Childcare Support and Public Capital in an Ultra-Declining Birthrate Society

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13 March 2021

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Abstract This paper analyzes whether public capital investment or childcare support maximize the growth rate in an ultra-declining birth rate society using a labor-augmented model with public capital. We clarify the global stability of the private capital-public capital ratio in the steady state. In addition, we analyze the effect of increasing the expenditure share of tax revenue on economic growth. The result of this analysis shows that an increased share of public capital investment brings higher economic growth. This means that if all tax revenue is allocated to public capital investment, the growth rate will be maximized. Furthermore, in the second case, the model is reconstructed in such a way that the child is regarded as a nominal consumer goods in the first period and the childcare cost is regarded as a price. In that case, the impact of increased public capital on growth is shown to be minor compared to the former case.

Keywords: Public capital investment・Childcare support・Income tax・Economic growth

JEL classification: D91・E62・O41

1. Introduction

The number of children born in Japan continues to decrease. The total fertility rate was 1.36[2] in 2019, the lowest level to date, as indicated by the Japanese Ministry of Health, Labor and Welfare (MHLW). The Cabinet Office continues to insist that Japan has been in a state of declining birth rates for many years, resulting in what is referred to as an "ultra-declining birth rate society." The demographic trends are such that, by 2050, one in 2.5 people will be elderly (aged 65 or older).[3] Viewing life in the long term, workers should determine their spending based on their estimated lifetime income. According to the overlapping generations (OLG) model proposed by Diamond (1965), the lifetime income of an individual is assumed to consist of earnings received in two periods: their working period and their later life. Individuals make decisions from a lifetime perspective while adhering to budgetary constraints. Becker (1981) and Becker and Lewis (1973) showed that the number of children in

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2 "Current population survey," MHLW website (https://www.mhlw.go.jp/toukei/list/81-1a.html) (accessed on September 20, 2020)

3 "Situations of Aging" (Japanese), Cabinet Office website (https://www8.cao.go.jp/kourei/whitepaper/w-2012/zenbun/pdf/1s1s_1.pdf) (accessed on June 15, 2020)

developed countries will decline; at first glance, this is seemingly a contradiction, considering that children are positive to societies, however, results from the fact that the cost of childcare is proportional in scale to its quantity multiplied by its quality. In this study, models are established based on a neoclassical theory that suggests that growth in capital boosts gross domestic product (GDP) and leads to a greater growth rate for the whole nation. The main portion of this study utilizes Romer's endogenous growth model (1986) to introduce the public capital models proposed by Barro (1990), Barro and Sala-i-Martin (1992), Futagami et al. (1993), Turnovsky (1997), Yakita (2008), and Maebayashi (2013). These models indicate that public capital stock boosts labor productivity. Investment in public capital is financed through the levying of income taxes (on labor income and capital income). Yakita (2008) used a birth rate internalization model that considers two public expenditures: public capital investment and public capital maintenance. Maebayashi (2013) showed the dynamics of the private-public capital ratio and confirmed the existence of a steady state and global stability. Furthermore, the author analyzed the optimal allocation of tax revenue between expenditure on public capital investment and public pension subsidies under a pay-as-you-go pension system. The study concluded that it was clear that the best policy for growth is to allocate all financial resources to public capital investment; however, from a social-welfare perspective, the optimal tax revenue allocation rate depends on the magnitude of the social discount rate.

