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Abstract

We study the existence of equilibrium when agents' preferences may not be convex. For some specific utility functions, we provide a necessary and sufficient condition under which there exists an equilibrium. The standard approach cannot be directly applied to our examples because the demand correspondence of some agents is neither single valued nor convex valued.

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1 Introduction

The existence of equilibrium is one of the fundamental issues in economic theory. A simple version of the existence theorem can be stated as follows. *Consider a pure exchange economy where there are m agents, L goods, and the consumption set of each agent is $\mathbb{R}^L_+$. If the utility function of each agent is continuous, strictly concave, strongly monotone, and the aggregate endowment of each good is strictly positive, then there exists a general equilibrium, i.e., a list of prices and allocations $(p_1, \dots, p_L, (\bar{c}_{i,1}, \dots, \bar{c}_{i,L})_{i=1}^m) \in \mathbb{R}^L_{++} \times \mathbb{R}^{mL}_+$ such that*

1. *For $(p_1, \dots, p_L)$ given, and for any agent $i = 1, \dots, m$, the allocation $(\bar{c}_{i,1}, \dots, \bar{c}_{i,L})$ is a solution of the agent i's maximization problem:*

$$(\bar{c}_{i,1}, \dots, \bar{c}_{i,L}) \in \underset{(c_{i,1},\dots,c_{i,L}) \in \mathbb{R}^L_+ : \sum_{l=1}^L p_l c_{i,l} \leq \sum_{l=1}^L p_l e_{i,l}}{\arg\max} u_i(c_{i,1}, \dots, c_{i,L}) \quad (1)$$

where u_i is the utility function and $(e_{i,1}, \dots, e_{i,L})$ is the endowment of agent i.

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2. *All markets clear:* $\sum_{i=1}^{m} \bar{c}_{i,l} = \sum_{i=1}^{m} e_{i,l} \forall l = 1, \ldots, L$ *(market clearing condition).*

Debreu (1982) and Florenzano (2003) offer excellent treatments of the existence of equilibrium, where they prove the existence of equilibrium in more much general frameworks (preferences of agents are considered instead of utility functions, productions may be introduced, the number of commodities and agents may be infinite,...).[1]

When proving the existence of general equilibrium, the standard approach requires the convexity of the preferences (or the (quasi)concavity of the utility function). The existence of equilibrium with non-convex preferences is an important issue but remains an open question. Although Aumann (1966) proves the existence of general equilibrium in an exchange economy consisting of a continuum of agents with non-convex preferences,[2] he also recognizes that such a result is not true when there are finitely many agents. Recently, Araujo et al. (2018) study the equilibrium existence in a model with two kinds of agents: one with strictly concave utility function and another with strictly convex one. A key condition ensuring the existence of equilibrium in Araujo et al. (2018) is that the aggregate endowment of convex-preference agents is sufficiently large in some state of nature compared to the aggregate endowment in other states.

The main aim of our paper is to study the issue of the existence of equilibrium when the agents' utility functions are neither concave nor convex. In general cases where preferences of some agents may not be convex, it would be interesting but difficult to find a necessary and sufficient condition for the equilibrium existence. By consequence, for the sake of tractability, we focus on a two-agent two-good exchange economy. The type A agent (risk averse agent) has logarithmic utility function $ln(c_1)+ln(c_2)$ and the type B agent's utility function is $\frac{c_1^2}{2}+\mathcal{D}\ln(c_2)$. The demand of agent A is single-valued and continuous. However, the demand of the type B agent may be multiple-valued because her utility function which is neither quasiconcave nor quasiconvex on the consumption set R^2_+. By the way, the standard approach (see, for instance, Mas-Colell et al. (1995) and references therein) cannot be directly applied to our example. Notice also that the results in Araujo et al. (2018) cannot be applied in our model. The reason is that agent B's utility function is neither concave nor convex (and hence agent B's demand may not be on the boundary of the budget set) while the second agent's utility function in Araujo et al. (2018) is strictly convex (hence her optimal consumption is on the boundary of their budget set).

Under our specifications, we manage to characterize all possible cases and find out a necessary and sufficient condition under which equilibrium exists (Proposition 1). We can also explicitly compute the equilibrium outcomes. Our necessary and sufficient condition allows us to understand how different parameters affect the equilibrium existence.

[1]See Becker et al. (1991), Becker et al. (2015), Le Van and Pham (2016) for the existence of intertemporal equilibrium in infinite-horizon economies with production and borrowing constraints. See Magill and Quinzii (1996), Florenzano (1999), Geanakoplos and Zame (2002) for the existence of equilibrium in a two-period stochastic economy with incomplete financial markets, and Araujo et al. (2002), Magill and Quinzii (2008), Bosi et al. (2018b) and references therein for the existence of equilibrium in infinite-horizon economies with collateral constraints and production.

[2]A key point in Aumann (1966) is that the aggregate preferred set is convex. He proves this by using a mathematical result which states that the integral of any set-valued function over a non-atomic measure space is convex (Aumann, 1965; Richter, 1963).

1. **Role of $\mathcal{D}$**. Given a distribution of endowments, when the agent B strongly wants to consume good 2 (i.e., $\mathcal{D}$ is high enough), then there exists a unique equilibrium. In this equilibrium, agent B does not consume good 1. Although we have the uniqueness of equilibrium, the demand of agent B may be multiple-valued.

   By contrast, if agent B strongly wants to consume good 1 ($\mathcal{D}$ is low enough), then there exists a unique equilibrium, and this equilibrium is interior. Interestingly, there is a room for the non-existence of equilibrium. This happens for some range of values of $\mathcal{D}$.

2. **Role of endowments**. If we fix all parameters excepted the endowments (e_1^A, e_2^A) of agent A (risk averse agent), then there exists an equilibrium if e_1^A or e_2^A is high enough (see Corollary 1). This result is consistent with the main finding in Araujo et al. (2018). However, while the equilibrium in Araujo et al. (2018) is a corner equilibrium, the equilibrium in our model may be interior or corner, depending on the distribution of endowments and the preferences of the agents.

   More interestingly, we complement Araujo et al. (2018) by pointing out, via examples, that: the equilibrium existence may fail even when the endowment of the agent B whose utility is neither concave or convex is very high.

The paper proceeds as follows. In Section 2, we present the structure of the economy and the concept of general equilibrium. In Section 3, we study the existence and properties of general equilibrium. Section 4 concludes. Technical proofs are presented in Appendix A.

2 A two-agent, two-good exchange economy

We consider a two-agent, two-good exchange economy with a risk averse agent and an agent who is neither risk loving nor risk averse.

