0.0053	0.0050	0.0057	0.0054	0.0046
$r_0 = 2$	500	0.0046	0.0037	0.0052	0.0040	0.0057	0.0036	0.0042	0.0051	0.0039	0.0056	0.0028	0.0032

Table 3.c

rank		Σ_1				Σ_2				Σ_3			
	N	*i*	*ii*	*iii*	*iv*	*i*	*ii*	*iii*	*iv*	*i*	*ii*	*iii*	*iv*
	$MA(1)$ $\theta = 0.8$ order 5												
$r_0 = 0$	50	1.0000	1.0000	0.9996	0.9996	0.9959	0.9939	0.9966	0.9901	0.9963	0.9941	0.9967	0.9879
$r_0 = 0$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	0.4886	0.4019	0.0575	0.1163	0.3855	0.3063	0.0632	0.1081	0.3956	0.3131	0.0658	0.1094
$r_0 = 1$	100	0.5686	0.4543	0.0589	0.1305	0.4479	0.3518	0.0638	0.1153	0.4618	0.3623	0.0652	0.1119
$r_0 = 1$	250	0.6196	0.4929	0.0571	0.1350	0.4959	0.3863	0.0629	0.1237	0.5035	0.3993	0.0619	0.1245
$r_0 = 1$	500	0.6389	0.5034	0.0591	0.1363	0.5014	0.4010	0.0599	0.1286	0.5202	0.4275	0.0603	0.1261
$r_0 = 2$	50	0.0929	0.0651	0.0043	0.0089	0.0599	0.0421	0.0058	0.0095	0.0648	0.0442	0.0066	0.0107
$r_0 = 2$	100	0.1076	0.0729	0.0048	0.0108	0.0700	0.0466	0.0050	0.0112	0.0722	0.0463	0.0044	0.0089
$r_0 = 2$	250	0.1192	0.0772	0.0036	0.0134	0.0713	0.0501	0.0059	0.0109	0.0780	0.0526	0.0045	0.0112
$r_0 = 2$	500	0.1264	0.0796	0.0043	0.0125	0.0759	0.0528	0.0043	0.0102	0.0887	0.0541	0.0045	0.0106
	$MA(1)$ $\theta = 0.8$ order 10												
$r_0 = 0$	50	0.9695	0.9521	0.8086	0.8699	0.8157	0.7992	0.8141	0.7898	0.8054	0.7932	0.8121	0.7958
$r_0 = 0$	100	1.0000	1.0000	0.9997	1.0000	0.9997	0.9998	0.9999	0.9999	0.9997	0.9996	0.9995	0.9995
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	0.1093	0.0930	0.0728	0.0650	0.0866	0.0840	0.0807	0.0637	0.0897	0.0776	0.0830	0.0675
$r_0 = 1$	100	0.1150	0.0966	0.0691	0.0629	0.0970	0.0883	0.0701	0.0605	0.1027	0.0903	0.0671	0.0621
$r_0 = 1$	250	0.1265	0.0992	0.0591	0.0628	0.1014	0.0917	0.0590	0.0622	0.1110	0.0889	0.0596	0.0578
$r_0 = 1$	500	0.1269	0.1032	0.0574	0.0606	0.1114	0.0941	0.0631	0.0601	0.1079	0.0908	0.0597	0.0550
$r_0 = 2$	50	0.0101	0.0068	0.0077	0.0047	0.0079	0.0086	0.0079	0.0053	0.0075	0.0072	0.0062	0.0061
$r_0 = 2$	100	0.0094	0.0084	0.0051	0.0044	0.0087	0.0081	0.0045	0.0044	0.0075	0.0068	0.0050	0.0037
$r_0 = 2$	250	0.0104	0.0086	0.0047	0.0042	0.0097	0.0065	0.0046	0.0043	0.0075	0.0070	0.0051	0.0044
$r_0 = 2$	500	0.0105	0.0078	0.0044	0.0046	0.0082	0.0072	0.0061	0.0043	0.0098	0.0073	0.0043	0.0048