In this study, we analyze the policy trade-off between public capital investment and childcare support and the effects on the growth rate under government budget constraints, where the government sources revenue only from income taxes on labor and capital. Furthermore, as an important point this study, the child-rearing support policy should be subsidized for the direct opportunity cost to workers (we will call this case, "Case A".) or the child should be regarded as a normal consumer good rather than a capital good, and a subsidy policy on the price should be implemented. (And let's call this case "Case B.) The point is that comparisons are made and explicitly derived the effect of policy on growth in these cases. In both cases, First of all, we prove the existence of a steady state, and confirm that the economy converges to the steady state globally and stably. We show that all variables: public capital, private capital, and GDP, grow at the same rate on the balanced growth path (BGP). Second, we analyze the effect of increasing the share of public capital investment on growth under constant tax revenue, and using a numerical example, we find that this growth is positive. Also, the elasticity of an increase in the relative share of public capital investment ratio on private-public capital and the labor share of GDP is considered. In the first case, the sign of this elasticity is positive, implying that the additional increase in public capital is pushing up private capital more than that increase. Clearly states that this is driving economic growth. In the second case, the sign turned out to be negative and the absolute value of elasticity is less than one for a marginal increase in public capital investment and the sign of effect on the relative value is negative. This means that the effect of increasing the wage rate due to the increase in public capital does not contribute much to the increase in savings. The reason for this was very clear. First, this depends on a shape of the utility function, as shown in the linear logarithm. The use of this function means that savings depend only on income in the first period and not on the interest rate. In other words, in this change in interest rate, the substitution effect and the income effect cancel each other out, and the effect on savings against changes in the interest rate becomes zero. The second thing to think about is that governmental childcare support measures do not contribute to an increase in the labor force. In this analysis, the public pension system and long-term care insurance system in the social security system are not considered, so there is no externality to the parent generation. Therefore, the incentive for parents to have children is related to them being considered consumer goods rather than capital goods, from an economic point of view. We constructed here using the Diamond model (1965), a two-period OLG model. We introduce public capital stock to construct a model that has labor-augmented production technology. Therefore, whether or not to have children depends on the preference rate for children as general consumer goods and consumption in the second period, so-called, how much deposit is required the second period because there is no public pension system in this model, Also, whether to leave for the second term depends greatly on the preference rate.

The remainder of this paper is organized as follows. The next section presents the model and it's dynamics in terms of (private and public) capital. The global stability of the dynamics in the steady state is then confirmed. The effects of governmental increases in income tax and public capital investment shares in the steady state are analyzed. The final section concludes the paper.

2. Model

The case A: the childcare cost is regarded as an opportunity cost.

2.1 Individuals

The two-period OLG model presented by Diamond (1965), with fully competitive markets, is considered. A homogeneous individual is assumed, who obtains utility from consumption in the working and later periods of life, and selects the number of children that they have. We consider a child to be a consumer good rather than a capital good, and there is no public pension (therefore, there is no "self-denial"). Individuals supply labor inelastically in only the first period, and it is assumed that every individual has one unit of labor to supply to the labor market. Individuals allocate income for consumption, saving, and childcare costs in the first period. The individual consumes all income, including saving and interest, in the first period, with no bequests in the second period. A logarithmic linear utility function and lifetime budget constraint, which must hold in order for the economy to be sustainable in the long term, are specified as follows:

$$max.\ \ u_t = logc_t + \rho logd_{t+1} + \varepsilon logn_t \tag{1}$$

$$s.t \quad w_t\,(1-\tau)[1-n_t(z-h_t)] = c_t + \frac{d_{t+1}}{r_{t+1}(1-\tau)} \tag{2}$$

$$c_t^* = \frac{(1-\tau)w_t}{[1+\varepsilon+\rho r_{t+1}(1-\tau)]} \tag{3}$$

$$n_t^* = \frac{\varepsilon}{[1+\varepsilon+\rho r_{t+1}(1-\tau)](z-h_t)} \tag{4}$$

$$d_{t+1}^* = \frac{\rho r_{t+1} w_t (1-\tau)^2}{[1+\varepsilon+\rho r_{t+1}(1-\tau)]} \tag{5}$$

$$s_t^* = \left\{1 - \frac{(1+\varepsilon)}{[1+\varepsilon+\rho r_{t+1}(1-\tau)]}\right\}(1-\tau)w_t \tag{6}$$

Where time preference, child preference, childcare cost, childcare support, and income tax are denoted as $\rho \in (0,1)$, $\varepsilon > 0$, $z \in (0,1)$, $h_t \in (0,1)$, $z \geq h_t$, and $n_t \geq 1$, respectively.