2.1 Individual demand

Assume that there are two goods and the consumption set is $\mathbb{R}^2_+$. We firstly compute the demand of each agent.

Agent A (risk averse). Assume that the utility of this agent is $U^A(c_1, c_2) = \ln(c_1) + \ln(c_2)$. This utility is strictly concave. Given prices p_1, p_2 and income w, the budget constraint of this agent is $p_1c_1 + p_2c_2 \leq w$. We can easily compute the demand of this agent as follows

$$p_1c_1^A = p_2c_2^A = w/2. \tag{2}$$

Agent B. Assume that agent B has utility function $U^B(c_1, c_2) = \frac{c_1^2}{2} + \mathcal{D}\ln(c_2)$. This agent is risk loving with good 1 but risk averse with good 2. Note that this function is neither quasiconcave nor quasiconvex on the consumption set R^2_+. With this specification, we can explicitly compute the demand for each good.

Lemma 1. *The demands for good 1 and 2 of agent B are given by*

$$c_1^B = \begin{cases} 0 & \text{if } w^2 \leq 4\mathcal{D}p_1^2 \\ 0 & \text{if } w^2 > 4\mathcal{D}p_1^2 \text{ and } V(w,p_1) < D\ln(w) \\ \in \left\{0, \frac{w+\sqrt{w^2-4\mathcal{D}p_1^2}}{2p_1}\right\} & \text{if } w^2 > 4\mathcal{D}p_1^2 \text{ and } V(w,p_1) = \mathcal{D}\ln(w) \\ \frac{w+\sqrt{w^2-4\mathcal{D}p_1^2}}{2p_1} & \text{if } w^2 > 4\mathcal{D}p_1^2 \text{ and } V(w,p_1) > \mathcal{D}\ln(w) \end{cases} \tag{3}$$

and $c_2^B = (w - p_1 c_1^B)/p_2$*, where*

$$V(w,p_1) \equiv \frac{1}{2}\Big(\frac{w+\sqrt{w^2-4\mathcal{D}p_1^2}}{2p_1}\Big)^2 + \mathcal{D}\ln\Big(\frac{w-\sqrt{w^2-4\mathcal{D}p_1^2}}{2}\Big).$$

Proof. See Appendix A. □

The demands of agent B deserve some comments.

First, notice that the three following conditions are not empty: (1) $w^2 > 4\mathcal{D}p_1^2$ and $V(w,p_1) < \mathcal{D}\ln(w)$, (2) $w^2 > 4\mathcal{D}p_1^2$ and $V(w,p_1) = \mathcal{D}\ln(w)$, (3) $w^2 > 4\mathcal{D}p_1^2$ and $V(w,p_1) > \mathcal{D}\ln(w)$.[3] Hence, the demand correspondences are single valued everywhere excepted when $w^2 > 4\mathcal{D}p_1^2$ and $V(w,p_1) = \mathcal{D}\ln(w)$. They are upper semi-continuous (usc), compact valued but not convex valued.

Second, to find the optimal value c_1^B (see Appendix A), we maximize the function $f(c_1) = \frac{c_1^2}{2} + \mathcal{D}\ln(w - p_1 c_1)$.

- The first case in (3) is related to the fact the the utility function for good 1 is not Inada at the origin. Indeed, consider another concave utility function $\tilde{U}^B(c_1,c_2) = \ln(1+c_1) + \mathcal{D}\ln(c_2)$. One can check that: if $w \leq \mathcal{D}p_1$ then $\tilde{c}_1^B = 0$. When the real revenue, in term of good 1, $\frac{w}{p_1}$ is smaller than the preference parameter $\mathcal{D}$ for good 2 then agent B consumes only good 2. The function $\ln(1+c_1) + \mathcal{D}\ln(w - p_1c_1)$ is strictly decreasing on $[0, w/p_1)$.

- The second case in (3): the function f is not decreasing but has two maxima in $[0, w/p_1)$, one is located at 0. The maximum at 0 is higher than the one at the interior point. When the real revenue w/p_1 is still low compared to the preference parameter $\mathcal{D}$, agent B drops her risk loving behavior for good 1.

- The third case in (3): the two maxima equal. Agent B is indifferent between consuming for good 1 or not ($c_1^B = 0$ or $c_1^B = \frac{w+\sqrt{w^2-4\mathcal{D}p_1^2}}{2p_1}$).

- The fourth case in (3): the real revenue w/p_1 is high enough to make agent B becoming both risk loving for good 1 and risk averse for good 2.

Remark 1. *Observe that good 2 is normal if $w^2 \leq 4\mathcal{D}p_1^2$ (income is low). When $w^2 > 4\mathcal{D}p_1^2$ and $V(w,p_1) > \mathcal{D}\ln(w)$, good 2 is inferior because $\frac{w-\sqrt{w^2-4\mathcal{D}p_1^2}}{2p_2} = \frac{4\mathcal{D}p_1^2}{2p_2(w+\sqrt{w^2-4\mathcal{D}p_1^2})}$ which is decreasing in w, but not Giffen.*

[3] Indeed, if $(p_1 - \frac{w}{2\sqrt{\mathcal{D}}}) \to 0_-$ then $w^2 > 4\mathcal{D}p_1^2$ and $V(w,p_1) < \mathcal{D}\ln(w)$. If $p_1 = \frac{w}{4\sqrt{\mathcal{D}}}$, then $w^2 > 4\mathcal{D}p_1^2$ and $V(w,p_1) > \mathcal{D}\ln(w)$. By continuity, there exists $p_1 \in \left(\frac{w}{4\sqrt{\mathcal{D}}}, \frac{w}{2\sqrt{\mathcal{D}}}\right)$ s.t. $w^2 > 4\mathcal{D}p_1^2$ and $V(w,p_1) = \mathcal{D}\ln(w)$.

2.2 Concept of general equilibrium

Assume that agent A has endowments $(e_1^A, e_2^A) \gg 0$ and agent B has endowments $(e_1^B, e_2^B) \gg 0$. The aggregate supply of good i is $e_i^A + e_i^B$ for $i = 1, 2$. We provide a standard definition of general equilibrium.

Definition 1. *A list $(p_1, p_2, c_1^A, c_2^A, c_1^B, c_2^B)$ of non-negative numbers is a general equilibrium if (i) given (p_1, p_2), for i=A,B, the allocation (c_1^i, c_2^i) is a solution to agent i's maximization problem*

$$\max_{(c_1,c_2)\geq 0: p_1c_1+p_2c_2\leq p_1e_1^i+p_2e_2^i} U^i(c_1, c_2) \tag{4}$$

and (ii) markets clear: $c_i^A + c_i^B = e_i^A + e_i^B$ for $i = 1, 2$.

