The grand dividends value

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Abstract

We introduce a new value for games with transferable utility, called grand dividends value. In the payoff calculation, the grand dividends value takes into account the worths of all subcoalitions of a player set. The concept of grand dividends, representing the surplus (which can also be non-positive) of the worth of the grand coalition over the worths of all coalitions that lack one player of the player set, is the initial point here. The grand dividends value satisfies many properties that we know from the Shapley value. Along with new axioms that have a similar correspondence to axioms that are also satisfied by the Shapley value, axiomatizations arise that have an analogous equivalent for the Shapley value, including the classics of Shapley and Young.

Keywords Cooperative game · (Harsanyi/Grand) Dividends · Shapley value · Grand dividends value

1. Introduction

The concept of a characteristic function or coalition function goes back to von Neumann and Morgenstern (1944). In Shapley (1953b), a (TU-)game is given by a finite subset N of the universe of all possible players and a superadditive set function (which we will call coalition function) from the subsets of N into the real numbers with the only condition that the worth of the empty set is zero. We will follow Shapley's approach but dispense with superadditivity. The coalition function can be used, for example, to model and analyze economic, political, or other social phenomena. In general, the worth of a coalition is the reward that this coalition can guarantee regardless of what players do outside the coalition.

In the model of Harsanyi (1959), the fundamental assumption is that each player is simultaneously a member of all possible different coalitions (Harsanyi uses the term 'syndicate') that contain it. Introducing the important concept of his (Harsanyi) dividends, he assumes that each coalition guarantees a certain payment, the Harsanyi dividend, which

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should be divided to the members of this coalition. Moreover, these dividends should be assumed in addition to any dividends that each member of the coalition may receive from other coalitions, particularly those coalitions that are subsets of that coalition. Under these assumptions, Harsanyi can show that his solution for TU-games that provides each player with an equal share of all Harsanyi dividends from coalitions containing it as a payoff coincides with the Shapley value. Thus, Harsanyi gives the Shapley value a justification inherent in the coalitional function, but only under the assumptions noted above.

Certainly, these assumptions are reasonable for many scenarios. But other situations are also conceivable. Harsanyi (1959) himself points out that in von Neumann and Morgenstern (1944) it is assumed that each player is a member of only one coalition of players from a player set. For the equal division value (see, e.g., Zou et al. (2021)), we can assume that the grand coalition (the coalition that contains all players) is the only coalition that actually forms. Consequently, only the grand coalition receives a dividend equal to its worth, which is then distributed equally among all players. If we consider the equal surplus division value, introduced in Driessen and Funaki (1991) as the center-of-gravity of the imputation-set value, the singletons and the grand coalition can be assumed to be the coalitions that actually form. Here, each player receives its stand-alone worth as a dividend, paid in full, plus an equal share of the surplus of the worth of the grand coalition over the worths of the singletons as a dividend of the grand coalition.

While the last two values take into account only a (small) part of the worths of all possible coalitions, this is not the case for the Shapley value and our new value presented in the following.

At first, we assume each player is a member of the grand coalition and is also a member of all coalitions that are subsets of the grand coalition that contain that player and one less player than the grand coalition. In addition, these coalitions should guarantee a payoff in the amount of the worth of the coalition, regardless of the other coalitions. Then we can consider the worth of the grand coalition minus the sum of all the worths of coalitions that are missing exactly one player of the player set as a 'grand dividend' for the grand coalition in this model.

But then, we can also examine the subgames on player sets where one player of the original player set is removed and get a grand dividend for each grand coalition of the subgames accordingly. Proceeding in this way, we obtain grand dividends for all coalitions which are subsets of the grand coalition of the original game with player set N, until in the end each player receives its stand-alone worth as a grand dividend for its singleton. Of course, we can also have non-positive grand dividends, just like the Harsanyi dividends. For player sets with only two players, grand dividends coincide with Harsanyi dividends.

With the concept of grand dividends, we can introduce a new TU-value, called 'grand dividends value'. The grand dividends value is given by the fact that each player receives an equal share of the grand dividends of all subgames where that player is a member of the player set as a payoff. Note, however, that, depending on the size of the player set and the number of members in a coalition, we may have to consider the same dividend several times, just as our assumption above would dictate.