2.2 Production

A Cobb-Douglas production technology in which labor increases with public capital investment, as in Romer (1986), is used. It is assumed that there are many firms in a goods market, and these firms have access to the same technology. The inputs are the private capital stock and labor. The production function of firm i is specified as follows:

$$Y_{it} = K_{it}^{\alpha}(A_t L_{it})^{1-\alpha} \tag{7}$$

$$A_t = \frac{G_t}{L_t} \tag{8}$$

$$Y_t = K_t^{\alpha} {G_t}^{1-\alpha} = \left(\frac{K_t}{G_t}\right)^{\alpha} G_t = x_t^{\alpha} G_t \tag{9}$$

The labor force in period t is determined as follows:

$$L_t = N_t[1-n_t(z-h_t)] \tag{10}$$

where N_t is the number of households in period t. We assume a perfectly competitive market and solve the profit maximization problem as follows:

$$(1-\alpha)\left(\frac{K_{it}}{L_{it}}\right)^{\alpha-1} {A_t}^{1-\alpha} = w_t \tag{11}$$

$$\alpha\left(\frac{K_{it}}{L_{it}}\right)^{\alpha-1} {A_t}^{1-\alpha} = r_t \tag{12}$$

From (14) and (15), the private capital-labor ratio will become the same value as in $K_{ti}/L_{ti} = K_t/L_t$. Also, $\sum_{i=1}^{\infty} L_{it} = L_{t,}$ $\sum_{i=1}^{\infty} K_{it} = K_t$ can be derived, where L_t and K_t denote the total labor supply and total private capital, respectively. By defining a new variable, $x = \frac{K}{G}$, to be the ratio of private and public capital, (14) and (15) can be rewritten as the following equations:

$$(1-\alpha)\left(\frac{K_t}{G_t}\right)^{\alpha} \frac{G_t}{L_t} = (1-\alpha)x_t^{\alpha}\frac{G_t}{L_t} = w_t \tag{13}$$

$$\alpha\left(\frac{K_t}{G_t}\right)^{\alpha-1} = \alpha x_t^{\alpha-1} = r_t \tag{14}$$

2.3 Government

The government taxes income and divides tax revenues between public capital investment, $E > 0$, and childcare support, $H > 0$. The share of spending on public capital investment and the income tax rate are respectively denoted $\varphi \in [0,1]$, $\tau \in [0,1]$. The depreciation rate of public and private capital is 1. The government budget constraint is shown in the following equations:

$$E_t + H_t = \tau Y_t = \tau x_t^{\alpha} G_t \tag{15}$$

$$E_t = G_{t+1} - G_t = \varphi\tau Y_t = \varphi\tau x_t^{\alpha} G_t \tag{16}$$

$$w_t h_t n_t N = (1-\varphi)\tau Y_t = (1-\varphi)\tau x_t^{\alpha} G_t \tag{17}$$

The per-capita childcare support is determined using (17), and is indicated by the following equation (the value of which will be constant):

$$h_t = \frac{(1-\varphi)(1-\varepsilon z)\tau}{\varepsilon[1-(1-\varphi)\tau]} \tag{18}$$

By using equation (10), equation (18), which indicates the labor force in period t, is rewritten as follows:

$$L_t = N_t\left[1 - \frac{\varepsilon}{[1+\varepsilon+\rho\alpha x_{t+1}^{\alpha-1}(1-\tau)]}\right] \tag{19}$$

This equation implies that the labor force in period t does not depend on childcare support, and furthermore, it also does not depend on the share of governmental tax revenue that is allocated to childcare support expenditure. This clearly means that the government can only intervene in childcare support through public capital investment. This suggests that the labor force will continue to decline in the future, even if late marriage is resolved and the preference for having children increases.
Next, we use equation (19) to derive an expression for labor growth:

$$g_L = \frac{L_{t+1}}{L_t} = \frac{[1+\rho\alpha x_{t+2}^{\alpha-1}(1-\tau)][1+\varepsilon+\rho\alpha x_{t+1}^{\alpha-1}(1-\tau)]}{[1+\rho\alpha x_{t+1}^{\alpha-1}(1-\tau)][1+\varepsilon+\rho\alpha x_{t+2}^{\alpha-1}(1-\tau)]}\frac{N_{t+1}}{N_t} \tag{20}$$

where the number of households in period t+1 is denoted by $N_{t+1} = N_t n_t$, and the number of children is constant in the steady state. Therefore, equation (20) can be written in the following form:

$$g_L = \frac{L_{t+1}}{L_t} = \frac{n_t N_t}{N_t} = n_t \tag{21}$$

which relates the growth in the labor force to the number of children in the steady state.