Table 4.a

rank	N	Σ_1 i	ii	iii	iv	Σ_2 i	ii	iii	iv	Σ_3 i	ii	iii	iv
	WN												
$r_0 = 0$	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 2$	50	0.0534	0.0535	0.0572	0.0519	0.0519	0.0550	0.0494	0.0528	0.0549	0.0548	0.0543	0.0536
$r_0 = 2$	100	0.0519	0.0572	0.0511	0.0499	0.0507	0.0502	0.0532	0.0493	0.0531	0.0516	0.0502	0.0540
$r_0 = 2$	250	0.0537	0.0554	0.0528	0.0505	0.0560	0.0518	0.0544	0.0483	0.0517	0.0537	0.0540	0.0497
$r_0 = 2$	500	0.0502	0.0512	0.0516	0.0551	0.0491	0.0528	0.0519	0.0562	0.0526	0.0518	0.0536	0.0551
	$AR(1)$ $\phi = 0.5$												
$r_0 = 0$	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 2$	50	0.0560	0.0531	0.0475	0.0464	0.0572	0.0514	0.0516	0.0474	0.0517	0.0537	0.0516	0.0492
$r_0 = 2$	100	0.0491	0.0534	0.0429	0.0478	0.0538	0.0556	0.0502	0.0461	0.0530	0.0499	0.0495	0.0474
$r_0 = 2$	250	0.0511	0.0498	0.0413	0.0502	0.0522	0.0580	0.0432	0.0512	0.0551	0.0509	0.0455	0.0473
$r_0 = 2$	500	0.0509	0.0526	0.0396	0.0475	0.0494	0.0520	0.0509	0.0460	0.0526	0.0515	0.0451	0.0474

Table 4.b

rank	N	Σ_1 *i*	*ii*	*iii*	*iv*	Σ_2 *i*	*ii*	*iii*	*iv*	Σ_3 *i*	*ii*	*iii*	*iv*
	$AR(1)\quad \phi = 0.8$												
$r_0 = 0$	50	0.9988	0.9986	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	0.8649	0.8640	0.9999	0.9790	0.9573	0.9632	0.9972	0.9938	0.9576	0.9602	0.9901	0.9915
$r_0 = 1$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 2$	50	0.0459	0.0494	0.0299	0.0346	0.0550	0.0562	0.0408	0.0403	0.0544	0.0507	0.0370	0.0391
$r_0 = 2$	100	0.0527	0.0540	0.0261	0.0370	0.0568	0.0501	0.0380	0.0385	0.0517	0.0515	0.0358	0.0397
$r_0 = 2$	250	0.0512	0.0479	0.0252	0.0357	0.0535	0.0507	0.0338	0.0361	0.0493	0.0523	0.0336	0.0371
$r_0 = 2$	500	0.0531	0.0494	0.0263	0.0317	0.0532	0.0541	0.0366	0.0356	0.0515	0.0536	0.0328	0.0390
	$MA(1)\quad \theta = 0.5$												
$r_0 = 0$	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	1.0000	1.0000	1.0000	1.0000	0.9997	1.0000	0.9999	1.0000	0.9999	0.9999	0.9999	0.9998
$r_0 = 1$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 2$	50	0.0618	0.0614	0.0495	0.0585	0.0552	0.0614	0.0559	0.0544	0.0627	0.0556	0.0508	0.0533
$r_0 = 2$	100	0.0655	0.0628	0.0504	0.0577	0.0596	0.0612	0.0512	0.0518	0.0603	0.0616	0.0551	0.0569
$r_0 = 2$	250	0.0608	0.0638	0.0570	0.0609	0.0649	0.0639	0.0512	0.0590	0.0636	0.0644	0.0522	0.0586
$r_0 = 2$	500	0.0642	0.0633	0.0521	0.0612	0.0609	0.0576	0.0522	0.0537	0.0575	0.0624	0.0562	0.0544