The grand dividends value satisfies many axioms that are also satisfied by the Shapley value. Remarkably, however, it also satisfies a set of new axioms that are analogous to ones also satisfied by the Shapley value. Thus we can give axiomatizations of the grand dividends value that are analogous to axiomatizations of the Shapley value in Shapley (1953b), Myerson (1980), and Besner (2020). In particular, the new grand dividends monotonicity which states that for a player the payoff does not decrease if the grand dividends do not decrease has interesting economic significance in our view, similar to strong monotonicity (Young, 1985). It offers, along with efficiency and symmetry, an analogous characterization of the grand dividends value to the axiomatization of the Shapley value in Young (1985).

The article is organized as follows. In Section 2 we give some preliminaries. Section 3 introduces the grand dividends and the grand dividends value. In Section 4, we propose a recursive formula of the grand dividends value and give two axiomatizations. In Sections 5 and 6, respectively, we provide axiomatizations that are similar to the classical axiomatizations of the Shapley value in Shapley (1953b) and Young (1985). Section 7 contains some concluding remarks. Finally, an Appendix (Section 8) shows the logical independence of the axioms in our characterizations.

2. Preliminaries

Let $\mathfrak{U}$ be a countably infinite set, the universe of all players and let $\mathcal{N}$ be the set of all non-empty and finite subsets of $\mathfrak{U}$. A cooperative game with transferable utility (**TU-game**) is a pair (N, v) with a player set $N \in \mathcal{N}$ and a **coalition function** $v\colon 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. Each subset $S \subseteq N$ is called a **coalition**, $v(S)$ is the **worth** of the coalition S and Ω^S denotes the set of all non-empty subsets of S. For each $S \in \Omega^N$, $|S|$ or s respectively denotes the cardinality of S, especially n denotes the cardinality of a player set N. $\mathbb{V}(N)$ denotes the set of all TU-games with player set N. The **restriction** of (N, v) to a player set $S \in \Omega^N$ is denoted by (S, v). A **unanimity game** (N, u_S), $S \in \Omega^N$, is defined for all $T \subseteq N$ by $u_S(T) = 1$, if $S \subseteq T$, and $u_S(T) = 0$, otherwise.

Let $N \in \mathcal{N}$ and $(N, v) \in \mathbb{V}(N)$. For all $S \in \Omega^N$ the **Harsanyi dividends** $\Delta_v(S)$ (Harsanyi, 1959) are defined inductively by

$$\Delta_v(S) := v(S) - \sum_{R \subsetneq S} \Delta_v(R). \tag{1}$$

The **marginal contribution** MC_i^v of a player $i \in N$ to $S \subseteq N \backslash \{i\}$ is given by $MC_i^v(S) := v(S \cup \{i\}) - v(S)$. A player $i \in N$ is called a **null player** in (N, v) if $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \backslash \{i\}$. Two players $i, j \in N$, $i \neq j$, are **symmetric** in (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.

For all $N \in \mathcal{N}$, a **TU-value** or **solution** φ is an operator that assigns to any $(N, v) \in \mathbb{V}(N)$ a payoff vector $\varphi(N, v) \in \mathbb{R}^N$.

For all $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, the **Shapley value** Sh (Shapley, 1953b), is given by

$$Sh_i(N, v) := \sum_{S \subseteq N,\, S \ni i} \frac{(s-1)!\,(n-s)!}{n!} \big[v(S) - v(S \backslash \{i\})\big] \text{ for all } i \in N. \tag{2}$$

We refer to the following well-known axioms for TU-values φ which hold for all $N \in \mathcal{N}$:

Efficiency, E. For all $(N, v) \in \mathbb{V}(N)$, we have $\sum_{i \in N} \varphi_i(N, v) = v(N)$.

Efficiency means that the worth of the grand coalition is fully shared among all players.

Null player, N. For all $(N, v) \in \mathbb{V}(N)$ and $i \in N$ such that i is a null player in (N, v), we have $\varphi_i(N, v) = 0$.

A null player receives nothing.

Additivity, A. For all $(N, v), (N, w) \in \mathbb{V}(N)$, we have $\phi(N, v) + \phi(N, w) = \phi(N, v + w)$.

Additivity requires that it is irrelevant whether one first adds the games and then applies the solution concept, or whether one first applies the solution concept to the individual games and then adds the payoffs.