We will investigate the existence of general equilibrium in our model. It should be noticed that we cannot directly apply Proposition 17.C.1 of Mas-Colell et al. (1995) to our example in order to prove the existence of general equilibrium because the demand correspondences of agent B are double-valued when $(w_B)^2 > 4\mathcal{D}p_1^2$ and $B(w_B, p_1) = \mathcal{D}\ln(w_B)$. Observe also that the aggregate demand correspondences are not convex valued. So, the method in McKenzie (1959) cannot be applied as well.

3 Existence and properties of general equilibrium

In order to prove the existence and to compute equilibrium, we need to introduce some notations. Let us denote

$$\pi^{cor} \equiv \frac{2e_1^B + e_1^A}{e_2^A} \tag{5}$$

and x^* the unique solution to the equation $g(x) = 0$ where

$$g(x) \equiv \frac{1}{8}(x + \sqrt{x^2 - 4\mathcal{D}})^2 + A\Big(\ln(1 - \sqrt{1 - 4\mathcal{D}x^{-2}}) - \ln(2)\Big). \tag{6}$$

We can check that the function g is increasing in x.

Notice that $x^* > 2\sqrt{\mathcal{D}}$ and x^* only depends on $\mathcal{D}$ (so we write $x^* = x^*(\mathcal{D})$). Moreover, $x^*(\mathcal{D})$ is an increasing function of $\mathcal{D}$ and it converges to 0 when $\mathcal{D}$ converges to 0.

Denote π^{int} the smallest root (if there exists a root) of the function

$$F(X) \equiv \big[(e_2^B + e_2^A)^2 - (e_2^B)^2\big]X^2 - 2\big[(e_1^B + e_1^A)(e_2^B + e_2^A) + e_1^B e_2^B\big]X + (e_1^B + e_1^A)^2 + 4\mathcal{D} - (e_1^B)^2.$$

Precisely, if $\Delta \equiv \big((e_1^B + e_1^A)(e_2^B + e_2^A) + e_1^B e_2^B\big)^2 - \big((e_2^B + e_2^A)^2 - (e_2^B)^2\big)\big((e_1^B + e_1^A)^2 + 4\mathcal{D} - (e_1^B)^2\big) \geq 0$, we have

$$\pi^{int} = \frac{(e_1^B + e_1^A)(e_2^B + e_2^A) + e_1^B e_2^B - \sqrt{\Delta}}{(e_2^B + e_2^A)^2 - (e_2^B)^2} \tag{7}$$

Observe that

$$0 < \pi^{int} < \frac{e_1^B + e_1^A}{e_2^B + e_2^A} < X^* < \frac{2e_1^B + e_1^A}{e_2^A} = \pi^{cor} \tag{8}$$

where $X^* \equiv \frac{(e_1^B+e_1^A)(e_2^B+e_2^A)+e_1^B e_2^B}{(e_2^B+e_2^A)^2-(e_2^B)^2}$ and $F'(X^*) = 0$.

Observe that, by Inada condition, c_1^A, c_2^A and c_2^B are strictly positive while c_1^B may be zero or strictly positive at equilibrium. The following results provide necessary conditions for the equilibrium existence.

Lemma 2. *Assume that there exists an equilibrium. Then there are two cases:*

1. *Agent B does not consume good 1 ($c_1^B = 0$). In this case, the relative price must be $\frac{p_2}{p_1} = \pi^{cor} \equiv \frac{2e_1^B+e_1^A}{e_2^A}$ and*

$$e_1^B + e_2^B \frac{2e_1^B + e_1^A}{e_2^A} \leq x^*(\mathcal{D}) \tag{9}$$

2. *Agent B consumes good 1 ($c_1^B > 0$). In this case, the equilibrium relative price p_2/p_1 must equal π^{int}, and*

$$e_1^B + e_2^B \pi^{int} \geq x^*(\mathcal{D}) \tag{10}$$

Proof. See Appendix A. □

We can now state the main result of the section.

Proposition 1 (existence and computation of equilibrium)**.** *Let us consider the exchange economy with two agents described at the beginning of this section.*

1. *There exists an equilibrium $(p_1, p_2, c_1^A, c_2^A, c_1^B, c_2^B)$ with $c_1^B = 0$ if and only if condition (9) holds. Such an equilibrium is unique, up to a positive scalar for the prices. The equilibrium relative price $p_2/p_1 = \pi^{cor}$.*

2. *There exists an equilibrium $(p_1, p_2, c_1^A, c_2^A, c_1^B, c_2^B)$ with $c_1^B > 0$ if and only if condition (10) holds. Such an equilibrium is unique, up to a positive scalar for the prices. The equilibrium relative price $p_2/p_1 = \pi^{int}$ and $c_1^B = \frac{1}{2}\Big(e_1^B + \pi^{int}e_2^B + \sqrt{(e_1^B + \pi^{int}e_2^B)^2 - 4\mathcal{D}}\Big) > 0$.*

3. *There is no equilibrium if and only if $e_1^B + e_2^B\pi^{cor} > x^*(\mathcal{D}) > e_1^B + e_2^B\pi^{int}$.*

Let us explain the intuition of Proposition 1. Condition (9), i.e., $e_1^B + e_2^B\pi^{cor} \leq x^*(\mathcal{D})$ means that the income in terms of good 1, when the relative price p_2/p_1 is π^{cor}, is low with respect to $x^*(\mathcal{D})$. Condition (10), i.e., $e_1^B + e_2^B\pi^{int} \geq x^*(\mathcal{D})$ means that the income in terms of good 1 of agent B, when the relative price p_2/p_1 is π^{int}, is high with respect to $x^*(\mathcal{D})$

Since $x^*(\mathcal{D})$ is an increasing function of $\mathcal{D}$, condition (9) (resp., (10)) is satisfied if, and only if, $\mathcal{D}$ is high enough (resp., low enough). The intuition is that when $\mathcal{D}$ is