Table 4.c

rank		Σ_1				Σ_2				Σ_3			
	N	i	ii	iii	iv	i	ii	iii	iv	i	ii	iii	iv
	$MA(1)$ $\theta = 0.8$ order 5												
$r_0 = 0$	50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	1.0000	1.0000	1.0000	1.0000	0.9999	1.0000	0.9998	1.0000	0.9998	0.9999	0.9999	0.9999
$r_0 = 1$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 2$	50	0.3282	0.2902	0.0545	0.0946	0.3017	0.2458	0.0568	0.0917	0.3045	0.2551	0.0565	0.0899
$r_0 = 2$	100	0.3712	0.3038	0.0536	0.1006	0.3226	0.2744	0.0519	0.0949	0.3515	0.2856	0.0553	0.0959
$r_0 = 2$	250	0.4005	0.3238	0.0564	0.1084	0.3563	0.2932	0.0558	0.1022	0.3648	0.3009	0.0559	0.1036
$r_0 = 2$	500	0.3931	0.3271	0.0628	0.1109	0.3587	0.3002	0.0508	0.1046	0.3731	0.3033	0.0582	0.1066
	$MA(1)$ $\theta = 0.8$ order 10												
$r_0 = 0$	50	1.0000	1.0000	0.9990	1.0000	0.9998	0.9999	0.9984	0.9995	0.9998	0.9999	0.9973	0.9998
$r_0 = 0$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 0$	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	50	0.9855	0.9873	0.8672	0.9720	0.8839	0.8941	0.8268	0.9006	0.8938	0.8955	0.8122	0.8898
$r_0 = 1$	100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	250	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 1$	500	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$r_0 = 2$	50	0.0916	0.0806	0.0542	0.0536	0.0847	0.0719	0.0530	0.0565	0.0870	0.0764	0.0531	0.0533
$r_0 = 2$	100	0.1095	0.0881	0.0591	0.0569	0.0971	0.0778	0.0530	0.0501	0.0971	0.0898	0.0533	0.0589
$r_0 = 2$	250	0.1099	0.0907	0.0525	0.0577	0.1013	0.0919	0.0529	0.0599	0.1009	0.0888	0.0542	0.0588
$r_0 = 2$	500	0.1110	0.0908	0.0519	0.0569	0.0997	0.0892	0.0551	0.0562	0.0997	0.0879	0.0516	0.0529

7 Apendix

Proof of Lemma 1:

First note that in (18) we have the cumulate sum $\sum_{j=1}^{\tau-1} U_{\tau-j}^{(3)}$ and that we could write:

$$\frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr \rfloor} U_j^{(3)} = \begin{bmatrix} \frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr \rfloor} U_\tau^1 \\ \frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr \rfloor} U_\tau^2 \\ \frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr \rfloor} U_\tau^3 \end{bmatrix} \Rightarrow \Psi^{(3)}(1)^{-1}\left[\mathbf{P}\otimes\mathbf{I}_S\right] W^{(3)}(r). \tag{48}$$

In order to prove (48) first note that the connection between $u_{s\tau}^j$ and $\varepsilon_{s\tau}^j$ for $j = 1, 2$ and 3 is the following $\left(1 - \psi_{1s}^j L - \cdots - \psi_{p-1,s}^j L^{p-1}\right) u_{s\tau}^j = \varepsilon_{s\tau}^j$. Also we assume that $E_{s\tau}^{(3)} = \begin{bmatrix} \varepsilon_{s\tau}^1 & \varepsilon_{s\tau}^2 & \varepsilon_{s\tau}^3 \end{bmatrix}'$ is a white noise vector with positive definite variance-covariance matrix $E\left[E_{s\tau}^{(3)} E_{s\tau}^{(3)\prime}\right] = \Sigma$ then for the $(3 \times S) \times 1$ vector $E_\tau^{(3)} = \left[E_\tau^{1\prime}, E_\tau^{2\prime}, E_\tau^{3\prime}\right]' = \left[\varepsilon_{1\tau}^1, \ldots, \varepsilon_{S\tau}^1, \varepsilon_{1\tau}^2, \ldots, \varepsilon_{S\tau}^2, \varepsilon_{1\tau}^3, \ldots, \varepsilon_{S\tau}^3\right]'$ we will have:

$$\frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr \rfloor} E_j^{(3)} \Rightarrow \left[\mathbf{P}\otimes\mathbf{I}_S\right] W^{(3)}(r).$$

Where $W^{(3)}(r)$ is a $(3S)\times 1$ multivariant Vector Brownian motion with variance covariance matrix $I_{(3S)\times(3S)}$ and $\mathbf{P}$ is a lower triangular matrix of order 3×3 such that such that $\Sigma = \mathbf{PP}'$. Hence $\left[\mathbf{P}\otimes\mathbf{I}_S\right] W^{(3)}(r)$ will have a variance covariance matrix $\Sigma\otimes\mathbf{I}_S$. Now note that ,as $\left(1 - \psi_{1s}^j L - \cdots - \psi_{p-1,s}^j L^{p-1}\right) u_{s\tau}^j = \varepsilon_{s\tau}^j$, there will be a vector of season representation for each $u_{s\tau}^j$ $j = 1, 2$ and 3, that is, a VAR representation of order $P = \lfloor (p-2)/S \rfloor + 1$ as follows:

$$\left(\Psi_0^j - \Psi_1^j L - \cdots - \Psi_P^j L^P\right) U_\tau^j = E_\tau^j.$$