Symmetry, Sym. For all $(N, v) \in \mathbb{V}(N)$ and $i, j \in N$ such that i and j are symmetric in (N, v), we have $\phi_i(N, v) = \phi_j(N, v)$.

Symmetry means that two players who contribute the same amount to each coalition should receive the same payoff.

Balanced contributions, BC (Myerson, 1980)**.** For all $(N, v) \in \mathbb{V}(N)$ and $i, j \in N$, $i \neq j$, we have $\varphi_i(N, v) - \varphi_i(N\backslash\{j\}, v) = \varphi_j(N, v) - \varphi_j(N\backslash\{i\}, v)$.

By this property, for two players the amount that one player would win or lose if the other player drops out of the game is the same for both players.

Strong monotonicity, SMon (Young, 1985)**.** For all $(N, v), (N, w) \in \mathbb{V}(N)$ and $i \in N$ such that $MC_i^v(S) \leq MC_i^w(S)$ for all $S \subseteq N\backslash\{i\}$, we have $\varphi_i(N, v) \leq \varphi_i(N, w)$.

Strong monotonicity states that a player's payoff should not decrease if the worth of the coalitions containing that player increases or stays the same compared to the worth of the coalitions that do not contain that player.

Marginality, Mar (Young, 1985)**.** For all $(N, v), (N, w) \in \mathbb{V}(N)$ and $i \in N$ such that $MC_i^v(S) = MC_i^w(S)$ for all $S \subseteq N\backslash\{i\}$, we have $\varphi_i(N, v) = \varphi_i(N, w)$.

By marginality, only a player's marginal contributions are relevant to the player's payoff. The following axiom states that the payoff differences of two players should be the same for different worths of the grand coalition.

Equal (aggregate) monotonicity, EMon (Béal et al., 2018)**.** For all $(N, v) \in \mathbb{V}(N)$ and $\alpha \in \mathbb{R}$, we have

$$\varphi_i(N, v) - \varphi_i(N, v + \alpha \cdot u_N) = \varphi_j(N, v) - \varphi_j(N, v + \alpha \cdot u_N) \text{ for all } i, j \in N,$$

3. The grand dividends value

Harsanyi (1959), in introducing Harsanyi dividends, assumes that all possible coalitions are formed simultaneously. The Harsanyi dividend of a singleton is just the stand-alone worth of it, and for all other coalitions we have recursively that their Harsanyi dividends are their worth minus the dividends of all proper subcoalitions.

As demonstrated in the introduction, we can also assume that only the grand coalition and all coalitions with one less player than the grand coalition actually form. Since we assume this for all games on all sets of players, the same applies to the subgames (S,v), $S \in \Omega^N$, of a game $(N,v) \in \mathbb{V}(N)$, $N \in \mathcal{N}$. Therefore, to distinguish our concept by name from that of Harsanyi dividends, we define as **grand dividend** $\Gamma_v(N)$, $N \in \mathcal{N}$, of the grand coalition N in the game (N,v) the (not necessarily positive) surplus of the worth of N over the sum of the worths of all subsets that contain one less player. Formally, we have, for all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$,

$$\Gamma_v(N) := v(N) - \sum_{j \in N} v(N \backslash \{j\}). \tag{3}$$

By Harsanyi (1959), equivalent to (2), for all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, the Shapley value Sh is given by

$$Sh_i(N,v) := \sum_{S \subseteq N,\, S \ni i} \frac{\Delta_v(S)}{s} \quad \text{for all} \quad i \in N. \tag{4}$$

The Shapley value assigns to each player an equal share of the Harsanyi dividends of all coalitions in which that player is a member. We now introduce a new TU-value that assigns to each player an equal share of the grand dividends of the grand coalitions of all subgames where that player is a member of the player set. However, since we successively consider all subgames when assigning dividends, depending on the size of the set of players, the respective coalitions are thus considered multiple times, so that we multiply each grand dividend by the number of times it occurs.

Definition 3.1. *For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, the **grand dividends value** Ψ is given by*

$$\Psi_i(N,v) := \sum_{S \subseteq N,\, S \ni i} \frac{(n-s)!}{s} \Gamma_v(S) \text{ for all } i \in N. \tag{5}$$

4. A recursive formula, inessential grand dividends, and balanced summarized contributions

We present a recursive formula for the grand dividends value.[1] Each player receives an equal share of the grand dividend and the sum of the payoffs received in all subgames in which one player of the player set is missing.