3. Equilibrium

There are three markets, and we consider only the capital market by Walras' law. The equilibrium condition is as follows:

$$s_t L_t = K_{t+1} \tag{22}$$

We substitute the optimal savings (6) for the equilibrium condition (22), and substitute for the wage rate (13) and the interest rate (14). These allow us to rewrite condition (23) as the next equation:

$$K_{t+1} = \left\{1 - \frac{(1+\varepsilon)}{[1+\varepsilon+\rho\alpha x_{t+1}^{\alpha-1}(1-\tau)]}\right\}(1-\tau)(1-\alpha)x_t^{\alpha}G_t \tag{23}$$

And we can get equation (24) by dividing both sides of equation (23) by K_t:

$$g_K = \frac{K_{t+1}}{K_t} = \left\{\frac{[\rho\alpha x_{t+1}^{\alpha-1}(1-\tau)^2](1-\alpha)x_t^{\alpha-1}}{[1+\varepsilon+\rho\alpha x_{t+1}^{\alpha-1}(1-\tau)]}\right\} \tag{24}$$

The dynamics of private capital are obtained in the following section.

4. Dynamics

The dynamics of public capital are indicated by equation (25):

$$g_G = \frac{G_{t+1}}{G_t} = \varphi\alpha\tau x_t^{\alpha} + 1 \tag{25}$$

The growth of x is indicated by the following equation, which combines the capital dynamic equations (24) and (25).

$$g_x = \frac{x_{t+1}}{x_t} = \frac{\frac{K_{t+1}}{K_t}}{\frac{G_{t+1}}{G_t}} = \frac{[\rho\alpha x_{t+1}^{\alpha-1}(1-\tau)^2](1-\alpha)x_t^{\alpha-1}}{(\varphi\alpha\tau x_t^{\alpha}+1)[1+\varepsilon+\rho\alpha x_{t+1}^{\alpha-1}(1-\tau)]} \tag{26}$$

Denote $[1+\varepsilon+\rho\alpha x_{t+1}^{\alpha-1}(1-\tau)]$ as $\emptyset > 0$. Then, equation (26) can be rewritten as follows:

$$g_x = \frac{x_{t+1}}{x_t} = \frac{\frac{K_{t+1}}{K_t}}{\frac{G_{t+1}}{G_t}} = \frac{[\rho\alpha x_{t+1}^{\alpha-1}(1-\tau)^2](1-\alpha)x_t^{\alpha-1}}{(\varphi\alpha\tau x_t^{\alpha}+1)\emptyset} \tag{27}$$

$$\frac{\partial x_{t+1}}{\partial x_t} = \frac{A(x_t, x_{t+1})}{B(x_t, x_{t+1})} = f(x_t, x_{t+1}) = +0.081 > 0 \tag{28}$$

$$A(x_t, x_{t+1}) = \rho\alpha^2 x_{t+1}^{\alpha-1} x_t^{\alpha-1}(1-\tau)^2(1-\alpha) - \emptyset x_{t+1}\varphi\alpha^2\tau x_t^{\alpha-1} = -0.12 < 0 \tag{29}$$

$$B(x_t, x_{t+1}) = \rho\alpha(1-\alpha)^2(1-\tau)^2 x_{t+1}^{\alpha-2} x_t^{\alpha} + \emptyset(\varphi\alpha\tau x_t^{\alpha}+1) - \rho\alpha(\varphi\alpha\tau x_t^{\alpha}+1)(1-\tau)(1-\alpha)x_{t+1}^{\alpha-1} = -1.49 < 0 \tag{30}$$

In order for the signs of “A” and “B” in equation (27) to be positive, the parameters in equations (29) and (30) are quantified concretely as $(\alpha, \varepsilon, \rho, \tau, x, z, \varphi) = (0.5, 0.05, 0.95, 0.3, 3.2, 0.06, 0.83)$. Here, extreme numerical examples such as $\varepsilon = 0.05, \rho = 0.95$, and so on, are presented. The reasons for this are that the engine of growth in this model is clearly public capital investment; therefore, in order to connect the wage rate and interest rate pushed up by public capital investment to higher growth, it is necessary to supply more labor time, that is, lower the opportunity cost for childcare, or raise the preference rate for future consumption. Next, we derive the second derivative of equation (27). When x_t approaches 0, the growth of x is zero in equation (27) $\left(\lim_{x_t \to 0} \frac{x_{t+1}}{x_t} = 0\right)$. In other words, the curve in the curve passes through the origin.