And in the case of the trivariant vector $U_\tau^{(3)} = \left[U_\tau^{1\prime}, U_\tau^{2\prime}, U_\tau^{3\prime}\right]'$ we will have:

$$\left(\Psi_0^{(3)} - \Psi_1^{(3)} L - \cdots - \Psi_P^{(3)} L^P\right) U_\tau^{(3)} = E_\tau^{(3)}$$
$$\left(\Psi_0^{(3)} - \Psi_1^{(3)} L - \cdots - \Psi_P^{(3)} L^P\right) = \Psi^{(3)}(L)$$

such that $\Psi_i^{(3)}$ $i = 0, 1, \ldots, P$ are block diagonal matrices with diagonal elements Ψ_i^j $j = 1, 2, 3$ for $i = 0, 1, \ldots, P$. Hence we have $U_\tau^{(3)} = \Psi^{(3)}(L)^{-1} E_\tau^{(3)}$ and it will be possible to write:

$$\frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr \rfloor} U_j^{(3)} = \frac{1}{\sqrt{T}}\sum_{j=1}^{\lfloor Tr \rfloor} \Psi^{(3)}(1)^{-1} E_j^{(3)} + o_p(1).$$

Hence (48) will come naturally. Next from (18) we have:

$$\frac{1}{\sqrt{T}} Y_{\lfloor Tr \rfloor}^{(3)} \Rightarrow \left(\mathbf{A}_0^{(3)}\right)^{-1} \mathbf{A}_1^{(3)} \left(\mathbf{A}_0^{(3)}\right)^{-1} \Psi^{(3)}(1)^{-1}\left[\mathbf{P}\otimes\mathbf{I}_S\right] W^{(3)}(r) \tag{49}$$
$$= \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1' & 0_{S\times S} & 0_{S\times S} \\ 0_{S\times S} & \mathbf{a}_2\mathbf{b}_2' & 0_{S\times S} \\ 0_{S\times S} & 0_{S\times S} & \mathbf{a}_3\mathbf{b}_3' \end{bmatrix} \Psi^{(3)}(1)^{-1}\left[\mathbf{P}\otimes\mathbf{I}_S\right] W^{(3)}(r).$$

In order to define the three scalar Brownian motions of lemma 1, that is, $w_1(r)$, $w_2(r)$ and $w_3(r)$. First we focus on $\left[\mathbf{P}\otimes\mathbf{I}_S\right] W^{(3)}(r)$, and note that the 3×3 lower triangular matrix $\mathbf{P}$ associated to 3×3 variance-covariance matrix Σ:

$$\Sigma = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 & \sigma_{13}^2 \\ \sigma_{12}^2 & \sigma_{22}^2 & \sigma_{23}^2 \\ \sigma_{13}^2 & \sigma_{23}^2 & \sigma_{33}^2 \end{bmatrix},$$

will be as follows:

$$\mathbf{P}=\begin{bmatrix} p_{11} & 0 & 0 \\ p_{12} & p_{22} & 0 \\ p_{13} & p_{23} & p_{33} \end{bmatrix}$$

$$=\begin{bmatrix} \sigma_{11} & 0 & 0 \\ \frac{\sigma_{12}^2}{\sigma_{11}} & \sqrt[2]{\sigma_{22}^2-\left(\frac{\sigma_{12}^2}{\sigma_{11}}\right)^2} & 0 \\ \frac{\sigma_{13}^2}{\sigma_{11}} & \frac{\sigma_{23}^2-\frac{\sigma_{12}^2\sigma_{13}^2}{\sigma_{11}^2}}{\sqrt[2]{\sigma_{22}^2-\left(\frac{\sigma_{12}^2}{\sigma_{11}}\right)^2}} & \sqrt[2]{\sigma_{33}^2-\left(\frac{\sigma_{13}^2}{\sigma_{11}}\right)^2-\left(\frac{\sigma_{23}^2-\frac{\sigma_{12}^2\sigma_{13}^2}{\sigma_{11}^2}}{\sqrt[2]{\sigma_{22}^2-\left(\frac{\sigma_{12}^2}{\sigma_{11}}\right)^2}}\right)^2} \end{bmatrix}.$$

Note that $W^{(3)}(r)$ is a $(3S)\times 1$ multivariant Vector Brownian motion with variance covariance matrix $I_{(3S)\times(3S)}$, hence we could write $W^{(3)}(r)=\left[W^1(r)', W^2(r)', W^3(r)'\right]'$, where each of the $W^j(r)'$ for $j=1$, 2 and 3 are $S\times 1$ multivariant Vector Brownian motions. So we could write:

$$[\mathbf{P}\otimes \mathbf{I}_S]\, W^{(3)}(r)=\mathbf{P}=\begin{bmatrix} p_{11}W^1(r) \\ p_{12}W^1(r)+p_{22}W^2(r) \\ p_{13}W^1(r)+p_{23}W^2(r)+p_{33}W^3(r) \end{bmatrix}.$$