Proposition 4.1. *For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, the grand dividends value Ψ is inductively given by*

$$\Psi_i(N,v) := \frac{\Gamma_v(N)}{n} + \sum_{j \in N,\, j \neq i} \Psi_i(N \backslash \{j\}, v) \text{ for all } i \in N.^{2} \tag{6}$$

[1] In Kongo and Funaki (2016), we find a recursive formula for the Shapley value which is proposed earlier in Hart and Mas-Colell (1989) and Sprumont (1990), given by

$$Sh_i(N,v) := \frac{1}{n}\big(v(N) - v(N \backslash \{i\}\big) + \frac{1}{n} \sum_{j \in N,\, j \neq i} Sh_i(N \backslash \{j\}, v) \text{ for all } i \in N.$$

[2] If $n = 1$, we use the convention that an empty sum evaluates to zero.

Proof. Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$. Each coalition $S \subsetneq N$, $S \ni i$, is a subset of $(n-s)$ different coalitions $T \subsetneq N$, $|T| = n-1$, $T \ni i$. Therefore, we have

$$\sum_{j\in N,\, j\neq i}\left[\sum_{S\subseteq N\backslash\{j\},\, S\ni i}\frac{(n-1-s)!}{s}\Gamma_v(S)\right] = \sum_{S\subsetneq N,\, S\ni i}\frac{(n-s)!}{s}\Gamma_v(S) \text{ for all } i\in N. \tag{7}$$

It follows, for all $i \in N$,

$$\begin{aligned}\Psi_i(N,v) &\underset{(5)}{=} \frac{\Gamma_v(N)}{n} + \sum_{S\subsetneq N,\, S\ni i}\frac{(n-s)!}{s}\Gamma_v(S)\\ &\underset{(7)}{=} \frac{\Gamma_v(N)}{n} + \sum_{j\in N,\, j\neq i}\left[\sum_{S\subseteq N\backslash\{j\},\, S\ni i}\frac{(n-1-s)!}{s}\Gamma_v(S)\right]\\ &\underset{(5)}{=} \frac{\Gamma_v(N)}{n} + \sum_{j\in N,\, j\neq i}\Psi_i(N\backslash\{j\},v).\end{aligned}$$

□

We call a TU-game $(N,v) \in \mathbb{V}(N)$ an **inessential grand dividend game** if $v(N) = \sum_{j\in N} v(N\backslash\{j\})$ which is, by (3), equivalent to $\Gamma_v(N) = 0$. The following property states that in an inessential grand dividend game, the payoff to a player is completely determined by the sum of the player's payoffs in all subgames in which one player of the player set is removed at a time.

Inessential grand dividend, IGD.[3] For all $N \in \mathcal{N}$ and all inessential grand dividend games $(N,v) \in \mathbb{V}(N)$, we have $\varphi_i(N,v) = \sum_{j\in N}\varphi_i(N\backslash\{j\},v)$ for all $i \in N$.

It follows our first axiomatization of the grand dividends value.

Theorem 4.2. *The grand dividends value Ψ is the unique TU-value that satisfies **E**, **IGD**, and **EMon**.*[4]

Proof. Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$.

I. Existence: **IGD** and **EMon** follow immediately by (6). We show **E** by induction on the size n.

Initialization: Let $n = 1$. Then, **E** is satisfied by (3) and (6).

Induction step: Let $n \geq 2$. Assume that Ψ satisfies **E** for all n', $n' < n$, (IH). We have

$$\sum_{i\in N}\Psi_i(N,v) \underset{(6)}{=} \sum_{i\in N}\left[\frac{\Gamma_v(N)}{n} + \sum_{j\in N,\, j\neq i}\Psi_i(N\backslash\{j\},v)\right] \underset{(IH)}{=} \Gamma_v(N) + \sum_{i\in N} v(N\backslash\{i\}) \underset{(3)}{=} v(N),$$

and **E** is shown.

II. Uniqueness: Let φ be a TU-value which satisfies all axioms from Theorem 4.2. We show uniqueness by induction on the size n.

Initialization: Let $n = 1$. Then, uniqueness is satisfied by **E**.

[3]This axiom is related to the inessential grand coalition property in Besner (2020).