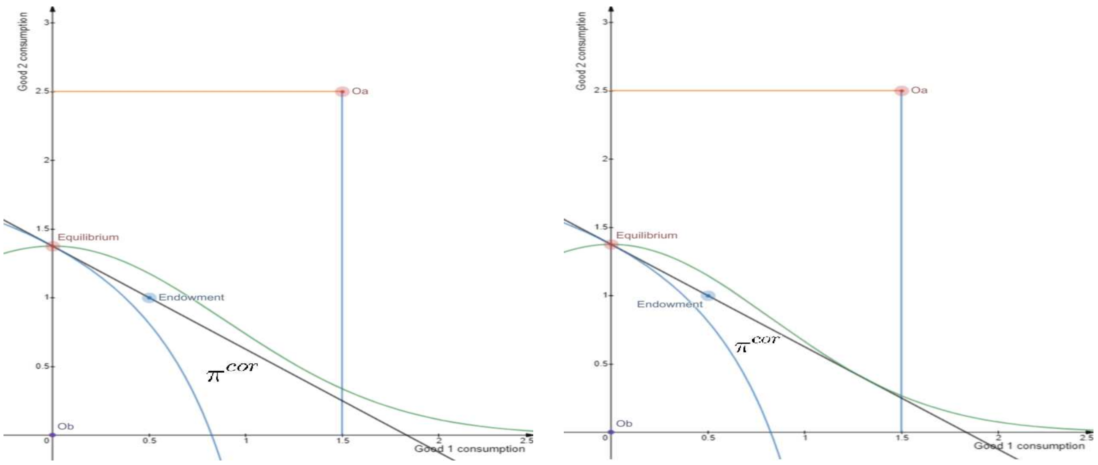

Figure 1: Edgeworth box with the **unique corner equilibrium**. LHS: the unique equilibrium. RHS: the unique equilibrium but the demand for good 1 of agent B is double-valued. The blue (resp., green) curve is the indifferent curve of agent A (resp., agent B) while the black line represents the equilibrium price.

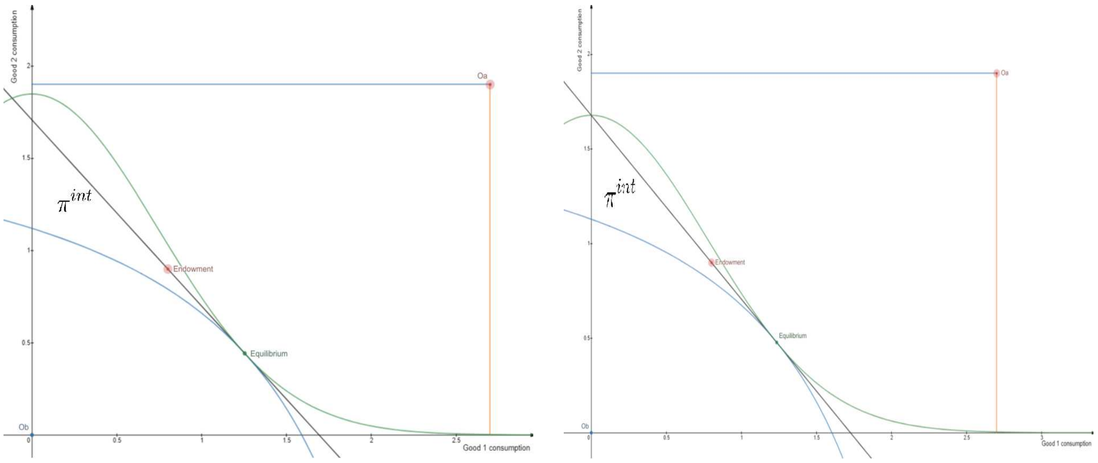

Figure 2: Edgeworth box with the **unique interior equilibrium**. LHS: the unique equilibrium, RHS: the unique equilibrium but the demand for good 1 of agent B is double-valued.

high enough, agent B strongly wants to consume good 2. In this case, there exists an equilibrium in which she does not buy good 1, i.e., $c_1^B = 0$. By contrast, when $\mathcal{D}$ is low enough but still strictly positive, agent B consumes good 1 at equilibrium.[4]

Figures 1 and 2 illustrate the unique equilibrium by using the Edgeworth box.[5] Although there is a unique equilibrium, the demand of agent B may be singleton or double-valued (see the right hand side of both Figures 1 and 2).

[4]Notice that when $\mathcal{D} = 0$ (agent B only wants to consume good 1), there is a unique equilibrium. At equilibrium, the relative price $p_2/p_1 = \pi^{int} = \frac{e_1^A}{2e_2^B + e_2^A}$.

[5]We numerically draw our figures by using the website https://www.desmos.com/calculator.

On the non-existence of equilibrium

Proposition 1 also provides a necessary and sufficient condition for the non-existence of equilibrium: $e_1^B + e_2^B\pi^{cor} > x^*(\mathcal{D}) > e_1^B + e_2^B\pi^{int}$ (notice that this happens if $\mathcal{D}$ has a middle level). The intuition is the following. If there exists an equilibrium, the equilibrium relative price must be either π^{cor} or π^{int}. However, π^{cor} cannot be an equilibrium price because it is not optimal for agent B if he does not consume good 1. In Figure 3 (dashed curves), we observe that point C is not an equilibrium.[6] We also see that π^{int} cannot be an equilibrium price. Indeed, continuous curves in Figure 3 indicates that point N is not an equilibrium.

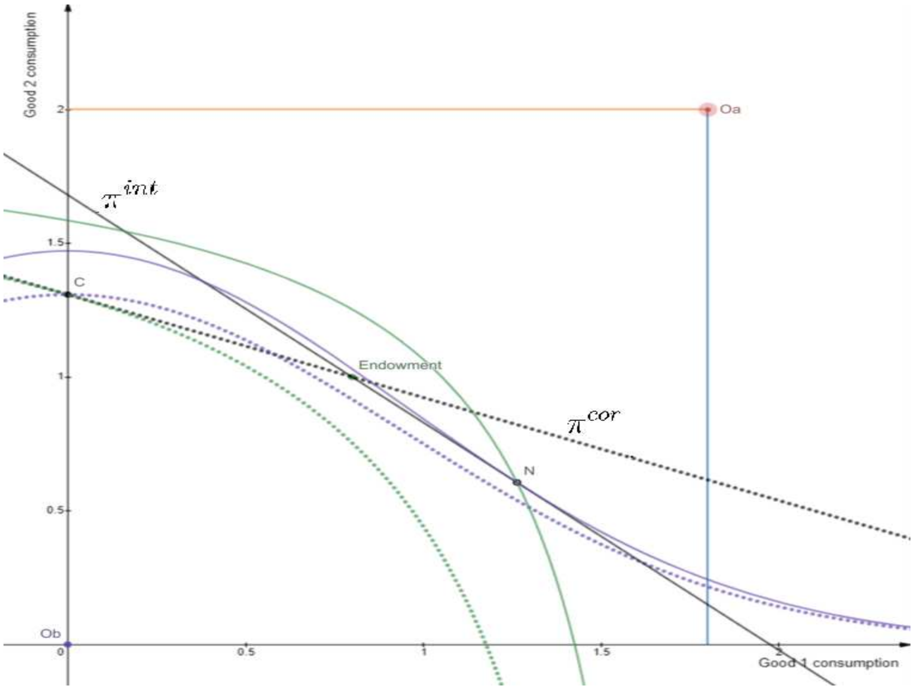

Figure 3: Edgeworth box. There is **no equilibrium**

Role of agents' endowments

Proposition 1 allows us to understand the roles of agents' endowments on the existence of equilibrium. We start with the following result.