And finally we could define the scalar Brownian motions $w_j(r)$ for $j=1$, 2 and 3 as:

$$\begin{aligned} w_1(r) &= \omega_1^{-1}p_{11}\mathbf{b}_1'\Psi^{(3)}(1)^{-1}W^1(r) \\ w_2(r) &= \omega_2^{-1}\mathbf{b}_2'\Psi^{(3)}(1)^{-1}\left[p_{12}W^1(r)+p_{22}W^2(r)\right] \\ w_3(r) &= \omega_3^{-1}\mathbf{b}_3'\Psi^{(3)}(1)^{-1}\left[p_{13}W^1(r)+p_{23}W^2(r)+p_{33}W^3(r)\right] \\ & \textit{with}: \end{aligned} \tag{50}$$

with:

$$\begin{aligned} \omega_1 &= \left(p_{11}^2\mathbf{b}_1'\Psi^{(3)}(1)^{-1}\Psi^{(3)}(1)'^{-1}\mathbf{b}_1\right)^{1/2} \\ \omega_2 &= \left(\left[p_{12}^2+p_{22}^2\right]\mathbf{b}_2'\Psi^{(3)}(1)^{-1}\Psi^{(3)}(1)'^{-1}\mathbf{b}_2\right)^{1/2} \\ \omega_3 &= \left(\left[p_{13}^2+p_{23}^2+p_{33}^2\right]\mathbf{b}_3'\Psi^{(3)}(1)^{-1}\Psi^{(3)}(1)'^{-1}\mathbf{b}_3\right)^{1/2}. \end{aligned} \tag{51}$$

■

Proof of Lemma 2:

Note that as in the previous lemma, here it also aplies (48) and from (29) and (30) we have:

$$\frac{1}{\sqrt{T}}Y_{\lfloor Tr\rfloor}^{(3)} \Rightarrow \left(\mathbf{A}_0^{(3)}\right)^{-1}\mathbf{A}_1^{(3)}\left(\mathbf{A}_0^{(3)}\right)^{-1}\Psi^{(3)}(1)^{-1}[\mathbf{P}\otimes\mathbf{I}_S]\,W^{(3)}(r) \tag{52}$$

$$=\begin{bmatrix} 0_{S\times S} & 0_{S\times S} & \alpha(\mathbf{a}_1\mathbf{b}_3') \\ 0_{S\times S} & 0_{S\times S} & \beta(\mathbf{a}_2\mathbf{b}_3') \\ 0_{S\times S} & 0_{S\times S} & \mathbf{a}_3\mathbf{b}_3' \end{bmatrix}\Psi^{(3)}(1)^{-1}[\mathbf{P}\otimes\mathbf{I}_S]\,W^{(3)}(r).$$

Finally the scalar ω_3 and the scalar Brownian motion $w_3(r)$ are as in lemma1 see (50) and (51).■

Proof of Lemma 3:

Note that as in the previous lemma, here it also aplies (48) and from (41) and (42) we have:

$$\frac{1}{\sqrt{T}}Y_{\lfloor Tr\rfloor}^{(3)} \Rightarrow \left(\mathbf{A}_0^{(3)}\right)^{-1}\mathbf{A}_1^{(3)}\left(\mathbf{A}_0^{(3)}\right)^{-1}\Psi^{(3)}(1)^{-1}[\mathbf{P}\otimes\mathbf{I}_S]\,W^{(3)}(r) \tag{53}$$

$$=\begin{bmatrix} 0_{S\times S} & \beta_1(\mathbf{a}_1\mathbf{b}_2') & \beta_2(\mathbf{a}_1\mathbf{b}_3') \\ 0_{S\times S} & \mathbf{a}_2\mathbf{b}_2' & 0_{S\times S} \\ 0_{S\times S} & 0_{S\times S} & \mathbf{a}_3\mathbf{b}_3' \end{bmatrix}\Psi^{(3)}(1)^{-1}[\mathbf{P}\otimes\mathbf{I}_S]\,W^{(3)}(r).$$

Finally the scalars ω_2 and ω_3 and the scalar Brownian motions $w_2(r)$ and $w_3(r)$ are as in lemma1 see (50) and (51).■