[4]A related axiomatization of the Shapley value can be found in Besner (2020) where the inessential grand dividend property is replaced by the inessential grand coalition property.

Induction step: Let $n \geq 2$. Assume that φ is unique for all n', $n' < n$, (IH). Then, by (IH) and **IGD**, φ is unique on the inessential grand dividend game $\big(N, v - \Gamma_v(N)u_N\big)$. By **EMon**, we have, for all $i, j \in N$,

$$\varphi_i(N,v) = \varphi_i\big(N, v - \Gamma_v(N)\cdot u_N\big) + \varphi_j(N,v) - \varphi_j\big(N, v - \Gamma_v(N)\cdot u_N\big)$$
$$\Leftrightarrow \sum_{k\in N} \varphi_k(N,v) = \sum_{k\in N} \varphi_k\big(N, v - \Gamma_v(N)\cdot u_N\big) + n \cdot \big[\varphi_j(N,v) - \varphi_j\big(N, v - \Gamma_v(N)\cdot u_N\big)\big]$$

and, by **E** and (IH), φ is unique for the player j. Since j is arbitrary, uniqueness and therefore also Theorem 4.2 is shown. □

The balanced contributions property **BC** states that for any two players, the amount that one player would win or lose if the other player drops out of the game is the same for both players. By the following property, the gain or loss for two players of a player set is the same if they would play the game with the entire player set instead of playing games with player sets each missing one of their players.

Balanced summarized contributions, BSC. For all $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and $i, j \in N$, $i \neq j$, we have

$$\varphi_i(N,v) - \sum_{k\in N,\, k\neq i} \varphi_i(N\backslash\{k\}, v) = \varphi_j(N,v) - \sum_{k\in N,\, k\neq j} \varphi_j(N\backslash\{k\}, v)$$

The balanced summarized contributions property has a strong connection to the grand dividends value. Similar as the Shapley value can be characterized by **E** and **BC** (Myerson, 1980), the grand dividends value can be characterized by **E** and **BSC**.

Theorem 4.3. *The grand dividends value Ψ is the unique TU-value that satisfies **E** and **BSC**.*

*P*roof. Since **E** is already shown in the proof of Theorem 4.2 and **BSC** follows immediately from (6), we only need to show uniqueness.

Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$, and φ be a TU-value which satisfies **E** and **BSC**. We show uniqueness by induction on the size n.

Initialization: Let $n = 1$. Then, uniqueness is satisfied by **E**.

Induction step: Let $n \geq 2$. Assume that φ is unique for all n', $n' < n$, (IH). By **BSC**, we have

$$\varphi_i(N,v) - \sum_{k\in N,\, k\neq i} \varphi_i(N\backslash\{k\}, v) = \varphi_j(N,v) - \sum_{k\in N,\, k\neq j} \varphi_j(N\backslash\{k\}, v)$$
$$\Leftrightarrow\ n \cdot \varphi_i(N,v) - n \cdot \sum_{k\in N,\, k\neq i} \varphi_i(N\backslash\{k\}, v) = \sum_{k\in N} \varphi_k(N,v) - \sum_{j\in N} \sum_{k\in N,\, k\neq j} \varphi_j(N\backslash\{k\}, v)$$

and, by **E** and (IH), φ is unique for the player i. Since i is arbitrary, uniqueness and, therefore, Theorem 4.3 is shown. □

5. An axiomatization in the spirit of Shapley

We pick the original axiomatization of the Shapley value as the starting point of this section.

Theorem 5.1 (Shapley, 1953b). *The Shapley value Sh is the unique TU-value that satisfies **E**, **A**, **N**, and **Sym**.*

We would like to point out that Nowak and Radzik (1994) also introduced their solidarity value with an axiomatization similar to this one. Their axiomatization differs from Shapley's by replacing the null player axiom **N** with their A-null player axiom. Further axiomatizations which differ only in the null player axiom from Shapley's axiomatization are the axiomatization of the equal division value in van den Brink (2007), using the nullifying player property, and the axiomatization of the equal surplus division value by Casajus and Huettner (2014), using the dummifying player property.

Our next axiomatization of the grand dividends value also differs from Shapley's only in the null player axiom. It is well-known and easy to prove that $i \in N$ is a null player in (N, v) if $\Delta_v(S) = 0$ for all $S \subseteq N$, $S \ni i$. Analogous, we call a player $i \in N$ a **grand dividends null player** in (N, v) if $\Gamma_v(S) = 0$ for all $S \subseteq N$, $S \ni i$. This yields the following property.