Corollary 1 (role of the risk averse agent's endowments)**.** *We observe that*

$$\lim_{e_2^A\to\infty} \pi^{int} = \lim_{e_2^A\to\infty} \pi^{cor} = 0 \tag{11}$$

$$\lim_{e_1^A\to\infty} \pi^{int} = \infty \ \textit{and} \ \lim_{e_1^A\to\infty} \pi^{cor} = \infty. \tag{12}$$

1. *There exists an equilibrium for any e_2^A high enough (because point 3 in Proposition 1 cannot happen when e_2^A is high enough). More precisely, we have that:*

[6]We draw Figure 3 with the following parameters: $e_1^A = e_2^A = 1$, $e_1^B = 0.8, e_2^B = 1$, $\mathcal{D} = 0.9$.

(a) *If $x^*(\mathcal{D}) > e_1^B$, then when e_2^A is very large, there exist a unique and $c_1^B = 0$ at equilibrium.*

(b) *If $x^*(\mathcal{D}) \leq e_1^B$, then when e_2^A is very large, there exist a unique and $c_1^B > 0$ at equilibrium.*

2. *When e_1^A is high enough, there exist a unique equilibrium and $c_1^B > 0$ at equilibrium.*

We now show the role of endowments of agent B whose utility is neither concave nor convex.

Corollary 2 (role of agent B's endowments)**.**

1. *When e_1^B is high enough, condition (10) holds. Thus, there exists a unique equilibrium and $c_1^B > 0$ at equilibrium.*

2. *Observe that $\lim_{e_2^B \to \infty} \left(e_1^B + e_2^B \pi^{int}\right) = \frac{e_1^A + 2e_1^B}{2} + \frac{2\mathcal{D}}{e_1^A + 2e_1^B}$. So, when e_2^B is high enough, we have that:*

   (a) *If $\frac{e_1^A + 2e_1^B}{2} + \frac{2\mathcal{D}}{e_1^A + 2e_1^B} < x^*(\mathcal{D})$,[7] then there is no equilibrium.*

   (b) *If $\frac{e_1^A + 2e_1^B}{2} + \frac{2\mathcal{D}}{e_1^A + 2e_1^B} > x^*(\mathcal{D})$, then there exists a unique equilibrium and $c_1^B > 0$ at equilibrium.*

According to Corollary 1, there exists an equilibrium if the endowment of good 1 or good 2 of the risk-averse agent is high enough. This point is consistent with the main finding in Araujo et al. (2018). Notice that Araujo et al. (2018) consider general utility functions but that of type A agent is concave and that of type B agent is convex. They prove that there exists an equilibrium when the endowment of risk averse agent e_2^A or e_1^A is high enough. Corollary 1 is distinct from Araujo et al. (2018) in two ways. First, although we work with specific preferences, the utility function of agent B is neither concave nor convex. By consequence, the method of Araujo et al. (2018) cannot be applied to our model. Second, in Araujo et al. (2018), the optimal allocation of the type B agent in equilibrium is always in the corner (which corresponds to the case $c_1^B = 0$ in our model) because this agent's utility is strictly convex. By contrast, in our model, when the risk-aversion agent's endowment is high enough, the equilibrium may be interior. Indeed, this happens if (i) e_2^A is high enough and $x^*(\mathcal{D}) \leq e_1^B$ (see point 1.b of Corollary 1) or (ii) e_1^A is high enough (see point 2 of Corollary 1).

More interestingly, point 2.a of Corollary 2 indicates that there may not exist an equilibrium when the good 2 endowment of agent B (whose utility function is neither concave nor convex) is very high, given that the remaining agent has concave utility function. This point complements the main finding of Araujo et al. (2018).

Remark 2. *We can prove that the equilibrium existence may fail even when (1) the utility function of the remaining agent is convex and (2) the good 2 endowment of the agent B is very high. Indeed, consider an economy consisting of agent B and agent D*

[7]This is satisfied if $e_1^A + 2e_1^B = 2\sqrt{\mathcal{D}}$ because $2\sqrt{\mathcal{D}} < x^*(\mathcal{D})$.

(who utility function is $U^D(c_1, c_2) = c_2$ which is convex). As we prove in Appendix that, there exists an equilibrium if and only if $e_1^B + e_1^D > \mathcal{D}$ and $e_1^B + e_1^D + \frac{\mathcal{D}}{e_1^B + e_1^D} \geq x^*(\mathcal{D})$. So, there exists an equilibrium when the good 1 endowment c_1^B of agent B is high enough but the equilibrium may fail when the good 2 endowment c_2^B of agent B is high enough. This is consistent with Corollary 2.

4 Conclusion

We have provided some examples where we can explicitly characterize a necessary and sufficient condition (based on fundamentals) for the (non-)existence of general equilibrium. Although the agent B's utility function is neither quasiconcave nor quasiconvex, there may exist a general equilibrium. We view our examples as part of the first steps to investigate a challenging issue - the existence of equilibrium in models with non-convex preferences.

A Appendix

Proof of Lemma 1. The budget constraint is binding $p_1c_1 + p_2c_2 = w$, which implies that $c_2 = \frac{w - p_1c_1}{p_2}$. So, we consider the following problem

$$\max_{0 \leq c_1 \leq w/p_1} \frac{c_1^2}{2} + \mathcal{D}\ln(w - p_1c_1) \tag{13}$$

in order to determine the demand for good 1 of agent A.

Let us denote $f(c_1) \equiv \frac{c_1^2}{2} + \mathcal{D}\ln(w - p_1c_1)$. We have that

$$f'(c_1) = c_1 - \mathcal{D}\frac{p_1}{w - p_1c_1} = \frac{-p_1c_1^2 + c_1w - \mathcal{D}p_1}{w - p_1c_1}$$

Observe that $f'(c_1) = 0 \Leftrightarrow p_1c_1^2 - wc_1 + \mathcal{D}p_1 = 0$.

Denote

$$x_1 \equiv \frac{w + \sqrt{w^2 - 4\mathcal{D}p_1^2}}{2p_1} \text{ and } x_2 \equiv \frac{w - \sqrt{w^2 - 4\mathcal{D}p_1^2}}{2p_1} \tag{14}$$

We consider all possible cases.