$$\frac{\partial^2 x_{t+1}}{(\partial x_t)^2} = \frac{\partial f(x_t, x_{t+1})}{\partial x_t} = \frac{A^{'} B - AB^{'}}{B^2} = -0.019 < 0 \tag{31}$$

$$A' = \frac{\partial A(x_t, x_{t+1})}{\partial x_t} = -\rho\alpha^2 x_t^{\alpha-2} x_{t+1}^{\alpha-1}(1-\tau)^2(1-\alpha) + \emptyset(1-\alpha)x_t^{\alpha-2}x_{t+1}\varphi\alpha^2\tau = 0.016 > 0 \tag{32}$$

$$B' = \frac{\partial B(x_t, x_{t+1})}{\partial x_t} = \rho\alpha^2(1-\alpha)^2(1-\tau)^2 x_{t+1}^{\alpha-2} x_t^{\alpha-1} + \emptyset\varphi\alpha^2\tau x_t^{\alpha-1} - \rho\alpha^3\varphi\tau x_t^{\alpha-1}(1-\alpha)(1-\tau)x_{t+1}^{\alpha-1} = 0.045 > 0 \tag{33}$$

$$\lim_{x_t \to 0} \frac{dx_{t+1}}{dx_t} = \infty \qquad \lim_{x_t \to \infty} \frac{dx_{t+1}}{dx_t} = 0 \tag{34}$$

The private-public capital ratio will increase, and the steady state of x is shown as x^*. If equation (35) is satisfied with x^* the growth rate of GDP, the private capital and public capital will be the same:

$$[\rho\alpha(x^*)^{\alpha-1}(1-\tau)^2](1-\alpha)(x^*)^{\alpha-1} = (\varphi\alpha\tau(x^*)^{\alpha} + 1)[1 + \varepsilon + \rho\alpha(x^*)^{\alpha-1}(1-\tau)] \tag{35}$$

Proposition 1. There is a unique value that shows the public-private capital ratio in the steady state. If the equation (35) is satisfied, public capital, private capital and GDP will grow at the same rate. That is, the growth path is balanced and globally stable.

$$\frac{\partial g}{\partial \varphi} = \tau\alpha^2(x^*)^{\alpha}\left[\frac{1}{\alpha} + \frac{\varphi}{x^*}\frac{dx^*}{d\varphi}\right] \tag{36}$$

$$\frac{dx^*}{d\varphi} = \frac{A}{B} > 0 \tag{37}$$

$$A = \varphi\alpha\tau(x^*)^{\alpha}\emptyset > 0 \tag{38}$$

$$B = -2(1-\alpha)^2(1-\tau)^2(x^*)^{2\alpha-3}\rho\alpha + [\varphi\alpha\tau(x^*)^{\alpha} + 1](1-\alpha)(x^*)^{\alpha-2}(1-\tau)\rho\alpha + \varphi\alpha^2\tau(x^*)^{\alpha-1}\emptyset > 0 \tag{39}$$

Where the second term in brackets indicates the elasticity of the share for the relative capital value, and the sign is positive. That is, an increase in the share of public capital investment raises the magnitude of the private-public capital ratio. This suggests that the economy will grow regardless of the private capital share of GDP or the size of the elasticity of the share.

Proposition 2. The economy will grow independently of the private capital share of GDP or the elasticity of the allocation rate to private and public capital.

$$\frac{\partial g}{\partial \varphi} = \tau\alpha^2\left[\frac{2(1-\alpha)^2(1-\tau)^2(x^*)^{2\alpha-2}\rho + (\varphi\alpha\tau(x^*)^{\alpha}+1)(1-\alpha)(x^*)^{\alpha-2}(1-\tau)\rho + [\tau(x^*)^{-1} - \varphi(x^*)^{\alpha}]\varphi\alpha\emptyset}{2\alpha(1-\alpha)^2(1-\tau)^2(x^*)^{\alpha-2}\rho + (\varphi\alpha\tau(x^*)^{\alpha}+1)(1-\alpha)(x^*)^{-2}(1-\tau)\rho + \varphi\alpha\tau(x^*)^{-1}\emptyset}\right] > 0 \tag{40}$$

An increase in the share of public capital investment boosts public capital. At the same time, it raises both the wage and interest rates. This leads to an upsurge in private capital through the two effects on income and price. A policy in which all tax revenue is spent on public capital investment yields the best results in terms of growth.