Grand dividends null player, GDNull. For all $(N, v) \in \mathbb{V}(N)$ and $i \in N$ such that i is a grand dividends null player in (N, v), we have $\varphi_i(N, v) = 0$.

It is not so far-fetched that a grand dividends null player i receives a payoff of zero. Each coalition containing player i has as its worth a sum of worths of coalitions that all do not contain player i. Player i does not contribute to any coalition, so the other players split the full payoff among themselves. We give a first axiomatization.

Theorem 5.2. *The grand dividends value Ψ is the unique TU-value that satisfies **E**, **A**, **GDNull**, and **Sym**.*

Proof. In unanimity games (N, u_S), $S \in \Omega^N$, which form a basis for $\mathbb{V}(N)$ (see Shapley (1953b)), we have $\Delta_{u_S}(S) = 1$ and $\Delta_{u_S}(T) = 0$, $T \in \Omega^N$, $T \neq S$. Analogos, we introduce another basis. For each coalition $S \in \Omega^N$, we use a TU-game $(N, z_S) \in \mathbb{V}(N)$ such that

$$\Gamma_{z_S}(T) := \begin{cases} 1, & \text{if } T = S, \\ 0, & \text{if } , T \in \Omega^N, T \neq S. \end{cases} \tag{8}$$

Due to (3), we have $z_S(S) = 1$ and all coalitions which are no supersets of S have a worth of zero. Each coalition T containing S as a proper subset, contains $\binom{t-s}{t-s-1} = t - s$ coalitions of the size $t - s - 1$ containing S and all other coalitions which are subsets of the same size have a worth of zero. Thus, each TU-game (N, z_S), $S \in \Omega^N$, is given, by

$$z_S(T) := \begin{cases} (t-s)!, & \text{if } S \subseteq T, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Since a $(2^n - 1) \times (2^n - 1)$ matrix A of the $2^n - 1$ entries of the $2^n - 1$ coalition functions z_S, correspondingly ordered, is a triangular matrix with *det* $A = 1 \neq 0$, we have found a basis for $\mathbb{V}(N)$.

Let now $N \in \mathcal{N}$, $(N, v), (N, w) \in \mathbb{V}(N)$ and $\alpha \in \mathbb{R}$.

I. Existence: **E** is shown in the proof of Theorem 4.2. By (5), Ψ obviously satisfies **GDNull** and **Sym**. Since we have, by (3), $\Gamma_{v+w} = \Gamma_v + \Gamma_w$, **A** is satisfied by (5).

II. Uniqueness: Let φ be a TU-value which satisfies all axioms from Theorem 5.2. For all $S \in \Omega^N$, $i \in N$, we have $\varphi_i(N, \alpha z_S) = 0$, by **Sym** and **E**, if $\alpha = 0$ and, by **GDNull**, if $i \in N \backslash S$. By **E**, **Sym**, and (9), it follows $\varphi_i(N, \alpha z_S) = \alpha \frac{(n-s)!}{s}$ for all $i \in S$. Therefore, φ is unique on all games $(N, \alpha z_S)$ for all $\alpha \in \mathbb{R}$ and all $S \in \Omega^N$. But then, by **A**, uniqueness is shown and the proof is complete. □

6. An axiomatization in the spirit of Young

Certainly, the following theorem is one of the most beautiful axiomatizations of the Shapley value.

Theorem 6.1 (Young, 1985). *The Shapley value Sh is the unique TU-value that satisfies **E**, **SMon**, and **Sym**.*

Thereby **SMon** can also be replaced by the weaker **Mar**. By (1), the condition '$MC_i^v(S) = MC_i^w(S)$ for all $S \subseteq N \backslash \{i\}$' in **Mar** can be equivalently replaced by '$\Delta_v(S) = \Delta_w(S)$ for all $S \subseteq N$, $S \ni i$', analogously in **SMon**. We replace marginal contributions or Harsanyi dividends respectively by grand dividends in both axioms and obtain two new properties.

Grand dividends independency, GDInd. For all $(N, v), (N, w) \in \mathbb{V}(N)$ and $i \in N$ such that $\Gamma_v(S) = \Gamma_w(S)$ for all $S \subseteq N$, $S \ni i$, we have $\varphi_i(N, v) = \varphi_i(N, w)$.