(1) If $w^2 - 4\mathcal{D}p_1^2 < 0$ then $f'(c_1) < 0, \forall c_1 \geq 0$. Then the demand is $c_1 = 0$.

(2) If $w^2 - 4\mathcal{D}p_1^2 = 0$ then $c_1 = x_1 = x_2$ and $f'(c_1) = \frac{-p_1(c_1 - \frac{w}{2p_1})^2}{w - p_1c_1} \leq 0$. Then the demand is again $c_1 = 0$.

(3) If $w^2 - 4\mathcal{D}p_1^2 > 0$, then we have $0 < x_2 < x_1 < w/p_1$. Moreover, we observe that $f'(c_1) > 0 \Leftrightarrow x_2 < c_1 < x_1$. In this case, we have $\max_{0 \leq c_1 \leq w/p_1} f(c_1) = \max\Big\{f(0), f(x_1)\Big\}$.

Observe that $f(x_1) - f(0) = V(w, p_1) - \mathcal{D}ln(w)$. From these properties, we can obtain the demand as in Lemma 1.

□

Proof of Lemma 2. Part 1. Consider a corner equilibrium where $c_1^B = 0$. We have

$$c_1^A = \frac{p_1 e_1^A + p_2 e_2^A}{2p_1}, \quad c_2^A = \frac{p_1 e_1^A + p_2 e_2^A}{2p_2}, \quad \text{and } c_1^B = 0, \quad c_2^B = \frac{p_1 e_1^B + p_2 e_2^B}{p_2}. \tag{15}$$

According to market clearing condition $c_1^A + c_1^B = e_1^A + e_1^B$, we have

$$\frac{p_1 e_1^A + p_2 e_2^A}{2p_1} = e_1^B + e_1^A \iff \frac{p_2}{p_1} = \frac{2e_1^B + e_1^A}{e_2^A} \tag{16}$$

According to Lemma 1, we have either (1) $(w_B)^2 \leq 4\mathcal{D}p_1^2$ or (2) $(w_B)^2 > 4\mathcal{D}p_1^2$ and $V(w_B, p_1) \leq \mathcal{D}\ln(w_B)$, where $w_B = p_1 e_1^B + p_2 e_2^B$. We consider two cases.

(1.a) $(w_B)^2 \leq 4\mathcal{D}p_1^2$, i.e., $(p_1 e_1^B + p_2 e_2^B)^2 \leq 4\mathcal{D}p_1^2$. This is equivalent to $e_1^B + \frac{p_2}{p_1} e_2^B \leq 2\sqrt{\mathcal{D}}$ or equivalently $e_1^B + e_2^B \frac{2e_1^B + e_1^A}{e_2^A} \leq 2\sqrt{\mathcal{D}}$. This implies (9) because $2\sqrt{\mathcal{D}} < x^*(\mathcal{D})$.

(1.b) $(w_B)^2 > 4\mathcal{D}p_1^2$ and $V(w_B, p_1) \leq \mathcal{D}\ln(w_B)$.

We see that $V(w_B, p_1) \leq \mathcal{D}\ln(w_B)$ if and only if

$$\frac{1}{2}\left(\frac{w_B + \sqrt{(w_B)^2 - 4\mathcal{D}p_1^2}}{2p_1}\right)^2 + \mathcal{D}\ln\left(\frac{w_B - \sqrt{(w_B)^2 - 4\mathcal{D}p_1^2}}{2}\right) \leq \mathcal{D}\ln(w_B)$$
$$\iff \frac{1}{2}\left(\frac{w_B + \sqrt{(w_B)^2 - 4\mathcal{D}p_1^2}}{2p_1}\right)^2 + \mathcal{D}\ln\left(\frac{w_B - \sqrt{(w_B)^2 - 4\mathcal{D}p_1^2}}{2w_B}\right) \leq 0.$$

Denote $x \equiv \frac{w_B}{p_1} = e_1^B + e_2^B \frac{p_2}{p_1}$. This inequality is equivalent to

$$g(x) \equiv \frac{1}{8}(x + \sqrt{x^2 - 4\mathcal{D}})^2 + \mathcal{D}\Big(\ln(1 - \sqrt{1 - 4\mathcal{D}x^{-2}}) - \ln(2)\Big) \leq 0. \tag{17}$$

When $x^2 > 4\mathcal{D}$, we can compute that

$$g'(x) \equiv \frac{1}{4}(x + \sqrt{x^2 - 4\mathcal{D}})\Big(1 + \frac{x}{\sqrt{x^2 - 4\mathcal{D}}}\Big) + \frac{\mathcal{D}}{1 - \sqrt{1 - 4\mathcal{D}x^{-2}}} \frac{-8Ax^{-3}}{2\sqrt{1 - 4\mathcal{D}x^{-2}}}$$
$$= \frac{x + \sqrt{x^2 - 4\mathcal{D}}}{\sqrt{x^2 - 4\mathcal{D}}}\Big(\frac{x + \sqrt{x^2 - 4\mathcal{D}}}{4} - \frac{\mathcal{D}}{x}\Big) > 0$$

Moreover, $g(2\sqrt{\mathcal{D}}) = \mathcal{D}(0.5 - ln(2)) < 0$ and $g(\infty) = \infty$. Hence, the equation $g(x) = 0$ has a unique solution x^* in the interval $(2\sqrt{\mathcal{D}}, \infty)$. Notice that this solution only depends on the parameter $\mathcal{D}$.

The above computation leads to the following observations:

$$(w_B)^2 > 4\mathcal{D}p_1^2 \Leftrightarrow e_1^B + e_2^B \frac{2e_1^B + e_1^A}{e_2^A} > 2\sqrt{\mathcal{D}} \tag{18}$$

$$V(w_B, p_1) \leq \mathcal{D}\ln(w_B) \Leftrightarrow g(e_1^B + e_2^B \frac{p_2}{p_1}) \leq g(x^*), \text{ i.e., } e_1^B + e_2^B \frac{p_2}{p_1} \leq x^*(\mathcal{D}) \tag{19}$$

$$\Leftrightarrow e_1^B + e_2^B \frac{2e_1^B + e_1^A}{e_2^A} \leq x^*(\mathcal{D}) \tag{20}$$

From this, we obtain (9).

Part 2: Let us consider an interior equilibrium. Since the good 1 consumption of agent B is positive, Lemma 1 implies that $(p_1e_1^B + p_2e_2^B)^2 > 4\mathcal{D}p_1^2$ and $V(w_B, p_1) \geq \mathcal{D}\ln(w_B)$.