Proposition 3. A policy in which all tax revenue is spent on public capital investment is the best policy in terms of growth.

5. The case B: the childcare cost is regarded as the price.

5.1. Households

Next, we consider the case where the children are the goods and pay the childcare cost as a price, like as consumption goods. The function of utility is indicated as same in the case A which shows log-linear type and the equation (2) which shows budget constraint is rewritten as follows:

$$w_t\,(1-\tau) = c_t + n_t(z-h_t) + \frac{d_{t+1}}{r_{t+1}(1-\tau)} \tag{41}$$

We solve the problem and the optimal solution like as follows.

$$c_t^* = \frac{1}{(1+\varepsilon+\rho)} w_t(1-\tau) \tag{42}$$

$$d_{t+1}^* = \frac{\rho}{(1+\varepsilon+\rho)} w_t r_{t+1}(1-\tau)^2 \tag{43}$$

$$n_t^* = \frac{\varepsilon}{(1-h_t)(1+\varepsilon+\rho)}(1-\tau)w_t \tag{44}$$

$$s_t^* = \frac{\rho}{(1+\varepsilon+\rho)}(1-\tau)w_t \tag{45}$$

5.2 Firms

The technology of firms can be drawn in the same way as the “case A” and the government budget constraint in the “case A” which is indicated by the equation (17) is rewritten as next equation.

$$h_t n_t L_t = (1-\varphi)\tau Y_t = (1-\varphi)\tau x_t^\alpha G_t \tag{46}$$

This can lead to the childcare support scale in the case B and we will compare about these characteristics of two equations. Let $\emptyset > 0$ be $(1+\varepsilon+\rho)$ here to simplify the formula. Here $\partial h_t/\partial\varphi < 0,$ that is in case B, the childcare support does not depend on relative scale of capitals.

$$h_t(\varphi) = \frac{\tau(1-\varphi)\emptyset}{\tau(1-\varphi)\emptyset + \varepsilon(1-\tau)(1-\alpha)} \tag{47}$$

5.3 Equilibrium

We substitute the optimal number of children (44) and the wage rate (13) for the above formula. The basic structure is the same as "the case A" in that it doesn't depends on the scale of x and only depend on the share of the public investment, $\varphi > 0$. The optimal savings can be written by using the profit maximizing condition, (13).

$$s_t^* = \frac{\rho}{(1+\varepsilon+\rho)}(1-\tau)(1-\alpha)x_t^{\alpha}\frac{G_t}{L_t} \tag{48}$$

Here, derivation of the dynamic equation of the private capital using the capital market equilibrium condition.

$$g_K = \frac{K_{t+1}}{K_t} = \frac{\rho(1-\tau)(1-\alpha)}{(1+\varepsilon+\rho)}x_t^{\alpha-1} \tag{49}$$

5.4 Dynamics

The above equation and the equation (25) can be combined to obtain the dynamic equation of capital which means the relative capital value between the private and public.

$$g_x = \frac{x_{t+1}}{x_t} = \frac{\frac{K_{t+1}}{K_t}}{\frac{G_{t+1}}{G_t}} = \frac{\rho(1-\tau)(1-\alpha)}{(\varphi\alpha\tau x_t^{\alpha}+1)(1+\varepsilon+\rho)}x_t^{\alpha-1} \tag{50}$$

Here we will try $(1+\varepsilon+\rho)$ as $\emptyset > 0$ for simplicity and that allows me to rewrite the above equation as follows equation:

$$g_x = \frac{x_{t+1}}{x_t} = \frac{\frac{K_{t+1}}{K_t}}{\frac{G_{t+1}}{G_t}} = \frac{\rho(1-\tau)(1-\alpha)}{(\varphi\alpha\tau x_t^{\alpha}+1)\emptyset}x_t^{\alpha-1} \tag{51}$$

Next, we will see if this economy converges to a steady state globally. For that purpose, when illustrating the above-mentioned dynamic equation of capital, we consider that the curve must trough the origin, then rise to the right, and finally have a concave shape with respect to the origin. We will do the total derivative with respect to x_{t+1} and x_t in the equation (51).