Grand dividends monotonicity, GDMon. For all $(N, v), (N, w) \in \mathbb{V}(N)$ and $i \in N$ such that $\Gamma_v(S) \leq \Gamma_w(S)$ for all $S \subseteq N$, $S \ni i$, we have $\varphi_i(N, v) \leq \varphi_i(N, w)$.

The grand dividends monotonicity states that the payoff to a player should not decrease if the grand dividends of all coalitions containing that player increase or stay the same. It is easy to show that **GDMon** implies **GDInd**. By this property, the payoffs remain the same if the grand dividends of all coalitions containing that player stay the same. Therefore, a players payoff depends only on the grand dividends of coalitions containing the player. Just as Young (1985) used **SMon** instead of **Mar** to axiomatize the Shapley value where the proof only used **Mar**, we could also use **GDMon** instead of **GDInd** in what follows, but we will not. We introduce **GDMon** only because it might seem more convincing to applications than **GDInd**. We formulate an axiomatization in the spirit of the characterization of the Shapley value just mentioned.

Theorem 6.2. *The grand dividends value Ψ is the unique TU-value that satisfies **E**, **GDInd**, and **Sym**.*

Proof. The proof is similar to the proof in Young (1985).

Since **E** is shown in the proof of Theorem 4.2 and **Sym** and **GDInd** follow immediately from (5), we only need to show uniqueness.

The games (N, z_S), $S \in \Omega^N$, defined by (9), form a basis of $\mathbb{V}(N)$. This means, we have for any $(N, v) \in \mathbb{V}(N)$ a unique representation of the coalition function v, given by

$$v = \sum_{S \in \Omega^N} \alpha_S z_S,\ \alpha_S \in \mathbb{R}. \tag{10}$$

Note, due to (8), that for all $S \in \Omega^N$, $c \in \mathbb{R}$, and two games $(N, v), (N, w) \in \mathbb{V}(N)$, $w := v + cz_S$, we have

$$\Gamma_v(T) = \Gamma_w(T) \text{ for all } T \subseteq N,\ T \neq S. \tag{11}$$

Therefore, **GDInd** implies

$$\varphi_i(N, v) = \varphi_i(N, w) \text{ for all } i \in N \backslash S. \tag{12}$$

Let $N \in \mathcal{N}$, $(N, v) \in \mathbb{V}(N)$, and φ be a TU-value which satisfies **E**, **Sym**, and **GDInd**. We use an induction on the size $r_v := |\{R \in \Omega^N : \Gamma_v(R) \neq 0\}|$.

Initialization: Let $r = 0$. We have $v(N) = 0$ and uniqueness is satisfied by **E** and **Sym**.

Induction step: Let $r \geq 1$. Assume that φ is unique for all TU-games (N, v'), $r_{v'} \leq r-1$, (IH). Let Q be the intersection of all coalitions $Q_k \in \Omega^N$, $\Gamma_v(Q_k) \neq 0$,

$$Q := \bigcap_{1 \leq k \leq r} Q_k.$$

Two cases can be distinguished: (a) $i \in N \backslash Q$ and (b) $i \in Q$.

(a) Each $i \in N \backslash Q$ is a member of at most $r - 1$ coalitions Q_k, $\Gamma_v(Q_k) \neq 0$ and we have at least one coalition $Q_i \in \Omega^N$, $\Gamma_v(Q_i) \neq 0$. Then, by (10), exists a coalition function v_i such that

$$v_i = \sum_{S \in \Omega^N,\, S \neq Q_i} \alpha_S z_S,$$

and, by (11), we have $\Gamma_v(S) = \Gamma_{v_i}(S)$ for all $S \subseteq N$, $S \ni i$. Therefore, by **GDInd**, (12), and (IH), φ is unique on (N, v) for all $i \in N \backslash Q$.