The first step is to determine the relative price $X \equiv p_2/p_1$. To do so, we consider market equilibrium for good 1.

$$\frac{p_1e_1^B + p_2e_2^B + \sqrt{(p_1e_1^B + p_2e_2^B)^2 - 4\mathcal{D}p_1^2}}{2p_1} + \frac{p_1e_1^A + p_2e_2^A}{2p_1} = e_1^B + e_1^A \tag{21a}$$

$$\Longleftrightarrow \sqrt{(p_1e_1^B + p_2e_2^B)^2 - 4\mathcal{D}p_1^2} = p_1(e_1^B + e_1^A) - p_2(e_2^B + e_2^A) \tag{21b}$$

Since $(p_1e_1^B + p_2e_2^B)^2 > 4\mathcal{D}p_1^2$, this equation is equivalent to

$$\begin{cases}(e_1^B + e_2^BX)^2 - 4\mathcal{D} = (e_1^B + e_1^A - (e_2^B + e_2^A)X)^2 \\ (e_1^B + e_1^A) - X(e_2^B + e_2^A) > 0\end{cases} \Leftrightarrow \begin{cases}F(X) = 0 \\ (e_1^B + e_1^A) - X(e_2^B + e_2^A) > 0\end{cases}$$

It means that the equation $F(X) = 0$ has at least one solution in $(0, \frac{e_1^B+e_1^A}{e_2^B+e_2^A})$. We find conditions under which this happens. Firstly, we compute

$$\begin{aligned}\Delta =&((e_1^B + e_1^A)(e_2^B + e_2^A) + e_1^Be_2^B)^2 - ((e_2^B + e_2^A)^2 - (e_2^B)^2)((e_1^B + e_1^A)^2 + 4\mathcal{D} - (e_1^B)^2) \\ =&\Big((e_2^B + e_2^A)e_1^B + e_2^B(e_1^B + e_1^A)\Big)^2 - 4\mathcal{D}\Big((e_2^B + e_2^A)^2 - (e_2^B)^2\Big).\end{aligned}$$

We observe that $F(0) = (e_1^B + e_1^A)^2 - (e_1^B)^2 + 4\mathcal{D} > 0$. We also compute that

$$F(\frac{e_1^B + e_1^A}{e_2^B + e_2^A}) = ((e_2^B + e_2^A)^2 - (e_2^B)^2)(\frac{e_1^B + e_1^A}{e_2^B + e_2^A})^2 - 2((e_1^B + e_1^A)(e_2^B + e_2^A) + e_1^Be_2^B)(\frac{e_1^B + e_1^A}{e_2^B + e_2^A})$$
$$+ (e_1^B + e_1^A)^2 + 4\mathcal{D} - (e_1^B)^2 = 4\mathcal{D} - (e_2^B\frac{e_1^B + e_1^A}{e_2^B + e_2^A} + e_1^B)^2.$$

We consider all possible cases.

(1) If $\Delta \equiv \Big((e_2^B+e_2^A)e_1^B+e_2^B(e_1^B+e_1^A)\Big)^2-4\mathcal{D}\Big((e_2^B+e_2^A)^2-(e_2^B)^2\Big) < 0$, the equation $F(X)$ has no solution, a contradiction.

(2) If $\Delta \equiv \Big((e_2^B+e_2^A)e_1^B+e_2^B(e_1^B+e_1^A)\Big)^2-4\mathcal{D}\Big((e_2^B+e_2^A)^2-(e_2^B)^2\Big) = 0$, the equation $F(X)$ has a unique solution and it is x^*. We see that

$$X^* \equiv \frac{(e_1^B + e_1^A)(e_2^B + e_2^A) + e_1^Be_2^B}{(e_2^B + e_2^A)^2 - (e_2^B)^2} > \frac{(e_1^B + e_1^A)(e_2^B + e_2^A)}{(e_2^B + e_2^A)^2} = \frac{e_1^B + e_1^A}{e_2^B + e_2^A} \tag{23}$$

a contradiction.

(3) We now assume that $\Big((e_2^B+e_2^A)e_1^B+e_2^B(e_1^B+e_1^A)\Big)^2-4\mathcal{D}\Big((e_2^B+e_2^A)^2-(e_2^B)^2\Big) > 0$

(3.a) If $\Big((e_2^B + e_2^A)e_1^B + e_2^B(e_1^B + e_1^A)\Big)^2 - 4\mathcal{D}(e_2^B + e_2^A)^2 \leq 0$, then the equation $F(X) = 0$ does not have solution in $(0, \frac{e_1^B+e_1^A}{e_2^B+e_2^A})$. Indeed, condition $\Big((e_2^B + e_2^A)e_1^B +$

$e_2^B(e_1^B + e_1^A)\Big)^2 - 4\mathcal{D}(e_2^B + e_2^A)^2 \leq 0$ is equivalent to $F(\frac{e_1^B+e_1^A}{e_2^B+e_2^A}) \geq 0$. We have

$$X^* \equiv \frac{(e_1^B + e_1^A)(e_2^B + e_2^A) + e_1^B e_2^B}{(e_2^B + e_2^A)^2 - (e_2^B)^2} > \frac{(e_1^B + e_1^A)(e_2^B + e_2^A)}{(e_2^B + e_2^A)^2} = \frac{e_1^B + e_1^A}{e_2^B + e_2^A}. \quad (24)$$

Since $F'(X^*) = 0$, the function $F(X)$ is strictly decreasing on $(0, \frac{e_1^B+e_1^A}{e_2^B+e_2^A})$. So, there is no solution in $(0, \frac{e_1^B+e_1^A}{e_2^B+e_2^A})$, a contradiction.

(3.b) If $\Big((e_2^B + e_2^A)e_1^B + e_2^B(e_1^B + e_1^A)\Big)^2 - 4\mathcal{D}(e_2^B + e_2^A)^2 > 0$, or equivalently,

$$e_1^B + \frac{e_1^B + e_1^A}{e_2^B + e_2^A} e_2^B - 2\sqrt{\mathcal{D}} > 0, \quad (25)$$

then the equation $F(X) = 0$ has a unique solution π^{int} in the interval $(0, \frac{e_1^B+e_1^A}{e_2^B+e_2^A})$. This is the smallest solution of the equation $F(X) = 0$.

We now look at the conditions. $(w_B)^2 > 4\mathcal{D}p_1^2$ and $V(w_B, p_1) \geq \mathcal{D}\ln(w_B)$, where $w_B = p_1 e_1^B + p_2 e_2^B$ is the income of agent B.