$$\frac{\partial x_{t+1}}{\partial x_t} = \frac{A(x_t, x_{t+1})}{B(x_t)} = f(x_t, x_{t+1}) > 0 \tag{52}$$

$$A(x_t, x_{t+1}) = [\alpha\rho(1-\tau)(1-\alpha)x_t^{\alpha-1} - \emptyset x_{t+1}\varphi\alpha^2\tau x_t^{\alpha-1}] = 0.01 > 0 \tag{53}$$

$$B(x_t) = (\varphi\alpha\tau x_t^{\alpha} + 1)\emptyset = 1.22 > 0 \tag{54}$$

Where the sign of the equation (54), that is the denominator of equation (52) will be clearly positive. But unfortunately the sign of numerator "A" is ambiguous. Therefore, we would like to derive this sign using real numbers. This numerical calculation revealed that the first derivative is positive, in other words, it became clear that the curve rising to the right. Next we want to investigate the stability of the economy, we further differentiate the equation (52) with respect to x_t.

$$\frac{\partial^2 x_{t+1}}{(\partial x_t)^2} = \frac{\partial f(x_t, x_{t+1})}{\partial x_t} = \frac{A'B - AB'}{B^2} = -0.024 < 0 \tag{55}$$

$$A' = \frac{\partial A(x_t, x_{t+1})}{\partial x_t} = -\alpha\rho(1-\tau)(1-\alpha)^2 x_t^{\alpha-2} + \emptyset(1-\alpha)x_{t+1}\varphi\alpha^2\tau x_t^{\alpha-2} = 0.02 > 0 \tag{56}$$

$$B' = \frac{\partial B(x_t)}{\partial x_t} = \emptyset\varphi\alpha^2\tau x_t^{\alpha-1} = 0.072 > 0 \tag{57}$$

As with the above, since the sign can not be explicity determined, we try to derive it using values. As a result, it can be found that the sign of the second derivative of the dynamic equation of capital is negative, and it turns out that x_{t+1} gradually decreases as x_t increases. Finally, consider wether this curve intersects the 45 degree line at one point. It has the same meaning as so-called "Inada-condition".

$$\lim_{x_t \to 0} \frac{dx_{t+1}}{dx_t} = \infty \qquad \lim_{x_t \to \infty} \frac{dx_{t+1}}{dx_t} = 0 \tag{58}$$

As shown as the equations (58), if x_t approaches 0 as much as possible, the value of the first derivative approximates infinity and conversely if x_t expands close to infinity, then the value of it approximates 0.

5.5 An analysis in the steady-state.

Now that the global stability of the economy has been proven, we focus our analysis on the steady-state economy. By using the equation (51) of the dynamic equation of capital and setting the left hand side of the equation to 1, it is possible to derive an equation that satisfies the steady-state.

$$[\varphi\alpha\tau(x^*)^{\alpha} + 1]\emptyset = \rho(1-\tau)(1-\alpha)(x^*)^{\alpha-1} \tag{59}$$

Where x^* indicates the relative value of capital in the steady-state. Next, we try to differentiate the equation (25) which shows the growth of public capital by the share on an expenditure for public capital investment to see the effect of increasing this share on the growth.

$$\frac{\partial g}{\partial \varphi} = \tau\alpha^2(x^*)^\alpha\left[\frac{1}{\alpha} + \frac{\varphi}{x^*}\frac{dx^*}{d\varphi}\right] \tag{60}$$

Here, the second item in parentheses indicates an elasticity of the share for relative value on capitals. In order to measure the magnitude of this elasticity, we will totally differentiate equation (59) with respect to φ and x^*. The result is as follows:

$$\frac{dx^*}{d\varphi} = \frac{A}{B} < 0 \tag{61}$$

$$A = -\varphi\alpha\tau(x^*)^\alpha\emptyset < 0 \tag{62}$$

$$B = [\varphi\alpha^2\tau(x^*)^{\alpha-1}\emptyset + \rho(1-\tau)(1-\alpha)(x^*)^{\alpha-2}] > 0 \tag{63}$$

As you can see, the sign of (61) has a negative. This means that the share rise pulling the relative value of capitals down. Because the expansion a public capital overtakes the rise of private capital. Here, this elasticity means the rate of change of x^* with respect to the rate of change of 1% of φ. It is synonymous with the well-known price elasticity of price. Therefore, the value is higher than 0 and lower than 1. Since the value of the share of private capital on GDP is between 0 and 1, the sigh of parentheses will be positive. The only question I have here is that why in case B the sign of elasticity is negative. This is because in case A, the optimal savings rate includes not only the income effect but also the price effect. Both of these effects play a role in boosting the private capital significantly.