(b) Each $i \in Q$ is a member of all coalitions Q_k, $\Gamma_v(Q_k) \neq 0$. Thus, all coalitions $S \in \Omega^N$, $Q \nsubseteq S$, have a grand dividend $\Gamma_v(S) = 0$. It follows, $v(S) = 0$ for all $S \in \Omega^N$, $Q \nsubseteq S$. Therefore, if $|Q| = 1$, by **E**, and (a), φ is unique for $i \in Q$. If $|Q| \geq 2$, we have $v(T \cup \{j\}) = v(T \cup \{k\})$ for all $j, k \in Q$ and $T \subseteq N \backslash \{j, k\}$. Hence, all players $i \in Q$ are symmetric in (N, v). By **Sym**, **E**, and (a), φ also is unique for all $i \in Q$ and the proof is complete. $\square$

7. Conclusion

Of course, the grand dividends value can be applied to all coalition functions, just like the Shapley value. However, in our view, if the respective assumptions are satisfied, the corresponding axioms and hence the associated TU-values are the most convincing. The same applies to the assumptions made in the introduction regarding the equal division value and the equal surplus division value. It may not be always appropriate, to give a nullifying or zero player (see van den Brink (2007) and Deegan and Packel (1978)), who causes any coalition containing that player to receive a worth of zero, a payoff of zero with no further penalty when the cooperation of the other players is actually present.

Therefore, when selecting a TU-value for a payoff calculation, each user should pay attention not only to the desired properties the value should have, i.e. the satisfied axioms, but also to which coalition formations actually occur.

Definition 3.1 or Proposition 4.1 immediately reveal various extensions of the grand dividends value. Analogous to the weighted Shapley values (Shapley, 1953a), each player could be assigned a personal weight and the summands in (5) would no longer be distributed equally among the members of the coalitions S but in proportion to these members' weights (see (13)). As in the case of the proportional Shapley value (Béal et al., 2018; Besner, 2019), these weights could also be replaced by the stand-alone worths of the individual members. An extension in the sense of the Harsanyi solutions (Hammer et al., 1977; Vasil'ev, 1978)[5] would also be conceivable where the weights of two players for different coalitions could be in different proportions.

The investigation and axiomatizations of these extensions are left to further research.

8. Appendix

We show the logical independence of the axioms in the theorems. The locical independence of the two axioms in Theorem 4.3 is obvious.

Remark 8.1. *The axioms in Theorems 4.2 and 6.2 are logically independent:*

- ***E***: *The null value ϕ^0, defined by $\phi^0_i(N,v) = 0$ for all $i \in N$, satisfies **IGD/Mar** and **EMon/GDInd** but not **E**.*
- ***IGD***: *The Shapley value Sh satisfies **E** and **EMon** but not **IGD**.*
- ***EMon***: *Let $W := \{f : \mathfrak{U} \to \mathbb{R}_{++}\}$, $w_i := w(i)$ for all $w \in W$, $i \in \mathfrak{U}$, be the collection of all positive weight systems on $\mathfrak{U}$ and $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$. For each $w \in W$, the **weighted grand dividends value** Ψ^w, given by*

$$\Psi^w_i(N,v) := \sum_{S \subseteq N,\, S \ni i} \frac{w_i(n-s)!}{\sum_{j\in S} w_j} \Gamma_v(S) \text{ for all } i \in N, \tag{13}$$

  *such that $w_j \neq w_k$ for at least two different players $j,k \in N$, satisfies **E** and **IGD** but not **EMon**.*

Remark 8.2. *The axioms in Theorem 5.2 are logically independent:*

- ***E***: *The null value ϕ^0 satisfies **A**, **GDNull**, and **Sym** but not **E**.*
- ***A***: *Let $N \in \mathcal{N}$, $(N,v) \in \mathbb{V}(N)$. The TU-value φ, given by*

$$\varphi_i(N,v) := \begin{cases} 0, & \text{if } i \text{ is a grand dividends null player,} \\ \dfrac{v(N)}{|\{j \in N : j \text{ is no grand dividends null player in } (N,v)\}|}, & \text{otherwise,} \end{cases}$$

[5]Detailed information can be found in Derks et al. (2000) and Vasil'ev and van der Laan (2002).

for all $i \in N$, satisfies ***E****,* ***GDNull****, and* ***Sym*** *but not* ***A****.*

- ***GDNull****: The Shapley value Sh satisfies* ***E****,* ***A****, and* ***Sym*** *but not* ***GDNull****.*
- ***Sym****: The TU-values Ψ^w, as defined in Remark 8.1, satisfy* ***E****,* ***A****, and* ***GDNull*** *but not* ***Sym****.*

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