As in the proof of part 1, we have

$$(w_B)^2 > 4\mathcal{D}p_1^2 \Leftrightarrow e_1^B + e_2^B \frac{p_2}{p_1} > 2\sqrt{\mathcal{D}} \Leftrightarrow e_1^B + e_2^B \pi^{int} > 2\sqrt{\mathcal{D}} \quad (26)$$

$$V(w_B, p_1) \geq \mathcal{D}\ln(w_B) \Leftrightarrow g(e_1^B + e_2^B \frac{p_2}{p_1}) \geq g(x^*(\mathcal{D})) \Leftrightarrow e_1^B + e_2^B \pi^{int} \geq x^*(\mathcal{D}). \quad (27)$$

Finally, we get that $e_1^B + e_2^B \pi^{int} \geq x^*(\mathcal{D})$.

Remark 3. *Since $x^*(\mathcal{D}) > 2\sqrt{\mathcal{D}}$, condition $e_1^B + e_2^B \pi^{int} \geq x^*(\mathcal{D})$ implies that $e_1^B + e_2^B \pi^{int} > 2\sqrt{\mathcal{D}}$.*

□

Proof of Proposition 1. We mainly use Lemmas 1 and 2. Let us firstly prove point 1. The necessary condition follows point 1 of Lemma 2. Assume now that (9) is satisfied. We will prove that there exists a unique equilibrium, the relative price is $\frac{p_2}{p_1} = \pi^{cor} \equiv \frac{2e_1^B + e_1^A}{e_2^A}$ and $c_1^B = 0$. First, suppose that there is another equilibrium relative price. According to Lemma 2, it must be π^{int}. In this case, we have $e_1^B + e_2^B \pi^{int} \geq x^*(\mathcal{D})$. Since $\pi^{int} < \pi^{cor}$, we get that $e_1^B + e_2^B \pi^{cor} \geq x^*(\mathcal{D})$, a contradiction. Therefore, we obtain the uniqueness of equilibrium relative price.

Second, we prove that π^{cor} is an equilibrium relative price. Indeed, let $\frac{p_2}{p_1} = \pi^{cor} \equiv \frac{2e_1^B + e_1^A}{e_2^A}$. Since $e_1^B + e_2^B \frac{2e_1^B + e_1^A}{e_2^A} \leq x^*(\mathcal{D})$, we have that $(c_1^B, c_2^B) = (0, p_1 e_1^B/p_2 + e_2^B)$ is an optimal solution to agent B's maximization problem. It is easy to see that $(\frac{p_1 e_1^A + p_2 e_2^A}{2p_1}, \frac{p_1 e_1^A + p_2 e_2^A}{2p_2})$ is the unique solution to agent A's maximization problem.

The market clearing condition for good 1 is

$$\frac{p_1 e_1^A + p_2 e_2^A}{2p_1} + c_1^B = e_1^B + e_1^A \quad (28)$$

which implies that $c_1^B = 0$ is the unique good 1 consumption of agent B at equilibrium.

We can prove point 2 by adopting a similar argument used in the proof of point 1. Point 3 is a direct consequence of Lemma 2.

□

Proof of Remark 2. Consider an equilibrium $(p_1, p_2, c_1^B, c_2^B, c_1^D, c_2^D)$. Since the utility function of agent D is $U^D(c_1^D, c_2^D) = c_2^D$, we have $c_1^D = 0, p_2 c_2^D = w_D \equiv p_1 e_1^D + p_2 e_2^D$. The good 1 market clearing condition implies that $c_1^B = e_1 \equiv e_1^B + e_1^D > 0$. Since the good 1 consumption of agent B is positive, Lemma 1 implies that $(p_1 e_1^B + p_2 e_2^B)^2 > 4\mathcal{D} p_1^2$ and $V(w_B, p_1) \geq \mathcal{D} \ln(w_B)$. Denote $X \equiv p_2/p_1$. Observe that

$$(w_B)^2 > 4\mathcal{D}p_1^2 \Leftrightarrow e_1^B + e_2^B \frac{p_2}{p_1} > 2\sqrt{\mathcal{D}} \Leftrightarrow e_1^B + e_2^B X > 2\sqrt{\mathcal{D}} \tag{29}$$

$$V(w_B, p_1) \geq \mathcal{D}\ln(w_B) \Leftrightarrow g(e_1^B + e_2^B \frac{p_2}{p_1}) \geq g(x^*(\mathcal{D})) \Leftrightarrow e_1^B + e_2^B X \geq x^*(\mathcal{D}). \tag{30}$$

We now determine the relative price $X \equiv p_2/p_1$. To do so, we consider market equilibrium for good 1.

$$\frac{p_1 e_1^B + p_2 e_2^B + \sqrt{(p_1 e_1^B + p_2 e_2^B)^2 - 4\mathcal{D}p_1^2}}{2p_1} + 0 = e_1^B + e_1^D \tag{31a}$$

$$\Longleftrightarrow \sqrt{(p_1 e_1^B + p_2 e_2^B)^2 - 4\mathcal{D}p_1^2} = p_1(e_1^B + 2e_1^D) - p_2 e_2^B \tag{31b}$$

$$\Longleftrightarrow \sqrt{(e_1^B + e_2^B X)^2 - 4\mathcal{D}} = (e_1^B + 2e_1^D) - e_2^B X \tag{31c}$$

Since $e_1^B + e_2^B X > 2\sqrt{\mathcal{D}}$, this is equivalent to

$$\begin{cases} (e_1^B + e_2^B X)^2 - 4\mathcal{D} = ((e_1^B + 2e_1^D) - e_2^B X)^2 \\ (e_1^B + 2e_1^D) - e_2^B X > 0 \end{cases} \Leftrightarrow e_2^B X = \frac{e_1^D(e_1^B + e_1^D) + \mathcal{D}}{e_1^B + e_1^D} < e_1^B + 2e_1^D$$

To sum up, parameters must satisfy

$$\begin{cases} \frac{e_1^D(e_1^B + e_1^D) + \mathcal{D}}{e_1^B + e_1^D} < e_1^B + 2e_1^D \\ e_1^B + e_2^B X \geq x^*(\mathcal{D}) \end{cases} \Leftrightarrow \begin{cases} \mathcal{D} < e_1^B + e_1^D \\ e_1^B + e_1^D + \frac{\mathcal{D}}{e_1^B + e_1^D} \geq x^*(\mathcal{D}) \end{cases}$$

Conversely, we can easily check that under these conditions, there exists an equilibrium whose relative price X is determined by $e_2^B X = \frac{e_1^D(e_1^B + e_1^D) + \mathcal{D}}{e_1^B + e_1^D}$.

□

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