Proposition 4. The sign of elasticity of the share on public capital investment for the relative value between the private and public capital is different in A and B cases because of the existence of the price effect as well as the income effect in the case A.

So we specifically derive the effect of rising share on growth and it is explicitly shown below.

$$\frac{\partial g}{\partial \varphi} = \tau\alpha^2\left\{\frac{[\varphi\alpha^2\tau(x^*)^\alpha\emptyset(1-\varphi) + \rho(1-\tau)(1-\alpha)(x^*)^{\alpha-1}]}{[\varphi\alpha^2\tau\emptyset + \rho(1-\tau)(1-\alpha)(x^*)^{-1}]}\right\} > 0 \tag{64}$$

It can be seen that the sigh in the above equation is explicitly positive.

7. Comparison of the impact of public capital on growth.

The next thing I'm interested in is whether case A or B has a higher growth rate. In equations (40) and (64) showing the effect of rising share on growth. Substitute numerical example for parameters to explicitly indicate these.

$$\frac{\partial g}{\partial \varphi}=\tau\alpha^2\left[\frac{2(1-\alpha)^2(1-\tau)^2(x^*)^{2\alpha-2}\rho+(\varphi\alpha\tau(x^*)^{\alpha}+1)(1-\alpha)(x^*)^{\alpha-2}(1-\tau)\rho+[\tau(x^*)^{-1}-\varphi(x^*)^{\alpha}]\varphi\alpha\emptyset}{2\alpha(1-\alpha)^2(1-\tau)^2(x^*)^{\alpha-2}\rho+(\varphi\alpha\tau(x^*)^{\alpha}+1)(1-\alpha)(x^*)^{-2}(1-\tau)\rho+\varphi\alpha\tau(x^*)^{-1}\emptyset}\right]=0.032 \tag{65}$$

$$\frac{\partial g}{\partial \varphi}=\tau\alpha^2\left\{\frac{[\varphi\alpha^2\tau(x^*)^{\alpha}\emptyset(1-\varphi)+\rho(1-\tau)(1-\alpha)(x^*)^{\alpha-1}]}{[\varphi\alpha^2\tau\emptyset+\rho(1-\tau)(1-\alpha)(x^*)^{-1}]}\right\}=0.078 \tag{66}$$

The above equations shows that Case B has a greater effect on growth. What is the main cause of this difference in effect? Intuitively, first of all, it is considered that the effect of the increase in the wage (income effect) due to the increase in public capital is very large in Case B. In both cases, wages rise. Furthermore, in Case A, the interest rate is also pushed up. At first glance, Case B is likely to have a higher growth rate, but the numerical numbers point to opposite. Second, in Case A, the increase in public capital should reduce the allocation for childcare support in government tax revenues, resulting in a decrease in the number of births and an increase in the opportunity cost of workers. Considering these and thinking that the impact on the growth of Case A is small, the government's policy for reducing the opportunity cost of workers, for example, in Japan, free childcare fees for nursery schools for children over 3 years old and a second or more children, and introduction of subsidies for the establishment of daycare centers in companies can be considered to be insignificant.

8. Concluding remarks

This study focused on the relative value of private-public capital in the presence of a childcare support policy. First, the global stability of economic growth and the unique steady state to which the economy convergence is clarified. In the steady state, the economy is on a balanced growth path in which private capital, public capital, and GDP grow at the same rate. Second, the effect of increasing the share of public capital investment on the steady-state growth rate was analyzed, and was found to depend on the absolute value of the elasticity of increasing the share of public capital investment relative to capital value or the labor share of GDP. More specifically, a smaller absolute elasticity value and larger labor share of GDP was found to be more likely to result in a positive growth rate. This is because the magnitude of the effect that increases public capital exceeds the effect of rising private capital, thus a larger increase in income is needed to increase savings.

Acknowledgements
The author is indebted to Tadashi Morita and participants in the Japan Association for Applied Economics seminar for their helpful comments and suggestions. He is grateful to many professors for their helpful comments and suggestions. He acknowledges the financial support received from the research expenses of Nippon Bunri University. He is responsible for all remaining errors.

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